ON ULTRAPRODUCTS OF BOOLEAN ALGEBRAS AND IRR SH703

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Annotated Content

This article continues Magidor Shelah [MgSh 433] and Shelah Spinas [ShSi 677], but does not rely on them: see [M2] for the background.

§1 Consistent inequality

1.1 Definition. Assume $\mu < \lambda, \lambda$ is strongly inaccessible Mahlo. Let $B^* = B_\lambda$ be the Boolean algebra freely generated by $\{x_\alpha : \alpha < \lambda\}$ and for $u \subseteq \lambda$ let B_u be the subalgebra of B^* generated by $\{x_\alpha : \alpha \in u\}$.
1) We define a forcing notion $\mathbb{Q} = \mathbb{Q}^1_{\mu,\lambda}$ as follows:

$p \in \mathbb{Q}$ iff: p has form (w^p, B^p), we may write $(w[p], B[p])$ for typographical reasons, satisfying:

- (i) $w^p = w[p] \subseteq \lambda$
- (ii) $B^p = B[p]$ is a Boolean algebra of the form $B_{w[p]}/I^p$ where $I^p = I[p]$ is an ideal of $B_{w[p]}$, so B^p is generated by $\{x_\alpha/I^p : \alpha \in w^p\}$
- (iii) $x_\alpha/I^p \notin \langle\{x_\beta/I^p : \beta \in w^p \cap \alpha\}\rangle_{B[p]}$, equivalently $x_\alpha \notin \langle\{x_\beta : \beta \in w^p \cap \alpha\} \cup I^p\rangle_{B_{w[p]}}$
- (iv) for every strongly inaccessible $\chi \in (\mu, \lambda]$ we have $|w^p \cap \chi| < \chi$.

The order is given by $p \le q$ iff $w^p \subseteq w^q$ and $I^p = I^q \cap B_{w[q]}$, so, abusing notation, we pretend that $B^p \subseteq B^q$, not distinguishing sometimes x_α from $x_\alpha/I^p \in B^p$ or (see below) from $x_\alpha/\underset{\sim}{I}$ in $\underset{\sim}{B}$.
2) We define $\underset{\sim}{I} = \cup\{I^p : p \in \underset{\sim}{G}_{\mathbb{Q}^1_{\mu,\kappa}}\}$ and $\underset{\sim}{B}$ is defined as $B_\lambda/\underset{\sim}{I}$.

1.2 Claim. *For $\mu < \lambda$ as in Definition 1.1, the forcing notion $\mathbb{Q}^1_{\mu,\lambda}$ is μ^+-complete (hence, adds no new subsets to μ), has cardinality λ, satisfies the λ-c.c., collapse no cardinal, changes no cofinality, so cardinal arithmetic which holds after the forcing is clear.*

Proof. Like the proof of the same facts for Easton forcing.

1.3 Claim. *For the forcing $\mathbb{Q} = \mathbb{Q}^1_{\mu,\lambda}$ with μ, λ as in Definition 1.1 we have*
1) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{B}$ is a Boolean Algebra generated by $\{x_\alpha : \alpha < \lambda\}$ such that $\alpha < \lambda \Rightarrow x_\alpha \notin \langle\{x_\beta : \beta < \alpha\}\rangle_{\underset{\sim}{B}}$, so $|\underset{\sim}{B}| = \lambda$ and $\lambda = \cup\{w^p : p \in \underset{\sim}{G}_{\mathbb{Q}}\}$".
2) $\Vdash_{\mathbb{Q}}$ "$irr^+(\underset{\sim}{B}) = \lambda = irr(\underset{\sim}{B})$", see Definition 1.4 below.
3) $\Vdash_{\mathbb{Q}}$ "if $y_\beta \in \underset{\sim}{B}$ for $\beta < \lambda$ then for some $\beta_0 < \beta_1 < \beta_2 < \lambda$ we have $\underset{\sim}{B} \models y_{\beta_1} \cap y_{\beta_2} = y_{\beta_0}$".
4) Let B^* be a finite Boolean algebra generated by $\{a^*, b^*, y^*_0, \ldots, y^*_{n(*)}\}$ such that $y^*_m \notin \langle\{y^*_\ell : \ell < m\} \cup \{a^*, b^*\}\rangle_{B^*}$ and $0 < a^* < y^*_m < b^* < 1$ for $m \in \{0, \ldots, n(*)\}$.

Then it is forced, ($\Vdash_{\mathbb{Q}^1_{\mu,\lambda}}$) *that:*

$\boxdot^{\lambda,n(*)}_{\underset{\sim}{B}}$ *if* $y_\beta \in \underset{\sim}{B}$ *for* $\beta < \lambda$ *and* $\beta \neq \gamma \Rightarrow y_\beta \neq y_\gamma$ *then we can find* a, b *in* $\underset{\sim}{B}$ *satisfying* $0 < a < b < 1$ *and* $\beta_0 < \ldots < \beta_{n(*)} < \lambda$ *such that*

(α) $\underset{\sim}{B} \models$ "$a < y_{\beta_\ell} < b$"

(β) *there is an embedding* f *of* B^* *into* $\underset{\sim}{B}$ *mapping* a^* *to* a, b^* *to* b *and* y_ℓ *to* $y^*_{\beta_\ell}$ *for* $\ell = 0, \ldots, n(*)$.

Recalling

1.4 Definition. For a Boolean algebra B let:
1) $X \subseteq B$ is called irredundant, if no $x \in X$ belongs to the subalgebra $\langle X \backslash \{x\} \rangle_B$ of B generated by $X \backslash \{x\}$.
2) $\mathrm{irr}^+(B) = \cup\{|X|^+ : X \subseteq B \text{ is irredundent}\}$.
3) $\mathrm{irr}(B) = \cup\{|X| : X \subseteq B \text{ is irredundent}\}$ so $\mathrm{irr}(B)$ is $\mathrm{irr}^+(B)$ if the latter is a limit cardinal and is the predecessor of $\mathrm{irr}^+(B)$ if the later is a successor cardinal.

Remark. Concerning 1.3, for the case $\kappa = \aleph_1$ see Rubin [Ru83], generally see [Sh 128], [Sh:e].

Proof of 1.3. 1) Should be clear.
2) Clearly for every $\chi < \lambda$ and $p \in \mathbb{Q}^1_{\mu,\lambda}$ we can find an $\alpha < \lambda$ such that $\alpha > \chi$ and $w^p \cap [\alpha, \alpha + \chi) = \emptyset$, hence we can find a q such that $p \leq q \in \mathbb{Q}^1_{\mu,\lambda}$ and $w^q = w^p \cup [\alpha, \alpha + \chi)$ and in B^q the set $\{x_\beta : \beta \in [\alpha, \alpha + \chi)\}$ is independent, hence $q \Vdash$ "$\mathrm{irr}^+(\underset{\sim}{B}) > \chi$". So we get $\Vdash$ "$\mathrm{irr}^+(B) \geq \lambda$. To prove equality use part (3).
3) Assume toward contradiction that $p \Vdash$ "$\langle \underset{\sim}{y}_\beta : \beta < \lambda \rangle$ is a counterexample". We can find for each $\beta < \lambda$ a quadruple $(p_\beta, n_\beta, \langle \alpha_{\beta,\ell} : \ell < n_\beta \rangle, \sigma_\beta)$ such that:

(*i*) $p \leq p_\beta \in \mathbb{Q}^1_{\mu,\lambda}$
(*ii*) $n_\beta < \omega$
(*iii*) $\alpha_{\beta,\ell} \in w^{p_\beta}$ increasing with ℓ
(*iv*) $\sigma_\beta(x_0, \ldots, x_{n_\beta - 1})$ is a Boolean term
(*v*) $p_\beta \Vdash$ "in $\underset{\sim}{B}$ we have $\underset{\sim}{y}_\beta = \sigma_\beta(x_{\alpha_{\beta,0}}, x_{\alpha_{\beta,1}}, \ldots, x_{\alpha_{\beta,n_\beta - 1}})$". Call the right-hand side y_β, so by part (1), without loss of generality, $\{\alpha_{\beta,\ell} : \ell < n_\beta\} \subseteq w^{p_\beta}$ hence y_β is a member of $B_{w[p_\beta]}$.

So we can choose a stationary $S \subseteq \{\chi : \chi \text{ strongly inaccessible}, \mu < \chi < \lambda\}$ and $n, \sigma, m, \langle \alpha_\ell : \ell < m \rangle, w, r$ such that for every $\beta \in S$ we have: $n_\beta = n$ & $\sigma_\beta = \sigma, \ell < m \Rightarrow \alpha_{\beta,\ell} = \alpha_\ell, \ell \in [m,n) \Rightarrow \alpha_{\beta,\ell} \geq \beta$ and $w^{p_\beta} \cap \beta = w$. Without loss of generality also $\alpha < \beta \in S \Rightarrow w^{p_\alpha} \subseteq \beta$. Without loss of generality

⊛ for β_0, β_1 in S the mapping $F_{\beta_0,\beta_1} = \mathrm{id}_w \cup \{\langle(\alpha_{\beta_0,\ell}, \alpha_{\beta_1,\ell}) : \ell < n\rangle\}$ induces an isomorphism g_{β_1,β_0} from the Boolean algebra $\langle\{x_\gamma : \gamma \in w\} \cup \{x_{\beta_0,\ell} : \ell < n\}\rangle_{B[p_{\beta_0}]}$ onto the Boolean algebra $\langle\{x_\gamma : \gamma \in w\} \cup \{x_{\beta_1,\ell} : \ell < n\}\rangle_{B[p_{\beta_1}]}$ that is g_{β_1,β_0} maps x_γ to x_γ for $\gamma \in w$ and maps $x_{\beta_0,\ell}$ to $x_{\beta_1,\ell}$ for $\ell < n$.

Choose in S three ordinals $\beta_0 < \beta_1 < \beta_2$ and we define $q \in \mathbb{Q}^1_{\mu,\lambda}$ such that $w^q = w[p_{\beta_0}] \cup w[p_{\beta_1}] \cup w[p_{\beta_2}]$ and B^q is the Boolean algebra generated by $\{x_\alpha : \alpha \in w[p_{\beta_0}] \cup w[p_{\beta_1}] \cup w[p_{\beta_2}]\}$ freely except the equations which hold in p_{β_ℓ} for each $\ell = 0,1,2$ and the equation $y_{\beta_1} \cap y_{\beta_2} = y_{\beta_0}$, in other words I^q is the ideal of B_{w^q} generated by $I[p_{\beta_0}] \cup I[p_{\beta_1}] \cup I[p_{\beta_2}] \cup \{y_{\beta_1} \cap y_{\beta_2} - y_{\beta_0}, y_{\beta_0} - y_{\beta_1} \cap y_{\beta_2}\}$. We should prove that $q \in \mathbb{Q}^1_{\mu,\lambda}$ and $I[q] \cap B_{w[p_{\beta_\ell}]} = I[p_{\beta_\ell}]$ for $\ell = 0,1,2$ (the rest: $p_{\beta_\ell} \leq q$ hence $p \leq q$ and $q \Vdash$ "$\underset{\sim}{y}_{\beta_\ell} = y_{\beta_\ell}$ for $\ell = 0,1,2$ and $y_{\beta_1} \cap y_{\beta_2} = y_{\beta_0}$" should be clear).

Let B_0 be the trivial Boolean algebra $\{0,1\}$.
For $w \subseteq \lambda$ and $f \in {}^w 2$ let $\hat{f}$ be the unique homomorphism from the Boolean algebra B_w freely generated by $\{x_\alpha : \alpha \in w\}$ to $\{0,1\}$ such that $\alpha \in w \Rightarrow \hat{f}(x_\alpha) = f(\alpha)$.
For $p^* \in \mathbb{Q}^1_{\mu,\lambda}$ let $\mathscr{F}[p^*] = \{f : f \in {}^{(w^{p^*})}2$ and $\{x_\alpha : f(\alpha) = 1\} \cup \{-x_\alpha : f(\alpha) = 0\}$ generates an ultrafilter of $B[p^*]\}$. For each $f \in \mathscr{F}[p^*]$ let $f^{[p^*]}$ be the homomorphism from $B[p^*]$ to B_0 induced by f, i.e., $f^{[p^*]}(x_\alpha) = f(\alpha)$ for every $\alpha \in w^{p^*}$. Clearly $\mathscr{F}[p^*]$ gives all the information on p^*. Define $u = w^{p_{\beta_0}} \bigcup w^{p_{\beta_1}} \bigcup w^{p_{\beta_2}}$ and let

$$\begin{aligned}\mathscr{F} = \big\{f : & f \in {}^u 2, \text{ and } \ell \leq 2 \Rightarrow f \restriction w[p_{\beta_\ell}] \in \mathscr{F}[p_{\beta_\ell}] \text{ and} \\ & B_0 \models \text{“}\hat{f}(\sigma(\langle x_{\beta_1,\ell} : \ell < n\rangle)) \cap \hat{f}(\sigma(\langle x_{\beta_2,\ell} : \ell < n\rangle)) \\ & = \hat{f}(\sigma(\langle x_{\beta_0,\ell} : \ell < n\rangle))\text{”}\big\}.\end{aligned}$$

We need to show that $\mathscr{F}$ is rich enough, clearly $\otimes_1 + \otimes_2$ below suffice.

$\bigotimes_1$ if $\ell \in \{0,1,2\}$ and $f_\ell \in \mathscr{F}[p_{\beta_\ell}]$ then there is an $f \in \mathscr{F}$ extending f_ℓ.

[Why? For $m = 0,1,2$ let p'_{β_m} be such that $B[p'_{\beta_m}]$ is the subalgebra of $B[p_{\beta_m}]$ generated by $\{x_\gamma : \gamma \in w[p_{\beta_m}]$ and $\gamma < \beta_m \vee \gamma \in \{\alpha_{\beta_m,0}, \dots, \alpha_{\beta_m,n-1}\}\}$. We define for $m = 0,1,2$ a homomorphism g_m from $B[p'_{\beta_m}]$ to B_0 such that: $\gamma \in w \Rightarrow g_m(x_\gamma) = f_\ell(\gamma)$ and $\gamma = \beta_{m,k} \Rightarrow g_m(x_\gamma) = f_\ell(\beta_{\ell,k})$. This is possible by ⊛ and let h_m be chosen as follows: it is $f_\ell^{[p_{\beta_\ell}]}$ if $\ell = m$ and it is chosen as any homomorphism from $B[p_{\beta_m}]$ to B_0 extending g_m if $m \in \{0,1,2\} \backslash \{\ell\}$, as $B[p'_{\beta_m}]$ is a subalgebra of $B[p_{\beta_m}]$ this clearly exists. Let $f_m \in {}^{w[p_{\beta_\ell}]}2$ for $m = 0,1,2$ be $f_m(\gamma) = h_m(x_\gamma)$; for $m = \ell$ the definitions are compatible; i.e., the definition of f_ℓ we have just given and the old one. Finally, let $f = f_0 \cup f_1 \cup f_2$. This is clearly a well defined function; now of the three conditions in the definition of $\mathscr{F}$, the first holds by the definition of u, the second by the choice of the h_m's and the third by the choice of the g_m's, it is easy to see $f_\ell \subseteq f \in \mathscr{F}$.]

$\bigotimes_2$ if $\ell \in \{0,1,2\}, \alpha \in w[p_{\beta_\ell}]$ then there are $f', f'' \in \mathscr{F}$ such that $f'(\alpha) \neq f''(\alpha)$ but $f' \restriction (\alpha \cap u) = f'' \restriction (\alpha \cap u)$.

[Why? As $p_{\beta_\ell} \in \mathbb{Q}^1_{\mu,\lambda}$ we can find $f'_\ell, f''_\ell \in \mathscr{F}[p_{\beta_\ell}]$ such that $f'_\ell(\alpha) \neq f''_\ell(\alpha)$ but $f'_\ell \restriction (\alpha \cap w[p_{\beta_\ell}]) = f'' \restriction (\alpha \cap w[p_{\beta_\ell}])$. Now for $m \in \{0,1,2,\} \backslash \{\ell\}$ recalling $\circledast$ above there are $f'_m \in \mathscr{F}[p_{\beta_m}]$ which extends $f'_\ell \circ F_{\beta_\ell,\beta_m}$ and $f''_m \in \mathscr{F}[p_{\beta_m}]$ which extends $f''_m \circ F_{\beta_\ell,\beta_m}$ in both cases this is shown as in the proof of $\otimes_1$. If $\ell = 0$, let $f' = f'_0 \cup f'_1 \cup f'_2 \in \mathscr{F}$ and let $f'' = f''_0 \cup f''_1 \cup f''_2 \in \mathscr{F}$; both memberships hold as in the proof of $\otimes_1$ and we are done. Also if $\alpha < \beta_\ell$ (so $\alpha \in w = \bigcap\limits_{m \le 2} w[p_{\beta_m}]$) the same proof works. So assume $\ell \neq 0, \alpha \notin w = \bigcap\limits_{m \le 2} w[p_{\beta_m}]$. If $(f'_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell}) = (f''_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell})$ let $f' = f'_0 \cup f'_1 \cup f'_2, f'' = f''_\ell \cup (f' \restriction (w[p_{\beta_0}] \cup w[p_{\beta_{3-\ell}}]))$, clearly O.K. So without loss of generality assume $(f'_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell}) = 0, (f''_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell}) = 1, \ell \in \{1,2\}$ and $\alpha \in w[p_{\beta_\ell}] \backslash w[p_{\beta_0}]$; and then choose $f' = f'_0 \cup f'_1 \cup f'_2$ as above and $f'' = f''_\ell \cup (f' \restriction (w[p_{\beta_0}] \cup w[\beta_{\beta_{3-\ell}}]))$. Now check; the main point is that as $\hat f'_{3-\ell}(y_{\beta_{3-\ell}}) = \hat f'_0(y_{\beta_0})$ we have $B_0 \models$ "$\hat f''(y_{\beta_1}) \cap \hat f''(y_{\beta_2}) = \hat f''(y_{\beta_\ell}) \cap \hat f''(y_{\beta_{3-\ell}}) = \hat f''_\ell(y_{\beta_\ell}) \cap \hat f'(y_{\beta_{3-\ell}}) = 1_{B_0} \cap \hat f'_{3-\ell}(y_{\beta_{3-\ell}}) = \hat f'_{3-\ell}(y_{\beta_{3-\ell}}) = \hat f'_0(y_{\beta_0}) = \hat f''(y_{\beta_0})$".

4) The proof is similar to that of the previous part (with a, b now in $p_{\beta_\ell} \restriction \beta_\ell$!). $\square_{1.3}$

1.5 Claim. *1) If $\mathbb{Q} = \mathbb{Q}^1_{\mu,\lambda} * \underset{\sim}{\mathbb{Q}}^2$ and $\Vdash_{\mathbb{Q}^1_{\mu,\lambda}}$ "$\underset{\sim}{\mathbb{Q}}^2$ satisfies the $(\lambda,3)$-Knaster condition (see below)", <u>then</u> $\Vdash_{\mathbb{Q}}$ "$irr^+(\underset{\sim}{B}) = \lambda$".*

2) If in $\mathbf{V}$ the condition $\boxdot^{\lambda,n(*)}_B$ from 1.3(4) holds and the forcing notion $\mathbb{Q}$ satisfies the (λ, n^*+1)-Knaster condition <u>then</u> also in $\mathbf{V}^{\mathbb{Q}}$ the condition $\boxdot^{\lambda,n(*)}_B$ holds. Hence if $\mathbb{Q} = \mathbb{Q}^1_{\mu,\lambda} * \underset{\sim}{\mathbb{Q}}^2$ and $\Vdash_{\mathbb{Q}_{\lambda,\mu}}$ "$\boxdot^{\lambda,n(*)}_{\underset{\sim}{B}}$ holds (see 1.3(4))" and $\Vdash_{\mathbb{Q}^1_{\lambda,\mu}}$ "$\underset{\sim}{\mathbb{Q}}^2$ satisfies the $(\lambda, n(*)+1)$-Knaster condition" <u>then</u> $\Vdash_{\mathbb{Q}^1_{\lambda,\mu} * \underset{\sim}{\mathbb{Q}}^2}$ "$\boxdot^{\lambda,n(*)}_{\underset{\sim}{B}}$".

3) In part (1) we even get the conclusion of Claim 1.3(3).

1.6 Definition. 1) The λ-Knaster condition says that among any λ members there is a set of λ members which are pairwise compatible. Recall that it is preserved by composition.
2) For $n^* \le \omega$, the (λ, n^*)-Knaster condition says that among any λ member there is a set of λ such that any $< 1 + n^*$ of them have a common upper bound.

Proof of 1.5. 1), 3) Clearly it suffices to prove (3).
This follows immediately by 1.3(3), in fact, just such $\mathbb{Q}^2$ preserves the properties mentioned there.
2) Similarly using 1.3(4). $\square_{1.5}$

1.7 Theorem. *Suppose*

- (a) $\mathbf{V}$ *satisfies GCH above* μ *(for simplicity)*
- (b) κ *is measurable,* $\kappa < \chi < \mu$
- (c) μ *is supercompact, Laver indestructible, more explicitly,*
  - $(*)$ *for some* $h_\ell : \mu \to \mathscr{H}(\mu)$, *(for* $\ell = 0, 1$*) we have* <u>*for every*</u> $(< \mu)$-*directed complete forcing* $\mathbb{Q}$, *cardinal* $\theta \ge \mu$ *and* $\mathbb{Q}$-*name* $\underset{\sim}{x}$ *of a subset of* θ, <u>*there is*</u> *in* $\mathbf{V}[G_{\mathbb{Q}}]$ *a normal ultrafilter* $\mathscr{D}$ *on* $[\theta]^{<\mu}$ *such that* $\prod\limits_{a \in [\theta]^{<\mu}} (h_1(a \cap \mu), h_2(a \cap \mu))/\mathscr{D} \cong (\theta, \underset{\sim}{x}[G_{\mathbb{Q}}])$
- (d) $\lambda > \mu$ *is strongly inaccessible, Mahlo and* λ^* *is such that* $\lambda^* = (\lambda^*)^\mu \ge \lambda$
- (e) D^* *is a normal ultrafilter on* κ.

<u>*Then*</u> *for some forcing notion* $\mathbb{P}$ *we have, in* $\mathbf{V}^{\mathbb{P}}$*:*

- (α) *forcing with* $\mathbb{P}$ *collapse no cardinal of* $\mathbf{V}$ *except those in the interval* (μ^+, λ)
- (β) *forcing with* $\mathbb{P}$ *adds no subsets to* χ, *preserves "*μ *is strong limit" and makes* $2^\mu = \lambda^*$
- (γ) μ *is strong limit of cofinality* κ *and* $\langle \mu_i : i < \kappa \rangle$ *is an increasing continuous sequence of strong limit cardinals with limit* μ
- (δ) *for each* $i < \kappa, \mu_i < \lambda_i \le \lambda^*_i = (\lambda^*_i)^{\mu_i} = 2^{\mu_i}$ *and we let* $\mu_\kappa = \mu, \lambda_\kappa = \lambda, \lambda^*_\kappa = \lambda^*$
- (ε) *for each* $i \le \kappa$ *we have:* B_i *is a Boolean algebra of cardinality* λ_i *and* $irr^+(B_i) = \lambda_i$
- (ζ) *for* $i < \kappa, \lambda_i$ *is a Mahlo cardinal even strongly inaccessible, but*
- (η) $\lambda = \lambda_\kappa$ *is* μ^{++} *(this in* $\mathbf{V}^{\mathbb{P}}$*)*

(θ) *$B = B_\kappa$ is isomorphic to $\prod\limits_{i<\kappa} B_i/D^*$, hence*

$\boxtimes$ $\text{irr}^+(B) = \lambda = \mu^{++}$ *so* $\text{irr}(B) = \mu^+$ *whereas* $\text{irr}(B_i) = \text{irr}^+(B_i) = \lambda_i$ *and* $\prod\limits_{i<\kappa} \lambda_i/D^* = \lambda$, *so* $\text{irr}(\prod\limits_{i<\kappa} B_i/D^*) < \prod\limits_{i<\kappa} \text{irr}(B_i)/D^*$.

Proof. Let $\mathbb{Q}_1 = \mathbb{Q}^1_{\mu,\lambda}$ and $\underset{\sim}{B}$ be from 1.2, let and for $Z \subseteq \lambda^*$ let $\mathbb{Q}_{2,Z}$ be $\{f : f$ a partial function from Z to $\{0,1\}$ with domain of cardinality $< \mu\}$ ordered by inclusion, let $\mathbb{Q}_2 = \mathbb{Q}_{2,\lambda^*}$ and let $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2$. Let $G = G_1 \times G_2 \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$ and let $\mathbf{V}_0 = \mathbf{V}, \mathbf{V}_1 = \mathbf{V}[G_1]$ and $\mathbf{V}_2 = \mathbf{V}[G] = \mathbf{V}_1[G_2]$.

$\boxtimes_0$ In $\mathbf{V}_2, \underset{\sim}{B}[G_1]$ is a Boolean algebra of cardinality λ with $\text{irr}^+(B) = \lambda$ and, for notational simplicity, with a set of elements λ.
[Why? In $\mathbf{V}_1, \underset{\sim}{B}[G_1]$ is like that by 1.3. Now as in $\mathbf{V}_1, \mathbb{Q}_2$ satisfies the (λ, n)-Knaster for every n hence clearly by 1.5 we are done.]

In $\mathbf{V}_2$ we have $2^\mu = \lambda^*$ and the cardinal μ is still supercompact, hence it is well known that

$\boxtimes_1$ for every $Y \subseteq 2^\mu$ for some normal ultrafilter $\mathscr{D}$ on μ and $\bar{Y} = \langle Y_i : i < \mu\rangle, Y_i \subseteq 2^{|i|}$ we have $\bar{Y}/\mathscr{D}$ is Y (i.e. $\bar{Y}/\mathscr{D} \in \mathbf{V}^\mu_2/\mathscr{D}$ and in the Mostowski Collapse of $\mathbf{V}^\mu_2/\mathscr{D}$ the element $\bar{Y}/\mathscr{D}$ is mapped to Y), hence $(2^\mu, Y, \mu, <)$ is isomorphic to $\prod\limits_{i<\mu} (2^{|i|}, Y_i, i, <)/\mathscr{D}$.

Again it is well known and follows from $\boxtimes_1$ that there is a sequence $\bar{\mathscr{D}}^0 = \langle \mathscr{D}^0_\zeta : \zeta < (2^\mu))^+\rangle$ of normal (fine) ultrafilters on μ satisfying: for each $\zeta < (2^\mu)^+$ the sequence $\bar{\mathscr{D}}^0 \restriction \zeta$ belongs to (the Mostowski collapse of) $\mathbf{V}^\mu_2/\mathscr{D}^0_\zeta$. In $\mathbf{V}_2$ we can code $\underset{\sim}{B} = \underset{\sim}{B}[G_1]$ and $\mathscr{P}(\mu)$ and $\bar{\mathscr{D}}^0 \restriction \kappa$ as a subset Y of $2^\mu = \lambda^*$ and get $\mathscr{D}, \bar{Y}$ as in $\boxtimes_1$ hence for some set $A \in \mathscr{D}$ of strongly inaccessible cardinals $> \chi$ there is a sequence $\langle (\mu_i, \lambda_i, B_i, \lambda^*_i) : i \in A\rangle$ such that:

$(*)_1$ for $i \in A$ we have $i = \mu_i < \lambda_i \le \lambda^*_i = (\lambda^*_i)^{\mu_i} < \mu, \lambda_i$ is weakly inaccessible, Mahlo, B_i is a Boolean algebra generated by $\{x_\alpha : \alpha < \lambda_i\}, x_\alpha \notin \langle\{x_\beta : \beta < \alpha\}\rangle_{B_i}$, $\text{irr}^+(B_i) = \lambda_i$ and, for notational simplicity, its sets of elements is λ_i

$(*)_2$ B is isomorphic to $\prod\limits_{i\in A} B_i/\mathscr{D}$ and $(\lambda^*, <) \cong \prod\limits_{i\in A} (\lambda^*_i, <)/\mathscr{D}$.

For $i \in \mu \backslash A$ choose $\mu_i, \lambda_i, \lambda^*_i, B_i$ such that $(*)_1$ holds such that $\mu_i \geq i$; why are there such λ_i, B_i? Just e.g. use $\lambda_{\mathrm{Min}(A\backslash i)}, B_{\mathrm{Min}(A\backslash i)}$.

Let $\mathscr{D}_i = \mathscr{D}^0_i$ for $i < \kappa$ and $\mathscr{D}_\kappa$ be the $\mathscr{D}$ as above. So $\mathscr{D}_i$ (for $i \leq \kappa$) is a normal ultrafilter on μ and we have $i < j \leq \kappa \Rightarrow \mathscr{D}_i \in \mathbf{V}^\mu_2/\mathscr{D}_j$, that is, there is a sequence $\bar g = \langle g_{i,j} : i < j \leq \kappa\rangle$ satisfying $g_{i,j} \in {}^\mu(\mathscr{H}(\mu))$ such that $\mathscr{D}_i$ is (the Mostowski collapse of) $g_{i,j}/\mathscr{D}_j \in \mathbf{V}^\mu_2/\mathscr{D}_j$.
All this was in $\mathbf{V}_2 = \mathbf{V}[G]$. So we have $\mathbb{Q}$-names $\underset{\sim}{\bar g} = \langle \underset{\sim}{g}_{i,j} : i < j \leq \kappa\rangle, \underset{\sim}{\bar{\mathscr{D}}} = \langle \underset{\sim}{\mathscr{D}}_i : i \leq \kappa\rangle$ and $\langle(\underset{\sim}{\mu}_i, \underset{\sim}{\lambda}_i, \underset{\sim}{B}_i, \underset{\sim}{\lambda}^*_i) : i < \mu\rangle$. As $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2, \mathbb{Q}_2$ satisfies the μ^+-c.c. and $\mathbb{Q}_1$ is μ^+-complete without loss of generality $\underset{\sim}{\bar g}$ is a $\mathbb{Q}_2$-name and $\bar g$ is from $\mathbf{V}[G_2]$. Hence without loss of generality $\bar g$ and similarly $\langle(\mu_i, \lambda_i, B_i, \lambda^*_i) : i < \mu\rangle$ belong to $\mathbf{V}[G_{2,Z}]$ where $G_{2,Z} = G_2 \cap \mathbb{Q}_{2,Z}$, as we could have forced first with $\{f \in \mathbb{Q}_2 : \mathrm{Dom}(f) \subseteq Z\}$ for some $Z \in [\lambda^*]^{\leq \mu}$. Let $\mathbb{P}(\underset{\sim}{\bar{\mathscr{D}}}, \underset{\sim}{\bar g})$ be (the $\mathbb{Q}$-name of the) Magidor forcing for $(\underset{\sim}{\bar{\mathscr{D}}}, \underset{\sim}{\bar g})$ (see [Mg4]). Let $\langle \underset{\sim}{\mu}_i : i < \kappa\rangle$ be the $\mathbb{P}(\underset{\sim}{\bar{\mathscr{D}}}, \underset{\sim}{\bar g})$-name of the increasing continuous κ-sequence converging to μ which the forcing adds and we can restrict ourselves to the case $\underset{\sim}{\mu}_0 > \chi$. Clearly clauses $(\alpha) - (\zeta)$ in the conclusion hold for $\mathbb{P} = \mathbb{Q} * \mathbb{P}(\underset{\sim}{\bar{\mathscr{D}}}, \underset{\sim}{\bar g})$. Now

$\boxtimes_2$ in $\mathbf{V}_2$, if $p \in \mathbb{P}(\bar{\mathscr{D}}, \bar g)$ and $p \Vdash$ "$\underset{\sim}{f} \in \prod\limits_{i<\kappa} \lambda_{\underset{\sim}{\mu}_i}$" <u>then</u> there are q, an extension of p in $\mathbb{P}(\bar D, \bar g)$ and $f \in \prod\limits_{i \in A^*} \lambda_i$ such that
$q \Vdash_{\mathbb{P}(\bar{\mathscr{D}}, \bar g)}$ "$\{i < \kappa : \underset{\sim}{f}(i) = f(\underset{\sim}{\mu}_i)\} \in D^*$".

[Why? By the properties of $\mathbb{P}(\bar{\mathscr{D}}, \bar g)$ there are a pure extension q_0 of p in $\mathbb{P}(\bar{\mathscr{D}}, \bar g)$ and/or sequence $\langle u_i : i < \kappa\rangle$ such that above q_0 we have: $\underset{\sim}{f}(i)$ depends just on $\langle \underset{\sim}{\mu}_j : j \in u_i \cup \{i\}\rangle$ where $u_i \subseteq i$ is finite. As D^* is a normal ultrafilter on κ, for some $a^* \in D^*$ and a finite $u \subseteq \kappa$ we have $i \in a^* \Rightarrow u_i = u$. So there is a q such that $\mathbb{P}(\bar{\mathscr{D}}, \bar g) \models q_0 \leq q$ and $q \Vdash$ "$\underset{\sim}{\mu}_j = \mu^*_j$" for $j \in u$, and so f is well defined.]

Let $G_3 \subseteq \mathbb{P}(\bar{\mathscr{D}}, \bar g)$ be generic over $\mathbf{V}_2$ and $\mathbf{V}_3 = \mathbf{V}_2[G_3]$ and let $\mu_i = \underset{\sim}{\mu}_i[G_3]$ so really $\langle \mu_i : i < \kappa\rangle$ is generic for $\mathbb{P}(\bar{\mathscr{D}}, \bar g)$. Now we shall show that:

$\boxtimes_3$ in $\mathbf{V}_3 = \mathbf{V}_2[G_3]$ we have

$$B \cong \prod_{i<\kappa} B_{\mu_i}/D^*.$$

[Why? In $\mathbf{V}_2$, by $(*)_2$ above there is an isomorphism F from B onto $\prod\limits_{i<\mu} B_i/\mathscr{D} = \prod\limits_{i\in A^*} B_i/\mathscr{D}_\kappa$, so let $F(x) = f_x/\mathscr{D}_\kappa$ with $f_x \in \prod\limits_{i\in A^*} \lambda_i$ for $x \in B$, i.e. $x \in \lambda$.
In $\mathbf{V}_3$ let $f'_x \in \prod\limits_{i<\kappa} \lambda_{\mu_i}$ be defined by $f'_x(i) = f_x(\mu_i)$ and we define a function F' from B, i.e. from λ to $\prod\limits_{i<\kappa} B_{\mu_i}/D^*$ by $F'(x) = f'_x/D^*$. Now $Y \in \mathscr{D} \Rightarrow \{i < \kappa : \mu_i \in Y\} = \kappa \mod J^{bd}_\kappa$ by the definition of $\mathbb{P}(\bar{\mathscr{D}}, \bar{g})$, so as F is one to one also F' is, and F' commute with the Boolean operations as F does; lastly F' is onto by $\boxtimes_2$.]

$\boxtimes_4$ if $i < \kappa$ then $\mathscr{H}(\mu_{i+1})^{\mathbf{V}_3}$ is the same as $\mathscr{H}(\mu_{i+1})^{\mathbf{V}_0^{\mathbb{P}_i}}$, for some μ_i-centered forcing notion from $\mathscr{H}(\mu_{i+1})$ (hence this forcing notion is λ_{μ_i}-Knaster).

[Why? Note that $\mathscr{H}(\mu_j)^{\mathbf{V}_2} = \mathscr{H}(\mu_j)^{\mathbf{V}_0}$ for $j \leq \kappa$. Also for each $i < \kappa$ in $\mathbf{V}_0$ there are $\mathscr{D}^i_j$, a normal ultrafilter on μ_i such that $(\bar{\mathscr{D}}^i, \bar{g}^i) = (\langle \mathscr{D}^i_j : j \leq i\rangle, \langle g_{j_1,j_2} \restriction \mu_i : j_1 < j_2 \leq i\rangle) \in \mathbf{V}$ is as above, i.e. $j_1 < j_2 \leq i \Rightarrow \mathscr{D}^i_{j_1} = g_{j_1,j_2}/\underset{\sim}{\mathscr{D}}^i_{j_2} \in \mathbf{V}^{\mu_i}/\underset{\sim}{\mathscr{D}}^i_{j_2}, g^i_{j_1,j_2} \in {}^{\mu_i}(\mathscr{H}(\mu_i))$ so $\mathbb{P}(\bar{\mathscr{D}}^i, \bar{g}^i)$ is as in [Mg4], and for some $G_{3,i} \subseteq \mathbb{P}(\langle \mathscr{D}^i_j : j \leq i\rangle, \langle g_{j_1,j_2} \restriction \mu_i : j_1 \leq \mu_2 \leq i\rangle)$ generic over $\mathbf{V}_0$ (equivalently over $\mathbf{V}_2$) we have $G_{3,i} \in \mathbf{V}_3$ and $\mathscr{H}(\mu_{i+1})^{\mathbf{V}_3} = \mathscr{H}(\mu_{i+1})^{\mathbf{V}_2[G_{3,i}]} = \mathscr{H}(\mu_{i+1})^{\mathbf{V}_0[G_{3,i}]}$. See [Mg4]. As $\mathbb{P}(\bar{D}^i, \bar{g}^i)$ is μ_i-centered, clearly $\boxtimes_4$ follows.]
So obviously (by 1.5)

$\boxtimes_5$ in $\mathbf{V}_3$, for each $i < \kappa$ we have B_{μ_i} is a Boolean algebra of cardinality λ_{μ_i}, $\mathrm{irr}^+(B_{\mu_i}) = \lambda_{\mu_i}, \lambda_{\mu_i}$ is weakly Mahlo.

Also in $\mathbf{V}[G_1]$, the forcing notion $\mathbb{Q}_2$ satisfies the λ-Knaster condition and in $\mathbf{V}_2 = \mathbf{V}[G_1, G_2]$, the forcing notion $\mathbb{P}(\bar{\mathscr{D}}, \bar{g})$ from [Mg4] is μ-centered hence satisfies the λ-Knaster hence

$\boxtimes_6$ in $\mathbf{V}_3, B$ is a Boolean Algebra of cardinality λ, a Mahlo cardinal and $\mathrm{irr}^+(B) = \lambda$.

Now let $\mathbb{R} = \mathrm{Levy}(\mu^+, < \lambda)^{\mathbf{V}} = \{f \in \mathbf{V} : \mathrm{Dom}(f) \subseteq \{(\alpha, \gamma) : \alpha < \lambda, \gamma < \mu^+\}, |\mathrm{Dom}(f)| \leq \mu$ and for $\gamma < \alpha$, we have $f(\alpha, \gamma) < 1 + \alpha\}$, ordered by inclusion. Clearly $\mathbb{R}$ satisfies the λ-Knaster condition, is μ^+-complete in $\mathbf{V}$ and also in $\mathbf{V}_1$. Let $G_{\mathbb{R}} \subseteq \mathbb{R}$ be generic over $\mathbf{V}_1$. Now in $\mathbf{V}[G_1, G_{\mathbb{R}}]$, the forcing notion $\mathbb{Q}_2$ has the same definition and same properties. Also (as in [MgSh 433], [ShSi 677]), in $\mathbf{V}[G_1, G_2, G_{\mathbb{R}}]$ the $\mathscr{D}_i (i \leq \kappa)$ are still normal ultrafilters on μ and the definition of $\mathbb{P}(\bar{\mathscr{D}}, \bar{g})$ gives the same forcing notion with the same properties and add the same family of subsets to κ (as $\mathscr{P}(\kappa)^{\mathbf{V}[G_1, G_2]} = \mathscr{P}(\kappa)^{\mathbf{V}[G_1, G_2, G_{\mathbb{R}}]}$).

So $G_{\mathbb{R}}$ is a subset of $\mathbb{R}$ generic over $\mathbf{V}[G_1, G_2, G_3]$. Also in $\mathbf{V}[G_1, G_2], \mathbb{R}$ satisfies the λ-Knaster condition and in $\mathbf{V}[G_1, G_2, G_{\mathbb{R}}], \mathbb{P}(\bar{\mathscr{D}}, \bar{g})$ is μ-centered hence satisfies the λ-Knaster condition. Let $\mathbf{V}_4 = \mathbf{V}_3[G_{\mathbb{R}}]$, so in $\mathbf{V}_4$ all the conclusions above holds but $\lambda = \mu^{++}$ hence $\text{irr}(B) = \mu^+$ whereas $\text{irr}^+(B)$ remains $\lambda = \mu^{++}$. So we are done. $\square_{1.7}$

1.8 Claim. *1) In the Theorem 1.7 we can replace*

> *"a Boolean algebra B of cardinality λ, $irr^+(B) = \lambda$" by e.g. "a λ-Souslin tree"*

The "λ strongly inaccessible Mahlo" is needed just for applying 1.3, etc., but for $\prod\limits_{i<\kappa} B_i/D^* \cong B$ it is not needed (any model M, with universe $\subseteq \lambda$ is O.K.)
2) We can apply the proof above to the proof in [Sh 128] hence to theories of cardinality $< \mu$ for simplicity in logics with Magidor Malitz quantifiers.

Proof. Similar to 1.7. $\square_{1.7}$

§2 Consistency for small cardinals

Theorem 2.1 generalizes 1.7 in some ways. First D^*, instead of being a normal ultrafilter on κ is just a normal filter which is large in an appropriate sense so later it will be applied to the case $\kappa = \aleph_1$ (after a suitable preliminary forcing). Second, we deal with a general model and properties. Thirdly, the forcing makes μ to $\beth_\kappa$ (and more)

2.1 Theorem. *Suppose*

- (a) $\mathbf{V}$ *satisfies GCH for every* $\mu' \geq \mu$ *(for simplicity)*
- (b) κ *is regular uncountable,* $\aleph_0 \leq \theta \leq \kappa < \chi < \mu < \vartheta < \lambda \leq \lambda^* = (\lambda^*)^\mu$, *say* $\vartheta = \mu^+$
- (c) μ *is supercompact, Laver indestructibly or just indestructibly* λ^**-hypermeasure (generally on such indestructibility see [GiSh 344], on the amount of hypermeasurable needed here see Gitik Magidor [GM])*
- (d) D^* *is a filter on* κ *including the clubs and if* f *is a pressing down function on* κ *then for some* $u \in [\kappa]^{<\theta}$ *we have* $\{\delta < \kappa : f(\delta) \in u\} \in D^*$
- (e) $\mathbb{Q}_1$ *is a* $(<\mu)$*-directed complete forcing,* $|\mathbb{Q}_1| \leq \lambda^*$ *and* $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{M}$ *is a model with universe* λ *and vocabulary* $\underset{\sim}{\tau} \in \mathscr{H}(\chi)$"
- (f) $\mathbb{R}$ *is a* μ^{++}*-complete forcing notion of cardinality* $\leq \lambda^*$
- (g) $\mathbb{Q}_2$ *is the forcing of adding* λ^* μ*-Cohen subsets to* μ *and* $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2$.

<u>Then</u> *for some forcing notion* $\mathbb{P}$ *we have* $\mathbb{Q}_1 \times \mathbb{Q}_2 \times \mathbb{R} \lessdot \mathbb{P}$ *and in* $\mathbf{V}^{\mathbb{P}}$:

- (α) *the forcing with* $\mathbb{P}$ *collapse no cardinal except those collapsed by* $\mathbb{Q}_1 \times \mathbb{R}$, *in fact* $\mathbb{P}/(\mathbb{Q}_1 \times \mathbb{Q}_2 \times \mathbb{R})$ *is* ϑ^-*-centered; i.e.,* $\mu^{+\alpha}$*-centered if* $\vartheta = \mu^{+\alpha+1}$
- (β) *forcing with* $\mathbb{P}$ *add no subset of* χ, *forcing with* $\mathbb{P}/(\mathbb{Q}_1 \times \mathbb{Q}_2 \times \mathbb{R})$ *satisfies* $\boxtimes^+_{\kappa,\mu,\vartheta,\lambda,\lambda^*}$ *from Definition 2.2 below as witnessed by* $\langle \underset{\sim}{\mu}_i : i < \kappa \rangle$
- (γ) $\mu_i = \underset{\sim}{\mu}_i[G_{\mathbb{P}}], \mu$ *is strong limit of cofinality* κ *and* $\langle \mu_i : i < \kappa \rangle$ *is an increasing continuous sequence of strong limit singulars with limit* μ *(and* $\mathscr{H}(\mu_{i+1})$ *satisfies a parallel of the statement* $\boxtimes_4$ *from the proof of 1.7),*
- (δ) *for each* $i < \kappa$ *we have* $\mu_i < \lambda_i \leq \lambda^*_i = (\lambda^*_i)^{\mu_i}$ *and* $\mu_\kappa = \mu, \lambda_\kappa = \lambda, \lambda^*_\kappa = \lambda^*$ *and* $(\mu_i, \lambda_i, \lambda^*_i)$ *is quite similar to* $(\mu, \lambda, \lambda^*)$ *(see proof), more specifically: in some intermediate universe* $\mathbf{V}_1$, *for some normal ultrafilter* $\mathscr{D}$ *on* μ *and* $F, F_* : \mu \to \mu$ *we have* $\prod\limits_{i<\mu}(F(i), <)/D \cong (\lambda, <), \lambda_i = F(\mu_i)$ *and*

$\prod_{i<\mu}(F_*(i),<)/\mathscr{D} \cong (\lambda^*,<)$ *and* $F_*(\mu_i) = \lambda^*_i$ *and we have* $\bar{M} = \langle M_i : i < \mu\rangle$ *and* M_i *a model with universe* λ_i *and vocabulary* τ*; and* $\prod_{i<\mu} M_i/\mathscr{D} \cong M$

(ε) *for* $i < \kappa$ *we have* $2^{\mu_i} = \lambda^*_i$ *and* $2^{\lambda^*_i} = \mu_{i+1}$

(ζ) $\prod_{i<\kappa} M_{\mu_i}/D^*$ *is isomorphic to* M *if* D^* *is a normal ultrafilter, in fact,* $\{\langle f(\mu_i) : i < \kappa\rangle/_{D^*} : f \in \mathbf{V}_1$ *and* $f \in \prod_{i<\mu} F(i)\}$ *is the universe of* $\prod_{i<\kappa} M_{\mu_i}/D^*$

(η) *for every* $f \in \prod_{i<\kappa} M_i/D^*$ *we can in* $\mathbf{V}_1$ *find* $\varepsilon(f) < \theta$ *and* $g_{f,\varepsilon} \in \prod_{i<\mu} F(i)$ *for* $\varepsilon < \varepsilon(f)$ *such that* $\{i < \kappa : \bigvee_{\varepsilon<\varepsilon(f)} f(i) = g_{f,\varepsilon}(\mu_i)\} \in D^*$

(θ) $\prod_{i<\kappa}(\lambda_i,<)/D^*$ *is* λ*-like linear ordering (not necessarily well ordering as possibly* $\theta > \aleph_0$*)*

(ι) *if* D^* *is a normal ultrafilter,* $\mathbb{Q}_1 = \mathbb{Q}^1_{\mu,\lambda}$ *(of 1.1) and* $\mathbb{R} = \text{Levy}(\mu,<\lambda)$*,* *<u>then</u> the conclusion on irr in 1.7 holds.*

2.2 Definition. 1) We say $\boxtimes_{\gamma,\mu,\vartheta,\lambda^*}(\mathbb{Q})$ or we say $\mathbb{Q}$ satisfies $\boxtimes_{\gamma,\mu,\vartheta,\lambda^*}$ (as witnessed by $(\underset{\sim}{\bar{\mu}},\mathscr{D})$ <u>if</u>:

(i) $\mathbb{Q}$ is a forcing notion of cardinality $\leq \lambda^*$

(ii) $\mathbb{Q}$ satisfies the ϑ-c.c.

(iii) $\mathbb{Q}$ (i.e. forcing with $\mathbb{Q}$) add a sequence $\langle \underset{\sim}{\mu}_i : i < \gamma\rangle$ of cardinals $< \mu$, strongly inaccessible in $\mathbf{V}$, strong limit in $\mathbf{V}^{\mathbb{Q}}$

(iv) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mu}_i\,(i < \gamma)$ is increasing continuous"

(v) $\mathscr{D}$ is a normal ultrafilter on μ

(vi) for every $p \in \mathbb{Q}$ for some $\beta < \gamma$ for every $A \in \mathscr{D}$ there is q satisfying $p \leq q \in \mathbb{Q}$ such that $q \Vdash$ "$\{\mu_i : \beta < i < \gamma\} \subseteq A$"

(vii) if γ is a limit ordinal then $\Vdash_{\mathbb{Q}}$ "$\mu = \bigcup_{i<\gamma} \underset{\sim}{\mu}_i$"

($viii$) in $\mathbf{V}^{\mathbb{Q}}$ we have $2^\mu = \lambda^*$ and μ is strong limit.

2) We say $\boxtimes^+_{\gamma,\mu,\vartheta,\lambda^*}(\mathbb{Q})$ or we say $\mathbb{Q}$ satisfies $\boxtimes^+_{\gamma,\mu,\vartheta,\lambda^*}$ (as witnessed by $(\underset{\sim}{\bar\mu}, f_\theta, f_{\lambda^*})$ if:

- (a) $\mathbb{Q}$ satisfies $\boxtimes_{\gamma,\mu,\vartheta,\lambda^*}$ as witnessed by $\underset{\sim}{\bar\mu} = \langle \underset{\sim}{\mu}_i : i < \gamma \rangle$
- (b) if $G \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$ then for every $\beta < \gamma$ we have $\mathscr{H}(\mu_{\beta+1})^{\mathbf{V}^{\mathbb{Q}}}$ is gotten from $\mathscr{H}(\mu_{\beta+1})^{\mathbf{V}}$ by a forcing $\mathbb{Q}_{\beta+1}$ which is like $\mathbb{Q}$ with (β, μ_β) here standing for (γ, μ) there.

Proof. Like the proof of 1.7 but we use [GM] instead of [Mg4]; note that $\vartheta = \mu^{+3}$ comes from making the forcing μ^{+3}-c.c. So the pure decision of $\mathbb{P}(\bar{\mathscr{D}}, \bar g)$ is changed accordingly. Of course, the change in the assumption on D^* also has some influence. $\square_{2.1}$

So we get e.g.
2.3 Conclusion: Assume $\mathbf{V}$ satisfies ZFC + μ is supercompact +"$\lambda > \mu$ is strong inaccessible".
1) For some forcing extension $\mathbf{V}^*$, for some ultrafilter D^* on ω_1 there is $\langle \lambda_i : i < \omega_1 \rangle$ such that:

- (i) for $i < \omega_1, \lambda_i$ is weakly inaccessible $< \beth_{\omega_1}$
- (ii) $\lambda = \beth^{++}_{\omega_1}$
- (iii) the linear order $\prod\limits_{i<\omega_1} (\lambda_i, <)/D^*$ is λ-like,
- (iv) λ_i is first weakly inaccessible $> \beth_i$.

2) In part (1) we have: for some sequence $\langle B_i : i < \omega_1 \rangle$ of Boolean algebras, each of cardinality $< \beth_{\omega_1}$ we have Length$(\prod\limits_{i<\omega_1} B_i/D^*) < \prod\limits_{i<\omega_1}$ Length$(B_i)/D^*$.
3) If λ in $\mathbf{V}, \lambda > \mu$ is Mahlo, replace (iv) by (iv)$'$ and we can demand in addition that for some sequence $\langle B_i : i < \omega_1 \rangle$ of Boolean algebra, $|B_i| =$ irr$(B_i) = \lambda_i$ we have irr$(\prod\limits_{i<\omega_1} B_i/D^*) = \beth^+_{\omega_1} < \prod\limits_{i<\omega_1} \lambda_i/D^*$ where

- $(iv)'$ λ_i is the first weakly inaccessible Mahlo cardinal $> \beth_i$.

Proof. 1) We start getting by forcing using a forcing notion from $\mathscr{H}(\mu)$ (see [Sh:f, Ch.XVI,2.5,p.793] and history there) a normal filter D^0 on ω_1 such that $\mathscr{P}(\omega_1)/D^*$

is layered[1] and $\diamondsuit_{\aleph_1} + 2^{\aleph_1} = \aleph_2$. Hence (see [FMSh 252] and history there) there is an ultrafilter D^* on ω_1 extending D as required in 2.1 clause (d) for $\kappa = \theta = \aleph_1$, that is: if $g \in {}^{\omega_1}\omega_1$ is pressing down on some member of D^* then for some $\alpha < \omega_1, \{\beta < \omega_1 : g(\beta) < \alpha\} \in D^*$. Next by forcing with some $\aleph_2$-complete μ-c.c. forcing notion of cardinality μ, we get Laver indestructibility (by [L]). Now apply 2.1 with $\kappa = \theta = \aleph_1, \mathbb{R} =$ Levy$(\mu^+, < \lambda), \mathbb{Q}_1$ trivial or for part (3) as in 1.1 recall that λ is inaccessible. Note that easily in $\mathbf{V}^{\mathbb{P}}, (\forall \lambda_1 < \lambda)(\lambda_1^{\aleph_0} < \lambda)$. The main new point is clause (iii) which follows by clause (η) of the conclusion of 2.1 and the previous sentence; see the proof of part (3).
2) The proofs in [MgSh 433] applies also in our changed circumstances.
3) But for irr the problem seems more involved. We use 2.5 below instead of 1.3 and note that $\mathbb{Q}_2, \mathbb{R}$ and the Gitik Magidor forcing $\mathbb{P}/(\mathbb{Q}_1 \times \mathbb{Q}_2 \times \underset{\sim}{\mathbb{R}})$ though not fully preserving $(*)_{\lambda,<\mu,\underset{\sim}{B}}$ of 2.5 below it still preserves enough as we now prove. So in $\mathbf{V}^{\mathbb{P}}$ let $f_\alpha/D^* \in \prod\limits_{i<\kappa} B_{\mu_i}/D^*$ so $f_\alpha \in \prod\limits_{i<\kappa} B_{\mu_i}$ for $\alpha < \lambda$. For each α we can find in $\mathbf{V}_2$ a sequence $\langle g_{\alpha,n} : n < \omega\rangle$ satisfying $g_{\alpha,n} \in \prod\limits_{i<\mu} B_i$ such that $\{i < \omega_1 : (\exists n)(f_\alpha(i) = g_{\alpha,n}(\mu_i))\} \in D^*$. Without loss of generality we have $A_{\alpha,n} = A_n$ where $A_{\alpha,n} = \{i < \omega_1 : f_\alpha(i) = g_{\alpha,n}(\mu_i)\}$, as $2^{\aleph_1} < \beth_{\omega_1} < \lambda =$ cf(λ). Now in $\mathbf{V}_1$, there is an isomorphism $\mathbf{j}$ from $\prod\limits_{i<\mu} B_i/\mathscr{D}$ onto B, so $\mathbf{j}(g_{\alpha,n}/\mathscr{D}) \in B$. In $\mathbf{V}_2[G_{\mathbb{R}}]$ we apply $(*)_{\lambda,\aleph_0,B}$ of 2.5 and find $\beta_0 < \beta_1 < \beta_2 < \beta_3 < \lambda$ such that $n < \omega \Rightarrow B \models \mathbf{j}(g_{\beta_0,n}/\mathscr{D}) = \sigma(\mathbf{j}(g_{\beta_0,n}/\mathscr{D}), \mathbf{j}(g_{\beta_0,n}/\mathscr{D}), \mathbf{j}(g_{\beta_3,n}/\mathscr{D}))$ where σ is the Boolean term $\sigma^*(x_0,x_1,x_2) = (x_0 \cap x_1) \cup (x_0 \cap x_2) \cup (x_1 \cap x_2)$. Hence

$$Y_n =^{df} \{\zeta < \mu : B_\zeta \models g_{\beta_0,n}(\zeta) = \sigma^*(g_{\beta_1,n}(\zeta), g_{\beta_2,n}(\zeta), g_{\beta_3,n}(\zeta))\} \in \mathscr{D}$$

hence $Y = \bigcap\limits_{n<\omega} Y_n \in \mathscr{D}$ hence for some $i^* < \kappa, (\forall i)[i^* \le i < \kappa \rightarrow \mu_i \in Y]$ but $\mu_i \in Y \Rightarrow (\forall n < \omega)[B_{\mu_i} \models g_{\beta_0,n}(\mu_i) = \sigma(g_{\beta_1,n}(\zeta), g_{\beta_i,n}(\zeta), g_{\beta_3,n}(\zeta))]$. As $A_{\beta_\ell,n} = A_n$ we are done. $\square_{2.3}$

2.4 Remark. 1) In 2.3(1),(2) without loss of generality $\beth_{\omega_1}$ is the limit of the first ω_1 (weakly) inaccessible.
2) In 2.3(3) without loss of generality $\beth_{\omega_1}$ is the limit of the first ω_1 Mahlo (weakly)

[1]it means that this Boolean algebra is $\bigcup\limits_{i<\omega_2} B_i^*, B_i^*$ is a Boolean algebra of cardinality $\aleph_1$, increasing continuous with i, and cf$(i) = \aleph_1 \Rightarrow B_i \lessdot \mathscr{P}(\omega_1)/D^*$

inaccessible. Can we omit Mahlo?
3) Of course, 2.3 is just one extreme variant.
4) If we would like to replace in 2.3, $\aleph_1$ by $\kappa = \kappa^{<\kappa} > \aleph_1$, we can use [FMSh 252], hence higher large cardinals.

2.5 Claim. *1) For $\mathbb{Q} = \mathbb{Q}^1_{\mu,\lambda}, \underset{\sim}{B}$ as in 1.3 we have, for $\tau < \mu$ it is forced ($\Vdash_{\mathbb{Q}^1_{\mu,\lambda}}$) that:*

$(*)_{\lambda,\tau,\underset{\sim}{B}}$ *if $y_{\alpha,\varepsilon} \in \underset{\sim}{B}$ for $\alpha < \lambda, \varepsilon < \tau$ <u>then</u> for some $\beta_0 < \beta_1 < \beta_2 < \beta_3$ we have $\varepsilon < \tau \Rightarrow y_{\beta_0,\varepsilon} = \sigma^*(y_{\beta_1,\varepsilon}, y_{\beta_2,\varepsilon}, y_{\beta_3,\varepsilon})$ where $\sigma^*(y_1,y_2,y_3) = (y_1 \cap y_2) \cup (y_1 \cap y_3) \cup (y_2 \cap y_3)$.*

2) If B is a Boolean algebra, $\tau < \lambda$ and $\mathbb{Q}^*$ is τ^+-complete (or just do not add new τ-sequence of ordinals $< |B|$) and satisfies the $(\lambda,4)$-Knaster property (i.e. among any λ conditions there are λ, any three of them has a common upper bound), <u>then</u> forcing by $\mathbb{Q}^*$ preserve $(*)_{\lambda,\tau,B}$.

Proof. 1) As in the proof of 1.3, again the point is checking $(*)_{\lambda,\tau,\underset{\sim}{B}}$ so let $p \Vdash$ "$\langle \underset{\sim}{y}_{\beta,\varepsilon} : \beta < \lambda, \varepsilon < \tau\rangle$ be a counterexample". For each $\alpha < \lambda$ choose p_α such that $p \le p_\alpha$ and $p_\alpha \Vdash$ "$\underset{\sim}{y}_{\alpha,\varepsilon} = y_{\alpha,\varepsilon}$" for $\varepsilon < \tau$ and without loss of generality $y_{\alpha,\varepsilon} \in B_{w[p_\alpha]}$ and choose $\alpha_{\beta,\zeta} \in w[p_\beta]$ for $\zeta < \zeta_\beta$ such that $y_{\beta,\varepsilon} \in \langle\{x_\gamma : \gamma \in \{\alpha_{\beta,\varepsilon} : \varepsilon < \zeta_\beta\}\rangle_{B[p_{\beta_\ell}]}$ for some $\zeta_\beta < \tau^+$ with $\alpha_{\beta,\varepsilon}$ increasing with ε, and let $\xi_\beta \le \zeta_\beta$ be such that $(\forall\varepsilon)[\alpha_{\beta,\varepsilon} < \beta \equiv \varepsilon < \xi_\beta]$. Let $y_{\beta,\varepsilon} = \sigma_{\beta,\varepsilon}(\ldots, x_{\alpha_{\beta,\varepsilon}}, \ldots)_{\varepsilon<\zeta_\beta}$ (so the term $\sigma_{\beta,\varepsilon}$ uses only finitely many of its variables). We choose S, w, r, etc., as in the proof there with $\xi \le \zeta, \langle \alpha_\varepsilon : \varepsilon < \xi\rangle, \langle \sigma_\varepsilon : \varepsilon < \tau\rangle$ replacing $m \le n, \langle \alpha_\ell : \ell < m\rangle, \sigma$.

We choose $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in S and it is enough to find $q \in \mathbb{Q}^1_{\mu,\lambda}$ such that $\ell < 4 \Rightarrow p_{\beta_\ell} \le q$ and $q \Vdash$ "$y_{\beta_0,\varepsilon} = \sigma(y_{\beta_1,\varepsilon}, y_{\beta_2,\varepsilon}, y_{\beta_3,\varepsilon})$ for $\varepsilon < \tau$". We define $u = \bigcup\limits_{\ell<4} w[p_{\beta_\ell}]$ and $\mathscr{F}$ as there, i.e.,

$$\begin{aligned}\big\{f :& f \in {}^u2, f \restriction w[p_{\beta_\ell}] \in \mathscr{F}[p_{\beta_\ell}] \text{ for } \ell < 4 \text{ and for some} \\ & \ell \in \{1,2,3\} \text{ we have} \\ & m \in \{0,1,2,3\}\backslash\{\ell\} \ \& \ \zeta < \mu \Rightarrow f(x_{\alpha_{\beta_0},\zeta}) = f(x_{\alpha_{\beta_m},\zeta})\big\}.\end{aligned}$$

Now check.
2) Straightforward. $\square_{2.5}$

Another case is, e.g.

2.6 Claim. *We can replace in all the results above irr*(B) *by* -cof(B).

Remark. Recall h-cof$^+(B) = \cup\{|Y|^+ : Y \subseteq B$ and $Y = \{a_\alpha : \alpha < |Y|\}$ satisfies $\alpha < \beta \Rightarrow \neg(a_\beta \le a_\alpha)$, see [M2, Th.18.1,p.226].

Proof. Similar just easier.

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