CLASSIFICATION THEORY FOR ELEMENTARY CLASSES WITH THE DEPENDENCE PROPERTY - A MODEST BEGINNING
SH715

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Abstract. Our thesis is that for the family of classes of the form $\mathrm{EC}(T), T$ a complete first order theory with the dependence property (which is just the negation of the independence property) there is a substantial theory which means: a substantial body of basic results for all such classes and some complimentary results for the first order theories with the independence property, as for the family of stable (and the family of simple) first order theories. We examine some properties.

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Annotated Content

[We consider indiscernible sequences $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ wondering whether they have an average type as in the stable case. We investigate for any such $\bar{\mathbf{b}}$ the set stfor($\bar{\mathbf{b}}$) of formulas $\varphi(\bar{x}, \bar{y})$ such that every instance $\varphi(\bar{x}, \bar{c})$ divide $\bar{\mathbf{b}}$ to a finite/co-finite sets. We also consider the set dpfor($\bar{\mathbf{b}}$) of formulas $\varphi(\bar{x}, \bar{y})$ which can divide $\bar{\mathbf{b}}$ only to finitely many intervals; this is always the case if T has the dependence property, i.e., dpfor($\bar{\mathbf{b}}$) $= \mathbb{L}_{\tau(T)}$. If T has the dependence property, indiscernible sequences behave reasonably while indiscernible sets behave nicely. Similar behavior occurs for $p \in \mathbf{S}(M)$ connected with indiscernible set $\bar{\mathbf{b}}$ which we call stable types. We then note the connection between unstable types, unstable $\varphi(x, y; \bar{c})$, and formulas $\varphi(x, y; \bar{c})$ with the independence property, i.e. on singletons.]

[Each indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$, has for each $\varphi = \varphi(\bar{x}, \bar{y})$ a characteristic number $n = n_{\bar{\mathbf{b}}, \varphi}$, the maximal number of intervals to which an instance $\varphi(\bar{x}, \bar{c})$ can divide $\bar{\mathbf{b}}$. We wonder what we can say about it.]

[For an indiscernible sequence $\langle \bar{b}_t : t \in I \rangle$ over a set A, if we increase the set a little, i.e. if $A' = A \cup B$ then not much indiscernability is lost. An easy case is: if I has cofinality $> |B| + |T|$ then for some end segment J of I the sequence $\langle \bar{b}_t : t \in J \rangle$ is an indiscernible sequence over A'.]

[We define perpendicularity and investigate its basic properties; any two mutually indiscernible sequences are perpendicular. E.g., (for theories with the dependence property) one indiscernible sequence can be perpendicular to at most $\geq |T|^+$ pairwise perpendicular indiscernible sequences. We then deal with $\mathbf{F}^{\rm sp}_{|T|^+}$-constructions.]

[Using constructions as above we show that we can build models controlling quite tightly the dual cofinality of such sequences where dual-cf($\bar{\mathbf{b}}, M$) $=$

Min$\{|B| : B \subseteq M$ and the average of $\bar{\mathbf{b}}$ over $B \cup \bar{\mathbf{b}}$ is not realized in $M\}$. That is, for any pairwise perpendicular $\langle \bar{\mathbf{b}}^\zeta : \zeta < \zeta^* \rangle$ we can find a model M including them with dual-cf$(\bar{\mathbf{b}}^\zeta, M)$ being any (somewhat large) pregiven regular cardinal).

Notation: 1) T is a first order theory in a vocabulary $\tau_T, \mathfrak{C}$ a monster model of T, $\mathbb{L}$ is first order logic so $\mathbb{L}_{\tau(T)}$ the first order language with vocabulary τ, i.e., the set of the first order formulas in that vocabulary. Let $\mathbb{L}(\tau(T)) = \mathbb{L}_{\tau(T)}$. We may write $\bar{a} \in A (\subseteq \mathfrak{C})$ for $\bar{a} \in {}^{(\ell g(\bar{a}))}A$.
2) Let φ, ψ, ϑ be first order formulas, $\varphi = \varphi(\bar{x})$ mean that $\bar{x}$ is a sequence of variables with no repetitions including all free variables of φ (usually $\bar{x} = \langle x_\ell : \ell < \ell g(\bar{x}) \rangle$). Let $\varphi(\bar{x}, \bar{y})$ mean that we have a sequence of variables $\bar{x}$ and parameters $\bar{y}$ where $\bar{x}\hat{\ }\bar{y}$ is with no repetitions; so $\varphi = \varphi(\bar{x}, \bar{y})$ is not exactly equality.
3) I, J denote linear orders (used to index indiscernible sequences or sets). We shall use $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ with I (or J) a linear order, and $\{\bar{b}_t : t \in I\} \subseteq {}^m\mathfrak{C}$ for some $m < \omega$. We use $\mathbf{I}, \mathbf{J}$ as subsets of ${}^m\mathfrak{C}$ for some m (not constant).
4) $\mathbf{t}$ denotes truth values, $\varphi^{\mathbf{t}}$ is φ if $\mathbf{t} =$ truth or one and is $\neg\varphi$ if $\mathbf{t} =$ false or 0. So $\varphi^{\text{if(statement)}}$ is $\varphi, \neg\varphi$ if the statement is true, false resp., and $\varphi^{\text{if}(i)}$ means $\varphi^{\text{if}(i=1)}$.
5) dcl(A) is the definable closure of A, acl(A) is the algebraic closure of A (inside $\mathfrak{C}$ or, here more interesting, $\mathfrak{C}^{\text{eq}}$).

* * *

Our main interest here is in (first order complete) theories with the dependence property, but in the beginning we do not always assume this.
This work is continued in [Sh 783].

§1 Indiscernible sequences and averages

We try to continue [Sh:c, Ch.II,4.13], but we do not rely on it so there is some repetition. In [Sh:c], the notion stable (complete first order) theories, the notions of indiscernible set and its average (and local versions of them) play an important role. In an unstable theory, indiscernible sequences are not necessarily indiscernible sets. Still for an indiscernible set $\mathbf{I}$ if T has the dependence property, the basic claim guaranteeing the existence of averages (any formula $\varphi(\bar{x}, \bar{a})$ divide $\mathbf{I}$ into a finite and a co-finite set) holds. Moreover, any $\varphi(\bar{x}, \bar{a})$ divides any indiscernible sequence into the union of $< n_\varphi$ convex sets. For any T, we can still look at the first order formulas $\varphi(\bar{x}, \bar{y})$ which behaves well, i.e. any $\varphi(\bar{x}, \bar{b})$ divide any indiscernible sequence $\bar{\mathbf{b}}$ to a finite/cofinite set.

In 1.3 - 1.9 + 1.9(c) + (d) we define the relevant notions: average type for an ultrafilter, $\mathrm{Av}(\mathbf{J}, D)$ or $\mathrm{Av}_\varphi(\mathbf{J}, D)$ or any average type for an indiscernible sequence $\mathrm{Av}(\mathbf{J}, \langle \bar{b}_t : t \in I \rangle)$ and majority (maj) for finite sequences (saying how the majority behave). We define the set of stable formulas for an indiscernible $\bar{\mathbf{b}}$ (stfor($\bar{\mathbf{b}}$); also dpfor($\bar{\mathbf{b}}$) which is $\mathbb{L}_{\tau(T)}$ for T with the dependence property), and state some basic properties.

We define a notion of distance ($< \omega$) between indiscernible sequences (as in [Sh:93]). Being of finite distance is an equivalence relation and this is related to canonical bases (of types, of indiscernible sets) which play important role for stable theories, hence we try to define parallels in 1.9, see 1.13(2).

Then we note a dichotomy for the types $p \in \mathbf{S}^m(M)$. Such a type p may be stable (see Definition 1.19, Claim 1.16 - 1.31); not only is then the type definable, but for every ultrafilter D on ${}^m M$ with $\mathrm{Av}(M, D) = p$, any indiscernible sequence constructed from D is an indiscernible set, and the definition of p comes from an appropriate finite large enough (Δ, k)-indiscernible set. If $p \in \mathbf{S}^m(M)$ is non-stable, that is not stable, then there is a partial order with infinite chains closely related to it. We note that if T is unstable with the dependence property, then some $\varphi(x, y, \bar{c})$ define a quasi order with infinite chains and also that if T is unstable some $\varphi(x, y; \bar{c})$ has the order property (though not necessarily the property (E) of Eherenfeucht, see [Eh57], some $\varphi(x_1, \ldots, x_n)$ is an irreflexive relation on some infinite set in some model of T). It may be hard now to see how he could have not defined stable, but without $\mathfrak{C}^{\mathrm{eq}}$, and no dichotomical theorem, looking for a reasonable property for which the proof works, it had been quite natural.

On the subject see [Sh:c, II].

By [Sh 10] the possible function $f_{T,\varphi(\bar{x},\bar{y})}(\lambda) = \ \mathrm{Sup}\{|\mathbf{S}^m_\varphi(A)| : |A| \le \lambda\}$ are characterizing (if $\varphi(\bar{x}; \bar{y})$ stable is: $\le \lambda$, if φ unstable without independence then $\mathrm{Ded}(\lambda)$ and if with the independence property 2^λ). Also $\varphi(\bar{x}, \bar{y})$ is unstable in T iff it has the order property iff $\varphi(\bar{x}; \bar{y})$ has the strict order property or the independence property. Hence T is unstable iff it has the order property or the strict order property even by a formula $\varphi'(x,\bar{y})$. But it follows there that even if T has both the independence property and the strict order property, if $\varphi(\bar{x},\bar{y})$ has the independence property some $\varphi'(x,\bar{y})$ has the independence property. Then Lachlan proves that if $\varphi(\bar{x},\bar{y})$ has the strict order property some $\varphi'(x,\bar{y})$ has the strict order property.

More on "$\varphi(\bar{x},\bar{y})$ with the independence property", see Laskowski [Lw92].

The above settles $f_{T,\varphi}$, now for $f_T(\lambda) = \sup\{|\mathbf{S}(A)| : |A| \le \lambda\}$ this gives only $(f_{T,\varphi}(\lambda))^{|T|}$, Keisler [Ke76] show that if a countable T is unstable without the independence property but if $\mathrm{Ded}(\lambda)$ does not bound for some $\lambda \ge 2^{|T|}$ then $f_T(\lambda) = \mathrm{Ded}(\lambda)^{\aleph_0}$ for $\lambda \ge 2^{|T|}$. In [Sh:c, III] it is proved (for T not necessarily countable, unstable without the independence property) that for some $\kappa = \kappa_{\mathrm{ord}}(T) \le |T|^+, f_T(\lambda) = \mathrm{Ded}(\lambda)^{<\kappa_{\mathrm{Ord}}(T)}$ for $\lambda \ge 2^{|T|}$.

In [Sh 10], [Sh:a, II,§4], contains some claims on averages when T (or φ) has the dependence property continued here. On other possible dividing lines among unstable theories see [Sh 500, §2], ($\mathrm{SOP}_n, n \ge 3$), [DjSh 692], ($\mathrm{NOP}_n, n = 1,2$), [DjSh 710] (the oak property), [Sh 702].

We may look at $\mathrm{Th}(M, P_1, \dots, P_n)$ with P_ℓ a unary relation, this is connected to investigating a model of T in logics with extra quantifiers; see Baldwin Shelah [BlSh 156], Shelah [Sh 205], [Sh 284b], [Sh 284c].

Baldwin Benedikt [BlBn00] deal with the following. Let N be a model of T, T with the dependence property, we expand N by adding a unary predicate P interpreted as an indiscernible sequence, $\mathbf{I}$. We extend $(N, \mathbf{I})$ to an $|\mathbf{I}|^+$-saturated model and ask is (M, P) benign (see, e.g., [BBSh 815]).

Grossberg Lessman [GrLe00, §4] deals with the context: let T be complete first order theory, $\mathfrak{C}$ its monster model, p say a 1-type, now we deal with the order property, independence property and strict order property when we restrict the parameters and variables, i.e., satisfaction, to $p(\mathfrak{C})$, mainly generalizing parallels of results in [Sh:c], in particular the average $\varphi(x,\bar{y})$ has the order property iff it has the strict order property or the independence property[1]. Compare with [Sh:c, II].

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1.1 Context. T a complete first order theory, its monster model being $\mathfrak{C} = \mathfrak{C}_T$ as usual in [Sh:c] the monster, $\mathfrak{C}^{\mathrm{eq}}$ is when we add elements to designate equivalence classes.

1.2 Definition. 1) T has the dependence property or is dependent means it does not have the independence property whose definition is repeated below.

[1]so though the names are similar this is not specially related to §1, e.g., the notion of stable types are not related

2) T has the independence property if some formula $\varphi(\bar{x},\bar{y})$ has the independence property (in T), which means that for every n

$$\boxtimes^n_\varphi \quad \mathfrak{C} \models (\exists \bar{y}_0,\dots,\bar{y}_{n-1}) \bigwedge_{\eta \in {}^n 2} (\exists \bar{x})(\bigwedge_{\ell<n} \varphi(\bar{x},\bar{y}_\ell)^{\text{if}(\eta(\ell))}).$$

3) The formula $\varphi(\bar{x};\bar{y})$ has the dependence property (in T) if it does not have the independence property.

We can use below just $\mathbf{J}$ just of the form ${}^{\omega>}B$.

1.3 Definition. Consider a set of $\mathbf{I}$ of finite sequences of length m from the monster model $\mathfrak{C}$ where $m < \omega$, and an ultrafilter D over $\mathbf{I}$.
1) Let $\text{Dom}(D) = \mathbf{I}$.
2) For $\mathbf{J}$ a set of finite sequences from $\mathfrak{C}$ we let $\text{Av}(\mathbf{J},D)$ be $\{\varphi(\bar{x},\bar{a}) : \bar{x} = \langle x_\ell : \ell < m\rangle, \bar{a} \in \mathbf{J}$ and $\{\bar{b} \in \mathbf{I} :\models \varphi(\bar{b},\bar{a})\} \in D\}\}$. It will be called the D-average over $\mathbf{J}$ or the average type over $\mathbf{J}$ by D. If $\mathbf{J} = {}^{\omega>}B$ abusing our notation we may write B instead of $\mathbf{J}$ (or M if $B = |M|$). (Av stands for average).
3) $\text{Av}_\varphi(A,D)$ where $\varphi = \varphi(\bar{x},\bar{y})$ is the set of formulas of the form $\varphi(\bar{x},\bar{a})$ or the form $\neg\varphi(\bar{x},\bar{a})$ belonging to $\text{Av}(A,D)$ and $\text{Av}_\Delta(A,D)$ is the union of $\text{Av}_\varphi(A,D)$ for $\varphi \in \Delta$. Similarly $\text{Av}_\varphi(\mathbf{J},D)$, $\text{Av}_\Delta(\mathbf{J},D)$.
4) Let D be an ultrafilter on ${}^nB$ and $B \subseteq A$ and I is an infinite linear order.

We say $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I\rangle$ is based on D over A or $\bar{\mathbf{b}}$ is an (A,D)-indiscernible sequence or is a D-indiscernible sequence over A <u>if</u> for each $t \in I$ the type $\text{tp}(\bar{b}_t, A \cup \{\bar{b}_s : s <_I t\})$ is the average by D over $A \cup \{\bar{b}_s : s <_I t\}$. If $A = \cup\{\bar{c} : \bar{c} \in \text{Dom}(D)\}$, i.e. A is the domain of D we may write "$\bar{\mathbf{b}}$ is a D-indiscernible sequence" or $\bar{\mathbf{b}}$ is based on D. This makes sense for I finite but we shall mention it in such a case.
5) Let $p \in \mathbf{S}^m(B)$ and $B \subseteq A$. We say $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I\rangle$ is based on p over A <u>if</u> for some ultrafilter D on ${}^mB$ the sequence $\bar{\mathbf{b}}$ is based on D over A and $p = \text{Av}(B,D)$.

<u>1.4 Comment</u>: 1) If D is a principal ultrafilter on $\mathbf{J}$, say $\{\bar{b}^*\} \in D$ <u>then</u> $\text{Av}(B,D) = \text{tp}(\bar{b}^*,B)$.
2) If $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I\rangle$ is based on D over A <u>then</u> D is a principal ultrafilter iff $(\forall s,t \in I)(\bar{b}_s = \bar{b}_t)$.

1.5 Claim. *1) For D an ultrafilter on $\mathbf{I} \subseteq {}^m\mathfrak{C}$ and $A \subseteq \mathfrak{C}$ the set $\text{Av}(A,D)$ is a complete m-type over A, i.e., $\in \mathbf{S}^m(A)$ (and it does not split over $\cup\mathbf{I}$, see Definition 4.23).*
2) Let $p \in \mathbf{S}^m(A)$. <u>Then</u> p is finitely satisfiable in B <u>iff</u> $p = \text{Av}(A,D)$ for some ultrafilter D on ${}^mA$.
3) If $p \in \mathbf{S}^m(M)$ <u>then</u> p is finitely satisfiable in M.

Proof. Easy.

1.6 Claim. *Let D be an ultrafilter on ${}^m B$ and $B \subseteq A$.*
0) Assume that $\bar{\mathbf{b}}^1 = \langle \bar b^1_t : t \in I_1\rangle$ is based on D over A and $\bar{\mathbf{b}}^2 = \langle \bar b^2_t : t \in I_2\rangle$ and for any $n < \omega, t_1 <_{I_1} \ldots <_{I_1} t_n$ and $s_1 <_{I_2} \ldots <_{I_2} s_n$ we have $\mathrm{tp}(\bar b^1_{t_1} \hat{} \ldots \hat{} \bar b^1_{t_n}, A) = \mathrm{tp}(\bar b^2_{s_1} \hat{} \ldots \hat{} \bar b^2_{s_n}, A)$. Then $\bar{\mathbf{b}}^2$ is based on D over A, (see part (5)); (if I_1 is finite we should add $|I_2| \le |I_1|$).
1) For any linear order I there is $\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$ based on D over A.
2) If I is a linear order and $\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$ is based on D over A, then $\bar{\mathbf{b}}$ is an indiscernible sequence over A.
3) If $\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$ is based on D over A and $J \subseteq I$, then $\bar b \restriction J = \langle \bar b_t : t \in J\rangle$ is based on D over A; here we allow that J is finite.
4) If $\bar{\mathbf{b}}^\ell = \langle \bar b^\ell_t : t \in I\rangle$ is based on D over A for $\ell = 1,2$ then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ realizes the same type over A (in $\mathfrak{C}$); again we allow I to be finite.
5) If $\bar{\mathbf{b}}^\ell = \langle \bar b^\ell_t : t \in I_\ell\rangle$ is based on D over A and $n < \omega$ and $I_\ell \models t^\ell_1 < \ldots < t^\ell_n$ for $\ell = 1,2$, then the sequences $\bar b^1_{t^1_1} \hat{} \ldots \hat{} b^1_{t^1_n}$ and $\bar b^2_{t^2_1} \hat{} \ldots \hat{} \bar b^2_{t^2_n}$ realize the same type over A.
6) $p = \mathrm{Av}(A,D)$ does not split over B, which means that: if $\bar b, \bar c \in {}^m A, M < \omega$ and $\mathrm{tp}(\bar b, B) = \mathrm{tp}(\bar c, B)$ and $\varphi = \varphi(\bar x, \bar y)$ a formula, then $\varphi(\bar x, \bar b) \in p \Rightarrow \varphi(\bar x, \bar c) \in p$.

Proof. Easy, E.g.
1) If I is well ordered, choose $\bar b_t$ by induction on t. By compactness this holds for any I. $\square_{1.6}$

1.7 Definition. 1) For an infinite linear order I such that for some[2] infinite linear orders I_1, I_2 we have $I = J_1 + I_2$ and an indiscernible sequence $\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$, having $\ell g(\bar b_t) = m$ for $t \in I$, we define:

- (a) $\mathrm{stfor}_{\mathrm{pa}}(\bar{\mathbf{b}}) = \{\varphi(\bar x, \bar y, \bar c) : \ell g(\bar x) = m$, and for every $\bar a \in {}^{\ell g(\bar y)}\mathfrak{C}$, the set $\{t \in I : \mathfrak{C} \models \varphi(\bar b_t; \bar a, \bar c)\}$ is finite or the set $\{t \in I : \mathfrak{C} \models \neg\varphi(\bar b_t; \bar a, \bar c)\}$ is finite$\}$
  (stfor stands for stable formulas, pa stands for parameters)
- (b) $\mathrm{stfor}(\bar{\mathbf{b}}) = \{\varphi(\bar x; \bar y) : \varphi(\bar x, \bar y) \in \mathrm{stfor}_{\mathrm{pa}}(\bar{\mathbf{b}})$, i.e., no parameters$\}$
- (c) $\mathrm{dpfor}(\bar{\mathbf{b}}) = \{\varphi(\bar x, \bar y) : \ell g(\bar y) = m$ and for every $\bar a \in {}^{\ell g(\bar y)}\mathfrak{C}$ the set $\{t \in I : \mathfrak{C} \models \varphi[\bar a, \bar b_t]\}$ is a finite union of convex subsets of $I\}$.
  Let $\mathrm{dpfor}_{\mathrm{pa}}(\bar{\mathbf{b}})$ be defined similarly allowing parameters, obviously dpfor stands for "formula with the (relevant) dependence property"

[2]or demand in each of the definitions below that if J is a linear order extending I and $\bar b_t$ for $t \in J \backslash I$ such that $\langle \bar b_t : t \in J\rangle$ is an indiscernible sequence the condition on $\varphi(\bar x, \bar y)$ holds

(d) $\mathrm{dpfor}^n(\bar{\mathbf{b}})$ is defined similarly when the union is of $\le n$ convex sets; similarly is the other cases

(e) writing $\mathrm{stfor}_{\mathrm{pa}}(\bar{\mathbf{b}}) = \mathbb{L}(\tau)$ we mean the set of "relevant" formulas.

2) For a sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t < k\rangle$ with $\bar{b}_t \in {}^m\mathfrak{C}$, and formula $\varphi = \varphi(\bar{y}, \bar{z}), \ell g(\bar{y}) = m$, we define

$$\mathrm{maj}_\varphi(A, \langle \bar{b}_\ell : \ell < k\rangle) = \{\varphi(\bar{y}, \bar{c})^{\mathbf{t}} : \mathbf{t} \in \{\mathbf{true}, \mathbf{false}\},$$
$$\bar{c} \in {}^{\ell g(\bar{z})}A, \text{ and } |\{\ell :\models \varphi(\bar{b}_\ell, \bar{c})^{\mathbf{t}}\}| > k/2\}.$$

Clearly "$\varphi(\bar{y}, \bar{c}) \in \mathrm{maj}_\varphi(\mathfrak{C}, \langle \bar{b}_\ell : \ell < k\rangle)$" is a first order property of $\bar{c}$ (with the parameters $\bar{b}_0\hat{\ } \ldots \hat{\ }\bar{b}_{k-1}$).
If not said otherwise we use k odd so that we have "completeness".
(Note that maj stands for majority; this is not necessarily a type, just a set of formulas).
3) $E = E^k_{\varphi(\bar{y}, \bar{z})}$, where $\varphi(\bar{y}, \bar{z})$ is a formula in $\mathbb{L}_{\tau(T)}$, written $\bar{z}_1 E \bar{z}_2$ with $\ell g(\bar{z}_1) = \ell g(\bar{z}_2) = (\ell g(\bar{x})) \times k$ (written $(\bar{x}_1, \ldots, \bar{x}_n)$ instead of $\bar{x}_1\hat{\ } \ldots \hat{\ }\bar{x}_n$, abusing notation; normally k is odd) is defined as follows: $(\bar{x}_0, \ldots, \bar{x}_{k-1}) E (\bar{x}'_0, \ldots, \bar{x}'_{k-1}) =:$
$(\forall \bar{z})(\bigvee\limits_{u \subseteq k, |u| > k/2} \bigwedge\limits_{\ell \in u} \varphi(\bar{x}_\ell, \bar{z}) \equiv \bigvee\limits_{u \subseteq k, |u| > k/2} \bigwedge\limits_{\ell \in u} \varphi(\bar{x}'_\ell, \bar{z}))$.
Of course, it is an equivalence relation.

1.8 Claim. *1) If $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I\rangle$ is an infinite indiscernible sequence, $\ell g(\bar{b}_t) = m$ and $\varphi(\bar{y}; \bar{z}) \in \mathrm{stfor}(\bar{\mathbf{b}})$ so $\ell g(\bar{y}) = m$, <u>then</u> for every k large enough we have:*

(a) *for any $\bar{c}$ of length $\ell g(\bar{z})$, for some truth value $\mathbf{t}$ the set $\{t \in I :\models \varphi(\bar{b}_t, \bar{c})^{\mathbf{t}}\}$ has $< k/2$ members*

(b) *if $t_0, \ldots, t_{k-1}$ are distinct members of I <u>then</u> $\mathrm{maj}_\varphi(A, \langle \bar{b}_{t_\ell} : \ell < k\rangle) \in \mathbf{S}^m_\varphi(A)$ for every $A \subseteq \mathfrak{C}$, in fact for every nonprincipal ultrafilter D over $\{\bar{b}_t : t \in I\}$ and set A we have $\mathrm{maj}_\varphi(A, \langle \bar{b}_{t_\ell} : \ell < k\rangle)$ is a subset of $\mathrm{Av}(A, D)$, in fact is equal to $\mathrm{Av}_\varphi(A, D)$*

(c) *if $t_0, \ldots, t_{k-1} \in I$ with no repetitions and $s_0, \ldots, s_{k-1} \in I$ with no repetition <u>then</u> $(\bar{b}_{t_0}, \ldots, \bar{b}_{t_{k-1}})\, E^k_{\varphi(\bar{x}, \bar{y})} (\bar{b}_{s_0}, \ldots, \bar{b}_{s_{k_1}})$*

(d) *for some finite Δ we have: if $I', I \subseteq J$ where J is a linear order, $\bar{\mathbf{b}}' = \langle \bar{b}'_t : t \in J\rangle, \bar{\mathbf{b}}' \restriction I = \bar{\mathbf{b}}$ and $\bar{\mathbf{b}}'$ is Δ-indiscernible sequence, $|I'| \ge k$, <u>then</u>*
 - (α) *(b),(c) holds for $\bar{\mathbf{b}}' \restriction I'$ and*
 - (β) *$\varphi(\bar{y}, \bar{z}) \in \mathrm{stfor}(\bar{\mathbf{b}}' \restriction I')$.*

2) Let $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ *be an indiscernible sequence* $\varphi(\bar{y}; \bar{z}) \in \mathrm{dpfor}(\bar{\mathbf{b}})$ *so* $\ell g(\bar{y}) = m$ *then for some* $k = k_{\varphi, \bar{\mathbf{b}}}$*:*

- (a) *for any* $\bar{c}$ *of length* $\ell g(\bar{z})$ *and* $\mathbf{t}$ *the set* $\{t \in I : \varphi[\bar{b}_t, \bar{c}]^{\mathbf{t}}\}$ *is the union of* $\leq k$ *intervals.*

3) The k *above depends only on* $\langle p_n : n < \omega \rangle$ *where* p_n *is* $\mathrm{tp}(\bar{b}_{t_1} \hat{} \ldots \hat{} \bar{b}_{t_n}, \emptyset)$ *for any* $t_1 <_I \ldots <_I t_n$*.*

Proof. 1)

- (a) By compactness
- (b) just think of the definitions
- (c) follows from clause (b)
- (d) by compactness.

2), 3) Similarly. $\square_{1.8}$

1.9 Definition. Let $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ be an infinite indiscernible sequence.
1) We define (Cb stands for canonical bases, working in $\mathfrak{C}^{\mathrm{eq}}$):

- (a) for $\varphi(\bar{y}; \bar{z}) \in \mathrm{stfor}(\bar{\mathbf{b}})$ let $\mathrm{Cb}_{\varphi(\bar{y};\bar{z})}(\bar{\mathbf{b}})$ be $(\bar{b}_{t_0}, \ldots, \bar{b}_{t_{k-1}})/E^k_{\varphi(\bar{y},\bar{z})} \in \mathfrak{C}^{\mathrm{eq}}$, with $k = k_{\varphi(\bar{y},\bar{z})}(\bar{\mathbf{b}})$ minimal as in 1.8(1)(a) and any pairwise distinct $t_0, \ldots, t_{k-1} \in I$
- (b) $\mathrm{Cb}(\bar{\mathbf{b}}) = \mathrm{dcl}\{\mathrm{Cb}_{\varphi(\bar{y},\bar{z})}(\bar{\mathbf{b}}) : \varphi(\bar{y}, \bar{z}) \in \mathrm{stfor}(\bar{\mathbf{b}})\} \subseteq \mathfrak{C}^{\mathrm{eq}}$.

2) If I has no last element then

$$\begin{aligned}\mathrm{Av}_\varphi(A, \bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{a})^{\mathbf{t}} :\ & \text{for every large enough } t \in I \text{ we have} \\ & \mathfrak{C} \models \varphi(\bar{b}_t, \bar{a})^{\mathbf{t}} \text{ where } \mathbf{t} \text{ is a truth value}\}\end{aligned}$$

$$\mathrm{Av}_\Delta(A, \bar{\mathbf{b}}) = \cup\{\mathrm{Av}_\varphi(A, \bar{\mathbf{b}}) : \varphi \in \Delta\}$$

$$\begin{aligned}\mathrm{Av}(A, \bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{a}) :\ & \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\tau(T)}, \bar{a} \in {}^{\omega>}A \\ & \text{and } \models \varphi(\bar{b}_t, \bar{a}) \text{ for every large enough } t \in I\}.\end{aligned}$$

3) Let $\mathrm{Av}_{\mathrm{stfor}}(A, \bar{\mathbf{b}})$ be $\mathrm{Av}_\Delta(\bar{\mathbf{b}}, A)$ for $\Delta =$ stfor$(\bar{\mathbf{b}})$, similarly for replacements to stfor.
4) If the order on $I =$ Dom$(\bar{\mathbf{b}})$ does not matter then we can replace $\bar{\mathbf{b}}$ by $\{\bar{b}_t : t \in I\}$.

1.10 Claim. *[T is dependent]*
Assume I is a linear order with no last element and $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I\rangle$ an indiscernible sequence, $\ell g(\bar{b}_t) = m$.
1) dpfor$(\bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{y})$*: any φ and $\bar{y}$ but $\bar{x} = \langle x_\ell : \ell < m\rangle\}$.*
2) $\mathrm{Av}_\varphi(A, \bar{\mathbf{b}}) \in \mathbf{S}^m_\varphi(A)$*, see Definition 1.9(2), i.e., for every $\varphi(\bar{x}, \bar{c})$, for some* $\mathbf{t}$ *for every large enough t we have $\models \varphi(\bar{b}_t, \bar{c})^{\mathbf{t}}$.*
3) $\mathrm{Av}(A, \bar{\mathbf{b}}) \in \mathbf{S}^m(A)$*, see Definition 1.9(2).*
4) $\mathrm{Av}_{\mathrm{stfor}}(A, \bar{\mathbf{b}})$ *does not depend on the order of I (so by 1.7 we can use I a set or infinite linear order which is not necessarily endless).*
5) If $\bar{\mathbf{b}}$ is an (infinite) indiscernible set, then stfor$(\bar{\mathbf{b}}) =$ dpfor$(\bar{\mathbf{b}}) = \mathbb{L}_{\tau(T)}$ *(and* $\mathrm{stfor}_{\mathrm{pa}}(\bar{\mathbf{b}}) = \mathrm{dpfor}_{\mathrm{pa}}(\bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{y}, \bar{c}) : \bar{c} \subseteq \mathfrak{C}$*, (and of course $\ell g(\bar{x}) = m$)$\}$.*

Proof. Left to the reader or see [Sh:c, II.4.13] (for part (5) see $(A) \Rightarrow (B)$ in 1.28).

To formalize clause (d) of 1.8(1) let

1.11 Definition. 1) For a set Δ of formulas and $k \le \omega$ we say that $\langle \bar{b}^1_t : t \in I_1\rangle, \langle \bar{b}^2_t : t \in I_2\rangle$ are immediate (Δ, k)-nb-s (or the first is an immediate (Δ, k)-nb of the second over A) if:

- (a) both are (Δ, k)-indiscernible sequences of length $\ge k$
- (b) for some (Δ, k)-indiscernible sequence $\langle \bar{b}_t : t \in I\rangle$ and order preserving functions h_1, h_2 from I_1, I_2 into I respectively we have $t \in I_\ell \Rightarrow \bar{b}^\ell_{h_\ell(t)} = \bar{b}_t$ for $\ell = 1, 2$

(nb stands for neighbors).
2) The relation "being (Δ, k)-nb-s" is the closure of being an "immediate (Δ, k)-nb" to an equivalence relation. We say "of distance $\le n$" if there is a chain of immediate (Δ, k)-nb-s of length $\le n$ starting with one ending in the other. We may write Δ instead of (Δ, ω) and if $\Delta = \mathbb{L}_{\tau(T)}$ we may omit Δ.
3) We can replace k by $< k$ (and even $< \omega$).
4) We write (A, Δ, k) if the indiscernibility is over A (also in part (2)).
5) If $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are infinite indiscernible sequences, we say they are "essentially nb-s (over A)" if for every finite $\Delta \subseteq \mathbb{L}_{\tau(T)}$ and $k < \omega$ they are (Δ, k)-nb-s (they are (A, Δ, k)-nb-s).
6) If $\bar{\mathbf{b}}$ is an infinite indiscernible sequence over A we let $C_A(\bar{\mathbf{b}}) = \{\bar{b}$: for some $\bar{\mathbf{b}}'$

an essentially nb of $\bar{\mathbf{b}}$ over $A, \bar{b}$ appears in $\bar{\mathbf{b}}'\}$.
7) If $\bar{\mathbf{b}}$ is an infinite indisernible sequence over A we let $C'_A(\bar{\mathbf{b}}) = \{\bar{b} : \bar{b}$ appears in some $\bar{\mathbf{b}}'$, an infinite indiscernible sequence over A which is an A-nb of $\bar{\mathbf{b}}\}$.

1.12 Remark. $C'_A(\bar{\mathbf{b}})$ was defined in [Sh:93, Def.5.1](4).

1.13 Claim. *1) If $\langle \bar{b}_t : t \in I\rangle$ is an infinite indiscernible sequence and $\varphi(\bar{y}, \bar{z}) \in$ $\mathrm{dpfor}^n(\langle \bar{b}_t : t \in I\rangle)$, <u>then</u> for some finite Δ and k, for any (Δ, k)-nb sequence $\langle \bar{b}'_t : t \in I'\rangle$ of $\langle \bar{b}_t : t \in I\rangle$ we have $\varphi(\bar{y}, \bar{z}) \in \mathrm{dpfor}^n(\langle \bar{b}'_t : t \in I'\rangle)$.*
2) The result in (1) holds also for $\mathrm{stfor}^n(\langle \bar{b}_t : t \in I\rangle)$. If $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I\rangle$ is an infinite indiscernible sequence and $\varphi(\bar{y}, \bar{z}) \in \mathrm{stfor}(\bar{\mathbf{b}})$, <u>then</u> for some finite Δ and k for any (Δ, k)-nb $\bar{\mathbf{b}}'$ of $\bar{\mathbf{b}}$ we have $\mathrm{Cb}_{\varphi(\bar{y},\bar{z})}(\bar{\mathbf{b}}') = \mathrm{Cb}_{\varphi(y,\bar{z})}(\bar{\mathbf{b}})$ and $\mathrm{Av}_{\varphi(\bar{y},\bar{z})}(\mathfrak{C}, \bar{\mathbf{b}}') =$ $\mathrm{Av}_{\varphi(\bar{y},\bar{z})}(\mathfrak{C}, \bar{\mathbf{b}})$.
3) If $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$ are (Δ, k)-nb-s of distance $\le n$ for every finite Δ and $k < \omega$, <u>then</u> they are $\mathbb{L}_{\tau(T)}$-nb-s of distance $\le n$.
4) Like part (3) with a fixed k, i.e., if $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$ are (Δ, k)-nb-s of distance $\le n$ for every finite Δ, <u>then</u> they are $(\mathbb{L}_{\tau(T)}; k)$-nb-s of distance $\le n$.
5) If $\bar{\mathbf{b}}$ is an infinite indiscernible sequence, <u>then</u> $\mathrm{stfor}(\bar{\mathbf{b}}) = \cup\{\mathrm{stfor}^n(\bar{\mathbf{b}}) : n < \omega\}$.

Proof. Easy. (Use compactness for (3) and (4).)

* * *

1.14 Definition. Let $p \in \mathbf{S}^m(B)$.
1) We say $p \restriction \varphi$ is definable where $\varphi = \varphi(\bar{x}, \bar{y})$ and $\ell g(\bar{x}) = m$, <u>if</u> some $\psi(\bar{y}, \bar{c})$ define it with $\bar{c} \subseteq B$, where
2) We say $\psi(\bar{y}, \bar{c})$ defined $p \restriction \varphi$ where $\varphi = \varphi(\bar{x}, \bar{y})$ <u>if</u>:

(∗) for every $\bar{a} \in {}^{\ell g(\bar{y})}B$ we have
$\varphi(\bar{x}, \bar{a}) \in p \Leftrightarrow \mathfrak{C} \models \psi[\bar{a}, \bar{c}]$.

3) We say $p \restriction \varphi$ is Δ-definable if it is definable by some $\psi(\bar{y}, \bar{c})$ with $\psi(\bar{y}, \bar{z}) \in \Delta$.
4) We say $p \in \mathbf{S}^m(B)$ is definable <u>if</u> every $p \restriction \varphi$ is definable.

1.15 Claim. *The number of definable $p \in \mathscr{S}^m(B)$ is at most $\le (|B| + 2)^{|T|}$.*

Proof. For every definable $p \in \mathbf{S}^m(B)$ choose a sequence $\langle \psi_{\varphi,p}(\bar{y}, \bar{c}_{\varphi,p}) : \varphi = \varphi(x_0, \ldots, x_{m-1}; \bar{y}) \in \mathbb{L}_{\tau(T)}\rangle$ such that $\psi_{\varphi,p}(\bar{y}, \bar{c}_{\varphi,p})$ define $p \restriction \varphi$ and $\bar{c}_{\varphi,p} \subseteq B$. Now

(a) the number of such sequences is $\leq (|B|+2)^{|T|}$

(b) if $p_1, p_2 \in \mathbf{S}^m(B)$ and $\langle \psi_{\varphi,p_1}(\bar y, \bar c_{\varphi,p_1}) : \varphi(x_0,\dots,x_{m-1};\bar y) \in \mathbb{L}_{\tau(T)}\rangle$ is equal to $\langle \psi_{\varphi,p_2}(\bar y, \bar c_{\varphi,p_2}) : \varphi(x_0,\dots,x_{m-1};\bar y) \in \mathbb{L}_{\tau(T)}\rangle$, <u>then</u> $p_1 = p_2$.

Together we are done. $\square_{1.15}$

1.16 Claim. *1) Assume that $B \subseteq \mathfrak{C}, p \in \mathbf{S}^m(B)$ and D is an ultrafilter on ${}^m B$ and $\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$ is an infinite D-indiscernible sequence over B (see **?**(4)) such*

$\to$ *scite{np.1} undefined*

that $\mathrm{Av}(B,D) = p$. <u>Then</u> there is a function $\mathscr{F}$ from $\{(A,\Delta) : A \subseteq B$ is finite and $\Delta \subseteq \mathbb{L}_{\tau(T)}$ is finite$\}$ into D, but we may write $\mathscr{F}_\Delta(A)$ instead of $\mathscr{F}((A,\Delta))$ and $\mathscr{F}_\Delta(\bar b_0,\dots,\bar b_{\ell-1},A)$ instead of $\mathscr{F}_\Delta(\bar b_0 \cup \ldots \cup \bar b_{\ell-1} \cup A)$, such that:

(∗) *if $\alpha \leq \omega$ and for each $\ell < \alpha$ we have $\bar b_\ell \in {}^m B, \bar b_\ell \in \mathscr{F}_\Delta(\bar b_0,\dots,\bar b_{\ell-1},A)(\in D)$ <u>then</u> the sequence $\langle \bar b_t : t \in I\rangle \hat{}\langle \bar b_\ell : \ell \in \alpha^*\rangle$ (where the superscript $*$ in α^* means invert the order) is a Δ-indiscernible sequence over A.*

2) Let $p \in \mathbf{S}^m(B), D, \bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle, A = \emptyset$ be as in part (1). For any $\varphi(\bar y,\bar z)$ there are a finite $\Delta_\varphi \subseteq \mathbb{L}_{\tau(T)}$ and $k_\varphi < \omega$ [also large enough as in 1.8] such that: <u>if</u> $\bar b_\ell \in {}^m B$ for $\ell < k$ where $k \geq k_\varphi$ are as in part (1) for $\Delta = \Delta_\varphi$, <u>then</u> we have:

⊛ *if $\varphi(\bar y,\bar z) \in \mathrm{stfor}(\bar{\mathbf{b}})$ <u>then</u>*

(i) $\mathrm{Av}_\varphi(\mathfrak{C},\bar{\mathbf{b}}) = \mathrm{maj}_\varphi(\mathfrak{C},\langle \bar b_\ell : \ell < k\rangle) = \mathrm{Av}_\varphi(\mathfrak{C},D)$

(ii) *$p \restriction \varphi(\bar y,\bar z)$ and even $p^+ = \mathrm{Av}_\varphi(\mathfrak{C},D)$ is definable by a first order formula with parameters from B (see Definition 1.7(2)).*

3) Assume that $B_\ell, p_\ell, D_\ell, \bar{\mathbf{b}}^\ell = \langle b^\ell_t : t \in I_\ell\rangle$ are as in part (1) for $\ell = 1,2$ and $B_1 = B = B_2, p_1 = p = p_2, I_1 = I = I_2$. <u>Then</u> $\mathrm{Av}_{\varphi(\bar y,\bar z)}(\mathfrak{C},D_\ell)$ does not depend on ℓ provided that $\varphi(\bar y,\bar z) \in \mathrm{stfor}(\bar{\mathbf{b}}^1) \cap \mathrm{stfor}(\bar{\mathbf{b}}^2)$ and $B = M \prec \mathfrak{C}$; recall that $\mathrm{Av}_{\varphi(\bar y,\bar z)}(\mathfrak{C},\bar{\mathbf{b}}^\ell) = \mathrm{Av}_{\varphi(\bar y,\bar z)}(\mathfrak{C},D)$.

Proof of 1.16. 1) There is no harm with increasing I, so without loss of generality, (as we can supply appropriate $\bar b_t$'s) I has no last element. For simplicity without loss of generality Δ is closed under permuting and identifying the variables. Next we shall choose $\mathscr{F}_\Delta(A)$.

Let $n < \omega$ be above the number of free variables in any formula in Δ and let $t_0 < \ldots < t_{n-1} < t$ be in I. Now $\mathrm{tp}_\Delta(\bar b_t, A \cup \{\bar b_{t_\ell} : \ell < n\},\mathfrak{C})$ is a finite set of formulas and let $\psi(\bar x,\bar c)$ be its conjunction. So $\psi(\bar x,\bar c) \in \mathrm{tp}(\bar b_t, B \cup \{\bar b_s : s < t\},\mathfrak{C}) = \mathrm{Av}(B \cup \{\bar b_s : s < t\}, D)$ hence $\mathbf{J} = \{\bar b \in {}^m B : \mathfrak{C} \models \psi(\bar b,\bar c)\} \in D$.

Choose $\mathscr{F}_\Delta(A)$ as any such $\mathbf{J}$; why is $\mathscr{F}$ as required? So suppose that $A \subseteq B$ is finite and let $\bar{b}_\ell \in \mathscr{F}_\Delta(\bar{b}_0 \cup \ldots \cup \bar{b}_{\ell-1} \cup A)$ for $\ell < \ell(*)$ or for $\ell < \omega$. Clearly

$(*)_1$ $\bar{b}' \in \mathscr{F}_\Delta(\bar{b}_0 \cup \ldots \cup \bar{b}_{\ell-1} \cup A)$ implies that $[t_{n-1} < t \in I \Rightarrow \bar{b}_t, \bar{b}'$ realizes the same Δ-type over $A \cup \{\bar{b}_{t_0}, \ldots, \bar{b}_{t_{n-1}}\} \cup \{b_{\ell-1}, \ldots, \bar{b}_0\}]$.

But as $\langle \bar{b}_s : s \in I \rangle$ is an indiscernible sequence over B, and $\bar{b}', \bar{b}_{\ell-1}, \ldots, \bar{b}_0 \in B$, we can replace $t_0 < \ldots < t_{n-1} < t$, by any $t'_0 < \ldots < t'_{n-1} < t'$ from I, so

$(*)_2$ if $\bar{b}' \in \mathscr{F}_\Delta(\bar{b}_0 \cup \ldots \cup \bar{b}_{\ell-1} \cup A)$ and $t'_0 < \ldots < t'_{n-1} < t'$ in I then $\bar{b}', \bar{b}_{t'}$ realizes the same Δ-type over $A \cup \{\bar{b}_{t'_0}, \ldots, \bar{b}_{t'_{n-1}}\} \cup \{\bar{b}_0, \ldots, \bar{b}_{\ell-1}\}$.

But by the choice of n,

$(*)_3$ for any $\bar{b} \in \mathscr{F}_\Delta(\bar{b}_0 \cup \ldots \cup \bar{b}_{\ell-1} \cup A)$ we have $\mathrm{tp}_\Delta(\bar{b}, A \cup \{\bar{b}_s : s \in I\} \cup \{\bar{b}_{\ell-1}, \ldots, \bar{b}_0\}) = \cup\{\mathrm{tp}_\Delta(\bar{b}, A \cup \{\bar{b}_{s_0}, \ldots, \bar{b}_{s_{n-1}}\} \cup \{\bar{b}_{\ell-1}, \ldots, \bar{b}_0\}) : s_0 < \ldots < s_{n-1}$ are in $I\}$.

By induction on $\ell < \omega$ we prove that

$(*)_4$ for $k < \omega$ for any $m^i_0 < \ldots < m^i_{\ell-1} < \ell$ and $t^i_0 <_I \ldots <_I t^i_{k-1}$ for $i = 1, 2$ the Δ-type which $\bar{b}_{m^1_0} {}^\frown \ldots {}^\frown \bar{b}_{m^1_{\ell-1}} {}^\frown \bar{b}_{t^1_{k-1}} {}^\frown \ldots {}^\frown \bar{b}_{t^1_0}$ and $\bar{b}_{m^2_0} {}^\frown \ldots {}^\frown \bar{b}_{m^2_{\ell-1}} {}^\frown \bar{b}_{t^2_{k-1}} {}^\frown \ldots {}^\frown \bar{b}_{t^2_0}$ realizes over A are equal.

So together we are done.
2) In Clause (i) the first equality holds by Claim 1.8, Clause (a); that is such Δ exists by it. For the second equality, if $\varphi(\bar{x}, \bar{c})^{\mathbf{t}} \in \mathrm{Av}(\mathfrak{C}, D)$, then we can find $\bar{\mathbf{b}}' = \langle \bar{b}'_t : t \in I \rangle$ based on $(B \cup \bar{c}, D)$, by Claim 1.6(1). Apply the first equality with $\mathbf{b}'$ standing for $\bar{\mathbf{b}}$. Note that $\mathrm{tp}(\bar{\mathbf{b}}', B) = \mathrm{tp}(\bar{\mathbf{b}}, B)$ hence there is an automorphism of $\mathfrak{C}$ over B mapping $\bar{\mathbf{b}}$ to $\bar{\mathbf{b}}'$ hence we can in clause (i) replace $\bar{\mathbf{b}}$ by $\bar{\mathbf{b}}'$ (not changing $\langle \bar{b}_\ell : \ell < 2k \rangle$.
Clause (ii) follows from clause (i) as

$\boxtimes$ for $\bar{c} \in {}^{(\ell g(\bar{z}))}\mathfrak{C}$ we have: $\varphi(x, \bar{c}) \in p^+$ <u>iff</u> $\varphi(\bar{x}, \bar{c}) \in \mathrm{Av}(B \cup \bar{c}, \bar{\mathbf{b}})$ iff $\varphi(\bar{y}, \bar{c}) \in \mathrm{Av}_\varphi(\mathfrak{C}, \langle \bar{b}_\ell : \ell < k \rangle)$ iff $\mathfrak{C} \models \vartheta[\bar{c}, b_0, \ldots, b_{k-1}]$ where $\vartheta(y, b_0, \ldots, \bar{b}_{k-1}) = \bigvee\limits_{u \subseteq k, |u| \geq k/r} \bigvee\limits_{\ell \in u} \varphi(y, \bar{b}_\ell)$.
[Why? The second "iff" holds as $\mathrm{Av}_\varphi(\mathfrak{C}, \bar{\mathbf{b}})$ restricted to $B \cup \bar{c}$ is $p^+ \restriction \varphi$, by an assumption of our claim and similarly the first. The third iff holds by clause (i) which we have proved, the fourth iff holds by Definition 1.7(2).]

3) Obvious from clause (i) of part (2). $\square_{1.16}$

The notions "$p \in \mathbf{S}^m(B)$ is finitely satisfiable in B", "is definable", "has uniqueness" and "is stable" are closely related and important here; we shall define now the two later ones and investigate their relationships.

1.17 Definition. We say $p \in \mathbf{S}^m(B)$ has unique indiscernibles (or has uniqueness) when it is finitely satisfiable in B and

$\boxtimes$ if $(*)(a)$ D_ℓ is an ultrafilter on ${}^m B$ for $\ell = 1,2$

(b) $p = \mathrm{Av}(B, D_\ell) \in \mathbf{S}^m(B)$ for $\ell = 1,2$

(c) $\bar{\mathbf{b}}^\ell = \langle \bar{b}^\ell_t : t \in I \rangle$ is a (B, D_ℓ)-indiscernible sequence, see Definition 1.3(4), i.e., $\bar{b}^\ell_t$ realizes $\mathrm{Av}(B \cup \{\bar{b}^\ell_s : s <_I t, D\})$ for $t \in I$ where of course I is an infinite linear order

then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ realizes the same type over B.

1.18 Claim. *If $p \in \mathbf{S}^m(B)$ is definable and finitely satisfiable in B (the second follows from the first if $B = M$), then p has unique indiscernibles.*

Proof. For each formula $\varphi = \varphi(\bar{x}, \bar{y}), \ell g(\bar{y}) = m$ let $\psi_\varphi(\bar{y}, \bar{c}_\varphi)$ be such that:

(α) $\bar{c}_\varphi \subseteq B$

(β) for every $\bar{a} \in {}^{\ell g(\bar{y})}B$ we have $\varphi(\bar{x}, \bar{a}) \in p \Leftrightarrow \mathfrak{C} \models \psi_\varphi(\bar{a}, \bar{c}_\varphi)$.

Let D_1, D_2 be ultrafilters on ${}^m B$ such that $\mathrm{Av}(B, D_1) = p = \mathrm{Av}(B, D_2)$ and let $\langle b^\ell_t : t \in I \rangle$ be as in Definition 1.17.

Now we prove that

$(*)_n$ if $t_n <_I \ldots <_I t_1$ and $\varphi = \varphi(\bar{x}_n, \ldots, \bar{x}_1, \bar{z}), \ell g(\bar{x}_\ell) = m, \bar{d} \in {}^{\ell g(\bar{z})}B$ then $\mathfrak{C} \models \varphi[\bar{b}^1_{t_n}, \ldots, \bar{b}^1_{t_1}, \bar{d}] \equiv \varphi[\bar{b}^2_{t_n}, \ldots, \bar{b}^2_{t_1}, \bar{d}]$.

Let $\varphi(\bar{x}_n, \ldots, \bar{x}_1, \bar{y})$ and $\bar{d} \in {}^{\ell g(\bar{y})}B$ be given.
We define the formula $\varphi_k(\bar{x}_k, \bar{x}_{k-1}, \ldots, \bar{x}_1, \bar{z}_k)$ and $\bar{d}_k \in {}^{\ell g(\bar{z}_k)}B$ by downward induction on $k \le n$.

Case 1: $k = n$.
$\varphi_n(\bar{x}_n, \ldots, \bar{x}_1, \bar{d}_0) = \varphi(\bar{x}_n, \ldots, \bar{x}_1, \bar{d})$ so $\bar{d}_0 = \bar{d}$.

Case 2: $k < n$.

We have $\varphi_{k+1}(\bar{x}_{k+1},\ldots,\bar{x}_1,\bar{d}_{k+1})$, now as p is definable (and fixing some of the parameters causes no harm) there is $\varphi_k(\bar{x}_k,\ldots,\bar{x}_1,\bar{d}_k)$ with $\bar{d}_k \subseteq B$ of course, such that:

$\boxtimes^\ell$ for every $\bar{a}'_k,\ldots,\bar{a}'_1 \in {}^m B$ we have:

$$\varphi_{k+1}(\bar{x},\bar{a}'_k,\ldots,\bar{a}'_1,\bar{d}_{k+1}) \in p \Leftrightarrow \mathfrak{C} \models \varphi_k(\bar{a}'_k,\ldots,\bar{a}'_1,\bar{d}_k).$$

So we have carried the induction.
Now let $\ell \in \{1,2\}$ and let $A = \cup\{\bar{d}_k : k \le n\}$, it is a finite subset of B, and let Δ be any finite subset of $\mathbb{L}_{\tau(T)}$ which includes $\{\varphi_k : k \le n\}$ and let $\mathscr{F}^\ell_\Delta$ be as in 1.16 with D_ℓ, B, A, Δ here standing for D, M, A, Δ there and let $\bar{a}^\ell_i \in \mathscr{F}^\ell_\Delta(\langle \bar{a}^\ell_j : j < i\rangle \cup A) \subseteq {}^n B$ for $i < \omega$ and $\ell \in \{1,2\}$.

So $\langle \bar{a}^\ell_i : i < \omega\rangle$ is Δ-indiscernible over A, and clearly by 1.16(1)

$\boxtimes_1$ if $i_1 < \ldots < i_n < \omega, t_n <_I \ldots <_I t_1$ then
$\mathrm{tp}_\Delta(\bar{a}^\ell_{i_1}\char`^ \ldots \char`^\bar{a}^\ell_{i_n}, A) = \mathrm{tp}_\Delta(\bar{b}^\ell_{t_1}\char`^ \ldots \char`^\bar{b}^\ell_{t_n}, A)$.

Easily

$\boxtimes_2$ for $\ell \in \{1,2\}, k \le n$ and $i_1 < \ldots < i_{k+1}$ we have

$$\models \varphi_{k+1}[\bar{a}^\ell_{i_{k+1}},\ldots,\bar{a}^\ell_{i_1},\bar{d}_{k+1}] \equiv \varphi_k[\bar{a}^\ell_{i_k},\ldots,\bar{a}^\ell_{i_1},\bar{d}_k]$$

[Why? It suffices to note that $\models \varphi_{k+1}[\bar{a}^\ell_{i_{k+1}},\bar{a}^\ell_{i_k},\ldots,\bar{a}^\ell_{i_1},\bar{d}_{k+1}]$, <u>iff</u> $\{\bar{a} \in {}^m B : \varphi_{k+1}[\bar{a}, a^\ell_{i_{k+1}},\ldots,\bar{a}^\ell_1,\bar{d}_{k+1}]\} \in D$, <u>iff</u> $\varphi_{k+1}(\bar{x},\bar{a}^\ell_{i_k},\ldots,\bar{a}^\ell_{i_1},\bar{d}_{k+1}) \in p$, <u>iff</u> $\models \varphi_k(a^\ell_{i_{k+1}},\ldots,a^\ell_{i_1},\bar{d}_k)$.
Why? The first iff holds by the choice of $\mathscr{F}^\ell_\Delta(\langle \bar{a}^\ell_0,\ldots,\bar{a}^\ell_{i_{k+1}-1}\rangle)$, the second <u>iff</u> as $p = \mathrm{Av}(B, D_\ell)$ and the last iff by the choice of $\varphi_k, \bar{d}_k$.]

hence by the transitivity of $\equiv$

$\boxtimes_3$ $\models \varphi_n[\bar{a}^\ell_{i_n},\ldots,\bar{a}^\ell_{i_1},\bar{d}_n] \equiv \varphi_0[\bar{d}_0]$

but $\varphi_n = \varphi$ and $\bar{d}_n = \bar{c}$ so

$\boxtimes_4$ $\models \varphi[\bar{a}^\ell_{i_n},\ldots,\bar{a}^\ell_{i_1},\bar{c}] \equiv \varphi_0[\bar{d}_0]$.

So the truth value of $\varphi[\bar{a}^\ell_{i_n},\ldots,\bar{a}^\ell_{i_1},\bar{c}]$ does not depend on $\ell \in \{1,2\}$ so together with 1.16(1) we get $(*)_n$ so we are done. $\square_{1.17}$

1.19 Definition. 1) We call $p \in \mathbf{S}^m(B)$ a stable type <u>if</u>:

- (a) p is finitely satisfiable in B
- (b) for some ultrafilter D on ${}^m B$ satisfying $p = \text{Av}(B, D)$ and sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ based on D (see Definition 1.3(4) so I is infinite) $\bar{\mathbf{b}}$ is an indiscernible set over B.

2) We call $p \in \mathbf{S}(B)$ a non-stable type <u>if</u> it satisfies (a) above but not (b).
3) An infinite indiscernible sequence $\bar{\mathbf{b}}$ is stable/nonstable <u>if</u> it is an indiscernible set/it is not an indiscernible set over the empty set (see 1.28).

1.20 Claim. *Assume T is dependent.*
1) If $p \in \mathbf{S}^m(B)$ is stable <u>then</u> p is definable and has uniqueness.
2) If $p \in \mathbf{S}^m(B)$ is stable, <u>then</u> every infinite $\bar{\mathbf{b}}$ based on p is an indiscernible set, that is for every $D, \bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ as in Definition 1.19 the sequence $\bar{\mathbf{b}}$ is an indiscernible set over B.
3) The number of stable $p \in \mathbf{S}^m(B)$ is $\leq (|B| + 2)^{|T|}$.

Proof. 1) The type p is definable by 1.16(2)(ii), 1.10(5) and so has uniqueness by 1.18.
2) Let $\bar{\mathbf{b}}$ be based on D over B and let $D^1 = D$ and $\bar{b}^1_t = \bar{b}_t$. As p is a stable type there are $D', \langle \bar{b}'_t : t \in I' \rangle$ as in Definition 1.19. Let $D_2 = D'$ and we can find $\langle \bar{b}^2_t : t \in I \rangle$ such that: $\bar{b}^2_t$ realizes $\text{Av}(B \cup \{\bar{b}'_s : s <_I t\}, D')$ (by 1.6). Now clearly $I \models t_1 < \ldots < t_n, I' \models s_1 < \ldots < s_n$ implies $\text{tp}(\bar{b}'_{t_n} {}^\frown \ldots {}^\frown \bar{b}'_{t_1}, B) = \text{tp}(\bar{b}^2_{s_n} {}^\frown \ldots {}^\frown \bar{b}^2_{s_1}, B)$ for every n (by 1.6(5)) hence also $\langle \bar{b}^2_t : t \in I \rangle$ is an indiscernible set over B. By part (1) the sequences $\langle \bar{b}^1_t : t \in I \rangle, \langle \bar{b}^2_t : t \in I \rangle$ realizes the same type over B hence also $\langle \bar{b}^1_t : t \in I \rangle$ is an indiscernible set over B.
3) By part (1) and 1.15. $\square_{1.20}$

1.21 Claim. *1) Assume $p \in \mathbf{S}^m(B)$ has uniqueness. If for $\ell = 1, 2, D_\ell$ is an ultrafilter on ${}^m B$ satisfying $p = \text{Av}(B, D_\ell)$ and $\bar{b}^\ell = \langle \bar{b}^\ell_t : t \in I \rangle$ is based on D_ℓ, <u>then</u> $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ has distance ≤ 2.*
2) In part (1), instead uniqueness it suffices to assume that $n < \omega, t_0 <_I \ldots <_I t_{n-1}$ the sequences $\bar{b}^1_{t_0} {}^\frown \ldots {}^\frown \bar{b}^1_{t_{n-1}}$ and $\bar{b}^2_{t_0} {}^\frown \ldots {}^\frown \bar{b}^2_{t_{n-1}}$ realizes the same type.

Proof. Should be easy.
1) By the definition of uniqueness (see Definition 1.17) this follows by part (2).
2) By 1.13 it is enough to prove, for any finite $\Delta \subseteq \mathbb{L}_{\tau(T)}$ and $k < \omega$ that $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are (Δ, k)-nb-s of distance ≤ 2. This in turn follows by 1.16. $\square_{1.21}$

The definability of $p \restriction \varphi$ proved in 1.16(2) for $\varphi \in$ stfor$(\bar{\mathbf{b}})$ say more than stated. The defining formulas are canonical (i.e., the formula depends on φ and T, the parameters depend also on p).

1.22 Definition. Let $\varphi(\bar{x}, \bar{y})$ be a formula $\in \mathbb{L}_{\tau(T)}$ which has the dependence property (holds if T has it).
1) Let $k_1[\varphi(\bar{x}, \bar{y})] = k_T[\varphi(\bar{x}, \bar{y})]$ be the minimal $k < \omega$ such that there is no sequence $\langle \bar{a}_\ell : \ell < k \rangle$ with $\bar{a}_\ell \in {}^{\ell g(\bar{y})}\mathfrak{C}$ for $\ell < k$ such that

$$\mathfrak{C} \models \wedge_{\eta \in {}^n 2} (\exists \bar{x}) \wedge_{\ell < k} \varphi(\bar{x}, \bar{a}_\ell)^{\eta(\ell)}.$$

2) We say (k, Δ) is suitable to $\bar{\mathbf{b}}, \varphi$ if:

- (a) $\bar{\mathbf{b}}$ is a Δ-indiscernible sequence
- (b) the formulas in Δ have arity $< \ell g(\bar{\mathbf{b}})$
- (c) if $\bar{\mathbf{b}}'$ is a Δ-indiscernible sequence of the same Δ-type as $\bar{\mathbf{b}}, \ell g(\bar{\mathbf{b}}') > 2k$ <u>then</u> for any $\bar{a} \in {}^{\ell g(\bar{y})}\mathfrak{C}$ for some truth value $\mathbf{t}$ the set $\{t \in \text{Dom}(\bar{\mathbf{b}}') : \mathfrak{C} \models \varphi[\bar{b}'_t, \bar{a}]^{\mathbf{t}}\}$ has $< k$ members.

3) We say $(k, \Delta, \bar{n})$ is strongly suitable to $\bar{\mathbf{b}}, \varphi$ <u>if</u> $\bar{n} = \langle n_\ell : \ell \leq k \rangle$, (a), (b), (c) as above hold and

- (d) if $\bar{\mathbf{b}}', \langle \bar{b}'_t : t \in I \rangle$ is a Δ-indiscernible sequence of the same Δ-type as $\bar{\mathbf{b}}$ and $|I| > \sum\limits_{i<k} n_i$ <u>then</u> for no $\bar{a} \in {}^{\ell g(\bar{y})}\mathfrak{C}$ are there $t_{\ell,m} \in I$ (for $\ell \leq k, m < n_\ell$) with no repetitions satisfying $[t_{\ell_1,m_1} < t_{\ell_2,m_2} \equiv (\ell_1 < \ell_2 \vee (\ell_1 = \ell_2 \wedge m_1 < m_2))]$ such that for any $m_1, m_2 < n_\ell$ we have $\varphi[\bar{b}'_{t_{\ell_1,m_1}}, \bar{a}] \equiv \varphi[\bar{b}'_{t_{\ell_2,m_2}}, \bar{a}]$ iff $\ell_1 - \ell_2$ is even.

4) $\psi(\bar{y}, \bar{z}) = \psi_{k,\varphi(\bar{x},\bar{y})}(\bar{y}, \bar{z})$ is the canonical definition <u>if</u> it is as in the proof of 1.16(2).
5) Similarly for $\varphi(\bar{x}, \bar{y}, \bar{c})$.

Trivially

1.23 Claim. *In 1.16(2), if $\varphi(\bar{x}, \bar{y}) \in$ stfor$(\bar{\mathbf{b}})$, <u>then</u> $p \restriction \varphi$ is defined by $\psi_{k,\varphi(\bar{x},\bar{y})}(\bar{y}, \bar{c})$ for some $\bar{c} \subseteq B$ (of length $\ell g(\bar{z}_{\varphi(\bar{x},\bar{y})})$).*

* * *

Now it should be clear that

1.24 Claim. *If $p \in \mathbf{S}^m(B)$ is stable and $\bar{\mathbf{b}}$ is an infinite indiscernible set based on p, <u>then</u> for any automorphism F of $\mathfrak{C}^{\rm eq}$ which is the identity over* $\mathrm{Cb}(p)$, *the sequences $\bar{\mathbf{b}}, f(\bar{\mathbf{b}})$ are nb-s of distance ≤ 2. [See Definition 1.11.]*

We shall show

1.25 Claim. *[T is dependent] If $\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$ is an infinite indiscernible set over $\emptyset$ and an indiscernible sequence over A, <u>then</u> $\bar{\mathbf{b}}$ is an indiscernible set over A.*

Proof. By 1.28 below.
But we first prove some "local" claim.

1.26 Claim. *Assume*

$(*)_\varphi$ *$\varphi = \varphi(\bar x_1, \dots, \bar x_n, \bar y), \ell g(\bar x_\ell) = m$ and for any permutation π of $\{1, \dots, n\}$ we let $\varphi_\pi = \varphi_\pi(\bar x_1, \dots, \bar x_m, \bar y) = \varphi(\bar x_{\pi(1)}, \dots, \bar x_{\pi(n)}, \bar y)$.*

1) If $\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$ is a $\{\varphi(\bar x_1, \dots, \bar x_n, \bar c)\}$-indiscernible sequence but not set and $I = I_0 + I_1 + I_2, |I_0| \ge n-2, |I_2| \ge n-2$, <u>then</u> for some permutation π of $\{1, \dots, n\}$ and $t_3, t_4, \dots, t_n \in I_0 \cup I_2$, we have: $\varphi_\pi(\bar x_1, \bar x_2, \bar d) =: \varphi(\bar x_1, \bar x_2, \bar b_{t_3}, \dots, \bar b_{t_1}, \bar c)$ linearly ordered $\langle \bar b_t : t \in I_1\rangle$ that is, for $t \ne s \in I$ we have $\models \varphi[\bar a_t, \bar a_s, \bar d] \Leftrightarrow t <_I s$.
2) In part (1), with I is infinite of course, $\varphi_\pi(\bar x_1; \bar x_2 {}^\frown \dots {}^\frown \bar x_n, \bar z) \notin$ $\mathrm{stfor}(\bar{\mathbf{b}} \restriction I)$; moreover $\varphi_\pi(\bar x_1, \bar x_2, \bar d) \notin$ $\mathrm{stfor}(\bar{\mathbf{b}} \restriction I_1)$.

Proof. See e.g. [Sh:c, II] and history there; the point being that the permuations exchanging $k, k+1$ for $k = 1, \dots, n-1$ generate the group of permutations of $\{1, \dots, n\}$. $\square_{1.26}$

More information concerning 1.26 is

1.27 Claim. *[T is dependent.]*
1) Assume

- (a) *$\bar{\mathbf{b}} = \langle \bar b_t : t \in I\rangle$ is an indiscernible sequence of m-tuples over $\bar d$ and $I = J_0 + J_1 + J_2$*
- (b) *$\varphi(\bar x, \bar y, \bar d) \notin$ $\mathrm{stfor}(\bar{\mathbf{b}})$ and let $\varphi'(\bar y, \bar x, \bar d) = \varphi(\bar x, \bar y, \bar d)$*
- (c) *$|J_0|, |J_2|$ are finite large enough, in fact just $\ge k_{\varphi(\bar x; \bar y, \bar z)}$ from 1.8(2).*

Then we can find $\psi(\bar{x},\bar{e})$ and truth value $\mathbf{t}$ such that

(α) *for $t \in J_1, \models (\exists \bar{y})(\psi(\bar{y},\bar{e}) \wedge \varphi'(\bar{y},\bar{a}_t,\bar{d})^{\mathbf{t}})$*

(β) *for $t \neq s \in J_1$ we have*
$t <_{J_1} s$ iff
$\psi(\bar{y},\bar{e}), \varphi'(\bar{y},\bar{a}_t,\bar{d})^{\mathbf{t}} \vdash \varphi'(\bar{y},\bar{a}_s,\bar{d})^{\mathbf{t}}$

(γ) *for some $n \leq k_{\varphi(\bar{x}_1,\bar{x}_2,\bar{z})}$ and $t_0 <_J \ldots <_J t_{n-1}$ from $J_0 \cup J_2$ and $\eta \in {}^n 2$ we have $\psi(\bar{y},\bar{e}) = \wedge_{\ell<n} \varphi'(\bar{y},\bar{b}_{t_\ell},\bar{d})^{\eta(\ell)}$.*

2) In part (1) we can deduce that

(δ) *$\vartheta(\bar{x}_1,\bar{x}_2,\bar{e})$ is a partial order and $\bar{\mathbf{b}} \restriction J_1$ is $\vartheta(\bar{x}_1,\bar{x}_2,\bar{e})$-increasing where $\vartheta(\bar{x}_1,\bar{x}_2,\bar{e}) = (\forall \bar{y})[\psi(\bar{y},\bar{e}) \wedge \varphi'(\bar{y},x_1,\bar{d}) \rightarrow \varphi'(\bar{y},\bar{x}_2,\bar{d})]$.*

3) Assume in part (1) that $J_2 = \emptyset$ and $|J_0|$ large enough. Then still we can find $\psi(\bar{x},\bar{e}), \mathbf{t}$ and $n(*)$ such that:

(α) *if $t_1 <_{J_1} \ldots <_{J_1} t_{n(*)}$ then*
$(\exists \bar{y})[\psi(\bar{y},\bar{e})$ & $\wedge_{i=1}^{n(*)} \varphi'(\bar{y},\bar{a}_{t_i},\bar{d})^{\mathbf{t}}$

(β) *if $s <_{J_1} t_1 <_{J_1} \ldots <_{J_1} t_{n(*)} <_{J_n} s$ then*
$\varphi'(\bar{y},\bar{a}_s,\bar{d})^{\mathbf{t}}, \psi(\bar{y},\bar{e}) \vdash \varphi'(\bar{y},\bar{a}_{t_1},\bar{d})^{\mathbf{t}} \vee \ldots \vee \varphi'(\bar{y},\bar{a}_{t_{n(*)}},\bar{d})^{\mathbf{t}}$

(γ) *as in part (1).*

Proof. 1) Without loss of generality I is dense without first and without last element. As $\varphi(\bar{x},\bar{y},\bar{d}) \notin \mathrm{stfor}_{\mathrm{pa}}(\bar{\mathbf{b}})$ for some sequence $\bar{a}$, for both truth value $\mathbf{t}$ the set $\{t \in I : \mathfrak{C} \models \varphi(\bar{b}_t,\bar{a},\bar{d})^{\mathbf{t}}\}$ is infinite. As we can replace φ by $\neg\varphi$ without loss of generality $\{t \in I : \mathfrak{C} \models \varphi(\bar{b}_t,\bar{a},\bar{d})\}$ is unbounded from above (in I). As $\bar{\mathbf{b}}$ is an indiscernible sequence over $\bar{d}$ easily

$(*)$ for every $s \in I$ the type $\{\varphi(\bar{b}_t,\bar{y},\bar{d})^{\mathrm{if}(s<t)} : t \in I\}$ is consistent.

Let $n < \omega$ be minimal such that for some $\eta \in {}^n 2$ and $t_0 <_I \ldots <_I t_{n-1}$ the type $p^{\eta}_{<t_0,\ldots,t_{n-1}>} = \{\varphi(\bar{b}_{t_\ell},\bar{y},\bar{d})^{\eta(\ell)} : \ell < n\} \cup \{\neg\varphi(\bar{b}_t,\bar{y},\bar{d}) : t <_I t_0\} \cup \{\varphi(\bar{b}_t,\bar{y},\bar{d}) : t_{n-1} <_I t\}$ is inconsistent. We let $\eta \in {}^n 2$ be as above such that for this n,η is minimal under the lexicographic order. Such n,η exists as $\varphi(\bar{x},\bar{y};\bar{d}) \in \mathrm{dpfor}_{\mathrm{pa}}(\bar{\mathbf{b}})$.

By $(*)$ above clearly $n \geq 1$. Also by the minimality on n clearly $\eta(0) \neq \mathbf{false}$, $\eta(n-1) \neq \mathbf{true}$, so $n \geq 2, \eta(0) = \mathbf{true}$ and $\eta(n-1) = \mathbf{false}$; so for some $k < n-1$ we have $\eta(0) = \eta(1) = \ldots = \eta(k) \neq \eta(k+1)$.

As $p^{\eta}_{<t_0,\ldots,t_{n-1}>}$ is inconsistent we can find finite sets $J^- \subseteq \{s : s <_I t_0\}, J^+ \subseteq \{s : t_{n-1} <_I s\}$ such that the set $\{\neg\varphi(\bar{b}_s,\bar{y},\bar{d}) : s \in J^-\} \cup \{\varphi(\bar{b}_{t_\ell},\bar{y},\bar{d})^{\eta(\ell)} : \ell <$

$n\} \cup \{\varphi(\bar{b}_s, \bar{y}, \bar{d}) : s \in J^+\}$ is inconsistent.
Let $\psi(\bar{x}, d^*)$ be the conjunction of $\{\neg\varphi(\bar{b}_s, \bar{y}, \bar{d}) : s \in J^-\} \cup \{\varphi(\bar{b}_{t_\ell}, \bar{y}, \bar{d})^{\eta(\ell)} : \ell < n, \ell \notin \{k, k+1\}\} \cup \{\varphi(\bar{b}_s, \bar{y}, \bar{d}) : s \in J^+\}$. So $\{\psi(\bar{y}, \bar{d}^*), \varphi(\bar{b}_{t_k}, \bar{y}, \bar{d}), \neg\varphi(\bar{b}_{t_{k+1}}, \bar{y}, \bar{d})\}$ is inconsistent, so

$$\mathfrak{C} \models (\forall \bar{y})[\psi(\bar{y}, \bar{d}^*) \wedge \varphi(\bar{b}_{t_k}, \bar{y}, \bar{d}) \rightarrow \varphi(\bar{b}_{t_{k+1}}, \bar{y}, \bar{d})].$$

However $\{\psi(\bar{y}, \bar{d}^*), \neg\varphi(\bar{b}_{t_k}, \bar{y}, \bar{d}), \varphi(\bar{b}_{t_{k-1}}, \bar{y}, \bar{d})\}$ is consistent (otherwise $\eta' \in {}^n2$ such that $(\eta'(\ell) = \eta(\ell)) \equiv \ell \notin \{k, k+1\}$ and $\langle t_\ell : \ell < n\rangle$ contradict the minimality of η by the lexicographic order). So by the indiscernibility

$\boxplus$ for $s_1, s_2 \in [t_k, t_{k+1}]_I$ we have $\psi(y, \bar{d}^*), \varphi(\bar{b}_{s_1}, \bar{y}, \bar{d}) \vdash \varphi(\bar{b}_{s_2}, \bar{y}, \bar{d})$ iff $s_1 < s_2$.

By indiscernibility this clearly finishes the proof of part (1), except the bounds on $|J_0| + |J_2|$, which we can get by redefining $k_{\varphi(\bar{x},\bar{y},\bar{z})}$ or repeating the proof.
(2), (3) Follows. $\square_{1.27}$

1.28 Claim. *[T is the dependent.]*
For an infinite indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I\rangle$ (over $\emptyset$) the following conditions are equivalent:

- (A) $\bar{\mathbf{b}}$ *is an (infinite) indiscernible set*
- (B) $\mathrm{stfor}(\bar{\mathbf{b}}) = \mathbb{L}_{\tau(T)}$
- (C) *for every set A, if $\bar{\mathbf{b}}$ is an indiscernible sequence over A, then $\bar{\mathbf{b}}$ is an indiscernible set over A.*

Proof. We shall prove three implications completing a "circle".
$(C) \Rightarrow (A)$:
Holds as (A) is a special case of (C), i.e. choosing $A = \emptyset$.
$(A) \Rightarrow (B)$:
Assume $\varphi(\bar{x}, \bar{y}) \notin \mathrm{stfor}(\bar{\mathbf{b}})$ then for every n for some $\bar{c}_n \in {}^{\ell g(\bar{y})}\mathfrak{C}$ the sets $I_n^{\text{true}} =: \{t \in I : \models \varphi[\bar{b}_t, \bar{c}_n]\}$ and $I_n^{\text{false}} = \{t \in I : \models \neg\varphi[\bar{b}_t, \bar{c}_n]\}$ has $\geq n$ members. Let $h : I \rightarrow \{\text{true,false}\}$, and define $p_h = \{\varphi(\bar{b}_t, \bar{y})^{h(t)} : t \in I\}$, and we shall show below that p_h is consistent, so T has the independence property, contradiction, so there is no $\varphi(\bar{x}, \bar{y}) \notin \mathrm{stfor}(\bar{\mathbf{b}})$, hence $\mathrm{stfor}(\bar{\mathbf{b}}) = \mathbb{L}_{\tau(T)}$, i.e., condition (B) holds.

So let q be a finite subset of p_h, so for some finite $J \subseteq I$ we have $q = \{\varphi(\bar{b}_t, \bar{y})^{h(t)} : t \in J\}$. Let $n = |J|$ and so there is a permutation π of the set I such that $t \in J \Rightarrow \pi(t) \in I_n^{h(t)}$, possible as $|I_n^{\mathbf{t}}| \geq n$. As $\bar{\mathbf{b}}$ is an indiscernible set there is an automorphism F of $\mathfrak{C}$ such that $t \in I \Rightarrow F(\bar{b}_t) = \bar{b}_{\pi(t)}$. Clearly $F(q) =$

$\{\varphi(\bar b_{\pi(t)}, \bar y)^{h(t)} : t \in J\} \subseteq \text{tp}(\bar c_n, \cup\{b_t : t \in I\})$, hence $F^{-1}(\bar c_n)$ realizes q, as promised.

<u>$(B) \Rightarrow (C)$</u>:

So assume that $\bar{\mathbf{b}}$ is an indiscernible sequence over A. Let $I^+ = I_0 + I + I_2$ with I_0, I_2 any infinite linear orders. Clearly there are $\bar b^*_t \in {}^m\mathfrak{C}$ for $t \in I_0 \cup I_2$ such that $\bar{\mathbf{b}}^+ = \langle \bar b_t : t \in I^+\rangle$ is an indiscernible sequence over A.

If $\bar{\mathbf{b}}^+$ is not an indiscernible set over A, then for some $\bar c \subseteq A$ and $\varphi = \varphi(\bar x_1, \ldots, \bar x_n, \bar c)$ the sequence $\bar{\mathbf{b}}^+$ is $\{\varphi\}$-indiscernible sequence but not a $\{\varphi\}$-indiscernible set, so by 1.26 some $\varphi' = \varphi'(\bar x_1, \bar x_2, \bar d) = \varphi_\pi(\bar x_1, \bar x_2, \bar d)$ linearly ordered $\{\bar b_t : t \in I\}$ where π is a permutation of $\{1, \ldots, n\}, t_3, \ldots, t_n \in I_0 \cup I_2$ and $\bar d = \bar b_{t_3} \hat{}\, \bar b_{t_4} \hat{}\, \ldots \hat{}\, \bar b_{t_n} \hat{}\, \bar c \subseteq \bar c \cup \{\bar b_s : s \in I_0 \cup I_2\}$; hence $\varphi'(\bar x_1, \bar x_2, \bar z)$ is not a stable formula for $\bar{\mathbf{b}}$. So $\text{stfor}(\bar{\mathbf{b}}) \neq \mathbb{L}_{\tau(T)}$, contradiction. So $\bar{\mathbf{b}}^+$ hence $\bar{\mathbf{b}}$ is an indiscernible set over A as required. $\square_{1.28}$

* * *

We now may think on $\varphi(\bar x, \bar y, \bar c)$ which are stable for $\bar{\mathbf{b}}$ which we get in the approximation of order in 1.25(4). We may wonder can we not by expanding p (with more variables, over the same B preserving finite satisfiability in B) get clearer picture. This may help in getting indiscernible sequences. (See more in concluding remarks).

1.29 Definition. If $p \in \mathbf{S}^m(B)$ is finitely satisfiable in B let

(a) $\text{stfor}(p) = \cap\{\text{stfor}(\bar{\mathbf{b}})$: for some ultrafilter D on ${}^mB$ satisfying $p = \text{Av}(B, D)$, the sequence $\bar{\mathbf{b}} = \langle \bar b_\alpha : \alpha < \omega\rangle$ an indiscernible sequence obeying $D\}$ (this does not matter if we take one or all such $\bar{\mathbf{b}}$ by 1.6(4),(5))

(b) $\text{Cb}(p) = \cap\{\text{Cb}(\mathbf{b})$: for some ultrafilter D on ${}^mB$ satisfying $\text{Av}(B, D) = p$ the sequence $\mathbf{b}$ is a (D, B)-indiscernible sequence (in $\mathfrak{C}^{eq}$, of course)$\}$.

Of course

1.30 Observation. For any $M \prec \mathfrak{C}$ and $p \in \mathbf{S}^m(M)$, or just $p \in \mathbf{S}^m(B)$ is finitely satisfiable in ${}^mB$ we have $\text{stfor}(p), \text{Cb}(p)$ are well defined as there are ultrafilters D on ${}^mM$ such that $\text{Av}(M, D) = p$.

<u>1.31 Observation</u> [T is dependent.]
If $\bar{\mathbf{b}}$ is a $\{\varphi(\bar x_\ell, \ldots, \bar x_1; \bar c)\}$-indiscernible sequence over A but is a Δ_φ-indiscernible set over $\emptyset$ and has $> k_\varphi$ members, <u>then</u> $\bar{\mathbf{b}}$ is a $\{\varphi(\bar x_0, \ldots, \bar x_{k-1}; \bar c)\}$-indiscernible set over A where $\Delta_\varphi = \{\exists \bar x \bigwedge\limits_{\ell < n} \varphi_\pi(\bar x, \bar y_\ell)^{\text{if}(\eta(\ell))} : \eta \in {}^n2$ and π is a permutation of $\{1, \ldots, \ell\}\}$ and $n = n_\varphi$ is such that $\boxtimes^n_{\varphi(\bar x, \bar y)}$ from 1.2 fail.

Proof. By the proof of $(B) \Rightarrow (C)$ inside the proof of 1.28 it is enough to prove for every permutation π of $\{1,\dots,\ell\}$ that $\varphi'_\pi(\bar{x}_1,\bar{y},\bar{d}) = \varphi_\pi(\bar{x}_1;\bar{x}_2,\dots,\bar{x}_\ell,\bar{d}) \in \text{stfor}_{\text{pa}}(\bar{\mathbf{b}})$.
By the proof of $(A) \Rightarrow (B)$ inside the proof of 1.28 this follows from "$\bar{\mathbf{b}}$ is Δ_φ-indiscernible over $\emptyset$. $\square_{1.31}$

1.32 Remark. Note that $p \in \mathbf{S}(M)$ may be definable but not stable, e.g. $M \prec N$ are models of the theory of $(\mathbb{R},<)$, and $a \in N \backslash M$ is above all $b \in N$, then $\text{tp}(a,M,N)$ is definable but not stable.

1.33 Conclusion: [T is dependent and is unstable.]
1) There are $M \prec \mathfrak{C}$ and non-stable $p \in \mathbf{S}^1(M)$ [in $\mathfrak{C}$, not just $\mathfrak{C}^{\text{eq}}$!].
2) There is an indiscernible sequence of elements which is not an indiscernible set of elements (over $\emptyset$!)

Proof. 1) As T is unstable, for some $M \prec \mathfrak{C}$ we have $|\mathbf{S}^1(M)| > \|M\|^{|T|}$ hence by 1.20(3) some type $p \in \mathbf{S}^1(M)$ is non-stable.
2) By part 1) and Definition 1.19. $\square_{1.33}$

Now 1.27(3) applies to 1.33(2) (where $\bar{x}_i$ is x_i) gives

1.34 Conclusion. If T has the dependence property but is unstable, then some formula $\varphi(x,y;\bar{c})$ define on $\mathfrak{C}$ a quasi order and even partial order with infinite chains, (so x,y singletons).

Proof. By 1.33(2) and 1.27. $\square_{1.34}$

1.35 Remark. So if T satisfies some version of $*$-stable (see [Sh 300, Ch.II] or [Sh 702]) then T is stable or T has the independence property.

So we may wonder
1.36 Question: 1) Does the "has the dependence property" case in 1.35 is needed?
2) If T has the independence property does some $\varphi(x,y,\bar{c})$ have the independence property?
3) Let $p \in \mathbf{S}^m(B)$ be non-stable, letting $D \in \{D' : D' \text{ an ultrafilter on } {}^m B \text{ satisfying } \text{Av}(B,D') = p\}$, how many $\bar{\mathbf{b}}^D$ can we find in 1.20 which are pairwise not nb-s, (for T with the dependence property)?

Note that

1.37 Observation. If T is unstable, <u>then</u> some formula $\varphi(x,y,\bar{c})$ has the order property (equivalently is unstable, hence some $\varphi(x,y,\bar{c})$) define a partial order with infinite chains or has the independence property.

Proof. We know that some $\varphi(x,\bar{y})$ is unstable so choose a formula $\varphi(x,\bar{y},\bar{c})$ with the order property, such that $\ell g(\bar{y})$ is minimal. So there is an indiscernible sequence $\langle \bar{a}_i\hat{\ } < b_i >: i < \omega 4\rangle$ such that $\mathfrak{C} \models \varphi[b_i,\bar{a}_j,\bar{c}]$ iff $j < i$. Clearly $\langle b_i : i < \omega 4\rangle$ is an indiscernible sequence over $\bar{c}$, if it is not an indiscernible set, say not (ϑ,k)-indiscernible set, $\vartheta = \vartheta(x_0,\dots,x_{k-1},\bar{c})$, then possibly permuting the variables of ϑ the formula $\vartheta(x,y,\bar{b}_0,\dots,b_{m-1},b_{n-2},b_{2\omega+1},\dots,b_{2\omega+k,m-3},\bar{c})$ linear orders $\langle a_{\omega+i} : i < \omega\rangle$, hence has the order property. So assume $\langle b_i : i < \omega\rangle$ is an indiscernible set over $\bar{c}$, and let a'_i be the first element of the sequence $\bar{a}_i$. If $\langle b_{2i+1} : i < \omega 4\rangle$ is not an indiscernible sequence over $\bar{c} \cup \{a'_{2\omega}\}$ then, by the indiscernibility of $\langle \bar{a}_i\hat{\ } < b_i >: i < \omega\rangle$ over $\bar{c}$, we can find a formula $\vartheta(x,y,\bar{c}'),\bar{c}' \subseteq \bar{c} \cup \{a_i : i < \omega \text{ or } \omega 3 \le i\}$ such that $\models \vartheta[b_{2\omega},a_{\omega+2i+1},\bar{c}']$ for $i < \omega$ but $\models \neg\vartheta[b_{2\omega},a_{\omega 2+2i+1},\bar{c}']$ for $i < \omega$ and we are done. So assume $\langle b_{2i+1} : i < \omega 4\rangle$ is an indiscernible sequence over $\bar{c} \cup \{a'_{2\omega}\}$, hence all $\{a'_{2j} : j < \omega 4\}$ realizes the same type over $\{b_{2i+1} : i < \omega 4\} \cup \bar{c}$ hence for $j < 2\omega$ we can find $\bar{a}^*_{2j}$ realizing $\text{tp}(\bar{a}_{2j},\{b_{2i+1} : i < \omega 4\} \cup \bar{c},\mathfrak{C})$ and the first element of $\bar{a}^*_{2j}$ is a'_0. This contradicts the choice of $\varphi(x,\bar{y},\bar{c})$ as having the order property with $\ell g(\bar{y})$ minimal as we can "move" a'_0 to $\bar{c}$. $\square_{1.37}$

* * *

1.38 Remark. Note that for indiscernible sets, the theorems on dimension in [Sh:c, III] holds for theories T with the dependence property, see §3.

§2 Characteristics of types

We continue to speak on canonical bases and we deal with the characteristics of types and of indiscernible sets. More elaborately, for any indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle, I$ an infinite linear order, we have a measure $\mathrm{Ch}(\bar{\mathbf{b}}) = \langle \mathrm{Ch}_{\varphi(\bar{x},\bar{y}_\varphi)}(\bar{\mathbf{b}}) : \varphi(\bar{x},\bar{y}_\varphi) \in \mathbb{L}_{\tau(T)} \rangle$ with $\bar{x} = \langle x_i : i < m \rangle, m = \ell g(\bar{b}_t)$ for $t \in J$, where $\mathrm{Ch}_{\varphi(\bar{x},\bar{y}_\varphi)}(\bar{\mathbf{b}})$ measure how badly $\varphi(\bar{x},\bar{y}_\varphi)$ fail to be in $\mathrm{stfor}(\bar{\mathbf{b}})$ (see Definition 2.5), we can find such $\bar{\mathbf{b}}$'s with maximal such $\mathrm{Ch}(\bar{\mathbf{b}})$ and wonder what can we say about them. The case of an indiscernible set is covered by §1.

2.1 Hypothesis. T has the dependence property.

2.2 Definition/Claim. Let $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ be an infinite indiscernible sequence, $k < \omega$. Then

(a) (Claim) if $t_i \in I$ and $i < j \Rightarrow t_i <_I t_j$ for $i < j < \omega$ and $\bar{\mathbf{b}}^k = \langle \bar{b}_{t_{ki}} \char`^ \bar{b}_{t_{ki+1}} \char`^ \ldots \char`^ \bar{b}_{t_{ki+k-1}} : i < \omega \rangle$ then

- (α) $\mathrm{Cb}(\bar{\mathbf{b}}) = \mathrm{Cb}(\bar{\mathbf{b}}^1) \subseteq \mathrm{Cb}(\bar{\mathbf{b}}^k)$,
- (β) if $\varphi'(\bar{x}_1,\ldots,\bar{x}_k;\bar{y}) = \varphi(\bar{x}_\ell,\bar{y})$ then: $\varphi'(\bar{x}_1,\ldots,\bar{x}_k;\bar{y}) \in \mathrm{stfor}(\mathbf{b}^k)$ iff $\varphi(\bar{x}_\ell;\bar{y}) \in \mathrm{stfor}(\bar{\mathbf{b}})$
- (γ) if $\bar{\mathbf{b}}^{k,1},\bar{\mathbf{b}}^{k,2}$ are related like $\bar{\mathbf{b}}^k$ above to our $\bar{\mathbf{b}}$ then $\mathrm{Cb}(\bar{\mathbf{b}}^{k,1}) = \mathrm{Cb}(\bar{\mathbf{b}}^{k,2})$ (even for the "local" version this is true)
- (δ) if $\varphi(\bar{x},\bar{y}) \in \mathrm{dpfor}(\bar{\mathbf{b}}), \varphi' = \varphi'(\bar{x}_1,\ldots,\bar{x}_k,\bar{y}) = \varphi(\bar{x}_\ell,\bar{y})$ & $\neg\varphi(\bar{x}_m,\bar{y})$ or $\varphi' = \varphi'(\bar{x}_1,\ldots,\bar{x}_k;\bar{y}) = (\varphi(\bar{x}_\ell,\bar{y}) \equiv \neg\varphi(\bar{x}_m,\bar{y}))$ then $\varphi' \in \mathrm{stfor}(\mathbf{b}^k)$

(b) (Definition) let $\mathrm{Cb}^k(\bar{\mathbf{b}}) = \mathrm{Cb}(\bar{\mathbf{b}}^k)$ and $\mathrm{Av}^k(\bar{\mathbf{b}},\mathfrak{C}) = \mathrm{Av}_{\mathrm{stfor}}(\bar{\mathbf{b}}^k,\mathfrak{C})$ for any $\bar{\mathbf{b}}^k$ as above

(c) (Definition) $\mathrm{Cb}^\omega(\bar{\mathbf{b}}) = \cup\{\mathrm{Cb}^k(\bar{\mathbf{b}}) : k < \omega\}$

(d) (Fact) if I_1, I_2 are infinite subsets of J and $\bar{\mathbf{b}} = \langle \bar{\mathbf{b}}_t : t \in J \rangle$ is an indiscernible sequence (recall J linear order) then $\mathrm{Cb}^\omega(\bar{\mathbf{b}} \restriction I_1) = \mathrm{Cb}^\omega(\bar{\mathbf{b}} \restriction I_2)$

(e) (Fact) if the infinite indiscernible sequences $\bar{\mathbf{b}}^1,\bar{\mathbf{b}}^2$ are nb-s, then $\mathrm{Cb}^\alpha(\bar{\mathbf{b}}^1) = \mathrm{Cb}^\alpha(\bar{\mathbf{b}}^2)$ for $\alpha \le \omega$

(f) (Definition) if $p \in \mathbf{S}^m(B)$ is finitely satisfiable in ${}^m B$ and $\alpha \le \omega$ then let $\mathrm{Cb}^\alpha(p) = \cap\{\mathrm{Cb}^\alpha(\bar{\mathbf{b}})$: for some ultrafilter D on ${}^m B$ satisfying $\mathrm{Av}(B,D) = p$, the sequence $\bar{\mathbf{b}}$ is a (D,B)-indiscernible sequence$\}$.

Proof. Easy.

Recall: non-forking is quite worthwhile for stable theory; and have several equivalent definitions. There is worthwhile generalization for simple theories; but of course, not all definitions for stable theories give it for simple T. We may look for a generalization of non-forking for dependent theories (see alternatives in [Sh 783]).

2.3 Definition. For $\alpha \le \omega$. We say $p \in \mathbf{S}^m(A)$ does not α-fork over $B \subseteq A$, <u>if</u> for every model $M \supseteq A$ for some $q \in \mathbf{S}^m(M)$ extending p we have $\mathrm{Cb}^\alpha(q) \subseteq \mathrm{acl}_{\mathfrak{C}^{\mathrm{eq}}}(B)$. Similarly we say that C/B does not α-fork over $A \subseteq B$ if $\bar{c} \subseteq C \Rightarrow \mathrm{tp}(C, B)$ does not α-fork over it.

2.4 Remark. Assume that T is a simple theory, $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ is an infinite indiscernible sequence. <u>Then</u> we cannot find $\langle \bar{a}_n : n < \omega \rangle$ indiscernible sequence, $\langle \varphi(\bar{x}, \bar{a}_n) : n < \omega \rangle$ pairwise contradictory (or just m-contradiction for some m) and

$$\bigwedge_{n<\omega} (\exists^\infty t \in I)(\varphi(\bar{b}_t, \bar{a}_n)).$$

Proof. Assume toward contradiction that $\varphi, \langle \bar{a}_n : n < \omega \rangle$ form a counterexample. By thinning and compactness without loss of generality the set $I_n = \{t \in I : \mathfrak{C} \models \varphi[b_t, \bar{a}_n]\}$ are pairwise disjoint, and each is a convex subset of I.
Now we can repeat and get the tree property. More fully, for any cardinals $\mu > \kappa$ we consider $J = {}^\kappa\mu$ as a linearly ordered set, ordered lexicographically and for $\rho \in {}^{\kappa>}\mu$ let $J_\rho = \{\nu \in J : \rho \triangleleft \nu\}$; without loss of generality let $I_0 \subseteq I$ has order type ω and let $h : I \to J$ be order preserving. We can find $\bar{c}_\eta \in \mathfrak{C}$ for $\eta \in J$ such that $\langle \bar{c}_\eta : \eta \in J \rangle$ is an indiscernible sequence satisfying $t \in I \Rightarrow \bar{c}_{h(t)} = \bar{b}_t$. By compactness, for each $\alpha < \kappa$ we can find $\langle a_\rho : \rho \in {}^\alpha\mu \rangle$ such that:

(α) $\langle \varphi(\bar{x}, \bar{a}_\rho) : \rho \in {}^\alpha\mu \rangle$ are pairwise contradictory (or just any m of them)

(β) $\eta \in J_\rho, \rho \in {}^\alpha\mu \Rightarrow \mathfrak{C} \models \varphi[\bar{c}_\eta, \bar{a}_\rho]$.

Now $\langle \varphi(\bar{x}, \bar{a}_\rho) : \rho \in {}^{\kappa>}\mu \rangle$ exemplified the tree property. $\square_{2.4}$

We have looked at indiscernible sequences which are stable. We now look after indiscernible sequences which are in the other extreme.

2.5 Definition. 1) For $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ an indiscernible sequence, we define its character

$$\mathrm{Ch}(\bar{\mathbf{b}}) = \langle \mathrm{Ch}_{\varphi(\bar{y},\bar{z})}(\bar{\mathbf{b}}) : \varphi(\bar{y},\bar{z}) \in \mathbb{L}_{\tau(T)} \rangle$$

where

$$\begin{aligned}\mathrm{Ch}_{\varphi(\bar{y},\bar{z})}(\bar{\mathbf{b}}) = \ & \mathrm{Max}\{n : \text{for some } \bar{c} \text{ the sequence } \langle \mathrm{TV}(\varphi(\bar{b}_t,\bar{c})) : t \in I \rangle \\ & \text{change sign } n \text{ times (i.e. } I \text{ is divided to } n+1 \text{ intervals)}\}\end{aligned}$$

(so it is $\leq 2k_{\varphi(\bar{y},\bar{z})}$).
2) For $p \in \mathbf{S}^m(A)$, let

- (a) $\mathrm{CH}(p) = \{\mathrm{Ch}(\bar{\mathbf{b}}) : \bar{\mathbf{b}}$ is an infinite indiscernible sequence such that every $\bar{b}_t$ realizes $p\}$
- (b) for a formula $\varphi = \varphi(\bar{x}_0,\dots,\bar{x}_{k-1})$ let $\mathrm{CH}(p,\varphi(\bar{x}_0,\dots,x_k)) = \{\mathrm{Ch}(\bar{\mathbf{b}}) : \bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ is an infinite indiscernible sequence such that $t_0 <_I t_1 <_I \dots <_I t_{k-1} \Rightarrow \mathfrak{C} \models \varphi[\bar{b}_{t_0},\dots,\bar{b}_{t_{k-1}}]$ and each $\bar{b}_t$ realizes $p\}$
- (c) $\mathrm{CH}^{\max}(p) = \{\bar{n} \in \mathrm{CH}(p) :$ there is no bigger such $\bar{n}' \in \mathrm{CH}(p)\}$, when "$\bar{n}'$ is bigger than $\bar{n}$" mean $(\forall\varphi)(n_\varphi \leq n'_\varphi)$ & $(\exists\varphi)(n_\varphi < n'_\varphi)$
- (d) $\mathrm{CH}^{\min}(p,\varphi(\bar{x}_0,\dots,x_{k-1})) = \{\bar{n} \in \mathrm{CH}(p,\varphi(\bar{x}_0,\dots,\bar{x}_{k-1}))$: there is no smaller $\bar{n}' \in \mathrm{CH}(p,\varphi(\bar{x}_0,\dots,\bar{x}_{k-1}))\}$.

Note: for the trivial φ, $\mathrm{CH}(p,\varphi) = \mathrm{CH}(p)$ hence $\mathrm{CH}^{\max}(p,\varphi) = \mathrm{CH}^{\max}(p)$.

2.6 Remark. 1) Instead of counting the number of interchanges of signs we can look at

- (a) $\mathrm{ch}_{\varphi(\bar{y},\bar{z})}(\bar{\mathbf{b}}) = \mathrm{Max}\{n$: for some sequence $\bar{c}$, if J is a cofinite subset of I then the sequence $\langle \mathrm{TV}(\varphi(\bar{b}_t,\bar{c})) : t \in J \rangle$ change signs $\geq n$ times$\}$ and then define $\mathrm{cH}(p)$, $\mathrm{cH}^{\max}(p)$, etc.

2) Alternatively use

$$(b)\quad \Gamma_{\varphi(\bar{y},\bar{z})}(\bar{\mathbf{b}}) = \{\eta \in {}^{\omega>}2 : \mathfrak{C} \models (\exists \bar{y}) \bigwedge_{\ell<\ell g(\eta)} \varphi(\bar{b}_{t_\ell},\bar{y})^{\eta(\ell)} \text{ for } t_0 < \dots < t_{\ell g(\eta)-1}\}$$

and/or other variants.
2) Clearly 2.5(2) is a try to get "maximal", "most general" extensions of p (as non-forking is for stable T)

2.7 Claim. *Let $p \in \mathbf{S}^m(A)$ be non-algebraic, $\bar{x} = \langle x_\ell : \ell < m\rangle$.*
1) If $\bar{n} = \langle n_\varphi : \varphi = \varphi(\bar{x},\bar{y})\rangle \in \mathrm{CH}(p)$, then there is $\bar{n}' \in \mathrm{CH}^{\max}(p)$ such that $\bar{n} \le \bar{n}'$.
2) $\mathrm{CH}^{\max}(p)$ is non-empty.
3) If $\mathrm{CH}(p,\varphi) \ne \emptyset$ then $\mathrm{CH}^{\min}(p,\varphi) \ne \emptyset$ and $\mathrm{CH}^{\max}(p,\varphi) \ne \emptyset$.

Proof. Let $R, <$ be an n-place and $2n$-place respectively predicate not in τ_T and let

$$\begin{aligned}\Gamma_p = \ & \mathrm{Th}(\mathfrak{C}_T, c)_{c\in A} \cup \{(\forall \bar{x})[R(\bar{x}) \to \vartheta(\bar{x},\bar{c})] : \vartheta(\bar{x},\bar{c}) \in p\} \\ & \cup \{(\exists \bar{x}_0,\dots,\bar{x}_{k-1})(\bigwedge_{\ell<k} R(\bar{x}_\ell) \ \& \bigwedge_{\ell_1<\ell_2} \bar{x}_{\ell_1} \ne \bar{x}_{\ell_2}) : k < \omega\} \\ & \cup \{\bar{x} < \bar{y} \to R(\bar{x}) \wedge R(\bar{y})\} \\ & \cup \{\text{“} \le \text{ linearly ordered } \{\bar{x} : R(\bar{x})\}\text{”}\} \\ & \cup \{(\forall \bar{x}_1),\dots,(\forall \bar{x}_k)(\forall \bar{y}_1)\dots(\forall \bar{y}_k)(\bar{x}_1 < \bar{x}_2 < \dots < \bar{x}_k \\ & \ \& \ \bar{y}_1 < \dots < \bar{y}_k \to \psi(\bar{x}_1,\dots,\bar{x}_k,\bar{c}) \equiv \psi(\bar{y}_1,\dots,\bar{y}_k,\bar{c}) : \\ & \bar{c} \subseteq A \text{ and } \psi \in \mathbb{L}_{\tau(T)} \text{ and } k < \omega\}\end{aligned}$$

(with $\bar{x}_i = \langle x_{i,\ell} : \ell < m\rangle$). For $\bar{n} = \langle n_{\varphi(\bar{x},\bar{y})} : \varphi(\bar{x},\bar{y}) \in \mathbb{L}(\tau_T)\rangle$ and $\bar{\varphi} = \langle \varphi_i(\bar{x},\bar{y}_i) : i < |T|\rangle$ listing the formulas for $\mathbb{L}_{\tau(T)}$ let

$$\begin{aligned}\Gamma_{\bar{n},\bar{\varphi}} = & \{\vartheta_{n_i,\varphi_i} : i < |T|\} \text{ where} \\ & \vartheta_{n,\varphi(\bar{x},\bar{y})} = (\exists \bar{y})(\exists \bar{x}_0,\dots,\exists \bar{x}_n)[\bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n \ \& \\ & \qquad \bigwedge_{\ell<n} (\varphi(\bar{x}_\ell,\bar{y})) \equiv \neg\varphi(\bar{x}_{\ell+1},\bar{y})].\end{aligned}$$

Now easily

- (a) Γ_p is a consistent type (using p being non algebraic and Ramsey theorem)
- (b) if $\bar{n} \le \bar{n}'$ then $\Gamma_{\bar{n},\bar{\varphi}} \subseteq \Gamma_{\bar{n}',\bar{\varphi}}$
- (c) $\Gamma_p \cup \Gamma_{\bar{n},\bar{\varphi}}$ is consistent iff $(\exists \bar{n}')[\bar{n} \le \bar{n}' \in \mathrm{CH}(p)]$
- (d) if J is a directed (partial) order, $\bar{n}_t = \langle n_{t,\varphi(\bar{x},\bar{y}_\varphi)} : \varphi(\bar{x},\bar{y}_\varphi) \in \mathbb{L}_{\tau(T)}\rangle \in \mathrm{CH}(p,\varphi)$ increases with $t \in J$ and $\bar{n}^* = \langle n^*_{\varphi(\bar{x},\bar{y}_\varphi)} : \varphi(x,\bar{y}_\varphi)\rangle$ and[3] $n^*_{\varphi(\bar{x},\bar{y}_\varphi)} = \max\{n_{t,\varphi(\bar{x},\bar{y}_\varphi)} : t \in J\}$, then $\bar{n}^* \in \mathrm{CH}(\bar{p},\varphi)$
- (e) like (d) inverting the order.

[3] well defined as T is dependent

Together we can deduce the desired conclusions. $\square_{2.7}$

2.8 Question: For $p \in \mathbf{S}(A)$ (or just $p \in \mathbf{S}(M)$), does indiscernible sequences $\bar{\mathbf{b}}$ of elements realizing p such that $\mathrm{Ch}(\bar{\mathbf{b}}) \in \mathrm{CH}^{\max}(p)$, $\mathrm{CH}(\bar{b}) \in \mathrm{CH}^{\min}(p, \varphi)$ play a special role?

§3 Shrinking indiscernibles

For stable theories we know that if $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible set over A and $\bar{c} \in {}^{\omega >}\mathfrak{C}$ then for some $J \subseteq I$ we have $|J| \le |T|$ and $\langle \bar{b}_t : t \in I \backslash J \rangle$ is an indiscernible set over $A \cup \bar{c} \cup \{\bar{b}_t : t \in J\}$. We look more closely at the generalization for theories with the dependence property (continuing [Sh:c, II]).
The case of indiscernible sets is easier so we delay it.

3.1 Notation. For a linear order I, let $\mathrm{comp}(I)$ be its completion.

3.2 Claim. *If $\bar{\mathbf{b}} = \langle \bar{\mathbf{b}}_t : t \in I \rangle$ is an indiscernible sequence over A and $\bar{c} \in {}^{\omega >}\mathfrak{C}$ (so finite), then*

- (a) *there is $J^* \subseteq \mathrm{comp}(I), |J^*| \le |T|$ such that*
  - $(*)$ *if $n < \omega$ and $\bar{s}, \bar{t} \in {}^n I, \bar{s} \sim_{J^*} \bar{t}$ (i.e., $\bar{s}, \bar{t}$ realize the same quantifier free type over J^* in the linear order $\mathrm{comp}(I)$) then $\bar{a}_{\bar{s}} = \langle \bar{a}_{s_\ell} : \ell < n \rangle, \bar{a}_{\bar{t}} = \langle a_{t_\ell} : \ell < n \rangle$ realize in $\mathfrak{C}$ the same type over $A \cup \bar{c}$*
- (b) *if we fix n and deal with φ-types we can demand $|J^*| < k_{\varphi,n} < \omega$*
- (c) *if in addition $\bar{\mathbf{b}}$ is an indiscernible set, then in $(*)$ of clause (a) we can weaken $\bar{s} \sim_{J^*} \bar{t}$ to $(\forall \ell, k)[(s_\ell = s_k \equiv t_\ell = t_k) \ \& \ s_\ell \in J^* \equiv t_\ell \in J^* \rightarrow s_\ell = t_\ell]$*
- (d) *if we replace $\bar{c}$ by $C \subseteq \mathfrak{C}$ in (a) we just use $|J^*| \le |C| + |T|$.*

Proof.

- (a) by (b)
- (b) follows by Claim 3.4 below
- (c) similarly
- (d) follows.

$\square_{3.2}$

* * *

The reader may restrict himself in 3.3, 3.4 to the case $n = 2$ so $\bar{\mathbf{a}}$ is just an indiscernible sequence; this suffices for 3.2.

3.3 Definition. 1) For $\Delta \subseteq \mathbb{L}_{\tau(T)}$, a linear order $I, m^* \le \omega, n \le \omega, \alpha_\ell$ an ordinal for $\ell < n$, a model M and a set $A \subseteq M$, we say that $\bar{\mathbf{a}} = \langle a_{u,\alpha,\ell} : \ell < n, u \in [I]^\ell, \alpha < \alpha_\ell = \alpha_{|u|}\rangle$ is (Δ, m^*)-indiscernible over A of the $\langle \alpha_\ell : \ell < n\rangle$-kind <u>if</u> the following holds:

- $(*)$ if $m < 1 + m^*, I \models t_0 < \cdots < t_{m-1}, I \models s_0 < \ldots < s_{m-1}$ and for every $v \subseteq m$ we let $u_v = \{t_\ell : \ell \in v\}, w_v = \{s_\ell : \ell \in v\}$ <u>then</u> $\langle a_{u_v,\alpha,\ell} : \ell < n, v \in [m]^\ell, \alpha < \alpha_\ell\rangle$ and $\langle a_{w_v,\alpha,\ell} : \ell \le n, v \in [m]^\ell, \alpha < \alpha_\ell\rangle$ realizes the same Δ-type over A in M.

2) If we omit Δ we mean all first order formulas, if we omit m^* we mean ω. Also in $a_{u,\alpha,\ell}$ we may omit ℓ (because it is $|u|$). Of course nothing changed if we allow $a_{u,\alpha,\ell}$ to be a finite sequence (with length depending on (α, ℓ) only) but we can instead increase α_ℓ.
3) We add "above J" where $J \subseteq I$ (or J is included in the completion of I) if in $(*)$ we demand $(\forall r \in J) \bigwedge\limits_\ell (r < t_\ell \equiv r < s_\ell \ \ \& \ \ r = t_\ell \equiv r = s_\ell \ \ \& \ \ t_\ell < r \equiv s_\ell < r)$. We say "almost above J" if we add $J \cap \{t_\ell : \ell < n\} = \emptyset$.

3.4 Claim. *1) Assume*

- (a) *Δ is a finite set of formulas*
- (b) *M a model of T and $A \subseteq M$*
- (c) *$\bar{\mathbf{a}} = \langle a_{u,k,\ell} : \ell < n, k < k_\ell, u \in [I]^\ell\rangle$ is indiscernible over A*
- (d) *$\bar c \in {}^{\omega>}M$.*

<u>Then</u> there is a finite subset J of I or of the completion $\mathrm{comp}(I)$ *of I such that $\langle a_{u,k,\ell} : \ell < n, k < k_\ell, u \in [I]^\ell\rangle$ is Δ-indiscernible over $A \cup \bar d$ above J.*
2) Moreover, there is a bound on $|J|$ which depend just on $\Delta, \langle k_\ell : \ell < n\rangle$ (and T), and so it is enough that $\bar{\mathbf{a}}$ is Δ_1-indiscernible for appropriate finite Δ_1.
3) So for every $C \subseteq \mathfrak{C}$ there is $J \subseteq$ $\mathrm{comp}(I)$ *of cardinality $\le |C| + |T|$ such that $\bar{\mathbf{a}}$ is indiscernible above J over $A \cup C$.*

Proof. 1) Toward contradiction assume that the conclusion fails. Let m^* be the maximal number of free variables in members of Δ times n. Without loss of generality I is a complete linear order with a first and a last element. For every finite $J \subseteq I$ we choose $\langle t^J_\ell : \ell < m_J\rangle, \langle s^J_\ell : \ell < m_J\rangle$ such that:

- (i) $m_J \le m^*$
- (ii) $t^J_0 < t^J_1 < \ldots < t^J_{m_J-1}$ and $s^J_0 < s^J_1 < \ldots < s^J_{m_J-1}$

(*iii*) for at least one $m < m_J$ we have $t^J_m, s^J_m \notin J$ (actually follows from the rest)

(*iv*) $\langle t^J_\ell : \ell < m_J\rangle, \langle s^J_\ell : \ell < m_J\rangle$ exemplify that J is not as required

(*v*) m_J is minimal.

Let $\bar b^0_J = \langle a^J_{u,k,\ell} : \ell < n, k < k_\ell, u \in [\{t^J_0, \dots, t^J_{m_J-1}\}]^\ell\rangle$ and $\bar b^1_J = \langle a^J_{u,k,\ell} : \ell < n, k < k_\ell, u \in [\{s^J_0, \dots, s^J_{m_J-1}\}]^\ell\rangle$.
So clearly

$(*)_1$ the Δ-types of $\bar c\char`^\bar b^0_J, \bar c\char`^\bar b^1_J$ over A are different
[why? by their choice].

For $J \in [I]^{<\aleph_0}$ let $t^J_\ell = s^J_{\ell-m_J}$ for $\ell = m_J, m_J+1, \dots, 2m_J-1$. Let D^* be an ultrafilter on $[I]^{<\aleph_0}$ such that $t \in I \Rightarrow \{J : t \in J \in [I]^{<\aleph_0}\} \in D^*$.

As $m_J \le m^* < \aleph_0$, and D^* is an ultrafilter, clearly for some $m(*) \le m^*$ we have $Y_0 = \{J \in [I]^{<\aleph_0} : m_J = m(*)\} \in D^*$. For $\ell < 2m(*)$, let

$$I^1_\ell = \{t \in I : \{J \in Y_0 : t^J_\ell <_I t\} \in D^*\}$$

$$I^0_\ell = \{t \in I : \{J \in Y_0 : t^J_\ell = t\} \in D^*\}$$

$$I^{-1}_\ell = \{t \in I : \{J \in Y_0 : t <_I t^J_\ell\} \in D^*\}.$$

Clearly $\langle I^{-1}_\ell, I^0_\ell, I^1_\ell\rangle$ is a partition of I, I^0_ℓ is a singleton or empty, I^{-1}_ℓ is an initial segment of I and I^1_ℓ is an end segment of I. As I is complete there is $t^*_\ell \in I$ such that $I^0_\ell \ne \emptyset \Rightarrow I^0_\ell = \{t^*_\ell\}$ and $\{t \in I : t \le_I t^*_\ell\} \supseteq I^{-1}_\ell$ and $\{t \in I : t^*_\ell \le_I t\} \supseteq I^1_\ell$. So there are functions $g, h : \{0, \dots, 2m(*)-1\} \times \{0, \dots, 2m(*)-1\} \to \{-1, 0, 1\}$ such that for each $\ell, k < 2m(*)$ we have

$$\{J \in Y_0 : t^J_\ell \in I^{h(\ell,k)}_k\} \in D^*$$

and

$$g(\ell,k) = 1 \Leftrightarrow \{J \in Y_0 : t^J_\ell <_I t^J_k\} \in D^*$$

$$g(\ell,k) = 0 \Leftrightarrow \{J \in Y_0 : t^J_\ell = t^J_k\} \in D^*$$

$$g(\ell,k) = -1 \Leftrightarrow g(k,\ell) = 1.$$

For any $\ell, k < 2(m(*))$ if there is t satisfying $t^*_\ell <_I t <_I t^*_k$ then choose such $t^*_{\ell,k}$. Now let

$$\begin{aligned} Y^* = \{J \in Y_0 : & \text{for every } \ell, k < 2m(*) \text{ we have } t^J_\ell \in I^{h(\ell,k)}_k, \\ & \text{and } t^J_\ell <_I t^J_k \Leftrightarrow g(\ell,k) = 1 \text{ and } t^J_\ell = t^J_k \Leftrightarrow g(\ell,k) = 0 \\ & \text{and if } t^*_{\ell,k} \text{ is well defined then} \\ & t^J_\ell < t^*_{\ell,k} < t^J_k\}. \end{aligned}$$

Clearly $Y^* \in D^*$.

For $\ell < 2m(*)$, let I^*_ℓ be the convex hull of $\{t^J_\ell : J \in Y^*\}$. We let $<_\ell$ be $<_I$ if $I^*_\ell \subseteq I^{-1}_\ell$ and be $>_I$ if $I^*_\ell \subseteq I^1_\ell$. If none of them hold, then $I^*_\ell = \{t^*_\ell\}, <_\ell = \emptyset$ and clearly

$\boxtimes_0$ if $<_\ell = \emptyset$ then in $(I^{h(\ell,i)}_\ell, <_\ell)$ there is no last element and $t \in I^{h(\ell,i)}_\ell \Rightarrow \{J : t <_\ell t^J_\ell \in I^{h(\ell,i)}_\ell\} \in D^*$

$\boxtimes_1$ $\mathbf{e} = \{(\ell,k) : \ell,k < 2m(*), I^*_\ell \cap I^*_k \neq \emptyset\}$ is an equivalence relation and $\ell \mathbf{e} k \Rightarrow <_\ell = <_k \;\&\; t^*_\ell = t^*_k$

$\boxtimes_2$ let $w = \{\ell : I^*_\ell = \{t^*_\ell\}\}$ then $\ell \mathbf{e} k \wedge \ell \in w \Rightarrow k \in w \;\&\; t^*_\ell = t^*_k$

$\boxtimes_3$ for each $J_0 \in Y^*$, the set $\{J \in Y^*$: if $\ell \mathbf{e} k, \ell \notin w$ then $t^{J_0}_k <_\ell t^J_\ell\}$ belongs to D^*.

We now choose $J_n \in Y^*$ by induction on n such that $\ell \mathbf{e} k \;\&\; \ell \notin w \Rightarrow t^{J_n}_k <_\ell t^{J_{n+1}}_\ell$. Now

$(*)_2$ if $i(*) < \omega, \eta \in {}^{i(*)}2$ <u>then</u> the types of $\bar b^0_{J_0} {}^\frown \bar b^0_{J_1} {}^\frown \ldots {}^\frown \bar b^0_{J_{i(*)-1}}$ and of $\bar b^{\eta(0)}_{J_0} {}^\frown \bar b^{\eta(2)}_{J_1} {}^\frown \ldots {}^\frown \bar b^{\eta(i(*)-1)}_{J_{i(*)-1)}}$ over A are equal.
[Why? By the indiscernibility, see Definition 3.3.]

Now by clauses (iv) + (v) in the choice of t^J_ℓ, s^J_ℓ (for $\ell < m_{J_\ell} = m(*)$) for each i there is $\varphi_i \in \Delta$ and $\bar d_i \subseteq A$ such that $\mathfrak{C} \models \varphi_i[\bar c, \bar b^0_{J_i}, \bar d_i] \wedge \neg\varphi_i[\bar c, \bar b^1_{J_i}, \bar d_i]$.
Now

$(*)_3$ the sequence $\langle \varphi_i(\bar y, \bar b^0_{J_i}, \bar d_i) : i < \omega\rangle$ of formulas is independent.
[Why? For each $i(*) < \omega$ and $\eta \in {}^{i(*)}2$ we need to prove that $\mathfrak{C} \models (\exists \bar y)[\bigwedge\limits_{i<i(*)} \varphi_i(\bar y, \bar b^0_{J_i}, \bar d_i)^{\eta(i)}]$. Now by $(*)_2$ it is enough to prove that $\mathfrak{C} \models (\exists \bar y)[\bigwedge\limits_{i<i(*)} \varphi_i(\bar y, \bar b^{\eta(i)}_{J_i}, \bar d^1_i)]$.
But $\bar c$ exemplifies the satisfaction of this formula so $(*)_3$ holds.]

As Δ is finite, one φ appears as φ_n for infinitely many n's (though not necessarily with the same $\bar{d}_n$ as we allow A to be infinite), so we get contradiction to "T has the dependence property".
2) Similar.
3) Follows. $\square_{3.4}$

3.5 Claim. *Assume $\bar{\mathbf{a}}^\ell = \langle \bar{a}^\ell_t : t \in I_\ell \rangle$ is an indiscernible sequence (with the linear order) I_ℓ of cofinality $\kappa > |T|$ for $\ell = 1,2$. Then we can find $s^\ell_\alpha \in I_\ell$ for $\ell = 1,2, \alpha < \kappa$ such that $\langle \bar{a}^1_{s^1_\alpha} \char`\^ \bar{a}^2_{s^2_\alpha} : \alpha < \kappa \rangle$ is an indiscernible sequence with $\langle s^\ell_\alpha : \alpha < \kappa \rangle$ being $<_{I_\ell}$-increasing unbounded in I_ℓ for $\ell = 1,2$.*

Proof. Easy by 3.3, just choose s^1_i, s^2_i by induction on i. $\square_{3.5}$

See more in 4.11.
As 3.5 deal with $\Delta = \mathbb{L}_{\tau(T)}$, we can derive the parallel result for finite $\Delta \subseteq \mathbb{L}_{\tau(T)}$.
3.6 Conclusion 1) Assume

- $(*)$ $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence over A, comp(I) the completion of I.

For every $C \subseteq \mathfrak{C}$ there are $\langle n_{\varphi(\bar{x},\bar{y})} : \varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\tau(T)} \rangle$, a sequence of finite numbers, $J \subseteq$ comp(I) of cardinality $\le |C| + |T|$ and $\langle J_{\varphi(\bar{x},\bar{y},\bar{c})} : \varphi \in \mathbb{L}_{\tau(T)} \rangle, J_{\varphi(\bar{x},\bar{y},\bar{c})}$ a finite subset of J such that:

- $(*)_0$ if J_* is an initial segment of J including $\cup \{J_{\varphi(\bar{x},\bar{y},\bar{c})} : \varphi \in \mathbb{L}_{\tau(T)}, \bar{c} \subseteq C\}$ then $\bar{\mathbf{b}} \restriction (J \backslash J_*)$ is an indiscernible sequence over $A \cup C \cup \{\bar{b}_t : t \in J_*\}$
- $(*)_1$ for every $\bar{a} \in {}^{\ell g(\bar{y})}A$ and $\varphi = \varphi(\bar{x},\bar{y},\bar{c}), \bar{c} \subseteq C$ there are $n \le n_{\varphi(\bar{x},\bar{y})}$ and $t_1 < \ldots < t_n$ from J_φ such that if $r,s \in I \backslash \{t_1,\ldots,t_n\}$ and $m \in [1,n] \Rightarrow (s <_I t_m) \equiv (r <_I t_m)$ then $\models \varphi[\bar{b}_s,\bar{a}] \equiv \varphi[\bar{b}_r,\bar{a}]$
- $(*)_2$ for every $k < \omega, \bar{a} \in {}^{\ell g(\bar{y})}A, \bar{c} \in {}^{\ell g(\bar{z})}A$ and $\varphi = \varphi(\bar{x}_1,\ldots,\bar{x}_k,\bar{z},\bar{y})$ there are $n \le n_\varphi$ and $t_1 < \ldots < t_n$ from J_φ such that if $s_1 <_I \ldots <_I s_k$ and $r_1 <_I \ldots <_I r_k$ are from J and $m \in [1,n]$ & $\ell \in [1,k] \Rightarrow (s_\ell <_I t_m \equiv r_\ell <_I t_m)$ & $(t_m <_I s_\ell \equiv t_m <_I r_\ell)$ then $\models \varphi[\bar{b}_{s_1},\ldots,\bar{b}_{s_k},\bar{c},\bar{a}] \equiv \varphi[\bar{b}_{r_1},\ldots,\bar{b}_{r_k},\bar{c},\bar{a}]$.

2) Assume

- $(*)_3$ $\langle \bar{b}_{u,\alpha,\ell} : \ell < n, u \in [I]^\ell, \alpha < \alpha_\ell \rangle$ is indiscernible over A and $\alpha_\ell < \omega$ for $\ell < n$ (and $n < \omega$).

For every $\bar{c}$ there are $J \subseteq I, |J| \leq |T|$ and finite $J_\varphi \subseteq J$ for $\varphi \in \mathbb{L}_{\tau(T)}$ such that the parallel of $(*)_1, (*)_2$ hold.

Proof. 1) This restates 3.2, 3.4.
2) Similar.

3.7 Question: If $<$ $(= \varphi(x, y, \bar{c}))$ is a partial order with infinite increasing sequences, we may consider κ-directed subsets, $\kappa = \operatorname{cf}(\kappa) > |T|$, they define a Dedekind cut.

What about orthogonality of those?

3.8 Conclusion. 1) Assume $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible set over A. For every B there is $J \subseteq I$ such that $|J| \leq |T| + |B|$ and $\langle \bar{b}_t : t \in I \backslash J \rangle$ is an indiscernible set over $A \cup B$.

Proof. Easy.

3.9 Claim/Definition. *Assume $p \in \mathbf{S}^m(B)$ is a stable type and $B \subseteq M$. Then* $\dim(p, M) = |\mathbf{I}| + |T|$ *for any* $\mathbf{I}$ *a maximal indiscernible set* $\subseteq {}^m M$ *base on p is well defined.*

§4 Perpendicular endless indiscernible sequences

Dimension and orthogonality play important role in [Sh:c], see in particular Ch.V. Now, as our prototype is the theory $\mathrm{Th}(\mathbb{Q},<)$, it is natural to look at cofinality, this is dual-cf$(\bar{\mathbf{b}},A)$ defined below (4.5(3)), measuring the cofinality of approaching $\bar{\mathbf{b}}$ from above (here $\bar{\mathbf{b}}$ is always indiscernible sequences with no last member). So a relative of orthogonality which we call perpencidularity suggest itself as relevant. It is defined in 4.5, as well as equivalence and dual-cf. Now perpendicularity is closely related to mutual indiscernibility (see 4.7(1), 4.11(2)), hence if T is unstable, then there are lots of pairwise perpendicular indiscernible sequences: if $\langle \bar{a}_\alpha : \alpha < \lambda\rangle$ is an indiscernible sequence, not an indiscernible set and $\bar{\mathbf{b}}^\alpha = \langle \bar{a}_{\omega\alpha+n} : n < \omega\rangle$ for $\alpha < \lambda$ then $\{\bar{\mathbf{b}}^\alpha : \alpha < \lambda\}$ are pairwise perpendicular. In this section we present basic properties of perpendicularity. In particular, it is preserved by equivalence (4.11(5)). For perpendicular sequences, we can more easily restrict them to get mutually indiscernible sets than in §3. In particular we show that if $\mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{b}}^1)) \neq \mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{b}}^2))$ then $\bar{\mathbf{b}}^1,\bar{\mathbf{b}}^2$ are perpendicular.

But for indiscernible sets perpendicularity does not become orthogonality, in fact it is trivial (see 4.15).

The case of looking at more than two indiscernible sequences reduced to looking at all pairs (4.14(2), 4.17(2)). Also, as in [Sh:c, V], if $\bar{\mathbf{b}}$ is not perpendicular to $\bar{\mathbf{a}}^\zeta$ for $\zeta < \zeta^*$ and the $\bar{\mathbf{a}}^\zeta$-s are pairwise perpendicular then $\zeta^* < |T|^+$ (see 4.19).

Lastly, we recall (from [Sh:c]) the density of "types not splitting over small sets" (for theories with the non independence property), hence the existence of a "quite constructible" model over any A.

We think

<u>4.1 Thesis</u>: First order T with the dependence property is somewhat like the theory of the rational order (or real closed fields).

If M is a model of $(\mathbb{Q},<)$ and $\langle (I^-_\alpha, I^+_\alpha) : \alpha < \alpha^*\rangle$ is a sequence of pairwise distinct Dedekind cuts of M, and N_α is a dense linear order for $\alpha < \alpha^*$ and N is M when in the cut (I^-_α, I^+_α) we insert N_α, then $M \prec N$; so we have total freedom of what we put in the cuts.

In the next section we shall prove that if $\langle \bar{\mathbf{a}}^\alpha : \alpha < \alpha^*\rangle$ is a sequence of pairwise perpendicluar endless indiscernible sequences that we have quite a total freedom in choosing $\langle \text{dual-cf}(\bar{\mathbf{a}}^\alpha, M) : \alpha < \alpha^*\rangle$, this is a parallel for the above property of $\mathrm{Th}(\mathbb{Q},<)$.

<u>4.2 Thesis</u>: If $\bar{\mathbf{b}}^1,\bar{\mathbf{b}}^2$ are endless indiscernible sequences, which are not perpendicular, then there is in a sense an inside definable function showing that $\mathbf{I}_\ell = \{\bar{c} \in M : \bar{c}$ realizes $\mathrm{Av}(\bar{\mathbf{b}}^\ell,\bar{\mathbf{b}}^\ell)\}$ has the same cofinality.

4.3 Hypothesis. T has the dependence property.

4.4 Definition. 1) We say the infinite sequences $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible over A if $\bar{\mathbf{b}}^\ell$ is an indiscernible sequence over $\cup\{\bar{b}_t^{3-\ell} : t \in \mathrm{Dom}(\bar{\mathbf{b}}^{3-\ell})\} \cup A$ for $\ell = 1, 2$. If we omit "over A" we mean $A = \emptyset$.
2) We say that the family $\{\bar{\mathbf{b}}^\zeta : \zeta < \zeta^*\}$ of sequences is mutually indiscernible over A, if for $\zeta < \zeta^*$ the sequence $\bar{\mathbf{b}}^\zeta$ is an indiscernible sequence over $\cup\{\bar{b}_t^\varepsilon : \varepsilon \neq \zeta, \varepsilon < \zeta^*, t \in \mathrm{Dom}(\bar{\mathbf{b}}^\varepsilon)\} \cup A$.
3) We say "$\mathbf{b}^1, \bar{\mathbf{b}}^2$ are mutually Δ-indiscernible over A" if $\mathbf{b}^\ell$ is a Δ-indiscernible sequence over $\cup\{\bar{b}_t^{3-\ell} : t \in \mathrm{Dom}(\bar{\mathbf{b}}^{3-\ell})\} \cup A$ for $\ell = 1, 2$. Similarly in part (2).

4.5 Definition. Let $\bar{\mathbf{a}}^\ell = \langle \bar{a}_t^\ell : t \in I_\ell \rangle$ be an indiscernible sequence which is endless (i.e., I_ℓ having no last element) for $\ell = 1, 2$.
1) We say that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular when:

> $(*)$ if $\bar{b}_n^\ell$ realizes $\mathrm{Av}(\{\bar{b}_m^k$: we have $m < n$ & $k \in \{1, 2\}$ or we have $m = n$ & $k < \ell\} \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^\ell)$ for $\ell = 1, 2$ then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible (see 4.4 above) where $\bar{\mathbf{b}}^\ell = \langle \bar{b}_n^\ell : n < \omega \rangle$ for $\ell = 1, 2$.

We define "Δ-perpendicular" in the obvious way.
2) We say $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are equivalent and write $\approx$ if for every $A \subseteq \mathfrak{C}$ we have $\mathrm{Av}(A, \bar{\mathbf{a}}^1) = \mathrm{Av}(A, \bar{\mathbf{a}}^2)$.
3) If $\bar{\mathbf{a}}^1 \subseteq A$ we let dual-cf$(\bar{\mathbf{a}}^1, A) = \mathrm{Min}\{|B| : B \subseteq A$ and no $\bar{c} \in {}^{\omega>}A$ realizes $\mathrm{Av}(B, \bar{\mathbf{a}}^1)\}$; we usually apply this when $A = M$.

4.6 Example: M a model of $\mathrm{Th}(\mathbb{Q}, <)$.
$\bar{\mathbf{b}}^\ell = \langle b_n^\ell : n < \delta_\ell \rangle$ is an increasing sequence in M.
Then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are not perpendicular iff they define the same cut of M.

4.7 Claim. *1) If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are endless mutually indiscernible sequences, then they are perpendicular.*
2) "Mutually indiscernible" and "perpendicular" are symmetric relations.
3) On the family of endless indiscernible sequences, being equivalent is an equivalence relation.
4) In Definition 4.5(1) in $(*)$ there, to say "for every such $\langle \bar{b}_n^\ell : n < \omega, \ell = 1, 2 \rangle$" and to say "for some $\langle \bar{b}_n^\ell : n < \omega, \ell = 1, 2 \rangle$" are equivalent.
5) If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are endless indiscernible sequences and Δ-mutually indiscernible sequence then they are Δ-perpendicular.
6) If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are endless indiscernible sequence, $\bar{\mathbf{a}}^2$ is indiscernible over $\bar{\mathbf{a}}^1$ then $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular.

4.8 Remark. By 4.12 below, in $(*)$ of Definition 4.5(1), for any set A we can add: $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible over A; that is for any A, if $\bar{b}^\ell_n$ realizes $\mathrm{Av}(\{\bar{b}^k_m : m < n \ \& \ k \in \{1,2\} \text{ or } m = n \ \& \ k < \ell\} \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup A, \bar{\mathbf{a}}^\ell)$ <u>then</u> $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible over A where $\bar{\mathbf{b}}^\ell = \langle \bar{b}^\ell_n : n < \omega \rangle$ for $\ell = 1, 2$.

Proof. 1) Let $\bar{b}^\ell_n$ for $\ell \in \{1,2\}, n < \omega$ be as in Definition 4.5(1).

Now we prove by induction on $k < \omega$ that

$(*)_k$ the sequences $\bar{\mathbf{a}}^{1,k} = \bar{\mathbf{a}}^1{}^\frown \langle \bar{b}^1_{k-1}, \dots, \bar{b}^1_0 \rangle, \bar{\mathbf{a}}^{2,k} = \bar{\mathbf{a}}^2{}^\frown \langle \bar{b}^2_{k-1}, \dots, \bar{b}^2_0 \rangle$ are mutually indiscernible.

For $k = 0$ this is assumed. For $k = m + 1$, by the choise of $\bar{b}^1_k$ as realizing $\mathrm{Av}(\bar{\mathbf{a}}^{1,k}{}^\frown \bar{\mathbf{a}}^{2,k}, \bar{\mathbf{a}}^1)$ clearly $\bar{\mathbf{a}}^{1,k+1}, \bar{\mathbf{a}}^{2,k}$ are mutually indiscernible. Similarly by the choice of $\bar{b}^2_k$, clearly $\bar{\mathbf{a}}^{1,k+1}, \bar{\mathbf{a}}^{2,k+1}$ are mutually indiscernible.

Now the statement "$\langle \bar{b}^1_n : n < \omega \rangle, \langle \bar{b}^2_n : n < \omega \rangle$ are mutually indiscernible" is a local condition, i.e., it is enough to check it for $\langle \bar{b}^1_n : n < k \rangle, \langle \bar{b}^2_n : n < k \rangle$ for each $k < \omega$, but this holds by $(*)_k$ above.

2) Read the definition and rename.

3) Let $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^3$ be endless indiscernible sequences. Clearly $\mathrm{Av}(A, \bar{\mathbf{a}}^1) = \mathrm{Av}(A, \bar{\mathbf{a}}^1)$ so $\bar{\mathbf{a}}^1 \approx \bar{\mathbf{a}}^1$ by Definition 4.5, so $\approx$ is reflexive. Also $\mathrm{Av}(A, \bar{\mathbf{a}}_1) = \mathrm{Av}(A, \bar{\mathbf{a}}_2) \Leftrightarrow \mathrm{Av}(A, \bar{\mathbf{a}}_2) = \mathrm{Av}(A, \bar{\mathbf{a}}_1)$ so $\approx$ is symmetric. Lastly, if $\bar{\mathbf{a}}^1 \approx \bar{\mathbf{a}}^2$ and $\bar{\mathbf{a}}^2 \approx \bar{\mathbf{a}}^3$ then for any A we have $\mathrm{Av}(A, \bar{\mathbf{a}}^1) = \mathrm{Av}(A, \bar{\mathbf{a}}^2) \wedge \mathrm{Av}(A, \bar{\mathbf{a}}^2) = \mathrm{Av}(A, \bar{\mathbf{a}}^3)$ hence $\mathrm{Av}(A, \bar{\mathbf{a}}^1) = \mathrm{Av}(A, \bar{\mathbf{a}}^3)$, as this holds for any A we can deduce that $\bar{\mathbf{a}}^1 \approx \bar{\mathbf{a}}^3$, i.e. $\approx$ is transitive. So $\approx$ is really an equivalence relation.

4) Suppose that for $i \in \{1,2\}$ we have $\langle \bar{b}^{i,\ell}_n : n < \omega, \ell \in \{1,2\} \rangle$ such that $\bar{b}^{i,\ell}_n$ realizes $\mathrm{Av}(\{\bar{b}^{i,k}_m : m < n \ \& \ k \in \{1,2\} \text{ or } m = n \ \& \ k < \ell\} \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^\ell)$. We can choose an increasing sequence of elementary mapping f^ℓ_n $(n < \omega, \ell < 2)$ such that $n_1 < n_2 \vee (n_1 = n_1 \wedge \ell_1 < \ell_2) \Rightarrow f^{\ell_1}_{n_1} \subseteq f^{\ell_2}_{n_2}, f^0_0$ is the identity on $\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2$, $\mathrm{Dom}(f^1_n) = \mathrm{Dom}(f^0_n) \cup \bar{b}^{1,1}_n$, $\mathrm{Dom}(f^0_{n+1}) = \mathrm{Dom}(f^1_n) \cup \bar{b}^{1,2}_n$, $f^1_n(\bar{b}^{1,1}_n) = \bar{b}^{2,1}_n$, $f^0_{n+1}(\bar{b}^{1,2}_n) = \bar{b}^{2,2}_n$. No problem to carry the induction and $f^* = \cup\{f^0_n : n < \omega\}$ can be extended to an automorphism of $\mathfrak{C}$ thus proving the claim.

5) Similar to (1).

6) Left to the reader (see 4.16). $\square_{4.7}$

4.9 Claim. *Assume that for $\ell = 1, 2$ we have:*

$(*)_\ell(a)$ *I_ℓ, J_ℓ are endless linear orders*

(b) *$\bar{\mathbf{b}}^\ell = \langle \bar{b}^\ell_t : t \in I_\ell \rangle$ is an indiscernible sequence*

(c) *$\bar{\mathbf{a}}^\ell = \langle \bar{a}^\ell_t : t \in J_\ell \rangle$ is an indiscernible sequence*

(d) *for $s \in J_\ell$ we have $\bar{a}^\ell_s = \bar{b}^\ell_{t(\ell,s,1)}{}^\frown \dots {}^\frown \bar{b}^\ell_{t(\ell,s,n_\ell)}$*

(e) $t(\ell,s,1) <_{I_\ell} t(\ell,s,2) <_{I_\ell} \ldots <_{I_\ell} t(\ell,s,n_\ell)$

(f) *if* $s_1 <_{J_\ell} s_2$ *then* $t(\ell,s_1,n_\ell) <_{I_\ell} t(\ell,s_2,1)$.

0) If $\bar{\mathbf{b}}^\ell$ is an indiscernible sequence, then so is $\bar{\mathbf{a}}^\ell$.
1) If $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible, then $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are mutually indiscernible.
2) Assume that $\{t(\ell,s,1) : s \in J_\ell\}$ is an unbounded subset of I_ℓ for $\ell = 1,2$. If $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are perpendicular, then $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular.
3) Like part (2) for "equivalent".

Proof. Just think.
(Concerning (2) see 4.11(4) below). $\square_{4.9}$

4.10 Claim. *Assume that $\bar{\mathbf{b}}^\ell = \langle \bar b^\ell_t : t \in I_\ell\rangle$ is an endless indiscernible sequence of m_ℓ-tuples, so $\bar b^\ell_t = \langle b^\ell_{t,m} : m < m_\ell\rangle$. Assume that $u_\ell \subseteq \{0,\ldots,m_\ell - 1\}$ and $\bar a^\ell_t = \bar b^\ell_t \restriction u_\ell$ for $t \in I_\ell, \ell = 1,2$.*
1) If $\langle \bar b^1_t : t \in I_1\rangle, \langle \bar b^2_t : t \in I_2\rangle$ are mutually indiscernible, then $\langle \bar a^1_t : t \in I_1\rangle, \langle \bar a^2_t : t \in I_2\rangle$ are mutually indiscernible.
2) If $\langle \bar b^1_t : t \in I_1\rangle, \langle \bar b^2_t : t \in I_2\rangle$ are perpendicular, then $\langle \bar a^1_t : t \in I_1\rangle, \langle \bar a^2_t : t \in I_2\rangle$ are perpendicular.
3) Of course, permuting, duplicating or renaming the indiscernible in $\langle b^\ell_{t,m} : m < m_\ell\rangle$, etc., also is O.K.

Proof. Easy.

4.11 Claim. *1) If $\bar{\mathbf{a}}^\ell = \langle a^\ell_t : t \in I_\ell\rangle$ is an indiscernible sequence for $\ell = 1,2$ and $|T| < \mathrm{cf}(I_1), |I_1| < \mathrm{cf}(I_2)$, then for some end segments J_1, J_2 of I_1, I_2 respectively, $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$ are mutually indiscernible; similarly with over A.*
1A) If Δ is finite to deduce just Δ-mutually indiscernible, we can omit $|T| < \mathrm{cf}(I_1)$.
2) If $\bar{\mathbf{a}}^\ell = \langle a^\ell_t : t \in I_\ell\rangle$ is an indiscernible sequence for $\ell = 1,2$ and $\mathrm{cf}(I_1), \mathrm{cf}(I_2)$ are infinite and distinct then $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular.
3) Assume that $\bar{\mathbf{a}}^\ell = \langle \bar a^\ell_t : t \in I_\ell\rangle$ is an endless indiscernible sequence for $\ell = 1,2$, δ is limit ordinal and $\bar b^\ell_\alpha$ realizes $\mathrm{Av}(\{\bar b^k_\beta : \beta < \alpha \ \ \& \ \ k \in \{1,2\}$ or $\beta = \alpha \ \ \& \ \ k < \ell\} \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^\ell)$ and $\bar{\mathbf{b}}^\ell = \langle \bar b^\ell_\alpha : \alpha < \delta\rangle$ for $\ell = 1,2$. Then the following are equivalent:

(α) $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ *are perpendicular*

(β) $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ *are perpendicular.*

4) If $\bar{\mathbf{a}}^\ell = \langle a^\ell_t : t \in I_\ell\rangle$ is an endless indiscernible sequence and $J_\ell \subseteq I_\ell$ is unbounded for $\ell = 1,2$, <u>then</u> $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular iff $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$ are perpendicular.
5) If $\bar{\mathbf{a}}^\ell = \langle a_t : t \in I^\ell\rangle$ are an endless indiscernible sequence for $\ell = 1,2,3,4$ and $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^3$ are equivalent and $\bar{\mathbf{a}}^2, \bar{\mathbf{a}}^4$ are equivalent, <u>then</u> $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular iff $\bar{\mathbf{a}}^3, \bar{\mathbf{a}}^4$ are perpendicular; so perpendicularity of $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ depend just on $\bar{\mathbf{a}}^1/\approx, \bar{\mathbf{a}}_2/\approx$.

Proof. 1) By 3.6(1) applied to $A = \cup\{\bar{a}^1_t : t \in I_1\}$ and $\bar{\mathbf{a}}^2$, there is an end segment J_2 of I_2 such that $\bar{\mathbf{a}}^2 \restriction J_2$ is an indiscernible sequence over A. Let J'_2 be a countable subset of J_2 and apply 3.6(1) to $A' = \cup\{\bar{a}^2_t : t \in J'_2\}$ and $\bar{\mathbf{a}}^1$, so there is an end segment J_1 of I_1 such that $\bar{\mathbf{a}}^1 \restriction J_1$ is an indiscernible sequence over A'. Reflecting on the meaning clearly $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$ are mutually indiscernible.
1A) Similar to (A); without loss of generality Δ is closed under permuting and identifying the variables, and we use the relevant variant of 3.6(1) or just 3.4.
2) Without loss of generality $\mathrm{cf}(I_1) < \mathrm{cf}(I_2)$. It is enough for every formula $\varphi = \varphi(\bar{x}_1,\dots,\bar{x}_m,\bar{y}_1,\dots,\bar{y}_k)$ with $\ell g(\bar{x}_\ell) = \ell g(\bar{a}^1_t)$ and $\ell g(\bar{y}_\ell) = \ell g(\bar{a}^2_t)$ to show that for some $t_1 \in I_1, t_2 \in I_2$:

- $\boxtimes$ for some truth value $\mathbf{t}$, for all $t_1 <_{I_1} t^1_1 <_{I_1} \dots <_{I_1} t^1_m, t_2 <_{I_2} t^2_1 <_{I_2} \dots <_{I_2} t^2_k$ we have $\models \varphi[\bar{b}^1_{t^1_1},\dots,\bar{b}^1_{t^1_m},\bar{b}^2_{t^2_1},\dots,\bar{b}^2_{t^2_k}]^{\mathbf{t}}$.

By (1A) and 4.7(5) this is easy (as for each finite Δ we can use suitable end segments).
3) If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular then by definition 4.5(1) we have $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible and by 4.7(1) this implies that $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are perpendicular. The other direction is even easier, though we have to use 4.7(4).
4) Let $\bar{b}^1_n, \bar{b}^2_n$ (let $n < \omega$ be as in $(*)$ of Definition 4.5(1)) for $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$. Now for every set A, the types $\mathrm{Av}(A, \bar{\mathbf{a}}^\ell)$, $\mathrm{Av}(A, \bar{\mathbf{a}}^\ell \restriction J_\ell)$ are equal (see 1.10(2)). As $(\bar{\mathbf{a}}^1 \restriction J_1) \cup (\bar{\mathbf{a}}^2 \restriction J_2)$ is included in $(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2)$ clearly $\langle \bar{b}^1_n : n < \omega\rangle, \langle \bar{b}^2 : n < \omega\rangle$ are as required in $(*)$ for $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$.
By 4.7(4) we have $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular iff $\langle \bar{b}^1_n : n < \omega\rangle, \langle b^2_n : n < \omega\rangle$ are mutually indiscernible iff $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$ are perpendicular; so we are done.
5) We choose $\bar{b}^\ell_n$ by induction on $2n+\ell$ for $n < \omega, \ell \in \{1,2\}$ as any sequence realizing $p^\ell_n = \mathrm{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \bar{\mathbf{a}}^3 \cup \bar{\mathbf{a}}^4 \cup \{\bar{b}^k_m : m < n \ \& \ k \in \{1,2\} \text{ or } m = n \ \& \ k < \ell\}, \bar{\mathbf{a}}^\ell)$. So $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular <u>iff</u> $\langle \bar{b}^1_n : n < \omega\rangle, \langle \bar{b}^2_n : n < \omega\rangle$ are mutually indiscernible (by 4.7(4)).

Now by the assumption (on the equivalence) the type p^ℓ_n is also equal to $\mathrm{Av}(\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \bar{\mathbf{a}}^3 \cup \bar{\mathbf{a}}^4 \cup \{\bar{b}^k_m : m < n \text{ or } m = n \ \& \ k < \ell\}, \bar{\mathbf{a}}^{2+\ell})$.

Using again 4.7(4) we have: $\bar{\mathbf{a}}^3, \bar{\mathbf{a}}^4$ are perpendicular iff $\langle \bar{b}^1_n : n < \omega\rangle, \langle \bar{b}^2_n : n < \omega\rangle$ are mutually indiscernible.
Together we get the desired conclusion. $\square_{4.11}$

4.12 Claim. *Let $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ be endless indiscernible sequences. The following are equivalent:*

- (A) *$\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular*
- (B) *there are $A, \bar{b}^1_n, \bar{b}^2_n$ (for $n < \omega$) such that*
  - (a) *$\bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \subseteq A$ and $\bar{b}^\ell_n$ realizes $\mathrm{Av}(A \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2 \cup \{\bar{b}^k_m : n < n \ \& \ k \in \{1,2\}$ or $m = n \ \& \ k < \ell\}, \bar{\mathbf{a}}^\ell)$ and*
  - (b) *the sequences $\bar{\mathbf{b}}^1 = \langle \bar{b}^1_n : n < \omega\rangle, \bar{\mathbf{b}}^2 = \langle \bar{b}^2_n : n < \omega\rangle$ are mutually indiscernible*
- (C) *for every $A \supseteq \bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ and $\bar{b}_1$ realizing $\mathrm{Av}(A, \bar{\mathbf{a}}^1)$ and $\bar{b}_2$ realizing $\mathrm{Av}(A, \bar{\mathbf{a}}^2)$ the sequence $\bar{b}_1$ realizes $\mathrm{Av}(A \cup \bar{b}_2, \bar{\mathbf{a}}^1)$*
- (D) *if $A \supseteq \bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ and $\bar{b}^1_n, \bar{b}^2_n$ are as in clause $(B)(a)$ then $\langle \bar{b}^1_n : n < \omega\rangle, \langle \bar{b}^2_n : n < \omega\rangle$ are mutually indiscernible over A.*

Proof.

$\underline{(D) \Rightarrow (B)}$:

We can find $A, \bar{b}^\ell_n$ for $n < \omega, \ell \in \{1,2\}$ which are as in clause (B) except possibly the mutual indiscernibility in the end, i.e., as in $(*)$ of Definition 4.5(1) but with $A \supseteq \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2$. By clause (D) this suffices.

$\underline{(C) \Rightarrow (D)}$:

Now we can prove by induction on $i < \omega$ that:

$\boxtimes$ if $m_1 < m_2 < n_1 < \ldots < n_i < \omega$ and $\ell_1, \ldots, \ell_i \in \{1,2\}$ then $\bar{b}^1_{m_1} \hat{}\, \bar{b}^2_{m_2} \hat{}\, \bar{b}^{\ell_1}_{n_1} \hat{}\, \ldots \hat{}\, b^{\ell_i}_{n_i}$ and $\bar{b}^1_{m_2} \hat{}\, \bar{b}^2_{m_1} \hat{}\, \bar{b}^{\ell_1}_{n_1} \hat{}\, \ldots \hat{}\, \bar{b}^{\ell_i}_{n_i}$ realized the same type over $A \cup \{\bar{b}^\ell_n : n < m_1, \ell \in \{1,2\}\}$.
[How? For $i = 0$ as we are assuming clause (C), for $i+1$, because the type $\mathrm{tp}(\bar{b}^{\ell_{i+1}}_{n_{i+1}}, A \cup \{b^\ell_n : n < n_{i+1}, \ell \in \{1,2\}\})$ does not split over $\bar{\mathbf{a}}^{\ell_{i+1}}$ by the definition of Av.]

By transitivity of equality of type from $\boxtimes$ we can prove that $\langle \bar{b}^1_{2n+1} : n < \omega\rangle, \langle \bar{b}^2_{2n+2} : n < \omega\rangle$ are mutually indiscernible over A; as in 4.7(4) this suffices.

$\underline{(B) \Rightarrow (A)}$:

If $A, \bar{b}^\ell_n$ for $n < \omega, \ell \in \{1,2\}$ are as in clause (B), then as $[B_1 \subseteq B_2 \Rightarrow \mathrm{Av}(B_1, \bar{\mathbf{a}}^\ell) \subseteq \mathrm{Av}(B_2, \bar{\mathbf{a}}^\ell)]$ they are as in $(*)$ of Definition 4.5. So by 4.7(4) the sequences $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^1$ are perpendicular, i.e., clause (A) holds.

$\underline{(A) \Rightarrow (C)}$:

Assume that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular but clause (C) fails for the[4] set A. So $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2 \subseteq A$, let $\lambda \ge \aleph_0$.
Choose $\bar{b}^\ell_\alpha$ for $\alpha < \lambda^+, \ell \in \{1,2\}$ by induction on α such that $\bar{b}^\ell_\alpha$ realizes $\mathrm{Av}(A \cup \cup\{\bar{b}^k_\beta : \beta < \alpha \vee (\beta = \alpha \ \ \& \ \ k < \ell\}, \bar{a}^\ell)$. Easily by the choice of A for some $\bar{c} \subseteq A$ and $\varphi(\bar{x},\bar{y},\bar{c})$ we have $\mathfrak{C} \models \varphi[\bar{b}^1_\alpha, \bar{b}^2_\beta, \bar{c}]$ iff $\alpha \le \beta$. By the mutual indiscernibility of $\langle \bar{b}^1_\alpha : \alpha < \lambda\rangle, \langle \bar{b}^2_\alpha : \alpha < \lambda\rangle$ (which holds as $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular) the set $\{\varphi(\bar{b}^1_\alpha, \bar{b}^2_\alpha, \bar{z}) : \alpha < \lambda\}$ of formulas is independent, contradiction. $\square_{4.12}$

4.13 Claim. *Assume $\bar{\mathbf{a}}^\ell = \langle \bar{a}^\ell_t : t \in I_\ell\rangle$ are endless indiscernible sequences for $\ell = 1,2$.*
1) If $\bar{\mathbf{a}}^1$ is an indiscernible sequence over A, <u>then</u>: $\bar{\mathbf{a}}^1$ is an indiscernible set over A <u>iff</u> $\bar{\mathbf{a}}^1$ is an indiscernible set over $\emptyset$.
2) Assume that $\bar{\mathbf{a}}^1$ is an infinite indiscernible sequence over A, <u>then</u>: $\bar{\mathbf{a}}^1$ is non-stable in $\mathfrak{C}$ <u>iff</u> $\bar{\mathbf{a}}^1$ is non-stable in $(\mathfrak{C}, c)_{c \in A}$.
3) If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are equivalent, <u>then</u> $\bar{\mathbf{a}}^1$ is non-stable <u>iff</u> $\bar{\mathbf{a}}^2$ is non-stable.
4) Assume that for $\ell = 1,2$, $J_\ell \subseteq I_\ell$ is convex and infinite and $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are mutually indiscernible <u>then</u> $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$ are mutually indiscernible over $\bigcup\limits_{\ell=1}^{2} (\bar{\mathbf{a}}^\ell \restriction (I_\ell \backslash J_\ell))$.
5) Let $k \in \{1,2\}$. In part (4) we can omit "J_k a convex (subset of I_k)" if $\bar{\mathbf{a}}^k$ is an indiscernible set.

Proof. 1) By 1.28.
2) Follows.
3), 4), 5) Check directly. $\square_{4.13}$

4.14 Claim. *1) Assume $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are endless indiscernible sequences. If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ has cofinality $> |T|$ and are mutually indiscernible and $\bar{b} \in {}^{\omega>}\mathfrak{C}$, <u>then</u> for some end-segments J_1, J_2 of $\mathrm{Dom}(\bar{\mathbf{a}}^1), \mathrm{Dom}(\bar{\mathbf{a}}^2)$ respectively $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$ are mutually indiscernible over $\bar{b}$.*
1A) Like (1) for mutual Δ-indiscernibility, when Δ_2 is finite, Δ_1 finite large enough, $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ just endless.
2) Assume $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2, \bar{\mathbf{b}}^3$ are endless indiscernible sequences $\subseteq A$ and I is an infinite linear order and $\bar{a}^\ell_t$ realizes $\mathrm{Av}(\{\bar{a}^k_s : s <_I t \ \ \& \ \ k \in \{1,2,3\}$ or $s = t \ \ \& \ \ k < \ell\} \cup A, \bar{\mathbf{b}}^\ell)$ for $\ell \in \{1,2,3\}$ and $t \in I$ and let $\bar{\mathbf{a}}^\ell = \langle \bar{a}^\ell_t : t \in I\rangle$, <u>then</u>:

(a) *$\langle \bar{a}^1_t {}^\frown \bar{a}^2_t {}^\frown \bar{a}^3_t : t \in I\rangle$ is an indiscernible sequence over A;*

---

[4] the choice of $\bar{b}^1, \bar{b}^2$ of course is immaterial as: if $\bar{b}', \bar{b}''$ realizes $\mathrm{Av}(A, \bar{\mathbf{a}}^1)$ and $\bar{b}^2$ realizes $\mathrm{Av}(A + \bar{b}' + \bar{b}'', \bar{\mathbf{a}}^2)$ then $\bar{b}^2 {}^\frown \bar{b}', \bar{b}^2 {}^\frown \bar{b}''$ realizes the same type over A

(b) *if* $\bar{\mathbf{a}}^1$ *is an indiscernible set* <u>then</u> $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ *are mutually indiscernible over* A

(c) *if any two of* $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^3$ *are mutually indiscernible and* I_1, I_2, I_3 *are disjoint*[5] *unbounded subsets of* I, <u>then</u> $\bar{\mathbf{a}}^1 \restriction I_1, \bar{\mathbf{a}}^3 \restriction I_3$ *are mutually indiscernible over* $A \cup (\bar{\mathbf{a}}^2 \restriction I_2)$

(d) *if* $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ *are mutually indiscernible then they are mutually indiscernible over* A.

Proof. 1) No new point so left to the reader.
2) Without loss of generality I is dense with no first, no last elements, and I is not a complete even restricted to an interval and every interval has cardinality $> |T|$. Now

<u>Clause (a)</u>:

Easy as in 1.6(2).

<u>Clause (b)</u>:

For any $s_1 <_I < \ldots <_I s_{n-1}$, stipulating $s_0 = -\infty, s_n = +\infty$ and letting $I_\ell = \{t \in I : s_\ell <_I t \leq_I s_{\ell+1}\}$, by the construction we know that: the sequences $\bar{\mathbf{a}}^1 \restriction I_0, \ldots, \bar{\mathbf{a}}^1 \restriction I_{n-1}$ are mutually indiscernible over $\bar{a}^2_{s_0} {}^\frown \ldots {}^\frown \bar{a}^2_{s_{n-1}}$. Recalling that every interval has cardinality $> |T|$, by 3.8 this implies that $\bar{\mathbf{a}}^1$ is an indiscernible set (see 4.13(1)) over $\bar{a}^2_{s_1} {}^\frown \ldots {}^\frown \bar{a}^2_{s_{m-1}}$; as this holds for any $n < \omega$ and $s_1 <_I \ldots <_I s_{n-1}$, we get that $\bar{\mathbf{a}}^1$ is an indiscernible set over $\bar{\mathbf{a}}^2$. But for any $t \in I$ the sequence $\langle a^2_s : s \in I \ \& \ t \leq_I s\rangle$ is an indiscernible sequence over $\cup\{\bar{a}^1_s {}^\frown \bar{a}^2_s : s <_I t\}$. So clearly if $\{s : s <_I t\}$ is infinite then by the last two sentences, $\bar{\mathbf{a}}^2 \restriction \{s \in I : t \leq_I s\}, \bar{\mathbf{a}}^1$ are mutually indiscernible (even over $\{\bar{a}^2_s : s <_I t\}$). By the assumption on I in the beginning of the proof we are done.

<u>Clause (c)</u>:

Note that by the assumption on $\bar{a}^\ell_t$, it is enough for any pairwise disjoint $I_\ell \subseteq I$ for $\ell = 1, 2, 3$, each as we assume in the beginning of the proof of part (2), to prove that $\bar{\mathbf{a}}^1 \restriction I_1, \bar{\mathbf{a}}^2 \restriction I_2, \bar{\mathbf{a}}^3 \restriction I_3$ are mutually indiscernible.

By transitivity of equality it is enough to prove:

(∗) if $\ell(1) \neq \ell(2) \in \{1, 2, 3\}, t_1 < t_2$ in I, then the sequences $\bar{a}^{\ell(1)}_{t_1} {}^\frown \bar{a}^{\ell(2)}_{t_2}$ and $\bar{a}^{\ell(1)}_{t_2} {}^\frown \bar{a}^{\ell(2)}_{t_1}$ realizes the same type over $A \cup \{\bar{a}^\ell_t : \ell \in \{1, 2, 3\}$ and $\neg(t_1 \leq_I t \leq_I t_2)\}$.

To prove (∗) it suffices for any n and $s_1 <_I \ldots <_I s_n$ with $t_1 <_I t_2 <_I s_1$ and $k \in \{0, 1, 2, 3\}$ to prove that $\bar{a}^{\ell(1)}_{t_1} {}^\frown \bar{a}^{\ell(2)}_{t_2}$ and $\bar{a}^{\ell(1)}_{t_2} {}^\frown \bar{a}^{\ell(2)}_{t_1}$ realize the same type over

[5]the disjointness can be omitted

$A \cup \cup\{\bar a^\ell_{s_m} : m \in \{1,\dots,n-1\}\ \ \&\ \ \ell \in \{1,2,3\}$ or $m = n\ \ \&\ \ \ell \le k\} \cup \{\bar a^\ell_s : s < t_1, \ell \in \{1,2,3\}\}$. We do it by induction on $4n+k$.

For $n=0$: The sequences $\bar{\mathbf a}^{\ell(1)}, \bar{\mathbf a}^{\ell(2)}$ are perpendicular by 4.7(1) hence the required conclusion holds by $(A) \Rightarrow (C)$ of 4.12.

For $n+1$: If $k=0$ this is known (being equivalent to the case $(n',k') = (n,3)$), otherwise this follows by the definition of average more exactly as $\mathrm{tp}(\bar a^k_{s_{n+1}}, A \cup \{\bar a^\ell_{s_m} : m \in \{1,\dots,n\}\ \ \&\ \ \ell \in \{1,2,3\}$ or $m = n+1\ \ \&\ \ \ell \in k\} \cup \{\bar a^\ell_s : s < t_1, \ell \in \{1,2,3\}\})$ does not split over $\bar{\mathbf a}^k$ by 1.6(6).

Clause (d):
By 4.12. $\square_{4.14}$

4.15 Conclusion. If $\bar{\mathbf b}^1, \bar{\mathbf b}^2$ are endless indiscernible sequences and $\bar{\mathbf b}^1$ is an indiscernible set, then $\bar{\mathbf b}^1, \bar{\mathbf b}^2$ are perpendicular.

Proof. By 4.14(2), clause (b).

4.16 Claim. *1) Assume $\bar{\mathbf a}, \bar{\mathbf b}$ are endless indiscernible sequences. Then $\bar{\mathbf a}, \bar{\mathbf b}$ are perpendicular sequences, iff for any $\varphi(\bar x, \bar y, \bar c)$ for some truth values $\mathbf t$ we have:*

- (a) *for every large enough $s \in \mathrm{Dom}(\bar{\mathbf a})$, for every large enough $t \in \mathrm{Dom}(\bar{\mathbf b})$ we have $\mathfrak C \models \varphi[\bar a_s, \bar b_t, \bar c]^{\mathbf t}$*
- (b) *for every large enough $t \in \mathrm{Dom}(\bar{\mathbf b})$ for every large enough $s \in \mathrm{Dom}(\bar{\mathbf a})$ we have $\mathfrak C \models \varphi[\bar a_s, \bar b_t, \bar c]^{\mathbf t}$.*

2) For any $\varphi = \varphi(\bar x, \bar y, \bar c)$, there is a truth value $\mathbf t = \mathbf t_{\varphi(\bar x, \bar y, \bar c)}$ for which clause (a) holds and there is a truth value $\mathbf t = \mathbf t_{\varphi(\bar x, \bar y, \bar c)}$ such that clause (b) holds.

Proof. By 4.12(C) (and 1.10(2)).
Part (2) is easy too. $\square_{4.16}$

4.17 Claim. *1) The parallel of the relevant earlier claims holds for several indiscernible sequences, that is, assuming $\bar{\mathbf a}^\zeta = \langle \bar a^\zeta_t : t \in I_\zeta \rangle$ is an endless indiscernible sequence for $\zeta < \zeta^*$*

- (A) *If the intervals $[\mathrm{cf}(I_\zeta), |I_\zeta|]$ are pairwise disjoint, $\mathrm{cf}(I_\zeta) > |T| + \zeta^*$, then for some end segment J_ζ of I_ζ for $\zeta < \zeta^*$, we have $\langle \bar{\mathbf a}^\zeta \restriction J_\zeta : \zeta < \zeta^* \rangle$ is mutually indiscernible, which means: each $\bar{\mathbf a}^\zeta \restriction \mathbf J_\zeta$ is indiscernible over*

$\cup\{\bar{\mathbf{a}}^\varepsilon \restriction J_\varepsilon : \varepsilon < \zeta^* \ \& \ \varepsilon \neq \zeta\}$ *(in fact we can get indiscernibility over* $\cup\{\bar{\mathbf{a}}^\varepsilon : \varepsilon < \zeta^* \ \& \ |I_\varepsilon| < \mathrm{cf}(I_\zeta)\} \cup \cup\{\bar{\mathbf{a}}^\varepsilon \restriction J_\varepsilon : \varepsilon \in (\zeta, \zeta^*)\})$

(B) *Assume* $\langle \bar{\mathbf{a}}^\zeta : \zeta < \zeta^* \rangle$ *are mutually indiscernible,* $\bar{b} \in {}^{\omega >}\mathfrak{C}$ *and* $I_\zeta = \mathrm{Dom}(\bar{\mathbf{a}}^\zeta)$ *and* $\mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{a}}^\zeta)) > |T| + \zeta^*$. *Then there are end segments* J_ζ *of* I_ζ *for* $\zeta < \zeta^*$ *such that* $\langle \bar{\mathbf{a}}^\zeta \restriction I_\zeta : \zeta < \zeta^* \rangle$ *is mutually indiscernible over* $\bar{b}$.

(C) *If* A *is a set,* J_ζ *is an infinite linear order disjoint to* $\cup\{I_\zeta : \zeta < \zeta^*\}$ *and* $\bar{a}^\zeta_t$ *realizes* $p^\zeta_t = \mathrm{Av}(A \cup \{\bar{a}^\varepsilon_s : \varepsilon < \zeta^* \ \& \ s \in I_\varepsilon$ *or* $\varepsilon = \zeta \ \& \ s \in J_\varepsilon \ \& \ s <_{J_\varepsilon} t$ *or* $s \in J_\varepsilon \ \& \ \varepsilon < \zeta\} \cup A, \bar{\mathbf{a}}^\zeta)$ *for any* $\zeta < \zeta^*, t \in J_\zeta$ *then* $\{\langle \bar{a}^\zeta_s : s \in J_\zeta \rangle : \zeta < \zeta^*\}$ *are mutually indiscernible over* $\cup\{\bar{a}^\varepsilon_s : \varepsilon < \zeta^*, s \in I_\varepsilon\} \cup A$.

(D) *If* $\langle \bar{\mathbf{a}}^\zeta : \zeta < \zeta^* \rangle$ *are pairwise perpendicular and* $J_\zeta = J$ *for* $\zeta < \zeta^*$ *then in clause (C),* $\bar{a}^\zeta_t$ *realizes* $q^\varepsilon_t = \mathrm{Av}(\{\bar{a}^\varepsilon_s : \varepsilon < \zeta^* \ \& \ s \in I$ *or* $\varepsilon < \zeta^* \ \& \ s \in J \ \& \ s <_J t$ *or* $s = t \ \& \ \varepsilon < \zeta\} \cup A, \bar{\mathbf{a}}^\zeta)$.

(E) *if* $J_\zeta = J$ *is an infinite linear order (disjoint to* $\cup\{\mathrm{Dom}(\bar{\mathbf{a}}^\varepsilon) : \varepsilon < \zeta^*\}$*) and* $\bar{a}^\zeta_t$ *for* $\zeta < \zeta^*, t \in J_\zeta$ *realizes the type* q^ε_t *from part (D), and for any* $\varepsilon < \zeta < \zeta^*, \langle \bar{a}^\varepsilon_s : s \in J_\varepsilon \rangle, \langle a^\zeta_s : s \in J_\zeta \rangle$ *are mutually indiscernible or just perpendicular, then* $\langle \langle a^\zeta_s : s \in J_\zeta \rangle : \zeta < \zeta^* \rangle$ *are mutually indiscernible, moreover even over* A.

2) If we weaken in the conclusion of clause (A) of part (1) the mutually indiscernible by mutually Δ*-indiscernible, then we can weaken* $\mathrm{cf}(I_\zeta) > |T| + |\zeta^*|$ *to* $\mathrm{cf}(I_\zeta) > |\zeta^*|$.

Proof. Similar to earlier proofs (4.11). $\square_{4.17}$

4.18 Claim. *1) If* $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ *is an indiscernible sequence,* $\bar{b} \in {}^{\omega >}\mathfrak{C}$ *then we can divide* I *to* $\leq 2^{|T|}$ *convex subsets* $\langle I_\zeta : \zeta < \zeta^* \rangle$ *such that* $\langle \bar{\mathbf{a}} \restriction I_\zeta : \zeta < \zeta^*, I_\zeta \text{ infinite} \rangle$ *is mutually indiscernible over* $\bar{b}$.
2) Similarly in 4.17: if $\bar{\mathbf{a}}^\zeta$ *is an endless indiscernible sequence over* A *for* $\zeta < \zeta^*$*, and they are mutually indiscernible and* $\bar{b} \in {}^{\omega >}\mathfrak{C}$ *then we can find* $w \subseteq \zeta^*, |w| \leq |T|$ *and for* $\zeta \in w$ *a partition of* I_ζ *to* $\leq 2^{|T|}$ *convex sets* $\langle I_{\zeta,\varepsilon} : \varepsilon < \varepsilon_\zeta \rangle$ *such that the family* $\{\bar{\mathbf{a}}^\zeta : \zeta \in \zeta^* \backslash w\} \cup \{\bar{\mathbf{a}}^\zeta \restriction I_{\zeta,\varepsilon} : \zeta \in w, \varepsilon < \varepsilon_\zeta\}$ *is mutually indiscernible over* $A \cup \bar{b}$ *(the partition of* I_ζ *is induced by some subset* I'_ζ *of* $\mathrm{comp}(I_\zeta)$ *of cardinality* $\leq |T|$*).*

Proof. 1) By 3.6.
2) Similarly. $\square_{4.18}$

Generalizing another claim for stable theories:

4.19 Claim. *Assume that*

(a) $\bar{\mathbf{b}}, \bar{\mathbf{a}}^\zeta$ *are endless indiscernible sequences for* $\zeta < \zeta^*$

(b) $\bar{\mathbf{a}}^\zeta, \bar{\mathbf{a}}^\varepsilon$ *are perpendicular for* $\zeta \neq \varepsilon$

(c) $\bar{\mathbf{b}}, \bar{\mathbf{a}}^\zeta$ *are not perpendicular.*

Then $\zeta^* < |T|^+$.

Proof. Assume toward contradiction that $\zeta^* \geq |T|^+$. We let $A = \bar{\mathbf{b}} \cup \bigcup\limits_\zeta \bar{\mathbf{a}}^\zeta$ and by induction on $n < \omega$, we choose $\langle \bar{a}^{\zeta,*}_n : \zeta < \zeta^* \rangle$ and $\bar{b}^*_n$ and for a fix $n < \omega$ we choose $\bar{a}^{\zeta,*}_n$ by induction on $\zeta < \zeta^*$ and then we choose $\bar{b}^*_n$ such that:

(a) $\bar{a}^{\zeta,*}_n$ realized the average of $\bar{\mathbf{a}}^\zeta$ over $A \cup \{\bar{b}^*_m : m < n\} \cup \{\bar{a}^{\varepsilon,*}_m : m < n \ \& \ \varepsilon < \zeta^*$ or $m = n \ \& \ \varepsilon < \zeta\}$

(b) $\bar{b}^*_n$ realizes the average of $\bar{\mathbf{b}}$ over $A \cup \{\bar{b}^*_m : m < n\} \cup \{\bar{a}^{\varepsilon,*}_m : m \leq n, \varepsilon < \zeta^*\}$.

For each ζ, as $\bar{\mathbf{b}}, \bar{\mathbf{a}}^\zeta$ are not perpendicular, we can find $n_\zeta < \omega, u^\ell_\zeta \in [\omega]^{n_\zeta}$ for $\ell = 0,1,2$ such that $\langle \bar{b}^*_n : n \in u^0_\zeta \rangle\hat{\ }\langle \bar{a}^{\zeta,*}_n : n \in u^1_\zeta \rangle$ and $\langle \bar{b}^*_n : n \in u^0_\zeta \rangle\hat{\ }\langle \bar{a}^{\zeta,*}_n : n \in u^2_\zeta \rangle$ does not realize the same type; say one satisfies $\varphi_\zeta(\bar{x}, \bar{y})$ the second not. As we can replace $\langle \bar{\mathbf{a}}^\zeta : \zeta < |T|^+ \rangle$ by any subsequence of length $|T|^+$, without loss of generality $\zeta < |T|^+ \Rightarrow n_\zeta = n_*, u^\ell_\zeta = u_\ell, \varphi_\zeta = \varphi$. Now for every $\mathscr{U} \subseteq |T|^+$ let $f_{\mathscr{U}}$ be the elementary mapping with domain $\cup\{\bar{a}^{\zeta,*}_n : n \in u_1, \zeta < |T|^+\}$, mapping $\bar{a}^{\zeta,*}_{n_1}$ to $\bar{a}^{\zeta,*}_{n_2}$ <u>iff</u> $\zeta \in \mathscr{U}, n_1 = n_2$ or $\zeta \in |T|^+ \backslash \mathscr{U}, n_1 \in u_1, n_2 \in u_2, |n_1 \cap u_1| = |n_2 \cap u_2|$. Let $g_{\mathscr{U}}$ be an automorphism of $\mathfrak{C}$ extending $f^{-1}_{\mathscr{U}}$. We have gotten the independence property for $\varphi(\bar{x}, \bar{y})$ as $g_{\mathscr{U}}(\langle \bar{b}^*_n : n \in u^0_0 \rangle)$ realizes $\{\varphi(\langle \bar{x}_n : n \in u_0 \rangle, \langle \bar{a}^{\zeta,*}_n : n \in u_1 \rangle)^{\text{if}(\zeta \in \mathscr{U})} : \zeta < |T|^+\}$, contradiction. $\square_{4.19}$

* * *

We can deal with perpenducularity of ultrafilters instead of indiscernible sequences.

4.20 Definition. Let D_ℓ be an ultrafilter on ${}^{m(\ell)}(B_\ell)$ for $\ell = 1,2$.
We say that D_1, D_1 are perpendicular <u>if</u>:

(∗) if $\bar{b}^\ell_n$ realizes $\text{Av}(\{\bar{b}^\ell_m : m < n \text{ or } m = n \wedge k < \ell\} \cup B_1 \cup B_2, D_\ell)$ for $n < \omega, \ell \in \{1,2\}$, <u>then</u> $\langle \bar{b}^1_n : n < \omega \rangle, \langle \bar{b}^2_n : n < \omega \rangle$ are mutually indiscernible.

Parallel claims hold, e.g.

4.21 Claim. *Let D_ℓ, B_ℓ, m_ℓ $(\ell = 1,2)$ be as in the definition 4.20 above.*
1) D_1, D_1 are perpendicular iff

(∗∗) *if $A \supseteq B_1 \cup B_2, \bar b^1$ realizes $\mathrm{Av}(A, D_1)$, and $\bar b^2$ realizes $\mathrm{Av}(A \cup \bar b^1, D_2)$ then $\bar b^1$ realizes $\mathrm{Av}(A \cup \bar b^2, D_1)$.*

2) Let $\bar{\mathbf{b}}^\ell = \langle \bar b^\ell_t : t \in I_\ell\rangle$ be endless indiscernible sequences, and let D_ℓ be an ultrafilter on $\{\bar b^\ell_t : t \in I_\ell\}$ containing $\{\bar b^\ell_t : t \in J\}$ for all the co-bounded subsets J of I_ℓ, for $\ell = 1,2$. Then D_1, D_2 are perpendicular iff $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are perpendicular.
3) In Definition 4.20 we can replace "mutually indiscernible" by "perpendicular".

Proof. No new point.

We can translate:

4.22 Claim. *1) Assume we are given set $B(\subseteq \mathfrak{C})$ and D is an ultrafilter on ${}^m B$ and I is an endless linear order. Then for some ultrafilter D^* on the cardinal $\lambda = |T| + |B|$, in $\mathfrak{C}^\lambda/D^*$ we can find an indiscernible sequence $\bar{\mathbf{b}} = \langle b_t : t \in I\rangle$ in B^λ/D such that:*

(∗) *if $A_1 \subseteq \mathfrak{C}, \bar a \in {}^m\mathfrak{C}$ then:*

$\odot$ *$\bar a$ realizes $\mathrm{Av}(A_1, D)$ iff $\bar a$ realizes $\mathrm{Av}(A_1, \bar{\mathbf{b}})$ (in $\mathfrak{C}^\lambda/D^*$) iff every $\bar b_t$ realizes $\mathrm{Av}(A_1, D) = \mathrm{Av}(A_1, \bar{\mathbf{b}}) = \mathrm{tp}(\bar a, A, \mathfrak{C})$.*

2) If $\bar b = \langle \bar b_t : t \in I\rangle$ is indiscernible, I endless, $\bar b_t \in {}^m B$ for $m < \omega$ and $B \subseteq A$, then there is an ultrafilter D on ${}^m B$ such that (∗) of part (1) holds.

Proof. Straightforward.

* * *

As background for the following note that for T a totally transcendental ($= \aleph_0$-stable), for every $A \subseteq \mathfrak{C}$ the set of isolated types in $\mathscr{S}(A)$ is dense (i.e. if $\mathfrak{C} \models (\exists x)\varphi(x, \bar a), \bar a \subseteq A$ then $\varphi(x, \bar a)$ belongs to some $q \in \mathbf{S}(A)$ which is isolated, i.e. such that for some $\psi(\bar x, \bar a') \in q$ we have $\psi(x, \bar a) \vdash q$). This gives that we can extend A to a model such that "few" types over A' are realized in it, so in some sense M is understood over A ([Mo65] or see [Sh:c]). This enables us to preserve much (e.g. "respecting", see the next section), this is fine but the assumption makes it irrelevant here.

For stable T we can replace isolated by $|T|^+$-isolated (see [Sh:c, IV]). Also we can replace isolated by "does not fork over some finite subset of A"; this looks like an opposite to being isolated (as a non-forking is an "opposite" to an isolated one) but is still managable (and helpful, called $\mathbf{F}^f_{\aleph_0}$-isolated). But all these seem under too strong assumptions so irrelevant here. We use a substitute: does not split over a small set, a precursor of non-forking (from [Sh 3]), still of interest when non-forking is not available.

Recall ([Sh:c, Ch.III,§7,IV])

4.23 Definition. 1) $p \in \mathbf{F}^{\rm sp}_\kappa(B)$ <u>if</u> for some set A we have $p \in \mathbf{S}^{<\omega}(A), B \subseteq A, |B| < \kappa$ and p does not split over B, see part (4) below. Let $p \in \mathbf{F}^{\rm sp}_\kappa$ mean that for some set B we have $p \in \mathbf{F}^{\rm sp}_\kappa(B)$.
2) $\mathscr{A} = (A, \langle \bar b_i, B_i : i < i^* \rangle)$ is an $\mathbf{F}^{\rm sp}_\kappa$-construction (or $\langle \bar b_i, B_i : i < i^* \rangle$ is an $\mathbf{F}^{\rm sp}_\kappa$-construction over A) <u>if</u> $\text{tp}(\bar b_i, A \cup \{\bar b_j : j < i\}) \in \mathbf{F}^{\rm sp}_\kappa(B_i)$, so $B_i \subseteq A^{\mathscr{A}}_i =: A \cup \{\bar b_j : j < i\})$ for every $i < i^*$.
3) Omitting B_i means for some B_i; let $i^* = \ell g(\mathscr{A})$.
4) Recall that $p \in \mathbf{S}^m(A)$ splits over $B \subseteq A$ <u>if</u> for some formula $\varphi(\bar x, \bar y)$ and sequences $\bar b, \bar c$ from ${}^{\ell g(\bar y)}A$ realizing the same type over B we have $\varphi(\bar x, \bar b), \neg\varphi(\bar x, \bar c) \in p$.

We give a proof of 4.24 for self-containment.

4.24 Claim. *1) If $B \subseteq A$ and p is an m-type over B, <u>then</u> there are $q \in \mathbf{S}^m(A)$ extending p and $B_1 \subseteq A, |B_1| \le |T|$ such that q does not split over $B \cup B_1$.*
2) For any A and $\kappa > |T|$ there is a model M and $\mathbf{F}^{\rm sp}_\kappa$-construction $\mathscr{A} = (A, \langle \bar b_i, B_i : i < i^* \rangle)$ such that:

(a) $M = A^{\mathscr{A}}_{i^*}$ *and* $\|M\| = |A|^{<\kappa} + \sum\limits_{\theta < \kappa} 2^{|T|+\theta}$

(b) M *is κ-saturated,*

(c) $\text{cf}(i^*) = \kappa$ *or κ singular,* $\text{cf}(i^*) = \kappa^+$.

3) If $\mathscr{A}$ is an $\mathbf{F}^{\rm sp}_\kappa$-construction, $\kappa = \text{cf}(\kappa), \bar{\mathbf{b}} \subseteq A^{\mathscr{A}}_{\ell g(\mathscr{A})}$ has length $< \kappa$, <u>then</u> $\text{tp}(\bar{\mathbf{b}}, A)$ does not split over some $B \subseteq A, |B| < \kappa$.
4) In part (2) we can add (d) if we replace (a) by (a)* where

(d) *if $p \in \mathbf{S}^m(M)$ does not split over $B \subseteq M, |B| < \kappa$ then $i^* = \sup\{i : B \subseteq A^{\mathscr{A}}_i$ and $\bar b_i$ realizes the type $p \restriction A^{\mathscr{A}}_i\}$*

(a)* $M = A^{\mathscr{A}}_{i^*}$ *and* $\|M\| = |A|^{<\kappa} + 2^{2^{\theta+|T|}}$.

Proof. 1) For any set $C \subseteq A$ let us define

$$p_C = p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b}) \equiv \varphi(\bar{x}, \bar{c}) : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\tau(T)} \text{ and } \bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}A \text{ realizes the same type over } B \cup C\}.$$

Now if there is $C \subseteq A$ of cardinality $\le |T|$ such that p_C is finitely satisfiable (in $\mathfrak{C}$), then choosing $B_1 = C$ and $q \in \mathbf{S}^m(A)$ any extension of p_C we are done. So assume toward contradiction

$(*)$ if $C \subseteq A$ has cardinality $\le |T|$, then p_C is not finitely satisfiable.

Now we choose by induction on $\zeta < |T|^+$ a set C_ζ and then a sequence $\langle \varphi_{\zeta,n}, \bar{b}_{\zeta,n}, \bar{c}_{\zeta,n} : n < n_\zeta \rangle$ such that

$(*)_1$ $\varphi_{\zeta,n} = \varphi_{\zeta,n}(\bar{x}, \bar{y}_{\zeta,n}) \in \mathbb{L}_{\tau(T)}$ and $\bar{b}_{\zeta,n}, \bar{c}_{\zeta,n}$ are sequences from A of length $\ell g(\bar{y}_{\zeta,n})$

as follows. In stage ζ we let $C_\zeta = \cup\{\bar{b}_{\varepsilon,n}{}^\frown \bar{c}_{\varepsilon,n} : \varepsilon < \zeta \text{ and } n < n_\varepsilon\}$.
So $C_\zeta \subseteq A$ and $|C_\zeta| < \aleph_0 + |\zeta|^+ < |T|^+$; of course, $C_0 = \emptyset$. Now by $(*)$ we know that p_{C_ζ} is not finitely satisfiable, hence we can find $n_\zeta < \omega$ and $\varphi_{\zeta,n}(\bar{x}, \bar{y}_{\zeta,n}), \bar{b}_{\zeta,n}, \bar{c}_{\zeta,n}$ as in $(*)_1$ such that

$(*)^2_\zeta$ $\bar{b}_{\zeta,n}, \bar{c}_{\zeta,n}$ realizes the same type over $B \cup C_\zeta$ for $n < n_\zeta$

$(*)^3_\zeta$ $p \cup \{\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n}) \equiv \varphi_{\zeta,n}(\bar{x}, \bar{c}_{\zeta,n}) : n < n_\zeta\}$ is not finitely satisfiable, that is $p(\bar{x}) \vdash \bigvee\limits_{n < n_\zeta} (\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n}) \equiv \neg\varphi_{\zeta,n}(\bar{x}, \bar{c}_{\zeta,n}))$.

Having carried the definition note that the number of possible sequences $\langle \varphi_{\zeta,n}(\bar{x}, \bar{y}_{\zeta,n}) : n < n_\zeta \rangle$ is $\le |T|$ hence for some unbounded $\mathscr{U} \subseteq |T|^+$ we have $\zeta \in \mathscr{U} \Rightarrow n_\zeta = n_* \ \& \bigwedge\limits_{n < n_*} \varphi_{\zeta,n}(\bar{x}, \bar{y}_{\zeta,n}) = \varphi_n(\bar{x}, \bar{y}_n)$.
Now note

$(*)_4$ if $p \subseteq q \in \mathbf{S}^m(C_\zeta)$ then for some $n < n_\zeta$ there are $q_0, q_1 \in \mathbf{S}^m(C_{\zeta+1})$ extending q such that $\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n}) \in q_0, \neg\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n}) \in q_1$.

[Why? Let $q \subseteq q' \in \mathbf{S}^m(C_{\zeta+1})$, now by $(*)^3_\zeta$, as $p \subseteq q \subseteq q'$ clearly for some $n < n_\zeta$ we have $[\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n}) \equiv \neg\varphi_{\zeta,n}(\bar{x}, \bar{c}_{\zeta,n})] \in q'$, and as $\bar{b}_{\zeta,n}, \bar{c}_{\zeta,n} \subseteq C_{\zeta+1}$ for some truth value $\mathbf{t}$ we have $\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n})^{\mathbf{t}} \in q'$ hence $\neg\varphi_{\zeta,n}(\bar{x}, \bar{c}_{\zeta,n})^{\mathbf{t}} \in q'$.

So $q \cup \{\neg\varphi_{\zeta,n}(\bar{x}, \bar{c}_{\zeta,n})^{\mathbf{t}}\}$ is finitely satisfiable, but by $(*)^2_\zeta$ the sequences $\bar{b}_{\zeta,n}, \bar{c}_{\zeta,n}$ realizes the same type over C_ζ hence also $q \cup \{\neg\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n})^{\mathbf{t}}\}$ is finitely satisfiable

hence can be extended to some $q'' \in \mathbf{S}^m(C_{\zeta+1})$. So $\{q', q''\}$ can serve as q_0, q_1 (in some order); so $(*)_4$ holds.]
Hence

$(*)_5$ for any finite set $u \subseteq |T|^+$, the following set has at least $2^{|u|}$ members

$$\begin{aligned}\{\eta : &\eta \text{ is a function from } \{(\zeta, n) : \zeta \in u, n < n_\zeta\} \\ &\text{to the truth values such that} \\ &p_\eta = p \cup \{\varphi_{\zeta,n}(\bar{x}, \bar{b}_{\zeta,n})^{\eta(\zeta,n)} : \zeta \in u, n < n_\zeta\} \\ &\text{is finitely satisfiable}\}.\end{aligned}$$

[Why? By induction on $|u|$ (or on $\sup(u)$) using $(*)_4$.]

Now let $\Delta = \{\varphi_n(\bar{x}; \bar{y}_n) : n < n_*\}$, so for every finite $u \subseteq \mathscr{U}$ by $(*)_5$ we have

$$\mathbf{S}^m_\Delta(\cup\{\bar{b}_{\zeta,n} : \zeta \in u, n < n^*\}) \text{ has at least } 2^{|u|} \text{ members}$$

whereas

$$\cup\{\bar{b}_{\zeta,n} : \zeta \in u, n < n^*\}$$

has at most $u \times m^*$ members where we let $m^* = \sum\limits_{n<n^*} \ell g(\bar{y}_n)$.
By [Sh:c, II,§4], T has the independence property.
2) Let κ' be κ if κ is regular and κ^+ if κ is singular. We shall choose by induction on $\zeta \le \kappa'$ the tuple $\mathscr{A}^\zeta = (A, \langle(\bar{b}^\zeta_i, B^\zeta_i) : i < i_\zeta\rangle)$ such that:

- (a) $\mathscr{A}^\zeta$ is an $\mathbf{F}^{\mathrm{sp}}_\kappa$-construction
- (b) $i_0 = 0$
- (c) if $\varepsilon < \zeta$ then $i_\varepsilon < i_\zeta$ and $(\bar{b}^\zeta_i, B^\zeta_i) = (\bar{b}^\varepsilon_i, B^\varepsilon_i)$ for every $i < i_\varepsilon$, so we call them (b_i, B_i)
- (d) if ζ is a limit ordinal then $i_\zeta = \cup\{i_\varepsilon : \varepsilon < \zeta\}$ and so $\mathscr{A}^\zeta$ is determined by clause (c)
- (e) $|i_\zeta| \le \lambda =: |A|^{<\kappa} + \sum\limits_{\theta<\kappa} 2^{\theta+|T|}$.

If $\mathscr{A}^\zeta$ is chosen, let $A^\zeta = A \cup \cup\{\bar{b}_i : i < i_\zeta\}$ and let $\mathscr{P}_\zeta = \{p$: for some $m < \omega, p$ is an m-type over some set $B \subseteq A_\zeta$ of cardinality $< \kappa$ hence of cardinality $< \kappa\}$. We know that $|\mathscr{P}_\zeta| \le \lambda$ and let $i_{\zeta+1} = i_\zeta + \lambda$ let $\langle p_i : i \in [i_\zeta, i_{\zeta+1})\rangle$ list $\mathscr{P}_\zeta$ (possibly with repetition). We choose $(A_i, B_i, q^+_i, \bar{b}_i)$ by induction on $i \in [i_\zeta, i_{\zeta+1})$ as follows.

Let $A_i = A^\zeta \cup \cup\{\bar b_j : j \in [i_\zeta, i)\}$; and let B_i, q_i^+ be such that $q_i^+ \in \mathbf{S}^{m_i}(A_i)$ be an extension of p_i which does not split over B_i, where $\mathrm{Dom}(p_i) \subseteq B_i \subseteq A_i$ & $|B_i| < \kappa$ where p_i is an m_i-type. Why can we find such B_i, q_i^+? by part (1) applied to A_i, p_i, $\mathrm{Dom}(p_i)$. Lastly, let $\bar b_i \in {}^{m_i}\mathfrak{C}$ be any sequence realizing q_i^+.

So we have carried the induction on $i \in [i_\zeta, i_{\zeta+1})$ hence $\mathscr{A}^{\zeta+1}$ is defined. As the case of limit ζ and $\zeta = 0$ were done we have finished the induction on ζ, so $\mathscr{A}^\zeta$ is defined also for $\zeta = \kappa'$ and $\mathscr{A}^{\kappa'}$ is as required.
3) Let $\mathscr{A} = (A, \langle \bar b_i, B_i : i < i^* \rangle)$ and let $B^* = A^{\mathscr{A}}_{\ell g(A)} = A \cup \cup\{\bar b_i : i < i^*\}$, and let $\mathbf{\bar b} \in {}^{\kappa>}(B^*)$. For each $i < i^*$ let $u_i \in [i]^{<\kappa}$ be such that $B_i \subseteq A \cup \cup\{\bar b_j : j \in u_i\}$. We can find $u_0^* \subseteq i^*$ of cardinality $< \kappa$ be such that $\mathbf{\bar b} \in {}^{\kappa>}(A \cup \cup\{\bar b_i : i \in u_0^*\})$, and defined $u_n^* \subseteq i^*$ of cardinality $< \kappa$ for $n < \omega$ by (u_0^* as above and) $u_{n+1}^* = u_n^* \cup \cup\{u_i : i \in u_n^*\}$. Let $u^* = \cup\{u_n^* : n < \omega\}$ and $B = A \cap [\cup\{B_i : i \in u^*\} \cup \mathbf{\bar b}]$, so $B \in [A]^{<\kappa}$. Now we can prove by induction on $i \in u^* \cup \{i^*\}$ that $\mathrm{tp}_*(\cup\{\bar b_j : j \in u^* \cap i\}, A)$ does not split over B. From the case $i = i^*$ we can deduce the desired conclusion.
4) Like the proof of part (2) let $\mathscr{P}'_\zeta = \{(p,B) : p \in \mathbf{S}^{<\omega}(A^\zeta), p$ does not split over some set $B \subseteq A^\zeta$ of cardinality $< \kappa\}$ and[6] let $\langle (p'_i, B'_i) : i \in [i_\zeta, i_{\zeta+1})\rangle$ listing $\mathscr{P}'_{i_j}$. But choosing $\bar b_i, B_i$ for $i \in [i_\zeta, i_\zeta + \lambda)$ we now have two cases.

Case 1: $i = i_\zeta + 2j$.

As in the proof of part (2) using $p_{i_\zeta + j}$ where $p_{i_\zeta+j}$ is as in the proof of part (2).

Case 2: $i = i_\zeta + 2j + 1$.

If there is $q \in \mathbf{S}^{<\omega}(A_{i_\zeta} \cup \{\bar b_\varepsilon : \varepsilon \in [i_\zeta, i)\}$ extending $p'_{i_\zeta+j}$ not splitting over the set $B'_{i_\zeta+j}$ which has cardinality $< \kappa$, choose q_i^+ as some such q. If not, act as in case 1. $\square_{4.24}$

Similarly, but if we like not to assume $\kappa > |T|$, we need to assume more on T.

4.25 Definition. 1) $p \in \mathbf{F}^{\mathrm{esp}}_\kappa(B)$ if for some A, m we have $p \in \mathbf{S}^m(A), B \subseteq A$, and for every $\varphi = \varphi(\bar x, \bar y)$ for some $\Delta \subseteq \mathbb{L}_{\tau(T)}$ and $B' \subseteq B$ both of cardinality $< \kappa$ the type p does not (φ,Δ)-split over B', see part (4) below.
1A) Let $\mathbf{F}^{\mathrm{esp}}(B) = \mathbf{F}^{\mathrm{esp}}_{\aleph_0}(B)$.
2) $\mathscr{A} = \langle A, \langle (\bar b_i, B_i) : i < i^* \rangle\rangle$ is an $\mathbf{F}^{\mathrm{esp}}_\kappa$-construction or $\langle (b_i, B_i) : i < i^*)\rangle$ is an $\mathbf{F}^{\mathrm{esp}}_\kappa$-construction over A <u>if</u> $\mathrm{tp}(\bar b_i, A \cup \{\bar b_j : j < i\}) \in \mathbf{F}^{\mathrm{esp}}_\kappa(B_i)$ so $B_i \subseteq \mathscr{A}^i =: A \cup \{\bar b_j : j < i\}$ for every $i < i^*$.
3) Omitting B_i means for some B_i; let $\ell g(\mathscr{A}) = i^*$.
4) Recall that $p \in \mathbf{S}^m(B)$ does (Δ_1, Δ_2)-split over A <u>if</u> for some $\varphi(\bar x, \bar y) \in \Delta_1$ with

[6] recall ([Sh 3] or [Sh:c]) that for $\kappa > |T|$, $\{p \in \mathbf{S}^m(A) : p$ does not split over some subset of A of cardinality $< \kappa\}$ is $\le |A|^{<\kappa} + \sum_{\theta<\kappa} 2^{2^\kappa}$; similarly for strong splitting

$\ell g(\bar{x}) = m$ and $\bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}B$ we have $\mathrm{tp}_{\Delta_2}(\bar{b}, A) = \mathrm{tp}_{\Delta_2}(\bar{c}, A)$ but $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$.

4.26 Claim. *1) Assume $\kappa \geq |T| + \aleph_1$.*

If $B \subseteq A, p$ is an m-type over B of cardinality $< \kappa$ and $|B| < \kappa$, <u>then</u> there are $B' \in [A]^{<\kappa}$ extending B and $q \in \mathbf{S}^m(A)$ from $\mathbf{F}^{\mathrm{esp}}_\kappa(B')$ extending p.
2) For any A and $\kappa = \mathrm{cf}(\kappa) \geq |T| + \aleph_1$ and is a model M and $\mathbf{F}^{\mathrm{esp}}_\kappa$-construction $\mathscr{A} = (A, \langle (\bar{b}_i, A_i) : i < i^* \rangle)$ such that:

- (a) $|M| = A^{\mathscr{A}}_{i^*}$
- (b) *M is κ-compact*
- (c) $\mathrm{cf}(i^*) \geq \kappa$.

3) If $\mathscr{A}$ is a $\mathbf{F}^{\mathrm{esp}}_\kappa$-construction, $\kappa = \mathrm{cf}(\kappa)$, <u>then</u> for any $\bar{b} \subseteq {}^{\omega>}(A^{\mathscr{A}}_{\ell g(\mathscr{A})})$ and $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\tau(T)}$ for some $B \subseteq A$ and $\Delta \subseteq \mathbb{L}_{\tau(T)}$ of cardinality $< \kappa$ the type $\mathrm{tp}(\bar{b}, A)$ does not (φ, Δ)-split over A.
4) The parallel of 4.24(4) holds.

The proof of 4.26 is similar to the proof of 4.24.

Proof. 1) Fix m and a m-type p over B such that $B \subseteq A, |B| < \kappa$. Without loss of generality $\kappa = |T|$.

Let $\bar{x} = \langle x_\ell : \ell < m \rangle$ and let $\{\varphi_i(\bar{x}, \bar{y}_i) : i < |T|\}$ be a list of all such formulas. For any set $C \subseteq A$ and $\Delta \subseteq \mathbb{L}_{\tau(T)}$ we define

$$\begin{aligned} q^i_{\Delta,C} = \{\varphi_i(\bar{x}, \bar{b}) \equiv \varphi_i(\bar{x}, \bar{c}) :& \bar{b}, \bar{c} \in {}^{\ell g(\bar{y}_i)}A \\ & \text{and } \mathrm{tp}_\Delta(\bar{b}, C) = \mathrm{tp}_\Delta(\bar{c}, C)\}. \end{aligned}$$

We now define by induction on $\zeta < |T|$, a pair (C_ζ, Δ_ζ) such that:

$\boxtimes_1$ (a) $C_\zeta \subseteq A$ is increasing continuous
- (b) $\Delta_\zeta \subseteq \mathbb{L}_{\tau(T)}$ is increasing continuous
- (c) Δ_ζ, C_ζ are of cardinality $\leq \aleph_0 + |B| + |\zeta|$
- (d) $C_0 = B, \Delta_0 = \emptyset$
- (e) $p \cup \cup\{q^\varepsilon_{\Delta_{\varepsilon+1}, C_{\varepsilon+1}} : \varepsilon < \zeta\}$ is finitely satisfiable.

If we succeed, clearly we are done, and there is no problem for $\zeta = 0$ and ζ limit. So assume $\zeta = \varepsilon + 1$. We now try to choose $\Delta_\zeta \supseteq \Delta_\varepsilon \cup \{\varphi_\varepsilon\}$ of the right cardinality close enough, e.g. of the form $\Delta_\zeta = \mathbb{L}_{\tau_\zeta}$ for some vocabulary $\tau_\zeta \subseteq \tau_T$. We now try to choose by induction on $i < \omega_1, C_{\varepsilon,i}, n_i, \bar b_{\varepsilon,i,\ell}, \bar c_{\varepsilon,i,\ell}$ for $\ell < n_i$ such that:

$\boxtimes_2(\alpha)$ $C_{\varepsilon,i}$ is increasing continuous

(β) $C_{\varepsilon,0} = C_\varepsilon$

(γ) $n_i < \omega$

(δ) $\bar b_{\varepsilon,i,\ell}, \bar c_{\varepsilon,i',\ell} \in {}^{\ell g(\bar y_i)}A$ realizes the same type over $C_{\varepsilon,i}$

(ε) $\bar b_{\varepsilon,i,\ell}, \bar c_{\varepsilon,i,\ell} \subseteq c_{\varepsilon,i}$ and $C_{\varepsilon,i+1} \backslash C_{\varepsilon,i}$ is finite

(ζ) $p \cup \cup\{p^\xi_{\Delta_\xi,c_\xi} \restriction c_{\varepsilon,i+1} : \xi < \varepsilon\} \cup \{\varphi_{\varepsilon,i,\ell}(\bar x, \bar b_{\varepsilon,i,\ell}) \equiv \varphi_{\varepsilon,i,\ell}(\bar x, \bar c_{\varepsilon,i,\ell}) : \ell < n_i\}$ is inconsistent.

For $i = 0, i$ limit no problem. For i successor, if the choice $C_{\varepsilon+1} = C_{\varepsilon,i}$ (and $\Delta_{\varepsilon+1}$ chosen above) is as required in $\boxtimes_1$ we are done choosing (C_ζ, Δ_ζ) thus finishing the proof. Otherwise $p \cup \cup\{q^\xi_{\Delta_{\xi+1},C_{\xi+1}} : \xi \leq \varepsilon\} \cup q^\varepsilon_{\Delta_{\varepsilon+1},C_{\varepsilon,i}}$ is inconsistent hence has a finite inconsistent subset $p'_{\varepsilon,i}$ and let $C_{\varepsilon,i+1} = C_{\varepsilon,i} \cup \mathrm{Dom}(p'_{\varepsilon,i})$, let $n_{\varepsilon,i} = |p'_{\varepsilon,i} \cap q^\varepsilon_{\Delta_{\varepsilon+1},C_{\varepsilon,i}}|$ and $\{\varphi_\varepsilon(\bar x, \bar b_{\varepsilon,i,\ell}) \equiv \varphi_\varepsilon(\bar x, \bar c_{\varepsilon,i,\ell}) : i < n_\varepsilon\}$ list $p'_{\varepsilon,i} \cap q^\varepsilon_{\Delta_{\varepsilon+1},C_{\varepsilon,i}}$.

So for some $n(*)$ the set $\mathscr{U} = \{i < \omega_1 : n_{\varepsilon,i} \leq n(*)\}$ is infinite. Now we prove

⊛ for every $i(*) < \omega_1$ and $u \subseteq \omega_1 \backslash i(*)$, the following set has at least $2^{|u|}$ members $\{\eta : \eta$ is a function from $\{(j,n) : j \in u$ and $n < n_\zeta\}$ to the truth values such that $p \cup \cup\{q^\xi_{\Delta_{\xi+1},c_{\xi+1}} \restriction C_{\varepsilon,i(*)} : \xi < \varepsilon\} \cup \{\varphi_\varepsilon(\bar x, \bar b_{\varepsilon,j,n})^{\eta(j,n)} : j \in u$ is finitely satisfiable$\}$.

We do this by induction on $|u|$; this gives that T has the independence property, contradiction.

We would have liked to look at all $\kappa = \aleph_0$, but we would get by the proof above less; say for a pregiven $k_0 < \omega$, say for $\varepsilon = 0$, we get every subset of $p \cup p_{\Delta_0,C_0}$ of cardinality $< k_0$ is satisfiable.
2) Similar to the proof of 4.24(2).
3) Like 4.24(3), only we have to take care of the Δ, too.
4) Like 4.24(4). $\square_{4.26}$

4.27 Claim. *If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular indiscernible sequences each of cofinality $> |T|$, <u>then</u> we can find $J_1 \subseteq \mathrm{Dom}(\bar{\mathbf{a}}^1), J_2 \subseteq \mathrm{Dom}(\bar{\mathbf{a}}^2)$ unbounded such that $\bar{\mathbf{a}}^1 \restriction J_1, \bar{\mathbf{a}}^2 \restriction J_2$ are mutually indiscernible.*

Proof. If $\mathrm{Dom}(\bar{\mathbf{a}}^1)$, $\mathrm{Dom}(\bar{\mathbf{a}}^2)$ has different confinalities we can apply 4.11 to $\bar{\mathbf{a}}^1 \restriction I_1, \bar{\mathbf{a}}^2 \restriction I_2$ where $I^\ell \subseteq \mathrm{Dom}(\bar{\mathbf{a}}^\ell)$ is unbounded and cofinal in $\mathrm{Dom}(\bar{\mathbf{a}}^\ell)$ and has cardinality $\mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{a}}^\ell))$.

Otherwise, let κ be the common cofinality, choose $\{t^\ell_\alpha : \alpha < \kappa\} \subseteq \mathrm{Dom}(\bar{\mathbf{a}}^\ell)$ increasing unbounded. As in 3.5 we can chooose $\alpha(i,\ell) < \kappa$ by induction on $2\alpha+\ell$, increasing such that $\alpha = \alpha(i,\ell)$ implies that $\bar{a}^\ell_{t^\ell_\alpha}$ realizes $\mathrm{Av}(\{\bar{a}^k_{t^k_\beta} : \beta < \alpha \ \& \ k \in \{1,2\}$ or $\beta = \alpha \wedge k = 1 < \ell = 2\}, \bar{\mathbf{a}}^\ell)$. So $\langle \bar{a}^1_{t^1_\alpha} {}^\frown \bar{a}^2_{t^2_\alpha} : \alpha < \kappa\rangle$ is an indiscernible sequence. By the perpendicularity easily $J^\ell = \{t^\ell_\alpha : \alpha < \kappa\}$ for $\ell = 1,2$ are as required. $\square_{4.27}$

§5 Indiscernible sequence perpendicular to cuts

Our aim is to show that for a set of $\{\bar{\mathbf{b}}_\zeta : \zeta < \zeta^*\}$ of pairwise perpendicular endless indiscernible sets, we can find a model $M \supseteq \cup\{\bar{\mathbf{b}}_\zeta : \zeta < \zeta^*\}$ with $\langle \text{dual-cf}(\bar{\mathbf{b}}_\zeta, M) : \zeta < \zeta^*\rangle$ essentially as we like, and other $\bar{\mathbf{b}}'$ in M has such dual cofinality iff this essentially follows. In fact we can demand $M \supseteq M_0$ for any given M_0. Toward this we define and investigate when an endless indiscernible sequence $\bar{\mathbf{c}}$ is perpendicular to a (Dedekind) cut (I_1, I_2) in an indiscernible sequence $\bar{\mathbf{a}}$.

We use

- $\boxtimes$ the Downward L.S. (on $M \prec N$) can replace $|T| < |P^N| < |Q^N|$ by $|P^M| = |Q^M|$ but in general cannot invert the inequality, however for cofinality it can.

For our purpose "respecting" defined in 5.2 is a central notion.

5.1 Discussion: 1) We can reformalize the aim as:

- $\boxtimes$ given $\bar{D} = \langle D_\zeta : \zeta < \zeta^*\rangle, D_\zeta$ an ultrafilter and $\cup\{\text{Dom}(D_\zeta) : \zeta < \zeta^*\} \subseteq M$ and given a sequence $\langle \lambda_\zeta : \zeta < \zeta^*\rangle$ of regular cardinals ($\geq \kappa = \text{cf}(\kappa) > |T|$) and, for simplicity, $\text{Dom}(D_\zeta)$ in $\mathfrak{C}$ an indiscernible sequence over $\cup\{\text{Dom}(D_\varepsilon) : \varepsilon \neq \zeta\}$, then there is a κ-saturated model $M \supseteq A$ such that $\zeta < \zeta^* \Rightarrow \lambda_\zeta = \text{dual-cf}(D_\zeta, M)$ defined naturally. This property is meaningful also for (complete first order theories) T with the independence property (and sequence $\bar{D}$). However, at least for some of them, e.g., for number theory
  - (a) assume $\zeta_1 \neq \zeta_2 < \zeta^*$ and F is a one to one function from $\text{Dom}(D_{\zeta_1})$ onto $\text{Dom}(D_{\zeta_2})$ maping D_{ζ_1} to D_{ζ_2} and is included in a function definable in $\mathfrak{C}$ with parameters from A.
    Then $A \subseteq M \prec \mathfrak{C} \Rightarrow \text{dual-cf}(D_{\zeta_1}, M) = \text{dual-cf}(D_{\zeta_2}, M)$.
    In other words if D_{ζ_1}, D_{ζ_2} are isomorphic as ultrafilters then D_{ζ_1}, D_{ζ_2} are not perpendicular in $\mathfrak{C}$ for $T = Th(\mathbb{N})$ because for every such F there is a definable function extending it.
    For dependent theory this gives just that definably isomorphic $\Rightarrow$ not perpendicular
  - (b) we can weaken the demands on D_{ζ_1}, D_{ζ_2}.

5.2 Definition. 1) We say $\bar{I} = (I_1, I_2)$ is a Dedekind cut (or just a cut) of the linear order I, if I is the disjoint union of I_1, I_2 and $s \in I_1 \ \ \& \ \ t \in I_2 \Rightarrow s <_I t$ and we write $I = I_1 + I_2$, and the cofinality of $\bar{I}$ is $(\text{cf}(I_1), \text{cf}(I_2^*))$ where I_2^* is I_2

inverted. If $I_1 + I_2$ is a convex subset of J and $I_1 \neq \emptyset \neq I_2$ we may abuse our notation saying "(I_1, I_2) is a Dedekind cut of J". We say (I_1, I_2) is a Dedekind cut of $\bar{\mathbf{a}}$ if it is a Dedekind cut of $\mathrm{Dom}(\bar{\mathbf{a}})$. If not say otherwise, $I_1 \neq \emptyset \neq I_2$, and the cut is non-trivial if both its cofinalities are infinite.
2) $(J_1, J_2) \leq (I_1, I_2)$ if J_1 is an end segment of I_1 and J_2 is an initial segment of I_2.
3) We say that the set A respects the Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$ <u>if</u> (I_1, I_2) is a Dedekind cut of $\bar{\mathbf{a}}$ and for every $\bar{b} \in {}^{\omega>}A$ for some $(J_1, J_2) \leq (I_1, I_2)$ the sequence $\bar{\mathbf{a}} \restriction (J_1 + J_2)$ is indiscernible over $\bar{b}$.
4) For indiscernible sequences $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ such that $\bar{\mathbf{b}}$ is endless and a Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$ we say that $\bar{\mathbf{b}}$ is perpendicular to the cut over A_0 when:

- (a) the set $A_0 \supseteq \bar{\mathbf{b}} \cup \bar{\mathbf{a}}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$
- (b) for any set $A \supseteq A_0$ respecting the Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$ and $\bar{c}$ realizing $\mathrm{Av}(A, \bar{\mathbf{b}})$ also the set $A \cup \bar{c}$ respects the Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$.

We also say "$\bar{\mathbf{b}}$ is perpendicular to $(\bar{\mathbf{a}} \restriction I_1, \bar{\mathbf{a}} \restriction I_2)$ over A_0". If we omit A_0 we mean $A_0 = \bar{\mathbf{b}} \cup \bar{\mathbf{a}}$.
4A) For indiscernible sequences $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ such that $\bar{\mathbf{b}}$ is endless and a Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$ we say that $\bar{\mathbf{b}}$ is truely perpendicular to the cut (I_1, I_2) of $\bar{\mathbf{a}}$ when for any set A, if $A \cup \bar{\mathbf{a}}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$ and $\bar{c}$ realizes $\mathrm{Av}(A \cup \bar{\mathbf{a}}, \bar{\mathbf{b}})$ then $A \cup \bar{\mathbf{a}} \cup \bar{c}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$.
5) For endless indiscernible sequence $\bar{\mathbf{a}}$ and $A \supseteq \bar{\mathbf{a}}$ we say an endless indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{\mathbf{b}}_t : t \in I \rangle$ over A is based on $\bar{\mathbf{a}}$ or $\bar{\mathbf{b}}$ is based on $(A, \bar{\mathbf{a}})$ if each $\bar{b}_t$ realizes $\mathrm{Av}(A \cup \{\bar{b}_s : s <_I t\}, \bar{\mathbf{a}})$.
6) We say that the set A weakly respects the Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$ <u>if</u> (I_1, I_2) is a Dedekind cut of $\bar{\mathbf{a}}$ and for every formula $\varphi(\bar{x}, \bar{b})$ with $\bar{b} \subseteq A$ for some $(J_1, J_2) \leq (I_1, I_2)$ and truth value $\mathbf{t}$ we have $s \in J_1 + J_2 \Rightarrow \mathfrak{C} \models \varphi[\bar{a}_s, \bar{b}]^{\mathbf{t}}$.

5.3 Claim. *1) If (I_1, I_2) is a nontrivial cut of the indiscernible sequence $\bar{\mathbf{a}}$ and $A = \cup\{\bar{a}_t : t \in I_1 \cup I_2\}$ <u>then</u> A respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$.*
2) For every A and indiscernible $\bar{\mathbf{a}} \subseteq A$ and endless I there is $\langle \bar{\mathbf{b}}_t : t \in I \rangle$ based on $(A, \bar{\mathbf{a}})$.
3) Assume that (I_1, I_2) is a nontrivial cut of the indiscernible sequence $\bar{\mathbf{a}}$ and $\bar{\mathbf{c}}$ is an endless indiscernible sequence. Then $(a) \Leftrightarrow (b) \Rightarrow (c)$ and if the cofinalities of the cut are $> \aleph_0$ then $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ where

- (a) *$\bar{\mathbf{c}}$ is perpendicular to the cut (I_1, I_2) of $\bar{\mathbf{a}}$*
- (b) *if $\bar{\mathbf{b}}$ is an indiscernible sequence based on $(\bar{\mathbf{c}} \cup \bar{\mathbf{a}}, \bar{\mathbf{c}})$ then the set $\bar{\mathbf{b}} \cup \bar{\mathbf{c}} \cup \bar{\mathbf{a}}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$*
- (c) *there is $\bar{\mathbf{b}}$ an indiscernible sequence based on $(\bar{\mathbf{c}} \cup \bar{\mathbf{a}}, \bar{\mathbf{c}})$ such that the set $\bar{\mathbf{b}} \cup \bar{\mathbf{c}} \cup \bar{\mathbf{a}}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$.*

4) If $A_i\,(i < \delta)$ is increasing, $A_0 \supseteq A$ and each A_i [truely, weakly] respects the cut (I_1, I_2) of $\mathbf{\bar a}$ then also $\bigcup\limits_{i<\delta} A_i$ does.
5) If $\mathbf{\bar b} \subseteq A_0 \subseteq A_1$ and the cofinalities (of I_1, I_2 and $Dom(\mathbf{\bar b})$) are $> |T|$ then: $\mathbf{\bar b}$ is perpendicular to the cut (I_1, I_2) of $\mathbf{\bar a}$ over A_0 then this holds over A_1.

Proof. 1) Let $\bar b \in {}^{\omega>}A$; hence for some $n < \omega$ and $t_0, \ldots, t_{n-1} \in I_1 \cup I_2$ we have $\bar b \subseteq \cup\{\bar a_{t_\ell} : \ell < n\}$ and define $J_1 = \{t \in I_1 : (\forall \ell < n \;\; \& \;\; t_\ell \in I_1 \Rightarrow t_\ell < t\}, J_2 = \{t \in I_2 : \ell < n \;\; \& \;\; t_\ell \in I_2 \Rightarrow t < t_\ell\}$. Clearly $(J_1, J_2) \le (I_1, I_2)$ and $\mathbf{\bar a} \restriction (J_1 + J_2)$ is indiscernible over $\cup\{\bar a_t : t \in (I_1 \cup I_2) \backslash (J_1 \cup J_2)\}$ hence over $\cup\{a_{t_\ell} : \ell < n\}$ hence over $\bar b$.
2) Also trivial.
3) (a) $\Rightarrow$ (c):

Let $\langle \bar b_n : n < \omega\rangle$ be an indiscernible sequence over $\cup \mathbf{\bar a} \cup \bigcup \mathbf{\bar c}$ based on $\mathbf{\bar c}$ and for $\alpha \le \omega$ let $A_\alpha = \cup\{\bar a_t : t \in I_1 \cup I_2\} \cup \mathbf{\bar c} \cup \bigcup\{\bar b_n : n < \alpha\}$. We can prove by induction on α that A_α respects the cut (I_1, I_2) of $\mathbf{\bar a}$; for $\alpha = 0$ by part (1), for $\alpha = n+1$ by clause (a), see Definition 5.2(4) for α limit by part (4) and is straightforward. Hence by the first phrase of part (2), $\langle \bar b_n : n < \omega\rangle$ exemplifies (a), as required.

(b) $\Rightarrow$ (c): Trivial.

(c) $\Rightarrow$ (b): Easy.

(b) $\Rightarrow$ (a): So we are assuming that the cofinalities are $> |T|$.

Let a set $A \supseteq \mathbf{\bar c} \cup \mathbf{\bar a}$ respecting the cut (I_1, I_2) be given and let $\bar b_n$ realize $\mathrm{Av}(\cup \mathbf{\bar c} \cup A \cup \{\bar b_m : m < n\}, \mathbf{\bar c})$ for $n < \omega$. By clause (b) we know that the set $B = \cup\{\bar b_n : n < \omega\} \cup \bigcup \mathbf{\bar a}$ respect the cut (I_1, I_2) of $\mathbf{\bar a}$. Let $\bar d$ be a finite sequence from A, then we shall prove

- $(*)_{\bar d}$ for some $(J_1, J_2) \le (I_1, I_2)$ the sequence $\mathbf{\bar a} \restriction (J_1 \cup J_2)$ is indiscernible over $\cup\{\bar b_n : n < \omega\} \cup \bar d$.

Clearly this sufices. Note that $\langle b_n : n < \omega\rangle$ is indiscernible over $A \supseteq \cup \mathbf{\bar a} \cup \bar d$.

As the set $\cup \mathbf{\bar b} \cup \cup \mathbf{\bar a}$ respects the cut (I_1, I_2) of $\mathbf{\bar a}$, for each n there is $(J^n_1, J^n_2) \le (I_1, I_2)$ such that $\mathbf{\bar a} \restriction (J^n_1 \cup J^n_2)$ is indiscernible over $\bar b_0{}^\frown \ldots {}^\frown \bar b_n$. As the cofinalities of (I_1, I_2) are $> \aleph_0$, also $(J_1, J_2) \le (I_1, I_2)$ where $J_\ell =: \bigcap\limits_{n<\omega} J^n_\ell$, and clearly $\mathbf{\bar a} \restriction (J_1 \cup J_2)$ is indiscernible over $\mathbf{\bar b}$. By the last two sentences $\mathbf{\bar a} \restriction (J_1 \cup J_2), \langle \bar b_n : n < \omega\rangle$ are mutually indiscernible. Also possibly replacing (J_1, J_2) by some $(J'_1, J'_2) \le (J_1, J_2)$, the sequence $\mathbf{\bar a} \restriction (J_1 \cup J_2)$ is indiscernible over $\bar d$ and similarly (as the cofinalities are large), $\mathbf{\bar a} \restriction J_\ell$ is indiscernible over $\bigcup\limits_n \bar b_n \cup \bar d \cup \mathbf{\bar a} \restriction J_{3-\ell}$ for $\ell = 1, 2$. Together if

$(*)_d$ fails, we get a contradiction to the assumption on the $\bar{b}_n$'s.
4),5) Check. $\square_{5.3}$

5.4 Claim. *Assume*

(a) (I_1, I_2) *is a non-trivial Dedekind cut of the indiscernible sequence* $\bar{\mathbf{a}}$
(b) J_1 *is an unbounded subset of* I_1
(c) J_2 *is a subset of* I_2 *unbounded from below*[7]

1) If $A \supseteq \bar{\mathbf{a}}$ *then: A respects the Dedekind cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$ *iff A respects the Dedekind cut* (J_1, J_2) *of* $\bar{\mathbf{a}} \restriction (J_1 \cup J_2)$.
2) If $\bar{\mathbf{b}}$ *is an endless indiscernible sequence and* $\bar{\mathbf{b}} \cup \bar{\mathbf{a}} \subseteq A_0$, *then* $\bar{\mathbf{b}}$ *is perpendicular to the Dedekind cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$ *over* A_0 *iff* $\bar{\mathbf{b}}$ *is perpendicular to the Dedekind cut* (J_1, J_2) *of* $\bar{\mathbf{a}} \restriction (J_1 \cup J_2)$ *over* A_0.
3) If J is an unbounded subset of $\mathrm{Dom}(\bar{\mathbf{b}})$, *in (2) we can replace the last* $\bar{\mathbf{b}}$ *by* $\bar{\mathbf{b}} \restriction J$.

Proof. 1), 2), 3) As T has the dependence property. $\square_{5.4}$

5.5 Claim. *1) Assume* $\bar{\mathbf{a}} = \bar{\mathbf{a}}^1 \hat{\ } \bar{\mathbf{a}}^2$ *is an indiscernible sequence and* $A \supseteq \bar{\mathbf{a}}^1 \hat{\ } \bar{\mathbf{a}}^2$ *respects the non-trivial cut* $(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2)$ *both cofinalities of which are* $> |T|$ *and* $\bar{\mathbf{c}} \subseteq A$ *is an endless indiscernible sequence perpendicular to* $\bar{\mathbf{a}}^1$ *and to the inverse of* $\bar{\mathbf{a}}^2$. *Then* $\bar{\mathbf{c}}$ *is perpendicular to* $(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2)$ *over A.*
2) Let (I_1, I_2) *be a Dedekind cut of the indiscernible sequence* $\bar{\mathbf{a}}$ *with cofinalities* $> |T|$ *and* $\bar{\mathbf{a}} \subseteq A$. *The set A respects the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$ *iff the set A weakly respected the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$.
3) Let (I_1, I_2) *be a Dedekind cut of the infinite indiscernible sequence* $\bar{\mathbf{a}}$ *with infinite cofinalities,* $\bar{\mathbf{b}}$ *an endless indiscernible sequence such that* $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ *are mutually indiscernible. Then* $\bar{\mathbf{b}}$ *is perpendicular to the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$.

Proof. 1) It is enough to show that if $\bar{c}$ realizes $\mathrm{Av}(A, \bar{\mathbf{c}})$ then $A \cup \bar{c}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$. By the second part (proved below) it is enough to show that $A \cup \bar{c}$ weakly respects $(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2)$. Let $\bar{d} \subseteq A \cup \bar{c}$, and consider $\varphi(\bar{x}, \bar{d})$. Without loss of generality for some $\bar{d}' \subseteq A, \bar{d} = \bar{c} \hat{\ } \bar{d}'$. By the choice of $\bar{c}$ and as $\bar{\mathbf{c}}, \bar{\mathbf{a}}^1$ are perpendicular for some truth value $\mathbf{t}$ we have:

(a) for every large enough $t \in \mathrm{Dom}(\bar{\mathbf{a}}_1)$ we have $\models \varphi[\bar{a}^1_t, \bar{c}, \bar{d}']^{\mathbf{t}}$

[7] we can omit this here (and many other places) if in Definition 5.2(3),(4), we say "for every finite Δ". But this does not help much because of "$\kappa > |T|$" in 4.23. We could replace $|T|^+$ by a kind of $\kappa^r_{\mathrm{ind}}(T)$.

(b) for every large enough $t \in \operatorname{Dom}(\bar{\mathbf{a}}_1)$ for every large enough $s \in \operatorname{Dom}(\bar{\mathbf{c}})$ we have $\models \varphi[\bar{a}^1_t, \bar{c}_s, \bar{d}']^{\mathbf{t}}$

(c) for every large enough $s \in \operatorname{Dom}(\bar{\mathbf{c}})$ for every large enough $t \in \operatorname{Dom}(\bar{\mathbf{a}}^1)$ we have $\models \varphi[\bar{a}^1_t, \bar{c}_s, \bar{d}']^{\mathbf{t}}$.

As $\bar{d}', \bar{c}_s \subseteq A$ and A respects $(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2)$ by clause (c) clearly

(d) for every large enough $s \in \operatorname{Dom}(\bar{\mathbf{c}})$ for every small enough $t \in \operatorname{Dom}(\bar{\mathbf{a}}^2)$ we have $\models \varphi[\bar{a}^2_t, \bar{c}_s, \bar{d}']^{\mathbf{t}}$.

As $\bar{\mathbf{c}}$ and the inverse of $\bar{\mathbf{a}}^2$ are perpendicular by clause (d) we have

(e) for every small enough $t \in \operatorname{Dom}(\bar{\mathbf{a}}^2)$ for every large enough $s \in \operatorname{Dom}(\bar{\mathbf{c}})$ we have $\models \varphi[\bar{a}^2_t, \bar{c}_s, \bar{d}']^{\mathbf{t}}$.

By the choice of $\bar{c}$ and clause (e)

(f) for every small enough $t \in \operatorname{Dom}(\bar{\mathbf{a}}^2)$ we have $\models \varphi[\bar{a}^2_t, \bar{c}, \bar{d}']^{\mathbf{t}}$.

Together we are done.
2) Trivially respect implies weakly respect. So assume $\bar{b} \subseteq A$. By 4.14(1) we can find $(J_1, J_2) \leq (I_1, I_2)$ such that $\bar{\mathbf{a}} \restriction J_1, \bar{\mathbf{a}} \restriction J_2$ are mutually indiscernible over $\bar{b}$. Toward contradiction assume that $\bar{\mathbf{a}} \restriction (J_1 \cup J_2)$ is not indiscernible over $\bar{b}$, so there is $\varphi(\bar{x}_1, \ldots, \bar{x}_n, \bar{b})$ witnessing it. We prove by induction on $m \leq n$ the natural statement: for some truth value $\mathbf{t}$ if $t_1 < \ldots < t_n$ are in $J_1 \cup J_2$ and $t_{m+1}, \ldots, t_n \in J_2$ then $\mathfrak{C} \models \varphi[\bar{a}_{t_1}, \ldots, \bar{a}_{t_n}, \bar{b}]^{\mathbf{t}}$.
For $m = 0$ this holds as $\bar{\mathbf{a}} \restriction J_2$ is an indiscernible sequence over $\bar{b}$ so we can choose the appropriate $\mathbf{t}$.
For $m+1$, if $t_m \in J_2$ uses the induction hypothesis, so assume $t_m \in J_1$, let t'_ℓ be t_ℓ if $\ell \neq m$ and any member t' of J_2 satisfying $t_m < t' < t_{m+1}$ if $\ell = m$ & $m+1 \leq n$ and t' any member of J_2 if $\ell = m = n$. Let $\bar{c} = \langle \bar{a}_{t_\ell} : \ell \neq m \rangle$ and $\psi(\bar{x}, \langle \bar{y}_\ell : \ell = 1, \ldots, n$ and $\ell \neq m \rangle, \bar{z}) = \varphi(\bar{y}_1, \ldots, \bar{y}_{m-1}, \bar{x}, \bar{y}_{m+1}, \ldots, \bar{y}_n, \bar{z})$. So $\models \psi[\bar{a}_{t'_m}, \bar{c}, \bar{b}]^{\mathbf{t}}$ by the induction hypothesis and $\models \neg\psi[\bar{a}_{t'_n}, \bar{c}, \bar{b}]^{\mathbf{t}}$ by the assumption toward contradiction. So by mutual indiscernibility $\psi(\bar{a}_{t'_s}, \bar{c}, \bar{b})^{\mathbf{t}}$ holds for every large enough $s \in J_1$ and fails for every small enough $s \in J_2$. But $\bar{c}, \bar{b} \subseteq A$ and this shows that "weakly respect" fails.
3) Easy, e.g., by part (1) and 4.7(1), clause (b) of Definition 5.2(4) holds and similarly to the proof of 5.3, clause (a) of Definition 5.2(4) holds.

$\square_{5.5}$

5.6 Definition. We say an endless indiscernible sequence $\bar{\mathbf{a}} \subseteq A$ has true dual cofinality κ inside A, tr-d-cf$(\bar{\mathbf{a}}, A) = \kappa$ <u>if</u> there is $\bar{\mathbf{b}} \subseteq A$ such that $\bar{\mathbf{a}}\hat{\ }\bar{\mathbf{b}}$ is an indiscernible sequence with the cut $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ being respected by A and Dom$(\bar{\mathbf{b}})$ has downward cofinality κ.

5.7 Observation. 1) tr-d-cf$(\bar{\mathbf{a}}, A)$ is well defined, i.e., has at most one value.
2) If tr-d-cf$(\bar{\mathbf{a}}, A)$ is well defined then it is equal to dual-cf$(\bar{\mathbf{a}}, A)$, see Definition 4.5(3).

Proof. Easy, as κ is unique.

5.8 Claim. *1) If δ is a limit ordinal, $\langle A_i : i < \delta\rangle$ is increasing, $\bar{\mathbf{a}} \subseteq A_0$ is an endless indiscernible sequence, $\bar{a}'_i \subseteq A_{i+1}$ realizes* $\mathrm{Av}(A_i, \bar{\mathbf{a}}), \bar{\mathbf{a}}' = \langle \bar{a}'_i : i < \delta\rangle$ *and $\bar{\mathbf{a}}''$ is the inverse of $\bar{\mathbf{a}}'$ <u>then</u>*

- (a) *$\bar{\mathbf{a}}\hat{\ }\bar{\mathbf{a}}''$ is an indiscernible sequence*
- (b) *the set $\bigcup\limits_{i<\delta} A_i$ respects the cut* $(\mathrm{Dom}(\bar{\mathbf{a}}), \mathrm{Dom}(\bar{\mathbf{a}}''))$ *of $\bar{\mathbf{a}}\hat{\ }\bar{\mathbf{a}}''$.*

2) If $\bar{\mathbf{a}}$ is a non-stable indiscernible sequence, $\bar{\mathbf{a}} \subseteq A$, the set A respects the non-trivial cut (I_1, I_2) of $\bar{\mathbf{a}}$ and the cofinalities of the cut are $> |T|$ <u>then</u> dual-cf$(\bar{\mathbf{a}} \restriction I_1, M) = \mathrm{cf}(I^*_2) =$ tr-d-cf$(\bar{\mathbf{a}} \restriction I_1, M)$.
3) [Not used] If $\bar{\mathbf{a}}$ is an indiscernible sequence with Dedekind cut (I_1, I_2) of cofinality $(\kappa_1, \kappa_2), \aleph_0 < \kappa_1, \kappa_2$ and $\bar{\mathbf{c}}$ an endless indiscernible sequence perpendicular to the cut (I_1, I_2) of $\bar{\mathbf{a}}$, <u>then</u>: for every formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and sequence $\bar{b}$ such that $\bar{\mathbf{a}} \cup \bar{b} \cup \bar{\mathbf{c}}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$ for some truth value $\mathbf{t}$ we have:

- $(*)$ *for some $(J_1, J_2) \le (I_1, I_2)$ and for every $t \in J_1 \cup J_2$, for every large enough $s \in \mathrm{Dom}(\bar{\mathbf{c}})$ we have $\mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}, \bar{c}_s]^{\mathbf{t}}$.*

4) If in part (3), $|T| < \kappa_1, \kappa_2$, <u>then</u> for some $(J_1, J_2) \le (I_1, I_2)$ and end segment J of Dom$(\bar{\mathbf{c}})$ we have: $\bar{\mathbf{a}} \restriction (J_1 + J_2), \bar{\mathbf{c}} \restriction J$ are mutually indiscernible.

Proof. 1), 2) Straightforward.
3) Let $\delta = |T|^+$ and let $\bar{d}_\gamma$ realize $\mathrm{Av}(\bar{\mathbf{a}} \cup \bar{b} \cup \bar{\mathbf{c}} \cup \{\bar{d}_\beta : \beta < \gamma\}, \bar{\mathbf{a}})$, for $\gamma < \delta$ so by the definition of "respect the Dedekind cut" and as $\kappa_1, \kappa_2 > \aleph_0$ there is $(J_1, J_2) \le (I_1, I_2)$ such that $\bar{\mathbf{a}} \restriction (J_1 \cup J_2)$ is indiscernible over $\cup\{\bar{d}_n : n < \omega\} \cup \bar{b}$ hence $\bar{\mathbf{a}} \restriction (J_1 + J_2)$ and $\bar{\mathbf{d}} = \langle \bar{d}_\gamma : \gamma < \delta\rangle$ are $\{\varphi\}$-mutually indiscernible over $\bar{b}$. So we have

truth value $\mathbf{t}$ such that $t \in J_\ell$ & $\gamma < \delta \Rightarrow \mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}, \bar{d}_\gamma]^{\mathbf{t}}$. Recall that (I_1, I_2) have cofinality (κ_1, κ_2) and for our purpose without loss of generality $\kappa_1, \kappa_2 > |T|$. Now clearly the three indiscernible sequences $\bar{\mathbf{a}} \restriction J_1$, the inverse of $\bar{\mathbf{a}} \restriction J_2$ and $\langle \bar{d}_\gamma : \gamma < \delta \rangle$ are mutually indiscernible, hence by 4.17, clause (B) without loss of generality they are mutually indiscernible over $\bar{b}$ (i.e., omitting an initial segment of each and renaming). By the choice of the $\bar{d}_\gamma$'s, for every $t \in J_1 + J_2$ for every large enough $s \in \operatorname{Dom}(\bar{\mathbf{c}})$ we have $\models \varphi[\bar{a}_t, \bar{b}, \bar{c}_s]^{\mathbf{t}}$.
4) Should be clear (compare with 1.30, 5.5, 5.8).

5.9 Claim. *Assume*

(a) $I = I_1 + I_2$ *and the Dedekind cut* (I_1, I_2) *has cofinality* (κ_1, κ_2)
(b) $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ *is an indiscernible sequence*
(c) $\bar{\mathbf{a}} \subseteq A$
(d) *the set A respects the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$
(e) $|T| < \kappa_1, \kappa_2$ *(not used in part (1)).*

1) If $\operatorname{tp}(\bar{d}, A) \in \mathbf{F}^{\rm sp}_\kappa$ *and* $\kappa \le \kappa_1, \kappa_2$, *then the set* $A \cup \bar{d}$ *respects the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$.
2) If $A^+ = A \cup \{a_i : i < i^*\}$ *and for each* $i, \operatorname{tp}(a_i, A \cup \{a_j : j < i\})$ *belongs to* $\mathbf{F}^{\rm sp}_{\min\{\kappa_1,\kappa_2\}}$ *or is* $\operatorname{Av}(A \cup \{a_j : j < i\}, \bar{\mathbf{b}})$ *where* $\bar{\mathbf{b}} \subseteq A \cup \{a_j : j < i\}$ *is an endless indiscernible sequence perpendicular to the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$ *over* $A \cup \{a_j : j < i\}$, *then* A^+ *respects the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$.

Proof. 1) As $p = \operatorname{tp}(\bar{d}, A)$ belongs to $\mathbf{F}^{\rm sp}_\kappa$, there is a subset B of A of cardinality $< \kappa$ such that p does not split over B.

Let $\bar{d}' \in {}^{\omega >}(A \cup \bar{d})$ and we should find $(J_1, J_2) \le (I_1, I_2)$ such that $\bar{\mathbf{a}} \restriction (J_1 + J_2)$ is indiscernible over $\bar{d}'$. As we can increase $\bar{d}'$ (this just makes our task harder) without loss of generality $\bar{d}' = \bar{d}{}^\frown\bar{e}$ with $\bar{e} \subseteq A$. Now for every $\bar{e}' \subseteq B \cup \bar{e}$ there is $(J^1_{\bar{e}'}, J^2_{\bar{e}'}) \le (I_1, I_2)$ such that $\bar{\mathbf{a}} \restriction (J^1_{\bar{e}'} + J^2_{\bar{e}'})$ is an indiscernible sequence over $\bar{e}'$. Let J_ℓ be $\cap\{J^\ell_{\bar{e}'} : \bar{e}' \subseteq B \cup \bar{e}\}$ for $\ell = 1, 2$ if $\kappa > \aleph_0$ and let $J_\ell = J^\ell_{\bar{e}^*}, \bar{e}^*$ listing $B \cup \bar{e}$ if $\kappa = \aleph_0$. As $\kappa \le \kappa_1, \kappa_2$ and $\operatorname{cf}(I_1, I_2) = (\kappa_1, \kappa_2)$ clearly $(J_1, J_2) \le (I_1, I_2)$ and $\bar{\mathbf{a}} \restriction (J_1 + J_2)$ is indiscernible over $B \cup \bar{e}$.

Now for any formula φ and $\bar{b} \in {}^{\omega >}B$ and $s \in J_1, t \in J_2$ we have

$(*)_1$ $\operatorname{tp}(\bar{a}^1_s{}^\frown\bar{e}, B) = \operatorname{tp}(\bar{a}^2_t{}^\frown\bar{e}, B)$.
[Why? By "$\bar{\mathbf{a}} \restriction (J_1 + J_2)$ indiscernible over $B \cup \bar{e}$".]

$(*)_2$ $\mathfrak{C} \models \varphi[\bar{d}, \bar{a}^1_s, \bar{e}, \bar{b}]$ iff $\varphi(x, \bar{a}^1_s, \bar{e}, \bar{b}) \in p$.
[Why? By the assumption on p as $\operatorname{tp}(\bar{d}, A)$.]

$(*)_3$ $\varphi(\bar{x}, \bar{a}^1_s, \bar{e}, \bar{b}) \in p$ iff $\varphi(\bar{x}, \bar{a}^2_t, \bar{e}, \bar{b}) \in p$.
[Why? By $(*)_1$ as p does not split over B.]

$(*)_4$ $\varphi(\bar x, \bar a^2_t, \bar e, \bar b) \in p$ iff $\mathfrak{C} \models \varphi(\bar d, \bar a^2_t, \bar e, \bar b)$.
[Why? By the definition of p as $\mathrm{tp}(\bar d, A)$.]

So by $(*)_2 + (*)_3 + (*)_4$ as they hold for every φ and $\bar b \in {}^{\omega>}B$

$(*)_5$ $\bar d \char94 \bar a^1_s \char94 \bar e$ and $\bar d \char94 \bar a^2_t \char94 \bar e$ realize the same type over B

hence

$(*)_6$ $\bar a^1_s, \bar a^2_t$ realized the same type over $\bar e \char94 \bar d$.

By 5.5(2) this suffices as $\kappa_1, \kappa_2 > |T|$ (or we could have repeated the proof for any increasing sequence $\langle s_1, \dots, s_n\rangle, \langle t_1, \dots, t_n\rangle$ from $J_1 + J_2$ so $\kappa_1, \kappa_1 > |T|$ will not be used).
2) Let $A_i = A \cup \{a_j : j < i\}$, and we prove by induction on i that the cut (I_1, I_2) of $\bar{\mathbf{a}}$ is respected by the set A_i. For $i = 0$ we use assumption (d). For i limit, we use 5.3(4). For $i = j+1$ we use part (1) if $\mathrm{tp}(\bar b_j, A_j) \in \mathbf{F}^{\mathrm{sp}}_\kappa$ and we use the definition 5.2(4) if not. For $i = i^*$ we have gotten the desired conclusion. $\square_{5.9}$

5.10 Claim. *Assume*

(a) (I_1, I_2) *is a non-trivial cut of the indiscernible sequence* $\bar{\mathbf{a}}$
(b) $\bar{\mathbf{b}}$ *is an endless indiscernible sequence*
(c) $\bar{\mathbf{a}} \restriction I_1, \bar{\mathbf{b}}$ *are perpendicular*
(d) *for each* $t \in I_2$ *the sequence* $\bar a^1_t$ *realizes* $\mathrm{Av}(\{\bar a^1_s : s \in I_1 \vee t <_I s \in I_2\} \cup \bar{\mathbf{b}}, \bar{\mathbf{a}}^1 \restriction I_1)$
(e) *both cofinalities of* (I_1, I_2) *and the cofinality of* $\bar{\mathbf{b}}$ *are* $> |T|$.

<u>Then</u> $\bar{\mathbf{b}}$ *is perpendicular to the cut* (I_1, I_2) *of* $\bar{\mathbf{a}}$.

Proof. Now by assumption (c) here and 4.27, for some unbounded $I'_1 \subseteq I_1$ and $I' \subseteq \mathrm{Dom}(\bar{\mathbf{b}})$, the sequences $\bar{\mathbf{a}} \restriction I'_1, \bar{\mathbf{b}} \restriction I'$ are mutually indiscernible. By 5.4(1)+(2) + 5.3(5), without loss of generality $I'_1 = I_1, I' = \mathrm{Dom}(\bar{\mathbf{b}})$, so without loss of generality $\bar{\mathbf{a}} \restriction I_1, \bar{\mathbf{b}}$ are mutually indiscernible. Easily also $\bar{\mathbf{b}}$ and $\bar{\mathbf{a}}$ are mutually indiscernible (by clauses (c) and (d)). Hence by 5.5(3) clearly $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ are perpendicular. $\square_{5.10}$

5.11 Theorem. *Assume*

(*a*) $\lambda = \lambda^{<\kappa_2} + 2^{2^{<\kappa_1}}$

(*b*) $|T| < \kappa_1 = \mathrm{cf}(\kappa_1) \le \kappa_2 \le \theta_2 = \mathrm{cf}(\theta_2), \kappa_1 \le \theta^1_\zeta = \mathrm{cf}(\theta^1_\zeta) \le \lambda$ *for* $\zeta < \zeta^*$

(*c*) $|A| \le \lambda$

(*d*) $\bar{\mathbf{a}}^\zeta \subseteq A$ *is endless, non-stable indiscernible for* $\zeta < \zeta^*$ *and* $\zeta^* \le \lambda$

(*e*) *the* $\bar{\mathbf{a}}^\zeta$ *for* $\zeta < \zeta^*$ *are pairwise perpendicular*

(*f*) $\mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{a}}^\zeta))$ *is* $\ge \kappa_1$.

Then we can find a model M *such that*

(α) $A \subseteq M$

(β) $\text{dual-cf}(\bar{\mathbf{a}}^\zeta, M) = \theta^1_\zeta + \mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{a}}^\zeta))$ *for every* $\zeta < \zeta^*$, *moreover* $\text{tr-d-cf}(\bar{\mathbf{a}}^\zeta, M) = \theta^1_\zeta$

(γ) *if* $\bar{\mathbf{a}} \subseteq M$ *is a non-stable endless indiscernible sequence of cardinality (hence cofinality)* $< \kappa_2$ *perpendicular to every* $\bar{\mathbf{a}}^\zeta$ *then* $\text{dual-cf}(\bar{\mathbf{a}}, M) = \theta_2$

(δ) M *is* κ_1*-saturated.*

Proof. Let $\theta^0_\zeta = \mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{a}}^\zeta))$. Note that without loss of generality

($*$) there is $\bar{\mathbf{b}}^\zeta$ such that $\bar{\mathbf{a}}^\zeta{}^\frown\bar{\mathbf{b}}^\zeta$ is an indiscernible sequence, with the cut $(\mathrm{Dom}(\bar{\mathbf{a}}^\zeta), \mathrm{Dom}(\bar{\mathbf{b}}^\zeta))$ having cofinality $(\theta^0_\zeta, \theta^1_\zeta)$ and this Dedekind cut being respected over the set A.
[Why? By using 5.10 and 5.3(4), of course.]

We can find $\bar{a}_i$ for $i < \delta^* =: \lambda \times \theta_2$ such that letting $A_i = A \cup \{\bar{a}_j : j < i\}$ we have

(i) for each $i < \delta^*$ we have $\mathrm{tp}(\bar{a}_i, A_i) \in \mathbf{F}^{\mathrm{sp}}_{\kappa_1}$ or $\mathrm{tp}(\bar{a}_i, A_i) = \mathrm{Av}(A_i, \bar{\mathbf{a}})$ for some non-stable endless indiscernible sequence $\bar{\mathbf{a}} \subseteq A_i$ of cardinality $< \kappa_2$ perpendicular to $\bar{\mathbf{a}}_\zeta$ for every $\zeta < \zeta^*$

(ii) if $p \in \mathbf{S}^{<\omega}(A_{\lambda\times(\varepsilon+1)}), p \in \mathbf{F}^{\mathrm{sp}}_{\kappa_1}$ then for λ ordinals $j < \lambda, p$ is realized by $\bar{b}_{\lambda\times\varepsilon+j}$

(iii) if $\bar{\mathbf{a}}$ is as in (i), $\bar{\mathbf{a}} \subseteq A_{\lambda\times\varepsilon}$ then for λ ordinal $j < \lambda, \bar{b}_{\lambda\times\varepsilon+j}$ realizes $\mathrm{Av}(A_{\lambda\times\varepsilon+j}, \bar{\mathbf{a}})$.

By bookkeeping, as in 5.9 this is straightforward and clauses (α) and (δ) obviously hold and in particular there is M with universe A_{δ^*}. By 5.9, M respects $(\bar{\mathbf{a}}^\zeta, \bar{\mathbf{b}}^\zeta)$ for each $\zeta < \zeta^*$ hence by 5.8(2), clause (β) holds.

As for clause (γ), let $\bar{\mathbf{a}} = \langle a_t : t \in I\rangle \subseteq M$ be a non-stable endless indiscernible sequence $|I| < \kappa_2, \bar{\mathbf{a}}$ perpendicular to every $\bar{\mathbf{a}}^\zeta$; it is also perpendicular to the inverse of $\bar{\mathbf{b}}^\zeta$ as their cofinalities are different. Hence it is also perpendicular to the cut $(\bar{\mathbf{a}}^\zeta, \mathbf{b}^\zeta)$ by 5.5(1).

As $\kappa_2 \le \theta_2 = \mathrm{cf}(\theta_2) = \mathrm{cf}(\delta^*)$ it follows that for some $\alpha < \delta^*, \bar{\mathbf{a}} \subseteq A_\alpha$ and so $u = \{i : i \in (\alpha, \delta^*)$ and $\bar{a}_i$ realizes $\mathrm{Av}(A_i, \bar{\mathbf{a}})\}$ is unbounded in δ^*, let J be $(u, >)$ and so $\bar{\mathbf{a}}\hat{\ }\langle a_t : t \in J\rangle$ is an indiscernible sequence. By 5.8(1) the set A_{δ^*} respect the Dedekind cut (I, J) of $\bar{\mathbf{a}}\hat{\ }\langle a_t : t \in J\rangle$ that is M respects it hence dual-cf$(\bar{\mathbf{a}}, M) = \mathrm{cf}(J^*) = \mathrm{cf}(\delta^*) = \theta_2$ as required. $\square_{5.11}$

§6 Concluding Remarks

We continue to deal with dependent T.
6.1 Conjecture: If $p \in \mathbf{S}(M)$ then there are at most $2^{|T|}$ infinite sequences of indiscernible, pairwise non nb-s based on some ultrafilter D on M with $\mathrm{Av}(M,D) = p$.

A major lack of this work is the absence of test questions.

A candidate is the problem of classifying first order theories by the existence of indiscernibles (raised by Grossberg and the author, see [Sh 702, §2]), e.g.:

6.2 Question: If $A \subseteq \mathfrak{C}_T, \kappa = |A| + |T|$ (or e.g. $\kappa = \beth_7(|A|+|T|)$ and $\lambda = \beth(2^\kappa)^+$ (or larger, but no large cardinals) and $a_i \in \mathfrak{C}_T$ for $i < \lambda$ then for some $w \in [\lambda]^{\kappa^+}$, the sequence $\langle a_i : i \in w\rangle$ is an indiscernible sequence over A(in $\mathfrak{C}_T$).

Now though this property cannot characterize the dependence property, it is quite natural in this context. Consider T_n, the model completion of the empty theory in the vocabulary $\{R_n\}, R_n$ an n-place relation. So if $\lambda = \beth_n(\kappa)^+$, we get a positive answer, but for $n \ge 2, T_n$ is independent. We may consider replacing indiscernible sequences $\bar{\mathbf{a}}$ by $\bar{\mathbf{a}} = \langle \bar a_t : t \in I\rangle$ as an index structure with $n(I) = \cup\{n(R)+1 : R$ an atomic relation of $I\} < \omega, \bar{\mathbf{a}}$ is I-indiscernible, i.e., $k < \omega, \bar s, t \in {}^k I, \bar s \sim_I \Rightarrow \bar a_{\bar t}, \bar a_{\bar s}$ realizes the same type. See also later.

Another direction is generalizing DOP, which in spite of its name is a non first order independence property.

On classification by Karp complexity see Laskowski and Shelah [LwSh 560], [LwSh 687] (let the κ-Karp complexity $\gamma_\kappa(M)$ of M be the least γ such that every $\mathbb{L}_{\infty,\kappa}(\tau_M)$-formula is equivalent in M to such a formula of quantifier depth $< \gamma$, and the (λ,κ)-Karp complexity of T is $\cup\{\gamma_\kappa(M)+1 : M$ a model of T of cardinality $\lambda\}$.
For elementary classes which are unstable but dependent the following parallel to DOT may help.

6.3 Definition. T has the dual-cf-$\bar\kappa$-dimensional independence property when: $\bar\kappa = (\kappa_0,\kappa_1,\kappa_2), \kappa_1 \ne \kappa_0 < \kappa_1, \kappa_0 < \kappa_2$ and for every λ and symmetric relation $R \subseteq \lambda \times \lambda$ we can find $M_R, \bar{\mathbf{b}}_\alpha, \bar{\mathbf{c}}_\alpha \in {}^{\kappa_0}(M_R)$ and an indiscernible sequences $\mathbf{I}_{\alpha,\beta} = \langle \bar a_{\alpha,\beta,i} : i < \kappa_0\rangle \subseteq M_R$ for $(\alpha,\beta) \in R, \alpha < \beta$ such that:

- (a) the type of $\bar{\mathbf{b}}_\alpha{}^\frown\bar{\mathbf{c}}_\beta{}^\frown\mathbf{I}_{\alpha,\beta}$ is the same for all pairs $(\alpha,\beta) \in R$
- (b) dual-cf$(\mathbf{I}_{\alpha,\beta}, M_R) = \kappa_1$ for $(\alpha,\beta) \in R$
- (c) if $\alpha < \beta$ and $\neg\alpha R_\beta$ and the sequence $\mathbf{I}'_{\alpha,\beta} = \langle \bar a'_{\alpha,\beta,i} : i < \kappa_0\rangle \subseteq M_R$ is such that for every $(\alpha_1,\beta_1) \in R$ there is an automorphism h of $\mathfrak{C}$ taking $\bar{\mathbf{b}}_{\alpha_1}$ to $\bar{\mathbf{b}}_\alpha, \bar{\mathbf{c}}_{\beta_1}$ to $\bar{\mathbf{c}}_\beta$ and $\bar a_{\alpha_1,\beta_1,i}$ to $\bar a_{\alpha,\beta,i}$ then dual-cf$(\mathbf{I}'_{\alpha,\beta}, M) = \kappa_2$

(*d*) M_R is κ_0^+-saturated.

Note that a sufficient condition for this is

$(c)^+$ if $\mathbf{I}'_{\alpha,\beta} = \langle \bar a'_{\alpha,\beta,i} : i < \kappa_0\rangle$ realizes the relevant type over $\bar{\mathbf{a}}_\alpha{}^\frown \bar{\mathbf{b}}_\beta$ and $(\alpha,\beta) \notin R, \alpha < \beta < \lambda$ and $\alpha_1 < \beta_1 < \lambda, (\alpha_1,\beta_1) \in R$ then $\mathbf{I}'_{\alpha,\beta}, \mathbf{I}_{\alpha_1,\beta_1}$ are perpendicular.

Recall

6.4 Definition. *M is κ-resplended when: if $\bar{\mathbf{c}} \in {}^{\kappa>}M, M \prec N, N$ the τ_M-reduct of $N^+, |\tau_{N^+} \backslash \tau_N| < \kappa$, then $(M,\bar{\mathbf{c}})$ can be expanded to a model of $\mathrm{Th}(N^+,\bar{\mathbf{c}})$.*

<u>6.5 Problem</u>: Characterize the λ-Karp complexities for μ-resplended models of a complete first order theory T of cardinality $> |T|$ say for $\lambda > \mu > |T|$ regular?

For T stable it is 0 (see [Sh:e, Ch.V]), as all such models are saturated. For T with the independence property we should consider combining [LwSh 687] (which constructs models of $T_1 \supseteq T$ with large λ-Karp complexity of the τ_T-reduct) and [Sh:e, Ch.V]. But here our concern is for unstable T with the dependence property.

If we look at an indiscernible sequence $\langle \bar b_t : t \in I\rangle$ inside a model M, we know that distinct Dedekind cuts with at least one side having large cofinality are quite unconnected. We shall show in subsequent work [Sh 783] that the large cofinality demand is not incidental.

<u>6.6 Question</u>: Investigate the graph $(\{\bar{\mathbf{b}}_t : \bar{\mathbf{b}}_t$ an endless indiscernible sequence$\}$, perpendicularly).

As in §5 we can show that many variants are equivalent (using $+\infty, -\infty$ to absorb). We can similarly discuss a parallel to deepness (see [Sh:f, X], recall that deepness is related to orthogonality).

<u>6.7 Discussion</u>: 1) It is known that e.g. (first theory of) the p-adics has the dependence property (and are unstable). Does this work tell us anything on them? Well, the construction in §5 gives somewhat more than what unstability gives: complicated models with more specific freedom. Note that instead dual-cf$(\mathbf{I},M)$ we can use more complicated invariants (see [Sh:e, Ch.III,§3] or earlier works).

We can, of course, (for the p-adic) characterize directly when indiscernible sequences are perpendicular.

2) We may like to define super dependence properties (and $\kappa_{\mathrm{nip}}(T)$) (parallel of superstable, i.e., $\kappa(T) = \aleph_0$ or super-simple $\kappa_{\mathrm{cdt}}(T) = \aleph_0$). There are some possibilities, one defined in [Sh:c, III], another in [Sh 783]. We may try the definition "$w(\mathbf{I}) < \aleph_0$", i.e., weight for every endless indiscernible sequence where

6.8 Definition. For an endless indiscernible sequence $\mathbf{I}$ let $w(\mathbf{I}) = \sup\{\alpha :$ there is a sequence of length α of pairwise perpendicular endless indiscernible sequences each non perpendicular to $\mathbf{I}\}$.
But $w(\mathbf{I})$ is not weight for superstable theory just of a variant of it hence exactly like dimension in the sense of algebraic manifolds.

6.9 Question: Assume $\mathbf{I}_\ell = \langle a^\ell_t : t \in I_\ell\rangle$ for $\ell = 1, 2$ are endless indiscernible sequences and they are non perpendicular.

- (a) Find a definable equivalence relation E such that $\langle a^2_t/E : t \in I_2\rangle$ is non-trivial and $a^2_t/E \in \operatorname{acl}(\mathbf{I}_1 \cup \{a^2_s : s <_{I_2} t\})$ for any large enough t (i.e., non-perpendicularty implies non-orthogonality for trivial reasons).
- (b) If $(\mathbf{I}_1, \mathbf{I}_2)$ is $(1, < \omega)$-mutual indiscernible (parallel to Hrushovski's theorem), can we define a derived group? More generally, it seems persuasive that groups appear naturally, particularly ordered groups.
- (c) Does the fact that putting elements together, make strong splitting implies dividing helps?
  [recall:
  - (i) p strongly splits over A if there is a sequence $\langle \bar a_t : i < \lambda\rangle$ indiscernible over A such that $\varphi(\bar x, \bar a_0)$ & $\neg\varphi(x, \bar a_1)$
  - (ii) p divides over A if there is an indiscernible sequence $\langle \bar b_i : i < \lambda\rangle$ over A and $\psi(x, \bar b_0) \in p$ such that $\{\psi(\bar x, \bar b_i) : i < \lambda\}$ is contradictory.

  So having (i), letting $\bar b_i = \bar a_{2i}{}^\frown \bar a_{2i+1}, \psi(x, \bar b_i) = \varphi(x, \bar a_{2i})$ & $\neg\varphi(x, \bar a_{2i+1})$ we have (ii).]
- (d) Can the canonical bases of §1 help?
- (e) What can we say on "$\bar{\mathbf{a}}$ perpendicular to a set (or model) A?"

6.10 Discussion Cherlin wonders on the place of parallel algebraic geometric dimension and the place of 0-minimal theory. In my perception probably if we succeed in 6.9(a), we may have a minimality notion which may then be characterized as some cases, but maybe it does not fit.

6.11 Question: If $B \subseteq C \subseteq \mathfrak{C}, p \in \mathbf{S}^m(B)$, then p has an extension q in $\mathbf{S}^m(C)$ which does not split strongly over B (and if p does not fork over A, then q does not fork over A).

6.12 Question: Given two non perpendicular types in $\mathbf{S}(A)$ (or ultrafilter on A) which are weakly perpendicular can we find naturally defined groups?

6.13 Concluding Remark. We can define when an endless indiscernible sequence is orthogonal to a set and the dimensional independence property and prove natural properties, we hope to pursue this.

6.14 Question: For an infinite indiscernible sequence $\langle \bar b_t : t \in I\rangle, \bar b_t \in {}^\alpha\mathfrak{C}$ can we find $A \subseteq \mathfrak{C}, |A| \le |\gamma| + |T|$ and $\bar b^+_t = \bar b_t \char`\^ \bar a_t, \ell g(\bar a_t) = (|\gamma| + |T|)$ for $t \in J$ such that $\langle \bar b^+_t : t \in J\rangle$ is an indiscernible sequence and letting $p(\bar x, \bar y) = \text{ tp}(\bar b^+_s \char`\^ \bar b^+_t, A)$ for $s <_J t$ we have $p(\bar x,\bar y) \cup p(\bar y, \bar z) \vdash p(\bar x, \bar z)$?

6.15 Question: 1) If $M \prec N, a \in N, (M,N,a)$ is λ-saturated $\lambda > 2^{|T|}$. Can we find for every $A \subseteq M, |A| < \lambda$ a set $B \subseteq M, |B| \le 2^{|T|}$ such that $p \restriction B \vdash p \restriction A$?
2) Fix finite set $\Gamma = \{\varphi(x,\bar a) : \bar a \in {}^m\mathfrak{C}\}$. Look at $\mathbf{S}(\Gamma)$, is it true that every $p \in \mathbf{S}(\Gamma)$ is determined uniquely by $q \subseteq p, |q| \le n^1_\varphi$ and a rank $< n^2_\varphi$?

Or another way to use having few types.

6.16 Problem: Let M be λ-saturated, $p \in \mathbf{S}(M)$ and for $\varphi = \varphi(x;\bar y)$ let $I_{\varphi(x;\bar y)}(p) = \{\psi(\bar y, \bar c) : \bar c \subseteq M$ and $\{\bar b \subseteq M :\models \psi(\bar b, \bar c), \varphi(x, \bar b) \in p\}$ is definable$\}$. Investigate this; we can prove that it is not too small.

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