A PARTITION RELATION USING STRONGLY COMPACT CARDINALS SH761

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Abstract. If κ is strongly compact and $\lambda > \kappa$ and λ is regular (or alternatively $\mathrm{cf}(\lambda) \geq \kappa$), then $\left(2^{<\lambda}\right)^+ \rightarrow (\lambda + \zeta)^2_\theta$ holds for $\zeta, \theta < \kappa$.

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§0 Introduction

The aim of this paper is to prove the following theorem.

0.1 Theorem. *If κ is a strongly compact cardinal, $\lambda > \kappa$ is regular and $\zeta, \theta < \kappa$ then the partition relation $\left(2^{<\lambda}\right)^+ \to (\lambda + \zeta)^2_\theta$ holds.*

0.2 Theorem. *In Theorem 0.1.*
Instead λ regular, $\mathrm{cf}(\lambda) > \kappa$ suffices.

We notice that our argument is valid in the case $\kappa = \omega$. As for the history of the problem we point out that Hajnal proved in an unpublished work, that $(2^\omega)^+ \to (\omega_1 + n)^2_2$ holds for every $n < \omega$. Then it was showed in [Sh 26, §6] that for $\kappa > \omega$ regular and $2^{|\alpha|} < \kappa$, the relation $(2^{<\kappa})^+ \to (\kappa+\alpha)^2_2$ is true. More recently Baumgartner, Hajnal, and Todorčević in [BHT93] extended this to the case when the number of colors is arbitrary finite. Earlier by [Sh 424], we have $(2^{<\lambda})^{+n} \to (\lambda \times m)^2_k$ for n large enough (this was complimentary to the main result there that $\aleph_0 < \lambda = \lambda^{<\lambda} + 2^\lambda$ arbitrarily large does not imply $2^\lambda \to (\lambda \times \omega)^2_2$). Subsequently [BHT93] improves n. We hope that the way the strong compactness was used will be useful elsewhere; see [Sh 666] for a discussion of a possible consistency of failure. I also thank Peter Komjath for improving the presentation.

Notation. If S is a set, κ a cardinal then $[S]^\kappa = \{a \subseteq S : |a| = \kappa\}, [S]^{<\kappa} = \{a \subseteq S : |a| < \kappa\}$. If D is some filter over a set S then $X \in D^+$ denotes that $S \setminus X \notin D$ and $X \subseteq S$. If $\kappa < \mu$ are regular cardinals then $S^\mu_\kappa = \{\alpha < \mu : \mathrm{cf}(\alpha) = \kappa\}$, a stationary set. The notation $A = \{x_\alpha : \alpha < \gamma\}_<$, etc., means that A is enumerated increasingly.

§1 The case of λ regular

1.1 Lemma. *Assume $\mu = \mu^\theta$. Assume that D is a normal filter on μ^+ and $A^* \in D^+$ satisfies $\delta \in A^* \Rightarrow$ cf$(\delta) > \theta$, and F' is a function with domain $[A^*]^2$ and range of cardinality θ. Then there are a normal filter D_0 on μ^+ extending $D, A_0 \in D_0$ with $A_0 \subseteq A^*$ and $C_0 \subseteq$ Rang(F') satisfying Rang$(F' \restriction [A_0]^2) = C_0$ such that: if $X \in D_0^+$ then Rang$(F' \restriction [X]^2) \supseteq C_0$.*

We first prove a claim

1.2 Claim. *Assume $\mu = \mu^\theta$ and $F' : [S^*]^2 \to C_*, |C_*| \leq \theta, D$ is a normal filter on $\mu^+, S^* \subseteq \mu^+$ belongs to D^+ and $\delta \in S^* \Rightarrow$ cf$(\delta) > \theta$. There is a set $A \in D^+$ such that $A \subseteq S^*$ and some $C \subseteq C_0$ satisfying Rang$(F' \restriction [A]^2) = C$ and: if $f : A \to \mu^+$ is a regressive function, then for some $\alpha < \mu^+$ we have Rang$(F' \restriction [f^{-1}(\alpha)]^2) = C$ and $f^{-1}(\alpha)$ is a subset of μ^+ from D^+.*

Proof. Toward contradiction assume that no such sets A, C exist. We build a tree T as follows. Every node t of the tree will be of the form

$$\begin{aligned} t &= \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle \\ &= \langle \langle A^t_\alpha : \alpha \leq \varepsilon \rangle, \langle f^t_\alpha : \alpha < \varepsilon \rangle, \langle i^t_\alpha : \alpha < \varepsilon \rangle \rangle \end{aligned}$$

for some ordinal $\varepsilon = \varepsilon(t)$ where $\langle A_\alpha : \alpha \leq \varepsilon \rangle$ is a decreasing, continuous sequence of subsets of μ^+; for every $\alpha < \varepsilon, f_\alpha$ is a regressive function on A_α; and $\langle i_\alpha : \alpha < \varepsilon \rangle$ is a sequence of distinct elements of C_*. It will always be true that if $t <_T t'$, then each of the three sequences of t' extend the corresponding one of t.

To start, we make the node t with $\varepsilon(t) = 0, A_0 = S^*$ the root of the tree.

At limit levels we extend (the obvious way) all cofinal branches to a node.

If we are given an element $t = \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle$ of the tree and the set A_ε is $= \emptyset$ mod D then we leave t as a terminal node. Otherwise, let $C = C_t =$ Rang $\left(F' \restriction [A_\varepsilon]^2\right)$ and notice that by hypothesis, toward contradiction, the pair A_ε, C_t cannot be as required in the Claim. There is, therefore, a regressive function $f = f_t$ with domain A_ε, such that for every $x < \mu^+$ the set Rang$\left(F' \restriction [f^{-1}(x)]^2\right)$ is a proper subset of C_t or $f^{-1}(x)$ is a $= \emptyset$ mod D subset of μ^+. We make the immediate extensions of t the sequences of the form a $t_x = \langle \langle A_\alpha : \alpha \leq \varepsilon + 1 \rangle, \langle f_\alpha : \alpha < \varepsilon + 1 \rangle, \langle i_\alpha : \alpha < \varepsilon + 1 \rangle \rangle$ where $A_{\varepsilon+1} = f^{-1}(x), f_\alpha = f_t$ and $i_\varepsilon \in C_t$ is some colour value such that: if $A_{\varepsilon+1} \neq \emptyset$ mod D then i_ε is not in the range of $F' \restriction [A_\varepsilon]^2$.

Having constructed the tree observe that every element $x \in S^* \subseteq \mu^+$ belongs to a set $A^{t(x)}_{\varepsilon(x)}$ for some (unique) terminal node $t(x)$ of T. Also, $\varepsilon(x) < \theta^+ (< \mu^+)$ holds by the selection of the i_β's as $\langle i^{t(x)}_\alpha : \alpha < \varepsilon(x)\rangle$ is a sequence of members of C_* with no repetitions while C_*, the set of colours, has $\leq \theta$ members. For some set $S \subseteq S^*$ of ordinals $x < \mu^+$ which belong to D^+ (by the normality of D) the value of $\varepsilon(x)$ is the same, say ε. For $x \in S$ we let $g_\alpha(x) = f^{t(x)}_\alpha(x)$ where $f^{t(x)}_\alpha$ is the α-th regressive function in the node $t(x) \in T$. Again, by $\mu^\theta = \mu \ \ \& \ \ (\forall \alpha \in S)[\mathrm{cf}(\alpha) > \theta]$ we have that $(\forall x \in S')(\forall \alpha < \varepsilon) g_\alpha(x) = \beta_\alpha$ holds for some sequence $\langle \beta_\alpha : \alpha < \varepsilon\rangle$ and subset $S' \subseteq S$ from D^+. But then we get that the set S' satisfies $x, y \in S' \Rightarrow (A^{t(x)}_\alpha, f^{t(x)}_\alpha, i^{t(x)}_\alpha) = (A^{t(y)}_\alpha, f^{t(y)}_\alpha, i^{t(y)}_\alpha)$ for every $\alpha < \varepsilon$; we can prove this by induction on α. We can then prove that $A^{t(x)}_\varepsilon = A^{t(y)}_\varepsilon$ for $x, y \in S'$. We can conclude that $x, y \in S' \Rightarrow t(x) = t(y)$, so $S' \subseteq A^t_{\varepsilon(t)}$ for some terminal node t, but this latter set is in D^+, a contradiction.

$\square_{1.2}$

Proof of Lemma 1.1. Apply Claim 1.2 with $S^* = A^*$ to get corresponding (C, A). Define the ideal I as follows. For $X \subseteq \mu^+$ we let $X \in I$ <u>iff</u> there are a member E of D and a regressive function $f : X \cap A \to \mu^+$ such that every $\mathrm{Rang}\big(F' \restriction [f^{-1}(\alpha)]^2\big)$ is a proper subset of C or $f^{-1}(\alpha)$ is a $= \emptyset \bmod D$ subset of μ^+.

Now:

1.3 Claim. *I is a normal ideal on μ^+ (and $A^* = \mu^+ \bmod I$).*

Proof. Straightforward.

Set D_0 to be the dual filter of I, let $A_0 = A$ and let $C_0 = C$; by 1.2 we are done. $\square_{1.1}$

1.4 Remark. 1) If Lemma 1.1 holds for some D_0, A_0, C_0 then it holds for D_1, A_1, C_0 when the normal filter D_1 extends D_0, and $A_1 \in D_1$ satisfies $A_1 \subseteq A_0$.
2) If D_0, A_0, C_0 satisfy Lemma 1.5, and $X \in D_0^+$ <u>then</u> X contains a homogeneous set of order type $\lambda + 1$ of color ξ for every $\xi \in C_0$.
3) Lemma 1.1 is closely related to the proof in [Sh 26].

Proof of Theorem 0.1. Let $\mu = 2^{<\lambda}$, and $F : [\mu^+]^2 \to \theta$ be a colouring; we apply 1.1 for $A^* = S^{\mu^+}_{\mathrm{cf}(\lambda)}$, $(F = F, \theta = \theta, \mu = \mu)$ and D the club filter. We shall write $F(\alpha, \beta)$ for $F(\{\alpha, \beta\})$ and 0 for $F(\alpha, \alpha)$.
We fix A_0, D_0, C_0 which we get by 1.1.

1.5 Lemma. *Almost every $\delta \in A_0$; (i.e. for all but a set $= \emptyset$ mod D_0) satisfies the following: if $s \in [A_0 \cap \delta]^{<\lambda}$ and $\{z_\alpha : \alpha < \gamma\}_< \subseteq A_0 \cap [\delta, \mu^+)$ with $\gamma < \kappa$ <u>then</u> there is $\{y_\alpha : \alpha < \gamma\}_< \subseteq A_0 \cap (\sup(s), \delta)$ such that:*

(a) $F(x, y_\alpha) = F(x, z_\alpha)$ *(for $x \in s, \alpha < \gamma$);*

(b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ *(for $\alpha < \beta < \gamma$).*

Proof. By simple reflection (using the regularity of λ).

1.6 Lemma. *There[1] is $A'_0 \subseteq A_0, A'_0 \in D_0$ such that: if $\delta \in A'_0, s \in [\delta]^{<\lambda}$ and $\xi \in C_0$, <u>then</u> there exists a $\delta_1 \in A_0, \delta < \delta_1$ such that*

(a) $F(x, \delta) = F(x, \delta_1)$ *(for $x \in s$);*

(b) $F(\delta, \delta_1) = \xi$.

Proof. Otherwise, there is some $X \subseteq A_0, X \in D_0^+$ such that for every $\delta \in X$ there are $s(\delta) \in [\delta]^{<\lambda}$ and $\xi(\delta) \in C_0$ such that there is no $\delta_1 > \delta$ satisfying (a) and (b). By normality and $\mu = \mu^{<\lambda}$ we can assume that $s(\delta) = s$ and $\xi(\delta) = \xi$ holds for $\delta \in X$. By Lemma 1.1, that is the choice of (A_0, D_0, C_0), there must exist $\delta < \delta_1$ in X with $F(\delta, \delta_1) = \xi$ and this is a contradiction.

$\square_{1.6}$

Continuation of the proof of Theorem 0.1. Let $A'_0 \subseteq A_0$ satisfy Lemmas 1.1 and 1.6 and pick some $\delta_1 \in A'_0$ and then let $T = A'_0 \setminus (\delta_1 + 1)$.

1.7 Lemma. *There exists a function $G : T \times T \to C_0$ such that: if $s \in [\delta_1]^{<\lambda}, \gamma < \kappa$, and $Z = \{z_\alpha : \alpha < \gamma\}_< \subseteq T$ <u>then</u> there is $\{y_\alpha : \alpha < \gamma\}_< \subseteq (\sup(s), \delta_1)$ such that*

(a) $F(x, y_\alpha) = F(x, z_\alpha)$ *(for $x \in s, \alpha < \gamma$);*

(b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ *(for $\alpha < \beta < \gamma$);*

(c) $F(y_\alpha, z_\beta) = G(z_\alpha, z_\beta)$ *(for $\alpha, \beta < \gamma$).*

[1] in fact, if $A_1^* \in D_0^+$ then for some $A'_0 \subseteq A_1 \cap A_0, A_1 \setminus A'_0 = \emptyset$ modulo D_0 and the conclusion holds for every $\delta \in A'_0$

Proof. As κ is strongly compact, it suffices to show that for every $Z \in [T]^{<\kappa}$ there exists a function $G : Z \times Z \to \theta$ as required. Clauses (a) and (b) are obvious by Lemma 1.5, and it is clear that, if we fix Z, then for every $s \in [\delta_1]^{<\lambda}$ there is an appropriate $G : Z \times Z \to \theta$. We show that there is some $G : Z \times Z \to \theta$ that works for every s. Assume otherwise, that is, for every $G : Z \times Z \to \theta$ there is some $s_G \in [\delta_1]^{<\lambda}$ such that G is not appropriate for s_G. Notice that the number of these functions G is less than κ. Then no G could be right for $s = \cup\{s_G : G$ a function from $Z \times Z$ to $\theta\} \in [\delta_1]^{<\lambda}$, a contradiction. $\square_{1.7}$

Continuation of the proof of Theorem 0.1. We now apply Lemma 1.1 to the colouring $\bar{G}\{x, y\} = \bar{G}(x, y) = \langle F(x, y), G(x, y)\rangle$ for $x < y$ in T and 0 otherwise, and the filter D_0 and the set T and get the normal filter $D_1 \supseteq D_0$, the set $A_1 \subseteq T \subseteq A'_0$ such that $A_1 \in D_1$ and the colour set $C_1 \subseteq \theta \times \theta$. Notice that actually $C_1 \subseteq C_0 \times C_0$. We can also apply Lemmas 1.5 and 1.6 and get some set $A'_1 \subseteq A_1$.

1.8 Lemma. *There is a set $a \in [A'_1]^{<\kappa}$ such that for every decomposition $a = \cup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ there is some $\bar{\xi} \in C_1$ such that*

- (α) *for every $\bar{\varepsilon} \in C_1$ there is an $\bar{\varepsilon}$-homogeneous subset for the colouring $\bar{G}$ of order type ζ in $a_{\bar{\xi}}$,*
- (β) *similarly for every $\varepsilon \in C_0$ and F.*

Proof. This follows from the strong compactness of κ as A'_1 itself has this partition property (or more details in 2.8). $\square_{1.8}$

Continuation of the Proof of 0.1. Fix a set a as in 1.8.

We now describe the construction of the required homogeneous subset. Let $\delta_2 \in A'_1$ be some element with $\delta_2 > \sup(a)$. For $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ let $a_{\bar{\xi}}$ be the following set:

$$a_{\bar{\xi}} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$

By Lemma 1.8, there is some $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ for which the statement in 1.8 above is true and necessarily (as $a \cup \{\delta_2\} \subseteq A'_1 \subseteq A_0$ and $a_{\bar{\xi}} \neq \emptyset$) we have $\xi_1, \xi_2 \in C_0$. Select some $b \subseteq a_{\bar{\xi}}$, $\mathrm{otp}(b) = \zeta$ such that F is constantly ξ_2 on b; this is possible by clause (β) of 1.8. This set b will be the ζ part of our homogeneous set of ordinals of order type $\lambda + \zeta$, so we will have to construct a set of order type λ below b. By induction on α we will choose x_α such that the set $\{x_\alpha : \alpha < \lambda\}_< \subseteq \delta_1$ satisfies the following conditions:

$(*)_1$ $F(x_\beta, x_\alpha) = \xi_2$ (for $\beta < \alpha$),
$(*)_2$ $F(x_\alpha, b \cup \{\delta_2\}) = \xi_2$, i.e. $F(x_\alpha, y) = \xi_2$ when $y \in b \cup \{\delta_2\}$.

Assume that we have reached step α, that is, we are given the set of ordinals with $\{x_\beta : \beta < \alpha\}_<$ and call this set s. Applying Lemma 1.6 for A_1, A_1', δ_2 and $s \cup b$ and the colouring $\bar{G}$ here standing for A_0, A_0', δ, s and the colouring F there (that is the choice of A_1') we get that there exists some $\delta_3 > \delta_2$ (standing for δ_1 there) such that

(i) $\delta_3 \in A_1$
(ii) $\bar{G}(x, \delta_3) = \bar{G}(x, \delta_2)$ for $x \in s \cup b$
(iii) $\bar{G}(\delta_2, \delta_3) = (\xi_1, \xi_2)$,

hence:

$(*)_3$ $F(x_\beta, \delta_3) = \xi_2$ (for $\beta < \alpha$).
[Why? As $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$ by (ii) and the choice of $\bar{G}$ and $F(x_\beta, \delta_2) = \xi_2$ by $(*)_2$ from the induction hypothesis.]
$(*)_4$ $G(b \cup \{\delta_2\}, \delta_3) = \xi_2$, i.e. $G(y, \delta_3) = \xi_2$ when $y \in b \cup \{\delta_2\}$.
[Why? If $y \in b$ then by (ii) and the definition of $\bar{G}$ we have $G(y, \delta_3) = G(y, \delta_2)$, but $b \subseteq a_{\bar{\xi}}$ so by the choice of $a_{\bar{\xi}}$ we have $G(y, \delta_2) = \xi_2$. For $y = \delta_2$ use clause (iii) that is $(\xi_1, \xi_2) = \bar{G}(\delta_2, \delta_3) = (F(\delta_2, \delta_3), G(\delta_2, \delta_3))$.]

By the choice of G this implies that there is some x_α as required; that is by the choice of $\bar{G}$ (see Lemma 1.7), applied to $Z = \{z_i : i < \gamma\}$ enumerating the set $b \cup \{\delta_2, \delta_3\}$ and s as above, we get $\{y_i : i < \gamma\}$, now necessarily $\delta_3 = z_{\gamma-1}$, and we can choose $y_{\gamma-1}$ as x_α. $\square_{1.1}$

§2 The case of λ singular

We prove version 0.2 of the main theorem.

Proof of Theorem 0.2. Let $\sigma = \operatorname{cf}(\lambda)$. Let $\lambda = \sum\limits_{\varepsilon<\sigma} \lambda_\varepsilon$ with $\lambda_\varepsilon > \sigma \geq \kappa > \theta$ strictly increasing. Let $\mu_\varepsilon = 2^{\lambda_\varepsilon}$ and $\mu = \Sigma\{\mu_\varepsilon : \varepsilon < \sigma\} = 2^{<\lambda}$. We also fix $F : [\mu^+]^2 \to \theta$.

2.1 Claim. *For some $\bar{\mathscr{C}}$ we have:*

- (a) $\bar{\mathscr{C}} = \langle \mathscr{C}_\alpha : \alpha \in S\rangle$
- (b) $S \subseteq \mu^+, \mathscr{C}_\delta \subseteq \delta$
- (c) $\operatorname{otp}(\mathscr{C}_\delta) \leq \sigma$
- (d) $S^* = \{\delta < \lambda : \operatorname{otp}(\mathscr{C}_\delta) = \sigma\}$ *is stationary*
- (e) $\mathscr{C}_\delta$ *unbounded in* δ *if* $\operatorname{otp}(\mathscr{C}_\delta) = \sigma$
- (f) $\alpha \in \mathscr{C}_\delta \Rightarrow \alpha \in S$ & $\mathscr{C}_\alpha = \mathscr{C}_\delta \cap \alpha$.

$\square_{2.1}$

Proof. By [Sh 420, §1] as $\sigma^+ < \mu^+, \sigma = \operatorname{cf}(\sigma)$.

<u>Continuation of the proof of 0.2</u>: Let D_0, A_0, C_0 be as given by Lemma 1.1 with the club filter of μ^+, S^* (from clause (d) of 2.1 above) here standing for D, A^* there so $A_0 \subseteq S^*$.

<u>Notation</u>: $\varepsilon(\alpha) = \operatorname{otp}(C_\alpha)$.

2.2 Claim. *Let $\chi > 2^\mu, <^*_\chi$ a well ordering of $\mathscr{H}(\chi)$). For any $x \in \mathscr{H}(\chi)$ we can find $\bar{\mathfrak{B}} = \langle \mathfrak{B}_\alpha : \alpha < \lambda\rangle$ such that:*

- (a) $\mathfrak{B}_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$
- (b) $\bar{\lambda}, \mu, F, \langle \lambda_\varepsilon : \varepsilon < \sigma\rangle, \bar{\mathscr{C}}, A_0, C_0, D_0$ *belong to* $\mathfrak{B}_\alpha$
- (c) $\langle \mathfrak{B}_\beta : \beta < \alpha\rangle \in \mathfrak{B}_\alpha$ *if* $\alpha \notin S^*$
- (d) $\|\mathfrak{B}_\beta\| = \mu_{\varepsilon(\beta)}$ *and* $[\mathfrak{B}_\beta]^{\leq \lambda_{\varepsilon(\beta)}} \subseteq \mathfrak{B}_\beta$ *and* $\mu_{\varepsilon(\beta)} + 1 \subseteq \mathfrak{B}_\beta$ *(actually follows)*
- (e) $\mathfrak{B}_\alpha = \cup\{\mathfrak{B}_\beta : \beta \in \mathscr{C}_\alpha\}$ *if* $\alpha \in S^*$.

Proof. Straightforward.

2.3 Observation. 1) We have $\varepsilon(\alpha) < \varepsilon(\beta)$ and $\mathfrak{B}_\alpha \in \mathfrak{B}_\beta$ and $\mathfrak{B}_\alpha \prec \mathfrak{B}_\beta$ if $\alpha \in \mathscr{C}_\beta$.

2.4 Claim. *There is a set $A'_0 \subseteq A_0$ such that*

(α) $A'_0 \in D_0$ *and* $\alpha < \delta \in A'_0 \Rightarrow \sup(\mathfrak{B}_\alpha \cap \mu^+) < \delta$

(β) *if* $\xi \in C_0$ *and* $\delta \in A'_0$ *and* $s \in \cup\{[\delta \cap \mathfrak{B}_\alpha]^{\le \lambda_{\varepsilon(\alpha)}} : \alpha \in \mathscr{C}_\delta\}$, *then there is* $\delta_1 \in A_0$ *such that* $\delta < \delta_1$ *and*

  (a) $F(x, \delta) = F(x, \delta_1)$ *for* $x \in s$

  (b) $F(\delta, \delta_1) = \xi$.

Proof. Requirement (α) holds for all but a non stationary set of $\delta \in A_0$. Requirement (β) is proved as in 1.6. $\square_{2.4}$

Now fix $A'_0 \subseteq A_0$ as in 2.4, and fix $\delta_1 \in A'_1$ and let $T = A'_0 \backslash (\delta_1 + 1)$. Recall $\delta_1 \in A'_0 \subseteq S^* = \{\delta : \mathrm{otp}(\mathscr{C}_\delta) = \sigma, \delta = \sup(\mathscr{C}_\delta)\} \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \sigma\}$.

2.5 Claim. *There is a function $G_\varepsilon : T \times T \to C_0$ such that:*

$\boxdot$ *if* $s \in [\delta \cap \mathfrak{B}_\alpha]^{\le \lambda_\varepsilon}$ *and* $\varepsilon = \varepsilon(\alpha)$ *and* $\alpha \in \mathscr{C}_{\delta_1}$ *and* $\gamma < \kappa$ *and* $Z = \{z_\beta : \beta < \gamma\}_< \subseteq T$, *then there is* $\{y_\beta : \beta < \gamma\}_< \subseteq \delta \cap \mathfrak{B}_\alpha = \mu^+ \cap \mathfrak{B}_\alpha$ *such that:*

  (a) $F(x, y_\beta) = F(x, z_\beta)$ *for* $x \in s, \beta < \delta$

  (b) $F(\delta, \delta_1) = \xi$
  $F(z_{\beta_1}, y_{\beta_2}) = G(y_{\beta_1}, y_{\beta_2})$ *for* $\beta_1, \beta_2 < \delta$

  (c) $F(z_{\beta_1}, z_{\beta_2}) = F(y_{\beta_1}, y_{\beta_2})$ *for* $\beta_1, \beta_2 < \gamma$

  (d) $y_0 > \sup(s)$.

Proof. Like 1.7.

2.6 Claim. *There exists a function $G : T \times T \to C_0$ such that if $s \in [T]^{<\kappa}$, then for arbitrarily large $\varepsilon < \sigma$ we have $G \restriction (s \times s) = G_\varepsilon \restriction (s \times s)$.*

Proof. Let D^* be a uniform κ-complete ultrafilter on σ and define G by $G(\alpha, \beta)$ is the unique $\xi \in C_0$ such that $\{\varepsilon < \sigma : G_\varepsilon(\alpha, \beta) = \xi\} \in D^*$. $\square_{2.6}$

Continuation of the Proof of 0.2. Now we apply Lemma 1.1 to the colouring $\bar{G}$ where $\bar{G}\{x, y\} = \bar{G}(x, y) = (F(x, y), G(x, y))$ for $x < y$ in T and zero otherwise and the filter D_0 and the set T. We get a normal filter D_1 and a set $A_1 \subseteq T \subseteq A'_0$ and a set of colours C_1. As $A_1 \subseteq A_0$ necessarily $C_1 \subseteq C_0 \times C_0$.

2.7 Claim. *There is $A'_1 \subseteq A_1$ such that:*

(α) $A_1 \backslash A'_1 = \emptyset$ *mod* D_1

(β) *if* $\delta \in A'_1, \alpha \in \mathscr{C}_\delta$ *and* $s \in [\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda_{\varepsilon(\alpha)}}$ *and* $\bar{\xi} \in C_1$, *then for some* δ_* *we have* $\delta < \delta_* \in A_1$ *and*

(a) $\bar{G}(x, \delta) = \bar{G}(x, \delta_1)$ *for every* $x \in s$

(b) $\bar{G}(\delta, \delta_*) = \bar{\xi}$.

Proof. Like the proof of 1.6 $\square_{2.7}$

2.8 Claim. *There is a set* $a \in [A'_1]^{<\kappa}$ *such that:*

$\boxdot$ *for every decomposition of* a *as* $\cup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ *there is* $\bar{\xi} \in C_1$ *such that*

(α) *for every* $\bar{\varepsilon} \in C_1$ *there is* $b \subseteq a_{\bar{\xi}}$ *of order type* ζ *such that* $\bar{G} \restriction [b]^2$ *is constantly* $\bar{\varepsilon}$

(β) *for every* $\varepsilon \in C_0$ *there is* $b \subseteq a_{\bar{\xi}}$ *of order type* ζ *such that* $F \restriction [b]^2$ *is constantly* ε.

Proof. The claim holds since A'_1 has this property and κ is strongly compact. If $A'_1 = \cup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ for some $\bar{\xi}, a_{\bar{\xi}} \in D^+_1$ hence clause (α) holds by the choice of D_1, C_1; and clause (β) holds as $D^+_1 \subseteq D^+_0$ (as $D_0 \subseteq D_1$) and the choice of D_0, C_0. $\square_{2.8}$

Continuation of the proof of 0.2. Now choose $\delta_2 \in A'_1$ such that $\delta_2 > \sup(a)$ and for $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ define $a_{\bar{\xi}}$ as

$$\bar{a}_{\bar{\xi}} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$

Clearly $\langle a_{\bar{\xi}} : \bar{\xi} \in C_1\rangle$ is a decomposition of a and so there is $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ as guaranteed by $\boxdot$ of 2.8. In particular, there is $b \subseteq a_{\bar{\xi}}$ of order type ζ such that $F \restriction [b]^2$ is constantly ξ_2 (note that $(\xi_1, \xi_2) \in C_1 \subseteq C_0 \times C_0$ so $\xi_2 \in C_0$). Now let $E = \{\varepsilon < \sigma : G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2)$ for every $\alpha \in b\}$. By the definition of G this is an unbounded subset of σ and clearly

(∗) if $\varepsilon \in E$ and $\alpha \in b$ then $G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2) = (\xi_1, \xi_2)$.

For $\alpha < \lambda$ let $\Upsilon(\alpha) = \text{ Min}\{\varepsilon \in E : \alpha < \lambda_\varepsilon\}$ and let $C_{\delta_1} = \{\gamma(\Upsilon) : \Upsilon < \sigma\}_<$.
Now we try to choose by induction on $\alpha < \lambda$ a element x_α satisfying

$(*)_0$ $x_\alpha < \delta_1$ and moreover $x_\alpha \in \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$, and $\beta < \alpha \Rightarrow x_\beta < x_\alpha$

$(*)_1$ $F(x_\beta, x_\alpha) = \xi_2$ for $\beta < \alpha$

$(*)_2$ $F(x_\alpha, \beta) = \xi_2$ for $\beta \in b \cup \{\delta_2\}$.

At step α, by 2.7, that is by the choice of A'_1 applying clause (β) there with $\{x_\beta : \beta < \alpha\} \cup b, \delta_2, \bar\xi$ here standing for $s, \delta, \bar\xi$ there, we can find δ_3 satisfying the requirement there on δ_1, so

(i) $\delta_2 < \delta_3 \in A_1$

(ii) $\bar G(x, \delta_3) = \bar G(x, \delta_2)$ for $x \in s \cup b$

(iii) $\bar G(\delta_2, \delta_3) = (\xi_1, \xi_2)$.

Now

$(*)_3$ $F(x_\beta, \delta_3) = \xi_2$ for $\beta < \alpha$.
[Why? By (ii) we have $\bar G(x_\beta, \delta_3) = \bar G(x_\beta, \delta_2)$ hence $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$ but the latter by $(*)_2$ is equal to ξ_2.]

$(*)_4$ $G(\beta, \delta_3) = \xi_2$ for $\beta \in b$
[Why? By (ii) and as $\beta \in b \Rightarrow \bar G(\beta, \delta_2) = (\xi_1, \xi_2) \Rightarrow G(\beta, \delta_2) = \xi_2$).]

$(*)_5$ $G(\delta_2, \delta_3) = \xi_2$
[Why? By clause (iii).]

$(*)_6$ $\{x_\beta : \beta < \alpha\}$ is a subset of $\delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$.

Let $\langle y_i : i < \zeta + 2\rangle$ list $b \cup \{\delta_2, \delta_3\}$ increasing order.

Now we use the choice of $G_{\Upsilon(\alpha)}$ to choose an increasing sequence $\langle z_i : i < \zeta + 2\rangle$ in $\delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}, z_0 > x_\beta$ for $\beta < \alpha$ such that $F(z_i, y_j) = G(y_i, y_j)$ for $i, j < \zeta + 2$ and $F(x_\beta, z_i) = F(x_\beta, y_i)$ for $i < \zeta + 2$. Let $x_\alpha = z_{\zeta+1}$ so $x_\alpha = \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$ is $> x_\beta$ for $\beta < \alpha$.
Also x_α satisfies $(*)_0$ of the recursive definition. Now $\beta < \alpha \Rightarrow F(x_\beta, x_\alpha) = F(x_\beta, z_{\zeta+1}) = F(x_\beta, y_{\zeta+1}) = F(x_\beta, \delta_3)$ which is ξ_2 by $(*)_3$ above, so for our choice of x_α, $(*)_1$ holds. Next if $\beta \in b \cup \{\delta_2\}$ then $F(x_\alpha, x_\beta) = F(x_\beta, z_{\zeta+1}) = G(x_\beta, \delta_3)$ which is ξ_2 by $(*)_4$ or $(*)_5$. So x_α is as required. $\square_{0.2}$

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