# FORCING AXIOM FAILURE FOR ANY $\lambda > \aleph_1$ SH784

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Annotated Content

§1 A forcing axiom for $\lambda > \aleph_1$ fails

> [The forcing axiom is: if $\mathbb{P}$ is a forcing notion preserving stationary subsets of any regular uncountable $\mu \le \lambda$ and $\mathscr{I}_i$ is dense open subset of $\mathbb{P}$ for $i < \lambda$ then some directed $G \subseteq \mathbb{P}$ meets every $\mathscr{I}_i$.
> We prove (in ZFC) that it fails for every regular $\lambda > \aleph_1$. In our counterexample the forcing notion $\mathbb{P}$ adds no new sequence of ordinals of length $< \lambda$).

§2 There are $\{\aleph_1\}$-semi-proper forcing notions

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## §1 A forcing axiom for $\lambda > \aleph_0$ fail

David Aspero asks on the possibility of, see Definition below, the forcing axiom $\mathrm{FA}(\mathfrak{K}, \aleph_2)$ for the case $\mathfrak{K}$ = the class of forcing notions preserving stationarily of subsets of $\aleph_1$ and of $\aleph_2$. We answer negatively for any regular $\lambda > \aleph_1$ (even demanding adding no new sequence of ordinals of length $< \lambda$), see 1.16 below.

**1.1 Definition.** 1) Let $\mathrm{FA}(\mathfrak{K}, \lambda)$, the $\lambda$-forcing axiom for $\mathfrak{K}$ mean that $\mathfrak{K}$ is a family of forcing notions and for any $\mathbb{P} \in \mathfrak{K}$ and dense open sets $\mathscr{I}_i \subseteq \mathbb{P}$ for $i < \lambda$ there is a directed $G \subseteq \mathbb{P}$ meeting every $\mathscr{I}_i$.
2) If $\mathfrak{K} = \{\mathbb{P}\}$ we may write $\mathbb{P}$ instead of $\mathfrak{K}$.

**1.2 Definition.** Let $\lambda$ be regular uncountable. We define a forcing notion $\mathbb{P} = \mathbb{P}^2_\lambda$ as follows:

$(A)$ if $p \in \mathbb{P}$ iff $p = (\alpha, \bar{S}, \bar{W}) = (\alpha^p, \bar{S}^p, \bar{C}^p)$ satisfying

- $(a)$ $\alpha < \lambda$
- $(b)$ $\bar{S}^p = \langle S_\beta : \beta \leq \alpha \rangle = \langle S^p_\beta : \beta \leq \alpha \rangle$
- $(c)$ $\bar{C}^p = \langle C_\beta : \beta \leq \alpha \rangle = \langle C^p_\beta : \beta \leq \alpha \rangle$

such that

- $(d)$ $S_\beta$ is a stationary subset of $\lambda$ consisting of limit ordinals
- $(e)$ $C_\beta$ is a closed subset of $\beta$
- $(f)$ if $\beta \leq \alpha$ is a limit ordinal then $C_\beta$ is a closed unbounded subset of $\beta$
- $(g)$ if $\gamma \in C_\beta$ then $C_\gamma = \gamma \cap C_\beta$
- $(h)$ $C_\beta \cap S_\beta = \emptyset$
- $(i)$ for every $\beta \leq \alpha$ and $\gamma \in C_\beta$ we have $S_\gamma = S_\beta$

$(B)$ order: natural
$p \leq q$ iff $\alpha^p \leq \alpha^q, \bar{S}^p = \bar{S}^q \restriction (\alpha^p + 1)$ and $\bar{C}^p = \bar{C}^q \restriction (\alpha^p + 1)$.

1.3 Observation: 1) $\mathbb{P}^2_\lambda$ is a (non empty) forcing notion of cardinality $2^\lambda$.
2) $\mathscr{I}_i = \{p \in \mathbb{P}^2_\lambda : \alpha^p \geq i\}$ is dense open for any $i < \lambda$.

*Proof.* 1) Obvious.
2) Given $p \in \mathbb{P}^2_\lambda$ if $\alpha^p \geq i$ we are done. So assume $\alpha^p < i$ and for $\gamma \in (\alpha^p, i]$ let $S^q_\gamma$ be $S^*$ for any stationary subset $S^*$ of $\{\delta < \lambda : \delta > i$ a limit ordinal$\}$ which does not belong to $\{S^p_\beta : \beta \leq \alpha^p\}$ and let $C^q_\gamma = \{j : \alpha^p < j < \gamma\}$ and

$q = (i, \bar S^p {}^\frown \langle S^q_\gamma : \gamma \in (\alpha^p, i]\rangle, \bar C^p {}^\frown \langle C^q_\gamma : \gamma \in (\alpha^p, i]\rangle)$.
It is easy to check that $p \le q \in \mathbb{P}^2_\lambda$ and $q \in \mathscr{I}_i$. $\square_{1.3}$

**1.4 Claim.** *Let $\lambda = \mathrm{cf}(\lambda)$ be regular uncountable and $\mathbb{P} = \mathbb{P}^2_\lambda$. For any stationary $S \subseteq \lambda$ and $\mathbb{P}^2_\lambda$-name $\underset{\sim}{f}$ of a function from $\gamma^* \le \lambda$ to the ordinals or just to $\mathbf{V}$ and $p \in \mathbb{P}$ there are $q, \delta$ such that:*

$\boxtimes(i)$ $p \le q \in \mathbb{P}$

$(ii)$ $\alpha^q = \delta + 1$

$(iii)$ $\delta \in S$ *if* $\gamma^* = \lambda$

$(iv)$ $q$ *forces a value to* $\underset{\sim}{f} \restriction (\delta \cap \gamma^*)$

$(v)$ *if* $\beta < \delta \cap \gamma^*$ *and* $\Vdash_{\mathbb{P}}$ "$\mathrm{Rang}(\underset{\sim}{f}) \subseteq \lambda$" *then* $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\beta) < \delta$".

*Proof.* Without loss of generality $S$ is a set of limit ordinals. We prove this by induction on $\gamma^*$, so without loss of generality $\gamma^* = |\gamma^*|$ and without loss of generality $\gamma^* < \lambda \Rightarrow \gamma^* = \mathrm{cf}(\gamma^*)$, but if $\gamma^* < \lambda$ the set $S$ is immaterial so without loss of generality

$$\circledast \ \gamma^* < \lambda \ \& \ \delta \in S \Rightarrow \mathrm{cf}(\delta) \ge \gamma^*.$$

Let $\chi$ be large enough (e.g. $\chi = (\beth_3(\lambda))^+$), $<^*_\chi$ is a well ordering of $\mathscr{H}(\chi)$ and choose $\bar N = \langle N_i : i < \lambda\rangle$ such that

$\circledcirc(a)$ $N_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is increasing continuous

$(b)$ $\lambda, p, \underset{\sim}{f}, S$ belongs to $N_i$ hence $\mathbb{P} \in N_i$

$(c)$ $\|N_i\| < \lambda$

$(d)$ $N_i \cap \lambda \in \lambda$

$(e)$ $\langle N_j : j \le i\rangle$ belong to $N_{i+1}$; hence $i \subseteq N_i$ so $\lambda \subseteq \cup\{N_i : i < \lambda\}$.

Let $\delta_i = N_i \cap \lambda$, and let $i(*) = \mathrm{Min}\{i : i < \lambda$ is a limit ordinal and $\delta_i \in S\}$, it is well defined as $\langle \delta_i : i < \lambda\rangle$ is strictly increasing continuous hence $\{\delta_i : i < \lambda\}$ is a club of $\lambda$; so by $\circledast$ we know that $\gamma^* < \lambda \Rightarrow \mathrm{cf}(i(*)) = \mathrm{cf}(\delta_{i(*)}) \ge \gamma^*$. Let $\alpha^*_i$ be $\delta_i$ for $i \le i(*)$ a limit ordinal and be $\delta_i + 1$ for $i < i(*)$ a non limit ordinal. Now by induction on $i \le i(*)$ choose $p^-_i$ and if $i < i(*)$ also $p_i$ and prove on them the following:

$(*)(i)$ $p_i, p^-_i \in \mathbb{P} \cap N_{i+1}$

$(ii)$ $p_i$ is increasing

(*iii*) $\alpha^{p_i} > \alpha^*_i$ (and $\delta_{i+1} > \alpha^{p_i}$ follows from $p \in \mathbb{P} \cap N_{i+1}$)

(*iv*) $S^{p_i}_{\alpha^*_i} = S$ and $C^{p_i}_{\alpha^*_i} = \{\alpha^*_j : j < i\}$

(*v*) $p^-_i$ is the $<^*_\chi$-first $q$ satisfying:
$q \in \mathbb{P}$
$j < i \Rightarrow p_j \le q$
$\alpha^q > \delta_i$
$S^q_{\alpha^*_i} = S$ and
$C^q_{\alpha^*_i} = \{\alpha^*_j : j < i\}$

(*vii*) $p_i$ is the $<^*_\chi$-first $q$ such that:
$q \in \mathbb{P}$
$p^-_i \le q$
$q$ forces a value to $\underset{\sim}{f}(i)$ if $\gamma^* < \lambda$
$q$ forces a value to $\underset{\sim}{f} \restriction \delta_i$ if $\gamma^* = \lambda$.

There is no problem to carry the definition, recalling the inductive hypothesis on $\gamma^*$ and noting that $\langle (p^-_j, p_j) : j < i\rangle \in N_{i+1}$ by the " $<^*_\chi$-first" being used to make our choices as $\langle N_j : j \le i\rangle \in N_{i+1}$ hence $\langle \delta_j : j \le i\rangle \in N_{i+1}$ and also $\langle \alpha^*_j : j \le i\rangle \in N_{i+1}$ (and $p, \underset{\sim}{f} \in N_0 \prec N_{i+1}$).

Now $p^-_{i(*)}$ is as required. $\square_{1.4}$

<u>1.5 Conclusion</u>: Let $\lambda =$ cf$(\lambda) > \aleph_0$. Forcing with $\mathbb{P}^2_\lambda$ add no bounded subset of $\lambda$ and preserve stationarity of subsets of $\lambda$ (and add no new sequences of ordinals of length $< \lambda$).

*Proof.* Obvious from 1.4.

**1.6 Claim.** *Let* $\lambda =$ cf$(\lambda) > \aleph_0$. *If* FA$(\mathbb{P}^2_\lambda)$, *(the forcing axiom for the forcing notion* $\mathbb{P}^2_\lambda, \lambda$ *dense sets) holds, <u>then</u> there is a witness* $(\bar{S}, \bar{C})$ *to* $\lambda$ *where*

**1.7 Definition.** 1) For $\lambda$ regular uncountable, we say that $(\bar{S}, \bar{C})$ is a witness to $\lambda$ or $(\bar{S}, \bar{C})$ is a $\lambda$-witness if:

(*a*) $\bar{S} = \langle S_\beta : \beta < \lambda\rangle$
(*b*) $\bar{C} = \langle C_\beta : \beta < \lambda\rangle$
(*c*) for every $\alpha < \lambda, (\alpha, \bar{S} \restriction (\alpha+1), \bar{C} \restriction (\alpha+1)) \in \mathbb{P}^2_\lambda$.

2) For $(\bar{S}, \bar{C})$ a witness for $\lambda$, let $F = F_{(\bar{S},\bar{C})}$ be the function $F : \lambda \to \lambda$ defined by

$$F(\alpha) = \text{ Min}\{\beta : S_\alpha = S_\beta\}.$$

3) For $\beta < \lambda$ let $W^\beta_{(\bar{S},\bar{C})} = \{\alpha < \lambda : F_{(\bar{S},\bar{C})}(\alpha) = \beta\}$.

*Proof of 1.6.* Let $\mathscr{I}_i = \{p \in \mathbb{P}^2_\lambda : \alpha^p \geq i\}$, by 1.3(2) this is a dense open subset of $\mathbb{P}^2_\lambda$, hence by the assumption there is a directed $G \subseteq \mathbb{P}^2_\lambda$ such that $i < \lambda \Rightarrow \mathscr{I}_i \cap G \neq \emptyset$. Define
$S_\alpha = S^p_\alpha, C_\alpha = C^p_\alpha$ for every $p \in G$ such that $\alpha^p \geq \alpha$.
Now check. $\square_{1.6}$

1.8 Observation: Let $(\bar{S}, \bar{C})$ be a witness for $\lambda$ and $F = F_{(\bar{S},\bar{C})}$.
1) If $\alpha < \lambda$ then $F(\alpha) \leq \alpha$.
2) If $\alpha < \lambda$ is limit then $F(\alpha) < \alpha$.
3) If $\alpha < \lambda$ then $\alpha \in W^{F(\alpha)}_{(\bar{S},\bar{C})}$.
4) If $\alpha < \lambda$ and $i = F(\alpha)$ and $\beta \in C_\alpha$ then $\beta \notin S_\alpha = S_i$.

*Proof.* Easy (for part (4) remember that each $S_\alpha$ is a set of limit ordinals $< \lambda$ and that for limit $\alpha \leq \alpha^p, p \in \mathbb{P}^2_\lambda$ we have $\alpha = \sup(C_\alpha)$ and $\alpha \in S_\beta \Rightarrow C_\alpha \cap S_\beta = \emptyset$).

**1.9 Claim.** *Assume $(\bar{S}, \bar{C})$ is a $\lambda$-witness and $S^* \subseteq \lambda$ satisfies $\delta \in S^* \Rightarrow \text{cf}(\delta) \geq \theta > \aleph_0$ and $F_{(\bar{S},\bar{C})} \restriction S^*$ is constant and $S^*$ is stationary. Then there is a club $E^*$ of $\lambda$ such that: $(\bar{S}, \bar{C}, S^*, E^*)$ is a strong $(\lambda, \theta)$-witness, where*

**1.10 Definition.** 1) We say that $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*)$ is a strong $\lambda$-witness if

- $(a)$ $(\bar{S}, \bar{C})$ is a $\lambda$-witness
- $(b)$ $S^* \subseteq \lambda$ is a set of limit ordinals and is a stationary subset of $\lambda$
- $(c)$ $E^*$ is a club of $\lambda$
- $(d)$ for every club $E$ of $\lambda$, for stationarily many $\delta \in S^*$ we have

$$\delta = \sup\{\alpha \in C_\delta : \alpha < \text{ Suc}^1_{C_\delta}(\alpha, E^*) \in E\}$$

where

$(*)(i)$ $\text{Suc}^0_{C_\delta}(\alpha) = \text{ Min}(C_\delta \backslash (\alpha + 1))$,

$(ii)$ $\text{Suc}^1_{C_\delta}(\alpha, E^*) = \sup(E^* \cap \text{ Suc}^0_{C_\delta}(\alpha))$.

2) We say $(\bar S, \bar C, S^*, E^*)$ is a strong $(\lambda, \theta)$-witness if in addition

$(e)$ $\delta \in S^* \Rightarrow \text{cf}(\delta) \geq \theta$.

3) For $(\bar S, \bar C, S^*, E^*)$ a strong $\lambda$-witness we let $\bar C' = \langle C'_\delta : \delta \in S^* \cap \text{acc}(E^*)\rangle, C'_\delta = C_\delta \cup \{\text{Suc}^1_{C_\delta}(\alpha, E^*) : \alpha \in C_\delta\}$; if $\mathbf{p} = (\bar S, \bar C, S^*, E^*)$ we write $\bar C' = \bar C'_{\mathbf{p}}$ and $\bar S_{\mathbf{p}} = \bar S, \bar C_{\mathbf{p}} = \bar C, S^*_{\mathbf{p}} = S^*, E^*_{\mathbf{p}} = E^*$. We call $(\bar S, \bar C, S^*, E^*, \bar C')$ an expanded strong $\lambda$-witness (or $(\lambda, \theta)$-witness).

*1.11 Observation.* In Definition 1.10(3) for $\delta \in S^* \cap \text{acc}(E^*)$ we have:

⊛ $C'_\delta$ is a club of $\delta$, $\text{Min}(C'_\delta) \geq \sup(E^* \cap \text{Min}(C_\delta))$ and if $\gamma_1 < \gamma_2$ are successive members of $C_\delta$ then $C'_\delta \cap (\gamma_1, \gamma_2)$ has at most one member (which necessarily is $\sup(E^* \cap \gamma_2)$) hence $\text{acc}(C'_\delta) = \text{acc}(C_\delta)$ and $\alpha \in C_\delta \wedge \alpha < \text{Suc}^1_{C_\delta}(\alpha) \Rightarrow \alpha \notin C'_\delta \backslash C_\delta$ and $\text{acc}(C_\delta) = \text{acc}(C'_\delta)$.

*Proof of 1.9.* As in [Sh:g, III], but let us elaborate, so assume toward contradiction that for no club $E^*$ of $\lambda$ is $(\bar S, \bar C, S^*, E^*)$ a strong $(\lambda, \theta)$-witness. We choose by induction on $n$ sets $E^*_n, E_n, A_n$ such that:

$(a)$ $E^*_n, E_n$ are clubs of $\lambda$

$(b)$ $E^*_0 = \lambda$

$(c)$ $E_n$ is a club of $\lambda$ such that the following set is not stationary (in $\lambda$)

$$A_n = \{\delta \in S^* : \delta \in \text{acc}(E^*_n) \text{ and } \delta = \sup\{\alpha \in C_\delta : \alpha < \text{Suc}^1_{C_\delta}(\alpha, E^*_n) \in E_n\}\}$$

$(d)$ $E^*_{n+1}$ is a club of $\lambda$ included in $\text{acc}(E^*_n \cap E_n)$ and disjoint to $A_n$.

For $n = 0$, $E^*_n$ is defined by clause (b).

If $E^*_n$ is defined, choose $E_n$ as in clause (c), possible by our assumption toward contradiction, also $A_n \subseteq S^*$ is defined and not stationary. So obviously $E^*_{n+1}$ as required in clause (d) exists.

So $E^* =: \cap\{E^*_n : n < \omega\}$ is a club of $\lambda$ and let $\alpha(*)$ be the constant value of $F_{(\bar S, \bar C)} \restriction S^*$, exists by an assumption of the claim. Recall that $S_{\alpha(*)}$ is a stationary subset of $\lambda$, so clearly $E^{**} =: \{\delta \in E^* : \delta = \sup(\delta \cap E^* \cap S_{\alpha(*)})\}$ is a club of $\lambda$. As $S^*$ is a stationary subset of $\lambda$, we can choose $\delta^* \in S^* \cap E^{**}$. For each $n < \omega$ we have $\delta^* \in S^* \cap E^{**} \subseteq E^{**} \subseteq E^* \subseteq E^*_{n+1}$ hence $\delta^* \notin A_n$ hence $\beta^*_n = \sup\{\beta \in C_{\delta^*} : \beta < \text{Suc}^1_{C_{\delta^*}}(\beta, E^*_n) \in E_n\}$ is $< \delta^*$ but $\in C_{\delta^*}$. But $\delta^* \in S^*$ so $\text{cf}(\delta^*) \geq \theta > \aleph_0$, hence $\beta^* = \sup\{\beta^*_n, \text{Min}(C_{\delta^*}) : n < \omega\}$ is $< \delta^*$ but $\geq \text{Min}(C_{\delta^*})$ and it belongs to $C_{\delta^*}$. As $\delta^* \in E^{**}$, we know that $\delta^* = \sup(\delta^* \cap E^* \cap S_{\alpha(*)})$ hence there is $\gamma^* \in E^* \cap S_{\alpha(*)} \cap (\text{Suc}^0_{C_{\delta^*}}(\beta^*), \delta^*)$. But $\delta^* \in S^* \subseteq S_{\alpha(*)}$ recalling by the choice of $\alpha(*)$ above $F_{(\bar{S},\bar{C})}(\delta^*) = \alpha(*)$ hence by Claim 1.8(4), i.e., Definition 1.2(1), clause (A)(h) and Definition 1.7(1) we have $C_{\delta^*} \cap S_{\alpha(*)} = \emptyset$ hence $\gamma^* \notin C_{\delta^*}$. But $\delta^* > \gamma^* > \beta^* \geq \text{Min}(C_{\delta^*})$ and $C_{\delta^*}$ is a closed subset of $\delta^*$ hence $\zeta^* = \max(C_{\delta^*} \cap \gamma^*)$ is well defined and so, recalling $\beta^* \in C_{\delta^*}$ we have

$$(\forall n < \omega)(\beta^*_n \leq \beta^* < \text{Suc}^0_{C_{\delta^*}}(\beta^*) \leq \zeta^* \in C_{\delta^*}).$$

Let $\xi^* = \text{Suc}^0_{C_{\delta^*}}(\zeta^*)$ so clearly $\gamma^* \in (\zeta^*, \xi^*)$. Now for every $n$ we have $\sup(\xi^* \cap E^*_n) \in [\gamma^*, \xi^*]$ as $\gamma^* \in E^* \cap S_{\alpha(*)} \subseteq E^* \subseteq E^*_n$.

So recalling $\zeta^* < \gamma^*$ clearly $\zeta^* < \sup(\xi^* \cap E^*_n)$; if also $\sup(\xi^* \cap E^*_n) \in E_n$ then recalling $\xi^* = \text{Suc}^0_{C_{\delta^*}}(\zeta^*)$, $\text{Suc}^1_{C_{\delta^*}}(\zeta^*, E^*_n) \equiv \sup(\xi^* \cap E^*_n)$ we have $\zeta^* \leq \beta^*_n$ (see its choice and see the choice of $\beta^*_n$ above), but this contradicts $\zeta^* \geq \text{Suc}^0_{C_{\delta^*}}(\beta^*) > \beta^* \geq \beta^*_n$ and the definition of $A_n$ (see clause (c) of $(*)$), contradiction. So necessarily $\sup(\xi^* \cap E^*_n)$ does not belong to $E_n$ hence does not belong to $E^*_{n+1}$, hence $\sup(\xi^* \cap E^*_n) > \sup(\xi^* \cap E^*_{n+1})$.

So $\langle \sup(\xi^* \cap E^*_n) : n < \omega \rangle$ is a strictly decreasing sequence of ordinals, contradiction. $\square_{1.9}$

**1.12 Definition.** Assume

- $(*)_1$ $(\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ is an expanded strong $\lambda$-witness so $\bar{C}' = \langle C'_\delta : \delta \in S^* \rangle, C'_\delta = C_\delta \cup \{\text{Suc}^1_{C_\delta}(\alpha, E^*) : \alpha \in C_\delta\}$ or just
- $(*)_2$ $S^* \subseteq \lambda$ is a stationary set of limit ordinals, $\bar{C}' = \langle C'_\delta : \delta \in S^* \rangle, C'_\delta$ is a club of $\delta$ and $E^*$ is a club of $\lambda$.

We define a forcing notion $\mathbb{P} = \mathbb{P}_{\bar{C}'}$

- $(A)$ $c \in \mathbb{P}$ iff
  - $(a)$ $c$ is a closed bounded subset of $\lambda$
  - $(b)$ if $\delta \in S^* \cap c$ then $\{\alpha \in C'_\delta : \text{Suc}^0_{C'_\delta}(\alpha) \in c\}$ is bounded in $\delta$

Let $\alpha^c = \sup(c)$.

- $(B)$ order: $c_1 \leq c_2$ iff $c_1$ is an initial segment of $c_2$.

**1.13 Claim.** *Let $\mathbb{P} = \mathbb{P}_{\bar{C}'}$ be as in Definition 1.12.*
*1) $\mathbb{P}$ is a (non empty) forcing notion.*
*2) For $i < \lambda$ the set $\mathscr{J}_i = \{c \in \mathbb{P} : i < \sup(c)\}$ is dense open.*

*Proof.* 1) Trivial.
2) If $c \in \mathbb{P}, i < \lambda$ and $c \notin \mathscr{J}_i$ then let $c_2 = c \cup \{i+1\}$, clearly $(c_2 \backslash c) \cap S^* = \emptyset$ as $S^*$ is a set of limit ordinals hence $c_2 \in \mathbb{P}$ and obviously $c \le c_2 \in \mathscr{J}_i$. $\square_{1.13}$

**1.14 Claim.** *Assume $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ is an expanded strong $\lambda$-witness.*
*Forcing with $\mathbb{P} = \mathbb{P}_{\bar{C}'}$ add no new bounded subsets of $\lambda$, no new sequence of ordinals of length $< \lambda$ and preserve stationarity of subsets of $\lambda$.*

*Proof.* Assume $p \in \mathbb{P}, \gamma^* \le \lambda$ and $\underset{\sim}{f}$ is a $\mathbb{P}$-name of a function from $\gamma^*$ to the ordinals or just to $\mathbf{V}$ and $S \subseteq \lambda$ is stationary and we shall prove that there are $q, \delta$ satisfying (the parallel of) $\boxtimes$ of 1.4, i.e.,

$\boxtimes(i)$ $p \le q \in \mathbb{P}$
$(ii)$ $\alpha^q = \delta + 1$
$(iii)$ $\delta \in S$ if $\gamma^* = \lambda$
$(iv)$ $q$ forces a value to $\underset{\sim}{f} \restriction (\delta \cap \gamma^*)$
$(v)$ if $\beta < \delta \cap \gamma^*$ and $\Vdash$ "$\underset{\sim}{p} : \gamma^* \to \lambda$" then $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\beta) < \delta$".

This is clearly enough for all the desired consequences. We prove this by induction on $\gamma^*$, so without loss of generality $\gamma^* = |\gamma^*|$ and without loss of generality $\gamma^* < \lambda \Rightarrow \gamma^* = \text{cf}(\gamma^*)$, but if $\gamma^* < \lambda$ then $S$ is immaterial so without loss of generality $\gamma^* < \lambda$ & $\delta \in S \Rightarrow \text{cf}(\delta) \ge \gamma^*$. Also we can shrink $S$ as long as it is a stationary subset of $\lambda$ and recall that $F_{(\bar{S},\bar{C})}$ is regressive on limit ordinals (see Observation 1.8(2)) so without loss of generality $F_{(\bar{S},\bar{C})} \restriction S$ is constantly say $\alpha(*)$.

Let $\chi$ be large enough and choose $\bar{N} = \langle N_i : i < \lambda \rangle$ such that

$\circledcirc(a)$ $N_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is increasing continuous
$(b)$ $\lambda, p, \underset{\sim}{f}, S$ belongs to $N_i$ hence $\mathbb{P} \in N_i$
$(c)$ $\|N_i\| < \lambda$
$(d)$ $N_i \cap \lambda \in \lambda$
$(e)$ $\langle N_j : j \le i \rangle$ belong to $N_{i+1}$ (hence $i \subseteq N_i$, so $\lambda \subseteq \cup\{N_i : i < \lambda\}$)
$(f)$ $N_{i+1} \cap \lambda \in S_{\alpha(*)}$ and $N_0 \cap \lambda \in S_{\alpha(*)}$.

Let $\delta_i = N_i \cap \lambda$ and $i(*) = \text{ Min}\{i : i < \lambda$ is a limit ordinal and $\delta_i \in S\}$, it is well defined as $\langle \delta_i : i < \lambda \rangle$ is (strictly increasing continuous) hence $\{\delta_i : i < \lambda\}$ is a club of $\lambda$, hence $\gamma^* < \lambda \rightarrow \text{ cf}(i(*)) = \text{ cf}(\delta_{i(*)}) \geq \gamma^*$.

Let[1] $W =: \{i \leq i(*)$: $i > 0$ and if $i < i(*)$ and $j < i$ then $C'_{\delta_{i(*)}} \cap \delta_{i+1} \nsubseteq \delta_{j+1}\}$. Clearly $W \cap i(*)$ is a closed subset of $i(*)$ and as $\delta_{i(*)} = \sup(C_{\delta_{i(*)}})$, also $W \cap i(*)$ is unbounded in $i(*)$. Also as by 1.11 we have $(\alpha \in \text{ acc } C'_{\delta_{i(*)}}) \Rightarrow \alpha \in C_{\delta_{i(*)}} \Rightarrow C'_\alpha = C'_{\delta_{i(*)}} \cap \alpha$ clearly

$(*)$ if $i \in W$ then $\langle N_j : j \in W \cap (i+1) \rangle \in N_{i+1}$.

Also note that

$(**)$ if $i < i(*)$ is nonlimit, then $\delta_i > \sup(C_{\delta_{i(*)}} \cap \delta_i)$ hence $\delta_i > \sup(C'_{\delta_{i(*)}} \cap \delta_i)$. [Why? By 1.8(4) as $\delta_{i(*)} \in S \subseteq S_{\alpha(*)}$ recalling the choice of $\alpha(*)$ clearly $C_{\delta_{i(*)}} \cap S_{\alpha(*)} = \emptyset$ but by clause $\circledcirc(f)$ we have $\delta_i \in S_{\alpha(*)}$ so $\delta_i \notin C_{\delta^*}$. But $C_{\delta_{i(*)}}$ is a closed subset of $\delta_{i(*)}$ hence $\delta_i > \sup(C_{\delta_{i(*)}} \cap \delta_i)$, and $C'_{\delta_{i(*)}} \cap \delta_i \backslash \sup(C_{\delta_{i(*)}} \cap \delta_i)$ has at most two members (see 1.11) so $C'_{\delta_{i(*)}} \cap \delta_i$ is a bounded subset of $\delta_i$ so we are done.]

Now by induction on $i \in W$ we choose $p_i, p^-_i$ and prove on them the following:

$(*)(i)$ $p_i, p^-_i \in \mathbb{P} \cap N_{i+1}$

$(ii)$ $p_i$ is increasing (in $\mathbb{P}$)

$(iii)$ $\max(p_i) > \delta_i$
(of course $\delta_{i+1} > \max(p_i)$ as $p_i \in \mathbb{P} \cap N_{i+1}$)

$(iv)$ $p^-_i = p \cup \{\sup(\delta_i \cup (C'_{\delta_{i(*)}} \cap \delta_{i+1})) + 1\}$ if $i = \text{ Min}(W)$

$(v)$ if $0 < i = \sup(W \cap i)$ and $\gamma_i = \max(C'_{\delta_{i(*)}} \cap \delta_{i+1})$ so $\delta_i \leq \gamma_i < \delta_{i+1}$ then $p^-_i = \cup\{p_j : j \in W \cap i\} \cup \{\delta_i, \gamma_i + 1\}$

$(vi)$ if $j < i$ are in $W$ then $p_j \leq p^-_i \leq p_i$

$(vii)$ $i \in W, i < i(*)$ and $j < i$ satisfies $j = \text{ Max}(W \cap i)$ and $\gamma_i = \max(\{\delta_i\} \cup (C'_{\delta^*} \cap \delta_{i+1}))$ so $\delta_i \leq \gamma_i < \delta_{i+1}$ then $p^-_i = p_j \cup \{\gamma_i + 1\}$

$(viii)$ $p_i$ is the $<^*_\chi$-first $q \in \mathbb{P}$ satisfying

- $(\alpha)$ $p^-_i \leq q \in \mathbb{P}$
- $(\beta)$ if $\gamma^* < \lambda$ then $q$ forces a value to $\underset{\sim}{f}(\text{otp}(\{j < i : j \in W$ and $\text{otp}(j \cap W)$ is a successor ordinal$\})$
- $(\gamma)$ if $\gamma^* = \lambda$ then $q$ forces a value to $\underset{\sim}{f} \restriction \delta_i$

[1] if $\text{cf}(\delta_{i(*)}) > \aleph_0$ then $W = \{i < i(*) : \delta_i \in C_{\delta_{i(*)}}\} \cup \{\delta_{i(*)}\}$ is O.K.

$(ix)$ $p_i^- \backslash \bigcup\limits_{j<i} p_j$ and $p_i \backslash p_i^-$ are disjoint to $C'_{\delta_{i(*)}} \backslash \operatorname{acc}(C'_{\delta_{i(*)}})$, which include the set $\{\operatorname{Suc}^1_{C_{\delta_{i(*)}}}(\alpha, E^*) : \alpha \in C_{\delta_{i(*)}}$ and $\alpha <$ $\operatorname{Suc}^1_{C_{\delta_{i(*)}}}(\alpha)\}$.

Note that clause (ix) follows from the rest; we now carry the induction.

Case 1: $i =$ Min$(W)$.
Choose $p_i^-$ just to fulfill clauses (iv), note that $\delta_i \le \gamma_i < \delta_{i+1}$ as $i \in W \cap i(*)$ and then choose $p_i$ to fulfill clause (viii).

Case 2: $i =$ Min$(W \backslash (j+1))$ and $j \in W$.
Choose $p_i^-$ by clauses (vii) and then $p_i$ by clause (viii).

Case 3: $0 < i = \sup(W \cap i)$.
A major point is $\langle p_j : j < i \rangle \in N_{i+1}$, this holds as $\langle p_j^-, p_j, j \in i \cap W \rangle$ is definable from $\bar N \restriction \delta_i, \underset{\sim}{f}, p, C'_{\delta_{i(*)}} \cap N_{i+1}$ all of which belong to $N_{i+1}$ and $N_{i+1} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$.
Let $p_i^-$ be defined by clause (v), note that $\delta_i \le \gamma_i < \delta_{i+1}$ as $i \in W$ and $p_i^- \in \mathbb{P}$ as:

$(\alpha)$ $(\forall j < i)[p_j \in \mathbb{P}]$ and
$(\beta)$ $\delta_i = \sup(\cup\{\delta_j : j < i$ and $j \in W\})$.
[Why? As $\delta_i <$ $\max(p_j) < \delta_{i+1}$ by clause (iii)] and
$(\gamma)$ $\alpha \in p_i^- \cap S^* \Rightarrow \sup(p_i^- \cap C'_\alpha \backslash \operatorname{acc}(C'_\delta)) < \alpha$.
[Why? If $\alpha < \delta_i$ then for some $j \in i \cap W$ we have $\alpha < \delta_j$ so $p_j$ is an initial segment of $p_i^-$ hence $\sup(p_i^- \cap C'_\alpha) =$ $\sup(p_j \cap C'_\alpha) < \alpha$. If $\alpha = \delta_i$ we can assume $\alpha \in S^*$ but clearly $\alpha = \delta_i \in C'_{\delta_{i(*)}}$ by the definition of $W$ and the assumption of case 3; so by $(\bar S, \bar C)$ being a $\lambda$-witness, $C'_{\delta_i} = C'_{\delta_{i(*)}} \cap \delta_i$ so by clause (ix) the demand (in $(\gamma)$) hold.]

So easily $p_i^-$ is as required. If $i < i(*)$ we can choose $p_i$ by clause (viii) using the induction hypothesis if $\gamma^* = \lambda$. So we have carried the definition and $p_{i(*)}^-$ is as required. $\square_{1.14}$

1.15 Conclusion: 1) If $\mathbf{p} = (\bar S, \bar C, S^*, E^*)$ is a strong $\lambda$-witness and $\bar C' = \bar C'_{\mathbf{p}}$ and $\mathbb{P} = \mathbb{P}_{\bar C'}$, then FA$(\mathbb{P}, \lambda)$ fails.
2) In part (1), $\mathbb{P}_{\bar C'}$ is a forcing of cardinality $\le 2^{<\lambda}$, add no new sequence of ordinals of length $< \lambda$ and preserve stationarity of subsets of any $\theta =$ cf$(\theta) \in [\aleph_1, \lambda]$.

*Proof.* 1) Recall that by Claim 1.13(2), $\mathscr{I}_i$ is a dense open subset of $\mathbb{P}$. Now if $G \subseteq \mathbb{P}_{\bar{C}'}$ is directed not disjoint to $\mathscr{I}_i$ for $i < \lambda$, let $E = \cup\{p : p \in G\}$. By the definition of $\mathbb{P}_{\bar{C}'}$ and $\mathscr{I}_i$ clearly $E$ is an unbounded subset of $\lambda$ and by the definition of $\mathbb{P}_{\bar{C}'}$ and $G$ being directed, $p \in G \Rightarrow E \cap (\max(p) + 1) = p$ and ($p$ is closed) hence $E$ is a closed unbounded subset of $\lambda$. So $E$ contradicts the definition of "$(\bar{S}, \bar{C}, \bar{S}^*, \bar{E}^*, \bar{C}')$ being a strong $\lambda$-witness".
2) Follows from 1.14 and direct checking. $\square_{1.15}$

1.16 Conclusion: Let $\lambda$ be regular $> \aleph_1$.
Then there is a forcing notion $\mathbb{P}$ such that:

- ($\alpha$) $\mathbb{P}$ of cardinality $\leq 2^\lambda$
- ($\beta$) forcing with $\mathbb{P}$ add no new sequences of ordinals of length $< \lambda$
- ($\gamma$) forcing with $\mathbb{P}$ preserve stationarity of subsets of $\lambda$ (and by clause ($\beta$) also of any $\theta = \text{cf}(\theta) \in [\aleph_1, \lambda)$)
- ($\delta$) $\text{FA}(\mathbb{P}, \lambda)$ fail.

*Proof.* We try $\mathbb{P}^2_\lambda$, it satisfies clause $(\alpha), (\beta), (\gamma)$ (see 1.3(1), 1.5, 1.6). If it satisfies also clause ($\delta$) we are done otherwise by Claim 1.6 there is a $\lambda$-witness $(\bar{S}, \bar{C})$. Let $S^* \subseteq \{\delta < \lambda : \text{cf}(\delta) > \aleph_0\}$ be stationary, so by 1.9 for some club $E^*$ of $\lambda$, the quadruple $\mathbf{p} = (\bar{C}, \bar{S}, S^*, E^*)$ is a strong $\lambda$-witness (see Definition 1.10), and let $\bar{C}' = \bar{C}'_{\mathbf{p}}$.

Now the forcing notion $\mathbb{P} = \mathbb{P}_{\bar{C}'}$ (see Definition 1.12) satisfies clauses $(\alpha), (\beta), (\gamma)$ by claims 1.15(2) and also clause ($\delta$) by claim 1.15(1). So we are done. $\square_{1.16}$

## §2 There are $\{\aleph_1\}$-semi-proper not proper forcing notion[2]

By [Sh:f, XII,§2], it was shown that if "remnant of large cardinal properties holds" (e.g. $\neg 0^\#$) then every quite semi-proper forcing is proper, more fully UReg-semi-properness implies properness. This leaves the problem

- $(*)$ is the statement $\big($for every forcing notion $\mathbb{P}$, "$\mathbb{P}$ is proper" follows from $\mathbb{P}$ is "semi-proper, i.e., $\{\aleph_1\}$-semi proper"$\big)$ consistent or is the negation provable in ZFC.

David Asparo raises the question and we answer affirmatively: there are such forcing notions. So the iteration theorem for semi proper forcing notions in [Sh:f, X] is not covered by the one on proper forcing notions even if $0^\#$ does not exist.

**2.1 Claim.** *There is a forcing notion $\mathbb{P}$ of cardinality $2^{\aleph_2}$ which is not proper but is $\{\aleph_1\}$-semi proper. This follows from 2.2 using $\kappa = \aleph_2$.*

**2.2 Claim.** *Assume $\kappa = \mathrm{cf}(\kappa) > \aleph_1, \lambda = 2^\kappa$. Then there is $\mathbb{P}$ such that*

- $(a)$ *$\mathbb{P}$ is a forcing notion of cardinality $2^\kappa$*
- $(b)$ *if $\chi > \lambda, p \in \mathbb{P} \in N \prec (\mathscr{H}(\chi), \in), N$ countable, then there is $q \in \mathbb{P}$ above $p$ such that*
  *$q \Vdash$ "$N \cap \kappa \triangleleft N[\underset{\sim}{G}_{\mathbb{P}}] \cap \kappa$" ($\triangleleft$ means initial segment); this gives $\mathbb{P}$ is $\{\aleph_1\}$-semi proper and more*
- $(c)$ *there is a stationary $\mathscr{S} \subseteq [\lambda]^{\aleph_0}$ such that $\Vdash_{\mathbb{P}}$ "$\mathscr{S}$ is not stationary"*
- $(d)$ *$\mathbb{P}$ is not proper.*

*Proof.* We give many details.

Stage A: Preliminaries.

Let $M^* = (\lambda, F_{n,m})_{n,m<\omega}$, with $F_{n,m}$ an $(n+1)$-place function be such that for every $n < \omega$ and $n$-place function $f$ from $\kappa$ to $\kappa$ there is $m < \omega$ such that $(\forall i_1, \ldots, i_n < \kappa)(\exists \alpha < \kappa)[f(i_1, \ldots, i_n) = F_{n,m}(\alpha, i_1, \ldots, i_n)]$.

Let $S_1, S_2$ be disjoint stationary subsets of $\kappa$ of cofinality $\aleph_0$ (i.e. $\delta \in S_1 \cup S_2 \Rightarrow \mathrm{cf}(\delta) = \aleph_0$).

Let

[2] done 2001/8/8

$$\mathscr{S} = \Big\{ a \in [\lambda]^{\aleph_0} : \text{for some } b \in [\lambda]^{\aleph_0} \text{ we have}$$
$$(\alpha) \quad a \subseteq b \text{ are closed under } F_{n,m} \text{ for } n,m < \omega,$$
$$(\beta) \quad \sup(a \cap \kappa) \in S_1, \sup(b \cap \kappa) \in S_2$$
$$(\gamma) \quad (a \cap \kappa) \triangleleft (b \cap \kappa) \; (\triangleleft \text{ is being an initial segment}) \Big\}$$

$$\mathbb{P} = \mathbb{P}_{\mathscr{S}} = \big\{ \bar a : \bar a = \langle a_i : i \le \alpha \rangle \text{ is an increasing continuous sequence of members of } [\lambda]^{\aleph_0} \backslash \mathscr{S} \text{ of length } \alpha < \omega_1 \big\}$$

Clearly clause (a) of 2.2 holds.

Stage B: $\mathscr{S}$ is a stationary subset of $[\lambda]^{\aleph_0}$.
Why? Let $N^*$ be a model with universe $\lambda$ and countable vocabulary, it is enough to find $a \in \mathscr{S}$ such that $N^* \restriction a \prec N$. Without loss of generality $N^*$ has Skolem functions and $N^*$ expands $M^*$. Choose for $\alpha < \kappa, N_\alpha \prec N^*, \|N_\alpha\| < \kappa, \beta < \alpha \Rightarrow N_\beta \subseteq N_\alpha, \alpha \subseteq N_\alpha, N_\alpha$ increasing continuous.
So $C =: \{\delta < \kappa : \delta$ a limit ordinal and $N_\delta \cap \kappa = \delta\}$ is a club of $\kappa$. Choose $\delta_1 < \delta_2$ from $C$ such that $\delta_1 \in S_1, \delta_2 \in S_2$. Choose a countable $c_1 \subseteq \delta_1$ unbounded in $\delta_1$, and a countable $c_2 \subseteq \delta_2$ unbounded in $\delta_2$.

Choose a countable $M \prec N_{\delta_2}$ such that $M \cap N_{\delta_1} \prec N_{\delta_1}$ and $c_1 \cup c_2 \subseteq \delta$. Let $a = M \cap N_{\delta_1}, b = M \cap N_{\delta_2}$. As $N^*$ expands $M^*$, clearly $a, b$ are closed under the functions of $M^*$. Also $c_1 \subseteq M \cap \delta_1 = M \cap (N_{\delta_1} \cap \kappa) = a \cap \kappa \subseteq N_{\delta_1} \cap \kappa = \delta_1$ hence $\delta_1 = \sup(c_1) \le \sup(a \cap \kappa) \le \delta_1$ so $\sup(a \cap \kappa) = \delta_1$. Similarly $\sup(b \cap \kappa) = \delta_2$. Lastly, obviously $a \cap \kappa \triangleleft b \cap \kappa$ so $b$ witnesses $a \in \mathscr{S}$, as required.

Stage C: $\Vdash_{\mathbb{P}}$ "$\mathscr{S}$ is not stationary".
Why? Define $\underset{\sim}{a}^*_\alpha = \{a_\alpha : \bar a \in \underset{\sim}{G}_{\mathbb{P}}, \ell g(\bar a) > \alpha\}$. Clearly

- $(*)_0$ $\mathbb{P} \ne \emptyset$.
  [Why? Trivial.]
- $(*)_1$ for $\alpha < \omega_1, \mathscr{I}^1_\alpha = \{\bar a \in \mathbb{P} : \ell g(\bar a) > \alpha\}$ is a dense open subset of $\mathbb{P}$.
  [Why? If $\langle a_i : i \le j\rangle \in \mathbb{P}, j < \gamma < \omega_1$ we let $a_i =: a_j$ for $i \in (j, \gamma]$ and then $\langle a_i : i \le j \rangle \le_{\mathbb{P}} \langle a_i : i \le \gamma \rangle$.]
  Also

$(*)_2$ for $\beta < \lambda, \mathscr{I}^2_\beta = \{\bar{a} \in \mathbb{P} : \beta \in a_\alpha$ for some $\alpha < \ell g(\bar{a})\}$ is a dense open subset of $\mathbb{P}$.
[Why? Given $\bar{a} = \langle a_i : i \le j\rangle$. Choose $\delta \in S_2$ such that $\delta > \sup(\kappa \cap (a_j \cup \{\beta\})$ let $c \subseteq \delta$ be countable unbounded in $\delta$ and let $a_{j+1} = a_j \cup \{\beta\} \cup c$; so trivially $\sup(a_{j+1} \cap \kappa) = \delta \in S_2$ hence $a_{j+1} \notin \mathscr{S}$. Now let $\bar{a}^+ = \langle a_i : i \le j+1\rangle$. Now check.]

So

$(*)_3$ $\Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{a}_i : i < \omega_1\rangle$ is an increasing continuous sequence of members of $([\lambda]^{\aleph_0})^{\mathbf{V}} \backslash \mathscr{S}$ whose union is $\lambda$"
hence

$(*)_4$ $\Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{a}_i : i < \omega_1\rangle$ witness $\mathscr{S}$ is not stationary (subset) of $[\lambda]^{\aleph_0}$".

So we have finished Stage C.

Stage D: Clauses (c),(d) of 2.2 holds.
Why? By Stage B and Stage C.

Stage E: Clause (b) of 2.2 holds.

So let $\chi > \lambda, N$ a countable elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ to which $\mathbb{P}$ and $p \in \mathbb{P}$ belong hence $M^*, \kappa, \lambda, S \in N$ (they are definable from $\mathbb{P}$ or demand it). In the next stage we prove

⊠ there is a countable $M \prec (\mathscr{H}(\chi), \in <^*_\chi)$ such that $N \prec M, (N \cap \kappa) \trianglelefteq (M \cap \kappa)$ and $M \cap \lambda \notin \mathscr{S}$.

Let $\langle \mathscr{I}_n : n < \omega\rangle$ list the dense open subsets of $\mathbb{P}$ which belong to $M$. Choose by induction on $n, p_n \in N \cap \mathbb{P} : p_0 = p, p_n \le_{\mathbb{P}} p_{n+1} \in \mathscr{I}_n$. So let $p_n = \langle a_i : i \le \gamma_n\rangle$, by $(*)_1$ of Stage C the sequence $\langle \gamma_n : n < \omega\rangle$ is not eventually constant. Define $q$ by: $q = \langle a_i : i \le \gamma\rangle$ where $\gamma = \cup\{\gamma_n : n < \omega\}$ and $a_\gamma = M \cap \lambda$. Trivially $a_i \subseteq M \cap \lambda$ and by $(*)_2$ of Stage C clearly $a_\gamma = \cup\{a_i : i < \gamma\}$ hence $\langle a_i : i \le \gamma\rangle$ is increasing continuous and $i \le \gamma \Rightarrow a_i \in [\lambda]^{\le \aleph_0}$ and $i < \gamma \Rightarrow a_i \in [\lambda]^{\aleph_0} \backslash \mathscr{S}$. So the only non trivial point is $a_\gamma \notin S$ which holds by ⊠.
Clearly $p \le q$ and $q$ is $(M, \mathbb{P})$-generic hence $q \Vdash$ "$N[G] \subseteq M[G]$ and $N \cap \kappa \subseteq (N[G] \cap \kappa) \subseteq M[G] \cap \kappa = M \cap \kappa$" so as $(N \cap \kappa) \triangleleft (M \cap \kappa)$ necessarily $(N[G] \cap \kappa) \trianglelefteq (N[G] \cap \kappa)$" as required.

Stage F: Proving ⊠.

If $N \cap \lambda \notin \mathscr{S}$ let $M = N$ and we are done so assume $M \cap \lambda \in \mathscr{S}$. Let $a = N \cap \lambda \in [\lambda]^{\aleph_0}$ and let $b \in [\lambda]^{\aleph_0}$ witness $a = N \cap \lambda \in \mathscr{S}$ [the rest should by now be clear but we elaborate]. Let $M$ be the Skolem Hull in $(\mathscr{H}(\chi), \in, <^*_\chi)$ of $N \cup (b \cap \kappa)$ (exists as $<^*_\chi$ is a well ordering of $\mathscr{H}(\chi)$ so $(\mathscr{H}(\chi), \in, <^*_\chi)$ has (definable) Skolem functions).

If $\gamma \in M \cap \kappa$ then we can find a definable function $f$ of $(\mathscr{H}(\chi), \in, <^*)$ and $x \in N$ (recall that in $N$ we can use $m$-tuple for every $m$) and $\alpha_1 \ldots \alpha_n \in b \cap \kappa$ such that $\gamma = f(x, \alpha_1, \ldots, \alpha_n)$. Fixing $x, f$ the mapping $(\alpha_1, \ldots, \alpha_n) \mapsto f(x, \alpha_1, \ldots, \alpha_n)$ is an $n$-place function from $\kappa$ to $\kappa$ definable in $N$ hence belong to $N$ and $M^* \in N$ hence for some $\beta \in N \cap \lambda$ and $m < \omega$ we have $(\forall \alpha_1, \ldots, \alpha_n < \kappa)[f(x, \alpha_1, \ldots, a_n) = F_{n,m}(\beta, \alpha_1, \ldots, \alpha_n)]$.

But $\alpha_1, \ldots, \alpha_n \in b \cap \kappa \subseteq b$ and $\beta \in N \cap \lambda \subseteq b \cap \lambda = b$ and as $b$ being in $\mathscr{S}$ is closed under $F_{n,m}$ clearly $\gamma = f(x, \alpha_1, \ldots, \alpha_n) = F_{n,m}(\beta, \alpha_1, \ldots, \alpha_n) \in b$ but $\gamma \in \kappa$ so $\gamma \in b \cap \gamma$. So $M \cap \kappa \subseteq b$ but of course $b \cap \kappa \subseteq M \cap \kappa$ so $b \cap \kappa = M \cap \kappa$. So $a \cap \kappa = (N \cap \lambda) \cap \kappa = N \cap \kappa$; but $a \cap \kappa \triangleleft b \cap \kappa$ by the choice of $b$ so $N \cap \kappa = a \cap \kappa \triangleleft b \cap \kappa = M \cap \kappa$.
Lastly, $\sup(M \cap \kappa) = \sup(b \cap \kappa) \in S_2$ hence $M \cap \kappa \notin S$. So $M$ is as required in $\boxtimes$ and we are done. $\square_{2.2}$

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