Paper Sh:837, version 2009-06-08_10. See https://shelah.logic.at/papers/837/ for possible updates.

MODEL THEORETIC STABILITY AND CATEGORICITY FOR COMPLETE METRIC SPACES

SH837

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We would like to thank Alice Leonhardt for the beautiful typing.
The first author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 242/03).
We thank the anonymous referee for very helpful comments and suggestions which greatly improved the exposition, especially in the last section of the paper.
Latest Revision - 09/May/18

Typeset by $\mathcal{AMS}$-TEX

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Abstract. We deal with systematic development of stability for the context of approximate elementary submodels of a monster metric space, which is not far, but still very different from first order model theory. In particular we prove the analogue of Morley's theorem for classes of complete metric spaces.

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§1 Introduction and Preliminaries

We work in the context of a compact homogeneous model $\mathfrak{C}$ which is also a complete metric space with a definable metric $\mathbf{d}(x, y)$ and all the predicates and function symbols respect the metric. Such a monster model will be called a "monster metric space" (a momspace), Definition 2.17. We investigate the class K of "almost elementary" complete submodels of $\mathfrak{C}$, Definition 2.19.

This paper is devoted to categoricty of such classes K in uncountable cardinalities (generalizing Morley's theorem to this context). Since we believe that isometry is too strong as a notion of isomorphism for classes of metric structures, we try to weaken the assumptions and work with ε-embeddings instead (Definitions 6.2, 6.4).

Several suggestions for a framework suitable for model theoretic treatment of classes arising in functional analysis and dynamics have been made in the last 40 years by Chang, Keisler, Stern, Henson, and more recently by Ben-Yaacov. All these attempts were concerned with allowing a certain amount of compactness (e.g. "capturing" ultra-products of Banach spaces introduced by Krivine), without having to deal with non-standard elements. In this paper the authors choose to work in the most general context which still allows compactness, therefore generalizing all the above frameworks. The main tools and techniques used here are borrowed from homogeneous model theory.

Homogeneous model theory was introduced and first studied by Keisler, developed further by the first author, Grossberg, Lessmann, Hyttinen and others. It investigates classes of elementary (sometimes somewhat "saturated") submodels of a big homogeneous model ("monster"), see precise definitions later.

In [Sh 54], the first author classified such "monsters" with respect to the amount of compactness they admit. Monsters which are compact in a language closed under negation are called "monsters of kind II". Later Hrushovski suggested the name "Robinson Theories" for such classes, see [Hrxz]. Monsters which are compact in a language not necessarily closed under negations are called "monsters of kind III". Recently Ben-Yaacov has studied this context in great detail. He called such monsters "compact abstract theories", in short CATS, see [BY03].

A simple generalization (replacing equality by definable metric) allows us to speak of a monster model of a class of metric spaces. We call such monsters "monster metric spaces", see Definition 2.17 below. Several results are proven in this general context, but some require compactness, Definition 2.15(2). Therefore, our main theorem (Theorem 8.2) holds in the metric analogue of "monsters of kind III", "metric cats" in Ben-Yaacov's terminology.

Independently of our work (and simultaneously), Ben-Yaacov investigated categoricity for metric cats under the additional assumption of the topology on the space of types being Hausdorff, which is the metric analogue of "monsters of kind

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II" (Robinson theories). In this context one can reconstruct most of classical stability theory (e.g., independence based on non-dividing), see [BY05]. These methods fail in the more general context we were working at. Therefore, techniques developed and used here are very different, and rely heavily on non-splitting and Ehrenfeucht-Mostowski constructions. These tools do not make any significant use of compactness, and we believe that this assumption can be eliminated by modifying our methods slightly. At this point we decided not to make the effort, but we try to mention where exactly compactness is used.

In a recent work [BeUs0y], Ben-Yaacov and the second author introduce a framework of continuous first order logic, closely related to [ChKe66] and show that once modified slightly, most model-theoretic approaches to classes of metric spaces (such as Henson's logic, see [HeIo02], Hausdorff metric cats, see [BY05]) are equivalent to continuous logic. Although if continuous model theory had been discovered earlier, this paper might have looked differently, we would still like to point out that equivalences shown in [BeUs0y] do not include monster metric spaces, not even compact ones. The assumption of the topology on the type space being Hausdorff is absolutely crucial in [BeUs0y] and [BY05]; it provides us with the ability to "approximate" negations, which makes continuous logic very similar to classical first order logic (of course, this has many advantages). Lacking Hausdorff assumption, one has to use more general methods in order to reobtain basic properties. This is why "monsters of kind III" (general cats) have been studied and understood much less than first order or Robinson theories, even in the discrete (non-continuous, classical first order) context.

Some work has been done, though: the first author proved the analogue of Morley's theorem for existentially closed models in [Sh 54], classes of existentially closed models were investigated further by Pillay in [Pi00], general cats were studied by Ben-Yaacov in [BY03], [BY03a] and other papers. Our work continues this investigation in the more general metric context (classical model theory can be viewed as a particular case with discrete metric).

Several words should be said also about the difference between the discrete and the metric context. For example, why could we have not simply modified the proof in [Sh 54] slightly and obtain Theorem 8.2? The answer is that our categoricity assumption is significantly weaker, as we assume categoricity only for complete structures. For instance, the class of infinite-dimensional Hilbert spaces is categorical in all densities, but not so is the class of inner-product spaces, not necessarily complete. Starting with a weaker assumption we aim for a weaker conclusion; but consequences of our assumption are not as powerful as what one gets in [Sh 54] (e.g. $\aleph_0$-stability is lost, we only have a topological version), which complicates life significantly.

This work was originally carried out as a Ph.D. thesis of the second author under the supervision of the first one. The paper is an expanded version of the

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thesis which was written in Hebrew and submitted to the Hebrew University.

* * *

The paper is organized as follows:

We introduce our context in §2. In particular, we define the class of models which is being studied ("almost elementary" submodels of $\mathfrak{C}, M \prec_1 \mathfrak{C}$). This notion generalizes Henson's approximate elementary submodels (see [HeIo02]). It has the following important property: if $M \prec_1 \mathfrak{C}$, then its completion (metric closure) N also satisfies $N \prec_1 \mathfrak{C}$. We will be interested mostly in complete (as metric spaces) "almost elementary" submodels of $\mathfrak{C}$.

The next section, §3, is devoted to different kinds of approximations to formulae and types. The importance of considering these approximations, i.e. topological neighborhoods of partial types, becomes clear later, when stability, isolation, and other central notions are discussed.

In §4 we generalize the notion of stability in a cardinal λ to our topological context. We say that $\mathfrak{C}$ is $0^+ - \lambda$-stable if for no $\varepsilon > 0$ can we find an ε-net of λ^+ types over a set of cardinality λ, i.e. the space of types over a set of cardinality λ has (in a sense) density λ. This is a generalization of λ-stability for Banach spaces studied by Iovino in [Io99]. It is equivalent to the definitions given by Ben-Yaacov for Hausdorff cats in [BY05], and in the context of continuous theories it coincides with the definitions given in [BeUs0y].

We prove equivalence of several similar definitions for $0^+ - \lambda$-stability, connect $0^+ - \aleph_0$-stability to non-splitting of types, classical stability in homogeneous model theory, existence of average types, saturation of a (closure of a) union of $(D, \aleph_1)$-homogeneous models. Notions of isolation are developed and density of strictly isolated types is proved under the assumption of $0^+ - \aleph_0$-stability.

In §5 we develop the theory of Ehrenfeucht-Mostowski models in our context. As we lack forking calculus, some basic facts (e.g. existence of $(D, \aleph_1)$-homogeneous models in all uncountable cardinalities) require a different approach, which is provided by the EM-models techniques. Also in the proof of the main theorem (§8) we take advantage of the representation of the homogeneous model as an EM-model in order to find inside it a converging sequence which is "close" to a subsequence of a given uncountable sequence.

§6 is devoted to notions of ε-embeddings and ε-isomorphisms. We try to weaken our assumptions as much as we can, and choose to work with the following notion of "weakly uncountably categorical" classes: for every $\varepsilon > 0$ there exists $\lambda > \aleph_0$ in which every two models and ε-isomorphic to each other. This property seems a priori weaker than uncountable categoricity in some $\lambda > \aleph_0$, and even than "there exists $\lambda > \aleph_0$ such that for every $\varepsilon > 0$ any two models of density λ are ε-isomorphic". But the main theorem (8.2) states the following: assume $\mathfrak{C}$ is

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weakly uncountably categorical. Then every model of density $\lambda > \aleph_0$ is (D, λ)-homogeneous (in particular, unique up to an isometry). So all the above notions turn out to be equivalent.

In §7 we prepare the ground for the proof of 8.2, showing that any wu-categorical (weakly uncountably categorical) momspace is uni-dimensional (in the sense of [Sh 3]), i.e., any $(D, \aleph_1)$-homogeneous model of density character λ is, in fact, (D, λ)-homogeneous.

The last section, §8, contains the proof of the main theorem, Theorem 8.2. We assume that $\mathfrak{C}$ is weakly uncountably categorical, but has a non-(D, λ)-homogeneous model in some density character $\lambda > \aleph_0$. By §7 this model is not $(D, \aleph_1)$-homogeneous. Applying the analysis done in §4 and §5 (and some infinitary combinatorics), we show that it is possible to construct non-$(D, \aleph_1)$-homogeneous models of arbitrarily large density characters. By §4 and §6, this contradicts weak uncountable categoricity.

* * *

We recall now basic definitions considering homogeneous model theory (see [Sh 3],[Sh 54] and [GrLe02]):

1.1 Definition. 1) Let τ be a vocabulary, D a set of complete τ-types over $\emptyset$ in finitely many variables. A τ-model M is called a <u>D-model</u> if $D(M) \subseteq D$ (where $D(M)$ = the set of complete finite τ-types over $\emptyset$ realized in M). M is called <u>(D, λ)-homogeneous</u> if $D(M) = D$ and M is λ-homogeneous.
2) We call D as in (1) a <u>finite diagram</u> if for some model $M, D = D(M)$.
3) A is a D-set in M if $A \subseteq M$ and $\bar{a} \in {}^{\omega>}A \Rightarrow \text{tp}(\bar{a}, \emptyset, M) \in D$. For A a D-set, $p \in \mathbf{S}^m_D(A, M)$ if there are $N, \bar{a}$ such that $M \prec N, \bar{a} \in N$ realizes p and $A \cup \bar{a}$ is a D-set in N.

1.2 Remark. Let D be a finite diagram. M is (D, λ)-homogeneous <u>iff</u> M is universal for D-models of cardinality $\leq \lambda$ (or just λ-sets when $\lambda < |\tau| + \aleph_0$) and λ-homogeneous <u>iff</u> $D(M) = D$ and M is (D, λ)-saturated (i.e. every D-type over a subset of cardinality $< \lambda$ is realized in M).

Proof. Easy (or see [GrLe02](2.3),(2.4)).

This motivates the following

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1.3 Definition. Let λ^* be big enough. A (D, λ^*)-homogeneous model $\mathfrak{C}$ will be called a <u>D-monster</u> (or a homomogeneous monster for D) . We usually assume $\|\mathfrak{C}\| = \lambda^*$.

Recall:

1.4 Definition. 1) A finite diagram D is called λ-<u>good</u> if there is a (D, λ)-homogeneous model M of cardinality $\geq \lambda$.
2) D is good if it is λ-good for every λ.

1.5 Remark. So D is good iff it has a monster.

1.6 Convention. In this paper we will fix a good finite diagram D and a D-monster model $\mathfrak{C}$.

1.7 Observation. 1) If (D is good, $\mathfrak{C}$ a D-monster model) $p \in \mathbf{S}_D(A), A \subseteq B$ <u>then</u> there is $q, p \subseteq q \in \mathbf{S}_D(B)$.

<u>Question</u>: Why can we use "good D"? There are several answers.

Basically, Claim 1.8 says that every D which is the finite diagram of a <u>compact</u> momspace (see Definitions 2.17, 2.21) is good. Claim 1.9 says that even without compactness, categoricity implies stability, which implies D is good by [Sh 3]. The reader can easily omit these claims in the first reading.

1.8 Claim. *Assume*

(a) *D is a finite diagram, i.e., a set of complete n-types in $\mathbb{L}(\tau_{\mathfrak{C}})$*
(b) *$\mathfrak{C}$ is $(D, \aleph_0)$-homogeneous*
(c) *Δ is full for $\mathfrak{C}$ (see Definition 2.15)*
(d) *$\mathfrak{C}$ is Δ-compact.*

<u>*Then*</u>

(∗) (α) *D is good*
(β) *if $\kappa(D) < \infty$ then $\kappa(D) \leq (|\tau_{\mathfrak{C}}| + \aleph_0)^+$ ($\kappa(D)$ as in [Sh 54]).*

Proof. See [Sh 54], the discussion of monsters of kind III or [BY03], existence of a universal domain.

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1.9 Claim. *1) Assume that*

(a) *τ a countable metric vocabulary and let $\delta(*) = (2^{\aleph_0})^+$*
(b) *$\tau = \tau_D$*
(c) *D is a finite diagram*
(d) *there is a $(D, \aleph_1)$-homogeneous model M*
(e) *there is a D-model of cardinality $\geq \beth_{\delta(*)}$ (probably less is enough) and*
(f) *$K^1_D = \{M : M \in K^c_1\}$ (see Definition 2.19) is categorical in $\lambda > \aleph_0$ or is wu-categorical (see Definition 6.4).*

Then D is stable, hence is good.

Proof. By 6.6, 4.10 hence we get stability, D is good follows by [Sh 3].

1.10 Notations.

λ, μ, χ	infinite cardinals
α, β, γ	infinite ordinals
δ	limit ordinals
ν, η	sequences of ordinals
φ, ψ, ϑ	formulae
$\mathfrak{C}$	the monster model
M, N	models (in the monster)
A, B, C	sets (in the monster)
ε, ζ, ξ	positive reals
I, J	order types
$\mathbf{I}, \mathbf{J}$	indiscernible sequences
$\mathbf{d}$	a metric

§2 Main context - monster metric spaces

In this section we discuss our main context. We start with some notations.

2.1 Definition. Let $(X, \mathbf{d})$ be a metric space.
1) We extend the metric to n-tuples: for $\langle a_i : i < n\rangle, \langle b_i : i < n\rangle$ we define $\mathbf{d}(\bar{a}, \bar{b}) = \max\{\mathbf{d}(a_i, b_i) : i < n\}$.
2) For $\bar{a}$ an n-tuple, A a set of n-tuples, $\mathbf{d}(\bar{a}, A) = \mathbf{d}(A, \bar{a}) = \inf\{\mathbf{d}(\bar{a}, \bar{b}) : \bar{b} \in A\}$.
3) For sets $A, B \subseteq X^n$, we define two versions of distances:

$$\mathbf{d}_1(A, B) = \inf\{\mathbf{d}(\bar{a}, \bar{b}) : \bar{a} \in A, \bar{b} \in B\}$$

$$\mathbf{d}_2(A, B) = \sup\{\{\mathbf{d}_1(\bar{a}, B) : \bar{a} \in A\} \cup \{\mathbf{d}_1(A, \bar{b}) : \bar{b} \in B\}\}.$$

4) We denote the density (the density character) of X by $\mathrm{Ch}(X)$. So $\mathrm{Ch}(X)$ is the minimal cardinality of a dense subset of X.
5) We denote the topological (metric) closure of a set A by $\bar{A}$ or $\mathrm{mcl}(A)$.

2.2 Definition. 1) We call a vocabulary τ metric if it contains predicates $P_{q_1,q_2}(x, y)$ for all $0 \leq q_1 \leq q_2$ rationals. We call the collection of these predicates a metric scheme.

2) Given a metric vocabulary τ, we call a τ-structure M semi-metric if for some (unique) $\mathbf{d}$:

- (i) M is a metric space with the metric $\mathbf{d}$
- (ii) $\mathbf{d}$ is definable in M by the τ-metric scheme, i.e., $\mathbf{d}(a, b) \in [q_1, q_2]$ iff $M \models P_{q_1,q_2}(a, b)$ for all rationals q_1, q_2 and $a, b \in M$.

3) We call a semi-metric structure complete if $(M, \mathbf{d})$ is complete as a metric space.

2.3 Remark. Given a semi-metric τ-structure M, we will usually write $M \models$ "$\mathbf{d}(a, b) \leq q$", etc., forgetting the τ-metric scheme.

Note that in a semi-metric structure for each $r_1, r_2 \in \mathbb{R}$, the property "$r_1 \leq \mathbf{d}(x, z) \leq r_2$" is 0-type-definable by a set of atomic formulas.

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2.4 Definition. 1) $(M, \mathbf{d})$ is a metric structure (or model) when:

(a) $\tau(M)$ is a metric vocabulary and $(M, \mathbf{d})$ is a semi-metric structure
(b) $P^M \subseteq {}^{\text{arity}(P)}M$ is closed (with respect to $\mathbf{d}$) for every predicate $P \in \tau_M$
(c) $F^M : {}^{\text{arity}(F)}M \to N$ is a continuous function for every function symbol $F \in \tau_M$.

2) $(\mathfrak{C}, \mathbf{d})$ is a (homogeneous) metric monster if: $(\mathfrak{C}, \mathbf{d})$ is a metric model, $\mathfrak{C}$ is a D-monster for some finite diagram D, constant here.

2.5 Remark. Each homogeneous monster admits the discrete definable metric, i.e., $\mathbf{d}(a, b) = 1$ for all $a \neq b \in \mathfrak{C}$, and it is definable by the equality and inequality. So each homogeneous monster is a metric monster with the discrete metric.

2.6 Notations. We will often identify $\varphi = \varphi(\bar{x}, \bar{a})$ (for φ a formula, $\bar{a} \in \mathfrak{C}$) with the set of the realizations in $\mathfrak{C}$ of $\varphi(\bar{x}, \bar{a})$, i.e., $\varphi(\mathfrak{C}, \bar{a}) = \varphi^{\mathfrak{C}} =: \{\bar{b} \in {}^{\ell g(\bar{x})}\mathfrak{C} : \mathfrak{C} \models \varphi[\bar{b}, \bar{a}]\}$. So $\mathbf{d}_1(\varphi, \psi)$ in fact means $\mathbf{d}_1(\varphi^{\mathfrak{C}}, \psi^{\mathfrak{C}})$, etc.

* * *

The following definition is the analogue of abstract elementary classes (see [Sh 88r]) in our context.

2.7 Definition. Let $(\mathfrak{K}, \leq_{\mathfrak{K}})$ be an ordered class of $\tau = \tau(\mathfrak{K})$ complete metric structures (τ is a metric vocabulary), $\mathfrak{K}$ closed under τ-isomorhisms. We call $(\mathfrak{K}, \leq_{\mathfrak{K}})$ an abstract metric class (a.m.c.) if

(1) $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathfrak{K}}$-increasing sequence (from $\mathfrak{K}$, of course), then $M = \text{mcl}(\bigcup_{i<\delta} M_i) \in \mathfrak{K}$. Moreover, $[i < \delta \Rightarrow M_i \leq_{\mathfrak{K}} M]$ and $[M_i \leq_{\mathfrak{K}} N$ for all $i < \delta \Rightarrow M \leq_{\mathfrak{K}} N]$
(2) for $M_1 \subseteq M_2 \subseteq M_3$ from $\mathfrak{K}, [M_1, M_2 \leq_{\mathfrak{K}} M_3] \Rightarrow M_1 \leq_{\mathfrak{K}} M_2$
(3) for some cardinal $\text{LS}(\mathfrak{K})$ we have the "downward Löwenheim-Skolem theorem", i.e., for each $M \in \mathfrak{K}, A \subseteq M$, there exists $N \in \mathfrak{K}, N \leq_{\mathfrak{K}} M, A \subseteq N$, $\text{Ch}(N) \leq |A| + \text{LS}(\mathfrak{K})$.

2.8 Definition. We say that a set of formula Δ for a homogeneous metric monster $\mathfrak{C}, \mathbf{d}$ is admissible if for each $\varphi(\bar{x}) \in \Delta$, the set $\varphi^{\mathfrak{C}} = \{\bar{a} \in \mathfrak{C}, \mathfrak{C} \models \varphi(\bar{a})\}$ is closed with respect to the metric $\mathbf{d}$ (topology induced by it).

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2.9 Definition. In $\mathfrak{C}, \varphi, \psi$ are contradictory if $\mathbf{d}_1(\varphi^{\mathfrak{C}}, \psi^{\mathfrak{C}}) > 0$.

2.10 Remark. Later (see 3.7) we show that in our context this is equivalent to $\varphi^{\mathfrak{C}} \cap \psi^{\mathfrak{C}} = \emptyset$.

2.11 Example. If $\mathbf{d}$ is discrete, then each subset of $\mathfrak{C}$ is closed, so the set of all formulas $\Delta = \mathbb{L}(\tau_{\mathfrak{C}})$ is admissible.

In order to give a non-trivial example, we define

2.12 Definition. For a metric model $(M, \mathbf{d})$ and a formula $\varphi(x, \bar{a})$ with parameters $\bar{a} \in M$, we say that $M \models \exists^* x \varphi(x, \bar{a})$ (there almost exists x such that $\varphi(x, \bar{a})$) if for every $\varepsilon > 0$ there exist $b, \bar{a}'$ such that $M \models \varphi(b, \bar{a}')$ and $\mathbf{d}(\bar{a}, \bar{a}') \leq \varepsilon$.

2.13 Definition. 1) We define positive formulae by induction: each atomic formula is positive, for φ, ψ positive, $\varphi \wedge \psi, \varphi \vee \psi, \exists^* x\varphi, \forall x\varphi$ are positive. So negation and implication are not allowed.
2) Positive existential formulae are defined similarly without allowing $\forall x\varphi$.

2.14 Observation. For $(M, \mathbf{d})$ a metric model and positive $\varphi(\bar{x}) \in \mathbb{L}(\tau_M), \varphi^M = \{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}]\}$ is a closed subset of ${}^{\ell g(\bar{x})}M$ (under $\mathbf{d}$), i.e., $\Delta =$ "the positive formulae" is admissible.

2.15 Definition. 1) For a homogeneous monster $\mathfrak{C}$, a set of formulae Δ is called full if

- (i) for all $\bar{a} \in \mathfrak{C}, A \subseteq \mathfrak{C}$, $\mathrm{tp}(\bar{a}, A, \mathfrak{C})$ is determined by the Δ-type $\mathrm{tp}_\Delta(a, A, \mathfrak{C})$
- (ii) if $\varphi \in \Delta, \bar{a} \in \mathfrak{C}$ and $\mathfrak{C} \models \neg\varphi(\bar{a})$, then there exists $\psi \in \Delta$ such that (φ, ψ) is contradictory (see 2.9) and $\mathfrak{C} \models \psi(\bar{a})$.

2) We call $\mathfrak{C}$ Δ-compact (where Δ is a set of formulae, i.e., $\subseteq \mathbb{L}(\tau_{\mathfrak{C}})$) if each set of Δ-formulae with parameters from $\mathfrak{C}$ of cardinality $< |\mathfrak{C}|$ which is finitely satisfiable in $\mathfrak{C}$, is realized in $\mathfrak{C}$. We omit Δ if constant, and abusing notation write $\Delta = \Delta(\mathfrak{C})$.

2.16 Definition. We say that Δ is full$^+$ if: as in 2.15 but in (ii) the quantifier depth of ψ is $\leq$ the quantifier depth of φ.

Now we make the main definition of this section, introducing the context of this paper.

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2.17 Definition. 1) A metric homogeneous class $(\mathfrak{K}, \leq_{\mathfrak{K}})$ (equivalently: its metric homogeneous monster $\mathfrak{C}$) is called Δ-momspace (monster metric space) if $\Delta = \Delta(\mathfrak{C})$ is a set of formulae containing the metric scheme, such that

- (a) $\Delta \subseteq \mathbb{L}(\tau_{\mathfrak{C}})$ is closed under $\wedge, \exists^*$ and subformulae
- (b) Δ is admissible
- (c) Δ is full for $\mathfrak{C}$.

2) Let "$\mathfrak{C}$ is momspace" mean "for some Δ" and choose such $\Delta = \Delta(\mathfrak{C})$ (well, it is not necessarily unique but we ignore this).
3) The metric $\mathbf{d} = \mathbf{d}_{\mathfrak{C}}$ can be defined from $\mathfrak{C}$ so we can "forget" to mention $\mathbf{d}$, but still will usually say $(\mathfrak{C}, \mathbf{d})$, e.g. to distinguish from $\mathfrak{C}$ when we use the (D, λ)-homogeneous context.

2.18 Convention. $(\mathfrak{C}, \mathbf{d})$ is a fixed momspace.

The class of models we are interested in is defined below.

2.19 Definition. 1) $K_1 = K^1_{\mathfrak{C}}$ is the class of M such that:

- (a) $M \subseteq \mathfrak{C}$
- (b) $M \prec^1_\Delta \mathfrak{C}$ which means:
  if $\varepsilon > 0, \mathfrak{C} \models \varphi[\bar{b}, \bar{a}], \varphi(\bar{y}, \bar{x}) \in \Delta, \bar{a} \in {}^{\ell g(\bar{x})}M$ then for some $\bar{b}' \in {}^{\ell g(\bar{y})}M$ we have $\mathfrak{C} \models (\exists \bar{x}, \bar{y})(\varphi(\bar{y}, \bar{x}) \wedge \mathbf{d}(\bar{y}\hat{\ }\bar{x}, \bar{b}'\hat{\ }\bar{a}) \leq \varepsilon)$.

2) K^c_1 is the class of members of K_1 which are complete.

2.20 Claim. *1) Assume $M \subseteq \mathfrak{C}$ and $N = \mathrm{mcl}(M)$. Then*

- (a) *$M \in K_1$ iff $N \in K_1$ iff $N \in K^c_1$.*

2) $(K^c_1, \subseteq)$ is an abstract metric class .

2.21 Definition. A momspace $(\mathfrak{C}, \mathbf{d})$ is called compact if $\mathfrak{C}$ is $\Delta(\mathfrak{C})$-compact.

2.22 Convention. We work in a compact Δ-momspace $(\mathfrak{C}, \mathbf{d}), M, N$ denote submodels which are from K_1; though really interested in the closed (complete) ones, i.e. versions of categoricity are defined using complete models. BUT we try to mention when we use compactness.

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2.23 Observation. [$\mathfrak{C}$ compact] For $\varphi(x,\bar{y}) \in \Delta, \bar{a} \in \mathfrak{C}$, we have $\mathfrak{C} \models \exists^* x\varphi(x,\bar{a}) \Leftrightarrow \mathfrak{C} \models \exists x\varphi(x,\bar{a})$ [so we can "forget" about the new existential quantifier and use the classical one].

Proof. Assume $\mathfrak{C} \models \exists^* x\varphi(x,\bar{a})$. Then the set $\{\varphi(x,\bar{y}), \mathbf{d}(\bar{y},\bar{a}) \leq \frac{1}{n} : n < \omega\}$ is consistent in $\mathfrak{C}$ (by compactness), so $\mathfrak{C} \models \exists x\varphi(x,\bar{a})$. $\square$

2.24 Examples. 1) Let T be first order, $\mathfrak{C}$ a big saturated model of T, then $\mathfrak{C}$ is a compact momspace (with discrete metric), $\Delta(\mathfrak{C}) = \mathscr{L}$ (all formulae).
2) Let T be a Robinson theory (see [Hrxz]), $\mathfrak{C}$ its universal domain. Then it is a compact momspace.
3) Consider the unit ball of a monster Banach space as in [HeIo02], Chapter 12. Then $(\mathfrak{C}, \|\cdot\|)$ is a compact momspace (we write the norm here instead of the metric) where $\Delta(\mathfrak{C})$ = positive formulae (more precisely positive bounded formulae, see [HeIo02], Chapter 5), $\mathfrak{K} = K_1^c = \{M : M \prec_A \mathfrak{C}\}$, see [HeIo02](6.2) for the definition of $\prec_A$. Compactness is proven in Chapter 9 there. One can easily show that $\prec_A = \prec^1_\Delta$, see also 3.9, [HeIo02](6.6).
4) Metric Hausdorff CATs, see [BY03] for a definition of a Hausdorff CAT. A CAT is metric if its monster is metric. See [BY05] for more on Hausdorff metric cats.
5) The main example we have in mind: $(\mathfrak{C}, \mathbf{d})$ is a compact momspace for $\Delta(\mathfrak{C})$ = positive existential formulae, $\mathfrak{K} = K_1^c$. So in particular it is a metric cat (not necessarily Hausdorff).

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§3 Approximations to formulas and types

The following notions of ε-approximations of formulae is of major importance. We give two different definitions and will use both for different purposes. Note that 3.1 simply defines topological neighborhoods, while in 3.2 by moving the parameters we allow the formula to change a little, see also 3.9. Let ε denote a non-negative rational number.

3.1 Definition. 1) For a formula $\varphi(\bar{x})$ possibly with parameters, we define $\varphi^{[\varepsilon]}(\bar{x}) = \exists \bar{x}'(\varphi(\bar{x}') \wedge \mathbf{d}(\bar{x},\bar{x}') \le \varepsilon)$. So $\varphi^{[\varepsilon]}(\bar{x},\bar{a}) = (\exists \bar{x}')[\mathbf{d}(\bar{x},\bar{x}') \le \varepsilon \wedge \varphi(\bar{x}',\bar{a})]$.
2) For a (partial) type p, define $p^{[\varepsilon]} = \{(\bigwedge\limits_{\ell<n} \varphi_\ell)^{[\varepsilon]} : n < \omega \text{ and } \varphi_0,\dots,\varphi_{n-1} \in p\}$.

3.2 Definition. 1) For a formula $\varphi(\bar{x},\bar{a})$ let $\varphi^{<\varepsilon>}(\bar{x},\bar{a}) = (\exists \bar{x}',\bar{y}')(\varphi(\bar{x}',\bar{y}') \wedge \mathbf{d}(\bar{x}',\bar{x}) \le \varepsilon \wedge \mathbf{d}(\bar{y}',\bar{a}) \le \varepsilon)$.
2) For a (partial) type $p, p^{<\varepsilon>}(\bar{x}) = \{(\bigwedge\limits_{\ell} \varphi_\ell(\bar{x},\bar{a}_\ell))^{<\varepsilon>} : n < \omega \text{ and } \varphi_\ell(\bar{x},\bar{a}_\ell) \in p$ for $\ell < n\}$.

3.3 Definition. 1) For a (partial) type (maybe with parameters) p, or set B and $\varepsilon > 0$, we say that $\bar{c} \in B$ realizes $p^{[\varepsilon]}$ if $\mathfrak{C} \models p^{[\varepsilon]}(\bar{c})$. We say that $p^{[\varepsilon]}$ is realized in B if some $\bar{c} \in B$ realizes it.
2) Similar for $p^{<\varepsilon>}$.

3.4 Remark. 1) Note that $\varphi^{[\varepsilon]}(\bar{x},\bar{a})$ is equivalent to $\varphi_1^{[\varepsilon]}(\bar{x})$, when we expand $\mathfrak{C}$ by individual constants for $\bar{a}$ and let $\varphi_1(\bar{x}) = \varphi(\bar{x},\bar{a})$.
2) $\varphi^{[\varepsilon]} \models \varphi^{<\varepsilon>}$ for all φ.
3) For a formula without parameters $\varphi, \varphi^{[\varepsilon]} = \varphi^{<\varepsilon>}$.
4) For $\zeta \ge \varepsilon \ge 0, \varphi^{[\varepsilon]} \models \varphi^{[\zeta]}, \varphi^{<\varepsilon>} \models \varphi^{<\zeta>}$.
5) For $\varepsilon = 0, \varphi^{[\varepsilon]} \equiv \varphi^{<\varepsilon>} \equiv \varphi$.

3.5 Observation. 1) If $\varphi = \varphi(\bar{x},\bar{b})$ (as usual is admissible), then $\varphi \equiv \bigwedge\limits_{\varepsilon>0} \varphi^{[\varepsilon]}$.
2) If $\varphi = \varphi(\bar{x},\bar{a})$ then $\varphi \equiv \bigwedge\limits_{\varepsilon>0} \bigvee\limits_{\zeta\in(0,\varepsilon)} \varphi^{[\zeta]}$.
3) Similar for $\varphi^{<\varepsilon>}$.

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Proof. 1) $[a \models \varphi \Rightarrow a \models \bigwedge\{\varphi^{[\varepsilon]} : \varepsilon > 0\}]$ is obvious.

Suppose $\bar a \models \varphi^{[\varepsilon]}$ for all ε, so for each n there exists $\bar a_n$ such that $\mathfrak{C} \models \varphi[\bar a_n]$ and $\mathbf{d}(\bar a, \bar a_n) \le \frac{1}{n}$. So $\langle \bar a_n : n \in \omega\rangle$ converges to $\bar a$, therefore $\bar a \models \varphi$, as $\varphi^{\mathfrak{C}}$ is a closed set.
2),3) Similar.

3.6 Claim. *1)* $(\theta^{<\zeta_1>}(\bar x))^{<\zeta_2>} \equiv \theta^{<\zeta_1+\zeta_2>}(\bar x)$.
2) $(p^{<\zeta_1>}(\bar x))^{<\zeta_2>} \equiv p^{<\zeta_1+\zeta_2>}$.
3) Assume $\mathbf{d}(\bar b, \bar b') \le \zeta_1 - \zeta_2$ *then* $\theta^{<\zeta_2>}(x, b') \models \theta^{<\zeta_1>}(x, b)$.

3.7 Claim. *1) If $\mathfrak{C}$ is Δ-compact <u>then</u> a pair (of admissible Δ-formulas) $\varphi = \varphi(\bar x, \bar a), \psi = \psi(\bar x, \bar b)$ is contradictory (see definition 2.9) iff $\varphi^{\mathfrak{C}} \cap \psi^{\mathfrak{C}} = \emptyset$.*
2) [$\mathfrak{C}$ not necessarily compact] A pair as in (1) φ, ψ is contradictory iff for some $\varepsilon > 0, \varphi^{[\varepsilon]} \cap \psi^{[\varepsilon]} = \emptyset$.
3) If φ, ψ in (2) are without parameters, we can add "iff $\varphi^{<\varepsilon>} \cap \psi^{<\varepsilon>} = \emptyset$ for some $\varepsilon > 0$".
4) If $\mathfrak{C}$ is compact, (3) holds for formulae with parameters.

Proof. 1) Obviously if (φ, ψ) is a contradictory pair then $\varphi^{\mathfrak{C}} \cap \psi^{\mathfrak{C}} = \emptyset$. Now assume $\mathbf{d}_1(\varphi, \psi) = 0$. So for each $n > 0$ there exists $\bar a_n, \bar b_n \in \mathfrak{C}$ such that $\mathbf{d}(\bar a_n, \bar b_n) < \frac{1}{n}$ and $\varphi(\bar a_n, \bar a), \psi(\bar b_n, \bar b)$ hold. Therefore the set $\{\varphi(\bar x, \bar a), \psi(\bar y, \bar b)\} \cup \{\mathbf{d}(\bar x, \bar y) \le \frac{1}{n} : n = 1, 2, \dots\}$ is finitely consistent. By compactness we obtain $\bar a^*, \bar b^*$ realizing it, but necessarily $\mathbf{d}(\bar a^*, \bar b^*) = 0$, so $\bar a^* = \bar b^*$ realizes both φ and ψ and we are done.
2) Trivial.
3) Follows from (2) and 3.4(3).
4) Assume $\varphi^{<\varepsilon>} \cap \psi^{<\varepsilon>} \ne \emptyset$ for all $\varepsilon > 0$, so the set $\{\varphi^{<\varepsilon>}(\bar x, \bar a) \wedge \psi^{<\varepsilon>}(\bar x, \bar b) : \varepsilon > 0\}$ is finitely satisfiable (by 3.4(4)) and therefore realized in $\mathfrak{C}$, now by 3.5(3) $\varphi \cap \psi \ne \emptyset$, therefore φ and ψ are not contradictory. The other direction is trivial. $\square_{3.7}$

3.8 Claim. *[Let $(\mathfrak{C}, \mathbf{d})$ be a compact momspace], p a type over a set A. Then $\bar a \in \mathfrak{C}$ realizes $p^{[\varepsilon]}$ iff there exists $\bar a' \in \mathfrak{C}$ realizing $p, \mathbf{d}(\bar a, \bar a') \le \varepsilon$ (so $p^{[\varepsilon]}$ is just the "ε-neighborhood" of p).*

Proof. The if direction is obvious. Now suppose $\bar a \models p^{[\varepsilon]}$, so $[\varphi(\bar a)]^{[\varepsilon]}$ holds for all $\varphi \in p$, but there is no $\bar a' \models p$ such that $\mathbf{d}(\bar a, \bar a') \le \varepsilon$. Then the set

$$\{\mathbf{d}(\bar x', a') \le \varepsilon\} \cup \{\varphi(\bar x') : \varphi \in p, \varphi \text{ is a } \Delta(\mathfrak{C})\text{-formula}\}$$

is inconsistent (as Δ is full for $\mathfrak{C}$), and by compactness we get a contradiction (because in the definition of $p^{[\varepsilon]}$ we essentially close p under conjunctions).

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We connect the relevant notion of submodel defined in 2.19 with the "logical" notion of approximation:

3.9 Observation. $M \prec^1_\Delta N$ (M, N structures, see 2.19 for the definition of $\prec^1_\Delta$) iff for every $\varphi(\bar{y}, \bar{x}) \in \Delta, \varepsilon > 0$, if $N \models \exists \bar{y} \varphi(\bar{y}, \bar{a}), \bar{a} \in M$, then $N \models \varphi^{<\varepsilon>}(\bar{b}, \bar{a})$ for some $\bar{b} \in M$.

Note that the assumption above of M being complete in fact follows from $(D, \aleph_1)$-homogeneous:

3.10 Observation. Let $M \in K_1$ be $(D, \aleph_1)$-homogeneous, then $M \in K_1^c$, i.e., is complete.

Proof. Let $\langle a_n : n < \omega \rangle$ converge to $a \in \mathfrak{C}, a_n \in M$. So the set $\{\mathbf{d}(x, a_n) \leq \mathbf{d}(a, a_n) : n \in \mathbb{N}\}$ is realized in M, so $\langle a_n : n < \omega \rangle$ converges in M (to the same limit, of course). $\square_{3.10}$

3.11 Definition. We call $M \in K_1^c$ pseudo (D, λ)-homogeneous if for every $A \subseteq M, |A| < \lambda$, for every $p \in S^m_D(A)$, for every $\varepsilon > 0, p^{<\varepsilon>}(\lambda)$ is realized in M (see Definition 3.3).
Note that replacing $p^{<\varepsilon>}$ above by $p^{[\varepsilon]}$, we get (D, λ)-homogeneous, see 3.13 below.

3.12 Definition. We say that a type $p \in S(A)$ is <u>almost realized</u> in B (or B almost realizes p) if for all $\varepsilon > 0$ there exists $b_\varepsilon \in B, b_\varepsilon \models p^{[\varepsilon]}$.

3.13 Claim. *[$(\mathfrak{C}, \mathbf{d})$ compact] Let $M \in K_1^c$ be such that every 1-type over a subset of M of cardinality less than λ is almost realized in M (λ infinite). Then M is (D, λ)-homogeneous.*

Proof. Let $p \in S^1_D(A), A \subseteq M, |A| < \lambda$. Choose by induction $b_n \models p^{[\frac{1}{2^n}]}, b_n \in M, \mathbf{d}(b_{n+1}, b_n) \leq \frac{1}{2^{n-1}}$.

Why is this possible? Let $q_n = p(x) \cup \{\mathbf{d}(x, b_n) \leq \frac{1}{2^n}\}$. As $b_n \models p^{[\frac{1}{2^n}]}$, some $b^* \in \mathfrak{C}$ realizes q_n (see 3.8) and let $q^*_n = \text{tp}_D(b^*/A \cup \{b_n\})$, so $q^*_n \in S^1_D(A \cup \{b_n\})$, and by the assumption there exists $b_{n+1} \in M$ realizing $(q^*_n)^{[\frac{1}{2^{n+1}}]}$. So $b_{n+1} \models p^{[\frac{1}{2^{n+1}}]}, \mathbf{d}(b_{n+1}, b_n) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} \leq \frac{1}{2^{n-1}}$, as required.

Now, $\langle b_n : n < \omega \rangle$ is a Cauchy sequence, let $b \in M$ be its limit ($M \in K_1^c$, so is complete). Now $b \models p^{[\varepsilon]}$ for all $\varepsilon > 0$, so $b \models p$ (see 3.5(1)), and we are done.

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3.14 Corollary. *Let $M \in K_1^c$ be non-(D,λ)-homogeneous. Then there exists $A \subseteq M, |A| < \lambda, p \in S^1_D(A)$ and $\varepsilon > 0$ such that $p^{[\varepsilon]}$ is omitted in M.*

Proof. This is just restating 3.13; but we prefer this form for later use.

3.15 Remark. We will not use the notion of pseudo homogeneity (Definition 3.11), in this paper (as a postoriori all the models will turn out to be (D,λ)-homogeneous), but it is interesting to point out what non-categorical classes can look like. See 4.23 later.

3.16 Claim. *If $A \subseteq \mathfrak{C}, p \in \mathbf{S}_D(\mathrm{mcl}(A))$ and $c \in \mathfrak{C}$ realizes $p \restriction A$ then it realizes p.*

Proof. If $\varphi(x,a_1,\dots,a_m) \in p$ ($\varphi \in \Delta$ of course, parameters not suppressed) then for each $n < \omega$ we can choose $b^n_\ell \in A$ (for $\ell = 1,\dots,m$) such that $\mathbf{d}(a_\ell, b^n_\ell) < \frac{1}{n+2}$. Hence $\varphi(x,b^n_1,\dots,b^n_m)^{<1/n+2>}$ belongs to p as $\varphi(x,a_1,\dots,a_m)$ implies it, hence it belongs to $p \restriction A$.

So if $c \in \mathfrak{C}$ realizes $p \restriction A$ then $\models \varphi(c,b^n_1,\dots,b^n_m)^{<1/n+2>}$ for each n. If $\mathfrak{C} \models \neg\varphi[c,a_1,\dots,a_m]$ then some $\psi, \psi = \psi(y,x_1,\dots,x_m) \in \Delta$ and $\varphi(y,x_1,\dots,x_n)$ are contradictory and $\mathfrak{C} \models \psi[c,a_1,\dots,a_n]$. So for all n, the tuples $cb^n_1,\dots,b^n_m$ lies in both $(\varphi^{<\frac{1}{n+2}>})^{\mathfrak{C}}$ and $(\psi^{<\frac{1}{n+2}>})^{\mathfrak{C}}$, therefore $\varphi^{<\varepsilon>} \cap \psi^{<\varepsilon>} \neq \emptyset$ for all $\varepsilon > 0$, which contradicts φ and ψ being a contradictory pair, see 3.7(4).

§4 Stability in momspaces

We define a topological version of stability. The intuition behind the definition is that there may be many types, but the density of the space of types is small. Our definition generalizes Iovino's stability for Banach spaces, see [Io99]. For Hausdorff cats and continuous theories, it coincides with definitions given in [BY05], [BeUs0y] respectively. Note that for elementary homogeneous class (i.e. discrete metric), this definition coincides with the usual one (so certainly for an elementary class, i.e., $\mathfrak{C}$ saturated). For a non-discrete metric the classical λ-stability of D (counting types, as in [Sh 3]) is stronger than the topological relative we define here, and is equivalent if and only if $\lambda = \lambda^{\aleph_0}$. Stability in our sense (i.e., λ-stable for some λ) is equivalent to stability for D, but for a specific λ (e.g. $\lambda = \aleph_0$) the notions differ.

4.1 Hypothesis. $(\mathfrak{C}, \mathbf{d})$ is a momspace for $\Delta = \Delta_{\mathfrak{C}}$.

4.2 Definition. 1) A momspace $(\mathfrak{C}, \mathbf{d})$ is called $\underline{0^+ - \lambda\text{-stable}}$ if for all $A \subseteq \mathfrak{C}$ of cardinality λ there exists $B \subseteq \mathfrak{C}$ of cardinality λ such that each $p \in \mathbf{S}_D(A)$ is realized in $\text{mcl}(B) = \bar{B}$ (i.e., the topological closure of B).

4.3 Claim. *[$(\mathfrak{C}, \mathbf{d})$ compact] For a compact momspace $\mathfrak{C}$, the following are equivalent:*

- (A) *$\mathfrak{C}$ is $0^+ - \lambda$-stable*
- (B) *for each $A \subseteq \mathfrak{C}, |A| \leq \lambda$, there exists $B, |B| \leq \lambda$ such that for each $\varepsilon > 0$ and $p \in \mathbf{S}_D(A), p^{[\varepsilon]}$ is realized in B (B almost realizes all types over A, see Definition 3.12)*
- (C) *there are no $A, |A| \leq \lambda, \langle p_i : i < \lambda^+ \rangle$ a sequence of members of $\mathbf{S}_D(A)$, i.e., complete D-types over A and $\varepsilon > 0$ such that $p_i^{[\varepsilon]} \cap p_j^{[\varepsilon]} = \emptyset$ for all $i, j < \lambda^+$.*

Proof. $(A) \Rightarrow (C)$.

Assume (A). Suppose (C) fails, so we have $\langle p_i : i < \lambda^+ \rangle$ over A as there. By (A) we can find B of cardinality $\leq \lambda$ such that each p_i is realized in $\text{mcl}(B) = \bar{B}$, by, say, a_i. As $\{a_i : i < \lambda^+\}$ form an ε-net (by the nature of the p_i's), it is obvious that density of $\bar{B}$ is at least λ^+, contradiction ($|B| = \lambda$).

$(C) \Rightarrow (B)$.

Let A be given. We construct B_n by induction:

$-B_0 = A$

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$-B_{n+1}$ realizes all complete D-types over B_n up to $\frac{1}{n+1}$, i.e.

if $p \in S_D(B_n)$ then for some $a \in B_{n+1}$, $\mathrm{tp}(a, B_n)^{[\frac{1}{n+1}]}(\mathfrak{C}) \cap p^{[\frac{1}{n+1}]}(\mathfrak{C}) \neq \emptyset$

$-|B_n| = \lambda$

$B = \cup B_n$ is obviously as required in (B).

How is B_{n+1} constructed? Let $\varepsilon = 1/2(n+1)$ and let $\langle p_i^k : i < \lambda \rangle$ be a maximal ε-disjoint set of complete types of κ-tuples over B_n, (I.e. $(p_i^k)^{[\varepsilon]} \cap (p_j^k)^{[\varepsilon]} = \emptyset$ for all i, j)

$B_{n+1} = B_n \cup \{\bar{a}_i^k : \bar{a}_i^k$ realizes p_i^k for some $i < \lambda, \kappa < \omega\}$.

Now each complete k-type p over B_n is $\frac{1}{n+1}$-realized in B_{n+1}, as there is $i < \lambda$ such that $(p_i^k)^{[\varepsilon]}(\mathfrak{C}) \cap p^{[\varepsilon]}(\mathfrak{C})$ is non-empty, let $\bar{b}$ be in the intersection. We can replace $\bar{b}$ by any $\bar{b}'$ realizing $\mathrm{tp}(b, B_n)$ hence without loss of generality $\mathbf{d}(\bar{b}, a_i^k) < \varepsilon$. Also we can find $\bar{c}$ realizing p such that $\mathbf{d}(\bar{b}, \bar{c}) < \varepsilon$. So $\bar{a}_i^k$ witnesses p is $\frac{1}{n+1}$-realized in B_{n+1}.

(B) $\Rightarrow$ (A).

Let A be given. Define $B_0 = A, B_{n+1}$ almost realizes all types over $B_n, B_n \subseteq B_{n+1}, |B_n| = \lambda$. Let $B = \bigcup_{n<\omega} B_n$. Let $p \in \mathbf{S}_D(A)$. Pick by induction on $n \geq 1, a_n \in B_n$ such that $a_n \models p^{[\frac{1}{n}]}$ and $\mathbf{d}(a_n, a_{n-1}) \leq \frac{1}{2^n}$ if $n > 1$ (possible by 3.8 as B_n almost realizes all types over $\bigcup_{i<n} B_n$). $\langle a_n : n < \omega \rangle$ is obviously a Cauchy sequence, let a be its limit in $\bar{B}$ and we are done by 3.5(1) as $a \models p^{[\frac{1}{n}]}$ for all n.

4.4 Observation. 1) If $\mathfrak{C}$ is λ-stable (in the sense of D, see [Sh 3]), then it is $0^+ - \lambda$-stable.
2) If $\mathfrak{C}$ is $0^+ - \lambda$-stable and $\lambda = \lambda^{\aleph_0}$ then $\mathfrak{C}$ is λ-stable.

Proof. 1) Trivial.
2) Recall $|\bar{B}| = |B|^{\aleph_0}$.

Recall

4.5 Definition. A (partial) type q <u>splits</u> over a set A if there are $\varphi(\bar{x}, \bar{a}), \psi(\bar{x}, \bar{b}) \in q, \varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})$ contradictory, and $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$.

4.6 Notation. We write $\bar{a} \equiv_A \bar{b}$ for $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$.

4.7 Observation. For a complete type $p, p^{[\varepsilon]}$ splits over $A \Leftrightarrow$ there exist $\varphi(\bar{x}, \bar{a}), \psi(\bar{x}, \bar{b}) \in p, \mathbf{d}_1(\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})) > 2\varepsilon$, $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$.

Proof. $p^{[\varepsilon]}$ splits over $A \Leftrightarrow$ there exist $\varphi^{[\varepsilon]}(\bar{x},\bar{a}),\psi^{[\varepsilon]}(\bar{x},\bar{b}) \in p^{[\varepsilon]}$ (where $\varphi(\bar{x},\bar{a}),\psi(\bar{x},\bar{b}) \in p$) such that $\bar{a} \equiv_A \bar{b}$ and $(\varphi^{[\varepsilon]}(\bar{x},\bar{y}),\psi^{[\varepsilon]}(\bar{x},\bar{y}))$ is a contradictory pair [so (φ,ψ) is a 2ε-contradictory pair].

4.8 Observation. 1) The type p splits over A iff for some $\varepsilon > 0$, the type $p^{[\varepsilon]}$ splits over A.

Proof. p splits over A iff some contradictory pair (φ,ψ) witnesses this, now pick $\varepsilon = \frac{\mathbf{d}_1(\varphi,\psi)}{3}$ and use 4.7.

4.9 Lemma. *Let $\mathfrak{C}$ be $0^+ - \aleph_0$-stable. Then for any $B \subseteq \mathfrak{C}, p \in \mathbf{S}_D(B)$ and $\varepsilon > 0, p^{[\varepsilon]}$ does not split over a finite subset of B.*

Proof. Suppose $p^{[\varepsilon]}$ splits over every finite subset of its domain. We construct finite sets A_n for $n < \omega$ and elementary maps F_η for $\eta \in {}^{\omega >}2$ as follows:

1. $A_0 = \emptyset$
2. $A_{n+1} = A_n \cup \{\bar{a}_n,\bar{b}_n\}$, where there are $\varphi_n(\bar{x},\bar{y}),\psi_n(\bar{x},\bar{y})$ such that $\varphi_n(\bar{x},\bar{a}_n),\psi_n(\bar{x},\bar{b}_n)$ exemplify $p^{[\varepsilon]}$ splits over A_n, i.e. $\bar{a}_n \equiv_{A_n} \bar{b}_n, \varphi_n(\bar{x},\bar{a}_n),\psi_n(\bar{x},\bar{b}_n) \in p^{[\varepsilon]}$ and $\varphi_n^{[\varepsilon]}(\mathfrak{C},\bar{a}_n) \cap \psi_n^{[\varepsilon]}(\mathfrak{C},\bar{b}_n) = \emptyset$
3. for $\eta \in {}^n 2, F_\eta : A_n \to \mathfrak{C}$ is an elementary mapping
4. for $\eta \in {}^n 2, F_{\eta\frown\langle 0\rangle}(\bar{a}_n) = F_{\eta\frown\langle 1\rangle}(\bar{b}_n)$ and $F_{\eta\frown\langle 0\rangle}, F_{\eta\frown\langle 1\rangle}$ extend F_η.

The construction is straightforward. Now denote for $\eta \in {}^\omega 2, F_\eta = \bigcup\limits_{n<\omega} F_{\eta\restriction n}, p^*_\eta = F_\eta(p), A = \cup\{\mathrm{Rang}(F_\eta) : \eta \in {}^{\omega>}2\}$ (so it is countable) and choose[1] $p_\eta, p^*_\eta \subseteq p_\eta \in \mathbf{S}_D(A)$. Obviously, $\eta \neq \nu \in {}^\omega 2 \Rightarrow p^{[\varepsilon]}_\eta \cap p^{[\varepsilon]}_\nu = \emptyset$, contradicting $0^+ - \aleph_0$-stability by 4.3 by an implication not using compactness.

4.10 Claim. *Assume $(\mathfrak{C},\mathbf{d})$ is $0^+ - \aleph_0$-stable. Then*

(a) *if $p \in \mathbf{S}_D(B)$ then p does not split over some countable $A \subseteq B$*

(b) *D is stable (in the sense of [Sh 3]), $\kappa(D) \leq \aleph_1$.*

---

[1] this uses "D is good"

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Remark. Note that it is close but not as in first order; there may be $\aleph_0$ exceptions.

Proof.

Clause (a):
By the previous claim for every $\varepsilon > 0$ for some finite $B_\varepsilon \subseteq A, p$ does not ε-split over B_ε. Let $B = \cup\{B_{1/(n+1)} : n < \omega\}$, so B is a countable subset of A and by 4.8 and the obvious monotonicity of non-ε-splitting, p does not split over B.

Clause (b): Follows from (a).

4.11 Definition. Given an uncountable indiscernible set $\mathbf{I} \subseteq {}^m\mathfrak{C}$ and a set $A \subseteq \mathfrak{C}$, define the average type of $\mathbf{I}$ over A, $\mathrm{Av}(A, \mathbf{I})$ as follows:

$$\mathrm{Av}(A,\mathbf{I}) = \{\varphi(\bar{x},\bar{a}) : \varphi(\bar{x},\bar{y}) \in \Delta, \ell g(\bar{x}) = m, \bar{a} \in A, \text{ and for infinitely many } \bar{c} \in \mathbf{I}, \varphi(\bar{c},\bar{a}) \text{ holds}\}.$$

4.12 Fact. If $\mathfrak{C}$ is stable, then any indiscernible sequence is an indiscernible set.

Proof. Standard.

We often say "$\mathbf{I}$ is indiscernible" meaning an indiscernible sequence, which is the same as indiscernible set.

4.13 Claim. *Let $\mathfrak{C}$ be $0^+ - \aleph_0$-stable.*
1) If $\mathbf{I} \subseteq {}^m\mathfrak{C}$ is indiscernible uncountable, $A \subseteq \mathfrak{C}$ a set, then $\mathrm{Av}(A, \mathbf{I}) \in S^m_D(A)$.
2) If $A, \mathbf{I}$ are as in (1), then

$$Av(A,\mathbf{I}) = \{\varphi(\bar{x},\bar{a}) : \varphi(\bar{x},\bar{y}) \in \Delta, \ell g(\bar{x}) = m, \bar{a} \in A \textit{ and for all but countably many } \bar{c} \in \mathbf{I} \textit{ does } \varphi(\bar{c},\bar{a}) \textit{ hold}\}.$$

Proof. Using the standard argument, one shows that for a given contradictory pair $(\varphi,\psi) = (\varphi(\bar{x},\bar{z}),\psi(\bar{x},\bar{z}))$ and $\bar{a} \in M$, one of the sets $\{\bar{c} \in \mathbf{I} : \models \varphi(\bar{c},\bar{a})\}, \{\bar{c} \in \mathbf{I} : \models \psi(\bar{c},\bar{a})\}$ is finite (otherwise, let $\varepsilon = \mathbf{d}_1(\varphi,\psi)$, and construct $2^{\aleph_0}$ ε-distant types over a countable set, contradictory $0^+ - \aleph_0$-stability; in fact, one constructs $2^{\aleph_0}$ pairwise distinct (φ,ψ)-types, i.e., types mentioning only φ and ψ).

Now given a formula $\varphi(\bar x,\bar a)$ over M, if $\{\bar c \in \mathbf{I} : \varphi(\bar c,\bar a) \text{ holds}\}$ is infinite, then for each $\psi(\bar x,\bar z)$ such that (φ,ψ) is contradictory, the set $J_\psi = \{\bar c \in \mathbf{I} : \psi(\bar c,\bar a) \text{ holds}\}$ is necessarily finite, so taking the union of J_ψ over all such ψ, we obtain a countable set of exceptions

$$J = \cup J_\psi = \{\bar c \in \mathbf{I} : \models \neg\varphi(\bar c,\bar a)\}.$$

This completes the proof of clause (2). For clause (1), let $\varphi(\bar x,\bar a)$ be a formula over $A, \ell g(\bar x) = m$. If for uncountably many $\bar c \in \mathbf{I}, \varphi(\bar c,\bar a)$ holds, then $\varphi(\bar x,\bar a) \in \text{ Av}(A,\mathbf{I})$, otherwise for some $\psi(\bar x,\bar a)$ such that (φ,ψ) is a contradictory pair, $\psi(\bar c,\bar a)$ holds for uncountably many $\bar c \in \mathbf{I}$, so $\psi(\bar x,\bar a) \in \text{ Av}(A,\mathbf{I})$. Clearly, only one of the two options above is possible, so (1) follows.

Discussion: Why countable and not finite in the definition of averages? Even if the majority satisfies $\varphi(\bar x,\bar a)$, for each ε there can be finitely many $\bar c \in \mathbf{I}$ such that $\mathfrak{C} \models \psi(\bar c,\bar a)$ and (φ,ψ) are ε-contradictory, and this finite number can increase when ε goes to 0.

4.14 Lemma. *If M is $(D,\aleph_1)$-homogeneous, $p \in \mathbf{S}^m(M)$ then for some uncountable $\mathbf{I} \subseteq {}^m M$, we have $p = \text{Av}(M,\mathbf{I})$.*

Proof. Let B be as in 4.10, clause (a). Let $m = 1$ for simplicity. Choose $a_\alpha \in M$ realizing $p \restriction B \cup \{a_\beta : \beta < \alpha\}$ by induction on $\alpha < \omega_1$. Now $\mathbf{I} = \langle a_\alpha : \alpha < \omega_1\rangle$ is indiscernible over B by [Sh:c, I,§2].

If $q = \text{Av}(M,\mathbf{I}) \neq p$ still $q \in \mathbf{S}_D(M)$ (by 4.13(1)) and we can find $\varphi(x,\bar b) \in q, \psi(x,\bar b) \in p$ and they are contradictory. So $u = \{\alpha < \omega_1 : \mathfrak{C} \models \varphi[a_\alpha,\bar b]\}$ is infinite let $v \subseteq u$ be of cardinality $\aleph_0$, let $v \subseteq \alpha(*) < \omega_1$ and choose $a'_\beta \in M(\beta \in [\alpha(*),\omega_1)$ realizing $p \restriction (B \cup \{a_i : i < \alpha(*)\} \cup \bar b \cup \{a'_\gamma : \gamma \in (\alpha(*),\beta)\}$.
Easy contradiction: for a given contradictory pair (φ,ψ) all but finitely many elements of $\mathbf{I}$ have to "make a choice", see also 4.13(1).

4.15 Definition. 1) A momspace $(\mathfrak{C},\mathbf{d})$ is called $(\mu,*)$-superstable if given $\langle M_i : i < \omega\rangle$ an increasing chain of (D,μ)-homogeneous models ($M_i \in K_1^c$, of course), $\text{mcl}(\bigcup\limits_{i<\omega} M_i)$ is (D,μ)-homogeneous.
2) We omit μ if this holds for every μ large enough.
3) Fully $*$-superstable means for every $\mu > |\tau_{\mathfrak{C}}| + \aleph_0$.

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Remark. This definition generalizes superstability for $\mathfrak{C}$ a saturated model of a first order theory, and $\mathfrak{C}$ a homogeneous monster.

The following claim will be mainly of interest for us when $\mu = \aleph_0$:

4.16 Claim. *Let $(\mathfrak{C}, \mathbf{d})$ be $0^+ - \aleph_0$-stable and compact. Then $(\mathfrak{C}, \mathbf{d})$ is $(\mu^+, *)$-superstable for every $\mu \geq \aleph_0$.*

Proof. Let $\langle M_n : n < \omega\rangle$ be an increasing sequence of (D, μ^+)-homogeneous models, and assume $p \in \mathbf{S}(A)$, $A \subseteq \overline{\bigcup_{n<\omega} M_n}$ of cardinality $\leq \mu$ is not realized in $M_\omega = \overline{\bigcup M_n}$.

By increasing A and the $0^+ - \aleph_0$-stability (i.e., for every $\varepsilon > 0$ trying to build a tree $\langle p_\eta : \eta \in {}^{\omega>}2\rangle$ of ε-contradictory types) without loss of generality p has a unique extension in $\mathbf{S}_D(M_\omega)$, call it q. By 3.14 as $(\mathfrak{C}, \mathbf{d})$ is compact we can add that for some $\varepsilon > 0, p^{[\varepsilon]}$ is not realized in M_ω. Without loss of generality $A \subseteq \mathrm{mcl}(A \cap \bigcup_{n<\omega} M_n)$ hence by 3.16 without loss of generality $A \subseteq \bigcup_{n<\omega} M_n$ (as p is determined by its restriction to $A \cap (\bigcup_{n<\omega} M_n)$).

Now by 4.9 there is a finite $B \subseteq A$ over which $q \restriction \cup\{M_n : n < \omega\}$ does not $(\varepsilon/5)$-split. Let $n < \omega$ be such that $B \subseteq M_n$ without loss of generality $q \restriction M_n$ does not split over $A \cap M_n$ (by increasing A and 4.10(a)). Let $q_n = q \restriction M_n$ and let $A_n = M_n \cap A$. As M_n is (D, μ^+)-homogeneous, by Lemma 4.14, there is an uncountable indiscernible sequence $\mathbf{I}_n$ in M_n with $\mathrm{Av}(M_n, \mathbf{I}_n) = q_n$; without loss of generality $|\mathbf{I}_n| = \mu^+$, and $\mathbf{I}_n$ is indiscernible over A (not just A_n!) (as for each finite type $\bar{b}$ from A, $\mathrm{tp}(\bar{b}/\mathbf{I}_n)$ does not split over a countable subset, so we can remove a subset of $\mathbf{I}_n$ of cardinality μ). Now since elements of $\mathbf{I}_n$ do not realize $p^{[\varepsilon]}$, for some formula $\varphi(\bar{x}, \bar{a}) \in p$ (really $\bar{x}$ is a singleton) there exists $\psi(\bar{x}, \bar{z})$ such that $\mathbf{d}_1(\varphi, \psi) > \varepsilon$ and $\forall \bar{c} \in \mathbf{I}_n, \psi(\bar{c}, \bar{a})$.

For $k < \omega$ let $\bar{c}_k \in \mathbf{I}_n$ be pairwise distinct. We can find $\bar{a}' \in M_n$ which realizes $\mathrm{tp}(\bar{a}, A_n \cup B \cup \cup\{\bar{c}_k : k < \omega\})$. So by Claim 4.13 we know that for all but countably many $\bar{c} \in \mathbf{I}_n$ we have $\mathfrak{C} \models \psi[\bar{c}, \bar{a}']$ (as this happens for $\{\bar{c}_k : k < \omega\} \subseteq I_n$), hence $\psi(\bar{x}, \bar{a}') \in q_n \subseteq q$.

Now we obtain: for some $m > n, \bar{a} \in M_m$ hence $(\varphi(\bar{x}, \bar{a}), \psi(\bar{x}, \bar{a}'))$ witness that $q \restriction \cup\{M_\ell : \ell < \omega\}$ does ε-split over B, hence q does, which is a contradiction to the choice of B. $\square$

We shall now proceed to proving an analogue of density of isolated types. As the "right" notion of a type in our context seems to be an ε-neighborhood of a complete type, the assumption of "non-isolated" will not be enough for us.

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4.17 Definition. We say that $M \in K_1$ $(< \varepsilon)$-omits $p(\bar{x})$, a type over $A \subseteq M$, when for no $\zeta \in [0, \varepsilon)_{\mathbb{R}}$ and $\bar{b} \in M$, does $\bar{b}$ realize $p^{[\zeta]}$.

4.18 Definition. 1) We say that a formula $\psi(\bar{x}, \bar{b})$ pseudo (ε, ζ) isolates a type $p(\bar{x})$ if $\psi^{<\zeta>}(\bar{x}, \bar{b}) \models p^{[\varepsilon]}(\bar{x})$ (note that the roles of ε, ζ are not symmetric and the different notions of approximation!). In other words, if $\mathfrak{C} \models \psi^{<\zeta>}[\bar{a}, \bar{b}]$ then for some $\bar{a}' \in p(\mathfrak{C})$ we have $\mathbf{d}(\bar{a}', \bar{a}) \leq \varepsilon$ (so $\bar{a}'$ realizes $p, \bar{a}$ realizes $\psi^{<\zeta>}(\bar{x}, \bar{b})$).
2) We say that A is a pseudo $(< \varepsilon)$-support for $p(\bar{x})$ or A pseudo $(< \varepsilon)$-supports p when there is a consistent $\psi(\bar{x}, \bar{b}), \bar{b} \subseteq A$ and positive ζ_1, ζ_2 such that $\psi^{<\zeta_1>}(\bar{x}, \bar{b}) \models p^{[\varepsilon - \zeta_2]}(\bar{x})$. So $\psi(\bar{x}, \bar{b})$ pseudo $(\varepsilon - \zeta_2, \zeta_1)$-isolates $p(\bar{x})$.
3) We say that A really $(< \varepsilon)$-omits $p(\bar{x})$ if it does not pseudo $(< \varepsilon)$-support $p(\bar{x})$.

4.19 Claim. *1) If $M \in K_1^c$ $(< \varepsilon)$-omits $p(\bar{x}), p(\bar{x}) \in \mathbf{S}^m(A), A \subseteq M$ <u>then</u> M really $(< \varepsilon)$-omits $p(\bar{x})$.*
2) If $p(\bar{x})$ is a type over M and M really $(< \varepsilon)$-omits $p(\bar{x})$, then $M (< \varepsilon)$-omits p.

Proof. 1) Assume M is a pseudo $(< \varepsilon)$-support for $p(\bar{x})$, i.e., there exist $\psi(\bar{x}, \bar{b})$ over M and $\zeta_1, \zeta_2 > 0$ such that $\psi^{<\zeta_1>}(\bar{x}, \bar{b}) \models p^{[\varepsilon - \zeta_2]}(\bar{x})$. $\mathfrak{C} \models \exists x \psi(\bar{x}, \bar{b}), M \in K_1^c$, so for some $\bar{a} \in M, \mathfrak{C} \models \psi^{<\zeta_1>}(\bar{a}, \bar{b})$ (see 3.9), therefore $p^{[\varepsilon - \zeta_2]}(\bar{a})$ holds, and we have $p^{(<\varepsilon)}(\bar{x})$ is realized in M.
2) Easier: assume $\bar{a} \models p^{[\varepsilon - \zeta]}(\bar{x})$ for $\zeta > 0, \bar{a} \in M$. Then the formula "$\bar{x} = \bar{a}$" is over M and pseudo $(\varepsilon - \zeta_2, \zeta_1)$-isolates $p(\bar{x})$ for $\zeta_2 = \zeta_1 = \frac{\zeta}{3}$.

4.20 Definition. 1) We say $\varphi(\bar{x}, \bar{b})$ strictly (ε, ζ)-isolates a type p if $\varphi(\bar{x}, \bar{b}) \in p$ and $\varphi(\bar{x}, \bar{b})$ pseudo (ε, ζ)-isolates p.
2) We say $\varphi(\bar{x}, \bar{b})$ is strictly (ε, ζ)-isolating over A when

- (a) $\bar{b} \subseteq A$
- (b) if $\varphi(\bar{x}, \bar{b}) \in p \in \mathbf{S}_D^{\ell g(\bar{x})}(A)$ then $\varphi(\bar{x}, \bar{b})$ strictly isolates p.

3) We say that $p \in S_D^m(A)$ is strictly (ε, ζ)-isolated if some φ strictly (ε, ζ)-isolates it.
3A) "Strictly ε-isolate" means "for some $\zeta > 0$, strictly (ε, ζ)-isolate".
4) We say that $p \in \mathbf{S}_D^m(A)$ is strictly isolated when for every $\varepsilon > 0$ for some $\varphi(\bar{x}, \bar{a}) \in p$ and some $\zeta > 0$ the formula $\varphi(\bar{x}, \bar{a})$ strictly (ε, ζ)-isolates the type p (i.e., p is ε-strictly isolated for all $\varepsilon > 0$).

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4.21 Claim. *[$(\mathfrak{C}, \mathbf{d})$ is compact]*
Assume that

(a) $p \in \mathbf{S}^m_D(A)$ *or just* p *is an* m*-type closed under conjunctions*
(b) $\psi(\bar{x}, \bar{b}) \in p$
(c) $\varepsilon > 0$ *and* $\zeta \geq 0$.

Then one of the following occurs

(α) *there is a pair* $(\psi_1(\bar{x}, \bar{y}), \psi_2(\bar{x}, \bar{y}))$ *of formulas and a sequence* $\bar{b}^*$ *from* A *such that* $\bar{b} \triangleleft \bar{b}^*, \ell g(\bar{b}^*) = \ell g(\bar{y})$ *such that* $\psi(\bar{x}, \bar{b}) \wedge \psi_1(x, \bar{b}^*), \psi^{<\zeta>}(\bar{x}, \bar{b}) \wedge \psi_2(\bar{x}, \bar{b}^*)$ *are* ε*-contradictory (and both consistent, of course), hence for no* $\bar{a}, \mathfrak{C} \models (\exists \bar{x})(\mathbf{d}(\bar{x}, \bar{a}) \leq \varepsilon/2 \wedge \psi(\bar{x}, \bar{b}) \wedge \psi_1(\bar{x}, \bar{b}^*))$ *and* $\mathfrak{C} \models (\exists \bar{x})[\mathbf{d}(\bar{x}, \bar{a}) \leq \varepsilon/2 \wedge \psi^{<\zeta>}(\bar{x}, \bar{b}) \wedge \psi_2(\bar{x}, \bar{b}^*)]$
(β) *for every* $\bar{a}'$ *such that* $\mathfrak{C} \models \psi^{<\zeta>}[\bar{a}', \bar{b}]$ *there is a sequence* $\bar{a}''$ *realizing* p *such that* $\mathbf{d}(\bar{a}', \bar{a}'') \leq \varepsilon$ *(so* $\psi(x, \bar{b})$ *strictly* (ε, ζ)*-isolates* p*, see 4.20).*

Proof. We can assume that clause (β) fails and let $\bar{a}'$ exemplify it. So for every $\bar{a}''$ realizing p we have $\mathbf{d}(\bar{a}'', \bar{a}') > \varepsilon$.

Let $q(\bar{y}) = \text{tp}(\bar{a}', A)$ so $\psi^{<\zeta>}(\bar{y}, \bar{b}) \in q$ as $\mathfrak{C} \models \psi^{<\zeta>}(\bar{a}', \bar{b})$, and let $r(\bar{x}, \bar{y}) = p(\bar{x}) \cup q(\bar{y}) \cup \{\mathbf{d}(\bar{x}, \bar{y}) \leq \varepsilon\}$. If $r(\bar{x}, \bar{y})$ is consistent, so is $r(\bar{x}, \bar{a}')$, and any $\bar{a}''$ realizing $r(\bar{x}, \bar{a}')$ is as required in clause (β), contradicting our assumption. So $r(\bar{x}, \bar{y})$ is inconsistent. As $(\mathfrak{C}, \mathbf{d})$ is compact and $p(\bar{x}), q(\bar{y})$ are closed under conjunctions, $\psi(\bar{x}, \bar{b}) \in p(\bar{x}), \psi^{<\zeta>}(\bar{y}, \bar{b}) \in q(\bar{y})$ and as we can add dummy variants, there are $\bar{b}^* \subseteq A, \bar{b} \triangleleft \bar{b}^*$ and $\psi_1(\bar{x}, \bar{b}^*) \in p(\bar{x}), \psi_2(\bar{y}, \bar{b}^*) \in q(\bar{y})$ such that $\{\psi(\bar{x}, \bar{b}) \wedge \psi_1(\bar{x}, \bar{b}^*), \psi^{<\zeta>}(\bar{x}, \bar{b}) \wedge \psi_2(\bar{y}, \bar{b}^*), \mathbf{d}(\bar{x}, \bar{y}) \leq \varepsilon\}$ is contradictory. $\square$

So we get clause (α).

4.22 Isolation Claim. *[$(\mathfrak{C}, \mathbf{d})$ is $0^+ - \aleph_0$-stable and compact]*
1) If $\bar{a} \subseteq A$ *and* $\varphi(\bar{x}, \bar{a})$ *is consistent and* $\varepsilon > 0$ *then we can find* $\varphi_1(\bar{x}, \bar{a}_1)$ *and* $\zeta > 0$ *such that* $\bar{a}_1 \subseteq A$ *and* $\varphi(\bar{x}, \bar{a}) \wedge \varphi_1(\bar{x}, \bar{a}_1)$ *is strictly* (ε, ζ)*-isolating over* A*, see Definition 4.20.*
1A) Similarly omitting ζ *getting strictly* ε*-isolating.*
2) The set of strictly isolated $p \in S^m_D(A)$ *is dense, i.e. for every* $\varphi(\bar{x}, \bar{a})$ *with* $\bar{a} \in A$*, there exists* $p \in S^m_D(A), \varphi(\bar{x}, \bar{a}) \in p, p$ *is strictly isolated.*

Proof of 4.22. Part (1A) is restating Part (1). Also part (2) follows from part (1) by choosing $\varphi_n(\bar{x}, \bar{a}_n)$ such that $\varphi(\bar{x}, \bar{a}) \wedge \varphi_1(\bar{x}, \bar{a}_1) \wedge \ldots \wedge \varphi_n(\bar{x}, \bar{a}_n)$ is $\frac{1}{n+1}$-isolating

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over A (iterating 1A) and applying compactness of $\mathfrak{C}$. So we concentrate on proving part (1) .

We can choose $\zeta_n > 0$ for $n < \omega$ such that, e.g. $\Sigma\{\zeta_n : n < \omega\} \le \varepsilon/5$

Assume that $\psi(\bar{x}, \bar{b}), \bar{b} \subseteq A$ is a counterexample. Now we choose $\langle \psi_\eta(\bar{x}, \bar{a}_\eta) : \eta \in {}^n 2\rangle$ by induction on n such that

$\boxtimes$ (a) $\bar{a}_\eta \subseteq A$

(b) $\psi_\eta(\bar{x}, \bar{a}_\eta)$ is consistent

(c) $\psi_{<>}(\bar{x}, \bar{a}_{<>}) = \psi(\bar{x}, \bar{b})$

(d) if $\nu^\frown\langle 0\rangle, \nu^\frown\langle 1\rangle \in {}^n 2$ then $\psi_{\nu^\frown\langle 0\rangle}(\bar{x}, \bar{a}_{\nu^\frown\langle 0\rangle}), \psi_{\nu^\frown\langle 1\rangle}(\bar{x}, \bar{a}_{\nu^\frown\langle 1\rangle})$ are ε-contradictory

(e) if $\eta = \nu^\frown\langle 0\rangle \in {}^n 2$ then $\psi_\eta(\bar{x}, \bar{a}_\eta) \models \psi_\nu(\bar{x}, \bar{a}_\nu)$

(f) if $\eta = \nu^\frown\langle 1\rangle \in {}^n 2$ then $\psi_\eta(\bar{x}, \bar{a}_\eta) \models \psi_\nu^{<\zeta_n>}(\bar{x}, \bar{a}_\nu)$.

By 4.21 there is no problem to carry the definition, i.e., having $\psi_\nu(\bar{x}, \bar{a}_\nu)$, clause (β) of 4.21 cannot hold (with $\psi_\nu, \bar{a}_\nu$ here standing for $\psi, \bar{b}$ there) as "$\psi(\bar{x}, \bar{a})$ is a counterexample". Hence clause (α) there holds, let us choose $\psi_{\nu^\frown<\ell>}(\bar{x}, \bar{a}_{\nu^\frown<\ell>})$ for $\ell = 0, 1$.

Now let $\xi_n = \Sigma\{\zeta_m : m \in [n, \omega)\}$, so clearly

(∗) if $n(1) < n(2) < \omega$ and $\eta_\ell \in {}^{n(\ell)}2$ for $\ell = 1, 2$ and $\eta_1 \triangleleft \eta_2$ then $\psi_{\eta_2}^{<\xi_{n(2)}>}(\bar{x}, \bar{a}_{\eta_2}) \models \psi_{\eta_1}^{<\xi_{n(1)}>}(\bar{x}, \bar{a}_{\eta_1})$
[Why? By 3.6 using clauses (e) + (f) of $\boxtimes$ we get $\psi_{\eta_2} \models \psi_{\eta_1}^{<\xi_{n(1),n(2)}>}$, where $\xi_{n(1),n(2)} = \Sigma\{\zeta_m : m \in [n(1), n(2))\}$. Now use 3.6 again.]

Now let $C = \cup\{\bar{a}_\eta : \eta \in {}^{\omega >}2\}$.
Hence

(∗) for $\eta \in {}^\omega 2$ the set $\{\psi_{\eta\restriction n}^{<\xi_n>}(\bar{x}, \bar{a}_{\eta\restriction n}) : n < \omega\}$ is consistent, hence is included in some $p_\eta \in \mathbf{S}(C)$

(∗) if $\nu^\frown\langle \ell\rangle \triangleleft \eta_\ell \in {}^\omega 2$ for $\ell = 0, 1$ then $p_{\eta_1}^{[\varepsilon/5]}(\bar{x}) \cup p_{\eta_2}^{[\varepsilon/5]}(\bar{x})$ is inconsistent.
[Why? By $\boxtimes(d)$ and the choice of ζ_n], a contradiction to $0^+ - \aleph_0$-stability.
$\square_{4.22}$

* * *

The following is not used at present but clarifies non-categoricity. Recall Definition 3.11 and Claim 3.13. Note that 3.14 says that a non-(D, λ)-homogeneuos model omits some $p^{[\varepsilon]}$. Here we clarify what happens in the case of pseudo (D, λ)-homogeneous non-(D, λ)-homogeneous model.

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4.23 Claim. *Assume $M \in K^c_1, \lambda > \aleph_0 + |\tau_{\mathfrak{C}}|$ and M is not (D,λ)-homogeneous. Then $(*)$ or $(**)$*

$(*)$ (a) $N \prec^1_\Delta M, |N| < \lambda$

(b) $p \in \mathbf{S}_D(N)$

(c) *if $\mathfrak{C}$ is $0^+ - \mu$-stable for some $\mu \in [\aleph_0 + |\tau_{\mathfrak{C}}|, \lambda)$ then p has a unique extension in $\mathbf{S}_D(M)$*

(d) *$\varepsilon > 0$ and M omits $p^{<\varepsilon>}$*

$(**)$ (a) *for every $B \subseteq M, |B| < \lambda$ and $q \in \mathbf{S}_D(B)$ and $\zeta > 0$, the type $q^{<\zeta>}$ is realized in M (so M is pseudo λ-saturated)*

(b) $N \prec^1_\Delta M, |\tau_{\mathfrak{C}}| + \aleph_0 \le \|N\| < \lambda$

(c) $p \in \mathbf{S}_D(N)$

(d) *if $\mathfrak{C}$ is $0^+ - \mu$-stable for some $\mu \in [|\tau_{\mathfrak{C}}| + \aleph_0, \lambda)$ then p has a unique extension in $\mathbf{S}_D(M)$*

(e) *$\varepsilon > 0, p^{[\varepsilon]}$ is omitted by M*

(f) *for every n, for some $c_n, \varepsilon_n, \zeta_n$ we have $1/(n+1) > \varepsilon_n > \zeta_n > 0, c_n \in M$ realizes $p^{<\varepsilon_n>}$ and $\mathbf{d}(c_n, p^{<\zeta_n>}(\mathfrak{C})) \ge 10 \times \varepsilon_n - \zeta_n$.*

Proof. Clearly there is $A \subseteq M$ such that $|A| < \lambda$ and $p \in \mathbf{S}^1_D(A)$ is omitted. Let $\mu = |A| + |\tau_{\mathfrak{C}}| + \aleph_0$ so $\mu < \lambda$. If $\mathfrak{C}$ is $0^+ - \mu'$-stable for some $\mu' \in [|\tau_{\mathfrak{C}}| + \aleph_0, \lambda)$, easily without loss of generality p has a unique extension in $\mathbf{S}_D(M)$ and $A = |N|, N \prec^1_\Delta M$. If for some $\varepsilon > 0, p^{<\varepsilon>}$ is also omitted by M, the case $(*)$ holds. So we may assume $(*)$ fails, in other words, clause (a) of $(**)$ holds.

Let $n^* < \omega, \varepsilon^* = \frac{1}{2}$ (any $\varepsilon^* > 0$ works). We are going to find $c_{n^*}, \varepsilon_{n^*}, \zeta_{n^*}$ as required in clause (f) above. First, we try to choose $b_n \in M$ by induction on $n < \omega$ such that

$\circledast_n$ (a) $b_n \in M$

(b) b_n realizes $p^{<\varepsilon^*/(n+1)>}$

(c) if $n = m+1$ then $\mathbf{d}(b_n, b_m) \le 10 \times \varepsilon^*/2^n$.

For $n = 0$ there is $b_0 \in M$ realizing $p^{<\varepsilon^*/1>} = p^{<\varepsilon^*>}$ by clause (a) of $(**)$. So we can begin.

Point 1: We cannot succeed to choose $\langle b_n : n < \omega \rangle$.
Why? Suppose we have succeeded. Then $\langle b_n : n < \omega \rangle$ is a Cauchy sequence and therefore converges to some $b^* \in M$. We will show that $b^* \models p$. If $\varphi(x, \bar{a}) \in p$, then for each $n, \mathfrak{C} \models \varphi^{<\varepsilon^*/(n+1)>}(b_n, \bar{a})$ so there is $\bar{a}_n \in {}^{\omega>}\mathfrak{C}, b'_n \in \mathfrak{C}$ such that

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$\boxtimes$ (a) $\mathbf{d}(b'_n, b_n) \le \varepsilon^*/(n+1)$

(b) $\mathbf{d}(\bar{a}_n, \bar{a}) \le \varepsilon^*/(n+1)$

(c) $\models \varphi[b'_n, \bar{a}_n]$.

Now

$$\langle b'_n : n < \omega \rangle \text{ converges to } b^*$$

$$\langle \bar{a}_n : n < \omega \rangle \text{ converges to } \bar{a}$$

hence $\models \varphi[b^*, \bar{a}]$. So $b^* \models p, b* \in M$, a contradiction.

Point 2:

So we are stuck in some $n = m+1$ so let $c_{n^*} := b_m, \varepsilon_{n^*} = \varepsilon^*/2^n, \zeta_{n^*} = \varepsilon^*/2^{n+2}$. If the demand in (f) of $(**)$ fails, then there is $b'_m \in \mathfrak{C}$ realizing $p^{<\zeta_{n^*}>}(\bar{x}) \cup \{\mathbf{d}(x, b_m) \le 10 \times \varepsilon_{n^*} - \zeta_{n^*}\}$ hence $p^{<\zeta_{n^*}>}(x) \cup \{\mathbf{d}(x, b_m) \le 10 \times \varepsilon_{n^*} - \zeta_{n^*}\}$ is consistent hence it is contained in some $q_n \in \mathbf{S}(N \cup \{b_m\})$.

So for every $\zeta > 0$ (we use ζ small enough) there is $b_n \in M$ realizing $q^{<\zeta>}(x)$ (recall we are assuming $(**)(a)$!). So b_n realizes $(p^{<\zeta_{n^*}>})^{<\zeta>}$ hence $p^{<\zeta_{n^*}+\zeta>}$ hence if ζ is small enough, $p^{<\varepsilon^*/(n+1)>}$. Also $\mathbf{d}(b_n, b_m) \le (10 \times \varepsilon_{n^*} - \zeta_{n^*}) + \zeta + \zeta < 10 \times \varepsilon_{n^*} = 10 \times \varepsilon^*/2^n$ (because if (a', b') realized $\mathbf{d}^{<\zeta>}(x, y) \le \xi$ then $\mathbf{d}(a', b') \le \xi + \zeta + \zeta$).

So b_n is as required in $\circledast_n(a) - (c)$ above, so we could have continued choosing the b_n. $\square_{4.22}$

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§5 Ehrenfeucht-Mostowski models

In this section we adapt the technique of constructing Ehrenfeucht-Mostowski models to our context. The reader should have a look at chapter 7 of [Sh:c] for the basic definitions (Φ proper, etc.). The basic idea is the following: we start with $\mathfrak{C}$ in vocabulary τ. Adding skolem functions, we obtain vocabulary τ'. Choosing an indiscernible sequence and taking its type (its EM - "blueprint") Φ, for each order type J we can construct $\mathrm{EM}(J,\Phi)$ (like in chapter 7 of [Sh:c]), which will be an elementary submodel of $\mathfrak{C}$ expanded to τ', therefore its restriction to τ, $\mathrm{EM}_\tau(J,\Phi)$ is an elementary submodel of $\mathfrak{C}$, although not necessarily complete. Taking the completion, we obtain a model in K_1^c. Adding more structure to the language we can make it (D,λ)-homogeneous, and more, see below.

Given a vocabulary τ^* with skolem functions, and a τ^*-diagram of indiscernibles Φ (EM-blueprint), we denote for each order-type I, the EM-model (the τ^*-skolem hull of a sequence $\langle a_i : i \in I\rangle$) by $\mathrm{EM}_{\tau^*}(I,\Phi)$ or $\mathrm{EM}(I,\Phi)$ if τ^* is clear from the context. We denote by $\mathrm{EM}_{\tau_0}(I,\Phi)$ the restriction of $\mathrm{EM}(I,\Phi)$ to the vocabulary $\tau_0 \subseteq \tau^*$.

Let $\mathfrak{C}$ be a momspace. Let τ be the vocabulary of $\mathfrak{C}$. It is clear that for any τ^* (with skolem functions) expanding τ, a τ^*-diagram of indiscernibles Φ (in $\mathfrak{C}$ expanded to τ^*), I an order, we can think of $\mathrm{EM}_\tau(I,\Phi)$ as an elementary submodel of $\mathfrak{C}$, so $\mathrm{EM}_\tau(I,\Phi) \prec_{\mathbb{L}(\tau(\mathfrak{C}))} \mathfrak{C}$. This is not necessarily true for the completion, but $\overline{\mathrm{EM}_\tau(I,\Phi)} \prec^1_\Delta \mathfrak{C}$ by 2.20.

5.1 Claim. *Let $(\mathfrak{C},\mathbf{d})$ be a momspace, τ the vocabulary of $\mathfrak{C}, |\tau| \leq \aleph_0, \tau' \supseteq \tau, \tau'$ with Skolem functions, Φ' a τ'-blueprint.*
0A) For every linear order J, $\mathrm{EM}_\tau(J,\Phi') \in K_1$
and $\mathrm{mcl}(\mathrm{EM}_\tau(J,\Phi')) \in K_1^c$.

0B) If $J_1 \subseteq J_2$ then $\mathrm{EM}_\tau(J_1,\Phi') \prec^1_\Delta \mathrm{EM}_\tau(J_2,\Phi')$*; moreover* $\mathrm{EM}_\tau(J_1,\Phi') \prec \mathrm{EM}_\tau(J_1,\Phi')$ *and* $\mathrm{mcl}(\mathrm{EM}_\tau(J_1,\Phi')) \prec^1_\Delta \mathrm{mcl}(\mathrm{EM}_\tau(J_2,\Phi'))$.
1) There exists τ^* expanding $\tau', |\tau^*| = 2^{\aleph_0}$ and a τ^*-diagram Φ^* such that for each finite order J, $EM_\tau(J,\Phi^*)$ is $(D,\aleph_1)$-homogeneous.
2) If $\mathfrak{C}$ is $(\aleph_1,*)$-superstable, then Φ^* as in (1) works for all orders J, but we have to take the closure, i.e., mcl($EM_\tau(J,\Phi^*)$) which $\in K_1^c$ is $(D,\aleph_1)$-homogeneous for all J.
3) If $\mathfrak{C}$ is $0^+ - \omega$-stable, then τ^* in (1) and (2) can be chosen of cardinality $\aleph_1$.

Proof. 0) (A),(B) straight.
1) Choose for $i < \aleph_1, \tau_i, \Phi_i, |\tau_i| = 2^{\aleph_0}$ with skolem functions expanding τ' increasing continuous such that for each τ_i-type p over a finite subset of the skeleton

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of $\mathrm{EM}(I,\Phi_i)$, say $a_1,\dots,a_n$, there exists a function symbol f_p in τ_{i+1} such that $f_p(a_1,\dots,a_n)$ realizes p.

More precisely, we do the following: for any consistent set p of formulas of the form $\varphi=\varphi(x,y_1,\dots,y_n)\in\tau_i$ ($y_1,\dots,y_n$ are the parameters; some of the y_i's may be dummy variables) such that p is closed under conjunctions and for every $\varphi\in p,\exists x\varphi(x,y_1,\dots,y_n)\in\Phi_i$, we add a function symbol f_p to τ_{i+1} such that for every $\varphi\in p$ the following formula is in Φ_{i+1} : $\exists x\varphi(x,y_1,\dots,y_n)\to\varphi(f_p(y_1,\dots,y_n),y_1,\dots,y_n)$.
Now let $\Phi^*=\bigcup\limits_{i<\aleph_1}\Phi_i$ and let $M=\mathrm{EM}_\tau(J,\Phi^*)$ for some finite J. Choose $A\subseteq M$ countable, $p\in S(A)$. As A is countable, it can be viewed as a countable subset of $\mathrm{EM}(J,\Phi_i)$ for some i,p is a type over the finite skeleton, so realized in $\mathrm{EM}(J,\Phi_{i+1})$, therefore in M, as required.
2) By induction on $|J|$. We just need to show that for an increasing sequence of linear orders $J_i,\bigcup\limits_i\mathrm{EM}(J_i,\Phi)=\mathrm{EM}(\bigcup\limits_i J_i,\Phi)$ and this is clear, since elements of $\mathrm{EM}(J,\Phi)$ have finite character, i.e., use only finitely many elements of the skeleton J. Of course, we then have to take metric closure.
3) Let J_n be a linear order with n elements. Similarly to (1), we choose by induction on $i<\omega_1$ countable $\tau_i\subseteq\tau^*$ increasing continuous, closed under skolem functions, and Φ_i such that each type over $\mathrm{EM}(J_n,\Phi_i)$ is almost realized in $\mathrm{EM}(J_n,\Phi_{i+1})$: we choose a countable set B which almost realizes all types over $\mathrm{EM}(J_n,\Phi_i)$, and for each such p and for each κ we have $f_{p,\kappa}\in\Phi_{i+1}$ such that $f_{p,\kappa}(\bar a)$ is $\frac{1}{k}$-close to a realization of φ for each $\varphi\in p$.

5.2 Corollary. *If $\mathfrak{C}$ is $0^+-\omega$-stable, it has a $(D,\aleph_1)$-homogeneous model in all uncountable density characters.*

Proof. Let $\lambda>\aleph_0$. Consider $\mathrm{mcl}(\mathrm{EM}_\tau(\lambda,\Phi^*)),\Phi^*$ as in 5.1(3) (note that $|\Phi^*|\le\aleph_1$) and use 5.1(2) + 4.16.

<u>5.3 Discussion</u>: If $(\mathfrak{C},\mathbf{d})$ is $0^+-\aleph_0$-stable, does it have a (D,λ) homogeneous model in every λ? By 7.6 this follows from categoricity, which is good enough for our purposes.

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§6 Embeddings, isomorphisms and categoricity

In this section we introduce notions of ε-embedding and ε-isomorphism which are weaker than isometry. This will lead us to the notion of weak uncountable categoricity that we investigate in §8.

6.1 Convention. Models are from $K = K_1$.

6.2 Definition. For two metric structures in the same vocabulary τ and $\varepsilon \geq 0$ we say

(1) $f : M_1 \to M_2$ is an ε-embedding if for every Δ-formula $\varphi, \bar{a} \in M_1, M_1 \models \varphi(\bar{a}) \Rightarrow M_2 \models \varphi^{[\varepsilon]}(\bar{a})$
(2) $f : M_1 \to M_2$ is an ε-isomorphism if it is an ε-embedding which is one-to-one and onto
(3) M_1, M_2 are ε^+-isomorphic if there exists a ζ-isomorphism $f_\zeta : M_1 \to M_2$ for all $\zeta > \varepsilon$

6.3 Observation. 1) 0-embedding is a regular notion of (elementary) embedding, 0-isomorphisms is regular isomorphism (in particular isometry).
2) If there exists a ζ-isomorphism $f_\zeta : M_1 \to M_2$ for all $\zeta > \varepsilon$, then there exists a ζ-isomorphism $g_\zeta : M_2 \to M_1$ for all $\zeta > \varepsilon$ (so clause (3) of the definition above makes sense).

Proof. Clear.

The following definition is the central one.

6.4 Definition. Let $(\mathfrak{C}, \mathbf{d})$ be a momspace, $\varepsilon \geq 0, \lambda$ a cardinal.
1) We say $\mathfrak{C}$ is ε^+-categorical in λ if every two complete $M_1, M_2 \in K_1^c$ of density λ are ε^+-isomorphic.
2) We say $\mathfrak{C}$ is categorical in λ if every two complete $M_1, M_2 \in K_1^c$ of density λ are isomorphic.
3) We say that $\mathfrak{C}$ is possibly categorical (ε^+-categorical) if it is categorical (ε^+-categorical) in some $\lambda > \aleph_0$.

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4) We say that $\mathfrak{C}$ is weakly uncountably categorical (wu-categorical) if the following holds: for each $\varepsilon > 0$ there exists a cardinal λ such that $\mathfrak{C}$ is ε^+-categorical in λ.

6.5 Observation. 1) If $\varepsilon \geq \zeta$ then ζ^+-categoricity implies ε^+-categoricity (for a specific λ).
2) Possible 0^+-categoricity implies weak uncountable categoricity.
3) Categoricity implies all the other notions (for a specific λ).

6.6 Theorem. *[$(\mathfrak{C}, \mathbf{d})$ compact]*
Let $K = K_1^c(\mathfrak{C})$ be a wu-categorical momspace with countable language. Then $\mathfrak{C}$ is $0^+ - \aleph_0$-stable.

Proof. Let τ be $\tau(K), \tau'$ is τ expanded with skolem functions, Φ-proper for K. So τ' is countable.

6.7 Subclaim. Under these assumptions, let I be a well-ordered set, $M_0 = \overline{\mathrm{EM}_\tau(I,\Phi)}$, $A \subseteq M_0$ is countable $\varepsilon > 0$. Then each ε-disjoint set $\mathscr{P}$ of types from $\mathbf{S}_D(A)$, $\frac{\varepsilon}{2}$-realized in M_0 (so $p_1, p_2 \in \mathscr{P} \Rightarrow p_1^{[\varepsilon]} \cup p_2^{[\varepsilon]}$ is contradictory and for each $p \in \mathscr{P}, p^{[\frac{\varepsilon}{2}]}$ is realized in M_0) is countable.

Proof of the Subclaim. If not, let $\langle p_i : i < \omega_1\rangle$ be ε-disjoint types over $A, p_i^{[\frac{\varepsilon}{2}]}$ realized in M_0 by $\bar{b}_i$. Pick $\bar{b}_i^0 \in \mathrm{EM}(I,\Phi), \mathbf{d}(\bar{b}_i^0, \bar{b}_i) \leq \frac{\varepsilon}{100}$. Without loss of generality $A = \mathrm{EM}(J,\Phi)$ for $J \subseteq I, |J| \leq \aleph_0$. As J is well ordered, by the standard argument, there are uncountably many b_i^0's satisfying the same type over A, but $b_i^0 \models p_i^{[\varepsilon]}$ and $p_i^{[\varepsilon]}, p_j^{[\varepsilon]}$ are contradictory for $i \neq j$, a contradiction. $\square_{6.7}$

Now we prove the theorem. Assuming $\mathfrak{C}$ is not $0^+ - \omega$-stable, we get $A \subseteq \mathfrak{C}$ countable and $\langle p_i : i < \omega_1\rangle$ ε-disjoint types over A for some $\varepsilon > 0$ (remember 4.3). Let λ be such that $\mathfrak{C}$ is δ^+-categorical in λ for $\delta << \varepsilon$. Now apply the usual argument: choose $M_1 \in K$ of density λ which includes A and $\langle b_i : i < \omega_1\rangle$ realizations of $\langle p_i : i < \omega_1\rangle$, and on the other hand consider $M_0 = \overline{\mathrm{EM}(\lambda,\Phi)}$. Applying $f : M_1 \to M_0$ which is a δ_1-embedding $\delta_1 < \frac{\varepsilon}{2}$, we get that $\langle f(b_i) : i < \omega_1\rangle$ contradict 6.7. $\square_{6.6}$

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6.8 Corollary. *Let $\mathfrak{C}$ and K be as in 6.6, then K has a $(D, \aleph_1)$-homogeneous model in λ (recall this means of density λ) for all $\lambda > \aleph_0$.*

Proof. By 6.6 and 5.2.

6.9 Observation. [$(\mathfrak{C}, \mathbf{d})$ compact].
1) Every two (D, λ)-homogeneous models at density character λ and isomorphic.
2) No (D, λ)-homogeneous model is 0^+-isomorphic to a non-(D, λ)-homogeneous model.

Proof. 1) Standard and does not require compactness.
2) By Corollary 3.14.

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§7 Uni-dimensionality

The following notion was explored in [Sh 3] but not defined there:

7.1 Definition. A (good) finite diagram D is uni-dimensional if for some regular λ there is no (D,λ)-homogeneous model of K which is not λ^+-homogeneous in cardinality $\geq \lambda^+$.

In [Sh 3] it is essentially proven (see [Sh 3, §6]) that:

7.2 Theorem. *Assume D is stable. Then the following are equivalent:*

(1) *D is not uni-dimensional*
(2) *there is some regular λ such that there are maximally (D,λ)-homogeneous models of arbitrary large cardinalities*
(3) *for all large enough regular $\lambda < \mu$, there is a (D,λ)-homogeneous model M and $\langle a_i : i < \mu\rangle, \langle b_i : i < \lambda\rangle$ mutually indiscernible sequences in M such that $\langle b_i : i < \lambda\rangle$ is a maximal indiscernible sequence in M, i.e., can not be extended in M*
(4) *there is a cardinal λ and a model M of cardinality λ which is $(D,\aleph_1)$-homogeneous, but not (D,λ)-homogeneous.*

7.3 Remark. In our context we will say "$\mathfrak{C}$ is uni-dimensional" or "K is uni-dimensional", meaning that D is.

7.4 Reminder. 1) For an indiscernible set $\mathbf{I} \subseteq \mathfrak{C}$ of cardinality $> |\tau_{\mathfrak{C}}| + \aleph_0$ and a set $A \subseteq \mathfrak{C}$, we define

$$\begin{aligned}\mathrm{Av}(\mathbf{I}, A) = \{\varphi(\bar{x},\bar{a}) : & \bar{a} \in A, \text{ infinitely many elements} \\ & \text{of } \mathbf{I} \text{ satisfy } \varphi(\bar{x},\bar{a})\}.\end{aligned}$$

We call this set the average type of $\mathbf{I}$ over A. (See 4.10, "all but finitely many" is wrong.)
2) For stable $\mathfrak{C}$, for any indiscernible $\mathbf{I}, |\mathbf{I}| > |\tau_{\mathfrak{C}}| + \aleph_0$ and set A, $\mathrm{Av}(\mathbf{I}, A)$ is a complete type, see 4.13(1).
3) Let $\mathbf{I} = \langle \bar{a}_i : i < \alpha\rangle$, where $\langle \bar{a}_i : i \leq \alpha\rangle$ is indiscernible. Then $\bar{a}_\alpha \models \mathrm{Av}(\mathbf{I}, \cup\mathbf{I})$.
4) Let $\mathbf{I}$ be indiscernible, $\mathbf{I} = \langle \bar{a}_i : i < \alpha\rangle$ and let $\bar{a}_\alpha \models \mathrm{Av}(\mathbf{I}, \cup\mathbf{I})$. Then $\langle \bar{a}_i : i \leq \alpha\rangle$

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is indiscernible.
5) It follows from (3) + (4) that $\mathbf{I} = \langle \bar{a}_i : i < \alpha \rangle \subseteq M$ is a maximal indiscernible sequence (set) in M iff $\mathrm{Av}(\mathbf{I}, \cup\mathbf{I})$ is omitted in M.
6) $\varphi(x, \bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) \in \mathrm{Av}(\mathbf{I}, \cup\mathbf{I})$ for $\mathbf{I} = \langle \bar{a}_i : i < \delta \rangle$ (δ-limit ordinal) iff $\bar{a}_j \models \varphi(x, \bar{a}_{i_1}, \ldots, \bar{a}_{i_n})$ for some/every $j \notin \{i_1, \ldots, i_n\}$.

7.5 Theorem. *1) Let $(\mathfrak{C}, \mathbf{d})$ be a momspace, $\tau(\mathfrak{C})$ countable, 0^+-categorical in $\lambda, \lambda > \aleph_0$. Then $\mathfrak{C}$ is uni-dimensional.*
2) The same is true if $\mathfrak{C}$ is wu-categorical.

Proof. 1) If not, choose $0 < \theta_1 << \theta_2$ and let M be a model $M \in K, |M| = \theta_2, M$ is θ_1-homogeneous, $\langle a_i : i < \theta_2 \rangle, \langle b_i : i < \theta_1 \rangle$ mutually indiscernible, $\langle b_i : i < \theta_1 \rangle$ cannot be extended in M, so (denoting $\mathbf{I} = \langle a_i : i < \theta_2 \rangle, \mathbf{J} = \langle b_i : i < \theta_1 \rangle$) $\mathrm{Av}(\mathbf{J}, \cup\mathbf{J})$ is omitted in M.

We now expand the language by a predicate P for J and skolem functions, call the new vocabulary τ'. Let $T' = \mathrm{Th}_{\tau'}(M')$ (where M' is M in the expanded language).

Note that

- $\circledast_0$ $T' \models$ "P is a τ-indiscernible set",i.e., for every τ-formula, T' implies that any two tuples from P behave the same.

The type $p = \mathrm{Av}_\tau(\mathbf{J}, \cup\mathbf{J})$ is omitted in M, therefore without loss of generality by 3.14, for some $\varepsilon, p^{[\varepsilon]}$ is omitted. If we choose θ_1, θ_2 carefully enough ($\theta_2 >> \theta_1$) in the beginning, then by the Erdös-Rado theorem (as in the proof of [Sh:c, VIII,5.3], or using [BY03a](1.2)) we can choose a diagram of indiscernibles (EM-blueprint) Φ in vocabulary τ' such that for any μ, denoting the skeleton of $M'_0 := \mathrm{EM}(\mu, \Phi)$ by $I' = \langle a'_i : i < \mu \rangle$, we have

- $\circledast_1$ $\mathbf{I}'$ is a τ'-indiscernible sequence (set), moreover, it is τ'-indiscernible over $P^{M'_0}$
- $\circledast_2$ for each $n < \omega$, for some $i_1, \ldots, i_n < \theta_2, a'_0, \ldots, a'_{n-1}$ has the same τ'-type as $a_{i_1}, \ldots, a_{i_n}$.

Now:

- $\circledast_3$ let $M_0 = \mathrm{EM}_\tau(\mu, \Phi)$, then $M_0 \prec \mathfrak{C}$, so $\mathrm{mcl}(M_0) \in K = K_1^c$ (as on the one hand τ' has skolem functions, and on the other hand M'_0 does not realize τ-types over $\emptyset$ that were not realized in M, so M_0 is a D-model)
- $\circledast_4$ $M'_0 \models T'$ (skolem functions, so $M'_0 \equiv M'$)

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$\circledast_5$ $P^{M'_0}$ is a τ-indiscernible set (by $\circledast_4$ and $\circledast_0$ above)

$\circledast_6$ $P^{M'_0}$ is countable. Why? Each $b \in P^{M'_0}$ is of the form $\sigma(a'_{i_1}, \dots, a'_{i_n})$ for some τ'-term σ. But as $\mathbf{I}'$ is τ'-indiscernible over $P^{M'_0}$, each such b depends only on σ, and there are countably many τ'-terms (τ is countable, and so is τ').

Denote $\mathbf{J}' = P^{M'_0}$, a τ-indiscernible set. Denote $p' = \mathrm{Av}_\tau(\mathbf{J}', \cup\mathbf{J}')$. Then $[p']^{[\varepsilon]}$ is omitted in M_0.
Why? Pick $\sigma(a'_{i_j}, \dots, a'_{i_n}) \in M'_0$. Let $j_1, \dots, j_n$ be such that $a'_{i_1}, \dots, a'_{i_n} \equiv a_{j_1}, \dots, a_{j_n}, \sigma(a_{j_1}, \dots, a_{j_n}) \in M'$ does not realize $p^{[\varepsilon]}$, so for some $\varphi(x) \in p, M' \models \mathbf{d}_1(\sigma(\bar{a}_j), \varphi) \geq \varepsilon$.
Note: $\varphi(x) = \varphi(x, \bar{c}), \bar{c} \in P^{M'}$. Call that $\varphi(x, \bar{c}) \in p \Leftrightarrow M' \models \varphi(d, \bar{c})$ for some/all $d \in P^{M'}, d \cap \bar{c} = \emptyset$ (as $P^{M'} = J$ is an indiscernible set, see 7.4(6)). So $M' \models$ "$\exists \bar{c} \in P$ such that $[\forall d \in P \backslash \bar{c}, \varphi(d, \bar{c})]$ & $[\mathbf{d}_1(\sigma(\bar{a}_j), \varphi(x, \bar{c})) \geq \varepsilon]$". Therefore, M'_0 satisfies the same formula with $\sigma(\bar{a}'_i)$, which obviously means that $\sigma(\bar{a}'_i)$ does not satisfy $[p']^{[\varepsilon]}$, as required.

We have finished now: let $\mu = \lambda$, so in $\bar{M}_0 = \overline{\mathrm{EM}_\tau(\lambda, \Phi)}$ we have a countable indiscernible set whose average is omitted (as $(p')^{[\varepsilon]}$ is omitted in M_0), so $\bar{M}_0$ is a non-$(D, \aleph_1)$-homogeneous model in $\mathfrak{K}$ of density λ, but by 6.8 we have a $(D, \aleph_1)$-homogenous model of density $\lambda > \aleph_0$. So categoricity in λ fails, moreover, 0^+-categoricity fails, as $\bar{M}_0$ is at least $\frac{\varepsilon}{2}$-distant from any $(D, \aleph_1)$-homogeneous model.
2) Repeat the proof of (1), and at the end choose λ in which, say, $(\frac{\varepsilon}{2})$-categoricity holds, and get the same contradiction.

7.6 Claim. *Let $\mathfrak{C}$ be a wu-categorical momspace, $K = K_1^c(\mathfrak{C})$.*
1) There exists a (D, λ)-homogeneous model in K for all $\lambda > \aleph_0$.
2) Each $(D, \aleph_1)$-homogeneous model in K of density λ is (D, λ)-homogeneous.
3) If K is 0^+-categorical in $\lambda > \aleph_0$, then each K-model of density $\geq \lambda$ is (D, λ)-homogeneous.

Proof. 1) By 6.8 and uni-dimensionality.
2) By the equivalence 7.2.
3) Otherwise by a Löwenheim-Skolem argument we will get a non-(D, λ)-homogeneous model of density λ, and together with (1) this will lead to a contradiction.

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§8 The main theorem

8.1 Hypothesis. $\mathfrak{C}$ is a compact momspace with countable vocabulary.

In this section we prove the main theorem of the paper. The proof is rather long and technical, but the ideas behind it are quite simple, so let us give an outline of the main steps.

We begin with the following situation; a model N, a countable subset B and a type $p \in S_D(B)$ which is really $(< \varepsilon)$-omitted in N. We would like to construct such a model of an arbitrarily large density character μ. The strategy is as follows: construct by induction on $\alpha < \mu$ tuples $\bar{c}_\alpha$ such that $A_\alpha = N \cup \{\bar{c}_\beta : \beta < \alpha\}$ still really $(< \varepsilon)$-omits p.

We have to make sure that A_μ is a model, and that A_μ has cardinality μ. Then the metric closure of A_μ, $\mathrm{mcl}(A_\mu)$, will be the desired model.

Making sure that A_μ is a model is not hard. For this purpose, whenever A_α is not in K_1, we choose $\bar{c}_\alpha$ to realize a formula over A_α whose realization is missing on A_α. In order not to create a pseudo-support for p while doing that, we choose $\bar{c}_\alpha$ carefully, namely, $\bar{c}_\alpha$ realizes a strictly isolated type over A_α. So that part of the proof is very similar to the classical proof of Morley's thoerem. The situation is slightly complicated by technical issues related to strict isolation, and some calculations are needed in order to show that p is still really $(< \varepsilon)$-omitted. This is the content of Claim 8.3.

Making sure that A_μ has the right density is more difficult. We take care of this requirement at stages α when $A_\alpha \in K_1$. At these stages we need to pick $\bar{c}_\alpha$ "close enough" to elements of A_α (to make sure it does not provide pseudo $(< \varepsilon)$-support for p) and yet "far from" A_α in terms of the metric in order to make sure that the density character increases. In classical model theory one would at this point take $\bar{c}_\alpha$ to continue an indiscernible sequence in A_α, more specifically, $c_\alpha \models \mathrm{Av}(A_\alpha, I)$ with $I \subseteq A_\alpha$. Unfortunately, we do not have true ω-stability, so existence of indiscernible sequences is not guaranteed.

What we do in Claim 8.4 is "immitating" existence of indiscernible sequences. Specifically, we recall that our model $M = \mathrm{mcl}(A_\alpha)$ can be thought of as a submodel of an Ehrenfeucht-Mostowski structure M^*. Using this representation, we can find an indiscernible sequence in M^* which is "close enough" to elements of M.

More precisely, we do the following. Since M has uncountable density character, we can find $\langle a_\alpha : \alpha < \omega_1 \rangle$ in M ε-distant from each other. If we knew that this sequence had an indiscernible subsequence, we would choose $b = c_\alpha$ to satisfy its average over A_α. Since it is not necessarily possible to do that, we use the representations of M as a subset of the closure of $\mathrm{EM}(I, \Phi^*)$ and choose a sequence $\langle b_{\alpha,n} : n < \omega \rangle$ of elements of $\mathrm{EM}(I, \Phi^*)$ converging to a_α for each α. Then using

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some infinitary combinations, we construct in an extension of M^* an indiscernible sequence which is "similar" to the diagonal of the matrix $\langle b_{\alpha,n} : \alpha < \omega_1, n < \omega\rangle$, hence "converging" to a subsequence of $\langle a_\alpha : \alpha < \omega_1\rangle$. Realizing the average type of that sequence will give us the desired new element $b = c_\alpha$.

Note that the second step would be simplified significantly if we assumed categoricity above the continuum. In that case, any sequence would have an indiscernible subsequence (since $\mathfrak{C}$ is truly stable in $\lambda = \lambda^{\aleph_0}$), and one could apply an argument along the lines of the classical ones.

8.2 Theorem. *Assume $K = K_1^c(\mathfrak{C})$ wu-categorical. Then K is λ-categorical for all $\lambda > \aleph_0$, moreover, any model of K of density $\lambda > \aleph_0$ is (D,λ)-homogeneous.*

Proof. Suppose not, so $\mathfrak{C}$ is $0^+ - \aleph_0$-stable, $(\aleph_1, *)$-superstable, uni-dimensional by 6.6, 4.16, 7.5, and there are $\lambda > \aleph_0, N \in K_1^c$, $\mathrm{ch}(N) = \lambda > \aleph_0, N$ is not (D,λ)-homogeneous. By 7.6, N is not $(D,\aleph_1)$-homogeneous.

Then there exists $B \subseteq N$, $|B| \le \aleph_0$, $p \in \mathbf{S}^1_D(B), p$ is omitted in N, and in fact by 3.14, $p^{[2\varepsilon]}$ is omitted in N for some $\varepsilon > 0$. Therefore (by 4.19) p has no pseudo $(< \varepsilon)$-support in N, see Definition 4.18.

Let $\mu > \lambda$ be a large enough regular cardinal.
We choose $\bar{c}_\alpha$ by induction on $\alpha < \mu$ such that

$\circledast$ $A_\alpha = N \cup \{\bar{c}_\beta : \beta < \alpha\}$ really $(< \varepsilon)$-omits $p(\bar{x})$.

<u>Case (a)</u>: If $A_\alpha =: N \cup \{\bar{c}_\beta : \beta < \alpha\}$ is not in K_1. Then first choose $\varphi(\bar{x}, \bar{y}) \in \Delta, \bar{a} \subseteq A_\alpha$ such that $\varphi(\bar{x}, \bar{a})$ witnesses $A_\alpha \notin K_1$, and second choose $\bar{c}_\alpha$ realize some strictly isolated $q \in \mathbf{S}_D^{\ell g(\bar{x})}(A_\alpha)$ which contains $\varphi(\bar{x}, \bar{a})$.

By 8.3 below, i.e., the next claim, this is possible and $\circledast$ is preserved.

<u>Case (b)</u>: Not (a), then $\bar{c}_\alpha \notin \mathrm{mcl}(A_\alpha)$ and $\circledast$ holds, using 8.4 below.
Having carried out the construction, let $M = \mathrm{mcl}(A_\mu)$.

Let us show that M is a non-$(D,\aleph_1)$-homogeneous model of density character μ. Since the proof is an easy version of some arguments which appear in Claim 8.4 below, we decided to give all the details.

First we note:

$(*)$ there is a club E of μ such that $\alpha \in E \Rightarrow A_\alpha \in K_1$.

In order to prove $(*)$, assume towards a contradiction that there is a stationary set $S \subseteq \mu$ such that $\alpha \in S \Rightarrow M_\alpha \notin K_1$. Hence for every $\alpha \in S$ there is a formula

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$\varphi_\alpha(\bar{x}_\alpha, \bar{y}_\alpha)$ and $\bar{a}_\alpha \in A_\alpha$ such that $\varphi_\alpha(\bar{x}, a_\alpha)$ is not realzied in A_α but $\varphi_\alpha(\bar{c}_\alpha, \bar{a}_\alpha)$ holds.

Since there are only countably many formulae, without loss of generality $\varphi_\alpha(\bar{x}_\alpha, \bar{y}_\alpha) = \varphi(\bar{x}, \bar{z})$ for all $\alpha \in S$. Now define $f : S \to \mu$ by $f(\alpha) = \min\{\beta : \bar{a}_\alpha \in A_{\beta+A}\}$. Clearly, f is regressive, so for some $S' \subseteq S$ stationary and $\beta < \mu$ we have $\bar{a}_\alpha \in A_\beta$ for all $\alpha \in S'$. Since $|A_\beta| < \mu$, it must be the case that $\bar{a}_\alpha = \bar{a}_{\alpha'}$ for some $\alpha < \alpha' \in S$, so the formula $\varphi_\alpha(\bar{x}, \bar{a}_\alpha)$ is taken care of twice in our construction, which is of course impossible ($\varphi_\alpha(\bar{x}, \bar{a}_\alpha)$ is realized by $\bar{c}_\alpha \in A_{\alpha'}$).

So we have shown $(*)$. It is now easy to derive:

- $(*)_1$ $A_\mu \in K_1$, hence $M \in K_1^c$
- $(*)_2$ $\mathrm{char}(M) = \mu$ (this is because by $(*)$ there is a club E of μ on which case (b) holds)
- $(*)_3$ M is not $(D, \aleph_1)$-homogeneous. In fact, it cannot be $(\frac{\varepsilon}{2})^+$-isomorphic to a $(D, \aleph_1)$-homogeneous model, because $p(x)$ is $(< \varepsilon)$-omitted by it (by $\circledast$ and 4.19(2)).

But for each $\mu > \aleph_0$ there is $M \in K_1^c$, $\mathrm{Ch}(M) = \mu$, M is $(D, \aleph_1)$-homogeneous (by 5.1(2)). So for arbitrarily large enough μ, there are two models of density character μ, which are not $(\frac{\varepsilon}{2})^+$-isomorphic (see $(*)_3$), a contradiction to wu-categoricity.

In order to complete the proof of the main theorem, we only need to show that the construction above (both Case (a) and Case (b)) is possible, which is done in the following two claims.

8.3 Claim. *[$(\mathfrak{C}, \mathbf{d})$ is $0^+ - \aleph_0$-stable, compact]*

Let $p \in \mathbf{S}^1_D(B), B$ countable, $A \supseteq B, A$ is not a pseudo $(< \varepsilon)$-support for p, see Definition 4.18. Let $\varphi(\bar{x}, \bar{a})$ be a consistent formula over A. Then there exists $\bar{b} \in \mathfrak{C}$ such that $\mathfrak{C} \models \varphi(\bar{b}, \bar{a})$ and $A \cup \bar{b}$ is not a pseudo $(< \varepsilon)$-support for p. In fact, it is enough to choose $\bar{b}$ such that $\mathrm{tp}(\bar{b}, A)$ is strictly isolated and $\mathfrak{C} \models \varphi(\bar{b}, \bar{a})$.

Proof. By 4.22(2) for some $\bar{b} \in \varphi(\mathfrak{C}, \bar{a})$, $\mathrm{tp}(\bar{b}, A)$ is strictly isolated. So assume toward contradiction

$(*)_1$ $A \cup \bar{b}$ is a pseudo $(< \varepsilon)$-support for $p(\bar{x})$.

Hence (by Definition 4.18) there are $\zeta(1), \zeta(2), \vartheta_1(\bar{x}, \bar{b}, \bar{c})$

$(*)_2$ $\vartheta_1^{<\zeta(1)>}(\bar{x}, \bar{b}, \bar{c}_1) \models p^{[\varepsilon - \zeta(2)]}(\bar{x})$ and $\bar{c}_1 \subseteq A$ and $\vartheta_1(\bar{x}, \bar{b}, \bar{c}_1)$ is consistent.

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As $\mathrm{tp}(\bar{b}, A)$ is strictly isolated (see Definition 4.20), there are $\zeta(3) > 0$ and $\psi(\bar{y}, \bar{c}_2)$ such that

$(*)_3$ (i) $\psi(\bar{y}, \bar{c}_2) \in \mathrm{tp}(\bar{b}, A)$ so $\bar{c}_2 \subseteq A$
(ii) $\psi(\bar{y}, \bar{c}_2)$ pseudo $(\zeta(1), \zeta(3))$-isolates $\mathrm{tp}(\bar{b}, A)$, i.e. $\psi^{<\zeta(3)>}(\bar{y}, c_2] \models \mathrm{tp}(\bar{b}, A)^{[\zeta(1)]}$.

Let

$(*)_4$ $\vartheta_2(\bar{x}, \bar{c}_1, \bar{c}_2) = (\exists \bar{y})[\psi(\bar{y}, \bar{c}_2) \wedge \vartheta_1(\bar{x}, \bar{y}, c_1)]$.

Clearly

$(*)_5$ $\vartheta_2(\bar{x}, \bar{c}_1, \bar{c}_2)$ is consistent.

Choose $\zeta(4)$ such that

$(*)_6$ $0 < \zeta(4) < \zeta(3)$ and $\zeta(4) < \zeta(1)$.

We shall now show that $\vartheta_2(x, \bar{c}_1, \bar{c}_2)$ pseudo $(\varepsilon - \zeta(2), \zeta(4))$-isolates p (and is over A), i.e.

$\boxtimes$ $\vartheta_2^{<\zeta(4)>}(\bar{x}, \bar{c}_1, \bar{c}_2) \models p^{[\varepsilon - \zeta(2)]}(\bar{x})$.

This will give a contradiction to the assumption that A is not a pseudo $(< \varepsilon)$-support for p. So assume

$(*)_7$ $\mathfrak{C} \models \vartheta_2^{<\zeta(4)>}[\bar{a}, \bar{c}_1, \bar{c}_2]$.

By the definition of $\vartheta_2^{<\zeta(4)>}$, there are $\bar{a}', \bar{c}_1', \bar{c}_2'$ such that

$(*)_8$ (i) $\mathfrak{C} \models \vartheta_2[\bar{a}', \bar{c}_1', \bar{c}_2']$
(ii) $\mathbf{d}(\bar{a}', \bar{a}) \leq \zeta(4) \leq \zeta(1)$
(iii) $\mathbf{d}(\bar{c}_1', \bar{c}_1) \leq \zeta(4) \leq \zeta(1)$
(iv) $\mathbf{d}(\bar{c}_2', \bar{c}_2) \leq \zeta(4) \leq \zeta(3)$.

By the choice of ϑ_2, i.e. $(*)_4$, for some $\bar{b}'$

$(*)_9$ (i) $\mathfrak{C} \models \psi[\bar{b}', \bar{c}_2']$
(ii) $\mathfrak{C} \models \vartheta_1[\bar{a}', \bar{b}', \bar{c}_1']$.

By the choice of $\psi(\bar{y}, \bar{c}_2)$, i.e., $(*)_3(ii)$ (note that as $0 < \zeta(4) \leq \zeta(3)$, we have $\psi^{<\zeta(3)>}(\bar{b}', \bar{c}_2)$) there is $\bar{b}'' \in \mathfrak{C}$ such that

$(*)_{10}$ (i) $\bar{b}''$ realizes $\mathrm{tp}(\bar{b}, A)$
(ii) $\mathbf{d}(\bar{b}'', \bar{b}') \leq \zeta(1)$.

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So $\mathbf{d}(\bar a, \bar a') \le \zeta(4) \le \zeta(1), \mathbf{d}(\bar b', b'') \le \zeta(1)$ and $\mathbf{d}(\bar c_1, \bar c_1') \le \zeta(4) \le \zeta(1)$, therefore by $(*)_9(ii)$ we get

$(*)_{11}$ $\mathfrak{C} \models \vartheta_1^{<\zeta(1)>}[\bar a, \bar b'', \bar c_1]$.

So by $(*)_2$, replacing $\bar b$ with $\bar b''$ which has the same type over A

$(*)_{12}$ $\bar a$ realizes $p^{[\varepsilon - \zeta(2)]}$

so we have finished proving ⊠, hence getting the desired contradiction. $\square_{8.3}$

8.4 Claim. *[$(\mathfrak{C}, \mathbf{d})$ is $0^+ - \aleph_0$-stable,uni-dimensional]*
1) Assume

- (a) $M = \text{mcl}(M) \subseteq \mathfrak{C}$
- (b) $\text{Ch}(M) > \aleph_0$
- (c) *$B \subseteq M$ is countable*
- (d) $p \in \mathbf{S}^m_D(B)$
- (e) *M really $(< \varepsilon)$-omits $p(\bar x)$, see Definition 4.18.*

Then for some $\bar b \in \mathfrak{C} \backslash M$, also $M \cup \{\bar b\}$ really $(< \varepsilon)$-omits p.
2) Assume clauses (a)-(e) and $\bar b \in \mathfrak{C}, \bar b \notin M$, then $M \cup \bar b$ really $(< \varepsilon)$-omits $p(x)$ when: for every $\bar c \in {}^{\omega>}M$ and $\zeta > 0$ there is $\bar b'$ realizing $\text{tp}(\bar b, B \cup \bar c)$ such that $\mathbf{d}(\bar b', {}^{\ell g(\bar b)}M) < \zeta$.
3) In clause (2) it is enough to assume that for every $\bar c \in {}^{\omega>}M$ and $\zeta > 0$ there exist $\bar b', \bar b^-$ such that $\mathbf{d}(\bar b, \bar b^-) < \zeta, \mathbf{d}(\bar b', M) < \zeta$ and $\bar b'$ realizes $tp(\bar b^-, B \cup \bar c)$.

Remark. Assuming just $|\tau_{\mathfrak{C}}| + \aleph_0 + |B| < \text{Ch}(M)$ suffices. Assuming $\text{Ch}(M) > |B| + 2^{\aleph_0} + |\tau_{\mathfrak{C}}|$, we can waive the assumption on $0^+ - \aleph_0$-stability.

Proof. 1) Let Φ^* be as in 5.1(3), so $\tau(\Phi^*)$ of cardinality $\aleph_1$. Recall

$(*)$ for every uncountable (large enough if we do not assume τ countable) linear order $I, M^*_I = \text{mcl}(\text{EM}_\tau(I, \Phi^*))$ is $(D, |I|)$-homogeneous.

[It is $(D, \aleph_1)$-homogeneous and use uni-dimensionality.] Choose $I = \lambda$ for λ regular, $\lambda >$ $\text{Ch}(M)$, and let $M^* = M^*_\lambda$.

So without loss of generality $M \subseteq M^*$. As $\text{Ch}(M) > \aleph_0$, we can by 8.7 below find $\varepsilon > 0$ and $a_\alpha \in M$ for $\alpha < \omega_1$ such that $\alpha < \beta < \omega_1 \Rightarrow \mathbf{d}(a_\alpha, a_\beta) > \varepsilon$. Let $\{a_{\omega_1 + n} : n < \omega\} \subseteq M$ include B. Now for each $\alpha < \omega_1 + \omega$ and $n < \omega$

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we can find $b_{\alpha,n} \in \mathrm{EM}_\tau(I,\Phi^*)$ such that $\mathbf{d}(a_\alpha, b_{\alpha,n}) < 1/(n+1)$. Let $b_{\alpha,n} = \sigma_{\alpha,n}(a_{t_{\alpha,n,0}},\dots,a_{t_{\alpha,n,k(\alpha,n)-1}})$.

Let (for $\alpha < \omega_1$)

$$S_\alpha = \{t_{\beta,n,\ell} : (\beta < \alpha) \vee \beta \in [\omega_1, \omega_1+\omega) \text{ and } n < \omega, \ell < k(\beta,n)\} \subseteq \lambda.$$

Let (for $\alpha < \omega_1, n < \omega, \ell < k(\alpha,n)$)

$$\gamma_{\alpha,n,\ell} = \mathrm{Min}\{\gamma \in S_\alpha \cup \{\lambda\} : t_{\alpha,n,\ell} \le \gamma\}.$$

8.5 Subclaim. Under these assumptions, there exists C, a club of ω_1, such that

(∗) if $\delta \in C$ and $m < \omega$ then the following set is stationary

$$W_{\delta,m} = \{\alpha \in C : \text{for every } n \le m, \sigma_{\alpha,n} = \sigma_{\delta,n}, \text{ hence } k(\alpha,n) = k(\delta,n) \text{ and } (\gamma_{\alpha,n,\ell} = \gamma_{\delta,n,\ell}) \wedge (\gamma_{\alpha,n,\ell} \in S_\alpha \equiv \gamma_{\delta,n,\ell} \in S_\delta) \text{ for } \ell < k(\delta,n)\}.$$

Proof. By a standard coding argument, there exist functions $f_n : \omega_1 \to \omega_1$ (for $n < \omega$) such that for $\alpha < \omega_1$, $f_n(\alpha)$ "encodes" the finite sequence

$$\langle \sigma_{\alpha,m} : m \le n \rangle \frown \langle (\beta^*_{\alpha,m,\ell}, m^*_{\alpha,m,\ell}, \ell^*_{\alpha,m,\ell}) : m \le n, \ell < k(\alpha,m) \rangle$$

where $\beta^*_{\alpha,m,\ell} < \omega_1 + \omega$, $m^*_{\alpha,m,\ell}$ and $\ell^*_{\alpha,m,\ell}$ are natural numbers, satisfying:

⊗ $(\beta^*_{\alpha,m,\ell}, m^*_{\alpha,m,\ell}, \ell^*_{\alpha,m,\ell})$ is the minimal (lexicographically) triple (β^*, m^*, ℓ^*) such that $t_{\beta^*,m^*,\ell^*} = \gamma_{\alpha,m,\ell}$.

In fact, there exists such coding $f_n : \omega_1 \to \omega_1$ such that on a club C'_n, f_n is regressive. Now by Födor's lemma, the following set contains a club (as its ω_1 complement cannot contain a stationary set):

$$C''_n = \{\delta \in C'_n : \{\alpha \in C'_n : f_n(\alpha) = f_n(\delta)\} \text{ is stationary}\}.$$

We call this club C_n and let $C = \bigcap_{n<\omega} C_n$, obviously C is as required. $\square_{8.5}$

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8.6 Subclaim. Under these assumptions, there exist $\delta(*) \in C$ such that $(\forall m < \omega)(\exists \delta \in \delta(*) \cap C)(\delta(*) \in W_{\delta,m})$.

Proof. Note that $\delta(*) \in W_{\delta,m} \Leftrightarrow \delta \in W_{\delta(*),m} \Leftrightarrow W_{\delta,m} = W_{\delta(*),m}$, so all we are looking for is $\delta(*)$ satisfying $\delta(*) > \min W_{\delta(*),m}$ for all m; now if for all $\delta < \omega_1 \exists m < \omega$ such that $\delta = \min W_{\delta,m}$ we set an easy contradiction (e.g. by Födor's lemma, although it is an overkill here). $\square_{8.6}$

By a similar argument we can find $\delta(*) \in C$ such that there exists a sequence $\langle \delta_n : n < \omega \rangle, \delta_n \in C, \delta_n < \delta_{n+1} < \delta(*)$ for all n and $W_{\delta(*),n} = W_{\delta_n,n}$ for all n, so in particular $m \leq n \Rightarrow W_{\delta_m,m} = W_{\delta_n,m} = W_{\delta(*),m}$.
We obtain (by the choice of $\delta_n, C, W_{\delta_n,m}$, etc.):

$\circledast_1$ $\langle b_{\delta_n,m} : n \geq m \rangle$ is an indiscernible sequence over $[B_m = B \cup \{b_{\alpha,i} : \alpha < \delta_m, i < \omega\}]$ for each m.

In fact, we can say more:

$\circledast_2$ for each $\bar{c} \in \mathrm{EM}(I, \Phi^*)$ finite and for each $m < \omega$, there exists $n^* < \omega$ such that $\langle b_{\delta_n,m}, n \geq n^* \rangle$ is indiscernible over $B_m \cup \bar{c}$.

We would like now to continue the indiscernible sequences above in a proper extension of M^*.

Let $J = I \times \mathbb{Q}$ and let us identify I with $I \times \{0\}$, so we think of I as a subset of J. Now for each $m < \omega, \ell < \kappa(\delta_m, m)$, look at the sequence $\langle t_{\delta_n,m,\ell} : n \geq m \rangle$. It is either constant or strictly increasing (recall $\circledast_1$), in the first case define $t^*_{m,\ell} = t_{\delta_m,m,\ell} \in I \subseteq J$, otherwise choose $t^*_{m,\ell} \in J$ such that $t_{\delta_n,m,\ell} < t^*_{m,\ell} < \sup\{t_{\delta_n,m,\ell} : n \geq m\}$.

Note:

$(**)$ for each $m < \omega$, the (quantifier free) type of the sequence $\langle t^*_{n,\ell} : n < m, \ell < k(\delta(*), n \rangle$ in the language $\{<\}$ (order) is the same as the type of $\langle t_{\delta(*),n,\ell} : n < m, \ell < k(\delta(*), n) \rangle$.

Let $M^+ = \mathrm{mcl}(\mathrm{EM}_{\tau(\mathfrak{C})}(J, \Phi^*)) \prec^1_\Delta \mathfrak{C}, M^+$ extends M^*. Let $b^*_n = \sigma_{\delta(*),n}(t^*_{n,0}, \ldots, t^*_{n,k(\delta(*),n)-1})$ so $b^*_n \in M^+, \langle b^*_n : n < \omega \rangle$ is a Cauchy sequence (as $\langle b_{\delta^*,n} : n < \omega \rangle$ is a Cauchy sequence, use the indiscernibility of J) with limit $b^* \in M^+$.

Note (by $\circledast_2$ and the choice of b^*_n)

$\circledast_3$ for each $\bar{c} \in \mathrm{EM}(I, \Phi^*)$ finite and for each $m < \omega$ there exists $n^* < \omega$ such that $\langle b_{\delta_n,m} : n \geq n^* \rangle \frown b^*_m$ is indiscernible over $B \cup \bar{c}$. In fact, by 3.16, the same is true for each $\bar{c} \in M^*$ finite.

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We also observe

$\circledast_4$ $b^* \in M^+ \backslash M^*$.

Why? Recall that $\langle a_\alpha : \alpha < \omega_1 \rangle$ form an ε-net, so for m big enough, $b_{\delta_{n_1},m}, b_{\delta_{n_2},m}$ can not be too close for $n_1 \neq n_2$. Combining this with $\circledast_3$ we see that b^*_m can not be $\frac{\varepsilon}{3}$ close to any $\bar{c} \in M^*$.

$\circledast_5$ b^* is as required in (3).

Why? Given $\bar{c} \in M$ finite and $\zeta > 0$ we choose m and n big enough such that b^*_m is close enough to $b^*, b_{\delta_n,m}$ is close enough to $a_{\delta_n} \in M_1$, and $\text{tp}(b_{\delta_1,m}, B \cup \bar{c}) = \text{tp}(b^*_m, B \cup \bar{c})$ [possible by $\circledast_3$].

2) Assume this fails, so there are $\zeta > 0, \bar{c} \subseteq M$ and a formula $\vartheta(\bar{x}, \bar{b}, \bar{c})$ such that $\vartheta^{<\zeta>]}(x, \bar{b}, \bar{c}) \models p^{[\varepsilon-\zeta]}(\bar{x})$.

Choose $\zeta_1, \zeta_2 > 0$ such that $\zeta_1 < \zeta_2 < \zeta$ and $\zeta_1 < \zeta - \zeta_2$. By the assumption there are $\bar{b}', \bar{b}''$ such that

$(*)$ (a) $\bar{b}''$ realizes $\text{tp}(\bar{b}, B \cup \bar{c})$
(b) $\bar{b}' \subseteq M$ of length $\ell g(\bar{b})$
(c) $\mathbf{d}(\bar{b}', \bar{b}'') < \zeta_1 < \zeta - \zeta_2$.

By clause (a) of $(*)$ as $\theta^{<\zeta>}(\bar{x}, \bar{b}, \bar{c}) \models p^{[\varepsilon-\zeta]}(\bar{x})$ also $\theta^{<\zeta>}(\bar{x}, \bar{b}'', \bar{c}) \models p^{[\varepsilon-\zeta]}(\bar{x})$. Hence (by $*(c)$ and 3.6(3)) $\theta^{<\zeta_2>}(\bar{x}, \bar{b}', \bar{c}) \models p^{[\varepsilon-\zeta]}(\bar{x})$, a contradiction to clause (e) of the assumptions of the Claim, that is, p being really $(< \varepsilon)$-omitted by M.
3) Similar proof to (2), first using 3.6(3) to show that $\theta^{<\zeta'>}(\bar{x}, \bar{b}^-, \bar{c}) \models p^{[\varepsilon-\zeta]}(\bar{x})$ for some ζ'. $\square_{8.4}$

8.7 Observation. Let $(X, \mathbf{d})$ be a non-separable metric space. Then there exist

$$\langle a_i : i < \omega_1 \rangle \subseteq X \text{ and } \varepsilon^* > 0 \text{ such that } \mathbf{d}(a_i, a_j) \geq \varepsilon^* \, \forall i, j < \omega_1.$$

Proof. Choose by induction on $i < \omega_1, a_i$ such that $a_i \notin \{a_j : j < i\}$. Choose $0 < \varepsilon_i \leq d(a_i, \{a_j : j < i\}), \varepsilon_i \in \mathbb{Q}$. Without loss of generality $\varepsilon_i = \varepsilon^*$ for all i, and we are done. $\square_{8.2}$

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