

On Weak and Strong Interpolation in Algebraic Logics

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Abstract

We show that there is a restriction, or modification of the finite-variable fragments of First Order Logic in which a weak form of Craig's Interpolation Theorem holds but a strong form of this theorem does not hold. Translating these results into Algebraic Logic we obtain a finitely axiomatizable subvariety of finite dimensional Representable Cylindric Algebras that has the Strong Amalgamation Property but does not have the Superamalgamation Property. This settles a conjecture of Pigozzi [12].

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1 Introduction

Formula interpolation in different logics is a classical and rapidly growing research area. In this note we give a modification of finite variable fragments of First Order Logic in which a weak version of Craig's Interpolation Theorem holds but a strong version of this theorem does not hold. To do this we will use classical methods and results of model theory of First Order Logic.

A traditional approach for investigating interpolation properties of logics is to "algebraize" the question, that is, after reformulating semantics in an algebraic way, interpolation, definability and related problems can also be considered as properties of the (variety of) algebras obtained by the above reformulation. Algebras obtained by algebraizing semantics are called "meaning algebras". As it is well known, the meaning algebras of first order logics are different classes of representable cylindric

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algebras (RCA_n will denote the class of n dimensional representable cylindric algebras; in this paper n will always be a finite number). For more details we refer to [6] and [7].

It turned out that interpolation properties on the logical side correspond amalgamation properties on the algebraic side (see, for example, [12] or Theorem 6.15 of [1] and references therein). Similarly, Beth Definability Property (on the logical side) corresponds to surjectiveness of the epimorphisms in the category of meaning algebras, see [11] or Theorem 6.11 of [1].

Our results can also be translated into Algebraic Logic. As we will show in Theorem 5.3, for $n \geq 3$, there is a subvariety U_n of RCA_n that has the Strong Amalgamation Property (*SAP* for short) but does not have the Superamalgamation Property (*SUPAP* for short). This settles a conjecture of Pigozzi in [12] (see page 313, Remark 2.1.21 therein). This was a long standing open problem in Algebraic Logic. So Theorem 5.3 can be considered as the main result of this note. We should mention the following earlier related results. Comer [5] proved that if $n \geq 2$ then RCA_n does not have the Amalgamation Property and Maksimova [10] has shown the existence of a BAO-type¹ variety that has *SAP* but doesn't have *SUPAP*. The essential difference between this result and our Theorem 5.3 is that in our case the variety U_n is a subvariety of RCA_n , as originally Pigozzi's question required. Indeed, as shown in Sections 3 and 4 this has immediate consequences of interpolation and definability properties for some modifications of First Order Logic restricted to finitely many variables.

In Section 2 we give some basic definitions about logics in general and then introduce and investigate the model theory of a certain modification of finite variable fragment of First Order Logic, this fragment will be called \mathcal{U}_n . In Section 3 we show that \mathcal{U}_n satisfies a weak version of Craig's Interpolation Theorem, but doesn't satisfy the strong version of it. In Section 4 we show that \mathcal{U}_n satisfies Beth's Theorem on implicit and explicit definability. Finally, in Section 5 we translate these results into algebraic form and in Theorem 5.3 we settle Pigozzi's conjecture: there is a finite dimensional, finitely axiomatizable subvariety of representable cylindric algebras, that has *SAP* but does not have *SUPAP*.

We conclude this section by summing up our system of notation.

Every ordinal is the set of smaller ordinals and natural numbers are identified with finite ordinals. Throughout, ω denotes the smallest infinite ordinal. If A and B are sets, then ${}^A B$ denotes the set of functions whose domain is A and whose range is a subset of B .

¹BAO stands for Boolean Algebra with Operators

2 A Portion of First Order Logic

In this note by a logic we mean a triplet $\mathcal{J} = \langle F, K, \models \rangle$ where F is the set of formulas of \mathcal{J} , K is the class of models of \mathcal{J} and \models is the satisfaction relation. Often, formulas have a structure: there are a *vocabulary* and a set of rules with which one can build formulas from elements of the vocabulary. Strictly speaking, in this case we obtain different logics for different vocabularies. Sometimes these families of logics have been regarded as a pair $\mathcal{J} = \langle S, K \rangle$ where S is a function on vocabularies associating the set of formulas $F = S(V)$ of \mathcal{J} and the satisfaction relation $\models_{S(V)}$ of \mathcal{J} to a given vocabulary V .

Particularly, when one deals with a concrete first order language, one should specify the names and arities of relation and function symbols to be used. Such a specification will also be called a vocabulary (for first order languages). Throughout this paper we will deal with variants, modifications and portions of First Order Logic. For a given vocabulary V , $F = F_V$ will be the set of formulas of First Order Logic restricted to individual variables $\{v_0, \dots, v_{n-1}\}$ (n is fixed and finite). In addition, in this note, the class K of models of a logic will always be a subclass of ordinary relational structures and the satisfaction relation will be the same as in ordinary First Order Logic.

If φ is a formula of such a logic \mathcal{J} then $\text{voc}(\varphi)$ denotes the smallest vocabulary for which φ is really a first order formula. If V is a vocabulary then $\mathcal{J}[V]$ denotes the logic in which the set of formulas consists of formulas of \mathcal{J} whose vocabularies are contained in V , the class of models of $\mathcal{J}[V]$ is the class of V -reducts of models of \mathcal{J} and the satisfaction relation of $\mathcal{J}[V]$ is the same as that of \mathcal{J} . Similarly, if $V \subseteq W$ are vocabularies and \mathcal{A} is a model for \mathcal{J} with vocabulary W then $\mathcal{A}|_V$ denotes the reduct of \mathcal{A} in which only elements of V interpreted as basic relations (or functions). In this note we will deal with vocabularies consisting relation symbols only.

Truth, meaning, and semantical consequence defined the obvious way, that is, these notions simply inherited from the first order case. Similarly, some concepts, methods, etc. of First Order Logic (like isomorphism, elementary equivalence, (generated) submodels of a structure) will be used the obvious way without any additional explanation.

As we mentioned, throughout the paper $n \in \omega$ is a fixed natural number. \mathcal{L}_n denotes usual First Order Logic restricted to the first n individual variables.

Definition 2.1 *Let A be a non-empty set, let $k \in \omega$ and let $\bar{s} \in {}^k A$. Then*

$$\ker(\bar{s}) = \{\langle i, j \rangle \in {}^2 k : s_i = s_j\}.$$

If $U_0 \subseteq A$ and $\bar{z} \in {}^k A$ then $\bar{s} \sim_{A, U_0, k} \bar{z}$ means that

- (i) $\ker(\bar{s}) = \ker(\bar{z})$ and*
- (ii) $(\forall i \in k)[s_i \in U_0 \Leftrightarrow z_i \in U_0]$.*

Sometimes we will simply write \sim_k or \sim in place of $\sim_{A, U_0, k}$.

Definition 2.2 A relational structure $\mathcal{A} = \langle A, U_0, R_i \rangle_{i \in V}$ is defined to be an U -structure (for the vocabulary $V = V[\mathcal{A}]$), if

- $U_0 \subseteq A$,
- $|U_0| \geq n, |A - U_0| \geq n$ and
- for any $i \in V$, if R_i is k -ary, $\bar{s} \in R_i, \bar{s} \sim_k \bar{z}$ then $\bar{z} \in R_i$.

A is the universe of \mathcal{A} and U_0 will be called the core of \mathcal{A} .

In Sections 2, 3 and 4 U -structures have been treated as special first order relational structures, that is, every relation has a finite arity, these arities may be different for different relations.

Let \mathcal{A} be an U -structure with core U_0 . It is easy to see that a permutation of the universe of \mathcal{A} mapping U_0 onto itself is an automorphism of \mathcal{A} .

It should be emphasized, that the core of an U -structure \mathcal{A} is not a basic relation of \mathcal{A} , that is, the core relation a priori doesn't have a name in the vocabulary of \mathcal{A} . Sometimes the core may be defined somehow, in some other U -structures the core cannot be defined by first order formulas. The core relation provides some extra structure for U -structures which will be used extensively below. The core of an U -structure \mathcal{A} will be denoted by $U_0^{\mathcal{A}}$ or simply by U_0 when \mathcal{A} is clear from the context.

Throughout this paper by a "definable relation" we mean a relation which is definable by a formula of $\mathcal{L}_n[V]$ without parameters (if the vocabulary V is clear from the context, we omit it).

If A is any set then $A^{*n} = \{s \in {}^n A : (\forall i \neq j \in n) s_i \neq s_j\}$. Clearly, this relation is definable from the identity relation. In order to keep notation simpler, we will identify this relation by one of it's defining formulas and sometimes we will write " A^{*n} " in the middle of another formula.

Definition 2.3 An U -structure \mathcal{A} is defined to be a strong U -structure if the following holds. If V is any sub-vocabulary of the vocabulary of \mathcal{A} such that

- $U_0^{\mathcal{A}}$ is not \mathcal{L}_n -definable in $\mathcal{A}|_V$ and
- R is an \mathcal{L}_n -definable m -ary relation of $\mathcal{A}|_V$ (for some $m \leq n$) such that

$R \subseteq A^{*m}$ and $i, j \in m, i \neq j$ then

$\mathcal{A} \models R \Leftrightarrow (\exists v_i R \wedge \exists v_j R) \wedge A^{*m}$ (more precisely, letting $\bar{v} = \langle v_0, \dots, v_{n-1} \rangle$ we require $\mathcal{A} \models (\forall \bar{v}) [R(\bar{v}) \Leftrightarrow \exists v_i R(\bar{v}) \wedge \exists v_j R(\bar{v}) \wedge A^{*m}(\bar{v})]$).

\mathcal{U}_n will denote the logic in which the set of formulas is the same as in \mathcal{L}_n and the class of models of \mathcal{U}_n is the class of strong U -structures. We will say that a relation (in an arbitrary structure) is \mathcal{U}_n -definable iff it is definable by a formula of \mathcal{U}_n . Similarly, two relational structures are called \mathcal{U}_n -elementarily equivalent iff they satisfy the same formulas of \mathcal{U}_n .

In the previous definitions "U" stand for "unary-generated", this choice of naming will be explained in Section 5 below. According to the previous definition, the

notion of strong U -structures depends on n , therefore strictly speaking, instead of "strong U -structure" we should write "strong U -structure for some n ". For simplicity we don't indicate n ; it will always be clear from the context.

We call the attention that in \mathcal{U}_n function symbols are not part of the vocabulary, that is, all the vocabularies contain relation symbols only.

It is easy to see that strong U -structures exist. We will show this in Theorem 2.6 below.

Lemma 2.4 *Suppose \mathcal{A} is an U -structure with universe A and core U_0 .*

(1) *If R is a definable unary relation of \mathcal{A} then $R \in \{\emptyset, A, U_0, A - U_0\}$. Thus, at most four definable unary relations exist in an U -structure.*

(2) *Let $\mathcal{A}' = \langle A, U_0 \rangle$ be the structure whose universe is the same as that of \mathcal{A} and whose unique basic relation is U_0 . If R is a \mathcal{U}_n -definable relation of \mathcal{A} then R is definable in \mathcal{A}' as well.*

(3) *If R is a definable relation in \mathcal{A} , $\bar{s} \in R$ and $\bar{z} \sim \bar{s}$ then $\bar{z} \in R$.*

Proof. First observe the following. If f is any permutation of A preserving U_0 (i.e. mapping it onto itself) then for any $k \in \omega$ and $\bar{s} \in {}^k A$ we have $\bar{s} \sim_k f(\bar{s})$. Therefore by Definition 2.2 f is an automorphism of \mathcal{A} . Now suppose $a \in U_0 \cap R$ and $b \in U_0$. Then there is an automorphism f of \mathcal{A} mapping a onto b . Since R is definable, f preserves R , thus $b \in R$. It follows, that if $R \cap U_0 \neq \emptyset$ then $U_0 \subseteq R$. Similarly, if $R \cap (A - U_0) \neq \emptyset$ then $A - U_0 \subseteq R$, whence (1) follows.

Now we turn to prove (2). Suppose that the arity of R is k . Since in \mathcal{U}_n there are only n individual variables, it follows that $k \leq n$.

For any equivalence relation $e \subseteq {}^2 k$ let $D_e = \{s \in {}^k A : \ker(s) = e\}$. For any $f : k/e \rightarrow 2$ let $At(f) = \{s \in D_e : (\forall i \in k) s_i \in U_0 \Leftrightarrow f(i/e) = 0\}$. Clearly, every D_e is definable in \mathcal{A}' (in fact, these relations are definable from the identity (equality) relation with a quantifier-free formula of \mathcal{U}_n which doesn't contain any other basic relation symbol). Similarly, every $At(f)$ is definable in \mathcal{A}' . Since ${}^k A$ is the disjoint union of the D_e 's, it is enough to show that for all e the relation $R \cap D_e$ is definable in \mathcal{A}' . Let e be fixed. If $R \cap D_e = \emptyset$ then it is definable in \mathcal{A}' , so we may assume $R \cap D_e \neq \emptyset$.

Suppose $\bar{s} \in R \cap D_e \cap At(f)$ for some $f : k/e \rightarrow 2$. We claim that in this case $At(f) \subseteq R \cap D_e$. To check this suppose $\bar{z} \in At(f)$. Let g be the partial function on A mapping each s_i onto z_i . Since $\ker(\bar{z}) = e = \ker(\bar{s})$, g is a well defined partial function and moreover g is injective. In addition, for every $i \in k$, $s_i \in U_0 \Leftrightarrow z_i \in U_0$. Therefore there is a permutation h of A extending g and preserving U_0 . As observed at the beginning of the proof of (1), h is an automorphism of \mathcal{A} . Since $R \cap D_e$ is definable in \mathcal{A} , it follows that h preserves $R \cap D_e$ and thus $\bar{z} = f(\bar{s}) \in R \cap D_e$, as desired.

Now let $S = \{At(f) : At(f) \cap R \cap D_e \neq \emptyset, f \in {}^{k/e} 2\}$ and let $P = \cup S$. Clearly, P is definable in \mathcal{A}' . We claim, that $P = R \cap D_e$. By the previous paragraph we have $P \subseteq R \cap D_e$. On the other hand, if $\bar{s} \in R \cap D_e$, then for the function $f : k/e \rightarrow 2$,

$f(i/e) = 0 \Leftrightarrow s_i \in U_0$ we have $\bar{s} \in At(f)$, therefore every element of $R \cap D_e$ is contained in an element of S and thus $R \cap D_e \subseteq P$.

For (3) observe that for any $At(f)$, if $\bar{s} \sim \bar{z}$ and $\bar{s} \in At(f)$ then $\bar{z} \in At(f)$. Now by the previous proof of (2), if $\bar{s} \in R$ then $\bar{s} \in At(f) \subseteq R$ for some f and therefore $\bar{z} \in At(f)$ whence $\bar{z} \in R$. \blacksquare

Below we will associate *Cylindric Set Algebras* with relational structures in the usual way. For completeness we recall here the details.

suppose \mathcal{A} is a relational structure. It's n -dimensional Cylindric Set Algebra will be denoted by $Cs_n(\mathcal{A})$. Roughly speaking, the elements of $Cs_n(\mathcal{A})$ are the \mathcal{L}_n -definable relations of \mathcal{A} . To be more precise, elements of $Cs_n(\mathcal{A})$ are n -ary relations. If $\varphi(v_0, \dots, v_{m-1})$ is a formula of \mathcal{L}_n in the vocabulary of \mathcal{A} with free variables as indicated then φ defines an m -ary relation in \mathcal{A} . The corresponding element of $Cs_n(\mathcal{A})$ is the n -ary relation $[\varphi] = \{\bar{s} \in {}^n A : \mathcal{A} \models \varphi[\bar{s}]\}$. The n -dimensional Cylindric Set Algebra $Cs_n(\mathcal{A})$ of \mathcal{A} is the following algebra $\mathcal{B} = \langle X; \cap, -, C_i, D_{i,j} \rangle_{i,j \in n}$. Here $X = \{[\varphi] : \varphi \text{ is a formula of } \mathcal{L}_n\}$ is the set of elements of \mathcal{B} . The operations \cap and $-$ are set-theoretic intersection and complementation (w.r.t. ${}^n A$), respectively. Then for any formulas φ, ψ of \mathcal{L}_n one has

$$[\varphi] \cap [\psi] = [\varphi \wedge \psi] \text{ and} \\ -[\varphi] = [\neg\varphi].$$

In addition, C_i is a unary operation and $D_{i,j}$ is a 0-ary operation for every $i, j \in n$. These operations correspond to the semantics of existential quantifier and the equality symbol of First Order Logic. In more detail, if $[\varphi] \in X$ is an element of \mathcal{B} then

$$C_i([\varphi]) = [\exists v_i \varphi] \text{ and} \\ D_{i,j} = \{s \in {}^n A : s_i = s_j\}.$$

Let V be the vocabulary of \mathcal{A} . Clearly, $\{[R_i] : i \in V\}$ is a set of generators of $Cs_n(\mathcal{A})$.

Lemma 2.5 *The class of (strong) U -structures is closed under ultraproducts.*

Proof. Let $\langle \mathcal{A}_i : i \in I \rangle$ be a system of U -structures and let \mathcal{F} be an ultrafilter on I . Then $\prod_{i \in I} \mathcal{A}_i / \mathcal{F}$ is an U -structure (with core $\prod_{i \in I} U_0^{\mathcal{A}_i} / \mathcal{F}$) because the requirements of Definition 2.2 can be expressed by first order formulas in the expanded vocabulary in which there is an extra symbol for the core relation.

Now let $\langle \mathcal{A}_i : i \in I \rangle$ be a system of strong U -structures, let \mathcal{F} be an ultrafilter on I and let $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i / \mathcal{F}$. Let $V = \{R_0, \dots, R_h\}$ be a finite sub-vocabulary of the common vocabulary of the previous system of structures.

Let $J = \{i \in I : U_0^{\mathcal{A}_i} \text{ is definable in } \mathcal{A}_i|_V\}$. We will show that $J \in \mathcal{F}$ implies

that the core $\Pi_{i \in I} U_0^{\mathcal{A}_i} / \mathcal{F}$ of \mathcal{A} is definable in $\mathcal{A}|_V$. So suppose $J \in \mathcal{F}$. For each $i \in J$ fix a $\mathcal{U}_n[V]$ -formula φ_i defining $U_0^{\mathcal{A}_i}$ in $\mathcal{A}_i|_V$. Observe, that there is a finite number N_0 such that for all $k \leq n$, for all A and for all U_0 the equivalence relation $\sim_{A, U_0, k}$ has at most N_0 equivalence classes. Hence, by Lemma 2.4 (3) there is a finite number N_1 such that $|C_{S_n}(\mathcal{A}_i|_V)| \leq N_1$ for all $i \in I$. For any $i \in I$ let $C_{S_n}^+(\mathcal{A}_i|_V) = \langle C_{S_n}(\mathcal{A}_i|_V), [R_i]_{i \in V} \rangle$, that is, $C_{S_n}^+(\mathcal{A}_i|_V)$ is $C_{S_n}(\mathcal{A}_i|_V)$ expanded with the relations corresponding to the interpretations of elements of V . The set $\{[R_i] : i \in V\}$ generates $C_{S_n}(\mathcal{A}_i|_V)$ therefore for each $i \in I$ there is a finite set T_i of cylindric terms such that for every $a \in C_{S_n}(\mathcal{A}_i|_V)$ there is a $t \in T_i$ with $a = t([R_0], \dots, [R_h])$. We may assume that $C_{S_n}^+(\mathcal{A}_i|_V) \cong C_{S_n}^+(\mathcal{A}_j|_V)$ implies $T_i = T_j$ for all $i, j \in I$. Since V is finite and $|C_{S_n}(\mathcal{A}_i|_V)| \leq N_1$ for all $i \in I$ there exist a finite set T of cylindric terms and $K \subseteq J$ such that $K \in \mathcal{F}$ and for every $i, j \in K$ we have $C_{S_n}^+(\mathcal{A}_i|_V) \cong C_{S_n}^+(\mathcal{A}_j|_V)$ and $T = T_i$. Hence there are a $t \in T$ and $L \subseteq K$ such that $L \in \mathcal{F}$ and for every $i \in L$ we have $[\varphi_i] = t([R_0], \dots, [R_h])$. Let φ be the formula corresponding to $t([R_0], \dots, [R_h])$. Then clearly, φ defines the core of \mathcal{A} .

Next we show that \mathcal{A} is a strong U -structure. Suppose W is (an arbitrary, not necessarily finite) sub-vocabulary of the vocabulary of \mathcal{A} such that the core of \mathcal{A} is not definable in $\mathcal{A}|_W$ and R is an \mathcal{L}_n -definable relation of $\mathcal{A}|_W$. Then there is a finite sub-vocabulary $V \subseteq W$ such that R is definable in $\mathcal{A}|_V$ and still, the core of \mathcal{A} is not definable in $\mathcal{A}|_V$. Applying the result of the previous paragraph to this V , it follows that $J \notin \mathcal{F}$. Finally observe that the required property of R (described in Definition 2.3) can be expressed by a formula of $\mathcal{L}_n[V]$ and this formula is true in \mathcal{A} since for every $i \in I$ \mathcal{A}_i is a strong U -structure. ■

Theorem 2.6 (1) *There exists a strong U -structure.*

(2) *There exists a structure $\mathcal{A} = \langle A, U_0, P, Q \rangle$ which is a strong U -structure with core U_0 such that P and Q are unary relations and $P = Q = U_0$.*

Proof. Since (2) implies (1), it is enough to prove (2). Let A be any countably infinite set, let $U_0 \subseteq A$ be such that $|U_0| = |A - U_0| = \aleph_0$ and finally let $P = Q = U_0$. We have to show that $\mathcal{A} = \langle A, U_0, P, Q \rangle$ is a strong U -structure. Since U_0 is infinite and the basic relations of \mathcal{A} are unary, \mathcal{A} satisfies Definition 2.2 for every $n \in \omega$. Thus, \mathcal{A} is an U -structure (for any $n \in \omega$).

Now suppose V is a sub-vocabulary of the vocabulary of \mathcal{A} such that U_0 is not \mathcal{U}_n -definable in $\mathcal{A}|_V$. It follows that V contains the equality symbol only. Suppose R is an m -ary relation \mathcal{U}_n -definable in $\mathcal{A}|_V$ such that $R \subseteq A^{*m}$. If $R = \emptyset$ then Definition 2.3 holds for R . Now suppose $\bar{s} \in R$ and $\bar{z} \in A^{*m}$. Then there is a permutation f of A mapping \bar{s} onto \bar{z} . Since permutations preserve the identity relation and R is definable in $\mathcal{A}|_V$, it follows that f preserves R and therefore $\bar{z} \in R$. Since $\bar{z} \in A^{*m}$ was arbitrary, $R = A^{*m}$. Clearly, this relation satisfies the requirements of Definition 2.3. So \mathcal{A} is a strong U -structure, as desired. ■

Theorem 2.7 *Suppose \mathcal{A} is a strong U -structure. If $\bar{a}, \bar{b} \in A$ satisfy the same \mathcal{U}_n -formulas in \mathcal{A} then there is an automorphism of \mathcal{A} mapping \bar{a} onto \bar{b} .*

Proof. Let U_0 be the core of \mathcal{A} . First suppose that U_0 can be defined in \mathcal{A} by a \mathcal{U}_n -formula. In this case (since \bar{a} and \bar{b} satisfy the same \mathcal{U}_n -formulas in \mathcal{A}) we have $\bar{a} \sim \bar{b}$. Then there is a permutation f of A preserving U_0 and mapping \bar{a} onto \bar{b} . Then f is an automorphism of $\mathcal{A}' = \langle A, U_0 \rangle$ hence it also preserves all the relations definable in \mathcal{A}' . Hence by Lemma 2.4 (2) f preserves every definable relation of \mathcal{A} as well, particularly, f is an automorphism of \mathcal{A} . (There is another way to prove that f is an automorphism of \mathcal{A} : since f preserves U_0 , for every tuple $\bar{s} \in A$ we have $\bar{s} \sim f(\bar{s})$ hence by Lemma 2.4 (3) it also follows that f is an automorphism of \mathcal{A} .)

Now suppose U_0 is not \mathcal{U}_n -definable in \mathcal{A} . We claim that every relation R definable in \mathcal{A} is definable using the identity relation only. This will be proved by induction on the arity of R . If R is unary then by Lemma 2.4 (1) R is either the empty set or $R = A$; in both cases R is \mathcal{U}_n -definable from the identity relation. Now suppose that $k < n$, R is $k + 1$ -ary, and that the claim is true for any relation with arity at most k . Again, for any equivalence relation $e \subseteq {}^2(k + 1)$ let $D_e = \{s \in {}^{k+1}A : \ker(s) = e\}$. Clearly, D_e is \mathcal{U}_n -definable from the identity relation for any e and $R = \cup_e (R \cap D_e)$. Therefore it is enough to show that $R \cap D_e$ is \mathcal{U}_n -definable from the identity relation. Let $m \subseteq k + 1$ be a set of representatives for e and for any $s \in A^{*m}$ let $s' \in {}^{k+1}A$ be the sequence for which $\ker(s') = e$ and $s = s'|_m$. Let $Q = \{s \in A^{*m} : s' \in R \cap D_e\}$. Then Q is \mathcal{U}_n -definable and $R \cap D_e$ is definable from Q and from the identity relation. If Q is at most unary then we are done because of the basic step of the induction. Otherwise there are distinct $i, j \in m$ and since \mathcal{A} is a strong U -structure, we have $\mathcal{A} \models Q \Leftrightarrow \exists v_i Q \wedge \exists v_j Q \wedge A^{*m}$. But the first two relations in the right hand side are at most k -ary, therefore by the induction hypothesis they are \mathcal{U}_n -definable from the identity relation. Hence Q and therefore $R \cap D_e$ is \mathcal{U}_n -definable in the same way, as well.

So suppose U_0 is not \mathcal{U}_n -definable in \mathcal{A} and \bar{a} and \bar{b} satisfy the same \mathcal{U}_n -formulas in \mathcal{A} . Then $\ker(\bar{a}) = \ker(\bar{b})$ and hence there is a permutation f of A mapping \bar{a} onto \bar{b} . Therefore f preserves the identity relation of \mathcal{A} and thus, by the previous paragraph, f preserves all the definable relations of \mathcal{A} . So f is the required automorphism of \mathcal{A} . ■

Suppose \mathcal{B} is a substructure of \mathcal{A} . If \bar{k}, \bar{k}' are tuples of A with the same length such that $k_j = k'_j$ for every $j \neq i$ then we will write $k \stackrel{i}{\cong} k'$. Recall that by the Tarski-Vaught test \mathcal{B} is an elementary substructure of \mathcal{A} if for any first order formula φ and tuple $\bar{k} \in B$ we have $\mathcal{A} \models \exists v_i \varphi[\bar{k}]$ if and only if there is another tuple $\bar{k}' \in B$ such that $\mathcal{A} \models \varphi[\bar{k}']$ and $\bar{k} \stackrel{i}{\cong} \bar{k}'$. It is also easy to check that \mathcal{B} is a \mathcal{U}_n -elementary substructure of \mathcal{A} if the previous condition holds for every \mathcal{U}_n -formula φ .

Theorem 2.8 (1) *Let \mathcal{A} be a U -structure with core U_0 and suppose $V \subseteq A$ is such that $|V \cap U_0|, |V - U_0| \geq n$. Let \mathcal{B} be the substructure of \mathcal{A} generated by V . Then \mathcal{B} is an U -structure (with core $U_0 \cap V$) which is a \mathcal{U}_n -elementary substructure of \mathcal{A} . \mathcal{A} is a strong U -structure if and only if so is \mathcal{B} .*

(2) *Suppose \mathcal{A} and \mathcal{B} are \mathcal{U}_n -elementarily equivalent U -structures with cores U_0, V_0 , respectively. Then any bijection $f : A \rightarrow B$ mapping U_0 onto V_0 is an isomorphism between \mathcal{A} and \mathcal{B} .*

Proof. To prove (1) we have to verify (the above recalled version of) the Tarski-Vaught test. To do this suppose $\bar{k} \in V$ and φ is a \mathcal{U}_n -formula such that $\mathcal{A} \models \exists v_i \varphi[\bar{k}]$. Let $\bar{k}' \in A$ be a tuple for which $\bar{k} \stackrel{i}{\cong} \bar{k}'$ and $\mathcal{A} \models \varphi[\bar{k}']$. By the condition on V , there is another tuple $\bar{h} \in V$ such that $\bar{h} \sim \bar{k}'$ and $\bar{h} \stackrel{i}{\cong} \bar{k}'$. Therefore it follows from Lemma 2.4 (3) that $\mathcal{A} \models \varphi[\bar{h}]$. This shows that \mathcal{B} is a \mathcal{U}_n -elementary substructure of \mathcal{A} . We claim that \mathcal{B} is an U -structure with core $U_0 \cap V$. To check this suppose R is an m -ary basic relation of \mathcal{B} , $\bar{s} \sim \bar{z} \in {}^m V$ and $\bar{s} \in R^{\mathcal{B}}$. Then $\bar{s} \in R^{\mathcal{A}}$ and since \mathcal{A} is an U -structure, $\bar{z} \in R^{\mathcal{A}}$ hence $\bar{z} \in R^{\mathcal{B}}$, as desired.

Now suppose \mathcal{A} is a strong U -structure. First observe that if W is a sub-vocabulary of the vocabulary of \mathcal{A} then by elementarity, if the core of \mathcal{A} is \mathcal{U}_n -definable in $\mathcal{A}|_W$ then the core of \mathcal{B} is also \mathcal{U}_n -definable in $\mathcal{B}|_W$. In addition, if the core of $\mathcal{A}|_W$ is not definable then by Lemma 2.4 (1) the only unary relations definable in $\mathcal{A}|_W$ are the empty set and the whole universe of \mathcal{A} ; thus, the same is true for $\mathcal{B}|_W$ and therefore in this case the core of $\mathcal{B}|_W$ is also not \mathcal{U}_n -definable. Now suppose W is such a sub-vocabulary that the core of \mathcal{B} is not \mathcal{U}_n -definable in $\mathcal{B}|_W$ and $R^{\mathcal{B}}$ is an m -ary \mathcal{U}_n -definable relation in $\mathcal{B}|_W$ such that $R^{\mathcal{B}} \subseteq V^{*m}$ and $i, j \in m, i \neq j$. Then by elementarity $R^{\mathcal{A}} \subseteq A^{*m}$ because this property of R can be described by a \mathcal{U}_n -formula. As observed, the core of \mathcal{A} cannot be defined in $\mathcal{A}|_W$. Therefore, since \mathcal{A} is a strong U -structure, $\mathcal{A} \models R \Leftrightarrow \exists v_i R \wedge \exists v_j R \wedge A^{*m}$. Again by elementarity the same formula is valid in \mathcal{B} , hence \mathcal{B} is indeed a strong U -structure. A similar argument shows that if \mathcal{A} is not a strong U -structure then \mathcal{B} is also not a strong U -structure.

To show (2) let $f : A \rightarrow B$ be any bijection mapping U_0 onto V_0 . Then clearly, f is an isomorphism between $\langle A, U_0 \rangle$ and $\langle B, V_0 \rangle$. Therefore f preserves any relation which can be defined by a \mathcal{U}_n -formula from U_0 . By Lemma 2.4 (2) every definable (particularly every basic) relation of \mathcal{A} can be defined from U_0 , thus f preserves them. ■

The following is an adaptation of Corollary 6.1.17 of [4].

Theorem 2.9 (*Separation Theorem.*)

Suppose K_0 and K_1 are disjoint classes of strong U -structures with same vocabularies such that both K_0 and K_1 are closed under ultraproducts and \mathcal{U}_n -elementary equivalence. Then there is a \mathcal{U}_n -formula φ with $K_0 \models \varphi$ and $K_1 \models \neg\varphi$.

Proof. Recall that by Lemma 2.5 any ultraproduct of strong U -structures is a strong U -structure.

Let Σ be the set of \mathcal{U}_n -formulas valid in K_0 . Suppose, seeking a contradiction, that there is no φ satisfying the requirements of the theorem. It follows that every finite subset of Σ also has a model in K_1 . Since K_1 is closed under ultraproducts, there is a strong U -structure $\mathcal{A}_1 \in K_1$ such that $\mathcal{A}_1 \models \Sigma$. In addition, if Ψ is a finite set of \mathcal{U}_n -formulas valid in \mathcal{A}_1 then Ψ has a model in K_0 (otherwise $K_0 \models \neg(\bigwedge \Psi)$ and hence $\neg(\bigwedge \Psi) \in \Sigma$ would follow, therefore we would have $\mathcal{A}_1 \models \neg(\bigwedge \Psi)$). Since K_0 is closed under ultraproducts there is an $\mathcal{A}_0 \in K_0$ which is \mathcal{U}_n -elementarily equivalent with \mathcal{A}_1 .

Summing up, $\mathcal{A}_0 \in K_0, \mathcal{A}_1 \in K_1$ and \mathcal{A}_0 and \mathcal{A}_1 are \mathcal{U}_n -elementarily equivalent. This is impossible because K_0 and K_1 are disjoint classes and both are closed under \mathcal{U}_n -elementary equivalence. \blacksquare

3 Interpolation

We start this section by recalling the weak and strong forms of Craig's Interpolation Theorem. Suppose \mathcal{L} is a logic (in the sense of the beginning of Section 2) and φ is a formula of \mathcal{L} . Then $\models \varphi$ means that for any model \mathcal{A} for \mathcal{L} , φ is valid in \mathcal{A} . If ψ is another formula of \mathcal{L} then, as expected, $\varphi \models \psi$ means that ψ is valid in every model in which φ is valid.

Definition 3.1 *A logic \mathcal{L} has the Strong Craig Interpolation Property if for any pair of formulas φ, ψ of \mathcal{L} the following holds. If $\varphi \models \psi$ then there is a formula ϑ such that $\varphi \models (\varphi \Rightarrow \vartheta) \wedge (\vartheta \Rightarrow \psi)$ and the relation symbols occurring in ϑ occur both in φ and in ψ .*

A logic \mathcal{L} has the Weak Craig Interpolation Property if for any pair of formulas φ and ψ of \mathcal{L} the following holds. If $\varphi \models \psi$ then there is a formula ϑ such that $\varphi \models \vartheta$ and $\vartheta \models \psi$ and the relation symbols occurring in ϑ occur both in φ and in ψ .

Lemma 3.2 *Suppose φ is a \mathcal{U}_n -formula and $V \subseteq \text{voc}(\varphi)$ is a vocabulary. Then the class K of V -reducts of \mathcal{U}_n -models of φ is closed under \mathcal{U}_n -elementary equivalence.*

Proof. Suppose $\mathcal{A}_0 \in K$ and \mathcal{A}_1 is \mathcal{U}_n -elementarily equivalent with \mathcal{A}_0 . Let U_0 and U_1 be the cores of \mathcal{A}_0 and \mathcal{A}_1 , respectively. Let \mathcal{A}_0^+ be an expansion of \mathcal{A}_0 which is a model of φ . Let C and $C_0 \subseteq C$ be sets such that $|C_0| \geq |U_0|, |U_1|$ and $|C - C_0| \geq |A_0 - U_0|, |A_1 - U_1|$. According to these cardinal conditions we may (and will) assume $U_0, U_1 \subseteq C_0$ and $A_0 - U_0, A_1 - U_1 \subseteq C - C_0$.

We will define three U -structures on C as follows. The core of these structures will be C_0 . For any k -ary basic relation $R^{\mathcal{A}_0^+}$ of \mathcal{A}_0^+ let

$$R^C = \{s \in {}^k C : (\exists z \in R^{\mathcal{A}_0^+}) s \sim z\}$$

and for any k -ary basic relation S^{A_1} of \mathcal{A}_1 let

$$S^{A_1} = \{s \in {}^k C : (\exists z \in S^{A_1}) s \sim z\}.$$

Finally let

$$\mathcal{C}_0^+ = \langle C, C_0, R^C \rangle_{R \in \text{voc}(\mathcal{A}_0^+)}, \quad \mathcal{C}_1 = \langle C, C_0, S^C \rangle_{S \in \text{voc}(\mathcal{A}_1)} \quad \text{and}$$

let \mathcal{C}_0 be the V -reduct of \mathcal{C}_0^+ .

By Theorem 2.8 (1) \mathcal{A}_0 and \mathcal{A}_1 are \mathcal{U}_n -elementary substructures of \mathcal{C}_0 and \mathcal{C}_1 , respectively. Therefore, since \mathcal{A}_0 and \mathcal{A}_1 are strong U -structures, by Theorem 2.8 (1) \mathcal{C}_0 and \mathcal{C}_1 are strong U -structures and moreover \mathcal{C}_0 and \mathcal{C}_1 are \mathcal{U}_n -elementarily equivalent. Similarly, \mathcal{C}_0^+ is a model of φ (and is a strong U -structure). Let f be the identity function on C . By Theorem 2.8 (2) f is an isomorphism between \mathcal{C}_0 and \mathcal{C}_1 . Let \mathcal{C}_1^+ be the expansion of \mathcal{C}_1 for which f remains an isomorphism between \mathcal{C}_0^+ and \mathcal{C}_1^+ (that is, for every $R \in \text{voc}(\mathcal{C}_0^+) - \text{voc}(\mathcal{C}_0)$ interpret $R^{\mathcal{C}_1^+}$ as $R^{\mathcal{C}_1^+} = R^{\mathcal{C}_0^+}$). Clearly, $\mathcal{C}_1^+ \models \varphi$ and \mathcal{C}_1^+ is a strong U -structure. Let \mathcal{A}_1^+ be the substructure of \mathcal{C}_1^+ generated by A_1 . Then by Theorem 2.8 (1) \mathcal{A}_1^+ is a \mathcal{U}_n -elementary substructure of \mathcal{C}_1^+ and therefore $\mathcal{A}_1^+ \models \varphi$ (and clearly, \mathcal{A}_1^+ is a strong U -structure by the last sentence of the statement of Theorem 2.8 (1)). In addition \mathcal{A}_1 is the V -reduct of \mathcal{A}_1^+ and therefore $\mathcal{A}_1 \in K$. ■

Theorem 3.3 *The logic \mathcal{U}_n has the Weak Craig Interpolation Property.*

Proof. Suppose φ and ψ are \mathcal{U}_n -formulas such that $\varphi \models \psi$. Let V be the vocabulary consisting of the relation symbols occurring both in φ and in ψ . Let K_0 be the class of V -reducts of models of φ and let K_1 be the class of V -reducts of models of $\neg\psi$. Clearly, K_0 and K_1 are closed under ultraproducts and by Lemma 3.2 K_0 and K_1 are closed under \mathcal{U}_n -elementary equivalence. Since $\varphi \models \psi$, it follows that K_0 and K_1 are disjoint. Therefore by the Separation Theorem 2.9 there is a \mathcal{U}_n -formula ϑ (in the common vocabulary V of K_0 and K_1) such that $K_0 \models \vartheta$ and $K_1 \models \neg\vartheta$. But then $\varphi \models \vartheta$ and $\vartheta \models \psi$, thus ϑ is the required weak interpolant. ■

Theorem 3.4 *If $n \geq 3$ then the logic \mathcal{U}_n doesn't have the Strong Craig Interpolation Property.*

Proof. Let P and Q be two distinct unary relation symbols. Throughout this proof we will use the vocabulary consisting the equality symbol, P and Q . Let $\varphi(x, y) = P(x) \Leftrightarrow \neg P(y)$ and let $\psi(x, y, z) = (Q(x) \Leftrightarrow Q(z)) \vee (Q(y) \Leftrightarrow Q(z))$.

First we show that in the class of strong U -structures

$$(*) \quad \models \varphi \Rightarrow \psi.$$

To do this assume $\mathcal{A} \models \varphi[a, b]$ where \mathcal{A} is a strong U -structure with core U_0 and $a, b \in A$. Since P is a unary definable relation of \mathcal{A} , it follows from Lemma 2.4 (1) that $P \in \{\emptyset, A, U_0, A - U_0\}$. According to our assumption $\mathcal{A} \models \varphi[a, b]$, either $P = U_0$ or $P = A - U_0$. In both cases it follows that exactly one of $\{a, b\}$ is in U_0 . Similarly, since Q is a unary definable relation in \mathcal{A} , by Lemma 2.4 (1) it follows that $Q \in \{\emptyset, A, U_0, A - U_0\}$. In the first two cases $\mathcal{A} \models \psi[a, b, c]$, for any $c \in A$. Now suppose Q is either U_0 or $A - U_0$. Then exactly one of $\{a, b\}$ is in Q . Therefore for any $c \in A$ we have $\mathcal{A} \models \psi[a, b, c]$. Thus, $(*)$ is true.

Now suppose, seeking a contradiction, that \mathcal{U}_n has the Strong Craig Interpolation Property. Then there exists a formula ϑ in which the only relation symbol may be the equality-symbol such that $\models (\varphi \Rightarrow \vartheta) \wedge (\vartheta \Rightarrow \psi)$. Now let $\mathcal{A} = \langle A, U_0, P, Q \rangle$ be the strong U -structure described in Theorem 2.6 (2). Let $a \in U_0, b \in A - U_0, a', b' \in U_0, a' \neq b', c \in A - U_0 - \{b\}$. Then $\mathcal{A} \models \varphi[a, b, c]$ therefore $\mathcal{A} \models \vartheta[a, b, c]$. Observe that there is a permutation f of A with $f(a) = a', f(b) = b', f(c) = c$. Since the only relation symbol that may occur in ϑ is the equality, it follows that $\mathcal{A} \models \vartheta[f(a), f(b), f(c)]$ and thus $\mathcal{A} \models \vartheta[a', b', c]$. Therefore, since ϑ is a strong interpolant, $\mathcal{A} \models \psi[a', b', c]$ would follow, but this contradicts to the choice of a', b', c . ■

Let \mathcal{U}_ω be the logic

- whose formulas are that of usual First Order Logic with ω many individual variables (but again, the vocabularies contain relation symbols only) and
- whose models are the strong U -structures.

Then \mathcal{U}_ω does not have the Strong Craig Interpolation Property because the proofs of Lemma 2.4 (1) and Theorem 3.4 can be repeated in this case, as well.

On the other hand \mathcal{U}_ω still has the Weak Craig Interpolation Property. To check this, suppose φ and ψ are formulas of \mathcal{U}_ω such that $\varphi \models \psi$. Then there exists an $n \in \omega$ for which φ and ψ are formulas of \mathcal{U}_n . It is easy to see that " $\varphi \models \psi$ in the sense of \mathcal{U}_ω " holds if and only if " $\varphi \models \psi$ in the sense of \mathcal{U}_n ". Hence by Theorem 3.3 the required interpolant exists in \mathcal{U}_n and consequently in \mathcal{U}_ω as well.

Thus, \mathcal{U}_ω is an example for a logic with infinitely many individual variables that has the Weak Craig Interpolation Property but does not have the Strong Craig Interpolation Property.

4 Definability

The goal of this section is to prove that \mathcal{U}_n has the Beth Definability Property. For completeness we start by recalling the relevant definitions.

Definition 4.1 Let \mathcal{L} be a logic, let $L \subseteq L^+$ be vocabularies for \mathcal{L} and suppose R is the unique relation symbol of L^+ not occurring in L . Suppose T^+ is a theory in $\mathcal{L}[L^+]$.

- We say that T^+ implicitly defines R over L if the following holds. If $\mathcal{A}, \mathcal{B} \models T^+$ and the L -reducts of \mathcal{A} and \mathcal{B} are the same (that is, the identity function on A is an isomorphism between them) then \mathcal{A} and \mathcal{B} are the same.
- We say that R can be explicitly defined in T^+ over L if there is a formula of $\mathcal{L}[L]$ which is equivalent with R in every model of T^+ .
- We say that \mathcal{L} has the Beth Definability Property if for any L, L^+, T^+ whenever T^+ implicitly defines R over L then R can be explicitly defined in T^+ over L .

Now we prove a Svenonius-type definability theorem for \mathcal{U}_n . The construction is essentially the same as Theorem 10.5.1 and Corollary 10.5.2 of [8].

Theorem 4.2 Suppose $L \subseteq L^+$ are vocabularies for \mathcal{U}_n , R is the unique relation symbol of L^+ not occurring in L and T^+ is a complete theory in L^+ . Then the following are equivalent.

- (1) R can be explicitly defined in T^+ over L .
- (2) If $\mathcal{A} \models T^+$ and f is an automorphism of $\mathcal{A}|_L$ then f preserves $R^{\mathcal{A}}$ as well.

Proof. Clearly, (2) follows from (1). To prove the converse implication suppose (2) holds and suppose, seeking a contradiction, that R cannot be explicitly defined in T^+ over L . Suppose that R is k -ary for some $k \leq n$. Expand L^+ by two k -tuples \bar{c}, \bar{d} which are new constant symbols and let $\Gamma = \{\varphi(\bar{c}) \Leftrightarrow \varphi(\bar{d}) : \varphi \text{ is a } \mathcal{U}_n[L]\text{-formula}\}$. Consider the following first order theory Σ (since in \mathcal{U}_n constant symbols are not part of the language, strictly speaking the following Σ is not a theory in \mathcal{U}_n).

$$\Sigma = T^+ \cup \Gamma \cup \{R(\bar{c}), \neg R(\bar{d})\}.$$

We claim that every finite subset Σ_0 of Σ has a model whose L^+ -reduct is a strong U -structure. To show this suppose, seeking a contradiction, that Σ_0 is a finite subset of Σ which doesn't have such a model. Let $\Gamma' = \Sigma_0 \cap \Gamma = \{\varphi_i(\bar{c}) \Leftrightarrow \varphi_i(\bar{d}) : i < m\}$. Observe that

$$(*) \text{ if } \mathcal{A} \models T^+ \text{ is a strong } U\text{-structure, } \bar{a}, \bar{b} \in A, \langle \mathcal{A}, \bar{a}, \bar{b} \rangle \models \Gamma' \text{ and } \mathcal{A} \models R(\bar{b}) \\ \text{then } \mathcal{A} \models R(\bar{a})$$

because otherwise $\langle \mathcal{A}, \bar{b}, \bar{a} \rangle$ would be a model of Σ_0 whose L^+ -reduct is a strong U -structure. Let $\Phi = \{\varphi_i(\bar{v}) : i < m\}$. Suppose $\mathcal{A} \models T^+$ and $\bar{a} \in A$. Then the Φ -type of \bar{a} in \mathcal{A} is defined as follows:

$$\Phi - tp^{\mathcal{A}}(\bar{a}) = \{\varphi_i(\bar{v}) : \mathcal{A} \models \varphi_i(\bar{a}), i < m\} \cup \{\neg\varphi_j(\bar{v}) : \mathcal{A} \not\models \varphi_j(\bar{a}), i < m\}.$$

Let $\varrho = \bigvee \{ \bigwedge \psi : \text{there are a strong } U\text{-structure } \mathcal{A} \models T^+ \text{ and } \bar{a} \in R^{\mathcal{A}} \text{ such that } \psi = \Phi - tp^{\mathcal{A}}(\bar{a}) \}$. Clearly, ϱ is a formula of $\mathcal{U}_n[L]$. We claim that ϱ defines explicitly R in T^+ over L . To verify this suppose $\mathcal{A} \models T^+$. If $\bar{a} \in R^{\mathcal{A}}$ then $\bigwedge(\Phi - tp^{\mathcal{A}}(\bar{a}))$ is a disjunctive component of ϱ therefore $\mathcal{A} \models \varrho(\bar{a})$. Thus, the relation defined by ϱ in \mathcal{A} contains $R^{\mathcal{A}}$. Conversely, suppose $\mathcal{A} \models \varrho(\bar{b})$. Then there is a disjunctive component $\bigwedge \psi$ of ϱ such that $\mathcal{A} \models \bigwedge \psi(\bar{b})$ and there are another strong U -structure $\mathcal{A}' \models T^+$ and $\bar{a}' \in R^{\mathcal{A}'}$ such that $\bigwedge \psi = \bigwedge(\Phi - tp^{\mathcal{A}'}(\bar{a}'))$. Thus, $\mathcal{A}' \models \exists \bar{v}(R(\bar{v}) \wedge \bigwedge \psi(\bar{v}))$. This last formula is a \mathcal{U}_n -formula, and since T^+ is complete, $\mathcal{A} \models \exists \bar{v}(R(\bar{v}) \wedge \bigwedge \psi(\bar{v}))$. Thus, there is $\bar{a} \in R^{\mathcal{A}}$ such that $\bigwedge(\Phi - tp^{\mathcal{A}}(\bar{a})) = \bigwedge \psi = \bigwedge(\Phi - tp^{\mathcal{A}}(\bar{b}))$. Therefore by (*) it follows that $\mathcal{A} \models R(\bar{b})$.

We proved that ϱ explicitly defines R in T^+ over L . This is impossible because we assumed that R cannot be explicitly defined. Hence every finite subset of Σ has a model whose L^+ -reduct is a strong U -structure.

Let $\langle \mathcal{A}, \bar{a}, \bar{b} \rangle$ be an ultraproduct of the above models of finite subsets of Σ for which $\langle \mathcal{A}, \bar{a}, \bar{b} \rangle \models \Sigma$. By Lemma 2.5 the L^+ -reduct of it (which is \mathcal{A}) is a strong U -structure. Since $\Gamma \subseteq \Sigma$, it follows that \bar{a} and \bar{b} satisfies the same $\mathcal{U}_n[L]$ -formulas. Therefore by Theorem 2.7 there is an automorphism of the L -reduct of \mathcal{A} mapping \bar{a} onto \bar{b} . This automorphism doesn't preserve $R^{\mathcal{A}}$, contradicting to (2). This proves that R can be explicitly defined in T^+ over L . \blacksquare

Theorem 4.3 *The logic \mathcal{U}_n has the Beth Definability Property.*

Proof. Let L, L^+, R and T^+ be as in Definition 4.1 and assume T^+ implicitly defines R over L . We have to show that R can be explicitly defined in T^+ over L .

First suppose that T^+ is a complete theory. Suppose \mathcal{A} is a model of T^+ and f is an automorphism of $\mathcal{A}|_L$. We claim that f preserves $R^{\mathcal{A}}$ as well. To see this, define another structure \mathcal{B} as follows. The universe of \mathcal{B} is A . For any subset X of (a direct power of) A the f -image of X will be denoted by $f[X]$. For every $P \in L$ let $P^{\mathcal{B}} = f[P^{\mathcal{A}}]$, let $R^{\mathcal{B}} = f[R^{\mathcal{A}}]$ and let $U' = f[U]$ where U is the core of \mathcal{A} . Since f is an automorphism of $\mathcal{A}|_L$, it follows that $\mathcal{A}|_L = \mathcal{B}|_L$. In addition, f is an isomorphism between $\langle \mathcal{A}, U \rangle$ and $\langle \mathcal{B}, U' \rangle$. Therefore \mathcal{B} is a strong U -structure with core U' and $\mathcal{B} \models T^+$. Since T^+ implicitly defines R over L , it follows that $R^{\mathcal{A}} = R^{\mathcal{B}}$, that is, f preserves $R^{\mathcal{A}}$. Since \mathcal{A} and f were chosen arbitrarily, it follows that every automorphism of the L -reduct of a model of T^+ also preserves the interpretation of R . Therefore by Theorem 4.2 R can be explicitly defined in T^+ over L .

Now let T^+ be an arbitrary (not necessarily complete) theory which implicitly defines R over L . We claim that there is a finite set $\Phi = \{\varphi_0, \dots, \varphi_{m-1}\}$ of $\mathcal{U}_n[L]$ -formulas such that if $\mathcal{A} \models T^+$ then

$$(*) \quad \mathcal{A} \models \bigvee_{i < m} (\forall v_0 \dots \forall v_{n-1} (R \Leftrightarrow \varphi_i)).$$

For if not, then for any finite set Φ of $\mathcal{U}_n[L]$ -formulas it would exist a model of

T^+ in which R would be different from all the relations defined by the members of Φ . Forming an ultraproduct of these models it would exist a strong U -structure $\mathcal{A} \models T^+$ in which $R^{\mathcal{A}}$ would not be definable in $\mathcal{A}|_L$. But then $T' = \{\varphi : \mathcal{A} \models \varphi, \varphi \text{ is a } \mathcal{U}_n[L^+]\text{-formula}\}$ would be a complete theory and since $T^+ \subseteq T'$, T' also implicitly defines R over L . Therefore by the second paragraph of this proof R would be explicitly definable in T' and particularly, there would be a $\mathcal{U}_n[L]$ -formula which would define $R^{\mathcal{A}}$ in \mathcal{A} ; a contradiction. Therefore $(*)$ is established.

Now for each $\mathcal{A} \models T^+$ let $\nu(\mathcal{A})$ be the smallest $i \in m$ for which $\mathcal{A} \models R \Leftrightarrow \varphi_i$ and let $K_i = \{\mathcal{A}|_L : \mathcal{A} \models T^+, \nu(\mathcal{A}) = i\}$. Clearly, the classes K_i are pairwise disjoint and closed under ultraproducts. In fact they are closed under \mathcal{U}_n -elementary equivalence because of the following. Suppose $\mathcal{A} \in K_i$ and \mathcal{A} and \mathcal{B} are \mathcal{U}_n -elementarily equivalent. Let U_0, V_0 be the cores of \mathcal{A} and \mathcal{B} , respectively. Let $C_0 \subseteq C$ be two sets such that $|C_0| \geq |U_0|, |V_0|$ and $|C - C_0| \geq |A - U_0|, |B - V_0|$. Then we may assume that $U_0, V_0 \subseteq C_0, A - U_0, B - V_0 \subseteq C - C_0$. We will define two U -structures on C as follows. If R is any m -ary relation symbol in L then let $R^{C_0} = \{s \in {}^m C : (\exists z \in R^{\mathcal{A}})s \sim z\}$ and let $R^{C_1} = \{s \in {}^m C : (\exists z \in R^{\mathcal{B}})s \sim z\}$. Then by Theorem 2.8 (1) \mathcal{A} and \mathcal{B} are \mathcal{U}_n -elementary substructures of \mathcal{C}_0 and \mathcal{C}_1 , respectively. Therefore \mathcal{C}_0 and \mathcal{C}_1 are strong U -structures and \mathcal{U}_n -elementarily equivalent with each other. Hence by Theorem 2.8 (2) the identity function on C is an isomorphism between \mathcal{C}_0 and \mathcal{C}_1 . Since $\mathcal{A} \in K_i$, i is the smallest number for which R and φ_i are equivalent in \mathcal{A} . Let R^{C_0} be the relation defined by φ_i in \mathcal{C}_0 . Since \mathcal{A} is a \mathcal{U}_n -elementary substructure of \mathcal{C}_0 , it follows that

$$(i) \quad \langle \mathcal{C}_0, R^{C_0} \rangle \models T^+.$$

Since T^+ implicitly defines R , this is the only way to extend \mathcal{C}_0 to a model of T^+ . In particular,

$$(ii) \quad \text{for every } j < i \text{ we have } \mathcal{C}_0 \not\models R^{C_0} \Leftrightarrow \varphi_j.$$

Since the identity function of C is an isomorphism between \mathcal{C}_0 and \mathcal{C}_1 , the above (i) and (ii) are true for \mathcal{C}_1 as well. Let $R^{\mathcal{B}} = R^{C_0} \cap {}^m B$. Then by Theorem 2.8 (1) $\langle \mathcal{B}, R^{\mathcal{B}} \rangle \models T^+$ and i is a smallest number for which φ_i defines $R^{\mathcal{B}}$ in \mathcal{B} . Thus, $\mathcal{B} \in K_i$, as desired.

Now by Theorem 2.9 for every $i \in m$ there is a $\mathcal{U}_n[L]$ -formula ϱ_i such that $K_i \models \varrho_i$ and $\bigcup_{j \in m - \{i\}} K_j \models \neg \varrho_i$. Finally let $\psi = \bigvee_{i \in m} (\varrho_i \wedge \varphi_i)$. It is easy to check that ψ is equivalent with R in every model of T^+ , thus R can be explicitly defined in T^+ over L , as desired. \blacksquare

5 Cylindric Algebraic Consequences

By translating the results of the previous sections to Algebraic Logic, in this section we prove that for finite $n \geq 3$, there is a (finitely axiomatizable) subvariety of RCA_n that has the Strong Amalgamation Property but doesn't have the Superalgamation Property (the definitions of these properties can be found for example in [1] before Definition 6.14). As we mentioned this settles a problem of Pigozzi in [12].

We assume that the reader is familiar with the theory of cylindric algebras. Some basic facts on this topic have been recalled before Lemma 2.5. For more details we refer to [6] and [7].

If K is a class of algebras then \mathbf{SK} and \mathbf{PK} denote the classes of (isomorphic copies of) subalgebras of members of K and (isomorphic copies of) direct products of members of K , respectively. Similarly, $\mathbf{Up}K$ denotes the class of (isomorphic copies of) ultraproducts of members of K . For other algebraic notions and notation we refer to [3].

Definition 5.1 US_n and U_n are defined to be the following subclasses of RCA_n :

$$\begin{aligned} US_n &= \{C_{S_n}(\mathcal{A}) : \mathcal{A} \text{ is a strong } U\text{-structure}\}. \\ U_n &= \mathbf{SP}US_n. \end{aligned}$$

Theorem 5.2 U_n is a finitely axiomatizable variety.

Proof. Lemma 2.4 (3) implies that there is a natural number N_1 such that for all strong U -structure \mathcal{A} we have $|C_{S_n}(\mathcal{A})| \leq N_1$ (we already observed this in the proof of Lemma 2.5). Therefore US_n is finite and hence $\mathbf{Up}US_n = US_n$. So $U_n = \mathbf{SP}US_n \subseteq \mathbf{SPUp}US_n = \mathbf{SP}US_n = U_n$. Hence U_n is the quasi-variety generated by US_n . The cylindric term $c_0 \dots c_{n-1}(x)$ is a switching-function in $\mathbf{S}US_n$ therefore the quasi-variety and the variety generated by US_n coincide. Thus U_n is the variety generated by US_n .

Finally observe that U_n is congruence-distributive since it has a Boolean reduct. Thus, U_n is a finitely generated congruence-distributive variety and hence by Baker's Theorem it is finitely axiomatizable (see [2] or [3]). ■

Now we return to the choice of naming our logic \mathcal{U}_n and the classes US_n and U_n . By Lemma 2.4 (2) every member of US_n is a subalgebra of the n -dimensional Cylindric Set Algebra generated by one UNARY relation: by the core of the corresponding structure. So " U " stands for "unary".

Now we are ready to prove the main theorem of the paper.

Theorem 5.3 (1) U_n has the Strong Amalgamation Property.

(2) U_n doesn't have the Superalgamation Property, if $n \in \omega, n \geq 3$.

Proof. (1) By theorem 4.3 \mathcal{U}_n has the Beth Definability Property and therefore by [11] the epimorphisms of U_n are surjective (see also [1], Theorem 6.11). By Theorem 3.3 \mathcal{U}_n has the Weak Craig Interpolation Property and by Theorem 5.2 U_n is a variety. Thus, by Theorem 6.15(i) of [1] (see also the beginning of Section 7 therein) U_n has the Amalgamation Property. Since U_n is a variety, it follows from Propositions 1.9 and 1.11 of [9] (see also Proposition 6.3 therein) that U_n indeed has the Strong Amalgamation Property.

(2) By Theorem 3.4 \mathcal{U}_n doesn't have the Strong Craig Interpolation Property and therefore by Theorem 6.15 (ii) of [1] U_n doesn't have the Superamalgamation Property. ■

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