EF EQUIVALENT NOT ISOMORPHIC PAIR OF MODELS SH907

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Abstract. We construct non-isomorphic models M, N, e.g. of cardinality $\aleph_1$ such that in the Ehrenfeucht-Fraissé game of any length $\zeta < \omega_1$ the isomorphism player wins.

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§0 Introduction

About 20 years ago, Heikki Tuuri in his thesis [Tur90] supervised by Väännänen, ask (for length $< \omega^3$ consistently the answer is yes).

0.1 Question: Are there models M, N, E.F. equivalent for the game of length ω^3 but not for the game of length ω_1, preferably M, N are of cardinality $\aleph_1$?

On the history see Väännänen [Va95], which ask me the question and get a fair amount of attention. Subsequently [Sh 836] showed that for most regular λ we have

- $(*)_\lambda$ there are models M, N of cardinality λ such that
  - (a) for any ordinal $\zeta < \lambda$ in the Ehrenfeucht-Fraissé game of length ζ for the pair (M, N), the isomorphism player wins.
  - (b) M, N are not isomorphic.

By "most regular λ" we mean $\lambda = \lambda^{\aleph_0}$. This was continued in Havlin Shelah [HvSh 866] which proved it for "almost" all regular λ: if $\lambda \geq \beth_\omega$ or if $\lambda > 2^{\aleph_0}$ assuming a very weak statement in pcf theory, quite possibly provable in ZFC. However, if $\lambda = \aleph_1 < 2^{\aleph_0}$ this does not help so the problem as stated remained open.

Here at last the question as stated is given a positive answer.

We construct a pair of non-isomorphic models of cardinality $\aleph_1$ which are equivalent for the EF-game of length ζ iff $\zeta < \omega_1$. We then prove $(*)_\lambda$ for every regular uncountable λ.

It is natural to assume that the proof would be more complicated than [Sh 836] but in fact it seems simpler and does not require any special background. It uses not just "abelian groups without zero" but also some derived objects giving more leeway in the game.

Note, however, that the method here is ad-hoc, whereas in [Sh 836], [HvSh 866] seem to me systematic. Hence their method should be helpful in more demanding related problems, in particualr hopefully for fat theories (see [Sh 897]).

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0.2 Definition. 1) We say that M_1, M_2 are EF-equivalent for the game of length α (or EF_α-equivalent) if M_1, M_2 are models (with same vocabulary) such that the isomorphism player has a winning strategy in the game $\Game^\alpha_1(M_1, M_2)$ defined below.
1A) Replacing α by $< \alpha$ means: for every $\beta < \alpha$; similarly below.
2) We say that M_1, M_2 are $\text{EF}_{\alpha,<\mu}$-equivalent when M_1, M_2 are models with the same vocabulary such that the isomorphism player has a winning strategy in the game $\Game^\alpha_\mu(M_1, M_2)$ defined below.

3) For M_1, M_2, α, μ as above and partial isomorphism f from M_1 into M_2 (e.g. the empty one) we define the game $\Game^\alpha_\mu(f, M_1, M_2)$ between the players ISO (the isomorphism player) and AIS (the anti-isomorphism player) as follows:

- (a) the play lasts α moves
- (b) after β moves a partial isomorphism f_β from M_1 into M_2 has been chosen increasing continuous with β
- (c) in the $(\beta+1)$-th move, the player AIS chooses $A_{\beta,1} \subseteq M_1, A_{\beta,2} \subseteq M_2$ such that $|A_{\beta,1}| + |A_{\beta,2}| < 1 + \mu$ and then the player ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that $A_{\beta,1} \subseteq \mathrm{Dom}(f_{\beta+1})$ and $A_{\beta,2} \subseteq \mathrm{Rang}(f_{\beta+1})$
- (d) if $\beta = 0$, ISO chooses $f_0 = f$; if β is a limit ordinal ISO chooses $f_\beta = \cup\{f_\gamma : \gamma < \beta\}$.

The ISO player loses if he had no legal move.
4) If $f = \emptyset$ we may write $\Game^\alpha_\mu(M_1, M_2)$. If μ is 1 we may omit it. We may write $\leq \mu$ instead of $< \mu^+$.

Recall

0.3 Observation. If $\lambda = \mu^+$ and M, N are τ-models and $\zeta < \lambda$ is divisible by λ^ω then

- (a) if ISO wins/does not lose in $\Game^\zeta_1(M, N)$ <u>then</u> it wins/does not lose in $\Game^\zeta_\lambda(M, N)$
- (b) if AIS wins/does not lose in $\Game^\zeta_1(M, N)$ <u>then</u> it wins/does not lose in $\Game^\zeta_\lambda(M, N)$ in fact
- (c) if $\mu_1 \leq \mu_2$ and AIS wins/does not lose in $\Game^\zeta_{\mu_1}(M, N)$ <u>then</u> it wins/does not lose in $\Game^\zeta_{\mu_2}(M, N)$.

§1 Models of cardinality $\aleph_1$

1.1 Choice: 1) Let G be a vector space of $\mathbb{Z}/2\mathbb{Z}$ of dimension (and cardinality) $\aleph_0$, with basis $\langle x_n : n < \omega \rangle$.
2) Let G^0_n be the subspace of G generated by $\{x_k : k < \omega, k \neq n\}$ and $G^1_n = x_n + G^0_n$.
3) Let $\mathscr{G} = \{G^\ell_n : n < \omega, \ell \in \{0,1\}\}$.

1.2 Observation. If $x \in G$ and $n < \omega, \ell \in \{0,1\}$ then:

- (a) $x + G^\ell_n := \{x + y : y \in G^\ell_n\} \in \{G^0_n, G^1_n\}$
- (b) x has a unique representation as $x = \Sigma\{x_k : k \in u\}, u \subseteq \omega$ finite, call $u =$ supp(x)
- (c) $x + G^\ell_n = G^\ell_n \Leftrightarrow n \notin$ supp(x).

1.3 Construction. We define a structure M:

(A) the universe of M is the disjoint union of:

- (a) $A_\alpha = \{\alpha\} \times G$ for $\alpha < \omega_1$
- (b) $B_\alpha = \{\eta : \eta \in {}^\alpha\mathscr{G}$ and for some $n = n_\eta$ we have
$$\aleph_0 > |\{\beta < \alpha : \eta(\beta) \neq G^0_n\}|\}$$
for $\alpha < \omega_1$ where $\mathscr{G}$ is from 1.1(3) (if $\alpha \geq \omega, n_\eta$ is unique, if $\alpha < \omega$ let $n_\eta = 0$).
That is $|M| = \cup\{A_\alpha \cup B_\alpha : \alpha < \omega_1\}$ and without loss of generality the A_α's, B_α's are pairwise disjoint

(B) relations (P_1, P_2 unary predicates, F_y unary function symbol for each $y \in G$ and E_1, E_2, R binary predicates):

- (a) $P^M_1 = \bigcup\limits_{\alpha<\omega_1} A_\alpha$
- (b) $E^M_1 = \{(a,b) : (\exists \alpha \leq \beta < \omega_1)(a \in A_\alpha \wedge b \in A_\beta)\}$
- (c) $P^M_2 = \bigcup\limits_\alpha B_\alpha$
- (d) $E^M_2 = \{(a,b) : (\exists \alpha \leq \beta < \omega_1)(a \in B_\alpha \wedge b \in B_\beta)\}$
- (e) for $y \in G$ the function F^M_y is defined as
  - (α) $F^M_y((\alpha, x)) = (\alpha, x + y)$ for $x \in G, \alpha < \omega_1$
  - (β) $F^M_y \restriction B_\alpha$ is the identity (for every $\alpha < \omega_1$ of course)

(f) $R^M = \{(\eta,(\alpha,x))$: for some $\beta < \omega_1$ we have
  (α) $\eta \in B_\beta \subseteq {}^\beta\mathscr{G}$
  (β) $\alpha < \beta$
  (γ) $x \in \eta(\alpha)\}$

1.4 Definition. 1) Let $M_{<\alpha} = M \restriction (\cup\{A_\beta \cup B_\beta : \beta < \alpha\})$.
2) If $\gamma < \omega_1$ and $\nu \in {}^\gamma G$ satisfies $\beta < \gamma \Rightarrow (\forall n)(\exists^{<\aleph_0}\alpha < \beta)[n \in \text{ supp}(\nu(\alpha))]$ <u>then</u> we define f_ν as a function with domain $M_{<\gamma}$ by:

(a) if $(\alpha,x) \in A_\alpha$ and $\alpha < \gamma$ then $f_\nu((\alpha,x)) = (\alpha, x + \nu(\alpha))$
(b) if $\eta_1 \in B_\beta$ and $\beta < \gamma$, then: $f_\nu(\eta_1) = \eta_2$ iff
  (α) $\eta_2 \in B_\beta$
  (β) $(\forall \alpha < \beta)(\eta_2(\alpha) = \nu(\alpha) + \eta_1(\alpha))$.

Remark. 1) We can use mainly $\nu \in {}^\gamma\{x_n : n < \omega\}$, a transparent case.
2) The assumption of 1.4 is needed to ensure that f_ν maps B_α into B_α.

1.5 Claim. *1) If $\gamma < \omega_1$ and $\nu \in {}^\gamma G$ is as in Definition 1.4(2), i.e. satisfies $\beta < \gamma \Rightarrow (\forall n)(\exists^{<\aleph_0}\alpha < \beta)(n \in \text{supp}(\nu(\alpha))$ <u>then</u>*

(A) f_ν (is well defined and) has domain $|M_{<\gamma}|$, the universe of $M_{<\gamma}$.
(B) f_ν is a function from $M_{<\gamma}$ into $M_{<\gamma}$.
(C) f_ν has range $|M_{<\gamma}|$ and is one to one.
(D) f_ν is an automorphism of $M_{<\gamma}$.

2) Conversely, if $\gamma < \omega_1$ and f is an automorphism of $M_{<\gamma}$ then $f = f_\nu$ for some $\nu \in {}^\gamma G$ satisfying the condition from 1.4(2).
3) If $\gamma(1) < \gamma(2) < \omega_1$ and $\nu_\ell \in {}^{\gamma(\ell)}G$ for $\ell = 1,2$ are as above and $\nu_1 \trianglelefteq \nu_2$ <u>then</u> $f_{\nu_1} \subseteq f_{\nu_2}$.

Proof. 1) <u>Clauses (A),(B)</u>:
Trivially f is a function with domain $\subseteq (\bigcup\limits_{\beta<\gamma} A_\beta) \cup (\bigcup\limits_{\beta<\gamma} B_\beta)$.

Clearly $\bigcup\limits_{\beta<\gamma} A_\beta \subseteq \text{Dom}(f_\nu)$ and f_ν maps A_β into $A_\beta \subseteq |M_{<\gamma}|$. Let $\eta \in B_\beta, \beta < \gamma$ then by the choice of B_β for some n_*

$$u_{\eta,n_*} = \{\alpha < \beta : \eta(\alpha) \neq G^0_{n_*}\} \in [\beta]^{<\aleph_0}.$$

Let $u_1 = \{\alpha < \beta : n_* \in \text{supp}(\nu(\alpha))\}$, also this set is finite by the condition in Definition 1.4(2).

Let $u = u_{\eta,n_*} \cup u_1$, so $u \in [\beta]^{<\aleph_0}$. We define $f_\nu(\eta)$ as $\langle \eta(\alpha) + \nu(\alpha) : \alpha < \beta \rangle \rangle$.

Now first, considering $\{\alpha < \beta : \eta(\alpha) \neq (f_\nu(\eta))(\alpha)\}$, recalling $\eta \in {}^\beta\mathscr{G}$ this set is $\subseteq u$ hence is finite.

Second, if $\eta(\alpha) = G^\ell_k$ then $(f_\nu(\eta))(\alpha) \in \{G^0_k, G^1_k\}$ hence $f_\nu(\eta) \in {}^\beta\mathscr{G}$. So together $f_\nu(\eta) \in B_\alpha$.
So $B_\beta \subseteq \text{Dom}(f_\nu)$ and f_ν maps B_β into B_β.

Clause (C):

In fact $f_\nu \circ f_\nu = \text{id}_{M_{<\gamma}}$ (the group has order 2, etc.), so should be clear.

Clause (D):

Check the relations as defined in 1.3, recalling Observation 1.2.

2) Let $f \in \text{Aut}(M_{<\gamma})$. The function f maps $P^{M_{<\gamma}}_1 = \bigcup\limits_{\alpha<\gamma} A_\alpha$ onto itself, and by the choice of $E^{M_{<\gamma}}_1$ (as a quasi well ordering with the A_α as its equivalence classes) for each $\alpha < \gamma$ it maps A_α onto itself, so in particular there is z_α such that $f((\alpha, 0_G)) = (\alpha, z_\alpha)$. Now, for every $y \in G$ by the choice of $F^{M_{<\gamma}}_y \restriction A_\alpha$ we have $M_{<\gamma} \models F^{M_{<\gamma}}_y((\alpha, 0_G)) = (\alpha, y)$. As f is an automorphism of $M_{<\gamma}$ we also have $M_{<\gamma} \models F^{M_{<\gamma}}_y(f((\alpha, 0_G))) = f((\alpha, y))$ and note $F^{M_{<\gamma}}_y((\alpha, z_\alpha)) = (\alpha, y + z_\alpha)$. We therefore have for every $y \in G$ that $f((\alpha, y)) = F^{M_{<\gamma}}_y((\alpha, z_\alpha)) = (\alpha, y + z_\alpha)$. Letting $\nu = \langle z_\alpha : \alpha < \gamma \rangle$ we have that $\nu \in {}^\gamma G$ and it is easily verified that $f = f_\nu$ and that ν satisfies the condition in Definition 1.4(2).

3) Check the definition of f_{ν_ℓ}. $\square_{1.5}$

1.6 Claim. *Let $a_1 = (0, x_1), a_2 = (0, x_0) \in A_0$ recalling $\langle x_n : n < \omega \rangle$ is a basis of G. If $\zeta < \omega_1$ then in the EF_ζ-game for $(M, a_1), (M, a_2)$ the isomorphism player wins (this is $\daleth^\zeta_1((M, a_1), (M, a_2))$).*

Proof. Let $\langle \mathscr{U}_\varepsilon : \varepsilon < \zeta \rangle$ be a partition of ω to infinite sets such that $0 \in \mathscr{U}_0$. In the strategy we define below, the isomorphism player does more than needed - he chooses in the ε-move an ordinal $\gamma(\varepsilon) > 0$ and an automorphism g_ε of $M_{<\gamma(\varepsilon)}$ mapping a_1 to a_2 such that the elements which the anti-isomorphism player chose so far belong to $M_{<\gamma(\varepsilon)}$ (no need to distinguish domain and range). The isomorphism player ISO satisfies the demands:

⊛ (a) g_ε is of the form f_{ν_ε} where for some $\gamma(\varepsilon) < \omega_1$ the sequence $\nu_\varepsilon \in {}^{\gamma(\varepsilon)}\{x_n$: $n < \omega\}$ satisfies: $\nu_\varepsilon(0) = x_0$ and $\mathrm{Rang}(\nu_\varepsilon) \subseteq \{x_n : n \in \mathscr{U}_\xi$ for some $\xi \le \varepsilon\}$ and $\langle \nu_\varepsilon(\alpha) : \alpha < \gamma_\varepsilon\rangle$ is with no repetitions

(b) $\langle \gamma(\xi) : \xi \le \varepsilon\rangle$ is increasing

(c) $\langle \nu_\xi : \xi \le \varepsilon\rangle$ is $\triangleleft$-increasing and $\nu(0) = x_1 + x_2 \in G$.

Clearly

$(*)_1$ $f_{\nu_\varepsilon}(a_1) = a_2$, see definition of f_{ν_ε} and the choice of a_1, a_2

$(*)_2$ f_{ν_ε} is a partial isomorphism by 1.5(1)

$(*)_3$ f_{ν_ε} extends f_{ν_ξ} for $\xi < \varepsilon$ by 1.5(2).

So ISO can satisfy the demands hence we are done.

$\square_{1.6}$

1.7 Claim. *If $f \in \mathrm{Aut}(M)$ then $f \restriction A_\alpha = \mathrm{id}_{A_\alpha}$ for every $\alpha < \omega_1$ large enough.*

Proof. Let $f \in \mathrm{Aut}(M)$.

As in the proof of 1.5(2), for each $\alpha < \omega_1$ there is $z_\alpha \in G$ such that

$$(\alpha, x) \in A_\alpha \Rightarrow f((\alpha, x)) = (\alpha, z_\alpha + x).$$

But G is countable, so for some $z_* \in G$ the set $\mathscr{U} := \{\alpha < \omega_1 : z_\alpha = z_*\}$ is unbounded in ω_1, and if possible choose z_* such that it is $\ne 0_G$; let u be such that $z_* = \Sigma\{x_n : n \in u\}$ so $u \subseteq \omega$ is finite.

Hence we can find $\gamma_* < \omega_1$ such that $\mathscr{U} \cap \gamma_*$ is infinite. If $z_* \ne 0_G$ let $n \in u = \mathrm{supp}(z_*)$, let $\eta \in B_{\gamma_*}$ be constantly G^0_n, such η exists by the definition of B_{γ_*}. Now $f(\eta)$ is "illegal", i.e. satisfies $\{\alpha < \gamma_* : f(\eta)(n) \ne G^0_n\}$ is infinite, contradicting Definition 1.3(A)(b). So $z_* = 0_G$ hence by the choice of z_* we have $z_\gamma = 0_G$ for every $\gamma < \omega_1$ large enough, say for $\gamma \in [\gamma_*, \omega_1)$, i.e. $f \restriction A_\gamma = \mathrm{id}_{A_\gamma}$, so we are done. $\square_{1.7}$

1.8 Definition. 1) For a sequence $\mathbf{p} = \langle(\beta_\alpha, g_\alpha) : \alpha < \omega_1\rangle$ satisfying $\beta_\alpha \le \alpha, g_\alpha \in \operatorname{Hom}(G,G)$ we define $M_{\mathbf{p}}$ as the expansion of M by $R_1^{M_{\mathbf{p}}} = \{((\beta_\alpha, y_2), (\alpha, y_1)) : \alpha < \omega_1, y_2, y_1 \in G, g_\alpha(y_1) = y_2\}$.
1A) Let $\mathbf{P}$ be the set of such $\mathbf{p}$'s.
2) For a sequence $\mathbf{p} = \langle(\beta_\alpha, h_\alpha) : \alpha < \omega_1\rangle$ satisfying $\beta_\alpha \le \alpha$ and $h_\alpha \in {}^\omega\omega$ the model $M_{\mathbf{p}}$ is defined as $M_{<(\beta_\alpha, \hat{h}_\alpha):\alpha<\omega_1>}$ where $\hat{h}_\alpha$ is the homomorphism from G to G defined by $\hat{h}_\alpha(\sum\limits_{n\in u} x_n) = \sum\limits_{n \in u} x_{h_\alpha(n)}$. Let $\mathbf{P}'$ be the set of such $\mathbf{p}$'s.
3) Let $M^{\mathbf{p}}_{<\gamma} = M_{\mathbf{p}} \restriction (|M_{<\gamma}|)$ for $\gamma < \omega_1$ and $\mathbf{p}$ as above.

1.9 Claim. *If $\mathbf{p} = \langle(\beta_\alpha, g_\alpha) : \alpha \in \omega_1\rangle \in \mathbf{P}$ is as in 1.8, $\gamma < \omega_1$ and $\nu \in {}^\gamma G$ as in Definition 1.4 and f_ν is as in 1.4, <u>then</u> f_ν is an automorphism of $M^{\mathbf{p}}_{<\gamma}$ iff*

$\circledast_{\mathbf{p},\nu}$ *if $\alpha < \gamma$ <u>then</u> g_α maps $\nu(\alpha)$ to $\nu(\beta_\alpha)$.*

Proof. Straight, for $\alpha < \gamma$ and $y_1, y_2 \in G$ we have

(a) $(f_\nu((\beta_\alpha, y_2)), f_\nu((\alpha, y_1))) \in R_1^{M_{\mathbf{p}}}$
<u>iff</u> (by f_ν's definition)

(b) $((\beta_\alpha, y_2 + \nu(\beta_\alpha)), (\alpha, y_1 + \nu(\alpha))) \in R_1^{M_{\mathbf{p}}}$
<u>iff</u> (by $R_1^{M_{\mathbf{p}}}$'s definition)

(c) $g_\alpha(y_1 + \nu(\alpha)) = (y_2 + \nu(\beta_\alpha))$
<u>iff</u> (as g_α is an endomorphism of G)

(d) $g_\alpha(y_1) + g_\alpha(\nu(\alpha)) = y_2 + \nu(\beta_\alpha)$.

So if $\circledast_{\mathbf{p},\nu}$ holds then clause (d) is equivalent to

(e) $g_\alpha(y_1) + \nu(\beta_\alpha) = y_2 + \nu(\beta_\alpha)$
<u>iff</u> (by cancellation)

(f) $g_\alpha(y_1) = y_2$
<u>iff</u> (by the definition of $R_1^{M_{\mathbf{p}}}$)

(g) $((\beta_\alpha, y_2), (\alpha, y_1)) \in R_1^{M_{\mathbf{p}}}$.

So $\circledast_{\mathbf{p},\nu}$ implies f_ν is an automorphism of $M^{\mathbf{p}}_{<\gamma}$. The inverse is easy, too. $\square_{1.9}$

1.10 Claim. *Let*

(a) $h_\alpha : \omega \to \omega$ *be* $h_\alpha(2n+\ell) = \ell$ *for* $n < \omega, \ell \in \{0,1\}$ *and* $\alpha < \omega_1$
(b) $\mathbf{p} = \langle (0, h_\alpha) : \alpha < \omega_1 \rangle \in \mathbf{P}'$, *so* $\beta_\alpha = 0$ *for every* α
(c) $N = M_{\mathbf{p}}$
(d) $\Lambda_\gamma = \{\nu \in {}^\gamma G : \nu(0) = x_0 + x_1$ *and if* $\beta \in (0,\gamma)$, *then* $\nu(\beta)$ *has the form* $x_{2n} + x_{2n+1}$ *and* $\langle \nu(\beta) : \beta < \gamma \rangle$ *is with no repetitions*$\}$.

<u>*Then*</u>

(A) *if* $\nu \in \Lambda_\gamma, \gamma < \omega_1$, <u>*then*</u> f_ν *is an automorphism of* $M^{\mathbf{p}}_{<\gamma}$ *extending* $f_{\nu \restriction \beta}$ *for* $\beta < \gamma$.
(A)′ *For* $\zeta < \omega_1$, *in the game* $\eth^\zeta((N,(0,x_0)),(N,(0,x_1))$, *the isomorphism player wins, i.e.* $(N,(0,x_1)),(N,(0,x_2))$ *are* EF$_\zeta$*-equivalent.*
(B) N *has no automorphism mapping* $a_1 = (0,x_0)$ *to* $a_2 = (0,x_1)$, *i.e. the models* $(N,(0,x_1)),(N,(0,x_2))$ *are not isomorphic.*
(B)′ $\|N\| = \aleph_1$, *so the isomorphism player loses* $\eth^{\omega_1}((M,a_1),(M,a_2)$.

Proof.
<u>Clauses (A),(A)′</u>: As in 1.6 using 1.9.

<u>Clause (B),(B)′</u>: Toward contradiction assume $f \in \text{Aut}(M_{\mathbf{p}}), f((0,x_0)) = (0,x_1)$. Continue as in the proof of 1.7, but $z_\alpha \neq 0$ for $\alpha < \omega_1$ (as $\hat{h}_\alpha(z_\alpha) = x_0 + x_1$) hence z_* is not zero and we get a contradiction. $\square_{1.10}$

1.11 Remark. It seems we can in 1.10 get a rigid M of interest.

1.12 Conclusion. There are non-isomorphic models of cardinality $\aleph_1$ which are EF$_\zeta$-equivalent for every $\zeta < \omega_1$.

§2 Other cardinals

2.1 Claim. *1) If $\lambda = \mu^+$ then there are models M, N of cardinality λ, EF_ζ-equivalent for every $\zeta < \lambda$ but not isomorphic hence not EF_λ-equivalent.*
2) Instead $\lambda = \mu^+$ just $\lambda = \mathrm{cf}(\lambda) = \mu > \aleph_0$ is enough.

Remark. For λ regular uncountable not a successor (i.e. weakly inaccessible), it makes a difference whether we allow the anti-isomorphism player to choose one element or $< \lambda$. In 2.1(2) we allow one element.
2) On 2.1 recall 0.3.

Proof. 1) Now let $G = \oplus\{(\mathbb{Z}/2\mathbb{Z})x_\varepsilon : \varepsilon < \mu\}$ and repeat 1.1 - 1.10 with the obvious changes: $\aleph_0, \aleph_1$ replaced by μ, λ but "finite" remains "finite" in particular in 1.3(A)(b).
2) So without loss of generality λ is not a successor cardinal (hence is weakly inaccessible). Define the abelian group G as in part (1), but now $\mu = \lambda$ and repeat 1.1 - 1.5, and also 1.8, 1.9 as above but now $h \in {}^\lambda\lambda$. But to immitate Claim 1.10 we choose $\mathbf{p}$ differently. Let $\mathscr{U}_\varepsilon = [\gamma_\varepsilon, \gamma_{\varepsilon+1})$ for $\varepsilon < \lambda$ where $\langle \gamma_\varepsilon : \varepsilon < \lambda \rangle$ is increasing continuous, $\gamma_0 = 0, \gamma_1 = 2$ each γ_ε is even and $\gamma_{\varepsilon+1} = \gamma_\varepsilon + 2\varepsilon$. So $\langle \mathscr{U}_\varepsilon : \varepsilon < \lambda \rangle$ be a partition of λ to sets such that $|\mathscr{U}_\varepsilon| = |2\varepsilon|$. Let $\mathbf{p} = \langle \beta_\alpha, h_\alpha : \alpha < \lambda \rangle$ be chosen as follows ($\beta_\alpha \le \alpha$ and $h_\alpha \in {}^\lambda\lambda$ of course and):

- (a) let $h_k(2\alpha + \ell) = \ell$ for $\alpha < \lambda, k < 2, \ell < 2$, and let $\beta_0 = \beta_1 = 0$.
- (b) for $\alpha \in [2, \lambda)$ let $\beta_\alpha = 1$ and h_α be such that for $\ell \in \{0, 1\}$ and $\gamma < \lambda$ we have $2\gamma + \ell \in \mathscr{U}_\varepsilon \Rightarrow h_\alpha(2\gamma + \ell) = 2\varepsilon + \ell$.

To prove the parallel of clause (B) of Claim 1.10 toward contradiction assume $(M_{\mathbf{p}}, a_1), (M_{\mathbf{p}}, a_2)$ are isomorphic. Let f be such an isomorphism, so f is an automorphism of $M_{\mathbf{p}}$ mapping a_1 to a_2. As in the proof of 1.7, we can find $z_\alpha \in G$ for $\alpha < \lambda$ such that $x \in G \Rightarrow f((\alpha, x)) = (\alpha, z_\alpha + x)$. Let $\nu = \langle z_\alpha : \alpha < \lambda \rangle \in {}^\lambda G$. So by the parallel to 1.9 we know that $\circledast_{\mathbf{p}, \nu \restriction \gamma}$ holds for $\gamma < \lambda$, i.e. $\hat{h}_\alpha(\nu(\alpha)) = \nu(\beta_\alpha)$.

But $f(a_1) = a_2$ so $z_0 = x_0 + x_1$, hence recalling $\beta_1 = 0$ we have $\nu(1) = z_1 \neq 0$ and let $\varepsilon < \lambda, \ell < 2$ be such that $2\varepsilon + 1 \in \mathrm{supp}(z_1)$. By the end of the previous paragraph

- $(*)$ if $2 \le \alpha < \lambda$ then $\mathrm{supp}(z_\alpha) \cap [\gamma_\varepsilon, \gamma_{\varepsilon+1}) \neq \emptyset$ hence we can find $\gamma^\alpha \in \mathrm{supp}(z_\alpha) \cap [\gamma_\varepsilon, \gamma_{\varepsilon+1})$.

So for some $\gamma_* < \lambda$ the set $\mathscr{U} = \{\alpha < \lambda : \gamma_* = \gamma^\alpha$ hence $\gamma_* \in \mathrm{supp}(z_\alpha)\}$ is an unbounded subset of λ. We continue as in the proof of 1.7.

Let $\varepsilon < \lambda$, without loss of generality $\varepsilon \geq \omega$ is a cardinal and we shall prove that $(M_{\mathbf{p}}, a_1), (M_{\mathbf{p}}, a_2)$ are EF_ε equivalent. We act as in the proof of 1.6 (though a_1, a_2 are different) but instead of $M_{<\gamma}$ there we use here $M^{\mathbf{p}}_u$ where $M^{\mathbf{p}}_u = M_{\mathbf{p}} \restriction \cup\{A_\alpha \cup B_\alpha : \alpha \in u\}$ for $u \subseteq \lambda$.

Let $u_0 = \{0, 1\}$. In stage $\zeta < \varepsilon$ of the game the AIS player chooses $\ell(\zeta) \in \{1, 2\}$ and $b^{\ell(\zeta)}_\zeta \in (M_{\mathbf{p}}, a_{\ell(\zeta)})$. So there is $\xi_\zeta < \lambda$ such that $b^{\ell(\zeta)}_\zeta \in A_{\xi_\zeta} \cup B_{\xi_\zeta}$. The ISO player chooses $u_{\zeta+1} \in [\lambda]^{<\varepsilon}$ and a sequence $\bar\beta_{\zeta+1} = \langle \beta_\alpha : \alpha \in u_{\zeta+1} \backslash \{0,1\} \wedge 2\beta_\alpha \in \mathscr{U}_\varepsilon \rangle$ satisfying the following:

- $(*)$ (a) $u_{\zeta+1} = u_\zeta \cup \{\xi_\zeta\}$
  - (b) $\bar\beta_\zeta = \bar\beta_{\zeta+1} \restriction u_\zeta \backslash \{0, 1\}$
  - (c) $\bar\beta_{\zeta+1}$ is with no repetitions.

For $\nu_{\zeta+1} \in {}^{u_{\zeta+1}}G$ defined by

$$\nu_{\zeta+1}(\alpha) = \begin{cases} x_0 + x_1 & \alpha = 0 \\ x_{2\varepsilon} + x_{2\varepsilon+1} & \alpha = 1 \\ x_{2\beta_\alpha} + x_{2\beta_\alpha+1} & \text{else} \end{cases}$$

$f_{\nu_{\zeta+1}}$ satisfies the requirement in 1.4(2) (with the modification $\forall \beta \in u_{\zeta+1}$ instead of $\forall \beta < \gamma$) and hence is an automorphism of $M^{\mathbf{p}}_{u_{\zeta+1}}$ and $f_{\nu_{\zeta+1}}(a_1) = a_2$.

$\square_{2.1}$

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