A NOTE
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Main Results

See around classification theorem: Springer 1986, October 3, 1985, A Note on κ-freeness.

Theorem. *If $\lambda > \aleph_1$ is regular, $|G| = \lambda$ and G is a λ-free abelian group* <u>*then*</u> *there is a free group $G' \subseteq G$, $|G'| = \lambda$ provided that*

> (∗) *λ is not the successor of a singular cardinal of cofinality $\aleph_0$.*

Proof. By [Sh:52,3.9], G is strongly μ-free for $\mu < \lambda$.

Let $G = \bigcup_{i<\lambda} G_i$, G_i strictly increasing continuous, $|G_i| = |i| + \aleph_1$. Additionally without loss of generality $G_{i+1} = \bigcup_{\alpha<|i+\omega_1|} G_{i,\alpha}$, $G_{i,\alpha}$ increasing continuous, $G/G_{i,\alpha+1}$ is λ-free.
[Why? Choose G_i by induciton on i and for $i = j+1$ choose $G_{j,\alpha}$ by induction on α; using the previous sentence and bookkeeping we are done.]

Without loss of generality $\forall i < j, \forall \alpha < |i+\omega_1|, \exists \beta < |j+\omega_1[G_{i,\alpha+1} \subseteq G_{j,\beta+1}]$. Without loss of generality

> ⊠ $\alpha < \lambda \Rightarrow G/G$ is $\aleph_1$-free (forgetting the "additionality").
> [Why? If λ is a limit cardinal we know G is strongly $|G_i|^+$-free so we can demand that G/G_{i+1} is $\aleph_1$-free. If $\lambda = \mu^+$, then $\mathrm{cf}(\mu) > \aleph_0$ by the hypothesis (∗), so if G/G_{i+1} is not $\aleph_1$-free then for $\alpha < |i+\omega_1|$ large enough, $G/G_{\alpha+1}$ is not $\aleph_1$-free, contradiction.]

Let $S = \{i < \lambda : \mathrm{cf}(i) = \omega_1\}$. Easily G/G_δ is $\aleph_1$-free for $\delta \in S$ (by ⊠). Let $y_\delta \in G_{\delta+1} \backslash G_\delta$, so there is $H_\delta \subseteq G_\delta$, $|H_\delta| \leq \aleph_0$, H_δ free, G_δ/H_δ free and the triple $(H_\delta, \langle H_\delta, y_\delta\rangle, G_\delta)$ is free. (See [Sh 161] so for some stationary $S_1 \subseteq S$ and $\alpha(*) < \delta, (\forall \delta \in S_1) H_\delta \subseteq G_{\alpha(*)}$. We are allowed to increase H_δ, so by (∗) without loss of generality $H_\delta = H$. Now $\langle H, y_\delta : \delta \in S_1\rangle$ is free.

Remark. The proof works for general classes.

REFERENCES.

[Sh 161] Saharon Shelah. Incompactness in regular cardinals. *Notre Dame Journal of Formal Logic*, **26**:195–228, 1985.