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No universal in singular

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Abstract
An old question is whether there is a countable complete first order theory T such that T has a universal model of cardinality $\lambda > \aleph_0 \underline{iff}\ \lambda = 2^{<\lambda} > \aleph_0$. We solve it here for the class singular cardinals.

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1 Introduction

The following question was asked in [7, §4]; so is by now quite old.

Question 1.1 Does there exist a countable complete first order T which has a universal model in a cardinal λ <u>iff</u> $\lambda = 2^{<\lambda} > \aleph_0$?

This essentially says that the existence results of Jonsson (for universal theories with JEP and amalgamation under embeddings) and Morley-Vaught (for complete first order T with elementary embeddings) are best possible. The parallel problem for universal-homogeneous and saturated was answered long ago, in [8, Ch.III]:

Theorem 1.2 *For a complete f.o. theory T and cardinal $\lambda > |T|$ the following conditions are equivalent:*

(a) *the theory T has a saturated model of cardinality λ*
(b) $\lambda^{<\lambda} = \lambda$ *<u>or</u> T is stable in λ*

The author thanks Alice Leonhardt for the beautiful typing. References like [4, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. First typed May 10, 2019. Supported by ISF 1838/19 (The Israel Science Foundation) and by ERC (European Research Council) grant 338821. Paper 1162 on author's list. The author states that there is no conflict of interest.

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(c) *at least one of the following hold:*

(α) $\lambda = \lambda^{<\lambda}$
(β) T *is a stable unsuperstable theory (so* $\aleph_0 < \kappa(T) \leq |T|^+$*), and* $\lambda = \lambda^{<\kappa(T)} \geq |\mathbf{D}(T)| + 2^{\aleph_0}$
(γ) T *is superstable,* $\lambda \geq |\mathbf{D}(T)| + 2^{\aleph_0}$ *and* $|\mathbf{S}(A, M)| \leq \lambda$ *for every* M *a model of* T *and countable* $A \subseteq M$

By Kojman-Shelah [2] for many cardinals λ (such that $\lambda < 2^{<\lambda}$) the answer is yes, even for the theory of dense linear order. Here we finish one of the main cases left: λ singular, see the survey [3] and new one [5]. The theory is quite simple to define. Note that for 1.1 it is enough to deal separately with each of finitely (or even countably many) cases, because e.g. for any complete theories T_1, T_2 there is a complete T such that for any $\lambda \geq |T_1| + |T_2|$ we have $\text{univ}(\lambda, T) = \max\{\text{univ}(\lambda, T_1) + \text{univ}(\lambda, T_2)\}$. On subsequent work see [6].

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2 Preliminaries

Recall that

Definition 2.1 (1) For a complete first order T, we let $\text{univ}(\lambda, T)$ be the minimal cardinal μ such that there is a sequence $\langle M_i : i < \mu\rangle$ of models of T of cardinality λ which is λ-universal; this mean that every model of T of cardinality λ can be elementarily embedded in some M_i.
(2) If T is not complete, we use usual embedding.
(3) If above $\mu = 1$ then we say that M_0 is universal for T.

3 The singular case

Our example is T^0_{elo}, a universal theory which has a finite relational vocabulary, has amalgamation and JEP hence a model completion called T_{elo} with elimination of quantifiers. For $M \models T_{\text{elo}}$, $<_M$ is a linear order, for $\ell = 1, 2$ we have R^M_ℓ is a three-place relation such that for each c, $R^M_{\ell,c} = \{(a, b) : (a, b, c) \in R^M_\ell\}$ is (essentially) a convex equivalence relation on some subset $\text{Dom}(R^M_{\ell,c})$ of $\{d : d <_M c\}$; also $\text{Dom}(R^M_{1,c})$, $\text{Dom}(R^M_{2,c})$ are disjoint. Formally

Definition 3.1 (1) Let $T^0_{1-\text{tr}}$ be the universal theory of trees, i.e. with vocabulary $\{<\}$ with $<$ a two-place relation such that $M \models T^0_{1-\text{tr}}$ <u>iff</u> $<_M$ is a partial order satisfying $M \models$ "$a < c \wedge b < c$" implies $M \models$ "$a = b \vee a < b \vee b < a$".
(2) Let T^0_{elo} be the universal theory with vocabulary $\{<, R_0, R_1\}$ where $<$ is a two-place predicate and R_0, R_1 are 3-place predicates such that:

($*$) $M \models T^0_{\text{elo}}$ <u>iff</u> (for $\ell = 0, 1$):

(a) $(|M|, <_M)$ is a linear order
(b)$_\ell$ if $(a, b, c) \in R^M_\ell$ then $a \leq_M b <_M c$
(c)$_\ell$ if $(a, b, c) \in R^M_\ell$ and $a \leq_M a' \leq_M b' \leq_M b$ then $(a', b', c) \in R^M_\ell$
(d) if $(a_1, a_2, c), (a_2, a_3, c) \in R^M_\ell$ then $(a_1, a_3, c) \in R^M_\ell$

(e) if $(a_\ell, b_\ell, c) \in R_\ell^M$ for $\ell = 0, 1$ then $b_0 <_M a_1$ or $b_1 <_M a_0$.

(3) Let T_{elo} be the model completion of T^0_{elo}, see below.

Definition 3.2 For a cardinal μ and regular $\kappa \le \mu$ let:

- $\text{trp}^+_\kappa(\mu) = \min\{\lambda$: there is no sub-tree $\mathscr{T}$ of $({}^{\kappa>}\mu, \lhd)$ of cardinality μ such that $\lim_\kappa(\mathscr{T}) = \{\eta \in {}^\kappa\mu : (\forall i < \kappa)(\eta{\restriction}i \in \mathscr{T})\}$ has cardinality $\ge \lambda\}$
- $\text{trp}_\kappa(\mu) = \sup\{\lambda : \lambda < \text{trp}^+_\kappa(\mu)\}$.

Remark 3.3 (1) Considering embeddings we may use positive formulas only.
(2) In [11], $\text{trp}_\kappa(\mu)$ was called $\mu^{\kappa,\text{tr}}$ or $\mu^{\langle\kappa\rangle}$.
(3) We intend to reconsider the oak property introduced in Dzamonja-Shelah [1], see more [12].

Claim 3.4 (1) *The theory T^0_{elo} has the disjoint* JEP *and disjoint amalgamation, is universal with predicates only and with a finite vocabulary, hence T_{elo} is well defined and $\mathbf{D}(T_{\text{elo}})$ is countable and T_{elo} is even $\aleph_0$-categorical.*
(2) *So T_{elo} have a universal and even a saturated countable model.*

Proof 3.4 Should be clear. □

We shall use (we can use linear orders or trees, it does not matter):

Claim 3.5 *Assume M is a tree, not necessarily well founded (in the model theoretic sense, that is $<_M$ is a partial order and $\{b \in M : b <_M c\}$ is linearly ordered for every $c \in M$) and M has universe μ.*

(1) *If $\kappa = \text{cf}(\kappa) \le \mu$ and $\text{trp}^+_\kappa(\mu) = \chi$, see Definition 3.2, <u>then</u> M has $< \chi$ initial segments of branches (not necessarily proper) which are of cofinality κ.*
(2) *If $\kappa = \text{cf}(\kappa) \le \mu$, <u>then</u> there is $\mathscr{P} \subseteq [\mu]^\kappa$ of cardinality $< \text{trp}^+_\kappa(\mu)$, see Definition 3.2(2) such that:*

$(*)$

(a) *each $u \in \mathscr{P}$ has order type κ by the order of M*
(b) *for any subset u of M of order type κ there is $v \in \mathscr{P}$ and a $<_M$-increasing sequence $\langle a_i : i < \kappa\rangle$ such that $i < \kappa \Rightarrow a_{2i} \in u \wedge a_{2i+1} \in v$.*

(3) *If $\theta < \kappa \le \mu$ where θ and κ are regular <u>then</u> there is $\mathscr{P}$ such that:*

$(*)$

(a) *$\mathscr{P}$ is a set of $\le \mu$ initial segments of branches of M*
(b) *if $B \in \mathscr{P}$ then some $\langle \beta_{B,i} : i < \theta\rangle$ increasing by $<_M$ and by $<$, forms an $<_M$-unbounded subset of B and for each $i < \theta$ we have $\{\beta_{B,j} : j \in (i, \theta)\}$ all realize the same cut over $\{\beta : \beta < \beta_{B,i}\}$ in M*
(c) *if $\langle a_i : i < \kappa\rangle$ is $<_M$-increasing then for some club E of κ we have:*

$\odot$ *if $j \in E$ has cofinality θ <u>then</u> there is $B \in \mathscr{P}$ such that $\{c \in M : (\exists i < j)(c <_M a_i)\} = \{c \in M : (\exists b \in B)(c <_M b)\}$.*

Proof 3.5 1) By (2) and the definitions.
2) Recall that the set of elements of M is μ; without loss of generality $(\forall \alpha < \mu)(\exists \beta)(\alpha < \beta < \mu \wedge \alpha <_M \beta)$. Let $<^*_M = \{(\alpha, \beta) : M \models$ "$\alpha < \beta$" and $\alpha < \beta\}$.
Now

$(*)_1$ $M' = (\mu, <^*_M)$ is a partial order with μ nodes

$(*)_2$ for each $\delta \leq \mu$ of cofinality κ

(a) choose an increasing continuous sequence $\bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \kappa \rangle$ of ordinals with limit δ such that $\alpha_{\delta,0} = 0$

(b) for $i < \kappa$ we define an equivalence relation:

(α) $E_{\delta,i} := \{(\beta_1, \beta_2) : \beta_1, \beta_2 \in [\alpha_{\delta,i}, \delta)$ and $(\forall \gamma < \alpha_{\delta,i})[\gamma <_M \beta_1 \equiv \gamma <_M \beta_2]\}$

(β) $A_{\delta,i}$ is the set of $\beta \in [\alpha_{\delta,i}, \alpha_{\delta,i+1})$ such that: $\beta = \min(\beta/E_{\delta,i})$

(c) let $A_\delta = \cup\{A_{\delta,i} : i < \kappa\}$

(d) define M'_δ (or pedantically $M'_{\delta,\bar{\alpha}_\delta}$) as the following partial order:

(α) the set of elements is A_δ

(β) $M'_\delta \models$ "$\alpha < \beta$" iff for some $i < j < \kappa$ we have $\alpha \in [\alpha_{\delta,i}, \alpha_{\delta,i+1}), \beta \in [\alpha_{\delta,j}, \alpha_{\delta,j+1})$ and $\alpha E_{\delta,i} \beta$

Now we investigate such M'_δ, fixing δ for a while:

$(*)_3$

(a) M'_δ is a (well founded) tree with $\leq \kappa$ levels and $\leq |\delta| \leq \mu$ nodes

(b) if $i < \kappa$ and $\alpha \in \delta \cap M \backslash \alpha_{\delta,i}, \beta = \min(\alpha/E_{\delta,i}) < \delta$ then for some $j \in [i, \kappa)$ we have $\beta \in [\alpha_{\delta,j}, \alpha_{\delta,j+1})$ and $\beta = \min(\beta/E_{\delta,j})$.

[Why? Check the definition. Note that there may be holes, that is $\alpha \in A_{\delta,i}$, and $j < i$ such that there is no $\beta \in A_{\delta,j}, \beta \in \alpha/E_{\delta,j}$.]

$(*)_4$ if B is an $<_M$-initial segment of a branch of $M \restriction \delta$, then at least one of the following occurs:

(a) there is $\gamma < \delta$ such that $B \cap \gamma$ is cofinal in B under $<_M$

(b) there is a $<^*_M$-increasing sequence $\langle \beta_i : i < \delta \rangle$ of ordinals from B with limit δ which is cofinal in $(B, <_M)$.

[Why? Should be clear.]

$(*)_5$ $\mathscr{P}_\delta$, the set of κ-branches of M'_δ, is a subset of $[\delta]^\kappa$ of cardinality $< \mathrm{trp}^+_\kappa(|\delta|) \leq \mathrm{trp}^+_\kappa(\mu) = \chi$.

[Why $(*)_5$ holds? By $(*)_3$ and the definition of $\mathrm{trp}^+_\kappa(-)$.]

$(*)_6$ if $B \subseteq M$ is $<_M$-linearly ordered of order type κ, then

(a) $B^+ = \{a \in M : (\exists b \in B)(a <_M b)\}$, necessarily linearly ordered (by $<_M$), is $<_M$-downward closed, and is of cofinality κ

(b) let $\delta = \delta_B \leq \mu$ be minimal such that $(\exists^\kappa b \in B)(\exists c \in B^+)(c < \delta \wedge b <_M c)$ hence δ has cofinality κ, clearly well defined

(c) for $i < \kappa$, let $\beta_i = \beta_{B,i} \in B^+ \cap \delta$ be minimal such that $(\forall \gamma \in B^+ \cap \alpha_{\delta,i})[\gamma <_M \beta_i]$, well defined by the choice of δ

(d) if $i < j < \kappa$ then:

(α) $(\forall \gamma < \beta_i)[(\gamma <_M \beta_i) \equiv (\gamma \in B^+ \cap \beta_i^+)]$

(β) $\beta_i \leq \beta_j$

(γ) $\beta_j \in \beta_i / E_{\delta,i}$

(e)

(α) the sequence $\langle \beta_{B,i} : i < \kappa \rangle$ is $\leq$-increasing not eventually constant
(β) there is $u = u_B \in \mathscr{P}_\delta$ such that:

if $\alpha \in u \cap [\alpha_{\delta,i}, \alpha_{\delta,i+1})]$ and $j \geq i$ then $\alpha E_{\delta,i} \beta_{B,j}$.

Hence

$(*)_7$ if B_1, B_2 are linearly ordered subsets of M of order type κ and $(\delta_{B_1}, u_{B_1}) = (\delta_{B_2}, u_{B_2})$ <u>then</u> $B_1^+ = B_2^+$.

This clearly suffices for part (2).
3) We rely on the proof of part (2); note that there $(*)_5$ do not apply, hence also $(*)_6(e)(\beta)$ and $(*)_7$.

For $\delta \leq \mu$ of cofinality θ we define $\mathscr{P}^*_\delta$ by:

$\oplus^1_\delta$ $B \in \mathscr{P}^*_\delta$ <u>iff</u> some α_* witnesses this which means:

(a) B is an initial segment of some branch of M of cofinality θ
(b) $B \cap \delta \setminus \alpha$ is $<_M$-cofinal in B but $B \cap \alpha$ is not, for every $\alpha < \delta$
(c) $\alpha_* \in [\delta, \mu)$
(d) for every $\beta < \delta$ for some $b \in B$ all members of $\{a \in B : b <_M a\}$ realize the same cut of $\{\gamma : \gamma < \beta\}$ in M as α_* does.

Note

$\oplus_2$ in $\oplus^1_\delta$, B is uniquely determined by the pair (α_*, δ).

[Why? Just read $\oplus^1_\delta$.]

Now

$\oplus_3$ let $\mathscr{P} = \bigcup\{\mathscr{P}^*_\delta : \delta$ be a limit ordinal $\leq \mu\}$

Obviously (by $\oplus_2 + \oplus_3$)

$\oplus_4$ $\mathscr{P}$ has cardinality $\leq \mu$ is a set of initial segments of branches of M of cofinality θ.

It suffice to prove that $\mathscr{P}$ is as required, Now clauses (a), (b) (of 3.5(3)) are clear by the choice of $\mathscr{P}$, but we have to prove also clause (c). So assume we are given $\langle a_i : i < \kappa \rangle$, a $<_M$-increasing sequence and let $B_\bullet = \{a \in M : (\exists i < \kappa)[a <_M a_i]\}$, and let δ be the minimal ordinal $\leq \mu$ such that $B_\bullet \cap \delta$ is cofinal in $(B_\bullet, <_M)$. Necessarily $\mathrm{cf}(\delta) = \kappa$ and let $\langle \alpha_{\delta,i} : i < \delta \rangle$ be an $<$-increasing sequence of ordinals $< \delta$ with limit δ. Let $E = \{i < \kappa : i$ is a limit ordinal and $(\forall j < i)(\exists b \in B_\bullet \cap \alpha_{\delta,i})[a_j <_M b]$ and $(\forall b \in B_\bullet \cap \alpha_{\delta,i})(\exists j < i)[b <_M a_j]\}$.

Now clearly E is a club of κ. Lastly

- for every $i \in E$ of cofinality θ the set B belongs to $\mathscr{P}$ where:
  $B = \{b \in B_\bullet : b <_M a_j$ for some $j < i\}$

[Why $B \in \mathscr{P}$? because B is as required in $\oplus^1_\delta$ with the pair $(\alpha_{\delta,i}, a_i)$ here standing for (δ, α_*) there.]

Theorem 3.6 *If λ is a singular cardinal satisfying $\lambda < 2^{<\lambda}$, <u>then</u> $T = T_{\mathrm{elo}}$ (and equivalently T^0_{elo}) has no universal model of cardinality λ; moreover,* $\mathrm{univ}_{T_{\mathrm{elo}}}(\lambda, T) \geq 2^{<\lambda}$.

Proof It suffices to prove it for $T = T^0_{\rm elo}$, so for embedding rather than elementary embeddings; toward contradiction assume:

$(*)_1$

(a) $\xi_* < 2^{<\lambda}$
(b) $\bar{M}^* = \langle M^*_\xi : \xi < \xi_* \rangle$
(c) M^*_ξ a model of T with universe λ
(d) $\bar{M}^*$ is universal, i.e. if $M \models T$ has cardinality λ, then M can be embedded into M^*_ξ for some $\xi < \xi_*$.

Next

$(*)_2$ choose κ, μ satisfying:

(a) $\kappa < \mu < \lambda$ are regular
(b) $\lambda < 2^\kappa$ and $\xi_* < 2^\kappa$
(c) $\mathrm{cf}(\lambda) < \mu$.

[Why? Recall that $\lambda < 2^{<\lambda}$ and $\xi_* + \lambda < 2^{<\lambda} = \Sigma\{2^\theta : \theta < \lambda\}$ hence for some $\theta < \lambda$ we have $\xi_* + \lambda < 2^\theta$ and let $\kappa = \theta^+$, $\mu = \mathrm{cf}(\lambda)^+ + \theta^{++}$.]

$(*)_3$

(a) let $\mathscr{P} \subseteq \mathscr{P}(\lambda)$ be $\cup\{\mathscr{P}_\xi : \xi < \xi_*\}$ where $\mathscr{P}_\xi$ is as in 3.5(3) with $(\lambda, \mu^+, \kappa, M_\xi)$ here standing for (μ, κ, θ, M) there
(b) so $\mathscr{P}$ is of cardinality $\leq \lambda + |\xi_*| < 2^\kappa$
(c) let $\bar{C}^1 = \langle C^1_\alpha : \alpha \in S^+_1 \rangle$ be such that:
    (α) $S^+_1 \subseteq \mu^+$ and $\alpha, \beta \in S^+_1 \wedge \alpha < \beta \Rightarrow \alpha + \omega \leq \beta$
    (β) C^1_δ is a closed subset[1] of some $\delta' \leq \delta$ of order type $\leq \kappa$
    (γ) $\alpha \in C^1_\beta \Rightarrow C^1_\alpha = C^1_\beta \cap \alpha$
    (δ) $S_1 := \{\delta \in S^+_1 : \mathrm{otp}(C^1_\delta) = \kappa\}$ is stationary
    (ε) $\bar{C}^1 \restriction S_1$ guesses clubs.
(d) let $\bar{C}^2 = \langle C^2_\delta : \delta \in S^+_2 \rangle$, S_2 be as in clause (c) with (μ^{+3}, μ^+) here standing for (μ^+, κ) there, so $S_2 = \{\delta \in S^+_2 : \mathrm{otp}(C_\delta) = \mu^+\}$
(e) let $\bar{g}^2 = \langle g^2_\alpha : \alpha \in S^+_2 \rangle$, g^2_α be an increasing function from $\mathrm{otp}(C^2_\alpha)$ onto C^2_α hence $\alpha \in C^2_\beta \Rightarrow g^2_\alpha \subseteq g^2_\beta$.
(f) let $\bar{g}^1 = \langle g^1_\alpha : \alpha \in S^+_1 \rangle$, g^1_α be an increasing function from $\mathrm{otp}(C^1_\alpha)$ onto C^1_α hence $\alpha \in C^1_\beta \Rightarrow g^1_\alpha \subseteq g^1_\beta$.
(g) Choose $\langle \bar{\alpha}_\delta : \delta \leq \mu^{+3}, \mathrm{cf}(\delta) = \mu^+ \rangle$ such that $\bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \mu^+ \rangle$ is increasing continuous with limit δ.

'[Why does such objects exists? First for clause (a) use $(*)_1(c)$, 3.5(3) + $(*)_2(b)$. Second clause (b) follows from clause (a). Third clauses (c),(d) hold by [9, §1] (for club guessing we can use [10, Ch.III]). Fourth clauses (e),(f) follows. Lastly choose the sequences as in clause (g).]

$(*)_4$ for any $\upsilon \subseteq \kappa$ we define a model $M = M_\upsilon$ of $T^0_{\rm elo}$ as follows:

(a) its universe is μ^{+3}

[1] Here the case $\delta' \neq \delta$ is not really needed, but in some other versions, it is helpful.

(b) $<_M$ is the standard order on the ordinals so M is linearly ordered
(c) for $\ell < 2$ let R^M_ℓ be the following set: $\{(\alpha,\beta,\delta)$: for some $\delta_2 \in S_2$ and $\delta_1 \in S_1$ we have $\delta = g^2_{\delta_2}(\delta_1)$ and $\alpha \le \beta < \delta$ and for some pair (ε,γ) we have $\varepsilon < \kappa$, $\varepsilon \in v \Leftrightarrow \ell = 1$, $\gamma \in C^1_\delta$, $\mathrm{otp}(C^1_\delta \cap \gamma) = \varepsilon + 1$ and $\sup(g''_{\delta_2}(C^1_{\delta_1}) \cap g_{\delta_2}(\gamma)) < \alpha \le \beta < g^2_{\delta_2}(\gamma)\}$

[Why? Note that it is easy to check that M_v is well defined and indeed a model of T^0_{elo}.]

By our assumption toward contradiction:

$(*)_5$ for every $v \subseteq \kappa$ there are $\xi_v = \xi(v)$, f^2_v and $u_v, \bar\alpha_v, \delta^2_v = \delta_2(v), \gamma_v$ such that:

(a) $\xi_v < \xi_*$
(b) f^2_v is an embedding of M_v into $M^*_{\xi_v}$ so a function from μ^{+3} into λ
(c)

(α) E^2_v is a club of μ^{+3} as in 3.5(3)(c) with $(\langle f^2_v(i) : i < \mu^{+3}\rangle, M^*_{\xi_v})$ here standing for $(\langle a_i : i < \kappa\rangle, M)$ there
(β) $\delta^2_v \in E^2_v \cap S_2$, moreover $C^2_{\delta_2(v)} \subseteq E^2_v$, note that $\mathrm{cf}(\delta^2_v) = \mu^+$ and $\delta^2_v < \mu^{+3}$
(γ) $u_v \in \mathscr{P}$, so $u_v \subseteq \lambda$ has order type μ^+ under $<_M$
(δ) $u_v \in \mathscr{P}$ is as in 3.5(3)(c) for δ^2_v, i.e. for the sequence $\langle f^2_v(i) : i < \delta^2_v\rangle$ with δ^2_v playing the role of j there
(ε) let $f_v = f^1_v = f^2_v \circ g^2_{\delta_2(v)}$, so $f^1_v : \mu^+ \to \lambda$
(ζ) $\alpha_v = \langle \alpha_{v,i} : i < \mu^+\rangle$ is equal to $\bar\alpha_{\delta_2(v)}$, see $(*)_3$(g)

(d) E^1_v satisfy: E^1_v is the set of limit ordinals $i < \mu^+$ such that: $(\forall j_1 < i)(\exists j_2)\big(j_1 < j_2 < i \wedge f^1_v(j_1) <_{M^*_{\xi(v)}} \alpha_{v,j_2} \wedge \alpha_{v,j_1} < f^1_v(j_2)\big)$
(e) $\delta^1_v = \delta_1(v)$, γ_v satisfy

(α) $\delta^1_v \in E^1_v$ satisfies $C^1_{\delta_1(v)} \subseteq E^1_v$ note that $\delta^1_v < \mu^+$, $\mathrm{cf}(\delta^1_v) = \kappa$
(β) $\gamma_v = f^1_v(\delta^1_v) < \lambda$

[Why? First, for clauses (a),(b) use the choice of $\bar M^*$ in $(*)_1$.

Second, for clause (c) we use the choice of $\mathscr{P}$, i.e. $(*)_3(a)$ and so 3.5(3)(c); that is, we apply the choice of $\mathscr{P}_\xi \subseteq \mathscr{P}$ to the sequence $\langle f^2_v(\alpha) : \alpha < \mu^{+3}\rangle$ (and the linear order (hence a tree) $M^*_{\xi(v)}$; so apply 3.5(3)(c), giving us a club E^2_v such that clause (c)(α) holds. Next, as as $S_2 \subseteq \mu^+$ is stationary we can choose $\delta^2_v \in E^2_v \cap S_2$ such that $C^2_{\delta_1(v)} \subseteq E^2_v$, note that necessarily $\mathrm{cf}(\delta^2_v) = \mu^+$. Now by the choice of E^2_v, δ^2_v there is a set $u = u_v$ as promised in clauses $(c)(\gamma)$, (δ). Lastly clause $(c)(\varepsilon)$ is straightforward.

Third, for clause (d) use clause (c)

Lastly, for clause (e) recall $(*)_3(c)(\varepsilon)$.]

$(*)_6$ there are $\xi_{**}, u_*, \gamma_*, \delta_*$ such that $\mathscr{V}$ has cardinality $> \lambda + |\xi_*|$ where $\mathscr{V}$ is the set of $v \subseteq \kappa$ such that:

(a) $\xi_v = \xi_{**} < \xi_*$
(b) $u_v = u_* \in \mathscr{P}$
(c) $\gamma_v = \gamma_* < \lambda$
(d) $\delta^2_v = \delta_2(*) \in S_2$ and $\delta^1_v = \delta_1(*) \in S_1$

Let $v_1 \neq v_2$ be from $\mathscr{V}$. As $v_1 \neq v_2$ there are ε_*, β_* such that:

$(*)_7$

(a) $\varepsilon_* < \kappa$ and $\varepsilon_* \in v_1 \Leftrightarrow \varepsilon_* \notin v_2$
(b) $\beta_* \in C^1_{\delta_1(*)} \subseteq \mu^+$ satisfies $\mathrm{otp}(C^1_{\delta_1(*)} \cap \beta_*) = \varepsilon_* + 1$.
(c) $\beta_{**} = g^1_{\delta_1(*)}(\beta_*) < \mu^{+3}$

Now easily

$(*)_8$ for $\ell \in \{1, 2\}$ we have:

(a) in M_{v_ℓ}, for some $\beta^*_\ell < \beta_{**}$ we have: if $\beta^*_\ell < \beta_1 < \beta_2 < \beta_{**}$ <u>then</u> $M_{v_\ell} \models R_\ell(\beta_1, \beta_2, \gamma_*)$ <u>iff</u> ℓ is the truth value of $\varepsilon_* \in v_\ell$.
[Why? By the choice of M_{v_ℓ}, see $(*)_2$, in particular, clause (c) there.]
(b) in $M_{\xi_{**}}$, for some unbounded subset B_ℓ of $B_* := \{\beta < \alpha_{\delta_2(*),\beta_{**}} : M_{\xi_{**}} \models$ "$\beta < \gamma''_*\}$ we have:

- if $\beta_1 < \beta_2$ are from B_ℓ then $M^*_{\xi_{**}} \models R_\ell(\beta_1, \beta_2, \gamma_*)$
[Why? Clearly "$\beta^*_\ell < \beta_1 < \beta_2 < \beta_{**}$ implies $M^*_{\xi_{**}} \models R_\ell(f^2_{v_\ell(}\beta_1), f^2_{v_k l}(\beta_2)$, $\gamma_*)''$, hence $B_2 = \{f^2_{v_\ell}(\beta) : \beta \in (\beta^*_\ell, \beta_*)\}$ is as required.

(c) B_* is linearly ordered in $M^*_{\xi_{**}}$ (with no last element) and does not depend on ℓ
[Why? Obvious]
(d) in M_{ξ_*}, for some end-segment B'_ℓ of B_* we have: if $\beta_1 \leq \beta_2$ are from B'_ℓ, then $M^*_{\xi_{**}} \models R_\ell(\beta_1, \beta_2, \gamma_*)$
[Why? the convex hull of B_ℓ is as required becuase B_ℓ is unbounded in B_* recalling clause (b) and the definition of T^0_{elo}.]

Note that

$(*)_9$ the statement in $(*)_8(d)$ does not depend on ℓ.

Now by $(*)_7$, $(*)_8(d)$, $(*)_9$ we get a contradiction. □

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