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ON SOME GENERAL PROPERTIES OF CHROMATIC NUMBERS

P. ERDŐS – A. HAJNAL – S. SHELAH

§1. INTRODUCTION

In his paper [7] Taylor introduced a generalization of chromatic number of graphs and stated several interesting problems. In this note we will be interested in one of his problems we will state below. We are going to formulate several pussible generalizations and quite a few related questions. Our main aim is to formulate the problems but we will write down some partial results we obtained trying to clear the problems up.

Let us start with the following remark

(1) Let $\varphi(x, \lambda)$ be any statement of set theory, A a set and $\psi(x)$ an operation such that $\forall x(\psi(x) \in A)$. Let us assume that $\sigma < \lambda$ and $\varphi(x, \lambda)$ imply $\varphi(x, \sigma)$. Then there is a λ such that for all x with $\varphi(x, \lambda)$ and for all $\sigma \geqslant \lambda$ there is a y such that $\varphi(y, \sigma)$ and $\psi(y) = \psi(x)$. To see this one defines $A' \subset A$ with the stipulation

$$A' = \{ y \in A : \exists \lambda \forall x (\psi(x) = y \Rightarrow \neg \varphi(x, \lambda))\}$$

i.e. the set of $y \in A$ for which "the λ-s of $\psi^{-1}(\{y\})$ form a bounded set," and denoting by

$$\lambda(y) = \min\{\lambda\colon \forall x(\psi(x) = y \Rightarrow \neg\varphi(x, \lambda))\} \quad \text{for} \quad y \in A'$$

i.e. the minimal bound for $y \in A'$, $\lambda = \sup\{\lambda(y)\colon y \in A'\}$ obviously satisfies the requirement of (1).*

It is obvious that in general one can not hope for the determination of the λ (depending on φ, and ψ). However, Taylor observed that in many cases it is quite natural to ask for the size of λ. The simplest interesting case of Taylor's general problem arises if we choose $\varphi(x, \lambda)$ to be the statement that X is a graph of chromatic number at least λ, B the set of finite graphs with vertices in ω, $A = P(B)$, and $\psi(x)$ the set of graphs in B isomorphic to a subgraph of X. Taylor's problem for chromatic numbers of graphs is to determine the minimal λ satisfying (1) in this case or to put it into words

(2) What is the minimal λ satisfying the following condition. For every graph $\mathcal{G}$ with chromatic number $\geqslant \lambda$ and for every $\sigma \geqslant \lambda$ there is a graph $\mathcal{G}'$ with chromatic number $\geqslant \sigma$ such that $\mathcal{G}$ and $\mathcal{G}'$ have the same finite subgraphs.

Taylor pointed out that known theorems imply $\lambda \geqslant \omega_1$ and he conjectured that probably $\lambda = \omega_1$.

This problem seems to be very difficult and so instead of solving it we will formulate variants of it which are probably even more difficult. We will not consider Taylor's generalization for relational structures but we will stick to set-systems. To have a brief notation we say that $\mathcal{H}$ is a set system if it consists of sets having at least two elements. For a set-system $\mathcal{H}$ we put

$$\chi(\mathcal{H}) = \min\{\chi\colon \exists \ \text{a function} \ f\colon \cup\mathcal{H} \to \chi \ \text{such that} \ \forall\rho\forall x \in \mathcal{H}(x \not\subset f^{-1}(\{\rho\}))\}.$$

$\chi(\mathcal{H})$ is the chromatic number of $\mathcal{H}$ and is the minimal cardinal

*This is the same proof which gives the existence of Hanf numbers in [6].

χ for which $\cup \mathscr{H}$ is the union of χ-sets none of which contains an element of $\mathscr{H}$ as a subset. $\mathscr{H}$ is said to be uniform with $\kappa(\mathscr{H}) = \kappa$ if $|X| = \kappa$ for $X \in \mathscr{H}$. A graph is a uniform set system with $\kappa(\mathscr{H}) = 2$. For further explanation of the terminology and for elementary results see e.g. [1]. Note that two set systems, $\mathscr{H}$, $\mathscr{H}'$ are considered isomorphic if there is a one-to-one mapping f of $\cup \mathscr{H}$ onto $\cup \mathscr{H}'$ such that for $X \subset \cup \mathscr{H}$

$$X \in \mathscr{H} \quad \text{iff} \quad f(X) = \{ f(u) \colon u \in X \} \in \mathscr{H}' .$$

We denote by $\mathscr{H} \cong \mathscr{H}'$ the fact that $\mathscr{H}$ and $\mathscr{H}'$ are isomorphic.

§2. STATEMENT OF SOME RESULTS ON CLASSES OF GRAPHS ADMITTING ARBITRARILY LARGE CHROMATIC NUMBERS

Definition. Let $\tau \geqslant \omega$ be a cardinal. Put

$$B(\tau) = \{ \mathscr{G} \colon \mathscr{G} \subset [\tau]^2 \wedge |\mathscr{G}| < \tau \} ; \qquad A(\tau) = P(B(\tau))$$

i.e. $B(\tau)$ is the set of subgraphs of cardinality $< \tau$ of the complete graph with set of vertices τ. Obviously, if $|\mathscr{G}| < \tau$, then $\mathscr{G}$ is isomorphic to an element of $B(\tau)$.

Let $\mathscr{G}$ be a graph. We denote by $\psi(\mathscr{G}, \tau)$ the set of $\mathscr{G}' \in B(\tau)$, $\mathscr{G}'$ is isomorphic to a subgraph of $\mathscr{G}$; $(\psi(\mathscr{G}, \tau) \in A(\tau))$.

Let $S \in A(\tau)$; We denote by $\mathscr{G}(S, \tau)$ the class of graphs $\mathscr{G}$ with $\psi(\mathscr{G}, \tau) \subset S$. $S \in A(\tau)$ is said τ-unbounded if

(3) For every λ there is $\mathscr{G} \in \mathscr{G}(S, \tau)$ with $\chi(\mathscr{G}) > \lambda$. An obvious approach to Taylor's problem would be first to characterize the $S \in A(\omega)$ which are ω-unbounded and then show that $\chi(\mathscr{G}) \geqslant \omega_1$ implies that $\psi(\mathscr{G}, \omega)$ satisfies this characterization.

This again seems to be hopeless at present. It is not quite easy to give nontrivial $S \in A(\omega)$ which are ω-unbounded. We now give the definition of some of them.

Let $R, \prec$ be an ordered set $i < \omega$. We will define two sorts of graphs $\mathscr{G}^\circ(R, i)$, $\mathscr{G}^1(R, i, t)$ for $i \geqslant 2$, or $i \geqslant 3$, $1 \leqslant t < i - 1$ respectively. The set of vertices will be the set of $\prec$ increasing sequences φ of length i of elements of R in both cases.

We put

$$\mathcal{G}^\circ(R, i) = \{\{\varphi, \varphi'\} : \varphi(j+1) = \varphi'(j) \quad \text{for} \quad j < i-1\} \quad \text{for} \quad i \geq 2 \quad \text{and}$$

$$\mathcal{G}^1(R, i, t) = \{\{\varphi, \varphi'\} : \varphi(j+t) < \varphi'(j) < \varphi(j+t+1) < \varphi'(j+1)$$
$$\text{for} \quad j < i-1-t\} \quad \text{for} \quad i \geq 3 .$$

We put

$$S^\circ(i) = \psi(\mathcal{G}^\circ(\omega, i), \omega) = \psi(\mathcal{G}^\circ(R, i), \omega) \quad \text{for} \quad |R| \geq \omega$$

and

$$S^1(i, t) = \psi(\mathcal{G}^1(\omega, i, t), \omega) = \psi(\mathcal{G}^1(R, i, t), \omega) \quad \text{for} \quad |R| \geq \omega .$$

The graphs $\mathcal{G}^1(R, i, t)$ we call Specker – Graphs, (Specker used first $\mathcal{G}^1(\omega, 3, 1)$ to show $\omega^3 \nrightarrow (\omega^3, 3)^2$) and the graphs $\mathcal{G}^\circ(R, i)$ we call with some abuse of terminology the "edge graphs" having in mind the special case $i = 2$.

The following are known about these graphs:

Old-lemmas (Erdős – Hajnal)

1/ *Let* $|R| \geq (\exp_{i-1}(\lambda))^+$; $\lambda \geq \omega$, $i \geq 2$. *Then* $\chi(\mathcal{G}^\circ(R, i)) \geq \lambda^+$. *As a corollary* $S^\circ(i)$ *is* ω-*unbounded for* $2 \leq i < \omega$.

2/ $S^\circ(i)$ *does not contain odd-circuits of length* $2j+3$ *for* $j \leq i-2$, $i \geq 2$,

3/ *Let* $\kappa \geq \omega$ *be a cardinal. Then* $\chi(\mathcal{G}^1(\kappa, i, t)) = \kappa$ *for* $3 \leq i < \omega$ *and* $1 \leq t < i-1$.

4/ $S^1(2i^2+1, i)$ *does not contain odd circuits of length* $2j+3$ *for* $j \leq i-1$, $1 \leq i < \omega$.

5/ *Assume* $\lambda \geq \omega$ *is a cardinal* $R, \prec$ *an ordered set with* $|R| \leq \exp(\lambda)$ *for* $2 \leq i < \omega$. *Then* $\chi(\mathcal{G}^\circ(R, i)) \leq \lambda$.

See [2] Theorem 1 for 1/, and 5/, [4] Theorem 7 for 2/, [1] Theorem 7.4 for 3/, and 4/.

The following inclusions hold:

(i) $S^\circ(i) \subsetneqq S^\circ(i+1)$ for $2 \leqslant i < \omega$

(*) (ii) $S^1(i, t) \subsetneqq S^1(i+1, t)$ for $3 \leqslant i < \omega$

(iii) $S^\circ(i) \subset S^1(i+1, 1)$ for $2 \leqslant i < \omega$.

We will give the proof of (i) on $p.$ We see now that the sets $S^\circ(i)$ corresponding to the "edge graphs" form a decreasing sequence. The members of the sequence are all ω-unbounded. The intersection $\cap\{S^\circ(i):$ $2 \leqslant i < \omega\}$ however by 2/ contains only graphs with $\chi(\mathcal{G}) = 2$, hence is not ω-unbounded.

One of our main points is that the $S^\circ(i)$ are not equally good as ω-unbounded classes.

To be able to formulate our result we need the following

Definition. Let $F(\lambda) \geqslant \lambda^+$ be an operation on cardinals.

We say that $S \in A(\omega)$ is ω-unbounded with the restriction F, if for all σ there is $\lambda \geqslant \sigma$, and a $\mathcal{G}$ with $\psi(\mathcal{G}, \omega) \subset S$ such that

(4) $$\chi(\mathcal{G}) > \lambda \quad \text{and} \quad |\mathcal{G}| \leqslant F(\lambda)\,.$$

We briefly say that S is ω-unbounded with the restriction ξ if it is ω-unbounded with the restriction

F_ξ where $F_\xi(\lambda) = \kappa$ iff $\lambda = \omega_a$, $\kappa = \omega_{a+1+\xi}$.

Theorem 1. *(a)* $S^1(i, t)$ *is* ω*-unbounded with the restriction* 0 *for* $3 \leqslant i < \omega$.

(β) $S^\circ(i)$ *is* ω*-unbounded with the restriction* $\exp_{i-1}(\lambda)^+$ *for* $2 \leqslant i < \omega$.

(γ) $S^\circ(i)$ *is not* ω*-unbounded with the restriction* $\exp_{i-1}(\lambda)$ *for* $2 \leqslant i < \omega$.

As a corollary if G.C.H holds then for every n there is an $S \in$

$\in A(\omega)$ which is ω-unbounded with the restriction $n+1$ but not with the restriction n.

Note that (α) follows from the old lemma 3/ (β) follows from 1/. We will prove (γ) in the next chapter. If G.C.H is assumed then $S^\circ(n+2)$ is ω-unbounded with $(\exp_{n+1}(\lambda))^+$, if $\lambda = \omega_\alpha$, $(\exp_{n+1}(\lambda))^+ = \omega_{\alpha+n+2}$, hence $S^\circ(n+2)$ is $n+1$ unbounded, and is not ω-bounded with $\exp_{n+1}(\lambda) = \omega_{\alpha+n+2}$, hence is not n-unbounded. Before giving the proof of (γ) in the next cl apter, it is time to state the first problem.

Problem. Does there exist an $S \in A(\omega)$ which is ω-unbounded but is not ω-unbounded with the restriction $\exp_n(\lambda)$ for every $n < \omega$?

Note that there is an obvious correlation with an old Erdős – Hajnal problem stated in 2 (Problem 1). This problem asks if there is a graph of $\chi(\mathcal{G}) > \lambda$ such that all subgraphs $\mathcal{G}'$ of $|\mathcal{G}'| \leqslant \exp_\omega(\lambda)$ have chromatic number $\leqslant \lambda$. The "edge graphs" $S^\circ(i)$ were used in [2] to establish a positive answer to the above problem when $\exp_\omega(\lambda)$ is replaced by $\exp_n(\lambda)$, $n < \omega$.

We finally mention that the definition of unboundedness with a restriction had to be done as in (4) because we have the following

Theorem 2. *Assume $S \in A(\omega)$ is ω-unbounded. Then for every σ there is $\lambda \geqslant \sigma$ and $\mathcal{G} \in \mathcal{G}(S, \omega)$ with*

$$\chi(\mathcal{G}) = |\mathcal{G}| = \lambda .$$

§3. PROOFS

First we prove Theorem 2.

Let S be ω-unbounded. For every λ choose $\mathcal{G}_\lambda \in \mathcal{G}(S, \omega)$ with $\chi(\mathcal{G}_\lambda) \geqslant \lambda$. Put $S_\lambda = \psi(\mathcal{G}_\lambda, \omega)$. Put $\hat{S}_\lambda = \{\mathcal{G} \in \psi(\mathcal{G}_\lambda, \omega)$: There are uncountably many subgraphs $\mathcal{G}' \subset \mathcal{G}_\lambda$ isomorphic to $\mathcal{G}$ with pairwise disjoint sets $\cup \mathcal{G}'\}$.

For each $\mathcal{G} \in S_\lambda - \hat{S}_\lambda$ let $\mathcal{F}(\mathcal{G})$ be a maximal system of subgraphs of $\mathcal{G}_\lambda$ satisfying the following conditions:

(i) $\mathscr{G}' \in \mathscr{F}(\mathscr{G}) \Rightarrow \mathscr{G}' \cong \mathscr{G}$

(ii) $\mathscr{G}' \neq \mathscr{G}'' \in \mathscr{F}(\mathscr{G}) \Rightarrow \cup\mathscr{G}' \cap \cup\mathscr{G}'' = \phi$.

By the definition of $\hat{S}_\lambda$ we have $|\mathscr{F}(\mathscr{G})| \leqslant \omega$ for $\mathscr{G} \in S_\lambda - \hat{S}_\lambda$. Hence $\cup\cup\mathscr{F}(\mathscr{G})$ is countable for $\mathscr{G} \in S_\lambda - \hat{S}_\lambda$. We now omit the vertices in $\cup\{\cup\cup\mathscr{F}(\mathscr{G}) \colon \mathscr{G} \in S_\lambda - \hat{S}_\lambda\} = T_\lambda$ from $\mathscr{G}_\lambda$ i.e. we consider $\hat{\mathscr{G}}_\lambda = \mathscr{G}_\lambda \cap [\cup\mathscr{G}_\lambda - T_\lambda]^2$. Then for $\lambda > \omega$ we have $\chi(\hat{\mathscr{G}}_\lambda) \geqslant \lambda$. It follows from the construction that $\psi(\hat{\mathscr{G}}_\lambda, \omega) = \hat{S}_\lambda$ and, using $|T_\lambda| \leqslant \omega$, for all $\mathscr{G}' \in \hat{S}_\lambda$ there are uncountably many $\mathscr{G}'' \subset \hat{\mathscr{G}}_\lambda$ isomorphic to $\mathscr{G}'$ with pairwise disjoint $\cup\mathscr{G}''$. It follows that $\hat{S}_\lambda$ has the following property

(5) Assume $\mathscr{G}_i \in \hat{S}_\lambda$, $i < n < \omega$ such that $\cup\mathscr{G}_i \cap \cup\mathscr{G}_j = \phi$ for $i \neq j < n$. Then $\bigcup_{i<n} \mathscr{G}_i \in \hat{\hat{S}}_\lambda$.

Now it follows that there is $S_\lambda \subset S$ S_λ ω-unbounded, such that S_λ satisfies (5). On the other hand, if S_λ is ω-unbounded and satisfies (5) then $\mathscr{G}(S_\lambda, \omega)$ is closed with respect to arbitrary unions of graphs with disjoint set of vertices. Let σ be given. We can choose $\mathscr{G}_0$, with $\chi(\mathscr{G}_0) \geqslant \sigma$, and $\mathscr{G}_{n+1}$ with $\chi(\mathscr{G}_{n+1}) > |\mathscr{G}_n|$ for $n < \omega$ such that $\mathscr{G}_n \in \mathscr{G}(S_\lambda, \omega)$ and the $\cup\mathscr{G}_n$ are disjoint. Then $\widetilde{\mathscr{G}} = \bigcup_{n<\omega} \mathscr{G}_n \in \mathscr{G}(S_\lambda, \omega) \subset \mathscr{G}(S, \omega)$ and $\chi(\widetilde{\mathscr{G}}) = |\widetilde{\mathscr{G}}| > \sigma$. This proves Theorem 2.

We now state the following.

Lemma. *Let $\mathscr{G}$ be a graph, $2 \leqslant i < \omega$, $\mathscr{G}$ is isomorphic to a subgraph of $\mathscr{G}^\circ(R, i)$ for some $(R, \prec)$ if the following conditions hold:*

Put $G = \cup\mathscr{G}$. There is a relation $\preceq$ on $G \times i$ such that

(α) *$\preceq$ is transitive.*

(β) *$\forall u, v \in G \times i\ (u \preceq v \vee v \preceq u)$ i.e. $\preceq$ is a preorder on $G \times i$ put $x \prec y$ for $x \preceq y \wedge y \npreceq x$; $x \vdash\!\dashv y$ for $x \preceq y \wedge y \preceq z$.*

(γ) *$\forall x \in G(\langle x, 0\rangle \prec \ldots \prec \langle x, i-1\rangle)$.*

(δ) *$\forall x, y \in G(\langle x, 0\rangle \not\vdash\!\dashv \langle y, 0\rangle \vee \ldots \vee \langle x, i-1\rangle \not\vdash\!\dashv \langle y, i-1\rangle \vee x = y)$.*

(ϵ) *$\forall x, y \in G(\{x, y\} \in \mathscr{G} \Rightarrow (\langle x, 1\rangle \vdash\!\dashv \langle y, 0\rangle \wedge \ldots \wedge \langle x, i-1\rangle \vdash\!\dashv \langle y, i-2\rangle) \vee (\langle y, 1\rangle \vdash\!\dashv \langle x, 0\rangle \wedge \ldots \wedge \langle y, i-1\rangle = \langle x, i-2\rangle)$.* The lemma is obvious.

Proof of Theorem 1. We only have to prove (γ) of Theorem 2. Let $\mathcal{G} \in \mathcal{G}(S^\circ(i), \omega)$, $2 \leq i < \omega$. Let $\lambda \geq \omega$ be arbitrary, and assume $|\mathcal{G}| \leq \exp_{i-1}(\lambda)$. By the assumption for every $\mathcal{G}' \subset \mathcal{G}$, $|\mathcal{G}'| < \omega$, $\mathcal{G}'$ is isomorphic to an element of $S^\circ(i)$. Hence by the lemma there is a preorder $\preceq$ satisfying the conditions $(a) \ldots (\epsilon)$ of the lemma for $\cup \mathcal{G}' \times i$. Then by the compactness theorem the same holds for $\cup \mathcal{G} \times i$. Hence by the lemma there is $R, \prec$ such that $\mathcal{G}$ is isomorphic to a subgraph of $\mathcal{G}^\circ(R, i)$. By $|\mathcal{G}| \leq \exp_{i-1}(\lambda)$ we may choose R with $|R| \leq \exp_{i-1}(\lambda)$. Then by the old lemma 5./ $\chi(\mathcal{G}) \leq \chi(\mathcal{G}^\circ(R, i)) \leq \lambda$. Thus $S^\circ(i)$ is not ω-unbounded with the restriction $\exp_{i-1}$. This proves Theorem 1.

Finally we prove $(*)$ (i).

It is sufficient to prove $S^\circ(i+1) \subset S^\circ(i)$ for $2 \leq i < \omega$. Let $\mathcal{G} \in S^\circ(i+1)$. We may assume $\mathcal{G} = \mathcal{G}^\circ(n, i+1)$, $\cup \mathcal{G} = {}^{i+1}n$ for some $n < \omega$. We define a partial order $\prec'$ on $({}^{i+1}n) \times i$ by the stipulation

$$(\varphi, j) \prec' (\psi, k) \quad \text{iff} \quad \varphi(j) < \psi(k) \vee (\varphi(j) = \psi(k) \wedge \varphi(j+1) < \psi(k+1))$$

and we extend $\prec'$ to an arbitrary preorder of ${}^{i+1}n \times i$. It is easy to see that the requirements $(a) \ldots (\epsilon)$ hold for $\preceq'$ hence by the Lemma, $\mathcal{G}$ is isomorphic to an element of $S^\circ(i)$.

§4. A THEOREM OF DIFFERENT TYPE

Old result. (see Erdős – Hajnal [1] (Corollary 5.6)).

Assume $\chi(\mathcal{G}) > \omega$. Then $\mathcal{G}$ contains a complete bypartite graph $[\kappa, \omega,]$ for all $\kappa < \omega$. As a corollary if $\mathcal{G}_0$ is a fixed finite bypartite graph and $\chi(\mathcal{G}) > \omega$, then $\mathcal{G}$ contains a subgraph isomorphic to $\mathcal{G}_0$ and again in another formulation if $S \in A(\omega)$. S is ω-unbounded, then S contains all bypartite finite graphs.

On the other hand, the old lemmas show that this statement is no longer true for any fixed nonbypartite graph.

However, it is still possible to prove statements of the following type:

(i) If $\chi(\mathcal{G}) > \lambda$ then $\psi(\mathcal{G}, \omega) \cap S \neq \phi$ for some fixed $S \in A(\omega)$.

(ii) If $\chi(\mathcal{G}) > \lambda$ *then there is* n_0 *such that* $S_n \in \psi(\mathcal{G}, \omega)$ *for* some fixed sequence $\langle S_n;\, n < \omega \rangle$, $S_n \in B(\omega)$, $n > n_0$.

Taylor's conjecture implies that if a statement like (i) or (ii) holds for some λ then it holds for $\lambda = \omega$ as well.

The following theorem is an example of a statement of this kind. In [1] we only could prove it in case $\chi(\mathcal{G}) > \omega_1$.

Theorem 3. *Assume* $\chi(\mathcal{G}) > \omega$. *Then there is* $n < \omega$ *such that* $\mathcal{G}$ *contains odd circuits of length* $2j + 1$ *for all* $n < j < \omega$.

Proof. Let $\chi(\mathcal{G}) > \omega$. Put $\cup \mathcal{G} = G$ for the set of vertices. We may assume $\mathcal{G}$ is connected. Let x be an arbitrary vertex of $\mathcal{G}$. Put $G_i = \{ y \in G\colon$ The length of the shortest path connecting x and y in $\mathcal{G}$ is $i \}$. Then $G_0 = \{x\}$, $G = \bigcup_{i<\omega} G_i$. Put $\mathcal{G}^i = \mathcal{G} \cap [G_i]^2$. Then there is $1 \leqslant i < \omega$ such that $\chi(\mathcal{G}^i) > \omega$. Let $\mathcal{G}^{i,m} = \{ \{u, v\} \in \mathcal{G}^i\colon$ There is a path of length $2(m+1)$ in $\mathcal{G}$ connecting u and v, all whose vertices but u and v do not belong to $G_i \}$. By the definition of G_i we have

$$\mathcal{G}^i = \bigcup_{m<i} \mathcal{G}^{i,m} .$$

Considering that then $\chi(\mathcal{G}^i) \leqslant \prod_{m<i} \chi(\mathcal{G}^{i,m})$ it follows that there is $m < i$ with $\chi(\mathcal{G}^{i,m}) > \omega$. By the old result, for all j, $2 \leqslant j < \omega$, there is an edge $\{u, v\} \in \mathcal{G}^{i,m}$ contained in an odd circuit of length $2j$ of $\mathcal{G}^{i,m}$. Omitting from this circuit the edge $\{u, v\}$ and adding to it the edges of the path of length $2(m+1)$ the existence of which is required by the definition of $\mathcal{G}^{i,m}$, we get an odd circuit of length $2(m+j)+1$ contained in $\mathcal{G}$ for $m + j \geqslant m + 2 = n$. This proves Theorem 3.

We have no counterexample to

Problem 2. Let $\chi(\mathcal{G}) > \omega$. Then there is i, with $2 \leqslant i < \omega$ such that

$$S^0(i) \subset \psi(\mathcal{G}, \omega) .$$

We think that the answer is no. A positive answer would yield the solution of all the problems mentioned so far, since, by the old lemmas, it would imply that if $\chi(\mathscr{G}) > \omega$, then $\psi(\mathscr{G}, \omega)$ is ω-unbounded with the restriction $\exp_n^+$ for some $n < \omega$. (i.e. that the answer to Taylor's problem (2) is yes and the answer to Problem 1 is no.)*

§5. FURTHER SPECULATIONS

1/ Let $F(\tau) \geqslant \tau^+$ be an operation on cardinals. Choosing the property $\varphi(x, \lambda)$ appearing in (1) to be $\exists \tau(\tau^+ \geqslant \lambda \wedge x$ is a graph $\wedge$ $\wedge\, \chi(x) > \tau \wedge |x| \leqslant F(\tau))$, we see that there is a Taylor number corresponding to each restriction. Obviously we can expect results only if F in some way reasonlable, (e.g. $F(\omega) = (2^\omega)^+$, $F(\lambda) = \lambda^+$ for $x \geqslant \omega_1$, is unreasonable.) Without going into details we state the simplest problems.

Problem 3. Is it true that $\chi(\mathscr{G}) > \omega$, $|\mathscr{G}| \leqslant (\exp_i(\omega))^+$ implies that $\psi(\mathscr{G}, \omega)$ is ω-unbounded with the restriction $(\exp_i(\lambda))^+$ for $i <$ $< \omega$?

There is no counterexample to the following stronger

Problem 4. Let $\lambda \geqslant \omega$, $i < \omega$. Assume that there is $\mathscr{G}$ with $\chi(\mathscr{G}) > \lambda$, $|\mathscr{G}| \leqslant (\exp_i(\lambda))^+$ then for every infinite τ there is $\mathscr{G}'$ with

$$\chi(\mathscr{G}') > \tau, \quad |\mathscr{G}'| \leqslant (\exp_i(\tau))^+, \quad \psi(\mathscr{G}, \omega) = \psi(\mathscr{G}', \omega).$$

(We emphasize again that $\chi(\mathscr{G}) = \lambda$, $|\mathscr{G}| = \lambda$ does not imply that there is $\mathscr{G}'$ with $\chi(\mathscr{G}') = |\mathscr{G}'| = \tau$

$$\psi(\mathscr{G}', \omega) = \psi(\mathscr{G}, \omega)$$

as is shown e.g. by the fact that $\chi(\mathscr{G}^\circ(\kappa, 2)) = \kappa$ for all strong limit κ.)

2/ The problems we mentioned so far had not been studied in detail for set systems, not even in case $\kappa(\mathscr{H}) = 3$.

Let us now extend for uniform set systems, with $\kappa(\mathscr{H}) = \kappa$, in a self-explanatory way the notation $B(\tau)$, $A(\tau)$, $\psi(\mathscr{G}, \tau)$, $\mathscr{G}(S, \tau)$ intro-

*Problem 2 has already been stated in Taylor [8].

duced in §2. We will use the notation $B_\kappa(\tau)$, $A_\kappa(\tau)$, $\psi_\kappa(\mathscr{H}, \tau)$, $\mathscr{H}_\kappa(S, \tau)$ respectively. Though we can not disprove the analogue of (2) which is

(2′) Let $\kappa(\mathscr{H}) = 3$, $\chi(\mathscr{H}) > \omega$. Then for each λ there is $\mathscr{H}'$ with $\chi(\mathscr{H}') > \lambda$,

$$\psi_3(\mathscr{H}', \omega) = \psi_3(\mathscr{H}, \omega)$$

we want to point out one new phenomenon.

As we explained in §4, the old lemmas even imply the following (trivial) statement.

(6) If $\mathscr{G}_0$ is such that $\mathscr{G}_0 \in S$ for all $S \in A(\omega)$ which is ω-unbounded with the restriction λ^+ (the strongest possible restriction) then $\mathscr{G}_0 \in S$ for all unbounded $S \in A(\omega)$. (Namely the $\mathscr{G}_0$ in question are the bipartite graphs only).

A result of Erdős – Hajnal – Rothschild [5] implies that the analogue of (6) does not hold true for uniform set systems with $\kappa(\mathscr{H}) = 3$. The following is true:

(7) Let $\mathscr{H}_0$ consist of two triples having two points in common. Then

(α) $\mathscr{H}_0 \in S$ for all $S \in A_3(\omega)$ which is ω-unbounded with the restriction $\lambda^+ = \exp_0(\lambda)^+$ but

(β) There is $S \in A_3(\omega)$, $\mathscr{H}_0 \notin S$, S is ω-unbounded with the restriction $\exp_1(\lambda)^+$.

We do not know what is the natural bound to this sort of counter-examples in case $\kappa(\mathscr{H}) = 3$.

Again we do not state the problem here in general context but we formulate a rather simple Taylor type problem.

Let $C_3(\omega) = \{\mathscr{H}_0 \in B_3(\omega)\colon \mathscr{H}_0$ does not occur in some $\psi_3(\mathscr{H}, \omega)$ for which $\chi(\mathscr{H}) > \omega\}$. For each $\mathscr{H}_0 \in C_3(\omega)$ there is a minimal $\tau = \tau(\mathscr{H}_0)$ such that there is $\mathscr{H}$, $|\mathscr{H}| = \tau$ $\mathscr{H}_0 \notin \psi_3(\mathscr{H}, \omega)$,

$\chi(\mathscr{H}) > \omega$. Put $\tau(3, \omega) = \sup\{\tau(\mathscr{H}_0): \mathscr{H}_0 \in C_3(\omega)\}$.

Problem 5. $\tau(3, \omega) \leqslant (\exp_1(\omega))^+$, (Or at least $\leqslant \exp_\omega(\omega)$.)

In a forthcoming Erdős – Galvin – Hajnal paper it will be proved that if $\mathscr{H}_0$ is the system consisting of three triangles, which have an empty intersection and pairwise one point in common then $\tau(\mathscr{H}_0) \leqslant \exp_2(\omega)^+$.

3/ Let $\mathscr{G}$ be a graph. Define a function $f_\mathscr{G}(n)$ for $n < \omega$ by $f_\mathscr{G}(n) = \max\{\chi(\mathscr{G}'): \mathscr{G}' \subset \mathscr{G} \wedge |\cup \mathscr{G}'| = n\}$ for $n < \omega$. We mention without proof that the example of the graphs $\mathscr{G}^\circ(\xi, i)$ shows that

(8) For every $i < \omega$ there are graphs $\mathscr{G}_i$ with $\chi(\mathscr{G}_i) > \omega$, $f_{\mathscr{G}_i}(n) < C_i \log_i(n)$ for $n < \omega$. We state

Problem 6. Does there exist a $\mathscr{G}$ with $\chi(\mathscr{G}) > \omega$ and such that

$$f_\mathscr{G}(n) < \log_i(n) \quad \text{for} \quad n > n(i), \quad i < \omega\,?$$

This should be compared with Problem 2.

4/ Interesting new problems arise if we investigate the case of uniform set systems with $\kappa(\mathscr{H}) = \omega$. We only mention

Problem 7. Does there exist a cardinal λ such that if $\mathscr{H}$ is a uniform set system with $\chi(\mathscr{H}) > \lambda$ and $\kappa(\mathscr{H}) = \omega$, then there always exists an $S \subset \cup \mathscr{H}$, $|S| = \omega$ for which $\chi(\mathscr{H} \cap P(S)) > 2$?

The answer to this question might turn out to be trivial, but certainly a lot of similar interesting problems could be raised.

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