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Why There Are Many Nonisomorphic Models for Unsuperstable Theories

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We review here some theorems from [S6], and try to show they are applicable in other contexts too.

1. Unsuperstable theories, in regular cardinalities. Let $PC(T_1, T)$ be the class of $L(T)$-reducts of models of T_1.

THEOREM 1.1. *If T is not superstable, $T \subseteq T_1$ (T complete), $\lambda > |T_1|$, λ regular, then in $PC(T_1, T)$ there are 2^λ models of cardinality λ, no one elementarily embeddable in another.*

This was mentioned in [S4], and in fact in [S2]. We shall first sketch the proof and then point out some applications of the theorem and the method.

We generalize the notion of indiscernibility used in Ehrenfeucht-Mostowski models (from [EM]). Let I be an (index) model, M a model and, for each $s \in I$, $\bar{a}_s$ is a (finite) sequence from M. For $\bar{s} = \langle s(0), \cdots, s(n-1)\rangle$, $s(l) \in I$, let $\bar{a}_{\bar{s}} = \bar{a}_{s(0)} \wedge \cdots \wedge \bar{a}_{s(n-1)}$. The indexed set $\{\bar{a}_s : s \in I\}$ is called indiscernible if whenever $\bar{s}$, $\bar{t}$ are finite sequences from I realizing the same quantifier-free type, $\bar{a}_{\bar{s}}$ and $\bar{a}_{\bar{t}}$ realize the same type in M.

Now as T is not superstable, by [S2], T has formulas $\varphi_n(\bar{x}, \bar{y}_n)$, a model M, and sequences $\bar{a}_\eta$, $\eta \in {}^{\omega \geqq}\lambda$ such that, for $\eta \in {}^\omega\lambda$, $\tau \in {}^n\lambda$, $M \models \varphi[\bar{a}_\eta, \bar{a}_\tau]$ iff τ is an initial segment of η. Clearly M has an elementary extension to a model M_1 of T_1. By using a generalization of Ramsey's theorem [Rm] to trees (a proof was in [S3]) and by compactness, we can assume $\{a_\eta : \eta \in I\}$ is indiscernible; where I is a model with universe ${}^{\omega \geqq}\lambda$, one place relations $P^I_\alpha = {}^\alpha\lambda$ ($\alpha \leqq \omega$), the lexicographical order $<_1$, and the function f, $f(\eta, \tau)$ = the lengthiest common initial segment.

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For each $\delta < \lambda$, cf $\delta = \omega$, choose an increasing sequence η_δ of ordinals converging to δ. For every set $w \subseteq \{\delta < \lambda: \text{cf } \delta = \omega\}$ let M^1_w be the Skolem hull of $\{\bar{a}_\eta: \eta \in {}^{\omega>}\lambda \text{ or } \eta = \eta_\delta,\ \delta \in w\}$, and let M_w be the $L(T)$-reduct of M^1_w. Now we can prove that if $M_{w(1)}$ can be elementarily embedded into $M_{w(2)}$ then $w(1) - w(2)$ is not stationary (using Fodor [**Fd**]). As every stationary subset of λ can be split to λ disjoint ones (see Solovay [**So**]), it is easy to finish.

We can apply this construction, e.g., to the theory of dense linear order. This was independently done by Baumgartner [**Ba**]. (Note that every unstable theory is unsuperstable, and T is unstable iff it has the order property, i.e., there is a formula $\varphi(\bar{x}, \bar{y})$ and sequences $\bar{a}_n$ is some model M of T such that $M \models \varphi[\bar{a}_n, \bar{a}_m] \Leftrightarrow n < m$.)

Fuchs [**Fu**] asked how many separable reduced p-groups of cardinality $\lambda > \aleph_0$ there are. The class of such groups is not elementary (we should omit the type $\{x \neq 0\} \cup \{(\exists y)(p^n y = x): n < \omega\}$). However, we can find suitable $\bar{a}_\eta$, φ_n. Hence there are 2^λ nonisomorphic ones of cardinality $\lambda\ (> \aleph_0)$. For let G be a group generated freely by x_η ($\eta \in {}^{\omega \geq}\lambda$) subject only to the conditions: If $\eta \in {}^n\lambda$, $p^{n+1} x_\eta = 0$; $x_\eta = \sum_{n<\omega} p^n x_{\eta \restriction n}$. (For λ singular, see §2; this solution appears in [**S5**].)

The first-order theory of any infinite Boolean algebra has the order property, hence is unstable and unsuperstable, so we can apply 1.1. Notice that, e.g., the theory of atomless Boolean algebras has elimination of quantifiers; hence "elementary embedding" can be replaced by embedding (the existence of 2^λ nonisomorphic Boolean algebras of cardinality λ was proved in [**S1**], [**X1**], [**X2**]).

The existence of a rigid model is somewhat more complex. Monk and McKenzie ask about the existence of rigid Boolean algebras of cardinality $\aleph_1$, when $2^{\aleph_0} > \aleph_1$ (in [**MM**], see there for references and results). Stepanek and Balcan [**SB**] show the consistency with ZFC of $2^{\aleph_0} > \aleph_1$ + there is a rigid Boolean algebra of cardinality $\aleph_1$ with rigid completion.

THEOREM 1.2. *For every $\lambda > \aleph_0$ there is a rigid Boolean algebra of power λ with a rigid completion. If λ is regular, the algebra satisfies the countable chain condition.*

PROOF. We prove it for regular λ (from that it is easy to prove for singular cardinalities). Let S_α $\{\alpha < \lambda\}$ be disjoint stationary subsets of $W^* = \{\delta < \lambda: \text{cf } \delta = \omega,\ \delta \text{ divisible by } |\delta|\}$. For each $\delta \in W^*$ choose an increasing sequence $\{\zeta(\delta, n): n < \omega\}$ which converges to it, so that $\delta \in S_\alpha \Rightarrow \alpha < \zeta(\delta, 0)$, and $\zeta(\delta, n)$ is odd.

Let B' be the free Boolean algebra generated by $\{x_\alpha: \alpha < \lambda\} \cup \{y_\delta: \delta \in W^*\}$, and let h be a function from λ onto B' which maps δ onto the subalgebra generated by $\{x_i, y_j: i, j < \delta\}$ for $\delta \in W^*$. Let $B = B'/J$ where J is the ideal generated by $y_\delta - x_{\zeta(\delta,n)}$ ($\delta \in W^*$) and $y_\delta - h(\alpha)$ (for $\delta \in S_\alpha$).

Let B_α be the subalgebra generated by $\{x_i: i < \alpha\} \cup \{y_\delta: \delta < \alpha\}$. For any $a \in B$, B^* a subalgebra of B, let $F(a, B^*)$ be the filter $\{b \in B^*: b \geqq a\}$. Let $T_a = \{\alpha \in W^*:$ there is $b \leqq a$ such that $F(b, B_\alpha)$ is not principal$\}$. Clearly T_a is not uniquely determined by the isomorphism type of (B, a), but it is uniquely determined modulo D_λ. Also, $S_\alpha \subseteq T_a$, if $a = h(\alpha)/J$, and if $b = h(\beta)/J$ is disjoint to a, then $S_\beta \cap T_a = \emptyset$. Now for any automorphism F of B, $T_a = T_{F(a)}$ mod D_λ, and, for some a, a, $F(a)$ are disjoint, except when F is the identity. Hence B is rigid. As B_λ is dense in

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its completion, it is not hard to prove that the completion is rigid. Also the proof of the c.c.c. is clear. Similarly we can prove

THEOREM 1.3. *For every $\lambda > \aleph_0$ there is a rigid order with a rigid completion of cardinality $\leqq 2^{\aleph_0} + \lambda$. This was well known for $\lambda \geqq 2^{\aleph_0}$.*

2. $I(\lambda, T_1, T)$ for other cardinals. Let $I(\lambda, T_1, T)$ be the number of nonisomorphic models in $PC(T_1, T)$ of cardinality λ.

THEOREM 2.1. *Let $\lambda \geqq |T_1| + \aleph_1$, $T_1 \supseteq T$ (T complete) and T unsuperstable. Then $I(\lambda, T_1, T) = 2^\lambda$ except possibly when all the following conditions hold.*

(1) $\lambda = |T_1|$;

(2) $T_1 \neq T$;

(3) *T is stable*;

(4) *for some $\mu < \lambda$, $\mu^{\aleph_0} = 2^\lambda$.*

We sketch the proof of the main cases. Of course for regular $\lambda > |T_1|$, the result follows by 1.1.

Case I. There is $\mu < \lambda \leqq \mu^{\aleph_0}$, $2^\mu < 2^\lambda$.

Let M_1, a_η ($\eta \in {}^{\omega \geqq}\mu$), φ_n be as in the proof of 1.1. For any $w \subseteq {}^\omega\mu$ let $M^1(w)$ be the Skolem hull of $\{a_\eta: \eta \in {}^{\omega >}\mu \text{ or } \eta \in w\}$, and $M(w)$ is the $L(T)$-reduct. Clearly $|w| = \lambda \Rightarrow \|M(w)\| = \lambda$, and $M(w_1) \simeq M(w_2)$ define an equivalence relation on $\{w: w \subseteq {}^\omega\mu, |w| = \lambda\}$. Each equivalence class has $\leqq 2^\mu$ members; hence there are 2^λ equivalence classes.

Case II. For some regular $\mu < \lambda$, $2^\mu = 2^\lambda$, and $\lambda > |T_1|$.

PROOF. Similar to 1.1.

Case III. $\lambda > |T_1|$, λ singular but not strong limit, and $\mu < \lambda \Rightarrow \mu^{\aleph_0} < \lambda$, $2^\mu < 2^\lambda$.

Choose regular $\mu < \lambda$ with $2^\mu \geqq \lambda$; and let M_1, $\bar{a}_\eta$ ($\eta \in {}^{\omega \geqq}\lambda$), φ_n be as in 1.1. For each sequence $\bar{w} = \langle w_i: i < \lambda \rangle$ of subsets of $\{\delta < \mu: \text{cf } \delta = \mu\}$, let $M^1(\bar{w})$ be the Skolem hull of

$$\{\bar{a}_\eta: \eta \in {}^{\omega >}\lambda, (\forall n > 0)\eta(n) > \mu\} \cup \{\eta \in {}^\omega\lambda: (\forall n)\eta(n+1) < \eta(n+2), \text{ and } \eta\text{'s limit} \in w_{\eta(0)}\}.$$

Let $M(\bar{w})$ be the $L(T)$-reduct of $M^1(\bar{w})$. Now we prove that if $M(\bar{w}^1)$, $M(\bar{w}^2)$ are isomorphic, where $\bar{w}^l = \langle \bar{w}^l_i: i < \lambda \rangle$ then for every $i < \lambda$ there are $n < \omega, j_1, \cdots, j_n < \lambda$ and closed unbounded $S \subseteq \lambda$ such that $w^1_i \subseteq S \cap \bigcap_{k=1}^n w_{j_k}$. (Again, variants of the Fodor theorem and downward Lowenheim-Skolem theorems are used.) The conclusion is now easy.

Case IV. $\lambda > |T_1|$, λ is a strong limit singular cardinal.

As the construction is somewhat complex, we describe a similar construction.

Let $\varphi_n(x, \bar{y}_n)$ be as in 1.1, let M, N be models of T, and we describe a game $G(M, N)$. In the nth move Player I chose a sequence $\bar{a}_n$ from M of the length of $\bar{y}_n$, and then Player II chose a sequence $\bar{b}_n$ from N of the length of $\bar{y}_n$. Player II wins if $\{\varphi_n(\bar{x}; \bar{a}_n): n < \omega\}$ is realized in M iff $\{\varphi_n(\bar{x}, \bar{b}_n): n < \omega\}$ is realized in N. Clearly if M, N are isomorphic, Player II has a winning strategy; hence if Player I has a winning strategy they are not isomorphic. Let M_i ($i < \alpha$) be models of T, and let

$\{\bar{b}_j: j < \kappa\}$ be an enumeration of the set of all sequences from some of the M_i's (so $\kappa = \sum_{i<\alpha} \|M_i\|$). Let M_1, $\bar{a}_\eta$ $(\eta \in {}^{\omega \geq}\kappa)$, φ_n be as in 1.1, and let M^1_α be the Skolem hull of $\{\bar{a}_\eta: \eta \in {}^{\omega>}\kappa\} \cup \{\bar{a}_\eta: \eta \in {}^{\omega}\kappa$, and there is an i, such that $\bar{b}_{\eta(n)} \in M_i$ has length of $\bar{y}_n$ and M_i omits $\{\varphi_n(\bar{x}, \bar{b}_{\eta(n)}): n < \omega\}\}$. Let M_α be the $L(T)$-reduct of M^1_α. Now in the game $G(M_\alpha, M_i)$ Player I has a winning strategy: He will choose $\bar{a}_0 = \bar{a}_{\langle\rangle}$, and if in the nth move he has chosen $\bar{a}_\eta$ and Player II has chosen $\bar{b}_j$, he will choose in the $(n+1)$th move $\bar{a}_{\eta\hat{}\langle j\rangle}$.

In this we can construct M_α, $\alpha < \lambda$, which are pairwise nonisomorphic, and $\|M_\alpha\| < \lambda$. With inessential changes we can have $\|M_\alpha\| = \lambda$. Now we can change the rules of the game so that each player chooses χ sequences each time; and then just as above we built one tower, we can build χ towers, and the place of each model in each of them is independent.

Case V. T unstable, $\lambda = |T_1|$.

The problem is more difficult for $\lambda = |T_1|$, because then it is harder to control the properties of the model. We can assume T is countable, and that $L(T_1)$ contains, except individual constants, only countably many nonlogical symbols. As T is unstable, there is a model M^1 of T_1, and an indiscernible sequence $\{\bar{a}_i: i < \mu\}$ in it (μ — a strong limit cardinal $> |T_1|$), such that $M_1 \models \varphi[a_i, a_j] \Leftrightarrow i < j$; $\varphi \in L(T)$. We expand M_1 by the one place predicate P^{M_1} = the set of individual constants in M_1, and Skolem functions, and we get M_2. So, by the Erdös-Rado theorem **[EHR]** and compactness we can have, for any ordered set I, a model $N_2(I)$ elementarily equivalent to M_2, and $\bar{a}_t \in M_2(I)$ for $t \in I$, and for $t, s \in I$,

$$M_2(I) \models \varphi[\bar{a}_s, \bar{a}_t] \Leftrightarrow s < t,$$

and $\{\bar{a}_s: s \in I\}$ is indiscernible over $P_I = \{a: M_2(I) \models P[a]\}$, and together they generate the model.

Let D be a good ultrafilter over μ (exists by Kunen **[Ku]**), $M_2'(I)$ be the elementary submodel of $M_2(I)^\mu/D$ with universe $\{f/D$: there are $n < \omega$, $s(1), \cdots, s(n) \in J$, such that, for every i, $f(i)$ belongs to the Skolem hull of $P_I \cup \bigcup_{s=1}^n \bar{a}_s(l)\}$. Now $M_2'(I)$ will be an elementary submodel of $M_2(I)$ of cardinality κ in a strong sense. (We chose κ so that $M_2'(I)$, T, $T_1\,\lambda \in H(\kappa)$ = the family of sets of hereditary power $< \kappa$ and takes an elementary submodel of $H(\kappa)$ to which i $(i \leqq \lambda)$, T, T_1, $M_2'(I)$ belong and the cofinality of the ordinals in it is ω. We take the intersection of this submodel with $M_2'(I)$ as our model.) Let $M(I)$ be the $L(T)$-reduct of $M_2''(I)$, and the rest is in the line of [S1].

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