COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
54. ALGEBRAIC LOGIC, BUDAPEST (HUNGARY), 1988

On a problem in cylindric algebra

SAHARON SHELAH

Abstract. Here we solved a problem of Tarski and his co-authors Andréka, Henkin, Monk, and Németi which appeared in [HMTAN], [B88]. We thank Biró for asking us about it. We prove that isomorphism does not imply base-isomorphism (the latter is the same as induced isomorphism or point-function isomorphism) for the most generic kinds of algebras in algebraic logic (both in Tarski's and Halmos' sense), even under very severe restrictions (restrictions which easily work in the propositional case). To this end we prove that there is a first order complete theory with two atomic minimal non-isomorphic models none of them interpretable in the other.

Section 0: Introduction and main result

Let $\mathfrak{A} \subseteq \mathcal{P}(U)$ and $\mathfrak{B} \subseteq \mathcal{P}(V)$ be two Boolean algebras (BA's). It is natural to ask the following question: For what kinds of BA's does $\mathfrak{A} \cong \mathfrak{B}$ imply the existence of $f : U \rightarrowtail\!\!\!\!\twoheadrightarrow V$ such that f induces an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$? Such induced isomorphisms are called *base-isomorphisms*, and if such a base-isomorphism exists, we call $\mathfrak{A}$ and $\mathfrak{B}$ *base-isomorphic*.

$\mathfrak{A}$ is called *base-minimal* if for no proper $Z \subsetneqq U$ is the function $\mathrm{rl}(Z) = \langle X \cap Z : x \in A \rangle$ an isomorphism on $\mathfrak{A}$. (This means that the base U of $\mathfrak{A}$

Large part of Sections 1 and 2.1 are a repetition of a part of [Sh189] and we thank the NDJFL for the permission to use them. This research was partially supported by BSF.

We thank Alice Leonhardt for the excellent typing of the manuscript.
Publication 246.

is minimal in the sense that if we omit any element from it then this will change the isomorphism type of $\mathfrak{A}$.) It is well-known that:

(∗) if the BA's $\mathfrak{A}$, $\mathfrak{B}$ are both atomic and base-minimal then $\mathfrak{A} \cong \mathfrak{B}$ implies that they are also base-isomorphic.

The question was investigated in [HMTAN], [HMTII], [N83], [B87], [BSh88], [S90], [S86], [F90], whether one can generalize statement (∗) above from BA's to the algebraic counterparts of first order logic i.e. to cylindric algebras. (The present paper *is* self-contained, but the basic definitions are recalled from [HMTII] in greater detail in §2 of the "Open problems" paper of this volume.)

The algebraic counterparts of models of first order logic are the so called locally finite regular cylindric set algebras, for short Lr's see [N90], [AS78] or [HMTII] §4.3. (The notation Lr was introduced in Andréka [A72] and [AGN77]; the above quoted books write $\mathrm{Cs}_\omega^{\mathrm{reg}} \cap \mathrm{Lf}_\omega$ instead of Lr. For brevity, we stick with Lr.)

An Lr is an algebra whose elements are, basically, finitary relations over some fixed set U. More precisely, if $\mathfrak{A} \in \mathrm{Lr}$ and $R \subseteq {}^nU$ then R is represented in $\mathfrak{A}$ by $R^+ = R \times {}^\omega U$. So $A \subseteq \mathcal{P}({}^\omega U)$, and every element of A is of the form $R \times {}^\omega U$ for some finitary relation R. Now $\mathfrak{A} = \langle A, \cup, \smallsetminus, c_i, d_{ij}, \rangle_{ij\in\omega}$ where $c_i R$ is the relation defined by $\exists v_i R(v_0, \dots, v_{n-1})$, and d_{ij} is the relation defined by $v_i = v_j$. Of course, $\mathfrak{A}$ is a BA and it has to be closed under the extra operations c_i and d_{ij}.

Such an Lr $\mathfrak{A}$ is never atomic except for the trivial cases. Therefore $\mathfrak{A}$ is called *neatly-atomic* if $Nr_n\mathfrak{A} = \{x \in A : (\forall i \geq n) c_i(x) = x\}$ is atomic for all $n \in \omega$ (cf. [HMTII]). Further, $\mathfrak{A}$ is *base-minimal* if for all $Z \subsetneqq U$, the function $\mathrm{rl}({}^\omega Z) = \langle x \cap {}^\omega Z : x \in A\rangle$ is not an isomorphism on $\mathfrak{A}$; cf. [HMTAN] p. 157.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Lr's, which are subalgebras of $\mathcal{P}({}^\omega U)$ and $\mathcal{P}({}^\omega V)$ respectively. They are called *base-isomorphic* if there is a function $f : U \rightarrowtail\!\!\!\!\!\twoheadrightarrow V$ inducing an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Beginning with [HMTAN], the question was investigated whether isomorphism implies base-isomorphism for neatly-atomic base-minimal Lr's. In other words, this amounts to asking whether (∗) generalizes from BA's to Lr's. See [B88] Problem 2(a) on p. 99 for an explicit formulation. Partial results (both negative and positive) were obtained in the above quoted works, see e.g. [HMTAN] Prop. II.3.4 (2), Theorem I.3.6 (a classical result of Monk).

Here we give a negative answer to the general question. Actually we will prove slightly more, we will prove that the answer remains in the negative even if we generalize (i.e. weaken) the notion of a base-isomorphism.

Theorem 0.1. *There are neatly-atomic base-minimal locally finite regular cylindric set algebras (i.e.* Lr*'s) which are isomorphic but not base-isomorphic.*

I.e. $(*)$ does not extent from BA's to Lr's.

The rest of this paper is devoted to proving this theorem. For the proof we will need to establish some model theoretical results first. We will return to finishing the proof of Theorem 0.1 at the very end of this paper after the proof of Theorem 3.2.

Before turning to the proof, let us strengthen Theorem 0.1.

Let $\mathfrak{A}$ be an Lr with greatest element ${}^{\omega}U$. Then $\mathfrak{A}^+$ is the extension of $\mathfrak{A}$ generated by $(A \cup \{\{u\} \times {}^{\omega}U : u \in U\})$ in the cylindric set algebra with universe $\mathcal{P}({}^{\omega}U)$. We note that $\mathfrak{A}^+$ was denoted by $\mathfrak{A}_U$ in [HMTII] §4.3.68(10) p. 178.

Theorem 0.2. *There are isomorphic base-minimal neatly-atomic* Lr*'s $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}$ is not base-isomorphic with any subalgebra of $\mathfrak{B}^+$ and similarly $\mathfrak{B}$ is not base-isomorphic with any subalgebra of $\mathfrak{A}^+$. Further $\mathfrak{A}$ and $\mathfrak{B}$ can be chosen in such a way that their greatest elements coincide (they have the same ${}^{\omega}U$).*

It is easy to see that Theorem 0.1 is an immediate corollary of Theorem 0.2 (but not vice versa). Actually we will prove an even stronger result than Theorem 0.2. Loosely speaking, we will prove that in Theorem 0.2, $\mathfrak{A}$ is not base-isomorphic to any subalgebra of even a relativized version of $\mathfrak{B}$ and the same with $\mathfrak{A}$ and $\mathfrak{B}$ interchanged. For a precise formulation recall from [HMTII] that $\mathfrak{Nr}_n\mathfrak{B}$ is the cylindric algebra with universe $Nr_n\mathfrak{B}$ and extra Boolean operators c_i, d_{ij} for $i, j < n$. For $x \in Nr_n\mathfrak{B}$, recall from [HMTII] that $\mathfrak{Rl}_x\mathfrak{Nr}_n\mathfrak{B}$ is the algebra obtained from $\mathfrak{Nr}_n\mathfrak{B}$ by relativizing it (both elements and operations) to x. That is, the universe of the new algebra is $\{y \in Nr_n\mathfrak{B} : y \leq x\}$ and $c_i(y) = x \cap c_i^{(\mathfrak{B})}(y)$ in the new algebra $\mathfrak{Rl}_x\mathfrak{Nr}_n\mathfrak{B}$. Now, we will prove that:

- $(**)$ The algebras $\mathfrak{A}$ and $\mathfrak{B}$ in Theorem 0.2 are such that $\mathfrak{Nr}_n\mathfrak{A}$ is not base-isomorphic to any subalgebra of $\mathfrak{Rl}_x\mathfrak{Nr}_n\mathfrak{B}^+$ for any $x \in Nr_n\mathfrak{B}^+$ and any $3 < n < \omega$ (and the same with $\mathfrak{A}$ and $\mathfrak{B}$ interchanged).

Of the above three results, Theorem 0.1 is the basic cylindric algebraic theorem. Its improvements Theorem 0.2 and statement $(**)$ serve to show that this negative result cannot be avoided by some of the standard cylindric algebraic "generalization methods" used e.g. in [HMTII].

The model theoretic definitions and theorems in the following sections were tailored for proving the above results. In particular Theorem 3.2 way below is the model theoretic counterpart of the above theorems according the standard connections between algebraic logic and model theory elaborated in [HMTII] §4.3, [N90], [AS78] and its accompanying paper.

Section 1: Building κ-systems

Definition 1.1. 1) A *δ-system* will mean here a model of the form $\mathfrak{A} = \langle G_i, h_{i,j}\rangle_{i\le j<\delta}$ where δ is an ordinal and:

- (i) G_i is an Abelian group such that $(\forall x \in G_i)(x + x = 0)$, the G_i's are pairwise disjoint.
- (ii) $h_{i,j}$ is a homomorphism from G_j into G_i when $i \le j$.
- (iii) $h_{i_1,i_2} \circ h_{i_2,i_3} = h_{i_1,i_3}$ when $i_1 \le i_2 \le i_3$.
- (iv) $h_{i,i}$ is the identity.

2) We denote δ-systems by $\mathfrak{A}$, $\mathfrak{B}$, and for a system $\mathfrak{A}$, we write $\delta = \delta^{\mathfrak{A}}$, $G_i = G_i^{\mathfrak{A}}$, $h_{i,j} = h_{i,j}^{\mathfrak{A}}$.

3) Let $\|\mathfrak{A}\| = \sum\limits_{i<\delta} \|G_i\|$.

4) Let $\mathfrak{A}\restriction\delta_1 = \langle G_i^{\mathfrak{A}}, h_{i,j}^{\mathfrak{A}}\rangle_{i\le j<\delta_1}$.

Definition 1.2. We say $\mathfrak{A} \le \mathfrak{B}$ if $\delta^{\mathfrak{A}} = \delta^{\mathfrak{B}}$, $G_i^{\mathfrak{A}}$ is a subgroup of $G_i^{\mathfrak{B}}$, $h_{i,j}^{\mathfrak{A}} \subseteq h_{i,j}^{\mathfrak{B}}$, and:

$(*)$ for every $j < \delta^{\mathfrak{A}}$, $a \in G_j^{\mathfrak{B}}$ there is a maximal $i \le j$ such that $h_{i,j}^{\mathfrak{B}}(a) \in G_i^{\mathfrak{A}}$.

Fact 1.3. *$\le$ is a transitive reflexive relation and if $\mathfrak{A}_\alpha(\alpha < \delta)$ is increasing then*

$$\bigwedge_{\alpha<\delta}\left[\mathfrak{A}_\alpha \le \bigcup_{\beta<\delta}\mathfrak{A}_\beta\right].$$

Definition 1.4.
$\mathrm{gr}(\mathfrak{A}) = \{\mathbf{a} = \langle a_{i,j} : i < j < \delta^{\mathfrak{A}}\rangle : a_{i,j} \in G_i$ and if $\alpha < \beta < \gamma < \delta^{\mathfrak{A}}$ then $a_{\alpha,\gamma} = a_{\alpha,\beta} + h_{\alpha,\beta}(\alpha_{\beta,\gamma})\}$. This is a group by coordinatewise addition.

Definition 1.5. For $\mathbf{a} = \langle a_i : i < \delta \rangle \in \prod_{i<\delta} G_i$, let $\mathrm{fact}(\mathbf{a}) = \langle a_{i,j} : i < j < \delta \rangle$ where $a_{i,j} = a_i - h_{i,j}(a_j)$. Let $\mathrm{Fact}(\mathfrak{A}) = \left\{ \mathrm{fact}(\mathbf{a}) : \mathbf{a} \in \prod_{i<\delta} G_i^{\mathfrak{A}} \right\}$.

Claim 1.6. *The mapping* $\mathbf{a} \mapsto \mathrm{fact}(\mathbf{a})$ *is from* $\prod_{i<\delta^{\mathfrak{A}}} G_i$ *into* $\mathrm{gr}(\mathfrak{A})$, *and is a homomorphism. So* $\mathrm{Fact}(\mathfrak{A})$ *is a subgroup of* $\mathrm{gr}(\mathfrak{A})$. ∎

Definition 1.7.
(1) $E(\mathfrak{A}) \stackrel{\mathrm{def}}{=} \mathrm{gr}(\mathfrak{A})/\mathrm{Fact}(\mathfrak{A})$.
(2) $\mathfrak{A}$ is called *smooth* if for every limit $\delta < \delta^{\mathfrak{A}}$, $E(\mathfrak{A} \restriction \delta)$ has power 1.

Remark 1.7A. We will not use smoothness. Note: in 1.11 we can demand also "$\mathfrak{A}$ is smooth" provided that $\mu = \mu^{|\alpha|}$ for $\alpha < \delta^{\mathfrak{A}}$.

Fact 1.8. *Let* $\mathfrak{A}$ *be a* δ^**-system:*

(1) *for every* $\delta < \delta^*$*:* $\mathrm{Fact}(\mathfrak{A} \restriction \delta) \subseteq \mathrm{gr}(\mathfrak{A} \restriction \delta)$.
(2) *If* $\mathbf{a} \in \mathrm{gr}(\mathfrak{A})$ *then for every* $\delta < \delta^{\mathfrak{A}}$,
$\langle a_{i,j} : i < j < \delta \rangle \in \mathrm{Fact}(\mathfrak{A} \restriction \delta)$.

Proof. (1) Easy. (2) Suppose $\mathbf{a} = \langle a_{i,j} : i < j < \delta^{\mathfrak{A}} \rangle \in \mathrm{gr}(\mathfrak{A})$, $\delta < \delta^{\mathfrak{A}}$. Now for $i < \delta$ we define $b_i \stackrel{\mathrm{def}}{=} a_{i,\delta}$. Now clearly $\mathbf{b} = \langle b_i : i < \delta \rangle \in \prod_{i<\delta} G_i$ and so it suffices to prove that $\mathbf{a} \restriction \delta = \mathrm{fact}(\mathbf{b})$.

Let $\mathrm{fact}(\mathbf{b}) = \langle b_{i,j} : i < j < \delta \rangle$, now for $i < j < \delta$,

$$\begin{aligned} b_{i,j} &= b_i - h_{i,j}(b_j) && \text{[by definition of fact}(\mathbf{b})] \\ &= a_{i,\delta} - h_{i,j}(a_{j,\delta}) && \text{[by definition of } b_i,\, b_j] \\ &= a_{i,j} && \text{[as } \mathbf{a} \in \mathrm{gr}(\mathfrak{A})]. \end{aligned}$$ ∎

Lemma 1.9. *Suppose* δ *is an ordinal* $\leq \mu$, *with cofinality* $> \aleph_0$, *and* T *is a set of sequences of ordinals* $< \mu$ *of length* $< \delta$, T *closed under initial segments. Let for* $i < \delta$, $T_i = \{\eta \in T : \eta$ *has length* $i\}$ *and let*

$$T_\delta = \left\{ \eta : \eta \text{ a sequence of ordinals of length } \delta \text{ such that } \bigwedge_{\alpha<\delta} \eta \restriction \alpha \in T \right\}$$

and assume $|\bigcup_{i<\delta} T_i| = \mu$. *Then there is a* δ*-system* $\mathfrak{A} = \mathfrak{A}(T)$ *such that:*

$$\|\mathfrak{A}\| = \sum_{i<\delta} \|G_i^{\mathfrak{A}}\| = \mu$$

$$|E(\mathfrak{A})| \geq |T_\delta|.$$

Remark. We shall use later how $\mathfrak{A}(T)$, $\mathbf{a}_\xi(\xi \in T_\delta)$ (see below) are defined.

Proof. Let G_i be the free Abelian group of order two generated by $W_\alpha = \{a^{\xi,\alpha}_{i,j} : \xi \in T_i,\ i < j < \delta \text{ and } i \le \alpha < \delta\}$. Let $a^\xi_{i,j} \overset{\text{def}}{=} a^{\xi,i}_{i,j}$. So we can identify G_i with the family of finite subsets of W_i, with addition being the symmetric difference except that for $i \ne j$ we consider the zero of G_i, $\emptyset_i =$ the empty subset of W_i, as $\ne \emptyset_j$. Now for $\alpha < \beta < \delta$, $h_{\alpha,\beta} : G_\beta \to G_\alpha$ is defined by:

(1) for $\xi \in T_\beta$, $i \le \beta < \delta$, $i < j < \delta$, $h_{\alpha,\beta}(a^{\xi,\beta}_{i,j})$ is $a^{\xi\restriction\alpha}_{\alpha,j} - a^{\xi\restriction\alpha}_{\alpha,i}$ if $\alpha < i$, and $a^{\xi,\alpha}_{i,j}$ if $\alpha \ge i$.

Check: For $\alpha < \beta < \gamma$, $h_{\alpha,\gamma} = h_{\alpha,\beta} \circ h_{\beta,\gamma}$, it is enough to check this for the generators of G_γ which are $a^{\xi,\gamma}_{i,j}$, $i \le \gamma < \delta$, $i < j < \delta$, $\xi \in T_\gamma$. Now
<u>if</u> $i = \gamma$ (so $a^{\xi,\gamma}_{i,j} = a^\xi_{\gamma,j}$):

$$h_{\alpha,\beta}\left(h_{\beta,\gamma}(a^\xi_{\gamma,j})\right) = h_{\alpha,\beta}\left(a^{\xi\restriction\beta}_{\beta,j} - a^{\xi\restriction\beta}_{\beta,\gamma}\right) = \left(a^{\xi\restriction\alpha}_{\alpha,j} - a^{\xi\restriction\alpha}_{\alpha,\beta}\right) - \left(a^{\xi\restriction\alpha}_{\alpha,\gamma} - a^{\xi\restriction\alpha}_{\alpha,\beta}\right)$$
$$= a^{\xi\restriction\alpha}_{\alpha,j} - a^{\xi\restriction\alpha}_{\alpha,\gamma} = h_{\alpha,\gamma}(a^\xi_{\gamma,j}).$$

<u>if</u> $\beta < i < \gamma$:

$$h_{\alpha,\beta}\big[h_{\beta,\gamma}(a^{\xi,\gamma}_{i,j})\big] = h_{\alpha,\beta}\big[h_{\beta,i}(a^{\xi,i}_{i,j})\big], \qquad h_{\alpha,\gamma}(a^{\xi,\gamma}_{i,j}) = h_{\alpha,i}(a^{\xi,i}_{i,j})$$

so this is reduced to the first case for $\alpha < \beta < i = \gamma'$.
<u>if</u> $\alpha < i \le \beta$:

$$h_{\alpha,\beta}\big[h_{\beta,\gamma}(a^{\xi,\gamma}_{i,j})\big] = h_{\alpha,\beta}(a^{\xi,\beta}_{i,j}) = a^{\xi\restriction\alpha}_{\alpha,j} - a^{\xi\restriction\alpha}_{\alpha,i} = h_{\alpha,\gamma}(a^{\xi,\gamma}_{i,j})$$

<u>if</u> $i \le \alpha$:

$$h_{\alpha,\beta}\big[h_{\beta,\gamma}(a^{\xi,\gamma}_{i,j})\big] = h_{\alpha,\beta}(a^{\xi,\beta}_{i,j}) = a^{\xi,\alpha}_{i,j} = h_{\alpha,\gamma}(a^{\xi,\gamma}_{i,j}).$$

In this context we define

Definition 1.9A. **(1)** $\mathfrak{A}(T) = \langle G_\alpha, h_{\alpha,\beta} : \alpha < \beta < \delta\rangle$.
(2) For $\xi \in T_\delta$, let $\mathbf{a}^\xi = \langle a^{\xi\restriction i}_{i,j} : i < j < \delta\rangle$.

Clearly $\mathbf{a}^\xi \in \mathrm{gr}(\mathfrak{A})$ (by the definition of the $h_{\alpha,\beta}$'s). We want to show $\mathbf{a}^\xi - \mathbf{a}^\zeta \notin \mathrm{Fact}(\mathfrak{A})$ for $\xi \ne \zeta$.

If not there are $w_i \in G_i$ such that $\mathbf{a}^\xi - \mathbf{a}^\zeta = \mathrm{fact}\langle w_i : i < \delta\rangle$ so:

$$a^{\xi\restriction i}_{i,j} - a^{\zeta\restriction i}_{i,j} = w_i - h_{i,j}(w_j) \text{ (for } i < j < \delta).$$

Clearly w_i is nothing but a finite subset of W_i. Let
(3) $a^*_{i,j} \stackrel{\text{def}}{=} a^{\xi\restriction i}_{i,j} - a^{\zeta\restriction i}_{i,j} = w_i - h_{i,j}(w_i)$.
Clearly $\mathbf{a}^* \stackrel{\text{def}}{=} \langle a^*_{i,j} : i < j < \delta\rangle$ belongs to $\mathrm{gr}(\mathfrak{A})$. For every limit ordinal $\alpha < \delta$, let $\epsilon(\alpha) < \alpha$ be such that:
$(*)$ if $a^{\eta,\alpha}_{i,j}$ appear in w_α then:

$$i < \alpha \Rightarrow i < \epsilon(\alpha)$$
$$j < \alpha \Rightarrow j < \epsilon(\alpha)$$
$$i \geq \alpha \ \&\ \eta\restriction\alpha \neq \xi\restriction\alpha \Rightarrow \eta\restriction\epsilon(\alpha) \neq \xi\restriction\epsilon(\alpha)$$

(remember $i = \ell g(\eta)$ by definition of w_α).
Let $w^0_\alpha = \sum\{a^{\xi,\alpha}_{i,j} : a^{\xi,\alpha}_{i,j}$ appear in w_α and $i < \alpha\}$, $w^1_\alpha = w_\alpha - w^0_\alpha$; so if $a^{\xi,\alpha}_{i,j}$ appear in w^1_α then $i \geq \alpha$, hence (by definition of W_α) $\alpha = i$; so it is of the form $a^{\xi}_{i,j}$.

Now as $\mathrm{cf}(\delta) > \aleph_0$, there is a stationary subset S of δ and $\epsilon(*) < \delta$ and n_0, $n_1 < \omega$ such that:

$$\alpha \in S \Rightarrow \alpha \text{ limit } \& \ \epsilon(\alpha) \leq \epsilon(*) \ \& \ |w^0_\alpha| = n_0 \ \& \ |w^1_\alpha| = n_1$$

and as $\xi \neq \zeta$ without loss of generality $[i \in S \Rightarrow \xi\restriction i \neq \zeta\restriction i]$.

Let $\alpha < \beta < \gamma$ be in S, by (3) $a^*_{\beta,\gamma} = w_\beta - h_{\beta,\gamma}(w_\gamma)$; apply $h_{\alpha,\beta}$ and get $a^*_{\alpha,\gamma} - a^*_{\alpha,\beta} = h_{\alpha,\beta}(w_\beta) - h_{\alpha,\gamma}(w_\gamma)$. So

$$a^*_{\alpha,\gamma} + h_{\alpha,\gamma}(w_\gamma) = a^*_{\alpha,\beta} + h_{\alpha,\beta}(w_\beta).$$

So for some $c_\alpha \in G_\alpha$ for every β, if $\alpha < \beta < \delta$ then
(4) $a^*_{\alpha,\beta} + h_{\alpha,\beta}(w_\beta) = c_\alpha$; i.e.
(5) $a^*_{\alpha,\beta} + h_{\alpha,\beta}(w^1_\beta) = c_\alpha - h_{\alpha,\beta}(w^0_\beta)$.
Let $U_\alpha = \{j : a^{\eta,\alpha}_{i,j}$ appear in c_α for some $\eta \in T_i$, $i < j < \delta$, and $i \leq \alpha\}$, remember c_α is a finite subset of W_α, so U_α is a finite subset of δ.

Without loss of generality, $\alpha \in S \wedge \beta \in S \wedge \alpha < \beta \Rightarrow \beta > \operatorname{Max} U_\alpha$. We look at the appearances of $u \stackrel{\text{def}}{=} \{a^{\xi\restriction\alpha}_{\alpha,\gamma} : \gamma \in [\beta,\delta)\}$ it appears in $a^*_{\alpha,\beta}$ (see (3), noting $\zeta\restriction\alpha \neq \xi\restriction\alpha$ because $\alpha \in S$). No appearance in c_α (as $\beta > \operatorname{Max} U_\alpha$), and no appearance in $h_{\alpha,\beta}(w^0_\beta)$ (as for any $a^{\eta,\beta}_{i,j}$ appearing in w^0_β, $i < \epsilon(*) < \alpha$ hence $h_{\alpha,\beta}(a^{\eta,\beta}_{i,j}) = a^{\eta,\alpha}_{i,j}$ is not of the right form). By equality (5), there has to be an odd number of appearances of members of u in $h_{\alpha,\beta}(w^1_\beta)$. But every member $a^{\eta,\beta}_{i,j}$ of w^1_β is necessarily of the form $a^{\eta}_{i,j}$, $\ell g(\eta) = i = \beta$. But by the definition of $h_{\alpha,\beta}(a^{\eta}_{i,j})$, $h_{\alpha,\beta}(a^{\eta}_{i,j}) = a^{\eta\restriction\alpha}_{\alpha,j} - a^{\eta\restriction\alpha}_{\alpha,i}$ contribute zero or two; i.e. an even number. ∎

We shall not use, but note

Fact 1.10. *Assume* $\mathrm{cf}(\delta^{\mathfrak{A}}) > \aleph_0$. *If* $\mathfrak{A}_\alpha$ $(\alpha < \delta)$ *is* $\leq$*-increasing continuous,* $\mathbf{a} \in \mathrm{gr}(\mathfrak{A}_0) \subseteq \mathrm{gr}(\mathfrak{A}_\alpha)$, $\mathbf{a} \notin \mathrm{Fact}(\mathfrak{A}_\alpha)$ *(for* $\alpha < \delta$*) then* $\mathbf{a} \notin \mathrm{Fact}\left(\bigcup_{\alpha<\delta} \mathfrak{A}_\alpha\right)$. ∎

Fact 1.11. *Let* T *be as in 1.9. There is a smooth* $\mathfrak{A}$, $|\mathfrak{A}| = \mu$ *with* $|E(\mathfrak{A})| \geq |T_\delta|$ *such that every* $h^{\mathfrak{A}}_{i,j}$ *is onto* $G^{\mathfrak{A}}_i$. ∎

Section 2

Hypothesis. $\mathfrak{A}$ is a δ-system where the $h^{\mathfrak{A}}_{i,j}$'s are onto.

Definition 2.1. For every $\mathbf{a} \in \mathrm{gr}(\mathfrak{A})$ we define a model $M_{\mathbf{a}} = M^1_{\mathbf{a}} = [M^1_{\mathbf{a}}]^{\mathfrak{A}} = M^1[\mathbf{a}]$:

(i) $|M_{\mathbf{a}}| = \bigcup_{i<\delta} G^{\mathfrak{A}}_i$

(ii) $P^{M_{\mathbf{a}}}_i = G^{\mathfrak{A}}_i$ for $i < \delta$

(iii) for every $i < \delta$, $c \in G_i$ we have a (partial) function $F_c : P^{M_{\mathbf{a}}}_i \to P^{M_{\mathbf{a}}}_i$: $F_c(x) = c + x$

(iv) for every $i < j$ we have a (partial) function $H_{i,j} : P^{M_{\mathbf{a}}}_j \to P^{M_{\mathbf{a}}}_i$:

$$H_{i,j}(x) = h_{i,j}(x) + a_{i,j}.$$

(v) for uniformity, we have a monadic $Q^{M_{\mathbf{a}}} = \emptyset$.

Definition 2.2. Now for every $\mathbf{b} \in \mathrm{gr}(\mathfrak{A})$ we define a model $M^2_{\mathbf{b}} = [M^2_{\mathbf{b}}]^{\mathfrak{A}}$.

<u>Its Universe:</u> $T \cup \delta \cup \bigcup_{i<\delta} G^{\mathfrak{A}}_i \cup \left\{\langle \mathbf{b}, x\rangle : x \in \bigcup_{i<\delta} G^{\mathfrak{A}}_i\right\}$.

<u>Relations:</u>

(i) P a two place relation $P^{M^2_{\mathbf{b}}} = \{\langle i, \langle \mathbf{b}, x\rangle\rangle : i < \delta, \text{ and } x \in G^{\mathfrak{A}}_i\}$.

(ii) F a partial two place function (make it a three place relation, if you want): $F(z, \langle \mathbf{b}, y\rangle) = \langle \mathbf{b}, y + z\rangle$ <u>if</u> $z \in G^{\mathfrak{A}}_i$, $y \in G^{\mathfrak{A}}_i$.

(iii) H a partial three place function $H(i, j, \langle \mathbf{b}, x\rangle) = \langle \mathbf{b}, h_{i,j}(x) + b_{i,j}\rangle$.

(iv) $<$ a well ordering of $\delta \cup \bigcup_{i<\delta} G^{\mathfrak{A}}_i \cup T$ (not depending on $\mathbf{b}$).

(v) Q, Q_0, Q_1, Q_2 one place relations: $Q = \delta \cup \bigcup_{i<\delta} G^{\mathfrak{A}}_i \cup T$, $Q_0 = \delta$, $Q_1 = \bigcup_{i<\delta} G^{\mathfrak{A}}_i$, $Q_2 = T$.
For later use we demand:
$G^{\mathfrak{A}}_i$ is the family of finite subsets of T_i.

(vi) individual constants for all members of G.

Definition 2.3. We let $M^3_{\mathbf{a}}$ be like $M^2_{\mathbf{a}}$ without the individual constants.

Fact 2.4. *$M^2_{\mathbf{a}}$ is isomorphic to $M^2_{\mathbf{b}}$ iff $M^1_{\mathbf{a}}$ is isomorphic to $M^1_{\mathbf{b}}$ ($\mathbf{a}, \mathbf{b} \in \mathrm{gr}(\mathfrak{A})$).*

Proof. Straightforward. ∎

Fact 2.5. *$M^3_{\mathbf{a}}$ is isomorphic to $M^3_{\mathbf{b}}$ iff $M^2_{\mathbf{a}}$ is isomorphic to $M^2_{\mathbf{b}}$.*

Proof. Easy. As $<^{M^3_{\mathbf{a}}}$, $<^{M^3_{\mathbf{b}}}$ are the same well ordering on $Q^{M^3_{\mathbf{a}}} = Q^{M^3_{\mathbf{b}}}$, an isomorphism from $M^3_{\mathbf{a}}$ onto $M^3_{\mathbf{b}}$ is the identity on Q, hence is an isomorphism from $M^2_{\mathbf{a}}$ onto $M^2_{\mathbf{b}}$. ∎

Fact 2.6. *$M^1_{\mathbf{a}} \cong M^1_{\mathbf{b}}$ iff $\mathbf{a} - \mathbf{b} \in \mathrm{Fact}(\mathfrak{A})$ (the subtraction is in $\mathrm{gr}(\mathfrak{A})$).*

Proof. Suppose $\mathbf{b} - \mathbf{a} = \mathrm{fact}(\mathbf{d})$ where $\mathbf{d} = \langle d_i : i < \delta \rangle$. We define an isomorphism $g = g_{\mathbf{d}}$ from $M_{\mathbf{a}}$ onto $M_{\mathbf{b}}$: for $x \in G_i^{\mathfrak{A}}$ let $g(x) \stackrel{\mathrm{def}}{=} x + d_i$.

Clearly g maps each $P_i^{M_{\mathbf{a}}}$ onto $P_i^{M_{\mathbf{b}}}$ hence it maps $|M^1_{\mathbf{a}}|$ onto $|M^1_{\mathbf{b}}|$. Also g is one–to–one.

Now for each $i < \delta$, $c \in G_i^{\mathfrak{A}}$, $x \in P_i^{M^1_{\mathbf{a}}} = G_i^{\mathfrak{A}}$

$$g\left(F_i^{M^1_{\mathbf{a}}}(x)\right) = g(c + x) = c + x + d_i = c + g(x) = F_c^{M^1_{\mathbf{b}}}\big(g(x)\big).$$

Lastly for $i < j$, $x \in P_i^{M^1_{\mathbf{a}}} = G_j^{\mathfrak{A}}$

$$\begin{aligned}
&g(H_{i,j}^{M^1_{\mathbf{a}}}(x)) = g(h_{i,j}(x) + a_{i,j}) = h_{i,j}(x) + a_{i,j} + d_i = \\
&h_{i,j}(x) + h_{i,j}(d_j) + b_{i,j} = h_{i,j}(x + d_j) + b_{i,j} = \\
&H_{i,j}^{M^1_{\mathbf{b}}}(x + d_j) = H_{i,j}^{M^1_{\mathbf{b}}}(g(x))
\end{aligned}$$

(the third equality is as $\mathbf{b} - \mathbf{a} = \mathrm{fact}(\mathbf{d})$ and $\mathrm{fact}(\mathbf{d})$'s definition).

For the other direction suppose g is an isomorphism from $M^1_{\mathbf{a}}$ onto $M^1_{\mathbf{b}}$. We let $d_i = g(x) - x$ for any (some) $x \in P_i^{M^1_{\mathbf{a}}}$ and $\mathbf{d} = \langle d_i : i < \delta \rangle$ and can check that $\mathbf{b} - \mathbf{a}{=}\mathrm{fact}(\mathbf{d})$. ∎

Fact 2.7. *If $\mathbf{a}, \mathbf{b} \in \mathrm{gr}(\mathfrak{A})$, $\ell \in \{1, 2, 3\}$, $i(*) < \delta$, $x \in P_{i(*)}^{M^\ell_{\mathbf{a}}}$, $y \in P_{i(*)}^{M^\ell_{\mathbf{b}}}$, then in the following game player II has a winning strategy:*

in stage n ($< \omega$): player I chooses successor ordinal i_n, such that $\max\{i(*), i_0, \dots, i_{n-1}\} < i_n < \delta^{\mathfrak{A}}$;
then player II chooses g_n such that:

(1) *g_n is a partial isomorphism from $M^\ell_{\mathbf{a}}$ to $M^\ell_{\mathbf{b}}$: it maps $Q^{M^\ell_{\mathbf{a}}} \cup \bigcup_{j<i_n} P_j^{M^\ell_{\mathbf{a}}}$ onto $Q^{M^\ell_{\mathbf{b}}} \cup \bigcup_{j<i_n} P_j^{M^\ell_{\mathbf{b}}}$ which is the identity on $Q^{M^\ell_{\mathbf{a}}}$ (and if $\ell = 1$, $Q \neq \emptyset$, of course)*

(2) *g_n extends $g_0, \dots, g_{n-1}$, and*

(3) *$g_n(x) = y$.*

Proof. We let (using the notation from the proof of Fact 2.6 and concentrating on the case $\ell = 2$).

$$F_\alpha = \left\{ \mathrm{id}_Q \cup g_{\mathbf{d}} : \mathbf{d} \in \prod_{i \in \alpha} G_i^{\mathfrak{A}}, \mathbf{a} \restriction \alpha - \mathbf{b} \restriction \alpha = \mathrm{fact}(\mathbf{d}) \right\}.$$

By 2.2 and 1.8(2) $F_\alpha \neq \emptyset$ and by the proof of 2.6, F_α is a set of isomorphisms from $M^\ell_{\mathbf{a}} \restriction Q^{M^\ell_{\mathbf{a}}} \cup \bigcup_{i<\alpha} G_i^{\mathfrak{A}}$ onto $M^\ell_{\mathbf{b}} \restriction Q^{M^\ell_{\mathbf{a}}} \cup \bigcup_{i<\alpha} G_i^{\mathfrak{A}}$. The strategy of player II is to use partial isomorphisms from F_{i_n} (if player I has chosen i_n).

<u>The first missing point</u> is: for successor $\alpha < \beta < \delta$, $g \in F_\alpha$, there is $g' \in F_\beta$, $g \subseteq g'$; equivalently, for $\mathbf{d}_0 \in \prod_{i<\alpha} G_i^{\mathfrak{A}}$ satisfying $\mathbf{a} \restriction \alpha - \mathbf{b} \restriction \alpha = \mathrm{fact}(\mathbf{d}_0)$, there is $\mathbf{d} \in \prod_{i<\beta} G_i^{\mathfrak{A}}$, $\mathbf{a} \restriction \beta - \mathbf{b} \restriction \beta = \mathrm{fact}(\mathbf{d})$, and $\mathbf{d}_0 = \mathbf{d} \restriction \alpha$. By 1.8 there are $\mathbf{d}_1, \mathbf{d}_2$ from $\prod_{i<\beta} G_i^{\mathfrak{A}}$ such that $\mathbf{a} \restriction \beta = \mathrm{fact}(\mathbf{d}_1)$, $\mathbf{b} \restriction \beta = \mathrm{fact}(\mathbf{d}_2)$. Let $\mathbf{d}_0 = \langle d_i^0 : i < \alpha \rangle$, $\mathbf{d}_1 = \langle d_i^1 : i < \beta \rangle$, $\mathbf{d}_2 = \langle d_i^2 : i < \beta \rangle$.

As $\mathbf{a} \restriction \alpha = \mathrm{fact}(\mathbf{d}_1 \restriction \alpha)$, $\mathbf{b} \restriction \alpha = \mathrm{fact}(\mathbf{d}_2 \restriction \alpha)$, and $\mathbf{a} \restriction \alpha - \mathbf{b} \restriction \alpha = \mathrm{fact}(\mathbf{d}_0)$, clearly for every $i < j < \alpha$

$$\left(d_i^1 - h_{i,j}(d_j^1)\right) - \left(d_i^2 - h_{i,j}(d_j^2)\right) = d_i^0 - h_{i,j}(d_j^0);$$

hence,

(a) $d_i^1 - d_i^2 - d_i^0 = h_{i,j}\left(d_j^1 - d_j^2 - d_j^0\right)$.

As $h_{\beta-1,\alpha-1}$ is from $G^{\mathfrak{A}}_{\beta-1}$ onto $G^{\mathfrak{A}}_{\alpha-1}$ (remember α, β are successor ordinals) for some $x \in G^{\mathfrak{A}}_{\beta-1}$:

(b) $h_{\alpha-1,\beta-1}(x) = d^1_{\alpha-1} - d^2_{\alpha-1} - d^0_{\alpha-1}$.

By (a) for every $i < \alpha$:

(c) $h_{i,\beta-1}(x) = d_i^1 - d_i^2 - d_i^0$.
Now define for i, $i < \beta$:

(d) $d_i = d_i^1 - d_i^2 - h_{i,\beta-1}(x)$.
By (c) for $i < \alpha$:

(e) $d_i = d_i^0$.

Let $\mathbf{d} = \langle d_i : i < \beta \rangle$, so $\mathbf{d} \restriction \alpha = \mathbf{d}_0$. We shall show that $\mathbf{a} \restriction \beta - \mathbf{b} \restriction \beta = \mathrm{fact}(\mathbf{d})$ thus finishing the proof of 2.7. For $i < j < \beta$

$$\begin{aligned} a_{i,j} - b_{i,j} &= \left(d_i^1 - h_{i,j}(d_j^1)\right) - \left(d_i^2 - h_{i,j}(d_j^2)\right) \\ &= \left(d_i^1 - d_i^2\right) - h_{i,j}\left(d_j^1 - d_j^2\right) \\ &= \left(d_i + h_{i,\beta-1}(x)\right) - h_{i,j}\left(d_j + h_{j,\beta-1}(x)\right) \\ &= d_i - h_{i,j}(d_j) + \left(h_{i,\beta-1}(x) - h_{i,j} \circ h_{j,\beta-1}(x)\right) \\ &= d_i - h_{i,j}(d_j). \end{aligned}$$

So $\mathbf{d}$ is as required and we finish the proof of the first point.

Second point: If $\alpha > i(*)$, $\alpha < \delta$, then there is $g_{\mathbf{d}} \in F_\alpha$ such that $g_{\mathbf{d}}(x) = y$. We know that for some $\mathbf{d}_0 = \prod_{i<\alpha} G_i^{\mathfrak{A}}$, $g_{\mathbf{d}_0}$ is an isomorphism from $\left[M^\ell_{\mathbf{a}\restriction\alpha}\right]^{\mathfrak{A}\restriction\alpha}$ onto $\left[M^\ell_{\mathbf{b}\restriction\alpha}\right]^{\mathfrak{A}\restriction\alpha}$.

Let $e_{i(*)} \in G_{i(*)}$ be such that $G^{\mathfrak{A}}_{i(*)} \models g_{\mathbf{d}_0}(x) + e_{i(*)} = y$. As $h^{\mathfrak{A}}_{i(*),\alpha}$ is onto $G^{\mathfrak{A}}_{i(*)}$ (a hypothesis of this section) there is $e_\alpha \in G^{\mathfrak{A}}_\alpha$ such that $h_{i(*),\alpha}(e_\alpha) = e_{i(*)}$.

Let $\mathbf{d} = \langle d_i + h_{i,\alpha}(e_\alpha) : i < \alpha \rangle$, $g_{\mathbf{d}}$ is as required. ∎

Conclusion 2.8. *For $k \in \{1,2\}$, $\mathbf{a} \in \mathrm{gr}(\mathfrak{A})$, $M^k_{\mathbf{a}}$ is atomic.*

Proof. By 2.7 if $x, y \in P_i^{M^k_{\mathbf{a}}}$ then $(M_{\mathbf{a}}, x)$, $(M_{\mathbf{a}}, y)$ (i.e. the models expanded by an individual constant) are elementarily equivalent (even in $\mathcal{L}_{\infty,\mathrm{cf}(\delta)}$). This is by Ehrenfeucht Games — see Chang and Keisler [CK].

So if $\Gamma = \mathrm{Th}(M_{\mathbf{a}})$ (the theory of $M_{\mathbf{a}}$), $\{P_i(x)\}$ is a complete type.

Why does it follow that $M^k_{\mathbf{a}}$ is atomic? Suppose $n < \omega$, $b_1, \dots, b_n \in M^k_{\mathbf{a}}$, $\mathbf{b} = \langle b_1, \dots, b_n \rangle$. Without loss of generality, no b_ℓ is an individual constant, so $b_\ell \notin Q$. Let $b_\ell \in P_{i_\ell}^{M_{\mathbf{a}}}$ and assume for simplicity that $i_1 \geq i_2 \geq \cdots \geq i_n$. Now as h_{i_ℓ,i_1} is onto $G^{\mathfrak{A}}_{i_\ell}$, there are $b'_\ell \in G^{\mathfrak{A}}_{i_1}$ such that $h_{i_\ell,i_1}(b'_\ell) = b_\ell$. So $H^{M^k_{\mathbf{a}}}_{i_\ell,i_1}(b'_\ell) = b_\ell + a_{i_\ell,i_1}$. So $F^{M_{\mathbf{a}}}_{-a_{i_\ell,i_1}} \circ H^{M_{\mathbf{a}}}_{i_\ell,i_1}(b'_\ell) = b_\ell$ (well, in $M^2_{\mathbf{a}}$, $F_{-a_{i_\ell,i_1}}$ is not a function symbol but it is equal to a term); also $b'_\ell = F^{M_{\mathbf{a}}}_{b'_\ell - b_1}(b_1)$. So $M^k_{\mathbf{a}} \models$ "$F_{-a_{i\ell,i_1}} H_{i_\ell,i_1} F_{b'_\ell - b_1}(a_1) = a_\ell$".

Clearly if $\{P_{i_1}(x_1)\}$ is complete type (for $\mathrm{Th}(M_{\mathbf{a}})$) then so is $\{P_{i_1}(x_1), \ldots, x_\ell = F_{-a_{i_\ell,i_1}} H_{i_\ell,i_1} F_{b'_\ell - b_1}(x_1) \ldots\}_{\ell=2,n}$ but $\langle b_1, \ldots, b_n\rangle$ satisfies those formulas, so its type in $M^k_{\mathbf{a}}$ is isolated by finitely many formulas so it is isolated. So $M^k_{\mathbf{a}}$ is atomic. ∎

Really we have proved

Fact 2.9. *If $x \in P_i^{M^k_{\mathbf{a}}}$, $y \in P_j^{M^k_{\mathbf{a}}}$, $i \le j$ and $k \in \{1,2\}$ then for some term τ in $\mathcal{L}(M_{\mathbf{a}})$, $M^k_{\mathbf{a}} \models x = \tau(y)$.* ∎

Conclusion 2.10. *If $\mathbf{a}, \mathbf{b} \in \mathrm{gr}(\mathfrak{A})$, $k \in \{1,2\}$ and g is an elementary embedding of $M^k_{\mathbf{a}}$ into $M^k_{\mathbf{b}}$ then g is an isomorphism from $M^k_{\mathbf{a}}$ onto $M^k_{\mathbf{b}}$ (hence each $M^k_{\mathbf{a}}$ is minimal).*

Proof. If not let $x \in M^k_{\mathbf{b}} - \mathrm{Rang}(g)$; if $k = 2$, g is necessarily the identity on Q so without loss of generality, $x \notin Q$; now $M^k_{\mathbf{b}} - Q = \bigcup_{i<\delta} P_i^{M^k_{\mathbf{b}}}$ so for some $i < \delta$, $x \in P_i^{M^k_{\mathbf{b}}}$; now choose $y \in P_i^{M^k_{\mathbf{a}}}$, so $g(y) \in P_i^{M^k_{\mathbf{b}}}$; by 2.9 applied to $M^k_{\mathbf{b}}$ there is a term τ such that $M^k_{\mathbf{b}} \models \tau(g(y)) = x$. But $\mathrm{Rang}(g)$ is closed under the functions of $M^k_{\mathbf{b}}$ hence under τ, so $x \in \mathrm{Rang}(g)$; contradiction. ∎

Section 3

Theorem 3.1. *Let $\mu > \aleph_0$. There is a first order complete theory Γ of power μ, and atomic minimal models M_1, M_2 of Γ such that:*

M_1 is not elementarily embeddable into M_2,

M_2 is not elementarily embeddable into M_1.

Proof. Let T be a tree as in 1.9 with $|T_\delta| > 1$, $\mathrm{cf}(\delta) > \aleph_0$, e.g.

$$T = \{\eta : \eta \text{ a sequence of length} < \mu \times \omega_1, \text{ which is constantly 0 or constantly 1}\}.$$

Let $\mathfrak{A}$ be a δ-system as guaranteed by 1.9.

So there are $\langle \mathbf{a}_\zeta : \zeta < \zeta(0)\rangle$ members of $\mathrm{gr}(\mathfrak{A})$, $\zeta \ne \xi \Rightarrow \mathbf{a}_\zeta - \mathbf{a}_\xi \notin \mathrm{Fact}(\mathfrak{A})$ and $\zeta(0) > 1$.

Now $M^1_{\mathbf{a}_\zeta}$ has μ non-logical symbols so $\Gamma = \mathrm{Th}(M^1_{\mathbf{a}_\zeta})$ is a complete theory of power μ. By 2.6 $M^1_{\mathbf{a}_\zeta} (\zeta < \zeta(0))$ are pairwise non-isomorphic. By 2.8 $M^1_{\mathbf{a}_\zeta}$ is an atomic model. By 2.10 $M^1_{\mathbf{a}_\zeta}$ cannot be elementarily embeddable into $M^1_{\mathbf{a}_\xi}$ when $\zeta \ne \xi$ (as they are not isomorphic). By 2.10 each $M^1_{\mathbf{a}_\zeta}$ is minimal. ∎

Theorem 3.2. *Let μ be a regular uncountable cardinal or just* $\mathrm{cf}(\mu) > \aleph_0$. *Then there is a first order complete theory Γ of power μ and models M_1, M_2 such that:*

(a) *M_1, M_2 are atomic models of Γ of power μ*
(b) *for $\ell = 1, 2$ M_ℓ cannot be interpreted in $M_{3-\ell}$*
(c) *M_1, M_2 are minimal.*

In the proof we will need the following:

Definition 3.2A. 1) A model M is *directly interpretable* in a model N if

(i) the universe of N has the form $\{\bar{a}/E : \bar{a} \in {}^n|N|, N \models \varphi[\bar{a}]\}$ where: $n < \omega$, φ a (first order) formula from $\mathcal{L}(N)$ and $E = E(\bar{x}, \bar{y})$ ($\ell g(\bar{x}) = \ell g(\bar{y})$) is also a formula, first order in $\mathcal{L}(N)$ which define an equivalence relation on $\{\bar{a} : \bar{a} \in {}^n|N|,\ N \models \varphi[a]\}$
(ii) for every relation R of M, m-place, $R = \{\langle \bar{a}^1/E, \dots, \bar{a}^m/E\rangle : N \models \varphi_R[\bar{a}^1, \dots, \bar{a}^m]\}$ for some first order formula φ from $\mathcal{L}(N)$
(iii) we treat (partial) functions of M as relations.

2) We add "*with parameters*" if we replace N by $(N, c)_{c \in N}$; i.e. allow all members of N to appear in the formulas.
3) M is *interpretable* (with parameters) in N if M is isomorphic to some M' which is directly interpretable in M.

Proof of 3.2. To be able to give a simple proof first, we first make a set theoretical hypothesis and prove 3.2 under it. We will later show how to eliminate it.

Hypothesis. There are T, μ, δ as in 1.9, $\mathrm{cf}(\delta) > \aleph_0$, $|T_\delta| > \mu^+$ (μ may be singular).

Remark. E.g. if $2^{\kappa^+} > (2^\kappa)^+$ then $T = \bigcup\limits_{\alpha<\kappa^+} {}^{\alpha>}2$, $\mu = 2^\kappa$ are like this.

Let $\mathfrak{A}$ be as constructed in 1.9, so let $\langle \mathbf{a}_\zeta : \zeta < |T_\delta|\rangle$ be from $\mathrm{gr}(\mathfrak{A})$, $[\zeta \neq \xi \implies \mathbf{a}_\zeta - \mathbf{a}_\xi \notin \mathrm{Fact}(\mathfrak{A})]$. So $\|M^2_{\mathbf{a}_\zeta}\| = \mu$. As in the proof of 3.1 $M^2_{\mathbf{a}_\zeta}$ are pairwise elementarily equivalent, atomic, minimal, pairwise non-isomorphic. But we need the non-interpretability.

Fatc 3.2B. *Let $\langle \mathbf{a}_\zeta : \zeta < |E(\mathfrak{A})|\rangle$ be a maximal list of members of* $\mathrm{gr}(\mathfrak{A})$ *with $\zeta \neq \xi \Rightarrow \mathbf{a}_\zeta - \mathbf{a}_\xi \notin$* $\mathrm{Fact}(\mathfrak{A})$. *For each ζ the set $S_\zeta = \{\xi : M^2_{\mathbf{a}_\xi}$ can be interpreted in $M^2_{\mathbf{a}_\zeta}\}$ has power $\leq \mu$.*

Proof. For each $\xi \in S_\zeta$, as we can interpret $M^2_{\mathbf{a}_\xi}$ in $M^2_{\mathbf{a}_\zeta}$ we certainly can interpret $M^3_{\mathbf{a}_\xi}$ in $M^2_{\mathbf{a}_\zeta}$. I.e. for each $\xi \in S_\zeta$, we can find a model $N_{\mathbf{a}_\xi}$ such that:

(i) $N_{\mathbf{a}_\xi}$ is isomorphic to $M^3_{\mathbf{a}_\xi}$

(ii) the universe, relations and functions of $N_{\mathbf{a}_\xi}$ are definable in $M^2_{\mathbf{a}_\zeta}$ (we may allow parameters).

If $|S_\zeta| > \mu$ for some $\xi_1 \neq \xi_2$ from S_ζ, $N_{\mathbf{a}_{\xi_1}} = N_{\mathbf{a}_{\xi_2}}$ [as: (α) — the language of the $M^3_{\mathbf{a}_\xi}$ is fixed and finite and (β) — $M^2_{\mathbf{a}_\zeta}$ has language of power $\leq \mu$ hence has $\leq \mu$ formulas (adding parameters does not change) now we use the pigeon hole principle]. Clearly $N_{\mathbf{a}_{\xi_1}} = N_{\mathbf{a}_{\xi_2}}$ implies $M^3_{\mathbf{a}_{\xi_1}} \cong M^3_{\mathbf{a}_{\xi_2}}$ which implies (by 2.5) $M^2_{\mathbf{a}_{\xi_1}} \cong M^2_{\mathbf{a}_{\xi_2}}$ by 2.4, 2.6 this implies $\xi_1 = \xi_2$. Hence necessarily $|S_\zeta| \leq \mu$. ∎

Returning to proving 3.2. By 3.2B, there is $\xi < |E(\mathfrak{A})|$ such that $\xi \notin \bigcup_{\zeta<\mu^+} S_\zeta$ (as $\left|\bigcup_{\zeta<\mu^+} S_\zeta\right| \leq \mu \times \mu^+ < |E(\mathfrak{A})|$). Similarly there is $\zeta \in \mu^+$, $\zeta \notin S_\xi$. Now $[M^2_{\mathbf{a}_\xi}]^{\mathfrak{A}(T)}$, $[M^2_{\mathbf{a}_\zeta}]^{\mathfrak{A}(T)}$ are as required (by their choice $\zeta \notin S_\xi$, $\xi \notin S_\zeta$, so we get the non-interpretability).

Note that we can get more:

Fact 3.2C. *There is a subset S of $|E(\mathfrak{A})|$ of power $|E(\mathfrak{A})|$ such that $\zeta \neq \xi \in S \Rightarrow \zeta \notin S_\xi$ (S_ξ — as defined in 3.2B).*

Proof. By 3.2B and Hajnal's free subset theorem (by Hajnal [4]). ∎

Now for each $\zeta < |T_\mu|$, let Γ_ζ be the first order theory of $M^2_{\mathbf{a}_\zeta}$. By 2.7 $\Gamma_\zeta = \Gamma$ for all ζ (Ehrenfeucht-Fraissé game, see [CK]). By 2.8 $M^2_{\mathbf{a}_\zeta}$ is atomic, by 2.10 minimal. By 3.2C we prove 3.2 under the hypothesis mentioned at the beginning of the proof of 3.2. We shall return to the proof of Theorem 3.2 later (i.e. to the proof without the extra assumptions).

Convention 3.3. Let χ be a large enough cardinal (e.g. $\beth_{\mu^+}$), $<^*_\chi$ a well ordering of $H(\chi)$, and as in Def.2.2 (iv), $<^{[M^2_{\mathbf{a}}]^{\mathfrak{A}}}$ is just the restriction of $<^*_\chi$ to $Q^{[M^2_{\mathbf{a}}]^{\mathfrak{A}}}$ and on the ordinals it is the usual ordering (when $\mathfrak{A} \in H(\chi)$, of course).

Definition 3.4. Let for a δ-system $\mathfrak{A} = \mathfrak{A}(T)$ and $\mathbf{a} \in \mathrm{gr}(\mathfrak{A})$, the model $M^4_{\mathbf{a}}$ be the expansion of $M^3_{\mathbf{a}}$ by:

(i) the function $x \mapsto \langle \mathbf{a}, x \rangle$ for $x \in \bigcup_{i<\delta} G^{\mathfrak{A}}_i$

(ii) G, G^1_ℓ, G^2_ℓ, G^3_ℓ ($\ell < \omega$) are partial unary functions from $G^{\mathfrak{A}}_\alpha$ to ω, $\delta^{\mathfrak{A}}$, $\delta^{\mathfrak{A}}$, T_α, respectively (remember: $Q_0 = \delta$, $Q_3 = T$), $x = \sum_{\ell<k} a^{\xi^\ell,\alpha}_{i_\ell,j_\ell}$ where $k = G(x)$, $i_\ell = G^1_\ell(x)$ for $\ell < k$, $j_\ell = G^2_\ell(x)$, $\xi^\ell = G^3_\ell(x)$, and the freedom we have is eliminated by the well order; i.e. $\ell < m \Rightarrow \langle \xi^\ell, i_\ell, j_\ell\rangle <^*_\chi \langle \xi^m, i_m, j_m\rangle$.

(iii) R a partial binary function, $R(\xi,\alpha) = \xi \restriction \alpha$ for $\alpha < \delta$, $\xi \in T$.

Claim 3.5. *Suppose $\mathfrak{A} = \mathfrak{A}(T) \in H(\chi)$ is a $\delta^{\mathfrak{A}}$-system and M is an elementary submodel of $[M^4_{\mathbf{a}}]^{\mathfrak{A}(T)}$ which includes $Q_0^{M_3^{\mathfrak{A}}}$. Then:*

(i) *$S = S^M \overset{\text{def}}{=} T \cap M = Q_2^M$ is as required in 1.9 for $\delta = \delta^{\mathfrak{A}}$; i.e. it is a set of sequences of ordinals of length $< \delta$ closed under initial segments;*

(ii) $Q_1^M = \bigcup_{i<\delta} G_i^{\mathfrak{A}(S)}$;

(iii) $\mathbf{a} \in \mathrm{gr}(\mathfrak{A}(S))$;

(iv) $M = [M^4_{\mathbf{a}}]^{\mathfrak{A}(S)}$;

(v) $[M^\ell_{\mathbf{a}}]^{\mathfrak{A}(S)} \prec [M^\ell_{\mathbf{a}}]^{\mathfrak{A}(T)}$ *for* $\ell = 1,2,3,4$.

Proof. Check. ∎

Claim 3.6. *If $\mathfrak{A} = \mathfrak{A}(T)$, $M \prec [M^4_{\mathbf{a}}]^{\mathfrak{A}(T)}$, $\delta^{\mathfrak{A}(T)} \subseteq M$, $S = Q_2^M$ then:*

(1) $\mathbf{b} \in \mathrm{gr}(\mathfrak{A}(S)) \Rightarrow \mathbf{b} \in \mathrm{gr}(\mathfrak{A}(T))$ *and*

(2) *when $\mathbf{b} = \langle b_{i,j} : i < j < \delta^{\mathfrak{A}(T)}\rangle$, $b_{i,j} \in [M^1_{\mathbf{a}}]^{\mathfrak{A}(S)}$, the inverse implication also holds.*

(3) $\mathbf{d} \in \mathrm{Fact}(\mathfrak{A}(S)) \Rightarrow \mathbf{d} \in \mathrm{Fact}(\mathfrak{A}(T))$

(4) $\mathrm{gr}(\mathfrak{A}(S))$, $\mathrm{Fact}(\mathfrak{A}(S))$ *is a subgroup of* $\mathrm{gr}(\mathfrak{A}(T))$, $\mathrm{Fact}(\mathfrak{A}(T))$, *respectively.*

(5) $\mathrm{Fact}(\mathfrak{A}(S)) = \mathrm{gr}(\mathfrak{A}(S)) \cap \mathrm{Fact}(\mathfrak{A}(T))$.

Proof. Check. ∎

Claim 3.7. *Suppose $\mathfrak{A} = \mathfrak{A}(T)$, $M \prec [M^4_{\mathbf{a}}]^{\mathfrak{A}(T)}$, $\delta^{\mathfrak{A}} \subseteq M$ and $\mathbf{b} = \langle b_{i,j} : i < j < \delta^{\mathfrak{A}(T)}\rangle$ belongs to $\mathrm{gr}(\mathfrak{A}, S)$ where $S = M \cap T$. Then $[M^3_{\mathbf{b}}]^{\mathfrak{A}(S)} \prec [M^3_{\mathbf{b}}]^{\mathfrak{A}(T)}$.*

Proof. By the Tarski-Vaught criterion (see [CK]) it is enough to prove that for any $b_1, \dots, b_n \in [M^3_{\mathbf{b}}]^{\mathfrak{A}(S)}$, $c \in [M^3_{\mathbf{b}}]^{\mathfrak{A}(T)}$ and first order φ such that $[M^3_{\mathbf{b}}]^{\mathfrak{A}(T)} \models \varphi[b_1, \dots, b_n, c]$ there is $c' \in [M^3_{\mathbf{b}}]^{\mathfrak{A}(S)}$ such that $[M^3_{\mathbf{b}}]^{\mathfrak{A}(T)} \models \varphi[b_1, \dots, b_n, c']$. Without loss of generality, none of $b_1, \dots, b_n, c$ is an individual constant hence none of them is in Q.

Let $\alpha < \mu$ be such that $b_1, \dots, b_n, c$ belong to $\bigcup_{\beta<\alpha} G_\beta^{\mathfrak{A}(T)}$ and, in particular, let $c \in G_{\beta(*)}^{\mathfrak{A}(T)}$ where $\beta(*) < \alpha$. We can find by 1.8 (2), $\mathbf{d} = \langle d_i : i < \alpha \rangle$ such that in $\mathrm{gr}(\mathfrak{A}(S) \restriction \alpha)$, $\langle a_{i,j} : i < j < \alpha \rangle - \langle b_{i,j} : i < j < \alpha \rangle = \mathrm{fact}\langle d_i : i < \alpha \rangle$. So easily $g_{\mathbf{d}} \in F_\alpha$ in the notation of the proof of 2.7, hence (by what we proved there on the family of F_γ's) $\langle [M_{\mathbf{b}}^3]^{\mathfrak{A}(T)}, b_1, \dots, b_n, c \rangle$ is elementarily equivalent to $\langle [M_{\mathbf{a}}^3]^{\mathfrak{A}(T)}, g_{\mathbf{d}}(b_1), \dots, g_{\mathbf{d}}(b_n), g_{\mathbf{d}}(c) \rangle$. Note that as $d_i \in [M_{\mathbf{a}}^3]^{\mathfrak{A}(S)}$, $b_\ell \in [M_{\mathbf{a}}^3]^{\mathfrak{A}(S)}$ clearly $g_{\mathbf{d}}(b_\ell) \in [M_{\mathbf{a}}^3]^{\mathfrak{A}(S)}$. As $M \prec [M_{\mathbf{a}}^4]^{\mathfrak{A}(T)}$ and $M_{\mathbf{a}}^4$ is an expansion of $[M_{\mathbf{a}}^3]^{\mathfrak{A}(T)}$ (and β, $g_{\mathbf{d}}(b_1), \dots, g_{\mathbf{d}}(b_n)$ belong to $[M_{\mathbf{a}}^3]^{\mathfrak{A}(S)}$) there is $c^1 \in G_\beta^{\mathfrak{A}(S)} \subseteq [M_{\mathbf{a}}^3]^{\mathfrak{A}(S)}$ such that $[M_{\mathbf{a}}^3]^{\mathfrak{A}(T)} \models \varphi[g_{\mathbf{d}}(b_1), \dots, g_{\mathbf{d}}(b_n), c^1]$ and by the definition of $g_{\mathbf{d}}$ (and as $d_\beta \in G_\beta^{\mathfrak{A}(S)}$ for $\beta < \alpha$) we can find $c^0 \in G_\beta^{\mathfrak{A}(S)}$ such that $c^1 = g_{\mathbf{d}}(c^0)$.

So $[M_{\mathbf{a}}^3]^{\mathfrak{A}(T)} \models \varphi[g_{\mathbf{d}}(b_1), \dots, g_{\mathbf{d}}(b_n), g_{\mathbf{d}}(c^0)]$. Again by what we have proved in the proof of 2.7 on the family of F_γ's, it follows that $[M_{\mathbf{b}}^3]^{\mathfrak{A}(T)} \models \varphi[b_1, \dots, b_n, c^0]$, as required. ∎

On logics with generalized quantifiers see

Definition 3.8. $\mathcal{L}^*$ is first order logic with the following additional quantifiers

$$(Q_{iw}x, y)[\varphi_1(x, y), \varphi_2(x, y)]$$

with the following interpretation:

$$M \models (Q_{iw}x, y)[\varphi_1(x, y), \varphi_2(x, y)] \quad \text{iff:}$$

for some ordinal α, for $\ell = 1, 2$ $(A_\ell/E_\ell, \leq_\ell)$ is a well ordering of order type α where:

- $A_\ell = \{a \in M : M \models \exists y \varphi_\ell(a, y)\}$,
- $E_\ell(x, y) \stackrel{\text{def}}{=} \varphi_\ell(x, y) \cap \varphi_\ell(y, x)$ is an equivalence relation on A_ℓ, and
- $\leq_\ell$ is defined by $a/E \leq_\ell b/E$ <u>iff</u> $a \in A_\ell$, $b \in A_\ell$, $(\exists x, y)[\varphi_\ell(xy) \cap E_\ell(x, a) \cap E_\ell(y, b)]$.

Fact 3.9. *<u>If</u> M is a model with $\leq \mu$ relations and functions and $A \subseteq M$ is infinite <u>then</u> there is $N \prec_{\mathcal{L}^*} M$, $A \subseteq N$, $\|N\| = \mu + |A| + \aleph_0$. In fact we can expand M to a model M^* by adding $\leq \mu$ functions such that for every $N \prec M^*$ we have $N \prec_{\mathcal{L}^*} M$.*

Proof. E.g. by [Sh11] (or: you can have Skolem functions witnessing E_ℓ is not an equivalence relation, $(A_c/E_\ell, \leq_\ell)$ is not a linear order, or is a linear order which is not a well ordering, or that one of $(A_1/E_1, \leq_1)$, $(A_2/E_2, \leq_2)$ is isomorphic to a proper initial segment of the other). ∎

Claim 3.10. *If (for a given δ-system $\mathfrak{A}$, $\mathbf{a}, \mathbf{b} \in \mathrm{gr}(\mathfrak{A})$) in $M^3_{\mathbf{a}}$ we can interpret $M^3_{\mathbf{b}}$, say N is directly interpretable in $M^3_{\mathbf{a}}$, $N \cong M^3_{\mathbf{b}}$ then:*

(i) $N \restriction Q^N \underset{h \restriction Q^N}{\cong} M^3_{\mathbf{a}} \restriction Q^{M^3_{\mathbf{a}}}$, *moreover*

(ii) *h is definable in $M^3_{\mathbf{a}}$ (in the logic $\mathcal{L}_{iw}$) from the same parameters that appear in the interpretation.*

Proof. (i) As $M^3_{\mathbf{a}} \restriction Q = M^3_{\mathbf{b}} \restriction Q$. (ii) As $h(x) = y$ iff the order types of $\langle [\{z \in N : z <^N x\}, <^N \rangle$, $\left\langle \{z \in M^3_{\mathbf{a}} : z <^{M^3_{\mathbf{a}}} y\}, <^{M^3_{\mathbf{a}}} \right\rangle$ are isomorphic. ∎

Claim 3.11. *If in 3.7 we assume also $M \prec_{\mathcal{L}^*} [M^4_{\mathbf{a}}]^{\mathfrak{A}(T)}$ then we can conclude also $[M^3_{\mathbf{b}}]^{\mathfrak{A}(S)} \prec_{\mathcal{L}^*} [M^3_{\mathbf{b}}]^{\mathfrak{A}(T)}$.*

Proof. As in 2.7. ∎

Claim 3.12. *Suppose $M \prec_{\mathcal{L}^*} [M^4_{\mathbf{a}}]^{\mathfrak{A}(T)}$, $S = Q^M_1$. <u>If</u> $\mathbf{b}, \mathbf{c} \in \mathrm{gr}(\mathfrak{A}(S))$, and $[M^2_{\mathbf{b}}]^{\mathfrak{A}(S)}$ is interpretable (with parameters) in $[M^2_{\mathbf{c}}]^{\mathfrak{A}(S)}$ <u>then</u> $[M^2_{\mathbf{b}}]^{\mathfrak{A}(T)}$ is interpretable in $[M^2_{\mathbf{c}}]^{\mathfrak{A}(T)}$.*

Proof. So there is a model M^S directly interpretable with parameters in $[M^2_{\mathbf{c}}]^{\mathfrak{A}(S)}$, $M^S \cong [M^3_{\mathbf{b}}]^{\mathfrak{A}(S)}$ (remember $[M^3_{\mathbf{b}}]^{\mathfrak{A}(S)}$ is a reduct of $[M^2_{\mathbf{b}}]^{\mathfrak{A}(S)}$). Let $h : M^S \to [M^3_{\mathbf{b}}]^{\mathfrak{A}(T)}$ be such an isomorphism. The same (finitely many) formulas which define M^S in $[M^3_{\mathbf{c}}]^{\mathfrak{A}(S)}$, define a model M^T in $[M^3_{\mathbf{c}}]^{\mathfrak{A}(T)}$. By 3.11 and the assumption $M \prec_{\mathcal{L}^*} [M^4_{\mathbf{a}}]^{\mathfrak{A}(T)}$ we have $[M^3_{\mathbf{c}}]^{\mathfrak{A}(S)} \prec_{\mathcal{L}^*} [M^3_{\mathbf{c}}]^{\mathfrak{A}(T)}$; $\langle [M^3_{\mathbf{c}}]^{\mathfrak{A}(S)}, M^S \rangle \prec_{\mathcal{L}^*} \langle [M^3_{\mathbf{c}}]^{\mathfrak{A}(T)}, M^T \rangle$. By 3.10, $h \restriction Q^{M^S}$ is definable in $[M^3_{\mathbf{c}}]^{\mathfrak{A}(S)}$ with parameters and is an isomorphism from $M^S \restriction Q^{M^S}$ onto $[M^3_{\mathbf{c}}]^{\mathfrak{A}(S)} \restriction Q$. So the same formula gives an isomorphism from $M^T \restriction Q^{M^T}$ onto $[M^3_{\mathbf{c}}]^{\mathfrak{A}(T)} \restriction Q$.

Choose $x_i \in G^{\mathfrak{A}(S)}_i$ and let, for $i < j < \delta$

$e_{i,j}$ =the unique member of $Q^{M^S}_1$ such that $e_{i,j} + h_{i,j}(x_j) = x_i$

$e'_{i,j} = h(e_{i,j})$

$\mathbf{e} = \langle e_{i,j} : i < j < \delta \rangle$, $\mathbf{e}' = \langle e'_{i,j} : i < j < \delta \rangle$.

Easily M^S is isomorphic to $[M^3_{\mathbf{e}'}]^{\mathfrak{A}(S)}$, so as $[M^3_{\mathbf{b}}]^{\mathfrak{A}(S)} \underset{h}{\cong} M^S$ we have $\mathbf{e}' - \mathbf{b} \in \mathrm{Fact}(\mathfrak{A}(S))$ by 2.6 hence (see 3.6(3)) $\mathbf{e}' - \mathbf{b} \in \mathrm{Fact}(\mathfrak{A}(T))$. So, it is now easy to check that $M^T \cong [M^3_{\mathbf{b}}]^{\mathfrak{A}(T)}$; remembering M^T is directly interpretable with parameters in $[M^3_{\mathbf{c}}]^{\mathfrak{A}(T)}$ we get the desired conclusion. ∎

Returning to proving 3.2. Let μ, θ be cardinals such that $\mu \geq \theta$, $\theta = \mathrm{cf}(\theta) > \aleph_0$. Choose a strong limit cardinal, λ, $\mu < \lambda$, $\mathrm{cf}(\lambda) = \mathrm{cf}(\theta)$ and a strong limit cardinal κ, $\lambda < \kappa$, $\mathrm{cf}(\kappa) = \mathrm{cf}(\theta)$. So $\lambda^\theta > \lambda$, $\kappa^\theta > \kappa$ but $\lambda^{<\theta} = \lambda$, $\kappa^{<\theta} = \kappa$, (e.g. $\lambda = \beth_{\mu+\theta}$, $\kappa = \beth_{\mu+\theta+\theta}$).

Let $T^0 = {}^{\theta>}\kappa = \{\eta : \eta$ a sequence of ordinals $< \kappa$ of length $< \theta\}$ so δ, the "height of T^0" is θ. We define the θ-system $\mathfrak{A}^{T^0}$ by the proof of 1.9; i.e. by 1.9A. Let $T^1 = {}^{\theta>}\lambda = \left\{\eta \in {}^{\theta>}\kappa : \bigwedge_{i<\ell g(\eta)} \eta(i) < \lambda\right\}$. So $T^1 \subseteq T$. Now for each $\nu \in T^0_\mu$ we know by 3.2B that

$$S^0_\nu \stackrel{\text{def}}{=} \left\{\rho \in T^0_\mu : M^2_{\mathbf{a}_\rho} \text{ interpretable with parameters in } M^2_{\mathbf{a}_\nu}\right\}$$

has cardinality $\leq \kappa$. So by 3.2B there is $\nu \in T^0_\mu$, $\nu \notin \bigcup\{S^0_\eta : \eta \in T^1_\mu\}$.

Let N be an $\mathcal{L}^*$-elementary submodel of cardinality λ of $M^4_{\mathbf{a}_\nu}$ which includes $Q_0 \cup T^1 \cup \{\nu \restriction \alpha : \alpha < \theta\}$.

Now define $T^2 \subseteq {}^{\theta>}\kappa$ by: for $i < \mu$, $T^2_i = T^0_i \cap N$, hence $T^2_\mu = \left\{\rho \in {}^\mu\lambda : \bigwedge_{i<\mu} \rho \restriction i \in T^2_i\right\}$. Remember 3.5, 3.6.

By 3.12 for every $\rho \in T^1_\mu (\subseteq T^2_\mu)$, $[M^2_{\mathbf{a}_\nu}]^{\mathfrak{A}(T^2)}$ is not interpretable with parameters in $[M^2_{\mathbf{a}_\rho}]^{\mathfrak{A}(T^2)}$. By 3.2B there is $\rho \in T^1_\theta$ such that $[M^2_{\mathbf{a}_\rho}]^{\mathfrak{A}(T^2)}$ is not interpretable in $[M^2_\nu]^{\mathfrak{A}(T^2)}$ even with parameters.

Let M be an $\mathcal{L}^*$-elementary submodel of N of cardinality μ which include

$$Q^N_0 \cup \{\nu \restriction \alpha, \rho \restriction \alpha : \alpha < \theta\}.$$

Using 3.5, 3.6 and letting $T^3 = T^2 \cap M$ we know that by 3.12 $[M^2_{\mathbf{a}_\nu}]^{\mathfrak{A}(T^3)}$, $[M^2_{\mathbf{a}_\rho}]^{\mathfrak{A}(T^3)}$ are not interpretable in each other (even with parameters). They satisfy the other requirements of 3.2 as in the proof of 3.1. ∎

Section 4: The concluding part of the cylindric algebraic proofs

From Theorem 3.2 above, we will prove our cylindric algebraic theorems by using the connections between cylindric algebras and model theory described in [HMTII] §4.3. From now on, CA abbreviates "cylindric algebra". The CA, $\mathfrak{Cs}^M$ associated to a model M was defined the natural way in [HMTII] p. 154. It was proved there that

$\binom{**}{*}$ If $\mathfrak{Cs}^M$ is base-isomorphic to a subalgebra of $\mathfrak{Cs}^N$ then M is interpretable in N. Moreover, by [HMTII] 4.3.65(10), Sh:246

$\binom{**}{**}$ If $\mathfrak{Cs}^M$ is base-isomorphic to a subalgebra of $(\mathfrak{Cs}^N)^+$ then M is interpretable with parameters in N.

Let μ be a cardinal satisfying the hypothesis of Theorem 3.2. Obviously there is such a μ. Then by Theorem 3.2 there are elementarily equivalent minimal models M and N such that none of them is interpretable with parameters in the other. (See the end of the proof of 3.2 for the part concerning parameters.)

Let $\mathfrak{A} = \mathfrak{Cs}^M$ and $\mathfrak{B} = \mathfrak{Cs}^N$. By $\binom{**}{**}$, $\mathfrak{A}$ is not base-isomorphic to any subalgebra of $\mathfrak{B}^+$ and the same holds with $\mathfrak{A}$ and $\mathfrak{B}$ interchanged.

Elementary equivalence of M and N implies $\mathfrak{A} \cong \mathfrak{B}$ by [HMTII] 4.3.68(7).

Since the cardinalities of M and N coincide, their universes can be identified hence the greatest elements of $\mathfrak{A}$ and $\mathfrak{B}$ coincide.

Minimality of M and N implies base-minimality of $\mathfrak{A}$ and $\mathfrak{B}$ by [HMTII] §4.3, see also [N90]. (By [HMTII] 4.3, $\mathfrak{A}$, $\mathfrak{B}$ are Lr's. But this is also very easy to check directly.)

These observations together prove Theorem 0.2 which in turn implies Theorem 0.1.

To see that we also proved (**) strengthening Theorem 0.2, recall that in Definition 3.2 A(i) when defining M's interpretability in N, we allowed the interpretation of the universe M to be a subset of N definable by a formula φ in N. Since φ corresponds to an element φ^N of $\mathfrak{Cs}^N$ we can choose $x = \varphi^N$ and relative with this x to obtain (**). (Restricting the universe of a model to a definable set is the model theoretic counterpart of the algebraic notion of relativization.) To be more precise, assume φ is of the form $\varphi(v_0)$ and the natural number n $(n > 3)$ in (**) is fixed. Then we choose $x = (\varphi(v_0) \wedge \varphi(v_1) \wedge \cdots \wedge \varphi(v_{n-1}))^N$. Clearly $x \in Nr_n(\mathfrak{Cs}^N)$. The rest of the argument is unchanged.

This finishes the proofs of the cylindric algebraic statements in §0.

References

[A72] H.Andréka, *Algebraic Logic*, Dissertation, Eötvös Loránd University, Budapest (1972).

[AGN77] H.Andréka, T.Gergely, and I.Németi, *On universal algebraic construction of logics*, Studia Logica Vol **36** (1977), 9-47.

[AS78] H.Andréka and I.Sain, *Connections between algebraic logic and initial algebra semantics of CF languages*, In: Mathematical Logic in Computer Science (Proc.

Sh:246

Conf. Salgótarján Hungary 1978) Eds.: B.Dömölki and T.Gergely, Colloq. Math. Soc. J. Bolyai — North-Holland Vol **26** (1981), 25–83.

[B87] B.Biró, *Isomorphic but not base-isomorphic base-minimal cylindric set algebras*, Algebra Universalis Vol **24** (1987), 292-300.

[B88] B.Biró, *Topics in algebraic logic*, Dissertation (Budapest, 1988), 113pp.

[BSh88] B.Biró and S.Shelah, *On isomorphic but not lower-base-isomorphic cylindric set algebras*, J. Symbolic Logic Vol **53, 3** (1988), 846-853.

[CK] C.C.Chang and H.J.Keisler, *Model Theory*, North Holland Publ. Co. (1973).

[F90] M.Ferenczi, *On inducing cylindric homomorphisms by point functions*, Notre Dame J. Formal Logic (to appear).

[H] A.Hajnal, *Proof of a conjecture of S. Ruziewicz*, Fund. Math Vol **50** (1961), 123–128.

[HMTII] L.Henkin, J.D.Monk, and A.Tarski, *Cylindric Algebras Part II*, North-Holland, Amsterdam (1985).

[HMTAN] L.Henkin, J.D.Monk, A.Tarski, H.Andréka, and I.Németi, *Cylindric Set Algebras*, Lecture Notes in Mathematics Vol **883**, Springer-Verlag, Berlin (1981), vi+323pp.

[N83] I.Németi, *Cylindric-relativised set algebras have strong amalgamation*, Preprint, expanded version of the paper appeared in J. Symbolic Logic **50,** 3 (September 1985), 689–700.

[N90] I.Németi, *Cylindric algebraic model theory*, In: Algebraic Logic and Universal Algebra in Computer Science (Proc. Conf. Ames 1988). Springer Lecture Notes in Computer Science (eds: C.H. Bergman, R.D.Maddux, and D.L.Pigozzi) **425** (1988), 37–75.

[S86] G.Serény, *Lower level connections between representations of relation algebras*, Bull. Section of Logic **15** (1986), 123–127.

[S90] G.Serény, *Neatly atomic cylindric algebras and representable isomorphisms*, this volume.

[Sh11] S.Shelah, *The number of non-isomorphic models of T in a power*, Pacific J. of Math., 811–818.

[Sh189] S.Shelah, *On the possible number no (M) — the number of non-isomorphic models $L_{\infty,\lambda}$ — equivalent to μ of power λ for λ singular*, Notre Dame J. of Formal Logic Vol **26** (1985), p. 36.

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Department of Mathematics
New Brunswick, NJ USA