New Nonfree Whitehead Groups by Coloring

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Abstract

We extend the method of uniformization by considering colorings which take values in a group. We use this to construct new non-free Whitehead groups, in particular answering an open problem in [1] by showing that it is consistent that there is a strongly $\aleph_1$-free $\aleph_1$-coseparable group of cardinality $\aleph_1$ which is not $\aleph_1$-separable.

0 INTRODUCTION

An abelian group A is called a Whitehead group, or W-group for short, if $\mathrm{Ext}(A, \mathbb{Z}) = 0$. For historical reasons, A is called an $\aleph_1$-coseparable group if $\mathrm{Ext}(A, \mathbb{Z}^{(\omega)}) = 0$, but for convenience we shall use non-standard terminology and say A is a W_ω-group when A is $\aleph_1$-coseparable. Obviously a W_ω-group is a W-group. In 1973–75, the second author proved that it is consistent with ZFC + GCH that every W-group is free and consistent with ZFC that there are non-free W_ω-groups of cardinality $\aleph_1$ ([8], [9]); he later showed that it is consistent with ZFC + GCH that there are non-free W_ω-groups of cardinality $\aleph_1$ ([10], [11]). Before 1973 it was known (in ZFC) that every W-group is $\aleph_1$-free, separable, and slender, and assuming CH, every W-group is strongly $\aleph_1$-free. (See, for example, [3, pp. 178–180].) These turned out, by the results of the second author, to be essentially all that could be proved without additional set-theoretic hypotheses.

However, new questions of what could be proved in ZFC arose, inspired by the consistency results and their proofs. One of the most intriguing was:

(0.1) Does every strongly $\aleph_1$-free W_ω-group of cardinality $\aleph_1$ satisfy the stronger property that it is $\aleph_1$-separable?

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(See [1, p. 454, Problem 5]). As we shall explain below, not only was the answer to this question affirmative in every known model of ZFC, but the nature of the known constructions of non-free Whitehead groups was such as to lead to the suspicion that the answer might be affirmative (provably in ZFC). However, in this paper we show that it is consistent that the answer is negative.

First we recall the key definitions. An abelian group A is *$\aleph_1$-free* if every countable subgroup of A is free; A is *strongly $\aleph_1$-free* if every countable subset is contained in a countable free subgroup B such that A/B is $\aleph_1$-free. A is *$\aleph_1$-separable* if every countable subset is contained in a countable free subgroup B which is a direct summand of A; so an $\aleph_1$-separable group is strongly $\aleph_1$-free. It is a consequence of CH (or even of $2^{\aleph_0} < 2^{\aleph_1}$) that there are strongly $\aleph_1$-free groups of cardinality $\aleph_1$ which are not $\aleph_1$-separable (see [12]). However, the existence of such groups is not settled by the hypothesis $2^{\aleph_0} = 2^{\aleph_1}$; specifically, in a model of MA + ¬CH every strongly $\aleph_1$-free group of cardinality $\aleph_1$ *is* $\aleph_1$-separable; but the methods of [6] show that it is consistent with $2^{\aleph_0} = 2^{\aleph_1}$ that there are strongly $\aleph_1$-free groups of cardinality $\aleph_1$ which are *not* $\aleph_1$-separable.

Now suppose A is strongly $\aleph_1$-free and is a W_ω-group. Consider a countable subgroup B of A such that A/B is $\aleph_1$-free. We have a short exact sequence

$$0 \to B \to A \to A/B \to 0$$

where the map of B into A is inclusion. Since B is a free group of countable rank, if we knew that A/B were a W_ω-group, then we would have $\mathrm{Ext}(A/B, \mathbb{Z}^{(\omega)}) = \mathrm{Ext}(A/B, B) = 0$ and we could conclude that this sequence splits and hence B is a direct summand of A. In every previously known model where there are non-free W_ω-groups, the construction of a W_ω-group A is such that A/B shares the properties of A closely enough that A/B is also a W_ω-group — when B is a countable subgroup such that A/B is $\aleph_1$-free; cf. the Remark after Theorem 2 below. Thus in these models the answer to (0.1) is affirmative. This motivates question (0.1) as well as the related question

> (0.2) If a group A of cardinality $\aleph_1$ is strongly $\aleph_1$-free and a W_ω-group, and B is a countable subgroup of A such that A/B is $\aleph_1$-free, is A/B a W_ω-group?

By what we have just remarked, a positive answer to (0.2) implies a positive answer to (0.1). The converse holds as well: if A and B are as in the hypotheses of (0.2) and A is $\aleph_1$-separable, $A = F \oplus A'$ where F is countable and contains B; then A/B is a W_ω-group because $A/B = F/B \oplus A'$ and F/B is free by hypothesis on B.

We shall prove that there is a model of ZFC in which not only the answer to (0.2) but also the answer to the following weaker question is "no":

> (0.3) If a group A of cardinality $\aleph_1$ is strongly $\aleph_1$-free and a W_ω-group, and B is a countable subgroup of A such that A/B is $\aleph_1$-free, is A/B necessarily a W-group?

The proof involves an extension of the method of uniformization which was first used by the second author in [10] and [11] to construct non-free Whitehead groups in a model of CH. We recall the method and describe the extended method in the next section. That section and the following one do not require a knowledge of forcing. In the last section we construct a model of ZFC with the desired properties; the construction is by means of an iterated forcing, with finite support, of c.c.c. posets.

It remains open whether negative answers to (0.1), (0.2) and (0.3) are consistent with CH.

1 COLORING METHODS

Throughout, E will be a stationary subset of ω_1 consisting of limit ordinals, with (for technical reasons) $\omega \notin E$. We begin with a general construction of a group, related to that in [1, XII.3.4].

Definition 1 *For each $\delta \in E$ let η_δ be a ladder on δ, that is, a strictly increasing function $\eta_\delta : \omega \to \delta$ whose range approaches δ. Let φ be a function from $E \times \omega$ to $\{0\} \cup \{x_k : k \in \omega\}$. Let F be the free abelian group with basis $\{x_\nu : \nu \in \omega_1\} \cup \{z_{\delta,n} : \delta \in E, n \in \omega\}$ and let K be the subgroup of F generated by $\{w_{\delta,n} : \delta \in E, n \in \omega\}$ where*

$$w_{\delta,n} = 2z_{\delta,n+1} - z_{\delta,n} - x_{\eta_\delta(n)} - \varphi(\delta, n). \tag{1}$$

Let $A = F/K$.

Clearly A is an abelian group of cardinality $\aleph_1$. Notice that because the right-hand side of (1) is 0 in A, we have for each $\delta \in E$ and $n \in \omega$ the following relation in A:

$$2^{n+1} z_{\delta,n+1} = z_{\delta,0} + \sum_{k=0}^{n} 2^k (x_{\eta_\delta(k)} + \varphi(\delta, k)) \tag{2}$$

where, in an abuse of notation, we write, for example, $z_{\delta,n+1}$ instead of $z_{\delta,n+1} + K$. If we let

$$A_\alpha = \langle \{x_\nu : \nu < \alpha\} \cup \{z_{\delta,n} : \delta \in E \cap \alpha, n \in \omega\} \rangle . \tag{3}$$

for each $\alpha < \omega_1$, then for each $\delta \in E$ $z_{\delta,0} + A_\delta$ is non-zero and divisible in $A_{\delta+1}/A_\delta$ by 2^n for all $n \in \omega$. Thus $A_{\delta+1}/A_\delta$ is not free and hence A is not free. (In fact $\Gamma(A) \supseteq \tilde{E}$; see [1, pp. 85f].) Moreover, A is strongly $\aleph_1$-free; in fact, for every $\alpha < \omega_1$, using Pontryagin's Criterion we can show that A/A_α is $\aleph_1$-free whenever $\alpha \notin E$.

The following definitions are, by now, standard. A *ladder system* on a stationary subset E of $\lim(\omega_1)$ is an indexed family of functions $\eta = \langle \eta_\delta : \delta \in E \rangle$ such that each $\eta_\delta : \omega \to \delta$ is a ladder on δ. The ladder system η is called *tree-like* if for all $\delta_1, \delta_2 \in E$ and all $n, m \in \omega$, $\eta_{\delta_1}(n) = \eta_{\delta_2}(m)$ implies $n = m$ and $\eta_{\delta_1}(k) = \eta_{\delta_2}(k)$ for all $k \leq n$. If λ is a cardinal, a *λ-coloring* of η is an indexed family, $c = \langle c_\delta : \delta \in E \rangle$, of functions $c_\delta : \omega \to \lambda$. (We are particularly interested in the cases when λ is 2 or ω.) We say that η has the *λ-uniformization property* provided that for every λ-coloring c of η, there exists a pair of functions (f, f^*) such that $f : \omega_1 \to \lambda$, $f^* : E \to \omega$, and for all $\delta \in E$, $f(\eta_\delta(n)) = c_\delta(n)$ for all $n \geq f^*(\delta)$.

The following theorem is due to the second author; a proof can be found, for example, in [1, pp. 371–374].

Theorem 2 *Suppose A is constructed as in Definition 1 with $\varphi(\delta, n) = 0$ for all δ, n.*

(1) If η has the 2-uniformization (respectively, ω-uniformization) property, then A is a W-group (respectively, a W_ω-group).

(2) If η is tree-like and A is a W-group (respectively, a W_ω-group), then η has the 2-uniformization (respectively, ω-uniformization) property. □

Since it is consistent with ZFC + GCH that there is a ladder system η with the ω-uniformization property, it is consistent with ZFC + GCH that there are non-free W_ω-groups (cf. [10]).

It is also true, though not needed for our present purposes, that if there exists any non-free W-group (respectively, W_ω-group) of cardinality $\aleph_1$, then there is a ladder system with the 2-uniformization (respectively, ω-uniformization) property; see [2, pp. 92–98] for a proof.

Remark. Note that if A and η are as in Theorem 2 (1) and $B = A_\alpha$ for $\alpha \notin E$, then A/B is $\aleph_1$-free and A/B is also a W-group (resp. W_ω-group), because Theorem 2 (1) applies to A/B since it is constructed as in Definition 1, using essentially the same ladder system η (with a countable initial segment deleted).

For our new construction, we shall need an extension of the notions of coloring and uniformization. Let M be a countable group. (We are particularly interested in the cases when M is $\mathbb{Z}$ or $\mathbb{Z}^{(\omega)}$.) Let η and φ be as above but such that φ takes values in $\{x_k : k \in \omega\}$. Denote by $\tilde{\varphi}(\delta, n)$ the unique $k \in \omega$ such that $\varphi(\delta, n) = x_k$. Define an *M-coloring* of (η, φ) to be an indexed family $c = \langle c_\delta : \delta \in E\rangle$ of functions $c_\delta : \omega \to M$. Say that the pair (f, f^*) *uniformizes* c if $f : \omega_1 \to M$ and $f^* : E \to \omega$ and for all $\delta \in E$ and all $n \geq f^*(\delta)$,

$$f(\eta_\delta(n)) + f(\varphi(\delta, n)) = c_\delta(n). \tag{4}$$

(Addition in the group M.) Say that (η, φ) has the *M-uniformization property* iff for every M-coloring c there is a pair (f, f^*) which uniformizes c.

Theorem 3 *Let A be constructed as in Definition 1, where φ takes values in $\{x_k : k \in \omega\}$. If (η, φ) has the M-uniformization property, then $Ext(A, M) = 0$.*

PROOF. It suffices to prove that every homomorphism $\psi : K \to M$ extends to a homomorphism $\theta : F \to M$. Given ψ, define $c_\delta(n) = -\psi(w_{\delta,n})$ for each $\delta \in E$, $n \in \omega$. Suppose that (f, f^*) uniformizes this coloring. Define $\theta(x_\nu) = f(\nu)$ and $\theta(z_{\delta,n}) = 0$ for $n \geq f^*(\delta)$. It then follows from equations (1) and (4) that $\theta(w_{\delta,n}) = \psi(w_{\delta,n})$ for all $n \geq f^*(\delta)$. Define $\theta(z_{\delta,n})$ for $n < f^*(\delta)$ by downward induction, using the equation

$$\theta(z_{\delta,n}) = 2\theta(z_{\delta,n+1}) - \theta(x_{\eta_\delta(n)}) - \theta(\varphi(\delta, n)) - \psi(w_{\delta,n}).$$

so that θ extends ψ. □

Corollary 4 *Suppose that there are a tree-like ladder system η and a function $\varphi : E \times \omega \to \{x_k : k \in \omega\}$ such that (η, φ) has the $\mathbb{Z}^{(\omega)}$-uniformization property but such that η does not have the 2-uniformization property. Then there is a strongly $\aleph_1$-free W_ω-group A of cardinality $\aleph_1$ with a countable subgroup B of A such that A/B is $\aleph_1$-free but is not a W-group.*

PROOF. Let A be constructed as in Definition 1, using the given η and φ. Then by Theorem 3, with $M = \mathbb{Z}^{(\omega)}$, A is a W_ω-group. Let B be A_ω. Then A/B is $\aleph_1$-free since $\omega \notin E$; moreover A/B is isomorphic to a group constructed as in Definition 1 but with $\varphi(\delta, n) = 0$ for all δ, n. Thus by the hypothesis and Theorem 2 (2), A/B is not a W-group. □

In what follows we prove that there is a model of ZFC in which the hypotheses of Corollary 4 are true. In the next section we describe the basic partial ordering used in the iterated forcing and prove that it is c.c.c. In the final section we describe the iteration and prove its properties.

2 A c.c.c PARTIAL ORDERING

Let M, η, and φ be as in the previous section where η is tree-like. Let c be a fixed M-coloring of (η, φ). We shall define a partial order which consists of finite approximations to a uniformization of c.

Definition 5 *Let Q_c be the set of all pairs (p, p^*) such that for some finite subset S of E, and some $r \in \omega$, $p^* : S \to \omega$ and p is the restriction to $\bigcup\{\{\eta_\delta(m), \bar\varphi(\delta, m)\} : \delta \in S,\ m < r\}$ of a function $f_p : \bigcup\{\{\eta_\delta(n), \bar\varphi(\delta, n)\} : \delta \in S,\ n \in \omega\} \to M$ such that for every $\delta \in S$, $f_p(\eta_\delta(n)) + f_p(\bar\varphi(\delta, n)) = c_\delta(n)$ if $n \geq p^*(\delta)$. Partially order Q_c by: $(p, p^*) \leq (q, q^*)$ if and only if q is an extension of p and q^* is an extension of p^*. Define $cont((p, p^*)) = S$ and $num((p, p^*) = r$.*

Lemma 6 *Q_c is c.c.c., that is, every antichain in Q_c is countable.*

PROOF. The proof is similar to that in [1, Proof of VI.4.6]. Let κ be large enough for Q_c. Identify M with ω. We shall show that 0 is N-generic whenever $N = \bigcup_{i \in \omega} N_i$ where $N_i \prec N_{i+1} \prec H(\kappa)$ and $N_i \cap \omega_1 < N_{i+1} \cap \omega_1$. Given such an N, let $\alpha_i = N_i \cap \omega_1$ and $\alpha = N \cap \omega_1$. Suppose that $(p, p^*) \in Q_c$ and $D \in N$ is a dense subset of Q_c. We need to show that there is an element of $D \cap N$ which is compatible with (p, p^*). Let f_p be as in the definition of Q_c, so that p is a restriction of f_p. Choose i so that $D \in N_i$, $\text{dom}(p) \cap \alpha \subseteq \alpha_i$, and $\text{dom}(p^*) \cap \alpha \subseteq \alpha_i$. Let

$$\begin{aligned} Y = \bigcup\{\{\eta_\delta(\ell), \bar\varphi(\delta, \ell)\} : \delta \in \text{dom}(p^*) \setminus \alpha_i, \eta_\delta(\ell) < \alpha_i\} \cup \\ \{\bar\varphi(\delta, k) : \eta_\delta(k) = \eta_\gamma(k) \text{ for some } \delta \neq \gamma \in \text{dom}(p^*)\}. \end{aligned}$$

Then Y is finite since $\delta > \alpha_i$ if $\delta \in \text{dom}(p^*) \setminus \alpha_i$ and since ladders on different ordinals have different limits and hence their ranges must have finite intersection. Since N_i is a model of set theory, there exists $q \in N_i$ such that $q = p \restriction (\text{dom}(p) \cap \alpha_i) \cup f_p \restriction Y$. Let $q^* = p^* \restriction (\text{dom}(p) \cap \alpha_i)$. Then $(q, q^*) \in N_i \cap Q_c$ so since $N_i \models$ "D is dense in Q_c", there exists $(\tilde q, \tilde q^*) \in D \cap N_i$ such that $(q, q^*) \leq (\tilde q, \tilde q^*)$. By choice of q and q^* and Y, we can modify f_p to agree with $f_{\tilde q}$ where necessary and show that for some extension q' of $p \cup \tilde q$, $(q', p^* \cup \tilde q^*) \in Q_c$. Therefore $(\tilde q, \tilde q^*) \in D \cap N$ is compatible with (p, p^*). □

We shall also need some results about denseness of certain subsets of Q_c. For $\delta \in E$, let $D_\delta = \{(p, p^*) \in Q_c : \delta \in \text{dom}(p^*)\}$. We claim that D_δ is dense in Q_c. Indeed, given any $(p, p^*) \in Q_c$ and $\delta \notin \text{dom}(p^*)$ we can find k such that if $n \geq k$, then

$$\eta_\delta(n) \notin \{\eta_\gamma(m) : \gamma \in \text{dom}(p^*),\ m \in \omega\} \cup \text{dom}(p).$$

Let $\tilde p^* = p^* \cup \{(\delta, k)\}$; we can easily extend f_p to show that $(\tilde p, \tilde p^*)$ belongs to Q_c for some $\tilde p$.

For any $\alpha \in \{\eta_\delta(n) : \delta \in E,\ n \in \omega\}$, let $D'_\alpha = \{(p, p^*) \in Q_c : \alpha \in \text{dom}(p)\}$. Then it is easy to see that D'_α is dense in Q_c.

Simply as an exercise, let us see how we can use these notions to show that non-free W_ω-groups exist in a model of MA + ¬CH.

Theorem 7 *Assume MA + ¬CH. Let A be constructed as in Definition 1. Then for any countable group M, $Ext(A, M) = 0$.*

PROOF. By Theorem 3, we need only prove that (η, φ) has the M-uniformization property. So let c be any M-coloring of (η, φ). Let Q_c be as in Definition 5. By Lemma 6, Q_c is c.c.c. Hence, by MA, there is a directed subset G of Q_c which meets each D_δ and each D'_α. Let $f^* = \cup\{p^* : (p, p^*) \in G$ for some $p\}$. Then f^* is a function — because G is directed — with domain E — because G meets every D_δ. Define $f' = \cup\{p : (p, p^*) \in G$ for some $p^*\}$. Then f' is a function with domain $\{\eta_\delta(n) : \delta \in E, n \in \omega\}$ — because G meets each D'_α — and we can arbitrarily extend f' to f so that (f, f^*) is a uniformization of c. □

Now η has the 2-uniformization property in any model of MA $+\neg$CH, so this is not the model we seek (satisfying the hypotheses of Corollary 4). However, in the next section we shall construct the desired model by imitating the proof of the consistency of MA: employing an iterated forcing with finite support, but instead of iterating over all c.c.c. posets, iterating over just those of the form Q_c defined in Definition 5.

3 THE FORCING CONSTRUCTION

In this section we prove the following theorem.

Theorem 8 *It is consistent with ZFC that there exists a tree-like ladder system η on a stationary set E and a function $\varphi : E \times \omega \to \{x_k : k \in \omega\}$ such that (η, φ) has the $Z^{(\omega)}$-uniformization property, but such that η does not have the 2-uniformization property.*

PROOF. We begin with a ground model where GCH holds and fix a stationary set E of limit ordinals such that $\omega \notin E$. We shall first do a forcing to create a generic family of ladders $\langle \eta_\delta : \delta \in E\rangle$ and a generic function φ. Let Q_0 be the set of all finite functions p such that dom(p) is a finite subset of E and for some $r^p \in \omega$, for all $\delta \in$ dom(p), $p(\delta)$ is a pair $(\eta^p_\delta, \varphi^p_\delta)$ where η^p_δ is a strictly increasing function: $r^p \to \delta$ and $\varphi^p_\delta : \{\delta\} \times r^p \to \{x_k : k \in \omega\}$; moreover, we require that if $\delta, \gamma \in$ dom(p), $n, m < r^p$, and $\eta^p_\delta(n) = \eta^p_\gamma(m)$, then $n = m$ and $\eta^p_\delta(\ell) = \eta^p_\gamma(\ell)$ for all $\ell < n$.

The group A is defined in V^{Q_0} as in Definition 1 using the generic ladders η_δ and the generic $\varphi : E \times \omega \to \{x_k : k \in \omega\}$. Then

$$P = \left\langle P_\alpha, \dot{Q}_\alpha : 0 \le \alpha < \omega_2 \right\rangle$$

will be constructed to be a finite support iteration of length ω_2 so that for each $\alpha \geq 1$ $\Vdash_{P_\alpha} \dot{Q}_\alpha = Q_{\dot{c}(\alpha)}$ where

$$\Vdash_{P_\alpha} \dot{c}^{(\alpha)} \text{is a } Z^{(\omega)}\text{-coloring of } (\eta, \varphi).$$

We choose our enumeration of names $\{\dot{c}^{(\alpha)} : 1 \leq \alpha < \omega_2\}$ so that if G is P-generic and $c \in V[G]$ is a $Z^{(\omega)}$-coloring of (η, φ), then for some $\alpha < \omega_2$, $\dot{c}^{(\alpha)}$ is a name for c in V^{P_α}. Then P is c.c.c. and it is clear from our construction that if G is P-generic, in $V[G]$ (η, φ) has the $Z^{(\omega)}$-uniformization property.

It remains to prove that in $V[G]$ η does not have the 2-uniformization property. In fact, we claim that in $V[G]$ the coloring c defined by

$$c_\delta(n) = \begin{cases} 0 & \text{if } \bar{\varphi}(\delta, n) \text{ is even} \\ 1 & \text{if } \bar{\varphi}(\delta, n) \text{ is odd} \end{cases}$$

cannot be uniformized.

Suppose, to the contrary, that there are names $\dot{f}$ and $\dot{f}^*$ and a $p \in G$ such that

$$p \Vdash (\dot{f}, \dot{f}^*) \text{ uniformizes } c.$$

For each $\delta \in E$, there exists $p_\delta \geq p$ and $m_\delta \in \omega$ such that

$$p_\delta \Vdash \dot{f}^*(\delta) = m_\delta.$$

Since ω is countable, there is a stationary $E' \subseteq E$ and $m \in \omega$ such that for all $\delta \in E'$, $m_\delta = m$. We claim

($\star$) *there exist* $\delta_1 < \delta_2$ *in* E', $r, k \in \omega$, *and* $q \geq p_{\delta_i}$ $(i = 1, 2)$ *such that* $r \geq m$ and q *forces the following:* $\eta_{\delta_1}(r) = \eta_{\delta_2}(r)$; $\varphi(\delta_1, r) = x_{2k}$*; and* $\varphi(\delta_2, r) = x_{2k+1}$.

Suppose that ($\star$) holds. We obtain a contradiction by considering what happens in a generic extension $V[\tilde{G}]$ where $q \in \tilde{G}$. In $V[\tilde{G}]$ we have: $f^*(\delta_i) = m$ for $i = 1, 2$, so $f(\eta_{\delta_i}(r)) = c_{\delta_i}(r)$ for $i = 1, 2$. But this is impossible by definition of c, since $\bar{\varphi}(\delta_1, r) = 2k$ and $\bar{\varphi}(\delta_2, r) = 2k + 1$.

So it remains to prove ($\star$). Without loss of generality, we can assume that for every $\delta \in E'$ and every $\beta > 0$, $p_\delta(\beta)$ is a finite function (in the ground model) and not just a name, and that $\delta \in \text{dom}(p_\delta(0))$. Moreover, we can assume that for all $\delta \in E'$, $p_\delta(0)$ determines $\eta_\sigma(n)$ for all $\sigma \in \text{cont}(p_\delta(\beta))$ and $n < \text{num}(p_\delta(\beta))$ for all $\beta \in \text{supp}(p_\delta) \setminus \{0\}$. We can also assume that there is an $r \geq m$ such that for all $\delta \in E'$ $r^{p_\delta(0)} = r$ and that for all $\beta \in \text{supp}(p_\delta) \setminus \{0\}$, $\text{num}(p_\delta(\beta)) = r$. Furthermore, by Fodor's Lemma, we can assume that $\langle \eta_\delta^{p_\delta}(n) : n < r \rangle$ and $\langle \varphi^{p_\delta}(\delta, n) : n < r \rangle$ are independent of δ. Finally, by the Δ-system lemma we can find $\delta_1 < \delta_2$ such that for all $\beta < \omega_2$, $p_{\delta_1}(\beta) \cup p_{\delta_2}(\beta)$ is a function.

Now we define $q \geq p_{\delta_i}$ $(i = 1, 2)$ by defining $q(\beta)$ for each $\beta < \omega_2$. In order to define $q(0)$, choose γ such that $\eta_{\delta_1}^{p_{\delta_1}}(r - 1) < \gamma < \delta_1$, and $k \in \omega$ such that x_{2k} and x_{2k+1} do not "occur in" any $p_{\delta_i}(\beta)$ $(i = 1, 2)$. Then define $q(0)$ extending $p_{\delta_1}(0)$ and $p_{\delta_2}(0)$ so that it forces $\eta_{\delta_1}(r) = \gamma = \eta_{\delta_2}(r)$, $\varphi(\delta_1, r) = x_{2k}$, and $\varphi(\delta_2, r) = x_{2k+1}$. We choose γ_1 and γ_2 so that $\gamma < \gamma_1 < \delta_1 < \gamma_2 < \delta_2$ and let $q(0)$ force $\eta_{\delta_i}(r + 1) = \gamma_i$ $(i = 1, 2)$; then we will have guaranteed that $q(0)$ forces $\eta_{\delta_1}(k) \neq \eta_{\delta_2}(\ell)$ and $\eta_{\delta_1}(\ell) \neq \eta_{\delta_2}(k)$ for all $k \geq r + 1$ and $\ell \in \omega$. Similarly we can define $q(0)$ to force values for $\eta_\sigma(j)$ for $j = r, r + 1$ for all other $\sigma \in \text{dom}(p_{\delta_1}(0)) \cup \text{dom}(p_{\delta_2}(0))$ so that $q(0)$ forces $\eta_{\sigma_1}(\ell) \neq \eta_{\sigma_2}(k)$ whenever $k \geq r$, $\ell \in \omega$, $\sigma_1 \neq \sigma_2 \in \text{dom}(p_{\delta_1}(0)) \cup \text{dom}(p_{\delta_2}(0))$, and either $k > r$ or $\sigma_2 \notin \{\delta_1, \delta_2\}$.

For any $\beta > 0$, if $\beta \in \text{supp}(p_{\delta_i}) \setminus \text{supp}(p_{\delta_j})$ for $i \neq j \in \{1, 2\}$, let $q(\beta) = p_{\delta_i}(\beta)$. If $\beta \in \text{supp}(p_{\delta_1}) \cap \text{supp}(p_{\delta_2})$, let $q(\beta)$ be a name $(\dot{q}, \dot{q}^*)$ for a condition extending $p_{\delta_1}(\beta) \cup p_{\delta_2}(\beta)$ so that (in V^{P_β}): $\text{num}(q(\beta)) = r + 2$; $\text{cont}(q(\beta)) = \text{cont}(p_{\delta_1}(\beta)) \cup \text{cont}(p_{\delta_2}(\beta))$; $\dot{q}(\gamma) + \dot{q}(2k) = \dot{c}_{\delta_1}^{(\beta)}(r)$; and $\dot{q}(\gamma) + \dot{q}(2k + 1) = \dot{c}_{\delta_2}^{(\beta)}(r)$. This is certainly possible because we have freedom to choose $\dot{q}(2k)$ and $\dot{q}(2k + 1)$ independently and because we have forced the ranges of the relevant ladders to be disjoint above the rth rung (except for $\gamma = \eta_{\delta_1}(r) = \eta_{\delta_2}(r)$). $\square$

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