APPENDIX 2 [Sh 345b]

ENTANGLED ORDERS AND NARROW BOOLEAN ALGEBRAS

On works on far linear orders see Galvin Shelah [GlSh23] and earlier works of Sierpinski [Sr]. On entangled linear orders see Bonnet [Bo], Abraham Shelah [AbSh106], Abraham Rubin Shelah [ARS153]; Bonnet Shelah [BSh210] proved their existence in cf $(2^{\aleph_0})$ (and more general in cf(2^λ) if $2^{<\lambda} \le \lambda$ or more generally there is a linear order of cardinality $\le \lambda$ with 2^λ Dedekind cuts). The aim was to show the existence of narrow Boolean algebras, in fact ordered ones (as befitted a work done during the Oberwolfach Conference on Boolean Algebra). Todorcevic [To] independently proved this for another application: a Boolean algebra may satisfy the cf(2^λ)-c.c. whereas its square fails this. This applies to topologies too, but if you want to apply it to non-productivity of cellularity you need cf$(2^{\aleph_0})$ being successor.

For the generalization to (μ,κ)-entangled, the parallel for Ens and more see [Sh462] and subsequently Rosłanowski Shelah [RoSh534].

Definition 2.1 (1) Ens(λ,μ,κ) means: there are linear orderings $\langle \mathcal{I}_\alpha : \alpha < \kappa\rangle$ such that:

(a) $\mathcal{I}_\alpha$ is a linear order of power λ

(b) <u>if</u> $n < \omega$, $\alpha_1 < \cdots < \alpha_n < \kappa$, $w \subseteq \{1,...,n\}$, $t^\ell_\zeta \in \mathcal{I}_{\alpha_\ell}$ for $\zeta < \mu$, $\ell = 1,...,n$ and $[\zeta_1 \neq \zeta_2 \Rightarrow t^\ell_{\zeta_1} \neq t^\ell_{\zeta_2}]$, <u>then</u> for some $\zeta < \xi < \mu$:

$$[\ell \in w \Rightarrow t^\ell_\zeta < t^\ell_\xi]$$

$$[1 \le \ell \le n \wedge \ell \notin w \Rightarrow t^\ell_\zeta > t^\ell_\xi]$$

(2) Ens(λ,μ,k) is defined similarly but $n \le k$.

(3) If we omit μ, this means $\lambda = \mu$.

(4) A linear order $\mathcal{I}$ is (μ,n)-entangled if: ($\mathcal{I}$ has cardinality $\ge \mu$ and) for every pairwise distinct $t^\ell_\zeta \in \mathcal{I}(1 \le \ell \le n, \zeta < \mu)$ such that $t^1_\zeta < t^2_\zeta < \cdots < t^n_\zeta$ and $w \subseteq \{1,...,n\}$, there are $\zeta < \xi < \mu$ such that:

$$(*) \quad 1 \le \ell \le n \Rightarrow [\ell \in w \Leftrightarrow t^\ell_\zeta < t^\ell_\xi].$$

(5) We omit μ if $|\mathcal{I}| = \mu$; we omit n if it holds for all $n < \omega$.

Fact 2.2 (1) $\langle \mathcal{I} \rangle$ witnesses $\mathrm{Ens}(\lambda, \mu, 1)$ <u>iff</u> $\mathcal{I}$ is a linear order of power λ, with no monotonic sequence of length μ.

(2) $\langle \mathcal{I}, \mathcal{J} \rangle$ witnesses $\mathrm{Ens}(\lambda, \mu, 2)$ <u>iff</u> $\mathcal{I}, \mathcal{J}$ are linear orders of power λ, with no monotonic sequence of length μ, and $\mathcal{I}, \mathcal{J}$ are μ-far i.e. have no isomorphic subsets of power μ and $\mathcal{I}, \mathcal{J}^*$ are μ-far where $\mathcal{J}^*$ is the reverse order on $\mathcal{J}$.

(3) If $\mathcal{I}$ has density $< \mu$, $\mu = \mathrm{cf}\mu$, <u>then</u> in the definition (2.1(4),(5)) of "$\mathcal{I}$ is μ-entangled" we can add:

$$(*)' \quad t^\ell_\zeta < t^{\ell+1}_\xi, \quad t^\ell_\xi < t^{\ell+1}_\zeta \text{ for } \ell = 1, ..., n-1.$$

(4) If $n \geq 2$, $\mathcal{I}$ is (μ, n)-entangled, <u>then</u> $\mathcal{I}$ has density $< \mu$.

(5) If $\mathcal{I}$ is μ-entangled, $|\mathcal{I}| = \lambda$ <u>then</u> $\mathrm{Ens}(\lambda, \mu, |\mathcal{I}|)$.

(6) If $\mathcal{I}$ is μ-entangled, $\chi = \lambda^+$ or at least there are $A_i \in [\lambda]^\lambda$ for $i < \chi$, $[i \neq j \Rightarrow |A_i \cap A_j| < \lambda]$ <u>then</u> $\mathrm{Ens}(\lambda, \mu, \chi)$.

Proof: (1), (2) Check.

(3) Let $\mathcal{J} \in [\mathcal{I}]^{<\mu}$ be dense in $\mathcal{I}$. Suppose that

$$\langle \langle t^\ell_\zeta : \ell = 1, ..., n \rangle : \ \ell < \mu \rangle$$

is as in 2.1(4), (5). For each $\ell \in \{1, ..., n\}$, $t^\ell_\zeta < t^{\ell+1}_\zeta$, and so there exists $s^\ell_\zeta \in \mathcal{J}$ such that $t^\ell_\zeta \leq s^\ell_\zeta \leq t^{\ell+1}_\zeta$ (and at least one inequality is strict). Define functions h_0, h_1 on μ by:

$$h_0(\zeta) =: \langle s^1_\zeta, ..., s^{n-1}_\zeta \rangle$$

$$h_1(\zeta) =: \langle < TV(t^\ell_\zeta = s^\ell_\zeta), TV(t^{\ell+1}_\zeta = s^\ell_\zeta) >: \ell = 1, ..., n \rangle$$

(where TV(-) is the truth value of -).

Now $\mathrm{Dom}(h_0) = \mu$ and $|\mathrm{Rang}(h_0)| \leq |\mathcal{J}|^{n-1} < \mu$. Similarly for h_1. Since $\mathrm{cf}(\mu) = \mu$, there exists $A \in [\mu]^\mu$ such that $h_0 \restriction A$ and $h_1 \restriction A$ are constant. That's to say, for some $s^1, ..., s^{n-1}$ in $\mathcal{J}$, $\forall \ell \in \{1, ..., n-1\}$, $\forall \zeta \in A$,

$$t^\ell_\zeta \leq s^\ell = s^\ell_\zeta \leq t^{\ell+1}_\zeta.$$

Since the t^ℓ_ζ are given as pairwise distinct, using $h_1 \restriction A$, one finds that

$$t^\ell_\zeta < s^\ell < t^{\ell+1}_\zeta.$$

Without loss of generality $A = \mu$ (relabelling); now applying 2.1(4), there exist $\zeta < \xi < \mu$ such that $1 \leq \ell \leq n \Rightarrow [\ell \in w \Leftrightarrow t^\ell_\zeta < t^\ell_\xi]$, and in addition, for $\ell = 1, ..., n-1$,

$$t^\ell_\zeta < s^\ell_\zeta = s^\ell = s^\ell_\xi < t^{\ell+1}_\xi$$

and

$$t^\ell_\xi < s^\ell_\xi = s^\ell = s^\ell_\zeta < t^{\ell+1}_\zeta$$

so that $(*)'$ holds.

(4) W.l.o.g. $n = 2$.

Suppose that $\mathcal{I}$ has density at least μ. By induction on $\zeta < \mu$, choose t^1_ζ, t^2_ζ such that:

(i) $t^1_\zeta < t^2_\zeta$

(ii) $t^1_\zeta, t^2_\zeta \notin \{t^1_\xi, t^2_\xi : \xi < \zeta\}$

(iii) $(\forall \xi < \zeta)(\forall \ell \in \{1,2\})(t^1_\zeta < t^\ell_\xi \Leftrightarrow t^2_\zeta < t^\ell_\xi)$.

Continue to define for as long as possible.

There are two possible outcomes.

Outcome (a): One gets stuck at some $\zeta < \mu$. Define $\mathcal{J} =: \{t^1_\xi, t^2_\xi : \xi < \zeta\}$. So $(\forall t^1 < t^2 \in \mathcal{I}\backslash\mathcal{J})(\exists s \in \mathcal{J})(\neg t^1 < s \Leftrightarrow t^2 < s)$. Since t^1, $t^2 \notin \mathcal{J}$; it follows that $t^1 < s \wedge t^2 > s$ or $t^1 > s \wedge t^2 < s$. So $\mathcal{J}$ is dense in $\mathcal{I}$ and is of power $2|\zeta| < \mu$-a contradiction.

Outcome (b): one can define t^1_ζ, t^2_ζ for every $\zeta < \mu$. Then $\langle t^1_\zeta, t^2_\zeta : \zeta < \mu\rangle$, $w = \{1,2\}$ constitute an easy counterexample to the $(\mu, 2)$-entangledness of $\mathcal{I}$.

(5) $\mathcal{I}$ has λ pairwise disjoint subsets each of power λ, say $\langle \mathcal{I}_i : i < \lambda\rangle$, we shall prove that this sequence witness $\mathrm{Ens}(\lambda, \mu, |\mathcal{I}|)$; for suppose $n < \omega$, $i_1, ..., i_n < \lambda$ are distinct and let $t^\ell_\zeta \in \mathcal{I}_{i_\ell}$ for $\zeta < \mu$ be distinct. For each $\zeta < \mu$ define a linear order $<_\zeta$ on $\{1, ..., n\}$: $\ell < m$ iff $t^\ell_\zeta < t^m_\zeta$ (they are distinct as the $\mathcal{I}'_i s$ are pairwise disjoint). As there are only finitely many such linear order without loss of generality $<_\zeta = <_0$, so by renaming without loss of generality $t^1_\zeta < ... < t^n_\zeta$ for each ζ. Now apply "$\mathcal{I}$ is μ-entangled".

(6) Similar to the proof of part (5). $\square_{2.2}$

Fact 2.3 For a linear order $\mathcal{I}$ and regular uncountable cardinal μ, the following are equivalent:

(a) $\mathcal{I}$ is μ-entangled.

(b) $B = BA_{\text{inter}}(\mathcal{I})$ (the interval Boolean algebra) is μ-narrow; i.e. with no μ pairwise incomparable elements.

Proof: [35]: (a) $\Rightarrow$ (b).

By 2.2(4) $\mathcal{I}$ has density $< \mu$.

Let $\langle \tau_\zeta : \zeta < \mu\rangle$ be distinct elements of B. We know that for each ζ there are: an even $n(\zeta) < \omega$ and $t^1_\zeta < \cdots < t^{n(\zeta)}_\zeta$ in $\mathcal{I}$ such that $\tau_\zeta =$

[35] A reader who is happy to have the proof should thank O. Kolman for asking for it.

$\bigcup_{\ell=1}^{n(\zeta)/2}[t_\zeta^{2\ell-1}, t_\zeta^{2\ell})$ (more exactly without loss of generality $\mathcal{I}$ has a first element and we allow $t_\zeta^{n(\zeta)} = \infty$). As $\mathrm{cf}\mu > \aleph_0$, without loss of generality $n(\zeta) = n(*)$; now by 2.1(4) and 2.2(3) (and the Δ-system lemma) for some $\zeta < \xi$, for $\ell = 1, ..., n(*)/2$, $t_\zeta^{2\ell-1} \le t_\xi^{2\ell-1} < t_\xi^{2\ell} \le t_\zeta^{2\ell}$, hence $B \models \tau_\xi \subseteq \tau_\zeta$ as required.
[(b) ⇒ (a):] Note that $\mathcal{I}$ has density $< \mu$. [36].

So let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a dense subset of $\mathcal{I}$ of cardinality $< \mu$. For $\mathcal{J} \subseteq \mathcal{I}$ and $s < t$ from $\mathcal{J}$, we let $(s,t)_\mathcal{J} = \{r \in \mathcal{J} : s < r < t\}$.
Let $\mathcal{J} = \{t \in \mathcal{I} : if\ \ \mathcal{I} \models s < t$ then $|(s,t)_\mathcal{I}| = \mu$ and if $\mathcal{I} \models t < s$ then $|(t,s)_\mathcal{I}| = \mu\}$.
Clearly

$(*)_1$ $|\mathcal{I} \backslash \mathcal{J}| < \mu$ and if $s < t$ are in $\mathcal{J}$ then $|\{r \in \mathcal{J} : s < r < t\}| = \mu$.
[why?

(a) if $|\mathcal{I} \backslash \mathcal{J}| = \mu$, let $t_\zeta \in \mathcal{I} \backslash \mathcal{J}$ be distinct for $\zeta < \mu$, so for each ζ there is $s_\zeta \in \mathcal{I}$ such that

$$s_\zeta < t_\zeta \ \&\ |(s_\zeta, t_\zeta)_\mathcal{I}| < \mu \text{ or } t < s_\zeta \ \&\ |(t_\zeta, s_\zeta)_\mathcal{I}| < \mu.$$

We can replace $\{t_\zeta : \zeta < \mu\}$ by any subset of the same cardinality so without loss of generality $s_\zeta < t_\zeta \Leftrightarrow s_0 < t_0$. By symmetry assume $s_0 < t_0$ otherwise look at $\mathcal{I}^*$. For each ζ, as $\mathcal{I}_0$ is a dense subset of $\mathcal{I}$ there is $r_\zeta \in \mathcal{I}_0$ such that $s_\zeta \le r_\zeta \le t_\zeta$. As $|\mathcal{I}_0| < \mu = \mathrm{cf}\mu$ without loss of generality $r_\zeta = r$ for each ζ. So for $\zeta < \mu$

$$|\,[r, t_\zeta]_\mathcal{I}\,| \le |(s_\zeta, t_\zeta)_\mathcal{I}| + 2 < \mu$$

hence for each $\zeta < \mu$,

$$|\{\xi < \mu : t_\xi \le t_\zeta\}| \le |[r_\zeta, t_\zeta]| < \mu.$$

Clearly there is $h(\zeta) < \mu$ such that:

$$[\xi < \mu \ \&\ \xi \ge h(\zeta) \Rightarrow t_\zeta < t_\xi]$$

and

$$C = \{\xi : \xi < \mu, (\forall \zeta < \xi)(h(\zeta) < \xi\}$$

is a club of μ, so $\langle t_\zeta : \zeta \in C\rangle$ is strictly increasing, contradicting "$\mathcal{I}$ has density $< \mu$."

[36] $\mathcal{I}$ has no well ordered subset of power μ nor an inverse well ordered subset of power μ. So if $\mathcal{I}$ has density $\ge \mu$, then there are disjoint close-open intervals $\mathcal{I}_0, \mathcal{I}_1$ with density $\ge \mu$. Now for each $\mathcal{I}_m$ we choose by induction on $\zeta <$ density(I_m) elements $a_\zeta^m < b_\zeta^m$ from $\mathcal{I}_m$ such that $[a_\zeta^m, b_\zeta^m]$ is disjoint from $\{a_\xi^m, b_\xi^m : \xi < \zeta\}$. So $\xi < \zeta \Rightarrow [a_\xi^m, b_\xi^m] \not\subseteq [a_\zeta^m, b_\zeta^m]$. Now $\langle [a_\zeta^0, b_\zeta^0) \cup (\mathcal{I}_1 \setminus [a_\zeta^1, b_\zeta^1)) : \zeta < \mu\rangle$ shows B is not μ-narrow.

(b) $s < t$ are in $\mathcal{J} \Rightarrow |(s,t)_{\mathcal{J}}| = \mu$ because $t \in \mathcal{J}$ implies

$$\mu \leq |(s,t)_{\mathcal{I}}| \leq |(s,t)_{\mathcal{J}}| + |\mathcal{I} \backslash \mathcal{J}|,$$

but $|\mathcal{I} \backslash \mathcal{J}| < \mu$ so $\mu = |(s,t)_{\mathcal{J}}|$.]

$(*)_2$ there is a dense subset $\mathcal{J}_0$ of $\mathcal{J}$ of cardinality $< \mu$ [even easier].

Now let $t^\ell_\zeta \in \mathcal{I}$ be distinct for $\zeta < \mu$, $\ell = 1, ..., n$ and $w \subseteq \{1, ..., n\}$ and we should find $\zeta < \xi$ such that:

$$[\ell \in w \Rightarrow t^\ell_\zeta < t^\ell_\xi],\ [\ell \in \{1, ..., n\} \backslash w \Rightarrow t^\ell_\zeta > t^\ell_\xi].$$

We, of course, can replace $\{(t^1_\zeta, ..., t^n_\zeta) : \zeta < \mu\}$ by any subset of cardinality μ. So without loss of generality

$(*)_3$ no t^ℓ_ζ is first or last, and every t^ℓ_ζ is in $\mathcal{J}$ (as $|\mathcal{I} \backslash \mathcal{J}| < \mu$).

The rest is easy, too, though tiring. So for each ζ we can find

$$r^1_\zeta, ..., r^{n+1}_\zeta \in \mathcal{J}_0$$

such that

$$r^1_\zeta < t^1_\zeta < r^2_\zeta < t^2_\zeta < \cdots < t^n_\zeta < r^{n+1}_\zeta.$$

As $|\mathcal{J}_0| < \mu = \mathrm{cf}(\mu)$ without loss of generality $r^\ell_\zeta = r_\ell$ for every ℓ.

Let for each $\zeta < \mu$,

$$u_\zeta =: \left\{\ell : \ell \in \{1, ..., n\} \text{ and } \quad t^\ell_{2\zeta} < t^\ell_{2\zeta+1}\right\}$$

u_ζ has $\leq 2^n$ possible values. Without loss of generality $u_\zeta = u^*$ for every $\zeta < \mu$.

Note

$$\left[\ell \notin u_\zeta \ \&\ \ell \in \{1, ..., n\} \Rightarrow t^\ell_{2\zeta} > t^\ell_{2\zeta+1}\right]$$

(as $t^\ell_{2\zeta} \neq t^\ell_{2\zeta+1}$). For each $\zeta < \mu$, $\ell \in \{1, ..., n\}$ there is $p^\ell_\zeta \in \mathcal{J}_0$ such that $t^\ell_{2\zeta} \leq p^\ell_\zeta \leq t^\ell_{2\zeta+1}$ or $t^\ell_{2\zeta+1} \leq p^\ell_\zeta \leq t^\ell_{2\zeta+1}$. Without loss of generality $p^\ell_\zeta = p_\ell$ and the inequalities are strict.

Now we define by induction on $\zeta < \mu$, for every $\ell = \{1, ..., n\}$, members $q^{\ell,1}_\zeta$, $q^{\ell,2}_\zeta$, $q^{\ell,3}_\zeta$, $q^{\ell,4}_\zeta$ of $\mathcal{J}$ such that:

(i) if $\ell \in u_\zeta$ (i.e. $t^\ell_{2\zeta} < t^\ell_{2\zeta+1}$) then

$$r_\ell < q^{\ell,1}_\zeta < t^\ell_{2\zeta} < q^{\ell,2}_\zeta < p_\ell < q^{\ell,3}_\zeta < t^\ell_{2\zeta+1} < q^{\ell,4}_\zeta < r_{\ell+1}$$

(ii) if $\ell \notin u_\zeta$ (but $\ell \in \{1, ..., n\}$, i.e. $t^\ell_{2\zeta} > t^\ell_{2\zeta+1}$) then

$$r_\ell < q^{\ell,1}_\zeta < t^\ell_{2\zeta+1} < q^{\ell,2}_\zeta < p_\ell < q^{\ell,3}_\zeta < t^\ell_{2\zeta+1} < q^{\ell,4}_\zeta < r_{\ell+1}$$

(iii) $q^{\ell,m}_\zeta (m \in \{1,2,3,4\})$ does not belong to

$$\Big\{q^{k,i}_\xi : \xi < \zeta, k \in \{1,...,n\}, i \in \{1,...,4\}\Big\} \cup \big\{t^\ell_\xi : \xi < \zeta, \ell \in \{1,...,n\}\big\}.$$

There are no problems by $(*)_1$. It is still possible that for some $\zeta < \xi$,

$$\emptyset \neq \{q^{\ell,m}_\zeta : \ell = 1,...,n \text{ and } m = 1,2,3,4\} \cap \{t^\ell_\xi : \ell = 1,...,n\}$$

for each ζ there are at most 4n such $\xi's$, so there is $h_1(\zeta) < \mu$ such that $h_1(\zeta) \leq \xi < \mu \Rightarrow \bigwedge_{\ell,m} \bigwedge_k q^{\ell,m}_\zeta \neq t^k_\xi$. So without loss of generality

$(*)_4$ the sets $\{q^{\ell,m}_\zeta, t^\ell_\zeta : \ell = 1,...,n \text{ and } m = 1,2,3,4\}$ for $\zeta < \mu$ are pairwise disjoint.

Now we define for every $\zeta < \mu$, a sequence $\langle s^\ell_\zeta : \ell = 1,...,4n\rangle$ by defining $s^{4\ell-3}_\zeta$, $s^{4\ell-2}_\zeta$, $s^{4\ell-3}_\zeta$, $s^{4\ell-4}_\zeta$ for each $\ell \in \{1,...,n\}$ as follows:

Case 1: $\ell \in w$, $\ell \in u^*$
$\langle s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta, s^{4\ell-1}_\zeta, s^{4\ell}_\zeta\rangle = \langle t^\ell_{2\zeta}, q^{\ell,2}_\zeta, q^{\ell,3}_\zeta, t^\ell_{2\zeta+1}\rangle$.

Case 2: $\ell \notin w$, $\ell \in u^*$
$\langle s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta, s^{4\ell-1}_\zeta, s^{4\ell}_\zeta\rangle = \langle q^{\ell,1}_\zeta, t^\ell_{2\zeta}, t^\ell_{2\zeta+1}, q^{\ell,4}_\zeta\rangle$.

Case 3: $\ell \in w$, $\ell \notin u^*$
$\langle s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta, s^{4\ell-1}_\zeta, s^{4\ell}_\zeta\rangle = \langle q^{\ell,1}_\zeta, t^\ell_{2\zeta+1}, t^\ell_{2\zeta}, q^{\ell,4}_\zeta\rangle$.

Case 4: $\ell \notin w$, $\ell \notin u^*$
$\langle s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta, s^{4\ell-1}_\zeta, s^{4\ell}_\zeta\rangle = \langle t^\ell_{2\zeta+1}, q^{\ell,2}_\zeta, q^{\ell,3}_\zeta, t^\ell_{2\zeta}\rangle$.

Clearly for $\zeta < \mu$, $s^1_\zeta < \cdots < s^{4n}_\zeta$ and the s^ℓ_ζ are pairwise distinct (by $(*)_4$) and

$$\mathcal{I} \models r_1 < s^1_\zeta < s^2_\zeta < p_1 < s^3_\zeta < s^4_\zeta < r_2 < s^5_\zeta < s^6_\zeta < p_2 < s^7_\zeta < s^8_\zeta < r_3 < \cdots.$$

Now for each ζ we define an element x_ζ of the Boolean algebra $BA(\mathcal{I})$:

$$x_\zeta = \bigcup_{\ell=1}^{2n} [s^{2\ell-1}_\zeta, s^{2\ell}_\zeta).$$

Note

$(*)_5$ for $\ell = 1,...,n$:
(a) $x_\zeta \cap [r_\ell, p_\ell) = [s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta)$
(b) $x_\zeta \cap (p_\ell, r_{\ell+1}) = [s^{4\ell-1}_\zeta, s^{4\ell}_\zeta)$.

So $\langle x_\zeta : \zeta < \mu\rangle$ is a sequence of μ members of the Boolean algebra $BA(\mathcal{I})$. By the assumption (we are proving (b) $\Rightarrow$ (a) in fact 2.3 for some $\zeta < \xi < \mu$. x_ζ, x_ξ are comparable members of $BA(\mathcal{I})$; i.e. $x_\zeta \subseteq x_\xi$ or $x_\xi \subseteq x_\zeta$. We derived our desired conclusion according to the case.

CASE A: $x_\zeta \subseteq x_\xi$.

In this case we shall prove that $2\zeta+1$, $2\xi+1$ are the ordinals we are looking for; i.e. conditions (α), (β), (γ) below holds, and we shall check those thus finishing this case.

Condition α : $2\zeta+1 < 2\xi+1$.

[Trivial by $\zeta < \xi$].

Condition β : if $\ell \in w$ then $t^\ell_{2\zeta+1} < t^\ell_{2\xi+1}$.

Possibility $\beta 1$: $\ell \in u^*$.

Then $t^\ell_{2\zeta+1} = s^{4\ell}_\zeta$, $t^\ell_{2\xi+1} = s^{4\ell}_\xi$ (by Case 1 in the definition of the s's), now by $(*)_5(b)$:

$$x_\zeta \cap [p_\ell, r_{\ell+1}) = [s^{4\ell-1}_\zeta, s^{4\ell}_\zeta)$$

hence (by Case 1 above)

$$x_\zeta \cap [p_\ell, r_{\ell+1}) = [q^{\ell,3}_\zeta, t^\ell_{2\zeta+1})$$

and

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [s^{4\ell-1}_\xi, s^{4\ell}_\xi)$$

hence (by Case 1 above)

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [q^{\ell,3}_\xi, t^\ell_{2\xi+1}).$$

But as we are in CASE A, $x_\zeta \subseteq x_\xi$ hence $x_\zeta \cap [p_\ell, r_{\ell+1}) \subseteq x_\xi \cap [p_\ell, r_{\ell+1})$ which means by the previous sentence $[q^{\ell,3}_\zeta, t^\ell_{2\zeta+1}) \subseteq [q^{\ell,3}_\xi, t^\ell_{2\xi+1})$ which implies $q^{\ell,3}_\xi \leq q^{\ell,3}_\zeta$ and $t^\ell_{2\zeta+1} \leq t^\ell_{2\xi+1}$. But $t^\ell_{2\zeta+1} \neq t^\ell_{2\xi+1}$ (as $\zeta \neq \xi$) so $t^\ell_{2\zeta+1} < t^\ell_{2\xi+1}$ as required.

Possibility $\beta 2$: $\ell \notin u^*$.

Then $t^\ell_{2\zeta+1} = s^{4\ell-2}_\zeta$, $t^\ell_{2\xi+1} = s^{4\ell-2}_\xi$ (by Case 3 in the definition of the s's) now by $(*)_5(a)$:

$$x_\zeta \cap [r_\ell, p_\ell) = [s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta)$$

hence (by Case 3 above)

$$x_\zeta \cap [r_\ell, p_\ell) = [q^{\ell,1}_\zeta, t^\ell_{2\zeta+1})$$

and

$$x_\xi \cap [r_\ell, p_\ell) = [s^{4\ell-3}_\xi, s^{4\ell-2}_\xi)$$

hence (by Case 3 above)

$$x_\xi \cap [r_\ell, p_\ell) = [q^{\ell,1}_\xi, t^\ell_{2\xi+1}).$$

But as we are in CASE A, $x_\zeta \subseteq x_\xi$ hence $x_\zeta \cap [r_\ell, p_\ell) \subseteq x_\xi \cap [r_\ell, p_\ell)$ which means by the previous sentence $[q^{\ell,1}_\zeta, t^\ell_{2\zeta+1}) \subseteq [q^{\ell,1}_\xi, t^\ell_{2\xi+1})$ which implies

$q_\zeta^{\ell,1} \geq q_\xi^{\ell,1}$ and $t^\ell_{2\zeta+1} \leq t^\ell_{2\xi+1}$. But $t^\ell_{2\zeta+1} \neq t^\ell_{2\xi+1}$ (as $\zeta \neq \xi$) so $t^\ell_{2\zeta+1} < t^\ell_{2\xi+1}$ as required.

Condition γ: If $\ell \notin w$ (but $\ell \in \{1, ..., n\}$ then $t^\ell_{2\zeta+1} > t^\ell_{2\xi+1}$.

Possibility $\gamma 1$: $\ell \in u^*$.

Then $t^\ell_{2\zeta+1} = s_\zeta^{4\ell-1}$, $t^\ell_{2\xi+1} = s_\xi^{4\ell-1}$ (by Case 2 in the definition of the s's). Now by $(*)_5$(b):

$$s_\zeta \cap [p_\ell, r_{\ell+1}) = [s_\zeta^{4\ell-1}, s_\zeta^{4\ell})$$

hence (by Case 2 above)

$$x_\zeta \cap [p_\ell, r_{\ell+1}) = [t^\ell_{2\zeta+1}, q_\zeta^{\ell,4})$$

and

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [s_\xi^{4\ell-1}, s_\xi^{4\ell})$$

hence (by Case 2 above)

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [t^\ell_{2\xi+1}, q_\xi^{\ell,4}).$$

But as we are in CASE A, $x_\zeta \subseteq x_\xi$ hence $x_\xi \cap [p_\ell, r_{\zeta+1}) \subseteq x_\xi \cap [p_\ell, r_{\ell+1})$ which means by the previous sentence $[t^\ell_{2\zeta+1}, q_\zeta^{\ell,4}) \subseteq [t^\ell_{2\xi+1}, q_\xi^{\ell,4})$ which implies $t^\ell_{2\zeta+1} \geq t^\ell_{2\xi+1}$ and $q_\xi^{\ell,4} \geq q_\zeta^{\ell,4}$. But $t^\ell_{2\zeta+1} \neq t^\ell_{2\xi+1}$ (as $\zeta \neq \xi$) so $t^\ell_{2\zeta+1} > t^\ell_{2\xi+1}$ as required.

Possibility $\gamma 2$: $\ell \notin u^*$.

Then $t^\ell_{2\zeta+1} = s_\zeta^{4\ell-3}$, $t^\ell_{2\xi+1} = s_\xi^{4\ell-3}$ (by Case 4 in the definition of the s's) now by $(*)_5$(a):

$$x_\zeta \cap [r_\ell, p_\ell) = [s_\zeta^{4\ell-3}, s_\zeta^{4\ell-2})$$

hence (by Case 4 above)

$$x_\zeta \cap [r_\ell, p_\ell) = [t^\ell_{2\zeta+1}, q_\zeta^{\ell,2})$$

and

$$x_\xi \cap [r_\ell, p_\ell) = [s_\xi^{4\ell-3}, s_\xi^{4\ell-2})$$

hence (by Case 4 above)

$$x_\xi \cap [r_\ell, p_\ell) = [t^\ell_{2\xi+1}, q_\xi^{\ell,2}).$$

But as we are in CASE A, $x_\zeta \subseteq x_\xi$ hence $x_\zeta \cap [r_\ell, p_\ell) \subseteq x_\xi \cap [r_\ell, p_\ell)$ which means by the previous sentence $[t^\ell_{2\zeta+1}, q_\zeta^{\ell,2}) \subseteq [t^\ell_{2\xi+1}, q_\xi^{\ell,2})$ which implies

$t^\ell_{2\xi+1} \le t^\ell_{2\zeta+1}$ and $q^{\ell,2}_\zeta \le q^{\ell,2}_\xi$. But $t^\ell_{2\zeta+1} \ne t^\ell_{2\xi+1}$ (as $\zeta \ne \xi$) so $t^\ell_{2\xi+1} < t^\ell_{2\zeta+1}$ as required.

<u>CASE B:</u> $x_\xi \subseteq x_\zeta$.

In this case we shall prove that 2ζ, 2ξ are a pair of ordinals we are looking for; i.e. conditions (α), (β), (γ) below holds and we shall check those, thus finishing this case (hence the proof of 2.3).

<u>Condition α:</u> $2\zeta < 2\xi$.

[Trivial by $\zeta < \xi$].

<u>Condition β :</u> if $\ell \in w$ then $t^\ell_{2\zeta} < t^\ell_{2\xi}$.

<u>Possibility $\beta 1$:</u> $\ell \in u^*$.

Then $t^\ell_{2\zeta} = s^{4\ell-3}_\zeta$, $t^\ell_{2\xi} = s^{4\ell-3}_\xi$ (by Case 1 in the definition of the s's); now by $(*)_5$(a):

$$x_\zeta \cap [r_\ell, p_\ell) = [s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta)$$

hence (by Case 1 above)

$$x_\zeta \cap [r_\ell, p_\ell) = [t^\ell_{2\zeta}, q^{\ell,2}_\zeta)$$

and

$$x_\xi \cap [r_\ell, p_\ell) = [s^{4\ell-3}_\xi, s^{4\ell-2}_\xi)$$

hence (by Case 1 above)

$$x_\xi \cap [r_\ell, p_\ell) = [t^\ell_{2\xi}, q^{\ell,2}_\xi).$$

But as we are in CASE B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [r_\ell, p_\ell) \supseteq x_\xi \cap [r_\ell, p_\ell)$ which means by the previous sentence $[t^\ell_{2\zeta}, q^{\ell,2}_\zeta) \supseteq [t^\ell_{2\xi}, q^{\ell,2}_\xi)$ which implies $t^\ell_{2\zeta} \le t^\ell_{2\xi}$ and $q^{\ell,2}_\xi \le q^{\ell,2}_\zeta$. But $t^\ell_{2\zeta} \ne t^\ell_{2\xi}$ (as $\zeta \ne \xi$), so $t^\ell_{2\zeta} < t^\ell_{2\xi}$ as required.

<u>Possibility $\beta 2$:</u> $\ell \notin u^*$ (but $\ell \in \{1, ..., n\}$.

Then $t^\ell_{2\zeta} = s^{4\ell-1}_\zeta$, $t^\ell_{2\xi} = s^{4\ell-1}_\zeta$ (by Case 3 in the definition of the s's); now by $(*)_5$(b):

$$x_\zeta \cap [p_\ell, r_{\ell+1}) = [s^{4\ell-1}_\zeta, s^{4\ell}_\zeta)$$

hence (by Case 3 above)

$$x_\zeta \cap [p_\ell, r_{\ell+1}) = [t^\ell_{2\zeta}, q^{\ell,4}_\xi)$$

and

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [s^{4\ell-1}_\xi, s^{4\ell}_\xi)$$

hence (by Case 3 above)

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [t^\ell_{2\xi}, q^{\ell,4}_\zeta).$$

But as we are in CASE B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [p_\ell, r_{\ell+1}) \supseteq x_\xi \cap [p_\ell, r_{\ell+1})$ which means by the previous sentence $[t^\ell_{2\zeta}, q^{\ell,4}_\zeta) \supseteq [t^\ell_{2\xi}, q^{\ell,4}_\zeta)$ which implies $t^\ell_{2\zeta} \leq t^\ell_{2\xi}$ and $q^{\ell,4}_\xi \leq q^{\ell,4}_\zeta$. But $t^\ell_{2\zeta} \neq t^\ell_{2\xi}$ (as $\zeta \neq \xi$), so $t^\ell_{2\zeta} < t^\ell_{2\xi}$ as required.

<u>Condition γ:</u> if $\ell \notin w$ (but $\ell \in \{1, ..., n\}$) then $t^\ell_{2\zeta} > t^\ell_{2\xi}$.

<u>Possibility $\gamma 1$:</u> $\ell \in u^*$.

Then $t^\ell_{2\zeta} = s^{4\ell-2}_\zeta$, $t^\ell_{2\xi} = s^{4\ell-2}_\zeta$ (by Case 2 in the definition of the s's); now by $(*)_5$(a):

$$x_\zeta \cap [r_\ell, p_\ell) = [s^{4\ell-3}_\zeta, s^{4\ell-2}_\zeta)$$

hence (by Case 2 above)

$$x_\zeta \cap [r_\ell, p_\ell) = [q^{\ell,1}_\zeta, t^\ell_{2\zeta})$$

and

$$x_\xi \cap [r_\ell, p_\ell) = [s^{4\ell-2}_\xi, s^{4\ell-2}_\xi)$$

hence (by Case 2 above)

$$x_\xi \cap [r_\ell, p_\ell) = [q^{\ell,1}_\xi, t^\ell_{2\xi}).$$

But as we are in CASE B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [r_\ell, p_\ell) \supseteq x_\xi \cap [r_\ell, p_\ell)$ which means by the previous sentence $[q^{\ell,1}_\zeta, t^\ell_{2\zeta}) \supseteq [q^{\ell,1}_\xi, t^\ell_{2\xi})$ which implies $q^{\ell,1}_\zeta \leq q^{\ell,1}_\xi$ and $t^\ell_{2\xi} \leq t^\ell_{2\zeta}$. But $t^\ell_{2\zeta} \neq t^\ell_{2\xi}$ (as $\zeta \neq \xi$), so $t^\ell_{2\xi} < t^\ell_{2\zeta}$ as required.

<u>Possibility $\gamma 2$:</u> $\ell \notin u^*$.

Then $t^\ell_{2\zeta} = s^{4\ell}_\zeta$, $t^\ell_{2\xi} = s^{4\ell}_\zeta$ (by Case 4 in the definition of the s's); now by $(*)_5$(b):

$$x_\zeta \cap [p_\ell, r_{\ell+1}) = [s^{4\ell-1}_\zeta, s^{4\ell}_\zeta)$$

hence (by Case 4 above)

$$x_\zeta \cap [p_\ell, r_{\ell+1}) = [q^{\ell,3}_\zeta, t^\ell_{2\zeta})$$

and

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [s^{4\ell-1}_\xi, s^{4\ell}_\xi)$$

hence (by Case 4 above)

$$x_\xi \cap [p_\ell, r_{\ell+1}) = [q^{\ell,3}_\xi, t^\ell_{2\xi}).$$

But as we are in Case B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [p_\ell, r_{\ell+1}) \supseteq x_\xi \cap [p_\ell, r_{\ell+1})$ which means by the previous sentence $[q^{\ell,3}_\zeta, t^\ell_{2\zeta}) \supseteq [q^{\ell,3}_\xi, t^\ell_{2\xi})$ which implies $q^{\ell,3}_\zeta \leq q^{\ell,3}_\xi$ and $t^\ell_{2\xi} \leq t^\ell_{2\xi}$. But $t^\ell_{2\zeta} \neq t^\ell_{2\xi}$ (as $\zeta \neq \xi$), so $t^\ell_{2\xi} < t^\ell_{2\zeta}$ as required.

So we finish the proof of 2.3.

Theorem 2.4 *(1) There is an entangled linear order $A \subseteq \mathbb{R}$ of power* $\mathrm{cf}\left(2^{\aleph_0}\right)$.

(2) Generalization to higher cardinals: if there is a linear order of power 2^λ *and density λ (for example λ strong limit), then there is an entangled linear order of power* $\mathrm{cf}(2^\lambda)$ *and density λ.*

Proof: Done independently by Bonnet Shelah [BSh210], Todorcevic [To]. As its use in the book is marginal we do not include a proof.