THE PAIR ($\aleph_n, \aleph_0$) MAY FAIL $\aleph_0$-COMPACTNESS

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Abstract. Let P be a distinguished unary predicate and $\mathbf{K} = \{M : M$ a model of cardinality $\aleph_n$ with P^M of cardinality $\aleph_0\}$. We prove that consistently for $n = 4$, that for some countable first order theory T we have: T has no model in $\mathbf{K}$ whereas every finite subset of T has a model in $\mathbf{K}$. We then show how we prove it also for $n = 2$.

ANNOTATED CONTENTS

§0. Introduction. Interest in two cardinal models comes from the early days of model theory, as generalizations of the Lowenheim-Skolem theorem. Already Mostowski [Mo57] considered a related problem concerning generalized quantifiers. Let us introduce the problem. Throughout the paper λ, μ and κ stand for infinite cardinals and n, k for natural numbers.

We consider a countable language = vocabulary τ with a distinguished unary relation symbol P and models M for τ; i.e., τ-models.

0.1. NOTATION. We let

$$K_{(\lambda,\mu)} =: \{M : \|M\| = \lambda \ \& \ |P^M| = \mu\}.$$

0.2. DEFINITION. (1) We say that $K_{(\lambda,\mu)}$ is $(< \kappa)$-compact when every first order theory T in the vocabulary τ (i.e., in the first order logic $\mathbb{L}(\tau)$) with $|T| < \kappa$, satisfies:

if every finite $t \subseteq T$ has a model in $K_{(\lambda,\mu)}$, then T has a model in $K_{(\lambda,\mu)}$.

We similarly give the meaning to $(\le \kappa)$-compactness. We say that (λ, μ) is $(< \kappa)$-compact if $K_{(\lambda,\mu)}$ is.

(2) We say that

$$(\lambda, \mu) \to'_\kappa (\lambda', \mu')$$

when for every first order theory T in $\mathbb{L}(\tau)$ with $|T| < \kappa$, if every finite $t \subseteq T$ has a model in $K_{(\lambda,\mu)}$, then T has a model in $K_{(\lambda',\mu')}$. Instead "κ^+" we may write "$\le \kappa$".

(3) We say that

$$(\lambda, \mu) \to_\kappa (\lambda', \mu')$$

when for every first order theory T of L with $|T| < \kappa$, if T has a model in $K_{(\lambda,\mu)}$, then T has a model in $K_{(\lambda',\mu')}$.

(4) In both $\to'_\kappa$ and $\to_\kappa$ we omit κ if $\kappa = \aleph_0$.

NOTE. Note that $\to_\kappa$ is transitive and $\to'_\kappa$ is as well. Also note that $\to_{\aleph_0}$ and $\to'_{\aleph_0}$ are equivalent.

We consider the problem of $K_{(\lambda,\mu)}$ being compact. Before we start, we review the history of the problem. Note that a related problem is the one of completeness, i.e., if

$$\{\psi: \psi \text{ has a model in } K_{(\lambda,\mu)}\}$$

is recursively enumerable and other related problems, see in the end. We do not concentrate on those problems here.

We review some of the history of the problem, in an order which is not necessarily chronological.

Some early results on the compactness are due to Furkhen [Fu65]. He showed that

(A) if $\mu^\kappa = \mu$, then $K_{(\lambda,\mu)}$ is $(\leq \kappa)$-compact.

The proof is by using ultraproducts over regular ultrafilters on κ, generalizing the well known proof of compactness by ultrapowers. Morley related result is

(B) (Morley [Mo68]) If $\mu^{\aleph_0} \leq \mu' \leq \lambda' \leq \lambda$, then $(\lambda,\mu) \to_{\leq\lambda} (\lambda',\mu')$.

Next result we mention is one of Silver [Si71] concerning Kurepa trees,

(C) (Silver [Si71]) From the existence of a strongly inaccessible cardinal, it follows that the following is consistent with ZFC:

$$GCH + (\aleph_3, \aleph_1) \not\to'_{\aleph_0} (\aleph_2, \aleph_0).$$

Using special Aronszajn trees Mitchell showed

(D) (Mitchell [Mi72]) From the existence of a Mahlo cardinal, it follows that it is consistent with ZFC to have

$$(\aleph_1, \aleph_0) \not\to_{\aleph_2} (\aleph_2, \aleph_1).$$

A later negative consistency result is the one of Schmerl in [Sc74]

(E) (Schmerl [Sc74]) Con(if $n < m$ then $(\aleph_n, \aleph_{n+1}) \not\to (\aleph_m, \aleph_{m+1})$).

Earlier, Vaught proved two positive results

(F) (Vaught [MV62]) $(\lambda^+, \lambda) \to'_{\aleph_1} (\aleph_1, \aleph_0)$.

Keisler [Ke66] and [Ke66a] has obtained more results in this direction.

(G) (Vaught [Va65]) If $\lambda \geq \beth_\omega(\mu)$ and $\lambda' > \mu'$, then $(\lambda,\mu) \to'_{\leq\mu'} (\lambda',\mu')$.

In [Mo68] Morley gives another proof of this result, using Erdős-Rado Theorem and indiscernibles.

Another early positive result is the one of Chang:

(H) (Chang [Ch]) If $\mu = \mu^{<\mu}$ then $(\lambda^+, \lambda) \to'_{\leq\mu} (\mu^+, \mu)$.

Jensen in [Jn] uses $\square_\mu$ to show

(I) (Jensen [Jn]) If $\mathbf{V} = \mathbf{L}$, then $(\lambda^+, \lambda) \to'_{\leq\mu} (\mu^+, \mu)$. (The fact that $0^\#$ does not exist suffices.)

Hence, Jensen's result deals with the case of μ is singular, which was left open after the result of Chang. For other early consistency results concerning gap-1 two cardinal theorems, including consistency, see [Sh:269], Cummings, Foreman and Magidor [CFM].

In [Jn] there is actually a simplified proof of (I) due to Silver. A further result of Jensen, using morasses, is:

(J) (Jensen, see [De73] for $n = 2$) If $\mathbf{V} = \mathbf{L}$, then $(\lambda^{+n}, \lambda) \to'_{\leq \mu} (\mu^{+n}, \mu)$ for all $n < \omega$.

Note that by Vaught's result [MV62] stated in (F) we have: the statement in (I), in the result of Chang etc., (λ^+, λ) can be without loss of generality replaced by $(\aleph_1, \aleph_0)$.

Finally, there are many more related results, for example the ones concerning Chang's conjecture. A survey article on the topic was written by Schmerl in [Sc74].

Note that many of the positive rsults above (F)–(J), their proof also gives compactness of the pair, e.g., $(\aleph_0, \aleph_1)$ by [MV62].

We now mention some results of the author which will have a bearing to the present paper.

(α) (Shelah [Sh:8] and the abstract [Sh:E17]). If $K_{(\lambda,\mu)}$ is $(\leq \aleph_0)$-compact, then $K_{(\lambda,\mu)}$ is $(\leq \mu)$-compact and $(\lambda, \mu) \to_{\leq \mu'} (\lambda', \mu')$ when $\lambda \leq \lambda' \leq \mu' \leq \mu$.

More than $(\leq \mu)$-compactness cannot hold for trivial reasons. In the same work we have the analogous result on $\to'$ and:

(β) (Shelah [Sh:8] and the abstract [Sh:E17]) $(\lambda, \mu) \to'_{\aleph_1} (\lambda', \mu')$ is actually a problem on partition relations, (see below), also it implies $(\lambda, \mu) \to'_{\leq \mu'} (\lambda', \mu')$ see 0.4(1) below.

We state a definition from [Sh:8] that will be used here too. We do not consider the full generality of [Sh:8], there problems like considering K with several λ_ℓ-like $(P_\ell^2, <_\ell)$ and $|P_\ell^1| = \mu_\ell$ were addressed.

(We can use below only ordered a and increase h, it does not matter much.)

0.3. DEFINITION. (1) An *identity*[1] is a pair (a, e) where a is a finite set and e is an equivalence relation on the finite subsets of a, having the property

$$b \, e \, c \Rightarrow |b| = |c|.$$

The equivalence class of b with respect to e will be denoted b/e.

(2) We say that $\lambda \to (a, e)_\mu$, if for every $f : [\lambda]^{<\aleph_0} \to \mu$, there is $h : a \xrightarrow{1-1} \lambda$ such that

$$b \, e \, c \Rightarrow f(h''(b)) = f(h''(c)).$$

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[1]identification in the terminology of [Sh:8].

(3) We define

$$ID(\lambda,\mu) =: \{(n,e)\colon n < \omega \ \& \ (n,e) \text{ is an identity and } \lambda \to (n,e)_\mu\}$$

and for $f\colon [\lambda]^{<\aleph_0} \to X$ we let

$$ID(f) =: \{(n,e)\colon (n,e) \text{ is an identity such that for some}$$
$$\text{one-to-one function } h \text{ from } n = \{0,\ldots,n-1\} \text{ to } \lambda$$
$$\text{we have } (\forall b, c \subseteq n)(b\,e\,c \Rightarrow f(h''(b)) = f(h''(c)))\}$$

0.4. Claim (Shelah [Sh:8] and the abstract [Sh:E17]). $(\lambda,\mu) \to'_{\aleph_1} (\lambda',\mu')$ *is equivalent to the existence of a function* $f\colon [\lambda']^{<\aleph_0} \to \mu'$ *such that*

$$ID(f) \subseteq ID(\lambda,\mu)$$

(*more on this see* [Sh:74, Theorem 3] *statement there on* $\to'_{\aleph_1}$, *see details in* [Sh:E28]).

0.5. Remark. The identities of $(\beth_\omega,\aleph_0)$ are clearly characterized by Morley's proof of Vaught's theorem (see [Mo68]). The identities of $(\aleph_\omega,\aleph_0)$ are stated explicitly in [Sh:37] and [Sh:49], when $\aleph_\omega \le 2^{\aleph_0}$ where it is also shown that $(\aleph_\omega,\aleph_0) \to' (2^{\aleph_0},\aleph_0)$. For $(\aleph_1,\aleph_0)$, the identities are characterized in [Sh:74] (for some details see [Sh:E28]). The identities for λ-like models, λ strongly ω-Mahlo are clear, see Schmerl and Shelah [ScSh:20] (for strongly n-Mahlo this gives positive results, subsequently sharpened (replacing $n+2$ by n) and the negative results proved by Schmerl, see [Sc85]).

By the referee request we indicate the proof for $(\aleph_\omega,\aleph_0)$ in 5.12.

We generally neglect here three cardinal theorems and λ-like models (and combinations, see [Sh:8], [Sh:18]), the positive results (like 0.4) are similar. Recently Shelah and Vaananen deal with recursiveness, completeness, and identities [ShVa:790] and see [ShVa:E47].

In Gilchrist, Shelah [GcSh:491] and [GcSh:583], we dealt with 2-identities.

0.6. Definition. (1) A two-identity or 2-identity[2] is a pair (a,e) where a is a finite set and e is an equivalence relation on $[a]^2$. Let $\lambda \to (a,e)_\mu$ mean $\lambda \to (a,e^+)_\mu$ where $b\,e^+c \leftrightarrow (bec) \vee (b = c \subseteq a)$ for any $b,c \subseteq a$.
(2) We defined

$$ID_2(\lambda,\mu) =: \{(n,e)\colon (n,e) \text{ is a 2-identity and } \lambda \to (n,e)_\mu\}$$

we define $ID_2(f)$ when $f\colon [\lambda]^2 \to X$ as

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[2]It is not an identity as e is an equivalence relation on too small set.

$\{(n, e):\ (n, e)$ is a two-identity such that for some h,

   a one-to-one function from $\{0, \ldots, n-1\}$ into λ

   we have $\{\ell_1, \ell_2\}e\{k_1, k_2\}$ implies that $\ell_1 \neq \ell_2 \in \{0, \ldots, n-1\}$,

   $k_1 \neq k_2 \in \{0, \ldots, n-1\}$ and

$$f(\{h(\ell_1), h(\ell_2)\}) = f(\{h(k_1), h(k_2)\})\}.$$

(3) Let us define

$$ID_2^{\circledast} =: \{(^n2, e):\ (^n2, e)\text{ is a two-identity and if}\{\eta_1, \eta_2\} \neq \{v_1, v_2\}\text{ are }\subseteq {}^n2,$$

$$\text{then }\{\eta_1, \eta_2\}e\{v_1, v_2\} \Rightarrow \eta_1 \cap \eta_2 = v_1 \cap v_2\}.$$

By [Sh:49], under the assumption $\aleph_\omega < 2^{\aleph_0}$, the families $ID_2(\aleph_\omega, \aleph_0)$ and $ID_2^{\circledast}$ coincide (up to an isomorphism of identities). In Gilchrist and Shelah [GcSh:491] and [GcSh:583] we considered the question of the equality between these $ID_2(2^{\aleph_0}, \aleph_0)$ and $ID_2^{\circledast}$ under the assumption $2^{\aleph_0} = \aleph_2$. We showed that consistently the answer may be "yes" and may be "no".

Note that $(\aleph_n, \aleph_0) \nrightarrow (\aleph_\omega, \aleph_0)$ so $ID(\aleph_2, \aleph_0) \neq ID(\aleph_\omega, \aleph_0)$, but for identities for pairs (i.e., ID_2) the question is meaningful.

The history of the problem suggested to me that there should be a model where $K_{(\lambda,\mu)}$ is not $\aleph_0$-compact for some λ, μ; I do not know about the opinion of others and it was not easy for me as I thought a priori. As mathematicians do not feel that a strong expectation makes a proof, I was quite happy to be able to prove the existence of such a model. This was part of my lectures in a 1995 seminar in Jerusalem and notes of the lecture were taken by Mirna Džamonja and I thank her for this, but because the proof was not complete, it was delayed.

The following is the main result of this paper (proved in 3.5):

0.7. Main Theorem. Con(the pair $(\aleph_n, \aleph_0)$ is not $\aleph_0$-compact $+ 2^{\aleph_0} \geq \aleph_n$) *for $n \geq 4$.*

Later in the paper we deal with the case $n = 2$ which is somewhat more involved. This is the simplest case by a reasonable measure: if you do not like to use large cardinals then assuming that there is no inaccessible in $\mathbf{L}$, all pairs (μ^+, μ) are known to be $\aleph_0$-compact and if $\mathbf{V} = \mathbf{L}$ also all logic $L(\exists^{\geq\lambda})$, $\lambda > \aleph_0$ are (by putting together already known results; $\mathbf{V} = \mathbf{L}$ is used just to imply that there is no limit, uncountable not strong limit cardinal).

How much this consistency result will mean to a model theorist, let us not elaborate, but instead say an anecdote about Jensen. He is reputed to have said: "When I started working on the two-cardinal problem, I was told it was the heart of model theory. Once I succeeded to prove something, they told me what I did was pure set theory, and were not very interested"; also, mathematics is not immune to fashion changes.

My feeling is that there are probably more positive theorems in this subject waiting to be discovered. Anyway, let us state the following

THESIS. Independence results help us clear away the waste, so the possible treasures can stand out.

Of course, I have to admit that, having spent quite some time on the independence results, I sometimes look for the negative of the picture given by this thesis.

The strategy of our proof is as follows. It seems natural to consider the simplest case, i.e., that of two-place functions, and try to get the incompactness by constructing a sequence $\langle f_k : k < \omega \rangle$ of functions from $[\aleph_n]^2$ into $\aleph_0$ such that for all n we have $ID_2(f_k) \supseteq ID_2(f_{k+1})$, yet for no $f : [\aleph_n]^2 \to \aleph_0$ do we have $ID_2(f) \subseteq \bigcap_{k<\omega} ID_2(f_k)$. This suffices. Related proofs to our main results were [Sh:522].

Note that another interpretation of 0.7 is that if we add to first order logic the cardinality quantifiers $(\exists^{\geq \lambda} x)$ for $\lambda = \aleph_1, \aleph_2, \aleph_3, \aleph_4$ we get a noncompact logic.

We thank the referee for many helpful comments and the reader should thank him also for urging the inclusion of several proofs.

This work is continued in [ShVa:790] and [Sh:824].

§1. **Relevant identities.** We commence by several definitions. For simplicity, for us all identities, colorings etc. will be 2-place.

1.1. DEFINITION. (1) For $m, \ell < \omega$ let

$$\mathrm{dom}_{\ell,m} = \{ \eta \in {}^{\ell+1}\omega : \eta \upharpoonright \ell \in {}^{\ell}2 \text{ and } \eta(\ell) < m \}$$

$$ID_{\ell,m}^1 = \big\{ (\mathrm{dom}_{\ell,m}, e) : e \text{ is an equivalence relation on } [\mathrm{dom}_{\ell,n}]^2$$
$$\text{such that } \{\eta_1, \eta_2\} e \{v_1, v_2\} \ \& \ \{\eta_1, \eta_2\} \neq \{v_1, v_2\}$$
$$\Rightarrow \eta_1 \cap \eta_2 = v_1 \cap v_2 \wedge lg(\eta, m_2) < \ell \big\}.$$

(2) Let

$$ID_\ell^1 = \cup \{ ID_{\ell,m}^1 : m < \omega \},$$
$$ID^1 = \cup \{ ID_\ell^1 : \ell < \omega \}.$$

(3) For $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e) \in ID_{\ell,m}^1$ and $v \in {}^{\ell \geq}2$ let

$$\mathrm{dom}_{\ell,m}^{[v]} = \{ \rho \in \mathrm{dom}_{\ell,m} : v \trianglelefteq \rho \}$$

and if $v \in {}^{\ell >}2$ we let

$$e_{\langle v \rangle}(\mathbf{s}) = e \upharpoonright \{\{\eta_0, \eta_1\} : v^\smallfrown\langle i \rangle \triangleleft \eta_i \text{ for } i = 0, 1\}.$$

We use $\mathbf{s}$ to denote identities so $\mathbf{s} = (\mathrm{dom}_{\mathbf{s}}, e(\mathbf{s}))$; and if $\mathbf{s} \in ID^1$ then let $\mathbf{s} = (\mathrm{dom}_{\ell(\mathbf{s}),m(\mathbf{s})}, e(\mathbf{s}))$.

(4) An equivalence class is nontrivial if it is not a singleton.

Note that it follows that every e-equivalence class is an $e_{\langle v \rangle}$-equivalence class for some v. We restrict ourselves to

1.2. **Definition.** (1) Let $ID^2_{\ell,m}$ be the set of $\mathbf{s} \in ID^1_{\ell,m}$ such that for every $v \in {}^{\ell >}2$ the equivalence relation $e_{\langle v \rangle}(\mathbf{s})$ has at most one non-singleton equivalence class, which we call $e_{[v]} = e_{[v]}(\mathbf{s})$.

     So we also allow $e_{\langle v \rangle} = $ empty, in which case we choose a representative equivalence class $e_{[v]}$ as the first one under, say, lexicographical ordering.

(2) $ID^2_\ell = \cup \{ID^2_{\ell,m} : m < \omega\}$.

1.3. **Definition.** (1) We define for $k < \omega$ when $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e)$ is k-nice: the demands are

    (a) $\mathbf{s} \in ID^1_{\ell,m}$,

    (b) if $v \in {}^\ell 2$ and $(v \restriction i)^\frown \langle 1 - v(i)\rangle \lhd \rho_i \in \mathrm{dom}_{\ell,m}$ for each $i < \ell$ then $\{\eta : v \lhd \eta \in \mathrm{dom}_{\ell,m}$ and for each $i < \ell$ the set $\{\rho_i, \eta\}/e$ is not a singleton$\}$ has at least two members,

    (c) the graph $H[e]$, see below, has no cycle $\leq k$ (for $k \leq 2$ this holds trivially),

    (d) the graph $H[e]$ has a cycle.

(2) We can interpret $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e)$ as the graph $H[\mathbf{s}]$ with set of nodes $\mathrm{dom}_{\ell,m}$ and set of edges $\{\{\eta, v\} : \{\eta, v\}/e$ not a singleton (and of course $\eta \neq v$ are from $\mathrm{dom}_{\ell,m}$)$\}$.

(3) We may write $e(\mathbf{s})$ instead of $\mathbf{s}$ if $\mathrm{dom}_{\ell,m}$ can be reconstructed from e (e.g., if the graph has no isolated point (e.g., if it is 0-nice, see clause (b) of part (1)). Saying nice we mean $[\log_2(m)]$-nice.

1.4. **Claim.** (1) *If (λ, μ) is $\aleph_0$-compact and $c_n : [\lambda]^{<\aleph_0} \to \mu$ and $\Gamma_n = ID(c_n)$ for $n < \omega$, then for some $c : [\lambda]^{<\aleph_0} \to \mu$ we have $ID(c) \subseteq \bigcap_{n<\omega} \Gamma_n$ (in fact equality holds).*

(2) *Similarly using ID_2.*

Remark. By the same proof, if we just assume $(\lambda_1, \mu_1) \to'_{\aleph_1} (\lambda_2, \mu_2)$ and $c_n : [\lambda_1]^{<\aleph_0} \to \mu_1$, then we can deduce that there is $c : [\lambda_2]^{<\aleph_0} \to \mu_2$ satisfying $ID(c) \subseteq \bigcap_{n<\omega} ID(c_n)$.

Proof. Straightforward.

(1) In details, let F_m be an m-place function symbol and P the distinguished unary predicate and let $T = \{\psi_n : n < \omega\} \cup \{\neg \psi_\mathbf{s} : c$ is an identity of the form (n, e) not from $\bigcap_{n<\omega} ID(c_n)\}$ where

    (a) $\psi_n = (\forall x_0)(\forall x_1) \ldots (\forall x_{n-1})(P(F_n(x_0, \ldots, x_{n-1}))$ &
$\wedge \{(\forall x_0) \ldots (\forall x_n) F_n(x_0, \ldots, x_{n-1}) = F_n(x_{\pi(0)}, \ldots, x_{\pi(n-1)}):$
π a permutation of $\{0, \ldots, n-1\}\}$,

(b) if $\mathbf{s} = (n, e)$ is an identity then $\psi_\mathbf{s} = (\exists x_0) \dots (\exists x_{n-1})[\ \bigwedge\limits_{\ell < m < n} x_\ell \neq x_m$ &
$\bigwedge\limits_{b,c \subseteq n, b \in c} F_{|b|}(\dots, x_\ell, \dots)_{\ell \in b} = F_{|b|}(\dots, x_\ell, \dots)_{\ell \in c}]$.

Clearly T is a (first order) countable theory so it suffices to prove the following two statements $\boxtimes_1, \boxtimes_2$.

$\boxtimes_1$ if $M \in K_{(\lambda,\mu)}$ is a model of T, then there is $c: [\lambda]^{<\aleph_0} \to \mu$ such that $ID(c) \subseteq \bigcap_{n<\omega} \Gamma_n$.

[Why does $\boxtimes_1$ hold? There is $N \cong M$ such that N has universe $|N| = \lambda$ and $P^N = \mu$. Now we define c: if $u \in [\lambda]^{<\aleph_0}$, let $\{\alpha_\ell^u : \ell < |u|\}$ enumerate u in increasing order and let $c(u) = F_{|u|}^N(\alpha_0^u, \alpha_1^u, \dots, \alpha_{|u|-1}^u)$. Note that because $N \models \psi_n$ for $n < \omega$ clearly c is a function from $[\lambda]^{<\aleph_0}$ into μ. Also because $N \models \psi_n$, if $n < \omega$ and $\alpha_0, \dots, \alpha_{n-1} < \lambda$ are with no repetitions then $F_n^N(\alpha_0, \dots, \alpha_{n-1}) = c\{\alpha_0, \dots, \alpha_{n-1}\}$. Now if $\mathbf{s} \in ID(c)$ let $\mathbf{s} = (n, e)$ and let $u = \{\alpha_0, \dots, \alpha_{n-1}\} \in [\lambda]^n \subseteq [\lambda]^{<\aleph_0}$ exemplify that $\mathbf{s} \in ID(c)$, hence easily $N \models \psi_\mathbf{s}$ so necessarily $\neg\psi_\mathbf{s} \notin T$ hence $\mathbf{s} \in \bigcap_{n<\omega} \Gamma_n$. This implies that $ID(c) \subseteq \bigcap_{n<\omega} \Gamma_n$ is as required.]

$\boxtimes_2$ if $T' \subseteq T$ is finite then T' has a model in $K_{(\lambda,\mu)}$.

[Why? So T' is included in $\{\psi_m : m < m^*\} \cup \{\neg\psi_{\mathbf{s}_k} : k < k^*\}$ for some $m^*, k^* < \omega$, $\mathbf{s}_k = (n_k, e_k)$ an identity not from $\bigcap_{\ell<\omega} ID(c_\ell)$, so we can find $\ell(k) < \omega$ such that $\mathbf{s}_k \notin ID(c_{\ell(k)})$. Let H be a one-to-one function from $^{k^*}\mu$ into μ. We define a model M:

(a) its universe $|M|$ is λ,

(b) $P^M = \mu$,

(c) if $n < \omega$, $\{\alpha_0, \dots, \alpha_{n-1}\} \in [\lambda]^n$ then
$F_n^M(\alpha_0, \dots, \alpha_{n-1}) = H\big(c_{\ell(0)}\{\alpha_0, \dots, \alpha_{n-1}\},$
$\quad c_{\ell(1)}\{\alpha_0, \dots, \alpha_{n-1}\}, \dots, c_{\ell(k^*-1)}\{\alpha_0, \dots, \alpha_{n-1}\}\big)$.

If $n < \omega$ and $\alpha_0, \dots, \alpha_{n-1} < \lambda$ is with repetitions we let $F_n^M(\alpha_0, \dots, \alpha_{n-1}) = 0$. Clearly M is a model from $K_{(\lambda,\mu)}$ of the vocabulary of T. Also M satisfies each sentence ψ_m by the way we have defined F_m^M. Lastly, for $k < k^*$, $M \models \neg\psi_{\mathbf{s}_k}$ because $(n_k, e_k) \notin ID(c_{\ell(k)})$ by the choice of the F_n's as H is a one-to-one function.]

$\square_{1.4}$

Of course

1.5. OBSERVATION. (1) *For every $\ell < \omega$, $k < \omega$ for some m there is a k-nice $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e)$.*

(2) *If $\mathbf{s}$ is k-nie and $m \leq k$, then $\mathbf{s}$ is m-nice.*

§2. Definition of the forcing.

We have outlined the intended end of the proof at the end of the introductory section. It is to construct a sequence of functions $\langle f_n : n < \omega \rangle$ with certain properties. As we have adopted the

decision of dealing only with 2-identities from ID_ℓ^1, all our functions will be colorings of pairs, and we shall generally use the letter c for them.

Our present Theorem 0.7 deals with $\aleph_4$, but we may as well be talking about some $\aleph_{n(*)}$ for a fixed natural number $n(*) \geq 2$. Of course, the set of identities will depend on $n(*)$. We shall henceforth work with $n(*)$, keeping in mind that the relevant case for Theorem 0.7 is $n(*) = 4$. Also we fix $\ell(*) = n(*) + 1$ on which the identities depend (but vary m). Another observation about the proof is that we can replace $\aleph_0$ with an uncountable cardinal κ such that $\kappa = \kappa^{<\kappa}$ replacing $\aleph_n$ by κ^{+n}. Of course, the pair (κ^{+n}, κ) is compact because $[\kappa = \kappa^{\aleph_0} < \lambda \Rightarrow (\kappa, \lambda)$ is $\leq \kappa$-compact], however, much of the analysis holds.

We may replace $(\aleph_n, \aleph_0)$ by $(\kappa^{+n(*)}, \kappa)$ if $\kappa^{+n(*)} \leq 2^{\aleph_0}$; we hope to return to this elsewhere.

To consider (κ^+, κ) we need large cardinals; even more so for considering $(\mu^+, \mu), \mu$ strong limit singular of cofinality $\aleph_0$, and even $(\kappa^{+n}, \kappa), \mu \leq \kappa < \kappa^{+n} \leq \mu^{\aleph_0}$.

We now describe the idea behind the definition of the forcing notion we shall be concerned with. Each "component" of the forcing notion is supposed to add a coloring

$$c : [\lambda]^2 \to \mu$$

preserving some of the possible 2-identities, while "killing" all those which were not preserved, in other words it is concerned with adding f_n; specifically we concentrate on the case $\lambda = \aleph_{n(*)}, \mu = \aleph_0$. Hence, at first glance the forcing will be defined so that to preserve an identity we have to work hard proving some kind of amalgamation for the forcing notion, while killing an identity is a consequence of adding a colouring exemplifying it. By preserving a set Γ of identities, we mean that $\Gamma \subseteq ID(c)$, and more seriously $\Gamma \subseteq ID_2(\lambda, \mu)$; we restrict ourselves to some ID^*, an infinite set of 2-identity.

We shall choose $ID^* \subseteq ID_2^{\circledast}$ below small enough such we can handle the identities in it.

We define the forcing by putting in its definition, for each identity that we want to preserve, a clause specifically assuring this. Naturally this implies that not only the desired identities are preserved, but also some others so making an identity be not in $ID(\lambda, \mu)$ becomes now the hard part. So, we lower our sights and simply hope that, if $\Gamma \subseteq ID^*$ is the set of identities that we want to preserve, than no identity $(a, e) \in ID^* \setminus \Gamma$ is preserved; this may depend on Γ.

How does this control over the set of identities help to obtain the non-compactness? We shall choose sets $\Gamma_n \subseteq ID^*$ of possible identities for $n < \omega$. The forcing we referred to above, let us call it $\mathbb{P}^{\Gamma_n}$, add a colouring $c_n : [\lambda]^2 \to \omega$ such that $ID_2(c_n)$ includes Γ_n and is disjoint to $ID^* \setminus \Gamma_n$; also it will turn out to have a strong form of the ccc. We shall force with $\mathbb{P} =: \prod_{n \in \omega} \mathbb{P}^{\Gamma_n}$, where the product is taken with finite support. Because of the strong version of ccc possessed by each $\mathbb{P}^{\Gamma_n}$, also $\mathbb{P}$ will have ccc. Now, in $V^{\mathbb{P}}$ we have for every n

a colouring $c_n\colon [\lambda]^2 \to \omega$ which preserves the identities in Γ_n, moreover $\mathbf{V}^\mathbb{P} \models \Gamma_n \subseteq ID(c_n) \cap ID^*$.

We shall in fact obtain that

$$ID^* = \Gamma_0 \supseteq \Gamma_1 \ \& \ \Gamma_1 \supseteq \Gamma_2 \ \& \ \cdots \ \& \ \bigcap_{n<\omega} \Gamma_n = \emptyset \ \& \ ID(c_n) \cap \Gamma_0 = \Gamma_n.$$

If we have $\aleph_0$-compactness for $(\lambda, \aleph_0)$, then by 1.4(2) there must be a colouring $c\colon [\lambda]^2 \to \omega$ in $\mathbf{V}^\mathbb{P}$ such that

$$ID_2(c) \cap \Gamma_0 \subseteq \bigcap_{n<\omega} \Gamma_n = \emptyset.$$

We can find a name $\underset{\sim}{c}$ in $\mathbf{V}$ for such c, so by ccc, for every $\{\alpha, \beta\} \in [\lambda]^2$, the name $\underset{\sim}{c}(\{\alpha, \beta\})$ depends only on $\aleph_0$ "coordinates". At this point a first approximation to what we do is to apply a relative of Erdős-Rado theorem to prove that there are an n, a large enough $w \subseteq \lambda$ and for every $\{\alpha, \beta\} \in [w]^2$ a condition $p_{\{\alpha,\beta\}} \in \prod_{\ell<n} \mathbb{P}^{\Gamma_l}$, such that $p_{\{\alpha,\beta\}}$ forces a value to $\underset{\sim}{c}(\{\alpha, \beta\})$ in a "uniform" enough way. We shall be able to extend enough of the conditions $p_{\{\alpha,\beta\}}$ by a single condition p^* in $\prod_{\ell<n} \mathbb{P}^{\Gamma_l}$, which gives an identity in $ID_2(\underset{\sim}{c})$ which belongs to $\bigcap_{\ell<n} \Gamma_\ell \setminus \Gamma_n$, contradiction.

Before we give the definition of the forcing, we need to introduce a notion of closure. The properties of the closure operation are the ones possible to obtain for $(\lambda, \aleph_0)$, but not for $(\aleph_\omega, \aleph_0)$. We of course need to use somewhere such a property, as we know in ZFC that $(\aleph_\omega, \aleph_0)$ has all those identities, i.e., $ID_2^\circledast = ID_2(\lambda, \aleph_0)$.

On a similar proof see [Sh:424] (for ω-place functions) and also (2-place functions), [Sh:522]. The definition of the closure in [GcSh:491] is close to ours, but note that the hard clause from [GcSh:491] is not needed here.

2.1. DEFINITION. Let $ID^*_{\ell(*)} =: \{\mathbf{s} \in ID^2_{\ell(*)} \colon \mathbf{s} \text{ is 0-nice}\}$.

REMARK. We can consider $\{\mathbf{s}_n \colon n < \omega\}$, which hopefully will be independent, i.e., for every $X \subseteq \omega$ for some c.c.c. forcing notion $\mathbb{P}$, in $\mathbf{V}^\mathbb{P}$ we have $\lambda \to (\mathbf{s}_n)_\mu$ iff $n \in X$. It is natural to try $\{\mathbf{s}_n \colon n < \omega\}$ where $\mathbf{s}_n = (\mathrm{dom}_{\ell(*),m_n}, e_n)$ where $m_n = n$ (or 2^{2^n} may be more convenient) and e_n is $[\log\log(n)]$-nice.

2.2. DEFINITION (λ is our fixed cardinal).   (1) Let M^* (or M^*_λ) be a model with universe λ, countable vocabulary, and its relations and functions are exactly those defined in $(\mathscr{H}(\chi), \in, <^*_\chi)$ for $\chi = \lambda^+$ (and some choice of $<^*_\chi$, a well ordering of $\mathscr{H}(\chi)$).

(2) For $\bar{a} \in {}^{\omega>}(M^*_\lambda)$ let $c\ell_\ell(\bar{a}) = \{\beta < \lambda \colon \text{for some first order } \varphi(y, \bar{x}) \text{ we have } M^*_\lambda \models \varphi[\beta, \bar{a}] \ \& \ (\exists^{\leq \aleph_\ell} x)\varphi(x, \bar{a})\}$ and $c\ell(\bar{a}) = \{\beta < \lambda \colon \text{for some first order } \varphi(y, \bar{x}) \text{ we have } M^*_\lambda \models \varphi[\beta, \bar{a}] \ \& \ (\exists^{\leq \aleph_0} x)\varphi(x, \bar{a})\}$.

(3) For a model M and $A \subseteq M$ let $c\ell_M(A)$ be the smallest set of elements of M including A and closed under the functions of M (so including the individual constants).

Note that

2.3. Fact. If $\beta_0, \beta_1 \in c\ell_{\ell+1}(\bar{\alpha})$ then for some $i \in \{0, 1\}$ we have $\beta_i \in c\ell_\ell(\bar{\alpha}^\frown \langle \beta_{1-i} \rangle)$.

PROOF. Easy.

The idea of our forcing notion is to do historical forcing (see [RoSh:733] for more on historical forcing and its history). That is, we put in only those conditions which we have to put in order to meet our demands, so every condition in the forcing has a definite rule of creation. In particular, (see below), in the definition of our partial colourings, we avoid giving the same color to any pairs for which we can afford this, if the rule of creation is to be respected. We note that the situation here is not as involved as the one of [RoSh:733], and we do not in fact need the actual history of every condition.

We proceed to the formal definition of our forcing.

Clearly Case 0 for $k \geq 0$ is not necessary from a historical point of view but it simplifies our treatment later; also Case 1 is used in clause (η) of Case 3.

Note that in Case 2 below we do not require that the conditions are isomorphic over their common part (which is natural for historic forcing) as the present choice simplifies clause $(\zeta)(iv)$ in Case 3.

2.4. Main Definition. Let $n(*) \geq 2, n(*) \leq \ell(*) < \omega, \lambda = \aleph_{n(*)}, \mu = \aleph_0$ be fixed. All closure operations we shall use are understood to refer to $M^*_{\aleph_{n(*)}}$ from 2.2(2). Let $\Gamma \subseteq ID^*_{\ell(*)}$ be given. For two sets u and v of ordinals with $|u| = |v|$, we let $OP_{v,u}$ stand for the unique order preserving 1–1 function from u to v.

We shall define $\mathbb{P} =: \mathbb{P}_\Gamma = \mathbb{P}^\lambda_\Gamma$, it is $\subseteq \mathbb{P}^*_\lambda$.

Members of $\mathbb{P}^*_\lambda$ are the pairs of the form $p = (u, c) =: (u^p, c^p)$ with

$$u \in [\lambda]^{<\aleph_0} \text{ and } c \colon [u]^2 \to \omega.$$

The order in $\mathbb{P}^*_\lambda$ is defined by

$$(u_1, c_1) \leq (u_2, c_2) \Leftrightarrow (u_1 \subseteq u_2 \ \& \ c_1 = c_2 \restriction [u_1]^2).$$

For $p \in \mathbb{P}^*_\lambda$ let $n(p) = \sup(\text{Rang}(c^p)) + 1$; this is $< \omega$.

We now say which pairs (u, c) of the above form (i.e., $(u, c) \in \mathbb{P}^*_\lambda$) will enter $\mathbb{P}$. We shall have $\mathbb{P} = \bigcup_{k<\omega} \mathbb{P}_k$ where $\mathbb{P}_k =: \mathbb{P}^{\lambda, \Gamma}_k$ are defined by induction on $k < \omega$, as follows.

Case 0. $k = 4\ell$. If $k = 0$ let $\mathbb{P}_0 =: \{(\emptyset, \emptyset)\}$.

If $k = 4l > 0$, a pair $(u, c) \in \mathbb{P}_k$ iff for some $(u', c') \in \bigcup_{m<k} \mathbb{P}_m$ we have $u \subseteq u'$ and $c = c' \restriction [u]^2$; we write $(u, c) = (u', c') \restriction u$.

CASE 1. $k = 4\ell + 1$. (This rule of creation is needed for density arguments.)
A pair (u, c) is in $\mathbb{P}_k$ iff (it belongs to $\mathbb{P}_\lambda^*$ and) there is a $(u_1, c_1) \in \bigcup_{m<k}\mathbb{P}_m$ and $\alpha < \lambda$ satisfying $\alpha \notin u_1$ such that:

(a) $u = u_1 \cup \{\alpha\}$,

(b) $c \restriction [u_1]^2 = c_1$ and

(c) For every $\{\beta, \gamma\}$ and $\{\beta', \gamma'\}$ in $[u]^2$ which are not equal, if $c(\{\beta, \gamma\})$ and $c(\{\beta', \gamma'\})$ are equal, then $\{\beta, \gamma\}, \{\beta', \gamma'\} \in [u_1]^2$. (Hence, c does not add any new equalities except for those already given by c_1.)

CASE 2. $k = 4\ell + 2$. (This rule of creation is needed for free amalgamation, used in the Δ-system arguments for the proof of the c.c.c..)
A pair (u, c) is in $\mathbb{P}_k$ iff (it belongs to $\mathbb{P}_\lambda^*$ and) there are $(u_1, c_1), (u_2, c_2) \in \bigcup_{m<k}\mathbb{P}_m$ for which we have

(a) $u = u_1 \cup u_2$.

(b) $c \restriction [u_1]^2 = c_1$ and $c \restriction [u_2]^2 = c_2$.

(c) c does not add any unnecessary equalities, i.e., if $\{\beta, \gamma\}$ and $\{\beta', \gamma'\}$ are distinct and in $[u]^2$ and $c(\{\beta, \gamma\}) = c(\{\beta', \gamma'\})$, then $\{\{\beta, \gamma\}, \{\beta', \gamma'\}\} \subseteq [u_1]^2 \cup [u_2]^2$.
Note that $[u_1]^2 \cap [u_2]^2 = [u_1 \cap u_2]^2$

(d) $c\ell_0(u_1 \cap u_2) \cap (u_1 \cup u_2) \subseteq u_1$ (usually $c\ell_0(u_1 \cap u_2) \cap (u_1 \cup u_2) \subseteq u_1 \cap u_2$) is O.K. too for present §2, §3 but not, it seems, in 4.6).

MAIN RULE.

CASE 3. $k = 4\ell + 3$. (This rule[3] is like the previous one, but the amalgamation is taken over a graph $\mathbf{s} = (\mathrm{dom}_{\ell(*),m}, e) \in \Gamma$).
A pair $(u, c) \in \mathbb{P}_k$ iff there are $\mathbf{s} = (\mathrm{dom}_{\ell(*),m(*)}, e) \in \Gamma$ and a sequence of conditions

$$\bar{p} = \langle p_y : y \in Y \rangle \text{ where } Y = \{y \in [\mathrm{dom}_\mathbf{s}]^2 : |y/e| > 1\}$$

from $\bigcup_{m<k}\mathbb{P}_m$ AND we have a sequence of finite sets $\bar{v} = \langle v_t : t \in Y^+ \rangle$ where

$$Y^+ = \{t : t \in Y \text{ or } t = \emptyset \text{ or } t = \{\eta\}, \text{ where } \eta \in \mathrm{dom}_\mathbf{s}\}$$

such that

(a) $u = \bigcup\{u^{p_y} : y \in Y\}$,

(b) $(u, c) \in \mathbb{P}_\lambda^*$ and $c \restriction [u^{p_y}]^2 = c^{p_y}$ for all $y \in Y$,

(c) if $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ are from u and $\{\alpha_1, \alpha_2\} \neq \{\beta_1, \beta_2\}$ and $c\{\alpha_1, \alpha_2\} = c\{\beta_1, \beta_2\}$ then $(\exists y)[\{\alpha_1, \alpha_2\} \subseteq u^{p_y}]$ and $(\exists y)[\{\beta_1, \beta_2\} \subseteq u^{p_y}]$,

(d) $v_t \cap v_s \subseteq v_{t \cap s}$ for $t, s, \in Y^+$,

(e) $c\ell_0(v_t) \cap u^{p_y} \subseteq v_t$ for all $y \in Y$ and $t \in \{\emptyset\} \cup \{\{\eta\} : \eta \in \mathrm{dom}_\mathbf{s}\}$,

(f) $u^{p_y} \subseteq v_y$ for all $y \in Y$,

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[3]You may understand it better seeing how it is used in the proof of 3.3.

(g) if $y_1, y_2 \in Y$ and $t \in \{\emptyset\} \cup \{\{\eta\} \colon \eta \in \mathrm{dom}_\mathbf{s}\}$ and $t = y_1 \cap y_2$, then $p_{y_1} \restriction v_t = p_{y_2} \restriction v_t$; equivalently: $\{p_\eta \colon \eta \in Y\}$ has a common upper bound in $\mathbb{P}_\lambda^*$.

2.5. Claim. (1) $\mathbb{P}_\Gamma^\lambda$ *satisfies the c.c.c. and even the Knaster condition.*

(2) *For each $\alpha < \lambda$ the set $\mathscr{I}_\alpha = \{p \in \mathbb{P}_\Gamma^\lambda \colon \alpha \in u^p\}$ is dense open.*

(3) $\Vdash_{\mathbb{P}_\Gamma^\lambda}$ "$\underset{\sim}{c} = \cup\{c^p \colon p \in \underset{\sim}{G}\}$ *is a function from* $[\lambda]^2$ *to* ω".

PROOF. (1) By Case 2.

In detail, assume that $p_\varepsilon \in \mathbb{P}_\Gamma^\lambda$ for $\varepsilon < \omega_1$ and let $p_\varepsilon = (u_\varepsilon, c_\varepsilon)$. As each u_ε is a finite subset of λ, by the Δ-system lemma without loss of generality for some finite $u^* \subseteq \lambda$ we have: if $\varepsilon < \zeta < \omega_1$ then $u_\varepsilon \cap u_\zeta = u^*$. By further shrinking, without loss of generality $\alpha \in u^* \Rightarrow \langle |u_\varepsilon \cap \alpha| \colon \varepsilon < \omega_1 \rangle$ is constant and $\varepsilon < \zeta < \omega_1 \Rightarrow |u_\varepsilon| = |u_\zeta|$. Also without loss of generality the set $\{(\ell, m, k) \colon$ for some $\alpha \in u_\varepsilon$ and $\beta \in u_\varepsilon$ we have $\ell = |\alpha \cap u_\varepsilon|, m = |\beta \cap u_\varepsilon|$ and $k = c_\varepsilon\{\alpha, \beta\}\}$ does not depend on ε. We can conclude that $\varepsilon < \zeta < \omega_1 \Rightarrow OP_{u_\zeta, u_\varepsilon}$ maps p_ε to p_ζ over u^*. Clearly for $\varepsilon < \omega_1$, the set $c\ell(u_\varepsilon)$ is countable hence for every $\zeta < \omega_1$ large enough we have $u_\zeta \cap c\ell_0(u_\varepsilon) = u^*$ so restricting $\langle p_\varepsilon \colon \varepsilon < \omega_1 \rangle$ to a club we get that $\varepsilon < \zeta < \omega_1 \Rightarrow c\ell_0(u_\varepsilon) \cap u_\zeta = u^*$ (this is much more than needed). Now for any $\varepsilon < \zeta < \omega_1$ we can define $q_{\varepsilon, \zeta} = (u_{\varepsilon, \zeta}, c_{\varepsilon, \zeta})$ with $u_{\varepsilon, \zeta} = u_\varepsilon \cup u_\zeta$ and $c_{\varepsilon, \zeta} \colon [u_{\varepsilon, \zeta}]^2 \to \omega$ is defined as follows: for $\alpha < \beta$ in $u_{\varepsilon, \zeta}$ let $c_{\varepsilon, \zeta}\{\alpha, \beta\}$ be $c_\varepsilon\{\alpha, \beta\}$ if defined, $c_\zeta\{\alpha, \beta\}$ if defined, and otherwise $\sup(\mathrm{Rang}(c_\varepsilon)) + 1 + (|u_{\varepsilon, \zeta} \cap \alpha| + |u_{\varepsilon, \zeta} \cap \beta|)^2 + |u_{\varepsilon, \zeta} \cap \alpha|$. Now $q \in \mathbb{P}_\Gamma^\lambda$ by Case 2, and $p_\varepsilon \le q_\varepsilon, p_\zeta \le q_{\varepsilon, \zeta}$ by the definition of order.

(2) By Case 1.

In detail, let $p \in \mathbb{P}_\Gamma^\lambda$ and $\alpha < \lambda$ and we shall find q such that $p \le q \in \mathscr{I}_\alpha$. If $\alpha \in u^p$ let $q = p$, otherwise define $q = (u^q, c^q)$ as follows $u^q = u^p \cup \{\alpha\}$ and for $\beta < \gamma \in u^q$ we let $c^q\{\beta, \gamma\}$ be: $c^p\{\beta, \gamma\}$ when it is well defined and $\sup(\mathrm{Rang}(c^p)) + 1 + (|\beta \cap u^q| + |\gamma \cap u^q|)^2 + |\beta \cap u^q|$ when otherwise. Now $q \in \mathbb{P}_\Gamma^\lambda$ by Case 1 of Definition 2.4, $p \le q$ by the order's definition and $q \in \mathscr{I}_\sigma$ trivially.

(3) Follows from part (2). $\hspace{2cm}\square_{2.5}$

§3. Why does the forcing work. We shall use the following claim for $\mu = \aleph_0$.

3.1. Claim. (1) *If $f \colon [\lambda]^2 \to \mu$ and M is an algebra with universe λ, $|\tau_M| \le \mu$ and $w_t \subseteq [\lambda], |w_t| < \aleph_0$ for $t \in [\lambda]^2$ and $\lambda \ge \beth_2(\mu^+)^+$, then for some $\langle v_t \colon t \in [W]^{\le 2} \rangle$ we have:*

(a) $W \subseteq \lambda$ *is infinite in fact* $|W| = \mu^+$,

(b) $f \restriction [W]^2$ *is constant,*

(c) $t \cup w_t \subseteq v_t \in [\lambda]^{<\aleph_0}$ *for* $t \in [W]^2$,

(d) $v_{t_1} \cap v_{t_2} \subseteq v_{t_1 \cap t_2}$ *when* $t_1, t_2 \in [W]^{\le 2}$ *but for no* $\alpha < \beta < \gamma$ *do we have* $\{t_1, t_2\} = \{\{\alpha, \beta\}, \{\beta, \gamma\}\}$,

(e) *if* $t_1, t_2 \in [W]^i$, *where* $i \in \{1, 2\}$ *then* $|v_{t_1}| = |v_{t_2}|$ *and* $OP_{v_{t_2}, v_{t_1}}$ *maps* t_1 *onto* t_2 *and* w_{t_1} *onto* w_{t_2} *and* v_{t_1} *onto* v_{t_2}, $w_{\{Min(t_1)\}}$ *onto* $w_{\{Min(t_2)\}}$ *and* $w_{\{Max(t_1)\}}$ *onto* $w_{\{Max(t_2)\}}$, $v_{\{Min(t_1)\}}$ *onto* $v_{\{Min(t_2)\}}$, *and* $v_{\{Max(t_1)\}}$ *onto* $v_{\{Max(t_2)\}}$,

(f) $v_{\{\alpha, \beta\}} \cap c\ell_M(v_{\{\gamma\}}) \subseteq v_\gamma$ *for* $\alpha, \beta, \gamma \in W$.

(2) *If* $[u \in [\lambda]^{<\aleph_0} \Rightarrow c\ell_M(u) \in [M]^{<\mu}$, *then* $\lambda = (\beth_2(\mu))^+$ *is enough.*

REMARK. See more in [Sh:289]; this is done for completeness.

PROOF. (1) Let $w_t \cup t = \{\zeta_{t,\ell} : \ell < n_t\}$ with no repetitions and we define the function c, c_0, c_1 with domain $[\lambda]^3$ as follows: if $\alpha < \beta < \gamma < \lambda$ then

$$c_0\{\alpha, \beta, \gamma\} = \{(\ell_1, \ell_2) : \ell_1 < n_{\{\alpha, \beta\}}, \ell_2 < n_{\{\alpha, \gamma\}} \text{ and } \zeta_{\{\alpha, \beta\}, \ell_1} = \zeta_{\{\alpha, \gamma\}, \ell_2}\},$$

$$c_1\{\alpha, \beta, \gamma\} = \{(\ell_1, \ell_2) : \ell_1 < n_{\{\alpha, \gamma\}}, \ell_2 < n_{\{\beta, \gamma\}} \text{ and } \zeta_{\{\alpha, \gamma\}, \ell_1} = \zeta_{\{\beta, \gamma\}, \ell_2}\},$$

$$c\{\alpha, \beta, \gamma\} = (c_0\{\alpha, \beta, \gamma\}, c_1\{\alpha, \beta, \gamma\}, f\{\alpha, \beta\}).$$

By Erdős-Rado theorem for some $W_1 \subseteq \lambda$ of cardinality and even order type μ^{++} for part (1), μ^+ for part (2) such that $c \upharpoonright [W_0]^3$ is constant. Let $\{\alpha_\varepsilon : \varepsilon < \mu^{+}\}$ list W_0 in increasing order. If $2 < i < \mu^{++}$, let

$$v_{\{\alpha_i\}} =: \{\zeta_{\{\alpha_i, \alpha_{i+1}\}, \ell_1} : \text{for some } \ell_2 \text{ we have } (\ell_1, \ell_2) \in c_0\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}\}$$
$$\cup \{\zeta_{\{\alpha_0, \alpha_i\}, \ell_1} : \text{for some } \ell_2 \text{ we have } (\ell_1, \ell_2) \in c_1\{\alpha_0, \alpha_1, \alpha_i\}\}$$

(clearly $\alpha_i \in v_{\{\alpha_i\}}$).

For $i < j$ in $(2, \mu^{++})$ let $v_{\{\alpha_i, \alpha_j\}} = v_{\{\alpha_i\}} \cup v_{\{\alpha_j\}} \cup w_{\{\alpha_i, \alpha_j\}}$. Now for some unbounded $W_2 \subseteq W_1 \setminus \{\alpha_0, \alpha_1\}$ and $Y \in [\lambda]^{\leq \mu}$ we have:

if $\alpha \neq \beta \in W_2$ then $c\ell_M(v_{\{\alpha\}}) \cap c\ell_M(v_{\{\beta\}}) \subseteq Y$.

Now by induction on $\varepsilon < \mu^+$ we can choose $\gamma_\varepsilon \in W_2$ strictly increasing with ε, γ_ε large enough. It is easy to check that $W = \{\gamma_\varepsilon : \varepsilon < \mu^+\}$ is as required.

(2) The same proof. $\square$ 3.1

3.2. CLAIM. *Let* $n(*), \ell(*), \lambda$ *be as in Definition 2.4, and see Definition 2.1. Assume that* $\Gamma_1, \Gamma_2, m^*, p^*$ *satisfies*:

(a) $\Gamma_1, \Gamma_2 \subseteq ID^*_{\ell(*)}$,

(b) *if* $(\text{dom}_{\ell(*), m}, e) \in ID^*_{\ell(*)}$ *and* $(\text{dom}_{\ell(*), m}, e)$ *is not* m^*-*nice then* $(\text{dom}_{\ell(*), m}, e) \in \Gamma_1 \Leftrightarrow (\text{dom}_{\ell(*), m}, e) \in \Gamma_2$,

(c) $p^* \in \mathbb{P}^*_\lambda$ *and* $|u^{p^*}| < m^*$.

Then $p^* \in \mathbb{P}^\lambda_{\Gamma_1} \Leftrightarrow p^* \in \mathbb{P}^\lambda_{\Gamma_2}$.

PROOF. We prove by induction on $k < \omega$ that

$(*)_k$ if $r' \in \mathbb{P}^{\lambda, \Gamma_1}_k$ (see Definition 2.4 before Case 0) and $r \leq r'$ and $|u^r| < m^*$, then $r \in \mathbb{P}^\lambda_{\Gamma_2}$.

This is enough by the symmetry in our assumptions.

For a fixed k we prove this by induction on $|u^r|$. The proof splits according to the Case in Definition 2.4 which hold for r'.

CASE 0. Trivial.

CASE 1. Easy.

CASE 2. Should be clear but let us check, so $r' = (u', c')$ is gotten from $(u'_1, c'_1), (u'_2, c'_2)$ as in clauses (a)–(d) of Case 2, and let $r = (u, c) \le r'$.

Let $u_\ell = u'_\ell \cap u$, $c_\ell = c'_\ell \upharpoonright [u_\ell]^2$. By the induction hypothesis $(u_\ell, c_\ell) \in \mathbb{P}^\lambda_{\Gamma_2}$ and it is enough to check that $(u, c), (u_1, c_1), (u_2, c_2)$ are in Case (2) of Definition 2.4 which is easy, e.g., in clause (d) we use monotonicity of $c\ell_0$.

CASE 3. So let r' be gotten from $\mathbf{s} = (\mathrm{dom}_{\ell(*),m}, e)$, $\langle p_y : y \in Y \rangle$, $\langle v_t :$ $t \in Y^+ \rangle$ as there. Of course, we have $(\mathrm{dom}_{\ell(*),m}, e) \in \Gamma_1$ and $p_y \in \bigcup_{\ell < k} \mathbb{P}^{\lambda, \Gamma_1}_\ell$ so by the induction hypothesis clearly $p_y \upharpoonright u^r \in \mathbb{P}^\lambda_{\Gamma_2}$.

SUBCASE 3A: $\mathrm{nice}(\mathrm{dom}_{\ell(*),m}, e) < m^*$ (see Definition 1.3(1)).

Hence $(\mathrm{dom}_{\ell(*),m}, e) \in \Gamma_2$ and the desired conclusion easily holds.

[Why? We can find $p^*_y = p_y \upharpoonright u^r = p_y \upharpoonright (u^{p_y} \cap u^r) = r' \upharpoonright (u^{p_y} \cap u^r)$ hence $|u^{p^*_y}| < m^*$.

By the induction hypothesis p^*_y belongs to $\mathbb{P}^\lambda_{\Gamma_2}$ for each $y \in Y$. Now r, $\langle p^*_y : y \in Y \rangle$ and $\langle v_t : t \in Y^+ \rangle$ satisfies clauses (a)–(g) of Case 3 of Definition 2.4. Hence by Case 3 of Definition 2.4 $r'' =: r' \upharpoonright (\bigcup \{ u^{p^*_y} : y \in Y \})$ belong to $\mathbb{P}^\lambda_{\Gamma_2}$ but $r = r''$ so $r \in \mathbb{P}^\lambda_{\Gamma_2}$.]

SUBCASE 3B: Not subcase 3A.

So $\mathrm{nice}(\mathrm{dom}_{\ell(*),m}, e) \ge m^* > |u^r|$. For $a \subseteq \mathrm{dom}_{\ell(*),m}$ let $u_a = \{\alpha \in u^r :$ $\alpha \in u^{p_y}$ for some $y \in Y$ satisfies $y \subseteq a$ or $\alpha \in v_{\{\eta\}}, \eta \in a$ or $\alpha \in v_\emptyset\}$. Now

$(*)_0$ if $u^r \subseteq v_{\{\eta\}}$ for some $\eta \in \mathrm{dom}_{\ell(*),m}$ then $r \in \mathbb{P}^\lambda_{\Gamma_2}$.

[Why? By applying Case 2 (and 0) of Definition 2.4.]

$(*)_1$ if for some $\eta \in \mathrm{dom}_{\ell(*),m}$ we have $[(\{\alpha_1, \beta_1\} \ne \{\alpha_2, \beta_2\} \in [u]^2)$ & $c^r\{\alpha_1, \beta_1\} = c^r\{\alpha_2, \beta_2\} \Rightarrow \{\alpha_1, \beta_1, \alpha_2, \beta_2\} \subseteq v_{\{\eta\}}]$ then $r \in \mathbb{P}^\lambda_{\Gamma_2}$.

[Why? By $(*)_0$ and uses of Case 1 of Definition 2.4.]

$(*)_2$ if $y \in Y$ and $u^r \subseteq v_y$ then $r \in \mathbb{P}^\lambda_{\Gamma_2}$.

[Why? Similarly.]

Now

$(*)_3$ It is enough to find $a, b \subseteq \mathrm{dom}_{\ell(*),m}$ such that:

$(*)^3_{a,b}$ $u_a \ne u_a \cap u_b, u_b \ne u_a \cap u_b, u^r \subseteq u_a \cup u_b, u^r \not\subseteq u_a, u^r \not\subseteq u_b$ and $[\eta_1 \in u_a \backslash u_b$ & $\eta_2 \in u_b \backslash u_a \Rightarrow (\{\eta_1, \eta_2\}/e)$ is a singleton] and $|a \cap b| \le 1$.

[Why is this enough? As then r is gotten by Case 2 of Definition 2.4 from $(u_a, c^p \upharpoonright [u_a]^2), (u_b, c^b \upharpoonright [u_b]^2)$. The main point is why clause (d) of this case holds; now we shall prove more $c\ell_0(u_a \cap u_b) \cap (u_a \cup u_b) \subseteq u_a \cap u_b$; now by clause (e) of Case 3 of Definition 2.4 letting $t = a \cap b$

(it $\in \{\emptyset\} \cup \{\{\eta\}: \eta \in \text{dom}_{\ell(*),m}\}$ by the last statement in $(*)^3_{a,b}$) we have $u_a \cap u_b = u_t$ (see $(d), (f)$ Definition 2.4, Case 3), hence $c\ell_0(u_a \cap u_b) = c\ell_0(u_t), u_t \subseteq v_t$ hence $c\ell_0(u_a \cap u_b) \subseteq c\ell_0(v_{a\cap b})$ which is disjoint to $u_a \setminus u_b$ and to $u_b \setminus u_a$ by clause (e) in Case 3 of Definition 2.4 as $u_a \cap v_t = u_t$ and $u_b \cap v_t = u_t$.]

So now why can we find such a, b?

We try to choose $a_i \subseteq \text{dom}_{\ell(*),m}$ for $i = 2, 3, \ldots$ or for $i = 1, 2, \ldots$, such that $|a_i| = i, a_i \subseteq a_{i+1}$ and $i \leq |u_{a_i}|$. First assume that we cannot find neither a_2 nor a_1, then $y \in Y \Rightarrow |u^{p_y} \cap u^r| \leq 1$ and $\eta \in \text{dom}_{\ell(*),m} \Rightarrow |v_{\{\eta\}} \cap u^r| = 0$. If $(*)_2$ applies we are done, so there are $\langle (y_\ell, \gamma_\ell): \ell < k \rangle$ satisfying $y_\ell = \{\eta_{1,\ell}, \eta_{2,\ell}\} \in Y$ such that $u^r \cap u^{p_{y_\ell}} \setminus v_{\{\eta_{1,\ell}\}} \setminus v_{\{\eta_{2,\ell}\}} = \{\gamma_\ell\}$ and $k \geq 2$ so $u^r \setminus v_\emptyset = \{\gamma_\ell: \ell < k\}$. Let $u_1 = (u^r \cap v_\emptyset) \cup \{\gamma_0\}, u_2 = u^r \setminus \{\gamma_0\}$, clearly r is gotten from $r \upharpoonright u_1, r \upharpoonright u_2$ as in Case 2 of Definition 2.4.

Second, assume a_1 or a_2 is defined. So we are stuck in $a_{i(*)}$ for some $i(*)$, i.e., $a_{i(*)}$ is chosen but we cannot choose $a_{i(*)+1}$. If $u_{a_{i(*)}} \neq u^r$, let $a = a_{i(*)}, b = \text{dom}_{\ell(*),m} \setminus a_{i(*)}$, so we get $(*)^3_{a,b}$ and we are done. So $u^r = u_{a_{i(*)}}$, hence $i(*) = |a_{i(*)}| = |u^r| < m^*$ and we can assume that $(*)_2$ does not apply. By the niceness of $(\text{dom}_{\ell(*),m}, e)$ the graph $H[e] \upharpoonright a_{i(*)}$ has no cycle so is a tree in the graph theoretic sense and so for some $c, b \subseteq a_{i(*)}$ we have $c \cap b = \{\eta\}, c \cup b = a_{i(*)}, b \neq \{\eta\}, c \neq \{\eta\}$ and $[\eta' \in b \setminus \{\eta\}$ & $\eta'' \in c \setminus \{\eta\} \Rightarrow \{\eta', \eta''\}$ not an $H[e]$-edge]; so we get $(*)^3_{a,b}$ and we are done. (So if we change slightly the claim demanding only $2|u^r| < m^*$, the proof is simpler). $\square_{3.2}$

3.3. THE PRESERVATION CLAIM. *Let* $n(*), \ell(*), \lambda, \mu = \aleph_0$ *be as in Definition 2.4 and assume* $\lambda > \beth_2(\mu^+)$.

(1) *If* $\mathbb{P} = \mathbb{P}^\lambda_\Gamma$ *and* $(\text{dom}_{\ell(*),m}, e) \in \Gamma \subseteq ID^*_{\ell(*)}$ *then in* $\mathbf{V}^\mathbb{P}$ *we have* $(\text{dom}_{\ell(*),m}, e) \in ID_2(\lambda, \aleph_0)$.

(2) *Assume that* $\mathbb{P} = \prod_{n<\gamma} \mathbb{P}^\lambda_{\Gamma_n}$ *where* $\Gamma_n \subseteq ID^*_{\ell(*)}$ *and* $\gamma \leq \omega$ *and* $p^* \in \mathbb{P}$ *forces that* $\underset{\sim}{c}$ *is a function from* $[\lambda]^2$ *to* ω. *Then for some finite* $d \subseteq \gamma$ *for any* $\mathbf{s} \in \bigcap_{n \in d} \Gamma_n$ *we have* $p^* \not\Vdash_\mathbb{P}$ "$\mathbf{s} \notin ID_2(\underset{\sim}{c})$".

PROOF. (1) Follows from (2), letting $\gamma = 1, \Gamma_0 = \Gamma$.

(2) Assume $p^* \in \mathbb{P}$ and $p^* \Vdash_\mathbb{P}$ "$\underset{\sim}{c}$ is a function from $[\lambda]^2$ to ω". Let $k(*) = 2^{\ell(*)} - 1$ and let $k(v) = |\{\rho \in {}^{\ell(*)>}2: \rho <_{\text{lex}} v\}|$ for $v \in {}^{\ell(*)>}2$. For $p \in \mathbb{P}$ let $u[p] = \cup\{u^{p(n)}: n \in \text{Dom}(p)\}$, so $u[p] \in [\lambda]^{<\aleph_0}$ and for any $q \in \mathbb{P}$ we let $n[q] = \sup(\cup\{\text{Rang}(c^{q(n)}): n \in \text{Dom}(q)\})$. For any $\alpha < \beta < \lambda$ letting $t = \{\alpha, \beta\}$ we define, by induction on $k \leq k(*)$ the triple $(n_{t,k}, w_{t,k}, d_{t,k})$ such that:

$(*)$ $n_{t,k} < \omega, w_{t,k} \in [\lambda]^{<\aleph_0}$ and $d_{t,k} \subseteq \gamma$ is finite.

CASE 1. $k = 0$: $n_{t,k} = n[p^*] + 2$ and $w_{t,k} = \{\alpha, \beta\} \cup u^{p^*}$ and $d_{t,k} = \text{Dom}(p^*)$.

CASE 2. $k + 1$:

Let $\mathscr{P}_{t,k} = \{q \in \mathbb{P} \colon p^* \leq q, u[q] \subseteq w_{t,k}$ and $n[q] \leq n_{t,k}$ and $\mathrm{Dom}(q) \subseteq d_{t,k}\}$; clearly it is a finite set, and for every $q \in \mathscr{P}_{t,k}$ we choose $p_{t,q}$ such that $q \leq p_{t,q} \in \mathbb{P}$ and $p_{t,q}$ forces a value, say $\zeta_{t,q}$ to $\underset{\sim}{c}(t)$. Now we let

$$w_{t,k+1} = w_{t,k} \cup \bigcup\{u[p_{t,k}] \colon q \in \mathscr{P}_{t,k}\},$$

$$d_{t,k+1} = d_{t,k} \cup \{\mathrm{Dom}(q_{t,p}) \colon p \in \mathscr{P}_{t,k}\},$$

$$n_{t,k+1} = \mathrm{Max}\{|w_{t,k+1}|^2, n_{t,k} + 1, n[p_q] + 1 \colon q \in \mathscr{P}_{t,k}\}.$$

We next define an equivalence relation E on $[\lambda]^2 \colon t_1 E t_2$ iff letting $t_1 = \{\alpha_1, \beta_1\}$, $t_2 = \{\alpha_2, \beta_2\}$, $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ and letting $h = OP_{w_{\{\alpha_2,\beta_2\},k(*)}, w_{\{\alpha_1,\beta_1\},k(*)}}$, we have

(i) $w_{t_1,k(*)}, w_{t_2,k(*)}$ has the same number of elements,

(ii) h maps α_1 to α_2 and β_1 to β_2 and $w_{t_1,k}$ onto $w_{t_2,k}$ for $k \leq k(*)$ (so h is onto),

(iii) $d_{t_1,k} = d_{t_2,k}$ for $k \leq k(*)$ (hence h maps $\mathscr{P}_{t_1,k}$ onto $\mathscr{P}_{t_2,k}$),

(iv) if $q_1 \in \mathscr{P}_{t_1,k}, k < k(*)$ then h maps q_1 to some $q_2 \in \mathscr{P}_{t_2,k}$ and it maps p_{t,q_1} to p_{t,q_2} and we have $\zeta_{t_1,q_1} = \zeta_{t_2,q_2}$.

Clearly E has $\leq \aleph_0$ equivalence classes. So let $c \colon [\lambda]^2 \to \aleph_0$ be such that $c(t_1) = c(t_2) \Leftrightarrow t_2 E t_2$ and let $w_t = w_{t,k(*)}$.

By Claim 3.1, recalling that we have assumed $\lambda > \beth_2(\aleph_1)$ we can find $W \subseteq \lambda$ of cardinality μ^+ and $\bar{v} = \langle v_t \colon t \in [W]^{\leq 2}\rangle$ as there; i.e., we apply it to an expansion of M_λ^* such that $c\ell_0(-) = c\ell_M(-)$.

Let $d_k^* = d_{t,k} \subseteq \omega$ for $t \in [W]^2$ and $k \leq k(*)$, now we choose $d = d_{k(*)}^* \subseteq \gamma$, and we shall show that it is as required in the claim. Let $\mathbf{s} = (\mathrm{dom}_{\ell(*),m(*)}, e) \in \bigcap_{\ell \in d}\Gamma_\ell$ and let $Y_v = Y_{e,v} = \{\{\eta_0, \eta_1\} \colon v^\frown\langle i\rangle \trianglelefteq \eta_i \in \mathrm{dom}_\mathbf{s}$ for $i = 0, 1$ and $\{\eta_0, \eta_1\}/e$ is not a singleton$\}$ for $v \in {}^{\ell(*)>}2$ and let $Y = \cup\{Y_{e,v} \colon v \in {}^{\ell(*)>}2\}$.

We now choose $\alpha_\eta \in W$ for $\eta \in \mathrm{dom}_\mathbf{s}$ such that $\eta_1 <_{\mathrm{lex}} \eta_2 \Rightarrow \alpha_{\eta_1} < \alpha_{\eta_2}$. Let $S = \{\alpha_\eta \colon \eta \in \mathrm{dom}_\mathbf{s}\}$. For $y \in Y$ let $t(y) = \{\alpha_\eta \colon \eta \in y\}$. Let $\langle v_\ell^* \colon \ell < k(*) = 2^{\ell(*)} - 1\rangle$ list ${}^{\ell(*)>}2$ in increasing order by $\leq_{\mathrm{lex}}$.

We now define q_ℓ and $q_{\eta,\ell}$ for $\eta \in \mathrm{dom}_{\ell(*),m(*)}$ and $p_{y,\ell}$ for $y \in Y$ by induction on $\ell \leq k(*)$ such that

(a) $p_{y,\ell} \in \mathscr{P}_{t(y),\ell}$ hence $u^{p_{y,\ell}} \subseteq w_{t,\ell}$ for every $y \in Y$,

(b) $\mathbb{P} \models ``p_{y,m} \leq p_{y,\ell}"$ for $m \leq \ell$,

(c) if $y \in Y$ and $\eta \in Y$ then $\mathrm{Dom}(q_{\eta,\ell}) = \mathrm{Dom}(p_{y,\ell})$ and for each $\beta \in \mathrm{Dom}(q_{\eta,\ell})$ we have $q_{\eta,\ell}(\beta) = p_{y,\ell}(\beta) \restriction v_{\{\eta\}}$ hence $m \leq \ell \Rightarrow q_{\eta,m} \leq q_{\eta,\ell}$ and $\mathrm{Dom}(q_\ell) = \mathrm{Dom}(q_{\eta,\ell}) \cap v_\emptyset$ and for each $\beta \in \mathrm{Dom}(q_\ell)$ we have $(q_{\eta,\ell}(\beta) \restriction v_\emptyset = q_\ell(\beta)$ so $m < \ell \Rightarrow q_m \leq q_\ell$,

(d) if $v_\ell^{*\frown}\langle i\rangle \trianglelefteq \eta_i \in \mathrm{dom}_{\ell(*),m(*)}$ for $i = 0, 1$ then $p_{y,\ell+1}$ forces a value to $\underset{\sim}{c}\{\alpha_\eta \colon \eta \in y\}$.

For $\ell = 0$ there is no problem. For $\ell + 1$ choose η_0^ℓ, η_1^ℓ such that $v_\ell^{*\frown}\langle i \rangle \trianglelefteq \eta_i^\ell \in$ $\mathrm{dom}_{\ell(*),m(*)}$ for $i = 0, 1$ and $\{\eta_0^\ell, \eta_1^\ell\}/e^{\mathbf{s}}$ is not a singleton and let $y_\ell = \{\eta_0, \eta_1\}$. As $p_{y_\ell, \ell} \in \mathscr{P}_{t(y_\ell), \ell}$ by the choice of $\mathscr{P}_{t(y), \ell+1}$ there is $p_{y_\ell}^\ell \in \mathscr{P}_{t(y_\ell), \ell+1}$ above $p_{y_\ell, \ell}$, which forces a value to $\underset{\sim}{c}(t(y_\ell))$. For $p \in \mathbb{P}$ and $u \subseteq \lambda$ let $q = p \restriction u$ means $\mathrm{Dom}(p) = \mathrm{Dom}(q)$ and $\beta \in \mathrm{Dom}(p) \Rightarrow q(p) = (p(\beta)) \restriction u$.

Now we define $\langle p_{y, \ell+1} \colon y \in Y_{v_i} \rangle$:

$$\text{if } y \in Y_{v_i} \text{ then } p_{y, \ell+1} = OP_{v_y, v_{y_\ell}}(p_{y_\ell}^\ell).$$

So necessarily

$(*)_1$ if $y'' \neq y'' \in Y_{v_\ell}$ then $p_{y', \ell+1} \restriction v_\emptyset = p_{y'', \ell+1} \restriction v_\emptyset$ is above (by $\leq_{\mathbb{P}}$) q_ℓ,

$(*)_2$ if $y' \neq y'' \in Y_{v_\ell}$ and $y' \cap y'' \neq \emptyset$ then for some $\eta \in \mathrm{dom}_{\mathbf{s}}$ we have $y' \cap y'' = \{\eta\}$ and $p_{y', \ell+1} \restriction v_{\{\eta\}} = p_{y'', \ell+1} \restriction v_{\{\eta\}}$ is above $q_{\eta, \ell}$.

[Why? As if let $y' = \{\eta_0', \eta_1'\}, y'' = \{\eta_0'', \eta_1''\}, v_\ell^{\frown}\langle i \rangle \trianglelefteq \eta_i', \eta_i''$ for $i = 0, 1$ then either $\eta_0' = \eta_0'', y' \cap y'' = \{\eta_0'\}$ or $\eta_1' = \eta_1'', y' \cap y'' = \{\eta_1'\}$ but $\eta_0' \neq \eta_1''$ & $\eta_1' \neq \eta_0''$. Now use the properties from 3.1 and clause (iv) above.]

Let $q_{\emptyset, \ell+1} = p_{y_\ell}^{\ell+1} \restriction v_\emptyset$. The $q_{\eta_i', \ell}$ is defined as $q_{\eta_i', \ell+1} = p_{\{\eta_0', \eta_1'\}, \ell+1} \restriction v_{\{\eta_i'\}}$ for $i = 0, 1$ if $v_\ell^{\frown}\langle i \rangle \trianglelefteq \eta_i'$ & $\{\eta_0', \eta_1'\} \in Y_{v_\ell}$.

Let $q_{\eta, \ell+1}$ be the result of free amalgamation (i.e., Case 2 of Definition 2.4) in each coordinate β of $q_{\eta, \ell}$ and $q_{\emptyset, \ell+1}$ if $\eta \in \mathrm{dom}_{\mathbf{s}} \wedge \neg(v_\ell \trianglelefteq \eta)$ and $\eta \in \mathrm{dom}_{\mathbf{s}}$.

Let $p_{y, \ell+1}$ be the result of free amalgamation (i.e., Case 2 of Definition 2.4) in each coordinate (twice) of $p_{y, \ell}, q_{\{\eta_0'\}, \ell+1}, q_{\{\eta_1\}, \ell+1}$ if $y = \{\eta_0, \eta_1\} \in Y \setminus Y_{v_\ell}$.

Of course, putting two conditions together using Case 2 of Definition 2.4, not repeating colours except when absolutely necessary.

Lastly, let p^+ be such that $\mathrm{Dom}(p^+) = d_{k(*)}^*$ and for each $\beta \in d_{k(*)}^*$

$$u^{p^+}(\beta) = \cup\{u^{p_{y,k(*)}} \colon y \in Y\};$$

if $u^{p_{y,k(*)}}(\beta)$ is not defined, it means $\emptyset$

$c^{p^+}(\beta)$ extend each $c^{p_{y,k(*)}}(\beta)$ otherwise is 1-to-1 with new values.

So $p^+ \geq p^*$ forces that $\{\alpha_\eta \colon \eta \in \mathrm{dom}_{\ell(*),m(*)}\}$ exemplify $\mathbf{s} = (\mathrm{dom}_{\ell(*),m(*)}, e) \in ID(f)$, a contradiction. $\qquad\square_{3.3}$

3.4. THE EXAMPLE CLAIM. *Let* $n(*), \ell(*), \lambda$ *be as in Definition 2.4. Assume*

(a) $(\mathrm{dom}_{\ell(*),m(*)}, e^*) \in ID_{\ell(*)}^*$,

(b) $\Gamma \subseteq ID_{\ell(*)}^*$,

(c) *if* $\mathbf{s} \in \Gamma$ *then* $\mathbf{s}$ *is* $(2^{\ell(*)}m(*))$-*nice*,

(d) $\mathbb{P} = \mathbb{P}_\Gamma^\lambda$,

(e) $\underset{\sim}{c}$ *is the* $\mathbb{P}$-*name* $\cup\{c^p \colon p \in G_{\mathbb{P}}\}$,

(f) $\ell(*) \geq n(*)$.

Then $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{c}$ *is a function from* $[\lambda]^2$ *to* μ *exemplifying* $(\mathrm{dom}_{\ell(*),m(*)}, e^*)$ *does not belong to* $ID_2(\lambda, \mu)$".

REMARK. The proof is similar to [GcSh:491].

PROOF. So assume toward contradiction that $p \in \mathbb{P}$ and $\alpha_\eta < \lambda$ for $\eta \in \mathrm{dom}_{\ell(*),m(*)}$ are such that p forces that $\eta \mapsto \alpha_\eta$ is a counterexample, i.e., $\langle \alpha_\eta : \eta \in \mathrm{dom}_{\ell(*),m(*)} \rangle$ is with no repetitions and p forces that $t_1 e^* t_2 \Rightarrow \underline{c}(\{\alpha_\eta : \eta \in t_1\}) = \underline{c}(\{\alpha_\eta : \eta \in t_2\})$. By 2.5(2) without loss of generality $\{\alpha_\eta : \eta \in \mathrm{dom}_{\ell(*),m(*)}\} \subseteq u^p$.

Let $Y = Y_{e^*} = \{y : y \in \mathrm{Dom}(e)$ and y/e is not a singleton$\}$ and for $v \in {}^{\ell(*)>}2$ let $Y_v = Y_{v,e^*} = \{\{\eta_0, \eta_1\} \in Y_{e^*} : v^\frown\langle i\rangle \trianglelefteq \eta_i$ for $i = 0, 1\}$ as in the previous proof. We now choose by induction on $\ell \leq n(*)$ the objects $\eta_\ell, v_\ell, Z_\ell$ and first order formulas $\varphi_\ell(x, y_0, \ldots, y_{\ell-1}), <^\ell_{y_0,\ldots,y_{\ell-1}}(x, y)$ in the vocabulary of M_λ^* such that:

$\boxtimes$ (a) $v_\ell \in {}^\ell 2, \eta_\ell \in \mathrm{dom}_{\ell(*),m(*)}$ and $M_\lambda^* \models (\exists^{\leq \aleph_{(n(*)-\ell)}} x)\varphi_\ell(x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}})$,

     (b) $<^\ell_{\alpha_{\eta_0}}, \ldots, \alpha_{\eta_{\ell-1}}$ is a well ordering of $\{x : M_\lambda^* \models \varphi_\ell[x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}}]\}$ of order type a cardinal $\leq \aleph_{n(*)-\ell}$,

     (c) $v_0 = \langle\rangle, \varphi_0 = [x = x]$,

     (d) $v_{\ell+1} = (\eta_\ell \restriction \ell)^\frown \langle 1 - \eta_\ell(\ell)\rangle$ and $v_\ell \triangleleft \eta_\ell$,

     (e) $Z_\ell = \{\eta : v_\ell \triangleleft \eta \in \mathrm{dom}_{\ell(*),m(*)}$ and $\{\eta_s, \eta\} \in e_{v\restriction s}$ for
$$s = 0, 1, \ldots, \ell - 1\},$$

     (f) $\eta \in Z_\ell \Rightarrow \alpha_\eta \in \{\beta : M_\lambda^* \models \varphi_\ell[\beta, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}}]\}$,

     (g) η_ℓ is such that:

         (α) $v_\ell \triangleleft \eta_\ell \in Z_\ell$,

         (β)   if $v_\ell \trianglelefteq \eta \in Z_\ell$ then $\alpha_\eta \leq^\ell_{\alpha_{\eta_0},\ldots,\alpha_{\eta_{\ell-1}}} \alpha_{\eta_\ell}$.

(See similar proof with more details in 4.3).

Let $v^* = v_{n(*)}, Z = Z_{n(*)}, Z^+ = \{\eta_\ell : \ell < n(*)\} \cup Z$; note that by Definition 1.3(1), clause (b) and Definition 2.1 we have $|Z| \geq 2$, i.e., this is part of $(\mathrm{dom}_{\ell(*),m(*)}, e^*)$ being 0-nice. For $v \in \{v_\ell : \ell < n(*)\}$ let s_v be such that: $\rho_1 \cap \rho_2 = v \;\&\; \rho_1, \rho_2 \in \{\eta_\ell : \ell < n(*)\} \cup Z \Rightarrow s_v = c\{\alpha_{\rho_1}, \alpha_{\rho_2}\}$ (clearly exists). By Case 0 in Definition 2.4, without loss of generality

$$u^p = \{\alpha_\eta : \eta \in Z^+\},$$

that is, we may forget the other $\alpha \in u^p$; by claim 3.2 we have $p \in \mathbb{P}_\emptyset^\lambda$ so for some k we have $r \in \mathbb{P}_k^{\lambda,\emptyset}$.

So we have

$\boxplus$ $\langle \eta_\ell : \ell < n(*)\rangle, Z, Z^+, \langle v_\ell : \ell \leq n(*)\rangle, \langle s_{\eta_\ell \restriction \ell} : \ell < n(*)\rangle$ and p are as above, that is

     (i) $\eta_\ell \in \mathrm{dom}_{\ell(*),m(*)}, v_0 = \langle\rangle, v_{\ell+1} = (\eta_\ell \restriction \ell)^\frown \langle(1 - \eta_\ell(\ell))\rangle, Z = \{\rho \in \mathrm{dom}_{\ell(*),m(*)} : v_{n(*)} \triangleleft \rho$ and $\{\eta_\ell, \rho\}/e$ is not a singleton for each $\ell < n(*)\}$ hence $|Z| \geq 2$ and $Z^+ = Z \cup \{\eta_\ell : \ell < n(*)\}$,

     (ii) $p \in \mathbb{P}_k^{\lambda,\emptyset}$,

     (iii) $\alpha_\eta \in u^p$ for $\eta \in Z^+$,

(iv) $\langle \alpha_\eta : \eta \in Z^+ \rangle$ is with no repetitions,

(v) $c^p \restriction \{\alpha_\eta : \eta \in Z^+\}$ satisfies:

  if $\ell < n(*)$ and $v \in Z \cup \{\eta_t : \ell < t < n(*)\}$ so $\eta_\ell \cap v = \eta_\ell \restriction \ell$ then $(\alpha_v \neq \alpha_{\eta_\ell}$ and) $c(\{\alpha_v, \alpha_{\eta_\ell}\}) = s_{\eta_\ell \restriction \ell}$,

(vi) $\{\alpha_\eta : \eta \in Z\} \subseteq c\ell_0\{\alpha_{\eta_\ell} : \ell < n(*)\}$,

(vii) Z has at least two members.

Among all such examples choose one with $k < \omega$ minimal. The proof now splits according to the cases in Definition 2.4.

CASE 0. $k = 0$.

Trivial.

CASE 1. Let p_1, α be as there, so recall that $\{\alpha, \beta\} e^{p_1} \{\alpha', \beta'\} \Rightarrow \{\alpha, \beta\} = \{\alpha', \beta'\}$. Hence obviously, by clauses (v) and (vii) above, $\eta \in Z^+ \Rightarrow \alpha_\eta \neq \alpha$, so $\{\alpha_\eta : \eta \in Z^+\} \subseteq u^{p_1}$, contradicting the minimality of k.

CASE 2. Let $p_i = (u_i, c_i) \in \bigcup_{\ell < k} \mathbb{P}_\ell^{\lambda, \emptyset}$ for $i = 1, 2$ be as there. We now prove by induction on $\ell < n(*)$ that $\alpha_{\eta_\ell} \in u_0 \cap u_1$. If $\ell < n(*)$ and it is true for every $\ell' < \ell$, but (for some $i \in \{1, 2\}$), $\alpha_{\eta_\ell} \in u_i \backslash u_{3-i}$, it follows by clause (v) of $\boxplus$ that the sequence $\langle c(\{\eta_\ell, v\}) : v \in Z_\ell^* \rangle$ is constant where we let $Z_\ell^* = \{\eta_{\ell+1}, \eta_{\ell+2}, \ldots, \eta_{n(*)-1}\} \cup Z$, hence $\{\alpha_v : v \in Z_\ell^*\}$ is disjoint to $u_{3-i} \backslash u_i$, so $\{\alpha_v : v \in Z^+\} \subseteq u_i$, so we get contradiction to the minimality of k.

As $\{\alpha_{\eta_\ell} : \ell < n(*)\} \subseteq u_2 \cap u_1$ necessarily (by clause (vi) of $\boxplus$) we have $\{\alpha_v : v \in Z_{n(*)}^*\} \subseteq c\ell_0\{\alpha_{\eta_\ell} : \ell < n(*)\} \subseteq c\ell_0(u_2 \cap u_1)$. But $\{\alpha_v : v \in Z_{n(*)}^*\} \subseteq u_2 \cup u_1$ by $\boxplus$(iii), and we know that $c\ell_0(u_2 \cap u_1) \cap (u_2 \cap u_1) \subseteq u_1$ by clause (d) of Definition 2.4, Case 2 hence $\{\alpha_v : v \in Z_0^*\} \subseteq u_1$ contradiction to "k minimal".

CASE 3. This case never occurs as $p \in \mathbb{P}_k^{\lambda, \emptyset}$. $\square_{3.4}$

3.5. THEOREM. (1) *Let* $n(*) = 4$ (*or just* $n(*) \geq 4$), $\lambda = \aleph_{n(*)}, \ell(*) = n(*) + 1$ *and* $2^{\aleph_\ell} = \aleph_{\ell+1}$ *for* $\ell < n(*)$.

  For some c.c.c. forcing $\mathbb{P}$ *of cardinality* λ *in* $\mathbf{V}^{\mathbb{P}}$ *the pair* $(\lambda, \aleph_0)$ *is not* $\aleph_0$-*compact.*

(2) *For given* $\chi = \chi^{\aleph_0} \geq \lambda$ *we can add* $\mathbf{V}^{\mathbb{P}} \models$ "$2^{\aleph_0} = \chi$".

PROOF. (1) Let $\Gamma_n = \{\mathbf{s} \in ID_{\ell(*)}^* : \mathbf{s} \text{ is } n\text{-nice}\}$, see Definition 2.1, clearly $\Gamma_{n+1} \subseteq \Gamma_n$ and $\Gamma_n \neq \emptyset$ (see 1.5) for $n < \omega$ and $\emptyset = \bigcap_{n < \omega} \Gamma_n$ and let $\mathbb{P}_n = \mathbb{P}_{\Gamma_n}^\lambda$ and let $c_n = \cup\{c^p : p \in G_{\mathbb{P}_m}\}$, it is a $\mathbb{P}_n$-name and $\mathbb{P}$ is $\prod_{n < \omega} \mathbb{P}_n$ with finite support. Now the forcing notion $\mathbb{P}$ satisfies the c.c.c. as $\mathbb{P}_n$ satisfies the Knaster condition (by 2.5(1)). By 3.4 we know that $\Vdash$ "$ID_2(c_n) \cap ID_{\ell(*)}^* \subseteq \Gamma_n$" for $\mathbb{P}_n$ hence for $\mathbb{P}$, in fact it is not hard to check that equality holds. If $\aleph_0$-compactness holds then in $\mathbf{V}^{\mathbb{P}}$ for some $c : [\lambda]^2 \to \omega$ we have $ID_2(c) \cap ID_{\ell(*)}^* \subseteq \bigcap_n \Gamma_n = \emptyset$ by claim 1.4.

But $\mathbf{V}^{\mathbb{P}}$, if $c : [\lambda]^2 \to \omega$ then by 3.3(2) it realizes some $\mathbf{s} \in \cup\{\Gamma_n : n < \omega\} \subseteq ID_{\ell(*)}^*$ (even k-nice one for every $k < \omega$).

Together we get that the pair $(\lambda, \aleph_0)$ is not $\aleph_0$-compact.

(2) We let $\mathbb{Q}$ be adding χ Cohen reals, i.e., $\{h \colon h$ a finite function from χ to $\{0, 1\}\}$ ordered by inclusion. Let $\mathbb{P}$ be as above and force with $\mathbb{P}^+ = \mathbb{P} \times \mathbb{Q}$, now it is easy to check that $\mathbb{P}^+$ is as required. $\square_{3.5}$

§4. Improvements and additions. Though our original intention was to deal with the possible incompactness of the pair $(\aleph_2, \aleph_0)$, we have so far dealt with $(\lambda, \aleph_0)$ where $2^{\aleph_0} \geq \lambda = \aleph_{n(*)}$ & $n(*) \geq 4$. For dealing with $(\aleph_3, \aleph_0), (\aleph_2, \aleph_0)$, that is $n(*) = 3, 2$ we need to choose M_λ^* more carefully.

What is the problem in §3 concerning $n(*) = 2$?

On the one hand in the proof of 3.4 we need that there are many dependencies among ordinals $< \lambda$ by M_λ^*; so if λ is smaller this is easier, but really just make us use larger $\ell(*)$ help.

On the other hand, in the proof of 3.3 we use 3.1, a partition theorem, so here if λ is bigger it is easier; but instead we can use demands specifically on M_λ^*. Along those lines we may succeed for $n(*) = 3$ using 3.1(2) rather than 3.1(1) but we still have problems for the pair $(\aleph_2, \aleph_0)$; here we change the main definition 2.4, in Case 3 changes $\langle v_y \colon y \in Y^+ \rangle$, i.e., for $\eta \in \mathrm{dom}_s$ we have $v_{\{\eta\}}^+, v_{\{\eta\}}^-$ instead $v_{\{\eta\}}$. For this we have to carefully reconsider 3.3, but the parallel of 3.1 is easier. Note that in §2, §3 we could have used a nontransitive version of $c\ell_M(-)$.

4.1. DEFINITION. We say that M^* is $(\lambda, < \mu, n(*), \ell(*))$-suitable if:

(a) M^* is a model of cardinality λ,

(b) λ is $> \mu, \leq \mu^{+n(*)}$ and $n(*) < \ell(*) < \omega$,

(c) τ_{M^*}, the vocabulary of M^*, is of cardinality $\leq \mu$,

(d) for every subset A of M^* of cardinality $< \mu$, the set $c\ell_{M^*}(A)$ has cardinality $< \mu$,

(e) for some $m^* < \omega$ we have:

if $\mathbf{s} = (\mathrm{dom}_{\ell(*), m}, e) \in ID_{\ell(*)}^*$ and $a_\eta \in M^*$ for $\eta \in \mathrm{dom}_{\ell(*), m}$ and s is m^*-nice, $m > m^*$, then we can find $\langle \eta_\ell \colon \ell < n(*) \rangle$ and $\langle v_\ell \colon \ell \leq n(*) \rangle$ such that

(α) $\eta_\ell \in \mathrm{dom}_{\ell(*), m}$,

(β) $v_0 = \langle \rangle, v_{\ell+1} = (\eta_\ell \restriction \ell)^\frown \langle 1 - \eta_\ell(\ell) \rangle$,

(γ) $v_\ell \lhd \eta_\ell$,

(δ) $Z = \{\rho \in \mathrm{dom}_{\ell(*), m} \colon v_{n(*)} \lhd \rho$ and in the graph $H[e]$, ρ is connected to η_ℓ for $\ell = 0, \ldots, n(*) - 1\}$,

(ε) $\{\alpha_\rho \colon \rho \in Z\} \subseteq c\ell_{M^*}\{\alpha_{\eta_\ell} \colon \ell < n(*)\}$.

4.2. DEFINITION. (1) We say that M is explicitly[1] $(\lambda, < \mu, n(*))$-suitable if:

(a) M^* is a model of cardinality λ,

(b) $\lambda = \mu^{+n(*)}$,

(c) τ_{M^*}, the vocabulary of M^*, is of cardinality $\leq \mu$,

(d) for $A \subseteq M^*$ of cardinality $< \mu$, the set $c\ell_{M^*}(A)$ has cardinality $< \mu$ and $A \neq \emptyset \wedge \mu > \aleph_0 \Rightarrow \omega \subseteq c\ell_{M^*}(A)$,

(e) for some $\langle R_\ell : \ell \leq n(*) \rangle$ we have

(α) R_ℓ is an $(\ell + 2)$-place predicate in τ_{M^*}; we may write $R_\ell(x, y, z_0, \ldots, z_{\ell-1})$ as $x <_{z_0,\ldots,z_{\ell-1}} y$ or $x <_{\langle z_0,\ldots,z_{\ell-1}\rangle} y$,

(β) for any $c_0, \ldots, c_{\ell-1} \in M^*$, the two place relation $<_{c_0,\ldots,c_{\ell-1}}$ (i.e., $\{(a,b): \langle a, b, c_0, \ldots, c_{\ell-1}\rangle \in R^{M^*}\}$) is a well ordering of $A_{c_0,\ldots,c_{\ell-1}} =: A_{\langle c_0,\ldots,c_{\ell-1}\rangle} =: \{b: (\exists x)(x <_{c_0,\ldots,c_{\ell-1}} b \vee b <_{c_0,\ldots,c_{\ell-1}} x)\}$ of order-type a cardinal,

(γ) $R_0^{M^*}$ is a well ordering of M^* of order type λ,

(δ) if $\bar{c} = \langle c_\ell : \ell < k \rangle$ and $<_{\bar{c}}$ is a well ordering of $A_{\bar{c}}$ of order type μ^{+m} then for every $c_k \in M_{\bar{c}}^*$ we have $A_{\bar{c}^\frown\langle c_k\rangle} = \{a \in A_{\bar{c}}: a <_{\bar{c}} c_k\}$ so is empty if $c_k \notin A_{\bar{c}}$, so if $lg(\bar{c}) = n(*)$ this is a definition of $A_{\bar{c}^\frown\langle c_k\rangle}$ as it is not covered by clause (β)

(ε) if $\bar{c} = \langle c_\ell : \ell < k \rangle \in {}^k(M^*)$ and $|A_{\bar{c}}| < \mu$ then $A_{\bar{c}} \subseteq c\ell_{M^*}(\bar{c})$.

(2) We say that M is explicitly[2] $(\lambda, < \mu, n(*))$-suitable if:

(a)–(d) as in part (1),

(e) for some $\langle R_\ell : \ell \leq n(*) \rangle$ we have (like (e) but we each time add z's and see clause (δ))

(α) R_ℓ is a $(2\ell + 2)$-place predicate in τ_μ; we may write $R_\ell(x, y, z_0, \ldots, z_{2\ell-1})$ or $x <_{z_0,\ldots,z_{2\ell-1}} y$ or $x <_{\langle z_0,\ldots,z_{2\ell-1}\rangle} y$,

(β) for any $c_0, \ldots, c_{2\ell-1} \in M^*$ the two-place relation $<_{c_0,\ldots,c_{2\ell-1}}$ (i.e., $\{(a,b): \langle a, b, c_0, \ldots, c_{2\ell-1}\rangle \in R_\ell^{M^*}\}$) is a well ordering of $A_{c_0,\ldots,c_{2\ell-1}} = A_{(c_0,\ldots,c_{2\ell-1})} = \{b: \text{for some } a, \langle a, b, c_0, \ldots, c_{2\ell-1}\rangle \in R_\ell^{M^*} \text{ or } \langle b, a, c_0, \ldots, c_{2\ell-1}\rangle \in R_\ell^{M^*}\}$,

(γ) $R_0^{M^*}$ is a well ordering of M^* of order type λ; for simplicity $R_0^{M^*} = c \restriction \lambda$,

(δ) if $\bar{c} = \langle c_\ell : \ell < 2k \rangle$ and $<_{\bar{c}}$ is a well ordering of $A_{\bar{c}}$ of order type μ^{+m} then for any $c_{2k}, c_{2k+1} \in M^*$ we have $A_{\bar{c}^\frown\langle c_{2k},c_{2k+1}\rangle}$ is empty if $\{c_{2k}, c_{2k+1}\} \not\subseteq A_{\bar{c}}$ and otherwise is $\{a \in A_{\bar{c}}: a <_{\bar{c}} c_{2k} \text{ and } a < c_{2k+1}\}$. If $k = n(*)$ this is a definition of $A_{\bar{c}^*\langle c_{2k},c_{2k+1}\rangle}$.

4.3. OBSERVATION. (1) *If M is an explicitly[1] $(\lambda, < \mu, n(*))$-suitable model, then M is a $(\lambda, < \mu, n(*) + 1, \ell(*))$-suitable model if $\ell(*) > n(*) + 1$.*

(2) *If M is an explicitly[2] $(\lambda, < \mu, n(*))$-suitable model, then M is a $(\lambda, < \mu, 2n(*) + 2, 2n(*) + 3)$-suitable model.*

PROOF. (1) Straightforward, similar to inside the proof of 3.4 and as we shall use part (2) only and the proof of (1) is similar but simpler, we do not elaborate.

(2) Clearly clauses (a)–(d) of Definition 4.1 holds, so we deal with clause (e). So assume $\ell(*) \geq 2n(*)$ and $\mathbf{s} = (\text{dom}_{\ell(*),m}, e) \in ID_{\ell(*)}^*$ and $\alpha_\eta \in M$ for $\eta \in \text{dom}_{\ell(*),m}$ are pairwise distinct. We choose by induction on $\ell \leq n(*)$ the

objects $\eta_{2\ell}$, $\nu_{2\ell+1}$, $Z_{2\ell}$, $\eta_{2\ell+1}$, $\nu_{2\ell+2}$, $Z_{2\ell+1}$ such that node $\nu_\ell = \langle \rangle$ and $\nu_{2\ell+2}$ is chosen in stage ℓ

$\boxtimes$ (a) $\nu_\ell \in {}^\ell 2, \eta_\ell \in \mathrm{dom}_{\ell(*), m(*)}$ and $M \models (\exists^{\leq \aleph_{(n(*)-\ell)}} x)\varphi_\ell(x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{2\ell-1}})$,

 (b) $<^\ell_{\alpha_{\eta_0}, \ldots, \alpha_{\eta_{2\ell-1}}}$ is a well ordering of
$$A_{\langle \alpha_{\eta_0}, \ldots, \alpha_{2\ell-1} \rangle} =: \{x : M \models \varphi_\ell[x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}}]\}$$
of order type a cardinal $\leq \aleph_{n(*)-\ell}$,

 (c) $\nu_0 = \langle \rangle$, $\varphi_0 = [x = x]$,

 (d) $\nu_{\ell+1} = (\eta_\ell \restriction \ell)^\frown \langle 1 - \eta_\ell(\ell) \rangle$,

 (e) $Z_\ell = \{\eta : \nu_{2\ell} \lhd \eta \in \mathrm{dom}_{\ell(*), m(*)}$ and $\{\eta_s, \eta\} \in e_{\nu \restriction s}$ for $s = 0, 1, \ldots, \ell - 1\}$,

 (f)' $\eta \in Z_\ell \Rightarrow \alpha_\eta \in A_{\langle a_{\eta_k} : k < 2\ell \rangle}$,

 (g) η_ℓ is such that:
  (α) $\nu_\ell \lhd \eta_\ell \in Z_\ell$,
  (β) if $\nu_\ell \trianglelefteq \eta \in Z_\ell$
   then [ℓ even $\Rightarrow \alpha_\eta \leq_{\alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}}} \alpha_{\eta_\ell}$] and [$\ell$ odd $\Rightarrow \alpha_\eta \leq \alpha_{\eta_\ell}$].

How do we do the induction step? Arriving to ℓ we have already defined $\langle \nu_k : k \leq 2\ell \rangle, \langle \eta_k : k < 2\ell \rangle$ and $\langle Z_k : k < 2\ell \rangle$, recalling $\nu_0 = \langle \rangle$. So by the definition of Z_k also $Z_{2\ell}$ is well defined and $\{\alpha_\eta : \eta \in Z_{2\ell}\}$ is included in $A_{\langle a_{\eta_k} : k < 2\ell \rangle}$ and let $\eta_{2\ell} \in Z_{2\ell}$ be such that $\eta \in Z_{2\ell} \Rightarrow a_\eta \leq_{\langle a_{\eta_k} : k < 2\ell \rangle} a_{\eta_{2\ell}}$ and $\nu_{2\ell+1} = \nu_{2\ell}^\frown \langle 1 - \eta_{2\ell}(2\ell) \rangle = (\eta_{2\ell} \restriction (2\ell))^\frown \langle 1 - \eta_{2\ell}(2\ell) \rangle$ so $Z_{2\ell+1}$ is well defined. Let $\eta_{2\ell+1} \in Z_{2\ell+1}$ be such that $\eta \in Z_{2\ell} \Rightarrow a_\eta \leq \alpha_{\eta_{2\ell+1}}$ and $\nu_{2\ell+2} = \nu_{2\ell+1}^\frown \langle 1 - \eta_{2\ell+1}(2\ell + 1) \rangle$ and we have carried the induction. $\square_{4.3}$

Are there such models? We shall use 4.4(2), the others are for completeness (i.e., part (3) is needed for $\lambda = \aleph_3$ and part (4) says concerning $\lambda = \aleph_2$ it suffices to use ID_3^*):

 4.4. OBSERVATION. (1) *For μ regular uncountable, there is an explicitly[1] $(\mu^{+2}, < \mu, 2)$-suitable model.*

 (2) *If $\mu = \aleph_0$, then there is an explictly[2] $(\mu^{+2}, < \mu, 2)$-suitable model.*

 (3) *If μ is regular uncountable, $t = 1$ or $\mu = \aleph_0$ & $t = 2$ and $n \in [3, \omega)$, then there is an explicitly[t] $(\mu^{+n}, < \mu, n)$-suitable model.*

 (4) *If $2^{\aleph_0} = \aleph_1, \mu = \aleph_0$ then for some $\aleph_2$-c.c., $\aleph_1$-complete forcing notion $\mathbb{Q}$ of cardinality $\aleph_2$ in $\mathbf{V}^\mathbb{Q}$ there is an explicitly $(\aleph_2, < \aleph_0, 2)$-suitable model.*

 4.5. REMARK. (1) It should be clear that if $\mathbf{V} = \mathbf{L}$ (or just $\neg 0^\#$), then this works also for singular μ *but* more reasonable is to use nontransitive closure.

 PROOF (1), (2). Let $t = 1$ for part (1) and $t = 2$ for part (2). Let $n(*) = 2$ and $\lambda = \mu^{+2}$. We choose M_α by induction on $\alpha \leq \lambda$ such that:

 (α) M_α is a τ^--model where $\tau^- = \{R_0, R_1, R_2\}$ with R_ℓ is $(t\ell + 2)$-predicate and $x <_{\bar{z}} y$ means $R_\ell(x, y, \bar{z})$,

 (β) M_α is increasing with α and has universe $1 + \alpha$,

 (γ) $R_0^{M_\alpha}$ is $< \restriction \alpha$ (and $A_{\langle \rangle}^{M_\alpha} = \alpha$),

(δ) for $\bar{c} \in {}^{tk}(M_\alpha), k = 0, 1, 2$ we have $<_{\bar{c}}$ is a well ordering of $A_{\bar{c}}^{M_\alpha} =:$
$\{a: M_\alpha \models (\exists x)(a <_{\bar{c}} x \lor x <_{\bar{c}} a)\}$ of order type a cardinal $<$
$\mu^{+(n(*)+1-k)}$,

(ε) (i) if $t = 1, \bar{c} \in {}^k(M_\alpha), k = 0, 1, 2$ and $d \in A_{\bar{c}}^{M_\alpha}$
then $A_{\bar{c}^{\frown}<d>}^{M_\alpha} = \{a \in A_{\bar{c}}^{M_\alpha}: M_\alpha \models a <_{\bar{c}} d\}$,

(ii) if $t = 2, \bar{c} \in {}^{2k}(M_\alpha), k = 0, 1, 2$ and $d_0, d_1 \in A_{\bar{c}}^{M_\alpha}$
then $A_{\bar{c}^{\frown}\langle d_0, d_1\rangle}^{M_\alpha} = \{a \in A_{\bar{c}}^{M_\alpha}: M_\alpha \models \text{``}a <_{\bar{c}} d_0 \ \& \ a < d_1\text{''}\}$,

(ζ) if A is a subset of M_α of cardinality $< \mu$
then $c\ell_{M_\alpha}^*(A)$ is of cardinality $< \mu$ and $c\ell_{M_\alpha}^*(c\ell_{M_\alpha}^*(A)) = c\ell_{M_\alpha}^*(A)$ where
$\boxtimes$ for $A \subseteq M_{\alpha'}, c\ell_{M_\alpha}^*(A)$ is the minimal set B such that: $A \subseteq B$
and $(\forall \bar{c} \in {}^{3t}B)(|A_{\bar{c}}^{M_\alpha}| < \mu \to A_{\bar{c}} \subseteq B)$; clearly B exists and
$c\ell_{M_\alpha}^*(\emptyset) = \emptyset$,

(η) for every $\beta < \alpha, k = 1, 2$ and $\bar{c} \in {}^k(M_\beta)$ we have $A_{\bar{c}}^{M_\alpha} = A_{\bar{c}}^{M_\beta}$,

(θ) if $A \subseteq \beta < \alpha$ then $c\ell_{M_\beta}^*(A) = c\ell_{M_\alpha}^*(A)$,

(ι) if $t = 2$ and $\mu = \aleph_0$ and $A \subseteq \alpha$ is finite, β is the last element in A, then
for some finite $B \subseteq \beta$ we have $c\ell_{M_\alpha}^*(A) = \{\beta\} \cup c\ell_{M_\beta}^*(B)$.

We leave the cases $\alpha < \mu$ and α a limit ordinal to the reader (for (ζ) we use (θ)) and assume $\alpha = \beta + 1$ and M_γ for $\gamma \leq \beta$ are defined. We can choose $\langle B_{\beta,i}: i < \mu^+\rangle$, a (not necessarily strictly) increasing sequence of subsets of β, each of cardinality $\leq \mu, B_{\beta,0} = \emptyset$ and $\cup\{B_{\beta,i}: i < \mu^+\} = \beta$ and $c\ell_{M_\beta}^*(B_{\beta,i}) = B_{\alpha,i}$.

For each $i < \mu^+$ let $\langle B_{\beta,i,\varepsilon}: \varepsilon < \mu\rangle$ be (not necessarily strictly) increasing sequence of subsets of $B_{\beta,i}$ with union $B_{\beta,i}$ such that $c\ell_{M_\beta}^*(B_{\beta,i,\varepsilon}) = B_{\beta,i,\varepsilon}$, $B_{\beta,0} = \emptyset$. Let $<_\beta^*$ be a well ordering of $\{\gamma: \gamma < \beta\}$ such that each $B_{\beta,i}$ is an initial segment so it has order type μ^+. For $\gamma \in B_{\beta,i+1}\backslash B_{\beta,i}$ let $<_{\beta,\gamma}^*$ be a well ordering of $A_{(\beta,\gamma)}^* = \{\xi: \xi <_\beta^* \gamma\}$ of order type $\leq \mu$ such that $(\forall \varepsilon < \mu)(B_{\beta,i+1,\varepsilon} \cap A_{(\beta,\gamma)}^*$ is an initial segment of $A_{(\beta,\gamma)}^*$ by $<_{\beta,\gamma}^*$.

Now we define M_α:

$$\text{universe is } \alpha$$
$$R_0^{M_\alpha} = < \upharpoonright \alpha$$

CASE 1. $t = 1$.
$R_1^{M_\alpha} = R_1^{M_\beta} \cup \{(a, b, \beta): a <_\beta^* b\}$

$R_2^{M_\alpha} = R_2^{M_\beta} \cup \{(a, b, \beta, \gamma): \gamma < \beta, \text{ and } a <_{\beta,\gamma}^* b \text{ hence } a <_\beta^* \gamma \ \& \ b <_\beta^* \gamma$
and $a, b \in B_{\beta,i+1}$ for the unique i such that $\gamma \in B_{\beta,i+1}\backslash B_{\beta,i}\}$.

CASE 2. $t = 2$.
$R_1^{M_\alpha} = R_1^{M_\beta} \cup \{(a, b, \beta, \gamma):$
$\quad\quad\quad\quad a <_\beta^* b$ and $a < \gamma, b < \gamma$ and, of course, $a, b, \beta \in \alpha\}$.

$$R_2^{M_\alpha} = R_2^{M_\beta} \cup \{(a, b, \beta, \gamma_0, \beta_1, \gamma_1) \colon a, b, \gamma_0, \beta_1 \in \alpha \text{ and } a < \beta,$$
$$b < \beta, a < \gamma_0, b < \gamma_0,$$
$$a, b, \beta_1, \gamma_1 \in A_{<\beta, \gamma_0>} \text{ and}$$
$$a <^*_{\beta, \gamma_0} b \text{ and } a < \gamma_1, b < \gamma_1\}.$$

To check for clause (ζ) is easy if $\mu = \mathrm{cf}(\mu) > \aleph_0$ and follows by clause (ι) if $\mu = \aleph_0$.

Having carried the induction we define M: it is M_λ expanded by $\langle F_i^M \colon i < \mu \rangle$ such that: if $\bar{c} \in {}^{3t}\lambda = {}^{3t}(M_\lambda)$ and $A_{\bar{c}}$ is a non empty well defined and of cardinality $< \mu$ (which follows) then $\{F_i^M(\bar{c}) \colon i < \mu\}$ list $A_{<c_0, c_1, c_2>} \cup \{0\}$ otherwise $\{F_i^M(\bar{c}) \colon i < \mu\}$ is $\{0\}$.

(3) Similar and used only for $(\aleph_3, \aleph_0)$ so we do not elaborate.

(4) Let $\mathbb{Q}$ be defined as follows:

$p \in \mathbb{Q}$ iff

(α) p is a τ^--model, as in (α) of the proof of part (1),

(β) the universe $\mathrm{univ}(p)$ of p is a countable subset of λ, we let $A_{\langle \rangle}^p = \mathrm{univ}(p)$,

(γ) $R_0^p = <\upharpoonright \mathrm{univ}(p)$ and $<_{\langle \rangle} = R_0^p$,

(δ) if $\bar{c} \in {}^{tk}(\mathrm{univ}(p)), k = 1, 2$ then $<_{\bar{c}} = <_{\bar{c}}^p$ is a well ordering of $A_{\bar{c}}^p = \{a \in p \colon p \models (\exists x)(a <_{\bar{c}} x \vee x <_{\bar{c}} a)\}$ and for $d \in A_{\bar{c}}^p$ we let $A_{\bar{c}^\frown <d>}^p = \{a \in A_{\bar{c}}^p \colon a <_{\bar{c}}^p d\}$,

(ε) $(A_{\bar{c}}^p, <_{\bar{c}})$ has order type ω if $k = 2$,

(ζ) if $A \subseteq \mathrm{univ}(p)$ is finite, then $c\ell_p^*(A)$ is finite (is defined as in(2)).

the order:

$\mathbb{Q} \models p \leq q$ iff

(i) p is a submodel of q,

(ii) if $\bar{c} \in {}^2(\mathrm{univ}(p))$ then $A_{\bar{c}}^p = A_{\bar{c}}^q$,

(iii) if $\bar{c} \in {}^1(\mathrm{univ}(p))$ then $A_{\bar{c}}^p$ is an initial segment of $A_{\bar{c}}^q$ by $<_{\bar{c}}$.

The rest should be clear. $\qquad\square_{4.4}$

4.6. CLAIM. *Assume (main case is $n(*) = 2$)*

(*) $2 \leq n(*) < \omega, \lambda = \aleph_{n(*)}, \ell(*) = 2n(*) + 3$ and $\lambda \leq \chi = \chi^{\aleph_0}$.

Then for some $\mathbb{P}^*$ we have

(a) $\mathbb{P}^*$ *is a forcing notion of cardinality χ,*

(b) $\mathbb{P}^*$ *satisfies the c.c.c.,*

(c) *in $\mathbf{V}^{\mathbb{P}^*}$ the pair $(\aleph_{n(*)}, \aleph_0)$ is not compact,*

(d) *in $\mathbf{V}^{\mathbb{P}^*}$ we have $2^{\aleph_0} = \chi$.*

REMARK. We intend to prepare a full version.

PROOF. We repeat §2, §3 with the following changes.

If $n(*) \geq 3$, we need change (A) below and using 3.1(2) instead of 3.1(1). For $n(*) = 2$ we need all the changes below

(A) we replace M_λ^* by any model as in 4.4(2) if $n(*) = 2$, 4.4(3) if $n(*) \geq 3$,

(B) in Definition 2.4, Case 3: we add $\langle v_{\{\eta\}}^+, v_{\{\eta\}}^- : \eta \in \mathrm{dom}_{\ell(*),m}\rangle, v_\emptyset^+, v_\emptyset^-$

   (d)′ (i) $v_y \supseteq u^{p_y}$ for $y \in Y$,

     (ii) if $\eta_1 <_{\mathrm{lex}} \eta_2 <_{\mathrm{lex}} \eta_3$ are from $\mathrm{dom}_{\ell(*),m}$ and $\{\eta_1, \eta_2\}, \{\eta_1, \eta_3\} \in Y$, then

$$v_{\{\eta_1, \eta_2\}} \cap v_{\{\eta_1, \eta_3\}} = v_\eta^+,$$

     (iii) if $\eta_1 <_{\mathrm{lex}} \eta_2 <_{\mathrm{lex}} \eta_3$ are from $\mathrm{dom}_{\ell(*),m}$ and $\{\eta_1, \eta_3\}, \{\eta_2, \eta_3\} \in Y$, then

$$v_{\{\eta_1, \eta_3\}} \cap v_{\{\eta_2, \eta_3\}} = v_\eta^-$$

$$p_{\{\eta_1, \eta_3\}} \restriction v_\eta^- = p_{\{\eta_2, \eta_3\}} \restriction v_\eta^-,$$

     (iv) $v_{\{\eta\}} = v_\eta^+ \cup v_\eta^-$,

     (v) if $\eta_1 \neq \eta_2$ then $v_{\eta_1}^+ \cap v_{\eta_2}^+ = v_\emptyset^+$ and $v_{\eta_2}^- \cap v_{\eta_2}^- = v_\emptyset^-$,

     (vi) if $\eta_1 <_{\mathrm{lex}} \eta_2 <_{\mathrm{lex}} \eta_2$ are from $\mathrm{dom}_{\ell(*),m}$, then $p \restriction v_\eta \in \bigcup_{r<k} \mathbb{P}_r$,

   (e) if $\eta_1 <_{\mathrm{lex}} \eta_2$ are from $\mathrm{dom}_{\ell(*),m}$ and $t = \{\eta_1, \eta_2\} \in Y$
     then $c\ell(v_{\eta_1}^+) \cap v_{\{\eta_1, \eta_2\}} = v_{\eta_1}^+, c\ell(v_{\eta_2}^-) \cap v_{\{\eta_1, \eta_2\}} = v_{\eta_2}^-$
     if $\{\eta_1, \eta_2\} \in Y, \eta \in \mathrm{dom}_{\ell(*),m} \setminus \{1, \eta_1\}$
     then $c\ell(v_\eta^\pm) \cap v_{\{\eta_1, \eta_2\}} \subseteq v_{\eta_1}^\pm$,

   (f) the functions $\langle c^{p_\eta} : \eta \in y\rangle$ are pairwise compatible,

(C) in 3.1

   (a) $\lambda \geq (2^\mu)^+, \mu = \mu^{<\mu}, (\forall A \in [M]^{<\mu})(|c\ell_M(A)| < \mu)$,

   (b) the conclusion: change as in Definition 2.4, Case 3,

   (c) proof:

CASE 1. $\mu = \aleph_0$: let $g : [\lambda]^2 \to \omega$ be $g(t) = |c\ell_M(\alpha, t \cup w_t)| < \omega$.

Let $W_1 \in [\lambda]^{\mu^+}$ be such that $g \restriction [W]^2$ is constant say $k(*)$ and $f \restriction [W]^2$ is constantly γ. Let $c\ell_M(t) = \{\zeta_{t,\ell} : \ell < g(t)\}$. By Ramsey theorem, there is an infinite $W \subseteq W_1$ such that:

   ⊛ the truth value on $\zeta_{\{\alpha_1, \beta_2\}, \ell_1} = \zeta_{\{\alpha_2, \beta_2\}, \ell_2}$ depend just on ℓ_1, ℓ_2, T.V.(α_i, β_j), T.V.$(\beta_j < \alpha_i)$ for $i, j \in \{1, 2\}$.

The conclusion should be clear.

(D) p in the proof of 3.2: only Case 3B need care, assuming $m(*) > 2|u^{p^*}|$, the relevant subgraph has no cycle by clause (e) of Case 3 of 2.4 we are done,

(E) in the proof of 3.3, we will have $q_{\eta,\ell}^+, q_{\eta,\ell}^-$ with domain $\subseteq v_\eta^-, v_\eta^+$ respectively and q_ℓ^+, q_ℓ^- such that if $\eta_1 <_{\mathrm{lex}} \eta_2$ and $\{\eta_1, \eta_2\} \in Y$ then $p_{\{\eta_1, \eta_2\}, \ell} \restriction v_{\eta_1}^+ = q_{\eta_1, \ell}^+$ and $p_{\{\eta_1, \eta_2\}, \ell} \restriction v_{\eta_2}^- = q_{\eta_2, \ell}^-$ and $q_{\eta, \ell}^+, q_{\eta, \ell}^-$ are compatible, $q_{\eta, \ell}^+ \restriction v_\emptyset^+ = q_\ell^+, q_{\eta, \ell}^- \restriction v_\emptyset = q_\ell^-$,

(F) 3.4: part of the work has already been done in 4.1–4.3.

§5. Open problems and concluding remarks. We finish the paper by listing some problems (some are old, see [CK]).

5.1. QUESTION. Suppose that λ is strongly inaccessible, $\mu > \aleph_0$ is regular not Mahlo and $\square_\mu$. Then $\lambda \to \mu$ in the λ-like model sense, i.e., if a first order ψ has a λ-like model then it has a μ-like model.

If λ is ω-Mahlo, the answer is yes, see [ScSh:20] by appropriate partition theorems. The assumption that μ is Mahlo is necessary by Schmerl, see [Sc85].

5.2. QUESTION. (Maybe under **V** = **L**.) Suppose that $\lambda^{\beth_\omega(\kappa)} = \lambda$ and $\lambda_1^{<\lambda_1} = \lambda_1 > \kappa_1$. Then $(\lambda^+, \lambda, \kappa) \to (\lambda_1^+, \lambda_1, \kappa_1)$.

5.3. QUESTION. (*GCH*) If λ and μ are strong limit singulars and λ is a limit of supercompacts, then $(\lambda^+, \lambda) \to (\mu^+, \mu)$.

5.4. QUESTION. Find a universe with $(\beth_2(\aleph_0), \aleph_0) \to (2^{2^\lambda}, \lambda)$ for every λ.

(The author has a written sketch of a result which is close to this one. He starts with $\aleph_0 = \kappa_0 < \kappa_1 < \cdots < \kappa_m$ which are supercompacts and let $\mathbb{P}_n$ be the forcing which adds κ_{n+1} Cohen subsets to κ_n in $V^{\mathbb{P}_0 * \mathbb{P}_1 \cdots \mathbb{P}_{n-1}}$ for $n < m$. The idea is using the partition on trees from [Sh:288, §4]).

5.5. QUESTION. Are all pairs in the set

$$\{(\lambda, \mu): 2^\mu = \mu^+ \;\&\; \mu = \mu^{<\mu} \;\&\; \mu^{+\omega} \le \lambda \le 2^{\mu^+}\}$$

such that there is μ^+-tree with $\ge \mu^+$ branches, equivalent for the two cardinal problem? More related to this particular work are

5.6. QUESTION. (1) Can we find $n < \omega$ and an infinite set Γ^* of identities (or 2-identities) such that for any $\Gamma \subseteq \Gamma^*$ for some forcing notion $\mathbb{P}$ in $V^{\mathbb{P}}$ we have $\Gamma = \Gamma^* \cap ID(\aleph_n, \aleph_0)$.

(2) In (1) we can consider (λ, μ) with $\mu = \mu^{\aleph_0}, \lambda = \mu^{+n}$, so we ask: can we find a forcing notion $\mathbb{P}$ not adding reals such that for every $\Gamma \subseteq \Gamma^*$ for some $\mu = \mu^{<\mu}$ we have $\Gamma = \Gamma^* \cap ID(\mu^{+n}, \mu)$.

5.7. QUESTION. (1) Can we get results parallel to 3.5 for $(\aleph_2, \aleph_1) + 2^{\aleph_0} \ge \aleph_2$ (so we should start with a large cardinal, at least a Mahlo).

(2) The parallel to 5.6(1),(2).

5.8. QUESTION. (1) Can we get results parallel to 3.5 for $(\aleph_{\omega+1}, \aleph_\omega) +$ G.C.H. (or (μ^+, μ), μ strong limit singular + G.C.H.

(2) The parallel to 5.6(1),(2).

5.9. QUESTION. How does assuming MA $+ 2^{\aleph_0} > \aleph_n$ influence $ID(\aleph_n, \aleph_0)$? (see below).

We end with some comments:

5.10. DEFINITION. (1) For $k \le \aleph_0$, we say (λ, μ) has k-simple identities when $(a, e) \subseteq ID(\lambda, \mu) \Rightarrow (a, e') \in ID(\lambda, \mu)$ whenever:

$(*)_k$ $a \subseteq \omega, (a, e)$ is an identity of (λ, μ) and e' is defined by $be'c$ iff $|b| = |c|$ & $(\forall b', c')[b' \subseteq b$ & $|b'| \leq k$ & $c' = OP_{c,b}(b') \rightarrow b'ec']$ recalling

$OP_{A,B}(\alpha) = \beta$ iff $\alpha \in A$ & $\beta \in B$ & $\mathrm{otp}(\alpha \cap A) = \mathrm{otp}(\beta \cap B)$.

5.11. Claim. (1) *If* (λ_1, μ_1) *has k-simple identities and there is f:* $[\lambda_2]^{\leq k} \rightarrow \mu_2$ *such that* $ID_{\leq k}(f) \subseteq ID_{\leq k}(\lambda_1, \mu_1)$, *then* $(\lambda_1, \mu_1) \rightarrow (\lambda_1', \mu_1')$.

(2) *If* $\mathrm{cf}(\lambda_1) > \mu$, *then we can use f with domain* $[\lambda_1']^{\leq k} \backslash [\lambda_1']^{\leq 1}$.

Proof. Should be easy.

5.12. Claim. (1) $[MA + 2^{\aleph_0} > \aleph_n]$. *The[4] pair* $(\aleph_n, \aleph_0)$ *has 2-simple identities.*

(2) *If* $\mu = \mu^{<\mu}$ *and* $\gamma \leq \omega$ *then for some* μ^+-*c.c.,* $(< \mu)$-*complete forcing notions,* $\mathbb{P}$ *in* $\mathbf{V}^{\mathbb{P}}$ *we have* $2^\mu \geq \mu^{+\gamma}$ *and* $n \leq \gamma$ & $n < \omega \Rightarrow (\mu^{+n}, \mu)$ *has 2-simple identities.*

(3) *If* $m < n < \omega, \mu = \mu^{<\mu}$, *then* $[\mu^{+n}, \mu^{+m})$ *has* $(m + 2)$-*simple identities in* $\mathbf{V}^{\mathbb{P}}$ *for appropriate* μ^+-*c.c.* $(< \mu)$-*complete forcing notion.*

Proof. (1) For any $c : [\aleph_n]^{<\aleph_0} \rightarrow \omega$ we define a forcing notion $\mathbb{P} = \mathbb{P}_c$ as follows:

$p \in \mathbb{P}$ iff:

(a) $p = (u, f) = (u^p, f^p)$,

(b) u is a finite subset of $\aleph_n$,

(c) f is a function from $[u]^2$ to ω,

(d) if $k < \omega, k \geq 2$ and $\alpha_0 < \cdots < \alpha_{k-1}$ are from $u, \beta_0 < \cdots < \beta_{k-1}$ are from u and $[\ell(1) < \ell(2) < k \Rightarrow f(\{\alpha_{\ell(1)}, \alpha_{\ell(2)}\}) = f(\{\beta_{\ell(1)}, \beta_{\ell(2)}\})]$, then $c(\{\alpha_0, \ldots, \alpha_{k-1}\}) = c(\{\beta_0, \ldots, \beta_{k-1}\})$.

The rest should be clear.

(2), (3) Similar (use e.g., [Sh:546]).   □ $_{5.12}$

We can give an alternative proof of [Sh:49], note that by absoluteness the assumption MA is not a real one; it can be eliminated and $(\mu^{+\omega}, \mu) \rightarrow' (2^{\aleph_0}, \aleph_0)$ can be deduced.

5.13. Claim. *Assume* $MA + 2^{\aleph_0} > \aleph_\omega$.

Then $(\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)$.

Proof. Let $\{\eta_\alpha : \alpha < 2^{\aleph_0}\}$ list $^\omega 2$, and define $f : [2^{\aleph_0}]^2 \rightarrow {}^{\omega >}2$ by:

$(*)$ $f\{\alpha_0, \alpha_1\} = \eta_{\alpha_0} \cap \eta_{\alpha_1} \in {}^{\omega >}2$ for $\alpha_0 \neq \alpha_1$.

So by 5.11, 5.12 it is enough to prove that $ID_2(f) \subseteq ID_2(\aleph_\omega, \aleph_0)$.

Clearly

$(*)_1$ if $\lambda \leq 2^{\aleph_0}, (u, e) \in ID_2(\lambda, \aleph_0)$ then $(u, e) \in ID_2(f \restriction \lambda)$ hence $(u, e) \in ID_2(f)$ hence for some $n, (u, e)$ can be embedded (in the natural sense) into $(^n 2, e_n^*)$ where $(\{\eta_1, \eta_2\} e_n^* \{\nu_1, \nu_2\}) \equiv (\eta_1 \cap \eta_2 = \nu_1 \cap \nu_2)$.

---

[4] Of course the needed version of MA is quite weak; going more deeply in [Sh:522].

So it is enough to prove

$(*)_2$ $({}^n2, e_n^*) \in ID_2(\mu^{+n}, \mu)$.

We prove this by induction on n.

$n = 0$: Trivial.

$n+1$: Let $c: [\mu^{+n+1}]^2 \to \mu$, choose $M \prec (\mathscr{H}(\mu^{+n+2}), \in)$ of cardinality μ^{+n} such that $\mu^{+n} + 1 \subseteq M, c \in M$, so let $\delta = M \cap \mu^{+n}$.

Define $c_n: \mu^{+n} \to \mu$ by $c_n\{\alpha, \beta\} = (c\{\alpha, \beta\}, c\{\delta, \alpha\}, c\{\delta, \beta\})$ for $\alpha < \beta < \mu^{+n}$. By the induction hypothesis there is a sequence $\langle \beta_\eta: \eta \in {}^n2 \rangle$ of distinct ordinal $< \mu^{+n}$ such that $\{\eta_1, \eta_2\} e_n^* \{\nu_1, \nu_2\} \Rightarrow c_n\{\beta_{\eta_1}, \beta_{\eta_2}\} = c_n\{\beta_{\nu_1}, \beta_{\nu_2}\}$.
Let

$A = \{\gamma < \mu^{+n+1}: \gamma \notin \{\beta_\eta: \eta \in {}^n2\}$ and

for every $\eta \in {}^n2$ we have $c\{\beta_\eta, \gamma\} = c\{\beta_\eta, \delta\}\}$.

Clearly $A \in M$ and $\delta \in A$ so $A \nsubseteq M$, hence necessarily $|A| = \mu^{+n+1}$. So by the induction hypothesis we can find a sequence $\langle \gamma_\eta: \eta \in {}^n2 \rangle$ of distinct members of $A \backslash \delta$ such that

$$\{\eta_1, \eta_2\} e_n^* \{\nu_1, \nu_2\} \Rightarrow c\{\gamma_{\eta_1, \eta_2}, \gamma_{\eta_2}\} = c\{\gamma_{\nu_1}, \gamma_{\nu_2}\}.$$

Now we define

$$\alpha_\eta = \begin{cases} \beta_{\langle \eta(1+\ell): \ell < n \rangle} & \text{if } \eta(0) = 0, \\ \gamma_{\langle \eta(1+\ell): \ell < n \rangle} & \text{if } \eta(0) = 1. \end{cases}$$

It is easy to check that $\langle \alpha_\eta: \eta \in {}^{n+1}2 \rangle$ is as required. $\square_{5.13}$

We further can ask:

5.14. QUESTION. Assume $\Gamma_i \subseteq ID^*$ for $i < i^*, \mathbb{P}$ is $\Pi\{\mathbb{P}_{\Gamma_i}^\lambda: i < i^*\}$ with finite support, $c: [\aleph_{n(*)}]^2 \to \omega$ in $\mathbf{V}^{\mathbb{P}}$ then $ID(c)$ is not too far from some $\bigcup_{i \in w}\Gamma_i, w \subseteq i^*$ finite.

5.15. DISCUSSION. We can look more at ordered identities (recall)

$(*)_1$ for $\mathbf{c}_i: [\lambda]^{<\aleph_0} < \mu$ let $OID(c) = \{(a, e): a$ a set of ordinals and there is an ordered preserving $f: a \to \lambda\}$ such that $b_1 e b_2 \Rightarrow \mathbf{c}\{f''(b_1)\} = \mathbf{c}(f''(b_2))$ and $OID(\lambda, \mu) = \{(n, e): (n, e) \in OID(\mathbf{c})$ for every $\mathbf{c}: [\lambda]^{<\aleph_0} \to \mu$, and similarly OID_2, OID_k.

Of course,

$(*)_2$ $ID(\lambda, \mu)$ can be computed from $OID(\lambda, \mu)$.

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