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On the cardinality and weight spectra of compact spaces, II

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Abstract. Let $B(\kappa,\lambda)$ be the subalgebra of $\mathcal{P}(\kappa)$ generated by $[\kappa]^{\le\lambda}$. It is shown that if B is any homomorphic image of $B(\kappa,\lambda)$ then either $|B|<2^\lambda$ or $|B|=|B|^\lambda$; moreover, if X is the Stone space of B then either $|X|\le 2^{2^\lambda}$ or $|X|=|B|=|B|^\lambda$. This implies the existence of 0-dimensional compact T_2 spaces whose cardinality and weight spectra omit lots of singular cardinals of "small" cofinality.

1. Introduction. It was shown in [J] that for every uncountable regular cardinal κ, if X is any compact T_2 space with $w(X)>\kappa$ (resp. $|X|>\kappa$) then X has a closed subspace F such that $\kappa\le w(F)\le 2^{<\kappa}$ (resp. $\kappa\le |F|\le\sum\{2^{2^\lambda}:\lambda<\kappa\}$). In particular, the weight or cardinality spectrum of a compact space may never omit an inaccessible cardinal; moreover, under GCH the weight spectrum cannot omit any uncountable regular cardinal at all.

In the present note we prove a theorem which implies that for singular κ, on the other hand, there is always a 0-dimensional compact T_2 space whose cardinality and weight spectra both omit κ.

We formulate our main result in a boolean-algebraic framework. The topological consequences easily follow by passing to the Stone spaces of the boolean algebras that we construct.

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2. The main result. We start with a general combinatorial lemma on binary relations. In order to formulate it, however, we need the following definitions.

Definition 1. Let $\prec$ be an arbitrary binary relation on a set X and τ, μ be cardinal numbers. We say that $\prec$ is *τ-full* if for every subset $a \subset X$ with $|a| = \tau$ there is some $x \in X$ such that $|\{y \in a : y \prec x\}| = \tau$. Moreover, $\prec$ is said to be *μ-local* if for every $x \in X$ we have $|\mathrm{pred}(x, \prec)| \le \mu$, where $\mathrm{pred}(x, \prec) = \{y \in X : y \prec x\}$.

Now, our lemma is as follows.

Lemma 2. *Let $\prec$ be a binary relation on the cardinal ϱ that is both τ-full and μ-local. Then for every almost disjoint family $\mathcal{A} \subset [\varrho]^\tau$ we have*

$$|\mathcal{A}| \le \varrho \cdot \mu^\tau.$$

Proof. For every set $a \in \mathcal{A}$ there is a $\xi_a \in \varrho$ such that $g(a) = a \cap \mathrm{pred}(\xi_a, \prec)$ has cardinality τ because $\prec$ is τ-full. This map g is clearly one-to-one for $\mathcal{A}$ is almost disjoint. But the range of g is a subset of $\bigcup\{[\mathrm{pred}(\xi, \prec)]^\tau : \xi \in \varrho\}$ whose cardinality does not exceed $\varrho \cdot [\mu]^\tau$, and this completes the proof.

Before we formulate our main result we need some notation. Given the cardinals κ and λ (we may assume $\lambda \le \kappa$) we denote by $B(\kappa, \lambda)$ the boolean subalgebra of the power set algebra $\mathcal{P}(\kappa)$ generated by all subsets of κ of size $\le \lambda$. In other words,

$$B(\kappa, \lambda) = [\kappa]^{\le\lambda} \cup \{x \subset \kappa : \kappa \setminus x \in [\kappa]^{\le\lambda}\}.$$

What we can show is that the size of a homomorphic image of $B(\kappa, \lambda)$ (as well as the size of its Stone space) has to satisfy certain restrictions, namely it is either "small" or cannot have "very small" cofinality.

Theorem 3. *Let $h : B(\kappa, \lambda) \to B$ be a homomorphism of $B(\kappa, \lambda)$ onto the boolean algebra B. Then*

(i) *either $|B| < 2^\lambda$ or $|B|^\lambda = |B|$;*

(ii) *if $X = \mathrm{St}(B)$ is the Stone space of B then either $|X| \le 2^{2^\lambda}$ or $|X| = |B| = |B|^\lambda$.*

Proof. Set $|B| = \varrho$ and assume that $\varrho \ge 2^\lambda$. Since $[\kappa]^{\le\lambda}$ generates $B(\kappa, \lambda)$ it follows that $A = h''[\kappa]^{\le\lambda}$ generates B and thus we have $|A| = \varrho$ as well. We claim that the relation $\le_B$ is

(a) τ-full on A for each $\tau \le \lambda$;

(b) 2^λ-local on A.

Indeed, if $a \in [A]^\tau$ where $\tau \le \lambda$ then there is a set $x \in [[\kappa]^{\le\lambda}]^\tau$ such that $a = h''x$. But then $b = \bigcup x \in [\kappa]^{\le\lambda}$ as well, hence $h(b) \in A$ and clearly

$a \subset \mathrm{pred}(h(b), \leq_B)$ because h is a homomorphism. This, of course, is much more than what we need for (a).

To see (b), first note that if $b, c \in [\kappa]^{\leq\lambda}$ and $h(b) \leq h(c)$ then $b \cap c \in [\kappa]^{\leq\lambda}$ as well and $h(b \cap c) = h(b) \wedge h(c) = h(b)$ using again the fact that h is a homomorphism. But this implies $\mathrm{pred}(h(c), \leq_B) = h''\mathcal{P}(c)$ for any $c \in [\kappa]^{\leq\lambda}$, consequently $|\mathrm{pred}(h(c), \leq_B)| \leq |\mathcal{P}(c)| \leq 2^\lambda$ and this completes the proof of (b).

Applying Lemma 2 we may now conclude that for every cardinal $\tau \leq \lambda$ and for every almost disjoint family $\mathcal{A} \subset [\varrho]^\tau$ we have

$$|\mathcal{A}| \leq \varrho \cdot (2^\lambda)^\tau = \varrho.$$

This, in turn, implies $\varrho^\lambda = \varrho$. Indeed, assume that $\varrho^\lambda > \varrho$ and τ be the smallest cardinal with $\varrho^\tau > \varrho$. Then $\tau \leq \lambda$ and $\varrho^{<\tau} = \varrho$, and as is well known, there is an almost disjoint family $\mathcal{A} \subset [{}^{<\tau}\varrho]^\tau$ of size $\varrho^\tau > \varrho$, namely $\mathcal{A} = \{A_f : f \in {}^\tau\varrho\}$ where $A_f = \{f \restriction \xi : \xi < \tau\}$ for any $f \in {}^\tau\varrho$.

Now, to prove (ii) first note that if $|B| \leq 2^\lambda$ then trivially $|X| \leq 2^{2^\lambda}$. So assume $|B| > 2^\lambda$ and in this case we prove that actually

$$|X| = 2^{2^\lambda} \cdot |B|.$$

We first show that $|X| \geq 2^{2^\lambda} \cdot |B|$, which, as $|X| \geq |B|$ is always valid, boils down to showing that $|X| \geq 2^{2^\lambda}$.

Using the fact that $|B| = |h''[\kappa]^{\leq\lambda}| = \varrho > 2^\lambda$ we may select a collection $\{a_\alpha : \alpha \in (2^\lambda)^+\} \subset [\kappa]^{\leq\lambda}$ such that $\alpha \neq \beta$ implies $h(a_\alpha) \neq h(a_\beta)$ and by a straightforward Δ-system argument we may also assume that $\{a_\alpha : \alpha \in (2^\lambda)^+\}$ is a Δ-system with root a. Then, as h is a homomorphism, we also have $h(a_\alpha) \wedge h(a_\beta) = h(a)$ for distinct α and β and so $\{h(a_\alpha) - h(a) : \alpha \in (2^\lambda)^+\}$ are pairwise disjoint and distinct elements of B, all but at most one of which are non-zero. However, the existence of 2^λ pairwise disjoint non-zero elements in a boolean algebra clearly implies the existence of 2^{2^λ} ultrafilters in it, hence we are done with showing $|X| \geq 2^{2^\lambda}$.

Next, to see $|X| \leq 2^{2^\lambda} \cdot |B|$ note that, again as h is a homomorphism, $h''[\kappa]^{\leq\lambda}$ is a (not necessarily proper) ideal in B, hence there is no more than one ultrafilter u on B such that $u \cap h''[\kappa]^{\leq\lambda} = \emptyset$. If, on the other hand, $u \in X$ is such that $b \in u \cap h''[\kappa]^{\leq\lambda}$ then u is generated by its subset $u \cap \mathrm{pred}(b, \leq_B)$. However, $\leq_B$ clearly is 2^λ-local on $h''[\kappa]^{\leq\lambda}$, and so we conclude that

$$|X| \leq 1 + \left|\bigcup\{\mathcal{P}(\mathrm{pred}(b, \leq_B)) : b \in h''[\kappa]^{\leq\lambda}\}\right| \leq 1 + 2^{2^\lambda} \cdot |B| = 2^{2^\lambda} \cdot |B|.$$

This completes the proof of our theorem.

Now let $X(\kappa, \lambda)$ be the Stone space of the boolean algebra $B(\kappa, \lambda)$. Using Stone duality and the notation of [J] the above result has the following reformulation for the weight and cardinality spectra of the 0-dimensional compact T_2 space $X(\kappa,\lambda)$.

Corollary 4. (i) *For every* $\mu \in \mathrm{Sp}(w, X(\kappa,\lambda))$ *we have either* $\mu < 2^\lambda$ *or* $\mu^\lambda = \mu$, *hence* $\mathrm{cf}(\mu) > \lambda$;

(ii) *if* $\mu \in \mathrm{Sp}(|\ |, X(\kappa,\lambda))$ *then either* $\mu < 2^{2^\lambda}$ *or* $\mu^\lambda = \mu$.

(*Note that both* $\mu = 2^\lambda$ *and* $\mu = 2^{2^\lambda}$ *imply* $\mu^\lambda = \mu$.) *In fact*, *for every closed subspace* Y *of* $X(\kappa,\lambda)$ *we have either* $w(Y) \le 2^\lambda$ *or* $w(Y)^\lambda = w(Y)$ *and* $|Y| = 2^{2^\lambda} \cdot w(Y)$.

It immediately follows from this that if $2^{2^\lambda} < \kappa$ then the cardinality and weight spectra of the space $X(\kappa,\lambda)$ omit every cardinal $\mu \in (2^{2^\lambda}, \kappa]$ with $\mathrm{cf}(\mu) \le \lambda$. In particular, if GCH holds then $\lambda < \kappa$ implies that both $\mathrm{Sp}(|\ |, X(\kappa,\lambda))$ and $\mathrm{Sp}(w, X(\kappa,\lambda))$ omit all cardinals $\mu \in (\lambda, \kappa]$ with $\mathrm{cf}(\mu) \le \lambda$.

Note that similar omission results were obtained by van Douwen in [vD] for the case $\lambda = \omega$ and κ strong limit.

An interesting open problem arises here that we could not settle: Can one find for every cardinal κ a compact T_2 space X such that the cardinality and/or weight spectra of X omit every singular cardinal below κ?

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