Uniquely Transitive Torsion-free Abelian Groups

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Abstract. We will answer a question raised by Emmanuel Dror Farjoun concerning the existence of torsion-free abelian groups G such that for any ordered pair of pure elements there is a unique automorphism mapping the first element onto the second one. We will show the existence of such a group of cardinality λ for any successor cardinal $\lambda = \mu^+$ with $\mu = \mu^{\aleph_0}$.

1. Introduction

We will consider the set $\mathfrak{p}G$ of all non-zero pure elements of a torsion-free abelian group G. Recall that $g \in G$ is pure if the equations $xn = g$ for natural numbers $n \neq 1$ have no solution $x \in G$. Clearly every element of the automorphism group $\operatorname{Aut} G$ of G induces a permutation on the set $\mathfrak{p}G$ and it is natural to consider groups where the action of $\operatorname{Aut} G$ on $\mathfrak{p}G$ is transitive: for any pair $x, y \in \mathfrak{p}G$ there is an automorphism $\varphi \in \operatorname{Aut} G$ such that $x\varphi = y$. In this case we will say that G is transitive, for short G is a T-group. (Transitive groups are A-transitive groups in Dugas, Shelah [**5**]). This kind of consideration is well-known for abelian p-groups. It was stimulated by Kaplansky and studied in many papers, see [**14, 15, 1**] for instance. If G is a free abelian group with two pure elements x and y, then there are two sets X and Y of free generators of G such that $x \in X$ and $y \in Y$. We

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can chose a bijection $\alpha\colon X \to Y$ with $x\alpha = y$ which extends to an automorphism $\alpha \in \operatorname{Aut} G$. Hence free groups are T-groups and a similar argument holds for a wide class of abelian groups. There are T-groups like $\mathbb{Z}^{\aleph_1}/\mathbb{Z}^{[\aleph_1]}$ with $\mathbb{Z}^{[\aleph_1]}$ the set of all elements in $\mathbb{Z}^{\aleph_1}$ of countable support, which are $\aleph_1$-free but not separable, see Dugas, Hausen [4]. The existence of $\aleph_1$-free, indecomposable T-groups in L for any regular, not weakly compact cardinality was also shown in Dugas, Shelah [**5**, Theorem (b), p. 192]. This was used to answer a problem in Hausen [**11**], see also [**12**] and [**5**, Theorem (a), p. 192].

Thus we may strengthening the action of $\operatorname{Aut} G$ on $\mathfrak{p}G$ and say that G is a U-group if any two automorphisms $\varphi, \varphi' \in \operatorname{Aut} G$ with $g\varphi = g\varphi'$ for some $g \in \mathfrak{p}G$ must be the same $\varphi = \varphi'$. Hence G is a UT-group if and only if G is both a T-group as well as a U-group. Thus G is a UT-group if and only if $\operatorname{Aut} G$ is transitive and (every non-trivial automorphism acts) fix point-free on $\mathfrak{p}G$. In connection with permutation groups such action is also called 'sharply transitive'. Note that $\mathfrak{p}G$ may be empty, if G is divisible for instance. In order to avoid trivial cases we also require that $0 \neq G \neq \mathbb{Z}$ and G is of type 0, hence G is torsion-free and every element of G is a multiple of an element in $\mathfrak{p}G$. If G is of type 0 and not finitely generated then $|\mathfrak{p}G| = |G|$ is large and the problem about the existence of UT-groups becomes really interesting. This question is related to problems in homotopy theory and was raised by Dror Farjoun. In response we want to show the following

Theorem 1.1. *For any successor cardinal $\lambda = \mu^+$ with $\mu = \mu^{\aleph_0}$ there is an $\aleph_1$-free abelian UT-group of cardinality λ.*

We will also determine the endomorphism rings of these groups. They are isomorphic to integral group rings $R = \mathbb{Z}F$ of groups F freely generated (as a non-abelian group) by λ elements (with λ as in the theorem). Since endomorphisms of a group G will act on the right (in accordance with used results from [**9, 10**]), we will also view G as a right R-module (and as a left or right $\mathbb{Z}$-module). Using classical results on group rings it will follow that $\operatorname{Aut} G = \pm F$, where $-1 \in \mathbb{Z}$ is scalar multiplication by -1, hence $\pm F$ is a direct product of a group of order 2 and F. Moreover $\mathbb{Z}F$ has no idempotents except 0 and 1, hence G in the theorem is indecomposable and obviously the center of $\mathbb{Z}F$ is just $\mathbb{Z}$, hence $\mathbb{Z}$ is also the center of $\operatorname{End} G$. Therefore Theorem 1.1 strengthens the Theorem in Dugas, Shelah [**5**, p. 192] substituting T-groups by UT-groups and removing the restriction $V = L$ to the constructible universe. Also note that it is straightforward to replace the ground ring $\mathbb{Z}$ by a p-reduced domain S for some prime p. Hausen's [**12**] problem can be answered also in ordinary set theory. Further applications can be found in Section 5.

First we would like to explain why constructing UT-groups is a hard task, much harder then finding suitable T-groups. Because R^+ above is a free abelian group, we can easily find groups G with $R = \operatorname{End} G$, see [**3, 2**]. But there are still two obstacles which must be taken into consideration. Often $|R| < |G|$ in realization theorems, thus the units of R which represent $\operatorname{Aut} G$ will never act transitive on a bigger group and G can't be a T-group. More importantly, inspecting the constructions in [**3, 2**], it is clear that they provide no control about the action, thus both U and

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T for UT-groups are a problem. Note that in many earlier constructions G has a free dense and pure R-submodule of rank > 1 mostly of rank $|G|$. But T-groups must obviously be cyclic over their endomorphism ring R, hence [**3, 2**] do not apply in principle. Inspecting the proof in [**5**] it is easy to see that G is not torsion-free over its endomorphism ring. This comes from the list of new variables $x, y, \ldots$ added to R in the construction in order to make R acting transitive on all pairs of pure elements. Even refining the list of pure pairs in [**5**] it seems hard to avoid clashes of related pairs such that $x - y$ for example has a proper annihilator. Thus the groups in [**5**] are T-groups and not UT-groups (even modifying the arguments).

Thus a new approach is need, which will be established in Section 3. We will use a geometric argument choosing carefully new partial automorphisms for making G transitive but with very small domain and image in order to preserve the U-property for the new monoid. Then we will feed the partial maps with pushouts to grow them up and become real automorphisms without destroying the UT-property. At the end we will have a suitable subgroup F of automorphisms of some group G, thus G becomes an R-module over the ring $\mathbb{Z}F =^{\mathrm{df}} R$.

Finally we have to fit these approximations to ideas getting rid of the endomorphisms outside R, see Section 4. We need the Strong Black Box as discussed and proven in terms of model theory in Eklof, Mekler [**6**, Chapter XIV]. Note that this prediction principle is stronger then (Shelah's) General Black Box, see [**2**, Appendix]. The Strong Black Box is also restricted to those particular cardinals mentioned in the theorem. However, here we will apply a version of the Strong Black Box stated and proven on the grounds of modules in ordinary, naive set theory, which can be found in a recent paper by Göbel, Wallutis [**10**], see also [**9**]. In order to show $\operatorname{End} G = R$ well-known arguments for realizing rings as endomorphism rings must be modified because the final ring and its action are only given to us at the very end of the construction: We will first replace the layers G_α from the construction by a new filtration only depending on the norm. Note that the members of the new filtration of the right R-module G must be right R_α-submodules for suitable subrings R_α of R to have cardinality less than $|G|$. But they are still good enough to kill unwanted endomorphisms referring to the prediction used during the construction. Moreover note that the two tasks indicated in the last two paragraphs must be intertwined and applied with repetition. While the final G is an $\aleph_1$-free abelian group, hence torsion-free, it is not hard to see that G is torsion as an R-module: If $0 \neq g \in G$, then we can choose distinct elements $g', x, y \in \mathfrak{p}G$ such that $ng' = g$, and $x - y \in \mathfrak{p}G$. Hence there are distinct unit elements $r_x, r_y, r_{xy} \in U(R) = \pm F$ such that $g'r_x = x, g'r_y = y, g'r_{xy} = x + y$. The endomorphism $r_x + r_y$ does not belong to $\pm F$, in particular it can not be r_{xy}. Hence $r = r_x + r_y - r_{xy} \neq 0$ but $g'r = x + y - (x + y) = 0$ and g is torsion. It is worth noting that the result can be strengthened under $V = L$, where we get strongly-λ-free groups of cardinality λ as in Theorem 1.1 for each regular, uncountable cardinal λ which is not weakly compact. In this case the approximations in Section 3 can be improved, replacing '$\aleph_1$-free' by 'free' at all obvious places. The main result of this section will then be a theorem on free groups G with a free (non-abelian) group

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$F \subseteq \operatorname{Aut} G$ acting uniquely transitive on G. Also Section 4 must be modified: The Strong Black Box 4.2 must be replaced by $\diamondsuit$ following arguments similar to [**3**].

2. Warming up: Construction of a special group

We begin with a particular case of an old theorem and discuss extra properties of the constructed group. Part of this proposition will be used in Section 3.

Proposition 2.1. *Let κ be a cardinal with $\kappa^{\aleph_0} = \kappa$, F be a free (non-abelian) group of rank $< \kappa$ and $R = \mathbb{Z}F$ be its integral group ring. Then there is a group G with the following properties.*

(i) *G is an $\aleph_1$-free abelian group of rank κ with* $\operatorname{End} G = R$.
(ii) *G is torsion-free as an R-module.*
(iii) $\operatorname{Aut} G = \pm F$
(iv) *If $\varphi \in \operatorname{End} G$, then φ is injective.*

Proof. Note that the integral group ring $R = \mathbb{Z}F$ has free additive group R^+ with basis F. We can apply a main theorem from [**2**] showing the existence of an $\aleph_1$-free abelian group G with $\operatorname{End} G = R$. The free group F is orderable (i.e. has a linear ordering which is compatible with multiplication by elements from the right), see Mura, Rhemtulla [**16**, p. 37]. However note, that torsion-free groups may be non-orderable, see [**16**, pp. 89 - 95, Example 4.3.1.]. The integral group ring $\mathbb{Z}F$ of any orderable group F satisfies the unit conjecture, this is to say that the units of $R = \mathbb{Z}F$ are the obvious ones, hence $U(\mathbb{Z}F) = \pm F$, see Sehgal [**17**, p. 276, Lemma 45.3]. Moreover, any group ring which satisfies the unit conjecture also satisfies the zero divisor conjecture, hence R has no zero-divisors, see Sehgal [**17**, p. 276, Lemma 45.2]. Therefore R is torsion-free as an R-module.

Now the remaining part of the proof is easy: $\operatorname{Aut} G = U(R) = \pm F$ and if $0 \neq g \in G$, then $g \in \oplus R \subseteq G$ because G is also an $\aleph_1$-free R-module by construction in [**2**]. If we consider multiplication of g by some $r \in R$ on a non-trivial component of g in this free direct sum, then $r = 0$ because R is torsion-free as an R-module. Hence G is torsion-free as an R-module. Any $\varphi \in \operatorname{End} G = R$ is scalar multiplication by a suitable $r \in R$ hence injective because G is a torsion-free R-module. □

We will use Proposition 2.1 in Section 3. We get more out of it if we know that a particular endomorphism is pure:

Lemma 2.2. *Let F be the free (non-abelian) group and* $\operatorname{End} G = \mathbb{Z}F$ *be the endomorphism ring of the $\aleph_1$-free abelian group G given by Proposition* 2.1. *If $\varphi \in \mathbb{Z}F \setminus \pm F$, then φ is a monomorphism and not onto. If $0 \neq \varphi \in \mathbb{Z}F$, then φ is pure in $\mathbb{Z}F^+$ if and only if* $\operatorname{Im} \varphi$ *is pure in G.*

Proof. All endomorphisms of G in Proposition 2.1 are monomorphisms as shown there. If $\varphi \in R = \mathbb{Z}F = \operatorname{End} G$ would be onto, then φ must be an automorphism, thus $\varphi \in U(R)$, which is $\pm F$; and this was excluded.

We come to the last assertion. We shall write $0 \neq \varphi = r \in R$ and suppose that $r = nr'$ ($n \neq \pm 1$) is not a pure element of R^+. Note that $nG \neq G$, hence $Gr' \neq Gnr'$ and we can pick an element $g \in Gr' \setminus Gr$ which is mapped into Gr

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under multiplication by n. Hence Gr is not pure in G. Conversely let r be pure in R and consider any $g \in G$ such that $gp \in Gr$ for some prime p. Hence $gp = g'r$ and by construction of G (just note that G is $\aleph_1$-free as R-module) there is $h \in G$ such that $g' \in hR$ and hR is a pure subgroup of G. Hence also $g \in hR$ and we can write $g = hr_g, g' = hr_{g'}$ which gives $hpr_g = hr_{g'}r$ and $pr_g = r_{g'}r$ because G is R-torsion-free. Using that p cannot divide r by purity in R and that $r, r_{g'}$ are elements of the group ring $\mathbb{Z}F$ we can write $r_{g'} = r'p$ for some $r' \in R$. Finally $gp = g'r = (hr'p)r$, hence $g = hr'r \in Gr$ and Gr is pure in G. □

If we replace [**2**] in the proof of Proposition 2.1 by [**3**], then we can strengthen Proposition 2.1 in the constructible universe L. We get a

Corollary 2.3. *Let κ be a regular, uncountable cardinal which is not weakly compact such that $\diamondsuit_\kappa$ holds and let F be a free (non-abelian) group of rank $< \kappa$ and $R = \mathbb{Z}F$ be its integral group ring. Then there is a strongly-κ-free abelian group G of rank κ with* $\operatorname{End} G = R$ *and properties (ii), (iii) and (iv) of Proposition* 2.1.

Recall that G is κ-free if all subgroups of cardinality $< \kappa$ are free, and G is strongly κ-free if also any subgroup of cardinality $< \kappa$ is contained in a subgroup U of cardinality $< \kappa$ such that G/U is κ-free as well.

3. Growing partial automorphisms

Besides the set $\mathfrak{p}G$ of pure elements of a group G we consider a particular subset $\operatorname{pAut} G$ of all partial automorphisms φ of G. Here φ is an isomorphism with domain $\operatorname{Dom}\varphi$ and range $\operatorname{Im}\varphi$ subgroups of G. The inverse isomorphism will be denoted by φ^{-1}. However note that φ^{-1} is not the inverse of φ as a member of $\operatorname{pAut} G$ because $\varphi\varphi^{-1} = \varphi^{-1}\varphi = 1$ only holds if $\operatorname{Dom}\varphi = \operatorname{Im}\varphi = G$. If we want to stress this point, then we call φ^{-1} a weak inverse element of φ. Surely $0 \in \operatorname{Dom}\varphi$ but it will happen often that $\varphi \neq 0$ but $\varphi^2 = 0$ for partial automorphisms φ. Here we denote with 0 the trivial partial automorphism with $\operatorname{Dom} 0 = 0 (= \{0\})$.

Because we are working exclusively with $\aleph_1$-free groups, we require that $\varphi \in \operatorname{pAut} G$ if and only if φ is a partial automorphism and $G/\operatorname{Dom}\varphi, G/\operatorname{Im}\varphi$ are $\aleph_1$-free abelian groups. The composition of partial automorphisms φ, ψ is again a partial automorphism with $\operatorname{Dom}(\varphi\psi) = (\operatorname{Im}\varphi \cap \operatorname{Dom}\psi)\varphi^{-1}$ and range $\operatorname{Im}(\varphi\psi) = (\operatorname{Im}\varphi \cap \operatorname{Dom}\psi)\psi$. Thus products of partial automorphisms of G act naturally on G as partial automorphisms and domain and range are well defined. If $\psi, \varphi \in \operatorname{pAut} G$, then we want to show that $\psi^{-1}, \varphi\psi \in \operatorname{pAut} G$. Hence it is enough to check the freeness condition. If we replace ψ by ψ^{-1}, then only domain and image are interchanged, thus trivially $\psi^{-1} \in \operatorname{pAut} G$. It remains to consider domain and range of $\varphi\psi$. Passing to an inverse, as just noted, it is enough to deal with $\operatorname{Dom}(\varphi\psi)$. We already observed that

$$\operatorname{Dom}(\varphi\psi) = (\operatorname{Im}\varphi \cap \operatorname{Dom}\psi)\varphi^{-1}. \tag{3.1}$$

From $\psi \in \operatorname{pAut} G$ follows that $G/\operatorname{Dom}\psi$ is $\aleph_1$-free, hence

$$\operatorname{Im}\varphi/(\operatorname{Im}\varphi \cap \operatorname{Dom}\psi) \cong (\operatorname{Im}\varphi + \operatorname{Dom}\psi)/\operatorname{Dom}\psi \subseteq G/\operatorname{Dom}\psi$$

is $\aleph_1$-free. We apply φ^{-1} and (3.1) to see that $\operatorname{Dom}\varphi/\operatorname{Dom}(\varphi\psi)$ is $\aleph_1$-free. Moreover $\varphi \in \operatorname{pAut} G$, and therefore $G/\operatorname{Dom}\varphi$ is $\aleph_1$-free, hence $G/\operatorname{Dom}(\varphi\psi)$ is $\aleph_1$-free as desired.

We arrive at our first

Lemma 3.1. *The set* $\operatorname{pAut} G$ *of all partial automorphism* φ *of* G *with* $G/\operatorname{Dom}\varphi$ *and* $G/\operatorname{Im}\varphi$ *both* $\aleph_1$*-free abelian groups is a submonoid of all partial automorphisms with* $1 = \mathrm{id}_G$ *and* $-1 = -\mathrm{id}_G$ *acting as multiplication by* 1 *and* -1 *respectively, which is closed under taking (weak) inverses.*

Moreover, if $\mathfrak{F} \subseteq \operatorname{pAut} G$, then $\langle\mathfrak{F}\rangle \subseteq \operatorname{pAut} G$ is the submonoid of all products taken from the set $\{\pm 1\} \cup \mathfrak{F} \cup \mathfrak{F}^{-1}$, where $\mathfrak{F}^{-1} = \{\psi^{-1} \mid \psi \in \mathfrak{F}\}$.

We begin with an observation which allows us to consider induced partial automorphisms on a factor group.

Observation 3.2. *If* $U \subseteq G$ *are abelian groups and* $\varphi \in \operatorname{pAut} G$ *with* $(\operatorname{Im}\varphi \cap U)\varphi^{-1} \subseteq U$ *and* $(\operatorname{Dom}\varphi \cap U)\varphi \subseteq U$*, then* φ *induces a partial automorphism* $\overline{\varphi}$ *of* $\overline{G}$ *where* $\overline{G} = \{\overline{g} = g + U \mid g \in G\}$ *taking* $\overline{g}$ *to* $\overline{g\varphi}$ *for any* $g \in \operatorname{Dom}\varphi$*. Moreover* $\operatorname{Dom}\overline{\varphi} = \overline{\operatorname{Dom}\varphi}$ *and* $\operatorname{Im}\overline{\varphi} = \overline{\operatorname{Im}\varphi}$.

Proof. If $g \in G$ and $\overline{g\varphi} = \overline{0}$ in $\overline{G}$, then $g\varphi = g' \in U$ and

$$g = g'\varphi^{-1} \in (\operatorname{Im}\varphi \cap U)\varphi^{-1} \subseteq U,$$

hence $\overline{g} = \overline{0}$ and $\overline{\varphi} \in \operatorname{pAut}\overline{G}$. The other assertions are also obvious. ☐

In order to show Proposition 3.9 we relate elements of free (non-abelian) groups and elements in $\operatorname{pAut} G$. It is important to be able to work with elements of $\operatorname{pAut} G$ acting on a partial free basis of G. To be precise, we will need the following definition extending freeness from G to $\operatorname{pAut} G$.

Definition 3.3. *Let* $\mathfrak{F} = \{\varphi_t \mid t \in u\} \subseteq \operatorname{pAut} G$ *be a finite set of partial automorphisms. Then* $(G, \mathfrak{F})$ *is called* $\aleph_1$*-free if any countable subset of* G *belongs to a countable subgroup* $X \subseteq G$ *with basis* B *and the following properties for any* $\varphi \in \langle\mathfrak{F}\rangle$.

(i) G/X *is* $\aleph_1$*-free.*
(ii) φ *induces a partial injection on* B*, that is, if* $b \in B \cap \operatorname{Dom}\varphi$*, then also* $b\varphi \in B$.
(iii) $X \cap \operatorname{Dom}\varphi = \langle B \cap \operatorname{Dom}\varphi\rangle$.

Passing to weak inverses, it follows from *(iii)* that also $X \cap \operatorname{Im}\varphi = \langle B \cap \operatorname{Im}\varphi\rangle$. Moreover $X \cap \operatorname{Dom}\varphi$ and $X \cap \operatorname{Im}\varphi$ are summands of X with free complements generated by $B \setminus \operatorname{Dom}\varphi$ and $B \setminus \operatorname{Im}\varphi$, respectively. It also follows that G is $\aleph_1$-free. We can ease arguments in Lemma 3.11 and Lemma 3.12 to note here that we only need a partial basis $b\langle\mathfrak{F}\rangle$ (a subset of B) with the property *(ii)* and $\langle b\langle\mathfrak{F}\rangle\rangle \cap \operatorname{Dom}\varphi = \langle b\langle\mathfrak{F}\rangle \cap \operatorname{Dom}\varphi\rangle$; see Proposition 3.9.

Next we relate basis elements of free non-abelian groups and partial automorphisms with care. Suppose the set $\mathfrak{F} = \{\varphi_t \mid t \in J\}$ generates a free group $\langle\mathfrak{F}\rangle$ and as in Definition 3.5 there is a map $\pi\colon \mathfrak{F} \to \operatorname{pAut} G$ (acting on the left), then this map can be extended to $\langle\mathfrak{F}\rangle$. The extension is unique if we restrict ourselves

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to **reduced elements** $\varphi = \varphi_1 \dots \varphi_n$ in $\langle \mathfrak{F} \rangle$ with $\varphi_i \in \mathfrak{F} \cup \mathfrak{F}^{-1}$ and define naturally $\pi(\varphi) = \pi(\varphi_1) \dots \pi(\varphi_n)$. However, if $\varphi_1, \varphi_2 \in \langle \mathfrak{F} \rangle$ are reduced and φ is the reduced element which coincides with the formal product $\varphi_1\varphi_2$ in $\langle \mathfrak{F} \rangle$, then only $\pi(\varphi_1)\pi(\varphi_2) \subseteq \pi(\varphi)$ holds as a graph and this means $\mathrm{Dom}(\pi(\varphi_1)\pi(\varphi_2)) \subseteq \mathrm{Dom}\,\pi(\varphi)$ and $\pi(\varphi) \restriction \mathrm{Dom}(\pi(\varphi_1)\pi(\varphi_2)) = \pi(\varphi_1)\pi(\varphi_2)$. Thus we have equality if also the formal product $\varphi_1\varphi_2$ is reduced.

Definition 3.4. *With* $\pi \colon \mathfrak{F} \to \mathrm{pAut}\, G$ *as above we say that* π *(or* $\pi(\mathfrak{F}) = \{\pi(\varphi) \mid \varphi \in \mathfrak{F}\}$*) satisfies the U-property if* $\varphi = \varphi'$ *for any reduced elements* $\varphi, \varphi' \in \langle \mathfrak{F} \rangle$ *with* $x\pi(\varphi) = x\pi(\varphi')$ *and some* $x \in \mathrm{Dom}(\varphi) \cap \mathrm{Dom}(\varphi') \cap \mathfrak{p}G$.

The last definition is crucial for this paper because it is the microscopic version of U-groups discussed in the introduction. We also must pass from groups $G^{\mathfrak{x}}$ with this U-property to suitable extension $G^{\mathfrak{y}}$ with the U-property. All this we encode into our main definition of quintuples $\mathfrak{x}$ and their extensions. Normally our maps will act on the right, but we allow three exceptions, the maps ε, π and h below. Also $\mathfrak{P}_{\aleph_0}(J)$ denotes all finite subsets of the set J.

Definition 3.5. *Let* $\mathfrak{K}$ *be the family of all quintuples*

$$\mathfrak{x} = (G, \mathfrak{F}, \varepsilon, \pi, h) = (G^{\mathfrak{x}}, \mathfrak{F}^{\mathfrak{x}}, \varepsilon^{\mathfrak{x}}, \pi^{\mathfrak{x}}, h^{\mathfrak{x}})$$

such that the following holds.

(i) G *is an* $\aleph_1$*-free abelian group.*

(ii) $\mathfrak{F} = \{\varphi_t \mid t \in J\}$ *is a set of free generators* φ_t *indexed by* $J = J^{\mathfrak{x}}$ *of a group* $\langle \mathfrak{F} \rangle$.

(iii) $\varepsilon \colon J \to \{1, -1\}$ *is a map.*

(iv) $\pi \colon \mathfrak{F} \to \mathrm{pAut}\, G$ *is a map which satisfies the U-property. We shall write* $\pi^{\mathfrak{x}}(\varphi_t) = \varphi_t^{\mathfrak{x}}$ *and omit* $\mathfrak{x}$ *if the meaning is clear from the context.*

(v) $h \colon \mathfrak{P}_{\aleph_0}(J) \to \mathrm{Im}(h)$ *is a partial function from* $\mathrm{Dom}\, h \subseteq \mathfrak{P}_{\aleph_0}(J)$. *If* $u \in \mathrm{Dom}\, h$ *and* $U = h(u)$, $\mathfrak{F} = \{\varphi_t \mid t \in u\}$, *then the following conditions must hold.*

  (a) U *is a countable subgroup of* G *and* $(\mathrm{Dom}\,\varphi^{\mathfrak{x}} \cap U)\varphi^{\mathfrak{x}} \subseteq U$ *for all* $\varphi \in \langle \mathfrak{F} \rangle$; *hence* $\varphi^{\mathfrak{x}}$ *induces* $\overline{\varphi}^{\mathfrak{x}} \in \mathrm{pAut}(G/U)$; *see Observation* 3.2. *Let* $\overline{G} = G/U$ *and* $\overline{\mathfrak{F}} = \{\overline{\varphi}^{\mathfrak{x}} \mid \varphi \in \mathfrak{F}\}$.

  (b) $(\overline{G}, \overline{\mathfrak{F}})$ *is* $\aleph_1$*-free; see Definition* 3.3.

It follows that $\overline{G}$ above is $\aleph_1$-free. From Definition 3.5 *(iv)* and Lemma 3.1 follows

Corollary 3.6. *If* $\mathfrak{x} \in \mathfrak{K}$ *then* $\langle \varphi_t^{\mathfrak{x}} \mid t \in J^{\mathfrak{x}} \rangle \subseteq \mathrm{pAut}\, G^{\mathfrak{x}}$.

Hence $G^{\mathfrak{x}} / \mathrm{Dom}\,\varphi^{\mathfrak{x}}$ is $\aleph_1$-free for all $\varphi \in \langle \mathfrak{F} \rangle$. We will carry on this condition inductively, just checking the generators in $\mathfrak{F}$ and using the following simple

Test Lemma 3.7. *If* $U \subseteq G$ *are groups and any countable subset of* G *is contained in a countable subgroup* X *with free generators* $B_1 \cup B_2$ *such that* $B_1 \subseteq U$ *and* $U \cap \langle B_2 \rangle = 0$ *then* G/U *is* $\aleph_1$*-free.*

The same test lemma will be used inductively for U in Definition 3.5 $(v)(b)$. We will pass from groups $G^{\mathfrak{x}}$ to larger groups $G^{\mathfrak{y}}$ related to $\mathfrak{x}, \mathfrak{y} \in \mathfrak{K}$ by taking pushouts

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or unions. This is reflected in the next definition (in particular condition (*iii*)) of an ordering on $\mathfrak{K}$. This is the final step before we can start working.

Definition 3.8. *Let* $\mathfrak{x} \leq \mathfrak{y}$ $(\mathfrak{x}, \mathfrak{y} \in \mathfrak{K})$ *if the following holds for* $\mathfrak{x} = (G^{\mathfrak{x}}, \mathfrak{F}^{\mathfrak{x}}, \varepsilon^{\mathfrak{x}}, \pi^{\mathfrak{x}}, h^{\mathfrak{x}})$ *and* $\mathfrak{y} = (G^{\mathfrak{y}}, \mathfrak{F}^{\mathfrak{y}}, \varepsilon^{\mathfrak{y}}, \pi^{\mathfrak{y}}, h^{\mathfrak{y}})$.

(i) $G^{\mathfrak{x}} \subseteq G^{\mathfrak{y}}$ *and* $G^{\mathfrak{y}}/G^{\mathfrak{x}}$ *is* $\aleph_1$*-free.*

(ii) $\pi^{\mathfrak{y}}$ *extends* $\pi^{\mathfrak{x}}$ *in the weak sense* $(\pi^{\mathfrak{x}} \preceq \pi^{\mathfrak{y}})$, *that is* $J^{\mathfrak{x}} \subseteq J^{\mathfrak{y}}$ *(equivalently* $\mathfrak{F}^{\mathfrak{x}} \subseteq \mathfrak{F}^{\mathfrak{y}}$*) and also* $\varphi_t^{\mathfrak{x}} \subseteq \varphi_t^{\mathfrak{y}}$ *extends for all* $t \in J^{\mathfrak{x}}$.

(iii) *If* $t \in J^{\mathfrak{x}}$, *then one of the following cases holds*

- (a) $\varepsilon^{\mathfrak{x}}(t) = \varepsilon^{\mathfrak{y}}(t)$ *and* $\varphi_t^{\mathfrak{x}} = \varphi_t^{\mathfrak{y}}$.
- (b) $G^{\mathfrak{x}} \subseteq \operatorname{Dom}\varphi_t^{\mathfrak{y}} \cap \operatorname{Im}\varphi_t^{\mathfrak{y}}$.
- (c) $\varepsilon^{\mathfrak{x}}(t) = 1 = -\varepsilon^{\mathfrak{y}}(t)$ *and* $G^{\mathfrak{x}} \subseteq \operatorname{Dom}\varphi_t^{\mathfrak{y}}$.
- (d) $\varepsilon^{\mathfrak{y}}(t) = 1 = -\varepsilon^{\mathfrak{x}}(t)$ *and* $G^{\mathfrak{x}} \subseteq \operatorname{Im}\varphi_t^{\mathfrak{y}}$.

(iv) $h^{\mathfrak{x}} \subseteq h^{\mathfrak{y}}$ *extends (i.e., if* $h^{\mathfrak{x}}(u) \subseteq G^{\mathfrak{x}}$, *then* $h^{\mathfrak{x}}(u) = h^{\mathfrak{y}}(u) \subseteq G^{\mathfrak{y}}$*).*

(v) *If* $u \in \operatorname{Dom} h^{\mathfrak{x}}$ *and* $\overline{G}^{\mathfrak{x}} =^{\mathrm{df}} G^{\mathfrak{x}}/h^{\mathfrak{x}}(u) \subseteq \overline{G}^{\mathfrak{y}} =^{\mathrm{df}} G^{\mathfrak{y}}/h^{\mathfrak{x}}(u)$, *then any basis* B *of a countable subgroup* $X \subseteq \overline{G}^{\mathfrak{x}}$ *as in Definition* 3.5 *extends to a basis* B' *of some countable subgroup* X' *of* $\overline{G}^{\mathfrak{y}}$ *which also satisfies Definition* 3.5.

Proposition 3.9. *Suppose that* $\mathfrak{x} < \mathfrak{y}$ *in* $\mathfrak{K}$ *and* $u \in \operatorname{Dom} h^{\mathfrak{y}}, |G^{\mathfrak{y}}| > \aleph_0, G^{\mathfrak{y}} = \operatorname{Dom}\varphi_t^{\mathfrak{y}} = \operatorname{Im}\varphi_t^{\mathfrak{y}}$ *for every* $t \in u$. *If* F *is the group freely generated by* $\{\varphi_t \mid t \in u\}$ *and* $\theta = \sum_{i\in I} a_i\theta_i \in \mathbb{Z}F$ *is an element of the integral group ring,* $a_i \neq 0$ *for all* $i \in I$ *and if the* θ_i*s are pairwise distinct (reduced) elements of* F, *such that* $\theta^{\mathfrak{y}} = \sum_{i\in I} a_i\theta_i^{\mathfrak{y}}$ *is bijective, then* I *is a singleton and its coefficient is* 1 *or* -1.

Proof. If $F = \langle \varphi_t \mid t \in u\rangle$, then by hypothesis $F^{\mathfrak{y}} = \langle \varphi_t^{\mathfrak{y}} \mid t \in u\rangle$ is a free subgroup of $\operatorname{Aut} G^{\mathfrak{y}}$. If also $\theta = \sum_{i\in I} a_i\theta_i \in \mathbb{Z}F$ is as above, then $\theta^{\mathfrak{y}} \in \mathbb{Z}F^{\mathfrak{y}}$ and we may assume that $\ker\theta^{\mathfrak{y}} = 0$. It remains to show that $\theta^{\mathfrak{y}}$ is surjective if and only if $I = \{0\}$ is a singleton and $a_0 = \pm 1$. If $I = \{0\}$, then it is clear that $\theta^{\mathfrak{y}}$ is surjective if and only if $a_0 = \pm 1$. Hence we may assume that $\theta^{\mathfrak{y}}$ is an isomorphism, and $|I| > 1$ for contradiction. In order to apply Definition 3.5 we pass to the quotient $\overline{G}^{\mathfrak{y}} = G^{\mathfrak{y}}/h(u)$ and to the induced maps $\overline{\varphi}_t$, which we rename again as $G^{\mathfrak{y}}, \varphi_t$. It follows that $h(u)\theta^{\mathfrak{y}} \subseteq h(u)$ and silently we assume that $h(u)$ is invariant under $(\theta^{\mathfrak{y}})^{-1}$; otherwise we must enlarge $h(u)$ by a back and forth argument such that the quotient satisfies again Definition 3.5 (v). Also $\overline{G}^{\mathfrak{y}} = G^{\mathfrak{y}}/h(u) \neq 0$ because $|h(u)| = \aleph_0 < |G^{\mathfrak{y}}|$.

If $X \neq 0$ is a countable subgroup of $G^{\mathfrak{y}}$, then X is free. We may assume without restriction that $X\theta^{\mathfrak{y}} \subseteq X$, $XF^{\mathfrak{y}} \subseteq X$ and $X(\theta^{\mathfrak{y}})^{-1} \subseteq X$ and $\theta_X =^{\mathrm{df}} \theta^{\mathfrak{y}} \restriction X \in \operatorname{End} X$. If $x \in X$, then $x = g\theta^{\mathfrak{y}} \in G^{\mathfrak{y}}\theta^{\mathfrak{y}} = G^{\mathfrak{y}}$, thus $g = x(\theta^{\mathfrak{y}})^{-1} \in X$ and θ_X is also surjective (on X). We can start with some X' with a special basis $B' \neq \emptyset$ as in Definition 3.5 (v) (the weak version mentioned after the Definition 3.3) and let X be its closure as above. Then X' will be a summand of X because G/X' is $\aleph_1$-free. Hence B' extends to a basis B of X: The maps $\varphi_X = \varphi^{\mathfrak{y}} \restriction X$, $(\varphi \in F)$ (by hypothesis) are total automorphisms of X, thus all automorphisms of $F_X = \{\varphi_X \mid \varphi \in F\}$ are permutations of B' when restricted further to B'.

Let G be the group given by Proposition 2.1. Note that $\varphi_t \in F$ $(t \in u)$ is given by Proposition 3.9 and $\mathbb{Z}F = \operatorname{End} G$, hence any φ_t can be viewed as an

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automorphism of G. In order to distinguish it from the element in F we will call the automorphism $\varphi_t^* \in \operatorname{Aut} G$ and F becomes F^*. The mapping * extends naturally to all of $\mathbb{Z}F$, (by the identification $\mathbb{Z}F = \operatorname{End} G$), thus $\theta^* = \sum_{i\in I} a_i\theta_i^* \in \operatorname{End} G$, where $\theta_i^* \in F^*$. From $|I| > 1$ and Lemma 2.2 it follows that θ^* is not surjective. Let $y \in G \setminus G\theta^*$ which will help us showing that θ_X can not be surjective either, in fact we want to show that $B' \cap X\theta_X = \emptyset$. Fix an element $c \in B'$ and define a map $\Phi\colon B \to G$ such that $c\Phi = y$. If $b \in B$ and there is $\varphi \in F$ such that $c\varphi_X = b$, then put $b\Phi = c\Phi\varphi^*$. If φ exists, then it is unique by the U-property. Hence Φ is defined on $cF_X(= cF^{\mathfrak{y}})$. If $b \in B' \setminus cF_X$, then let $b\Phi = 0$. Hence Φ is well-defined on B and extends uniquely to an homomorphism $\Phi\colon X \to G$. It follows $b\varphi_X\Phi = b\Phi\varphi^*$, hence $b\theta_X\Phi = b\Phi\theta^*$ for all $b \in B$, i.e. commuting with Φ replaces the $_X$ by *. If $c \in X\theta_X$, then there is $x = \sum_{b\in B} x_b b \in X$ with $x\theta_X = c$. Thus $c = x\theta_X = \sum_{b\in B} x_b(b\theta_X)$ and we apply Φ to this equation to get the contradiction

$$y = c\Phi = \sum_{b\in B} x_b(b\theta_X)\Phi = \sum_{b\in B} x_b(b\Phi)\theta^* \in G\theta^*.$$

Thus θ_X, hence $\theta^{\mathfrak{y}}$ is not surjective. □

It is convenient to check the U-property by the following simple characterization.

Proposition 3.10. *Let $\langle\mathfrak{F}\rangle$ be the group freely generated by $\mathfrak{F}$ and $\pi\colon \mathfrak{F} \to \operatorname{pAut} G^{\mathfrak{x}}$ be a map as in Definition 3.5 with $\pi(\varphi_t) = \varphi_t^{\mathfrak{x}}$ for all $t \in J$. Then $\mathfrak{F}$ satisfies the U-property if and only if any reduced product $\varphi \in \langle\mathfrak{F}\rangle$ with $x\varphi^{\mathfrak{x}} = x$ for some $x \in \operatorname{Dom}\varphi^{\mathfrak{x}} \cap \mathfrak{p}G$ is $\varphi = 1 \in \langle\mathfrak{F}\rangle$.*

Proof. If we can choose a reduced element $\varphi \in \langle\mathfrak{F}\rangle$ with $x\varphi^{\mathfrak{x}} = x = (x1^{\mathfrak{x}})$ for some $x \in \operatorname{Dom}\varphi^{\mathfrak{x}} \cap \mathfrak{p}G$, then $\varphi = 1$ follows by the U-property of $\mathfrak{F}$. Conversely, if there are reduced elements $\varphi, \psi \in \langle\mathfrak{F}\rangle$ with $x\varphi^{\mathfrak{x}} = x\psi^{\mathfrak{x}}$ for some $x \in \operatorname{Dom}\varphi^{\mathfrak{x}} \cap \operatorname{Dom}\psi^{\mathfrak{x}} \cap \mathfrak{p}G$, then we can write $x = x\varphi^{\mathfrak{x}}(\psi^{\mathfrak{x}})^{-1}$. Hence $x \in \operatorname{Dom}\varphi^{\mathfrak{x}}(\psi^{\mathfrak{x}})^{-1}$ and we can cancel $\varphi\psi^{-1}$ to get a reduced $\theta \in \langle\mathfrak{F}\rangle$ with $\theta = \varphi\psi^{-1}$ in $\langle\mathfrak{F}\rangle$. From $x \in \operatorname{Dom}\varphi^{\mathfrak{x}}(\psi^{\mathfrak{x}})^{-1} \subseteq \operatorname{Dom}\theta^{\mathfrak{x}}$ it follows $x = x\theta^{\mathfrak{x}}$. We have $\theta = 1$ by hypothesis, and $\varphi = \psi$ follows. □

The last proposition shows that the U-property is a strong restriction on $\pi(\mathfrak{F})$. If only $x\varphi^{\mathfrak{x}} = x$ for a reduced φ and pure $x \in G$, then $\varphi = 1$. However note that if $\varphi \in \langle\mathfrak{F}\rangle \setminus \{1\}$, then $x\varphi^{\mathfrak{x}}(\varphi^{\mathfrak{x}})^{-1} = x$ for some $x \in \mathfrak{p}G$, hence $\varphi^{\mathfrak{x}}(\varphi^{\mathfrak{x}})^{-1} \subseteq \operatorname{id}_G$ but not $\varphi^{\mathfrak{x}}(\varphi^{\mathfrak{x}})^{-1} = \operatorname{id}_G$ because $\varphi\varphi^{-1}$ is not reduced.

The next lemma will be used to make the desired group 'more transitive'. We want to isolate the argument on the existence of $h(u)$: If $\mathfrak{x} = (G, \mathfrak{F}, \varepsilon, \pi, h) \in \mathfrak{K}$, $\varphi_0 \in \operatorname{pAut} G$ with $0 \notin J$ as in Lemma 3.11, then we extend $h\colon \mathfrak{P}_{\aleph_0}(J) \to \operatorname{Im}(h)$ to $h'\colon \mathfrak{P}_{\aleph_0}(J') \to \operatorname{Im}(h')$ where $J' = J \cup \{0\}$. If $u \in \operatorname{Dom} h$, then $h'(u) = h^{\mathfrak{x}}(u)$. If x, y and $\mathfrak{x}$ are from Lemma 3.11, then let $h(u')$ for $u' = u \cup \{0\}$ be a countable subgroup $U' \subseteq G$ containing $\{x, y\} \cup h(u)$ such that $\overline{G} = G/U'$ and the induced maps $\overline{\mathfrak{F}}^{\mathfrak{y}}$ satisfy Definition 3.5 (v) (the weak version mentioned after Definition 3.3 will suffice). Note that we only use extensions of groups $G^{\mathfrak{x}}$ as in Lemma 3.11 or Lemma 3.12 and unions of ascending chains of such groups. By a back and forth argument, and a moments reflection about the action of the extended partial automorphisms by the pushouts, it follows that such a countable subgroup U' exists.

Lemma 3.11 (to get more partial automorphisms). *If $\mathfrak{x} = (G, \mathfrak{F}, \varepsilon, \pi, h) \in \mathfrak{K}$ and $x, y \in \mathfrak{p}G$ such that $x\varphi^{\mathfrak{x}} \neq \pm y$ for all $\varphi \in \langle \mathfrak{F} \rangle$, then let $\varphi_0^{\mathfrak{y}}\colon x\mathbb{Z} \to y\mathbb{Z}(x \to y)$ be the natural isomorphism. If $\mathfrak{F}^{\mathfrak{y}} = \mathfrak{F} \cup \{\varphi_0\}, J^{\mathfrak{y}} = J \cup \{0\}, \varepsilon^{\mathfrak{y}} = \varepsilon \cup \{(0,1)\}, \pi^{\mathfrak{y}} = \pi^{\mathfrak{x}} \cup \{(\varphi_0, \varphi_0^{\mathfrak{y}})\}$ and $h^{\mathfrak{y}} = h'$ as above, then $\mathfrak{x} < \mathfrak{y} = (G, \mathfrak{F}^{\mathfrak{y}}, \varepsilon^{\mathfrak{y}}, \pi^{\mathfrak{y}}, h^{\mathfrak{y}}) \in \mathfrak{K}$*

Proof. Obviously $\mathfrak{x} \leq \mathfrak{y}$. Also $\varphi_0^{\mathfrak{y}} \in \mathrm{pAut}\, G$ because $x\mathbb{Z}, y\mathbb{Z}$ are pure subgroups of G and G is $\aleph_1$-free, hence $G/x\mathbb{Z}$ and $G/y\mathbb{Z}$ are $\aleph_1$-free. But it is not clear at the beginning that $\mathfrak{y}$ satisfies the U-property. We will check this with Proposition 3.10.

Let be $\mathfrak{F}^{\mathfrak{y}} = \mathfrak{F}', \varphi_0 = \eta$ and $\varphi \in \langle \mathfrak{F}' \rangle$. We write $\varphi = \varphi_1 \eta^{\varepsilon_1} \varphi_2 \dots \eta^{\varepsilon_{k-1}} \varphi_k$ with $0 \neq \varepsilon_i \in \mathbb{Z}$ and $\varphi_i \in \langle \mathfrak{F} \rangle$ reduced and assume that all φ_i's are different from ± 1, except possibly φ_1, φ_k. Now we assume that $z\varphi^{\mathfrak{y}} = z$ for some $z \in \mathrm{Dom}\, \varphi^{\mathfrak{y}} \cap \mathfrak{p}G$ and want to show that $\varphi = 1$.

However next we claim, that the product φ must be very special and show first that $\varepsilon_i = \pm 1$ for all $i < k$. If this is not the case, then some η^2 or η^{-2} is a factor of φ. We may assume that η^2 appears. Note that $\mathrm{Dom}(\eta^{\mathfrak{y}})^2 = (\mathrm{Im}(\eta^{\mathfrak{y}}) \cap \mathrm{Dom}\, \eta^{\mathfrak{y}})(\eta^{\mathfrak{y}})^{-1}$, and $\mathrm{Im}\, \eta^{\mathfrak{y}} \cap \mathrm{Dom}\, \eta^{\mathfrak{y}} = \mathbb{Z}y \cap \mathbb{Z}x = 0$ by the choice of x, y. Hence $(\eta^{\mathfrak{y}})^2 = 0$ and $\varphi^{\mathfrak{y}} = 0$ is a contradiction, because $0 \neq z \in \mathrm{Dom}\, \varphi^{\mathfrak{y}}$, so the first claim follows. Next we show that

$$\varphi = \pm \varphi_1 \eta^{\varepsilon_1} \varphi_2. \tag{3.2}$$

We look at the path of z, the set $[z]$ of all consecutive images of z:

$$z_0 = z, z_1 = z\varphi_1^{\mathfrak{y}}, z_2 = z_1(\eta^{\mathfrak{y}})^{\varepsilon_1}, \dots, z_{2k-1} = z_{2k-2}\varphi_k^{\mathfrak{y}}$$

In order to apply $(\eta^{\mathfrak{y}})^{\varepsilon_1}$ to z_1 we must have $z_1 \in \mathrm{Dom}(\eta^{\mathfrak{y}})^{\varepsilon_1}$, but $\mathrm{Dom}(\eta^{\mathfrak{y}})^{\varepsilon_1}$ is either $\mathbb{Z}x$ or $\mathbb{Z}y$, hence z_1 is one of the four elements of the set $V = \{\pm x, \pm y\}$ by purity. Inductively we get $z_i \in V$ for all $0 < i < 2k-1$. Suppose that η^{ε_2} appears in φ, then $z_3 = z_2\varphi_2^{\mathfrak{y}} \in V$ because $z_4 = z_3(\eta^{\mathfrak{y}})^{\varepsilon_2}$ is defined and z_3 is pure. However $x \in \mathrm{Dom}\, \varphi_2^{\mathfrak{y}}$ or $y \in \mathrm{Dom}\, \varphi_2^{\mathfrak{y}}$, respectively. Hence φ_2 is multiplication by ± 1 on $\mathbb{Z}x$ or on $\mathbb{Z}y$ or $x\varphi_2 = \pm y$. The last case was excluded by our hypothesis on x, y and the first two cases and the U-property would give $\varphi_2 = \pm 1$ which also was excluded. Hence (3.2) follows.

Our assumption is reduced to $z = \pm z\varphi_1^{\mathfrak{y}}(\eta^{\mathfrak{y}})^{\varepsilon_1}\varphi_2^{\mathfrak{y}}$ for some $z \in \mathfrak{p}G$. We may replace η by η^{-1}, hence $\varepsilon_1 = 1$ without loss of generality and $z = \pm z\varphi_1^{\mathfrak{y}}\eta^{\mathfrak{y}}\varphi_2^{\mathfrak{y}}$. We consider the path $[z]$ and have $z_1 = z\varphi_1^{\mathfrak{y}} = \pm x$ from purity of $z_1 \in \mathrm{Dom}\, \eta^{\mathfrak{y}}$. It follows $z_2 = \pm y$ and $z_3 = y\varphi_2^{\mathfrak{y}} = \pm z$ from our assumption. Thus $y\varphi_2^{\mathfrak{y}}\varphi_1^{\mathfrak{y}} = \pm x$ and $\varphi_2\varphi_1 \in \langle \mathfrak{F} \rangle$, which contradicts our choice of x, y. Hence only $\varphi = 1$ is possible and $\mathfrak{y} \in \mathfrak{K}$ follows. □

The next lemma will increase domain and image of partial automorphisms, respectively.

Lemma 3.12 (growing the partial automorphisms). *If $\mathfrak{x} = (G, \mathfrak{F}, \varepsilon, \pi, h) \in \mathfrak{K}$ with $\mathfrak{F} = \{\varphi_s \mid s \in J\}$ and $t \in J$, then there is $\mathfrak{x} \leq \mathfrak{y} = (G^{\mathfrak{y}}, \mathfrak{F}^{\mathfrak{y}}, \varepsilon^{\mathfrak{y}}, \pi^{\mathfrak{y}}, h) \in \mathfrak{K}$ such that the following holds.*

(i) $\mathfrak{F}^{\mathfrak{x}} = \mathfrak{F}^{\mathfrak{y}}$, $\varepsilon^{\mathfrak{x}} \restriction (J \setminus \{t\}) = \varepsilon^{\mathfrak{y}} \restriction (J \setminus \{t\})$, $\varepsilon^{\mathfrak{x}}(t) = -\varepsilon^{\mathfrak{y}}(t)$ *and* $G^{\mathfrak{y}} = G_0 + G_1$ *is a pushout with* $D = G_0 \cap G_1$ *and* $G^{\mathfrak{x}} = G_0 \cong G_1$.

(ii) (a) *If* $\varepsilon^{\mathfrak{x}}(t) = 1$, *then* $G_0 = \operatorname{Dom} \varphi_t^{\mathfrak{y}}$, $G_1 = \operatorname{Im} \varphi_t^{\mathfrak{y}}$ *and* $D = \operatorname{Dom} \varphi_t^{\mathfrak{x}}$.
(b) *If* $\varepsilon^{\mathfrak{x}}(t) = -1$, *then* $G_0 = \operatorname{Im} \varphi_t^{\mathfrak{y}}$, $G_1 = \operatorname{Dom} \varphi_t^{\mathfrak{y}}$ *and* $D = \operatorname{Im} \varphi_t^{\mathfrak{x}}$.

Proof. The set J and h do not change when passing from $\mathfrak{x}$ to $\mathfrak{y}$. Thus we consider $\pi^{\mathfrak{x}}$ next and restrict to $\varepsilon^{\mathfrak{x}} = 1$ (the case $\varepsilon^{\mathfrak{x}} = -1$ follows if we replace φ_t by φ_t^{-1}). For the pushout we let $G^{\mathfrak{y}} = (G \times G)/H$ with $H = \{(x\varphi, -x) \mid x \in \operatorname{Dom} \varphi_t\}$. If we also say that $U_0 = (U \times 0) + H/H \subseteq G^{\mathfrak{y}}$ and $U_1 = (0 \times U) + H/H$ for any $U \subseteq G$, then in particular $G^{\mathfrak{y}} = G_0 + G_1$ and $D =^{\mathrm{df}} G_0 \cap G_1 = (\operatorname{Im} \varphi_t^{\mathfrak{x}})_0 = (\operatorname{Dom} \varphi_t^{\mathfrak{x}})_1$ by the pushout. Moreover we identify $G_0 = G^{\mathfrak{x}}$, hence $\operatorname{Dom} \varphi_t^{\mathfrak{x}} = (\operatorname{Dom} \varphi_t^{\mathfrak{x}})_0 =^{\mathrm{df}} D'$ and $D = \operatorname{Im} \varphi_t^{\mathfrak{x}}$. The canonical map

$$\varphi_t^{\mathfrak{y}} \colon G^{\mathfrak{y}} \to G^{\mathfrak{y}} \quad ((x,0) + H \to (0,x) + H)$$

extends $\varphi_t^{\mathfrak{x}}$ because $((x,0) + H)\varphi_t^{\mathfrak{y}} = (0,x) + H = (x\varphi_t^{\mathfrak{x}}, 0) + H$ for all $x \in \operatorname{Dom} \varphi_t^{\mathfrak{x}}$. Clearly $G_0 = \operatorname{Dom} \varphi_t^{\mathfrak{y}}$ and $G_1 = \operatorname{Im} \varphi_t^{\mathfrak{y}}$. Moreover $G^{\mathfrak{y}}/D = G_0/\operatorname{Dom} \varphi_t^{\mathfrak{x}} \oplus G_1/\operatorname{Im} \varphi_t^{\mathfrak{x}}$ is $\aleph_1$-free, hence also $G^{\mathfrak{y}}$ and $G^{\mathfrak{y}}/G^{\mathfrak{x}} \cong G_1/D$ are $\aleph_1$-free, and the maps $\varphi_s^{\mathfrak{x}} = \varphi_s^{\mathfrak{y}}$ $(t \neq s \in J)$ remain the same. It follows that $\pi^{\mathfrak{y}} \colon \mathfrak{F}^{\mathfrak{y}} \to \operatorname{pAut} G^{\mathfrak{y}}$. The existence of a partial basis satisfying Definition 3.3 was discussed before Lemma 3.11. So for $\mathfrak{x} \leq y \in \mathfrak{K}$ we only must check the U-property for $\mathfrak{F}$ with the new partial automorphisms from $\varphi_s^{\mathfrak{y}}$ $(s \in J)$ and apply Proposition 3.10:

Consider a reduced product $\varphi = \varphi_1 \varphi_t^{\delta_1} \varphi_2 \ldots \varphi_t^{\delta_{n-1}} \varphi_n$, where $1 \neq \varphi_i \in \langle \mathfrak{F} \setminus \{\varphi_t\} \rangle$ except possibly $\varphi_1 = \pm 1$ and $\varphi_n = 1$.

Suppose that $z\varphi^{\mathfrak{y}} = z$ for some $z \in \mathfrak{p}G^{\mathfrak{y}}$ and let

$$z_0 = z, z_1 = z_0\varphi_1^{\mathfrak{y}}, t_1 = z_1(\varphi_t^{\mathfrak{y}})^{\delta_1}, z_2 = t_1\varphi_2^{\mathfrak{y}}, t_2 = z_2(\varphi_t^{\mathfrak{y}})^{\delta_2}, \ldots, z_n = t_{n-1}\varphi_n^{\mathfrak{y}}$$

be the path $[z]$ of z. Thus $z_n = z_0$ by assumption on φ. First we note that $\varphi_i^{\mathfrak{y}} = \varphi_i^{\mathfrak{x}}$ for all $i \leq n$ with the possible exceptions for $\varphi_1 = \pm 1$ or $\varphi_n = 1$. If also all the $(\varphi_t^{\mathfrak{y}})^{\delta_i}$ can be replaced by $(\varphi_t^{\mathfrak{x}})^{\delta_i}$, then $[z] \subseteq G^{\mathfrak{x}}$ and by the U-property of $\mathfrak{F}$, $z\varphi^{\mathfrak{y}} = z\varphi^{\mathfrak{x}} = z$ it follows $\varphi = 1$. We will consider the two cases $z_0 \in G_0$ and $z_0 \in G_1 \setminus G_0$.

First we reduce the second case $z_0 \in G_1 \setminus G_0$ to the first case. Since $z_0 \in \operatorname{Dom} \varphi_1$ it follows $\varphi_1 = \pm 1$, hence $z_1 = \pm z_0 \in G_1 \setminus G_0$. From $z_1 \in \operatorname{Dom}(\varphi_t^{\mathfrak{y}})^{\delta_1}$ it follows $\delta_1 \leq -1$ and $t_1' =^{\mathrm{df}} z_1(\varphi_t^{\mathfrak{y}})^{-1} \in G_0$. From $z_n = z_0$ it follows $z_{n-1}(\varphi_t^{\mathfrak{y}})^{\delta_{n-1}}\varphi_n^{\mathfrak{y}} = z_0 \in G_1 \setminus G_0$ and therefore $\varphi_n = 1$ and $\delta_{n-1} \geq 1$. The equation $z_0\varphi^{\mathfrak{y}} = z_0$ reduces to

$$\pm t_1'(\varphi_t^{\mathfrak{y}})^{\delta_1+1}\varphi_2^{\mathfrak{x}}(\varphi_t^{\mathfrak{y}})^{\delta_2} \cdots \varphi_{n-1}^{\mathfrak{x}} = t_1' \text{ with } t_1' \in G_0,$$

which is the first case for a new $z = t_1' \in \mathfrak{p}G_0$.

If $z_0 \in G_0$, then also $z_0 \in \operatorname{Dom} \varphi_1^{\mathfrak{x}}$ and $z_1 = z_0\varphi_1^{\mathfrak{x}} \in G_0$. We will continue along the path step by step having two subcases each time, but one of them will lead to a contradiction. In the first step either $z_1 \in D'$ or $z_1 \notin D'$. In any case $\delta_1 = 1$ and in the second case $t_1 = z_1\varphi_t^{\mathfrak{y}} \in G_1 \setminus D$, but $t_1 \in \operatorname{Dom} \varphi_2^{\mathfrak{y}}$ so necessarily $\varphi_2 = 1$ and $n = 2$. We get $t_1\varphi_2^{\mathfrak{y}} = z_2 = t_1$, and $t_1 = z_0 \in G_0$ contradicting $t_1 \in G_1 \setminus D$. We arrive at the other case $z_1 \in D'$. Hence $t_1 = z_1\varphi_t^{\mathfrak{x}} \in D$ and we must have $t_1 \in \operatorname{Dom} \varphi_2^{\mathfrak{x}}$. Therefore also $z_2 = t_1\varphi_2^{\mathfrak{x}} \in G_0$, and $\delta_2 = 1$. Again we have two cases

$z_2 \in D'$ and $z_2 \notin D'$ with $\delta_2 = 1$, where the second case leads to a contradiction. Hence $t_2 = z_2\varphi_t^{\mathfrak{x}} \in D$ and we continue until we reach n and all the $\varphi_t^{\mathfrak{y}}$s are replaced by the $\varphi_t^{\mathfrak{x}}$s. Now our first remark applies and $\varphi = 1$ follows from the U-property for $\mathfrak{F}$. □

Lemma 3.13. *Let α be a limit ordinal. Then any increasing continuous chain $\mathfrak{x}^j = (G^j, \mathfrak{F}^j, \varepsilon^j, \pi^j, h^j)_{j\in\alpha}$ in $(\mathfrak{K}, <)$ obtained by applications of Lemma 3.11 and Lemma 3.12 has a natural supremum $\mathfrak{x} = (G, \mathfrak{F}, \varepsilon, \pi, h)$ in $\mathfrak{K}$, where $G = \bigcup_{j\in\alpha} G^j$, $\mathfrak{F} = \bigcup_{j\in\alpha} \mathfrak{F}^j$, $J = \bigcup_{j\in\alpha} J^j$ and π, h are defined below.*

Proof. The map h extends uniquely, because h is defined on finite subsets of J. Similarly we can handle π. We define $\pi\colon \mathfrak{F} \longrightarrow \operatorname{pAut} G$ by taking unions: If $t \in J$, then $t \in J^j$ for all $i < j \in \alpha$ and $i \in \alpha$ large enough. Therefore we can let $\varphi_t^{\mathfrak{x}} =^{\mathrm{df}} \bigcup_{i<j\in\alpha} \pi^j(\varphi_t^j)$, and $\operatorname{Dom}\varphi_t^{\mathfrak{x}} = \bigcup_{i<j\in\alpha} \operatorname{Dom}\pi^j(\varphi_t^j)$. We found a well-defined partial automorphism $\varphi_t^{\mathfrak{x}}\colon \operatorname{Dom}\varphi_t^{\mathfrak{x}} \to G$ and also want that $\pi(\varphi_t) = \varphi_t^{\mathfrak{x}} \in \operatorname{pAut} G$. Hence we must show that $G/\operatorname{Dom}\varphi_t^{\mathfrak{x}}$ and $G/\operatorname{Im}\varphi_t^{\mathfrak{x}}$ are $\aleph_1$-free. Passing to an inverse it is enough to consider $G/\operatorname{Dom}\varphi_t^{\mathfrak{x}}$. Either there is a strictly increasing chain $j_i \in \alpha$ $(i \in I)$ cofinal to α such that $\pi^{j_i}(\varphi_t^{j_i}) \neq \pi^{j_i+1}(\varphi_t^{j_i+1})$ for all $i \in I$ or the sequence $\pi^j(\varphi_t^j)$ becomes stationary, say at $j_0 \in \alpha$. In the first case Lemma 3.12 applies and $\operatorname{Dom}\pi^{j_i+1}(\varphi_t^{j_i+1}) = G_0^{\mathfrak{x}j_i}$ and $\bigcup_{i\in I} G_0^{\mathfrak{x}j_i} = G^{\mathfrak{x}}$ by cofinality, hence $\operatorname{Dom}\varphi_t^{\mathfrak{x}} = G$ and $G/\operatorname{Dom}\varphi_t^{\mathfrak{x}} = 0$ is trivially $\aleph_1$-free. In the other case we have $\operatorname{Dom}\varphi_t^{\mathfrak{x}} = \operatorname{Dom}\varphi_t^{j_0}$, hence $G^{j_0}/\operatorname{Dom}\varphi_t^{\mathfrak{x}}$ is $\aleph_1$-free by induction. Moreover G/G^{j_0} is $\aleph_1$-free because $\aleph_1$-free is of finite character by Pontryagin's theorem (speaking about subgroups of finite rank). Hence $G/\operatorname{Dom}\varphi_t^{\mathfrak{x}}$ is $\aleph_1$-free, as required. Now it is easy to see that $\mathfrak{x} \in \mathfrak{K}$: Definition 3.3 is easily verified by taking union of partial bases. The U-property carries over to limit ordinals. □

The next main result of this section follows by iterated application of the Lemma 3.11, Lemma 3.12 and Lemma 3.13. Without danger we now can identify the free groups F' and $\langle\mathfrak{F}'\rangle$ by the isomorphism π' in the theorem.

Theorem 3.14. *If $\mathfrak{x} = (G, \mathfrak{F}, \varepsilon, \pi, h)$ is in $\mathfrak{K}$, then we can also find a quintuple $\mathfrak{x}' = (G', \mathfrak{F}', \varepsilon', \pi', h')$ in $\mathfrak{K}$ with $\mathfrak{x} \leq \mathfrak{x}'$ such that the following holds.*

- (a) *$\mathfrak{F}'$ is a set of automorphisms of G' which freely generates $\langle\mathfrak{F}'\rangle \subseteq \operatorname{Aut} G$, hence G' is a module over $R' = \mathbb{Z}\langle\mathfrak{F}'\rangle$.*
- (b) *$\langle\mathfrak{F}'\rangle$ acts uniquely transitive on $\mathfrak{p}G'$.*
- (c) $|G'| + |\mathfrak{F}'| = |G| + |\mathfrak{F}|$

Proof. We will proceed by induction to get a chain $\mathfrak{x} \leq \mathfrak{x}_n \in \mathfrak{K}$ $(n \in \omega)$ taking several steps each time. Finally $\mathfrak{x}'$ will be the supremum of the $\mathfrak{x}_n$s.

In the first step we apply $|G_0|$-times Lemma 3.11 and Lemma 3.13 to $\mathfrak{x} = \mathfrak{x}_0$ taking care of all appropriate pairs of elements in $\mathfrak{p}G_0$ and let $\mathfrak{x}_0 \leq \mathfrak{x}_1 = (G_1, \mathfrak{F}_1, \varepsilon_1, \pi_1, h_1)$ be the union of this chain such that $\langle\mathfrak{F}_1\rangle$ acts transitive on $\mathfrak{p}G_0$. In this case $G_0 = G_1$ but the other parameters in $\mathfrak{x}_0$ increase (along a chain) by Lemma 3.11. In the next step we apply $|G_1|$-times Lemma 3.12 and Lemma 3.13 to get $\mathfrak{x}_1 \leq \mathfrak{x}_2 = (G_2, \mathfrak{F}_2, \varepsilon_2, \pi_2, h_2)$ such that $G_0 \subseteq \operatorname{Dom}\varphi_t^{\mathfrak{x}_2}$ and $G_0 \subseteq \operatorname{Im}\varphi_t^{\mathfrak{x}_2}$ for all $t \in J^{\mathfrak{x}_2} = J^{\mathfrak{x}_1}$. We continue this way for each n. From Lemma 3.13 follows $\mathfrak{x} \leq \mathfrak{x}' \in \mathfrak{K}$, and by construction $\langle \mathfrak{F}' \rangle \subseteq \operatorname{Aut} G'$ acts uniquely transitive on $\mathfrak{p}G'$ such that also (c) of Theorem 3.14 holds. □

Now we can restrict elements in $\mathfrak{K}$ to triples and say that $\mathfrak{x} = (G^{\mathfrak{x}}, \mathfrak{F}^{\mathfrak{x}}, \pi^{\mathfrak{x}})$ belongs to $\mathfrak{K}^*$ if and only if $G^{\mathfrak{x}}$ is an $\aleph_1$-free abelian group, $\mathfrak{F} = \{\varphi_t \mid t \in J^{\mathfrak{x}}\}$ freely generates a (non-abelian) group $\langle \mathfrak{F} \rangle$ and $\pi^{\mathfrak{x}} \colon \mathfrak{F} \to \operatorname{Aut} G^{\mathfrak{x}}$ is an injective map such that $\pi^{\mathfrak{x}}(\mathfrak{F}) = \{\pi^{\mathfrak{x}}(\varphi_t) = \varphi_t^{\mathfrak{x}} \mid t \in J^{\mathfrak{x}}\}$ satisfies the U-property. Still we can view $\mathfrak{K}^*$ as a subset of $\mathfrak{K}$ and have an induced ordering, compare Definition 3.8: We have $\mathfrak{x} \leq \mathfrak{y}$ $(\mathfrak{x}, \mathfrak{y} \in \mathfrak{K}^*)$ if (v) and the following holds.

(i) $G^{\mathfrak{x}} \subseteq G^{\mathfrak{y}}$ and $G^{\mathfrak{y}}/G^{\mathfrak{x}}$ is $\aleph_1$-free.
(ii) $\pi^{\mathfrak{x}} \subseteq \pi^{\mathfrak{y}}$.

The other conditions in Definition 3.8 are vacuous.

4. Construction of uniquely transitive groups

In this section we will sharpen Theorem 3.14 which shows that a group ring $R = \mathbb{Z}F$ for some free group F can be represented as $R \subseteq \operatorname{End} G$ of some $\aleph_1$-free group G (hence G is an R-module), such that the units $U(R) = \pm F$ act uniquely transitive on G. If $\pm F \subseteq \operatorname{Aut} G$ is not necessarily transitive but satisfies the uniqueness property $((\varphi, \varphi' \in \pm F \text{ and } \exists x \in \mathfrak{p}G, x\varphi = x\varphi') \Rightarrow \varphi = \varphi')$, then we will also say that the pair (G, F) has the U-property. We will modify G such that equality holds, that is $R = \operatorname{End} G$. Thus we have to kill unwanted endomorphisms, which is done using the Strong Black Box 4.2. We need some preparation to work with this prediction principle. ,

First we need an $\mathbb{S}$-adic topology. Let $\mathbb{S} = \{q_n \mid n \in \omega\}$ be an enumeration of a multiplicatively closed set generated by 1 and at least one more natural number different from 1 and define $s_0 = 1, s_{n+1} = s_n \cdot q_n$ for all $n \in \omega$ which obviously defines a Hausdorff $\mathbb{S}$-topology on the groups G under consideration. Let $\widehat{G}$ be the $\mathbb{S}$-adic completion of G and $G \subseteq_* \widehat{G}$ naturally, where purity "$\subseteq_*$" is $\mathbb{S}$-purity.

Next we must formulate the Strong Black Box and adjust our notations; see Strong Black Box 4.2. We rely on the version adjusted to and proven for modules using only naive set theory as explained in the introduction, see [**10, 9**]. Thus we have to fix a few parameters next. To do so we choose an enumeration by ordinals $\alpha \in \lambda$, with $\lambda = \mu^+$ a successor cardinal such that $\mu^{\aleph_0} = \mu$: Let F_α be a free non-abelian group with a set of μ free generators $\mathfrak{F}_\alpha$, the $\mathfrak{F}_\alpha$s constitute a strictly increasing, continuous chain in $\alpha < \lambda$. We will write $R_\alpha = \mathbb{Z}F_\alpha$ with $R = \bigcup_{\alpha \in \lambda} R_\alpha$ and note that R_α^+ and R^+ are free abelian groups. By Theorem 3.14 there is an R_α-module G_α of cardinality μ which is cyclic as R_α-module and $\aleph_1$-free as abelian group and F_α acts sharply transitive on $\mathfrak{p}G_\alpha$. We can choose a free abelian and $\mathbb{S}$-dense subgroup $B_\alpha \subseteq_* G_\alpha \subseteq_* \widehat{B_\alpha}$. Also the B_αs constitute a strictly increasing, continuous chain in $\alpha < \lambda$. Passing to $\alpha + 1$ we can let $B_{\alpha+1} = B_\alpha \oplus A_\alpha$ with $A_\alpha = \bigoplus_{i<\rho} \mathbb{Z}a_i$ (and $\rho = \mu$). It helps (when using [**10, 9**]) not to identify ρ and μ in the formulation of the Strong Black Box 4.2.

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Thus the construction is based on a free abelian group B of rank λ and its $\mathbb{S}$-adic completion $\widehat{B}$, in fact the desired group G will be sandwiched as $B \subseteq_* G \subseteq_* \widehat{B}$. Put $B = \bigoplus_{\alpha<\lambda} e_\alpha A_\alpha$. Then, writing $e_{\alpha,i}$ for $e_\alpha a_i$, we have $B = \bigoplus_{(\alpha,i)\in\lambda\times\rho} \mathbb{Z}e_{\alpha,i}$. For later use we put the lexicographic ordering on $\lambda \times \rho$; since ρ, λ are ordinals $\lambda \times \rho$ is well ordered.

For any $g = (g_{\alpha,i}e_{\alpha,i})_{(\alpha,i)\in\lambda\times\rho} \in \widehat{B} \subseteq \prod_{(\alpha,i)\in\lambda\times\rho} \widehat{\mathbb{Z}}e_{\alpha,i}$ we define the *support* of g by $[g] = \{(\alpha,i) \in \lambda\times\rho \mid g_{\alpha,i} \neq 0\}$ and the support of $H \subseteq \widehat{B}$ by $[H] = \bigcup_{g\in H}[g]$; note $|[g]| \leq \aleph_0$ for all $g \in \widehat{B}$. Moreover, we define the λ-*support* of g by $[g]_\lambda = \{\alpha \in \lambda \mid \exists i \in \rho : (\alpha,i) \in [g]\} \subseteq \lambda$ and the A-*support* of g by $[g]_A = \{i \in \rho \mid \exists \alpha \in \lambda : (\alpha,i) \in [g]\} \subseteq \rho \subseteq \lambda$. Recall that $e_{\alpha,i} = a_i e_\alpha$, where $a_i \in A_\alpha$, which explains the use of the notion "A-support".

Next we define a *norm* on λ, respectively on $\widehat{B}$, by $\|\{\alpha\}\| = \alpha + 1$ ($\alpha \in \lambda$), $\|H\| = \sup_{\alpha\in H}\|\alpha\|$ ($H \subseteq \lambda$), hence $\|\alpha\| = \alpha$, and $\|g\| = \|[g]_\lambda\|$ ($g \in \widehat{B}$), i.e. $\|g\| = \min\{\beta \in \lambda \mid [g]_\lambda \subseteq \beta\}$. Note, $[g]_\lambda \subseteq \beta$ holds if and only if $g \in \widehat{B_\beta}$ for $B_\beta = \bigoplus_{\alpha<\beta} e_\alpha A_\alpha$. We also define an A-*norm* of g by $\|g\|_A = \|[g]_A\|$. Also let $\lambda^o := \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \omega\}$.

Finally, we need to define canonical homomorphisms used in the Strong Black Box for predictions. For this we fix bijections $g_\gamma \colon \mu \to \gamma$ for all γ with $\mu \leq \gamma < \lambda$ where we put $g_\mu = \mathrm{id}_\mu$ and so $|\gamma| = |\mu| = \mu$ for all such γ's. For technical reasons we also put $g_\gamma = g_\mu$ for $\gamma < \mu$.

Definition 4.1. *Let the bijections g_γ ($\gamma < \lambda$) be as above and put $\gamma_{\alpha,i} = \gamma_\alpha \times \gamma_i$ for all $(\alpha,i) \in \lambda \times \rho$. We define P to be a* canonical summand *of B if $P = \bigoplus_{(\alpha,i)\in I} \mathbb{Z}e_{\alpha,i}$ for some $I \subseteq \lambda \times \rho$ with $|I| \leq \aleph_0$ such that:*

- *if $(\alpha,i) \in I$, then $(i,i) \in I$; if $(\alpha,i) \in I, \alpha \in \rho$ then $(i,\alpha) \in I$ and*
- *if $(\alpha,i) \in I$, then $(I \cap (\mu\times\mu))\, g_{\alpha,i} = I \cap \alpha \times i$.*

Accordingly, a homomorphism $\phi \colon P \to \widehat{B}$ is a canonical homomorphism *if P is a canonical summand of B and $\mathrm{Im}\,\phi \subseteq \widehat{P}$; we put $[\phi] = [P]$, $[\phi]_\lambda = [P]_\lambda$ and $\|\phi\| = \|P\|$.*

Note, by the above definition, a canonical summand P satisfies $\|P\|_A \leq \|P\|$. Let $\mathfrak{C}$ denote the set of all canonical homomorphisms. From assumption $\mu^{\aleph_0} = \mu$ follows $|\mathfrak{C}| = \lambda$. We are now ready to formulate the Strong Black Box:

The Strong Black Box 4.2. *Let $\lambda = \mu^+$ and $\mu = \mu^{\aleph_0}$ be as before and let $E \subseteq \lambda^o$ be a stationary subset of λ.*

Then there exists a family $\mathfrak{C}^*$ of canonical homomorphisms with the following properties:

(1) *If $\phi \in \mathfrak{C}^*$, then $\|\phi\| \in E$.*

(2) *If ϕ, ϕ' are two different elements of $\mathfrak{C}^*$ of the same norm α, then $\|[\phi]_\lambda \cap [\phi']_\lambda\| < \alpha$.*

(3) Prediction: *For any homomorphism $\psi \colon B \to \widehat{B}$ and for any subset I of $\lambda \times \rho$ with $|I| \leq \aleph_0$ the set*

$$\{\alpha \in E \mid \exists \phi \in \mathfrak{C}^* : \|\phi\| = \alpha, \phi \subseteq \psi, I \subseteq [\phi]\}$$

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is stationary.

We will enumerate $\mathfrak{C}^*$ taking care of the order of the norm and can write $\mathfrak{C}^* = \{\phi_\alpha \mid \alpha < \lambda\}$ such that $\|[\phi_\alpha]_\lambda\| \leq \|[\phi_\beta]_\lambda\|$ for all $\alpha < \beta < \lambda$. Note that we distinguish between $\varphi \in \mathfrak{F}$ and ϕ given by the Strong Black Box 4.2.

The enumeration is now used to find a continuous, ascending chain G_α $(\alpha < \lambda)$ such that $B_\alpha \subseteq_* G_\alpha \subseteq_* \widehat{B_\alpha}$ and the corresponding $\mathfrak{x}_\alpha = (G_\alpha, \mathfrak{F}_\alpha, \pi_\alpha) \in \mathfrak{K}^*$. Let $E \subseteq \lambda^o$ be a fixed stationary subset of λ^o and choose $\mathfrak{x}_0 = (G_0, \mathfrak{F}_0, \pi_0)$ to be any triple given by Theorem 3.14 for cardinality μ. For any $\beta < \lambda$, let $P_\beta = \operatorname{Dom}\phi_\beta$ and suppose that the triples $\mathfrak{x}_\beta = (G_\beta, \mathfrak{F}_\beta, \pi_\beta)$ are constructed for all $\beta < \alpha$. If α is a limit ordinal, then by continuity we let $\mathfrak{x}_\alpha = \sup_{\beta<\alpha} \mathfrak{x}_\beta$, thus $\mathfrak{x}_\alpha = (G_\alpha, \mathfrak{F}_\alpha, \pi_\alpha)$ is well defined and belongs to $\mathfrak{K}^*$ by Proposition 4.3 and Theorem 3.14; in particular $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ is an R_α-module over the integral group ring $R_\alpha = \bigcup_{\beta<\alpha} R_\alpha = \mathbb{Z}F_\alpha$ of the free group $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ and $B_\alpha \subseteq_* G_\alpha \subseteq_* \widehat{B_\alpha}$.

We may assume that $\alpha = \beta + 1$ and distinguish two cases, the trivial situation and the ordinal α where we have to work.

The first case arrives if we do not want to work or cannot work: If β is also a discrete ordinal, then we apply two steps. First let $G'_\alpha = G_\beta \oplus G^1_\beta$ where $G^1_\beta = e_\beta R_\beta$ is a "new" free summand. In the second step we apply Theorem 3.14 to get $G'_\alpha \subseteq_* G_\alpha$ and $\mathfrak{x}_\alpha = (G_\alpha, \mathfrak{F}_\alpha, \pi_\alpha) \in \mathfrak{K}^*$. Note that in particular G_α is an R_α-module with $R_\beta \subseteq R_\alpha$ as integral group rings (because $F_\beta \subseteq F_\alpha$) and G_α/G_β is $\aleph_1$-free. Using A_β, we can arrange that $B_{\beta+1} \subseteq_* G_{\beta+1} \subseteq_* \widehat{B_{\beta+1}}$. If β is a limit ordinal and not in E, we proceed similarly. If $\beta \in E$, then we also apply the trivial extension if ϕ_β is scalar multiplication by some $r \in R_\beta$ when restricted to P_β or if $\operatorname{Im}\phi_\beta \not\subseteq G_\beta$. Otherwise we will meet the condition of the Step Lemma 4.4 and must work:

Suppose that $\alpha = \beta + 1$ with $\beta \in E$ and $\operatorname{Im}\phi_\beta \subseteq G_\beta$, $\phi_\beta \notin R_\beta$. In this case we try to 'kill' the undesired homomorphism ϕ_β which comes from the black box prediction. (However note that it could be that $\phi_\beta \in R$ later on, so in this case ϕ_β is a good candidate which should survive the massacre. This we must and will see clearly at the marked place near the end of the proof!) Recall that $\|\phi_\beta\| \in \lambda^o$, hence there are $(\beta_n, i_n) \in [\phi_\beta]$ $(n \in \omega)$ such that $\beta_0 < \beta_1 < \cdots < \beta_n < \cdots$ and $\sup_{n\in\omega} \beta_n = \|\phi_\beta\|$. Without loss of generality we may assume that $\beta_n \notin E$ for all $n \in \omega$ and hence $G'_{\beta_n+1} = G_{\beta_n} \oplus e_{\beta_n} R_{\beta_n}$. We put $I = \{(\beta_n, i_n) \mid n < \omega\}$. Then $I_\lambda \cap [g]_\lambda$ is finite for all $g \in G_\beta$. We apply the Step Lemma 4.4 to $I, P = \operatorname{Dom}\phi_\beta$ and $H = G_\beta$. Therefore there exists an extension $\mathfrak{x}_\alpha = \mathfrak{x}_{\beta+1}$ of $\mathfrak{x}_\beta$ and an element $y_\beta \in G_\alpha$ such that $y_\beta\phi_\beta \notin G_\alpha$ and $\|y_\beta\| = \|\phi_\beta\| = \|P_\beta\|$. Thus the chain G_α $(\alpha < \lambda)$ is constructed up to the used Step Lemma 4.4. Finally let $G = \bigcup_{\alpha\in\lambda} G_\alpha$.

It remains to show $R = \operatorname{End} G$, the Step Lemma 4.4 and the following proposition which ensures that the U-property can be extended when applying the step lemma. We will use obvious simplified notations:

Proposition 4.3. *Let $G = \bigcup_{n\in\omega} G_n$, $x_n \in G^1_n$ be pure (R_n-torsion-free) elements, where $G'_{n+1} = G_n \oplus G^1_n \subseteq_* G_{n+1}$ and G_{n+1} is obtained from G'_{n+1} by Theorem 3.14. If $b \in \widehat{G_0}, x = \sum_{n\in\omega} x_n s_{n-1} + b$ and $G' = \langle G, xR_\omega\rangle_*$, then $U(R_\omega) = \pm F$*

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for the free group $F = \bigcup_{n\in\omega} F_n$, the pair (G', F) satisfies the U-property and G' is $\aleph_1$-free.

Proof. It is well-known that G' is $\aleph_1$-free, see [**2**] or [**6, 9**] for instance. We want to show that the pair (G', F) satisfies the U-property.

If $r \in R_\omega$ and $\varphi \in F$, then $r \in R_n$ and $\varphi \in F_n$ for some $n \in \omega$ large enough. Hence there is no restriction to write any $0 \neq y \in G'$ as $y = x^k r + g$ with $x^k = \sum_{n>k} \frac{s_{n-1}}{s_k} x_n + b^k$, $r \in R_n$, $b - b^k \in G_0$, $g = g_0 + g_1$, $g_0 \in \widehat{G_0}$, $0 \neq g_1 \in G_n$ and $[G_0] \cap [g_1] = [x^k r] \cap [g] = \emptyset$.

Suppose $y\varphi = y$, hence $(x^k r)\varphi + g\varphi = x^k r + g$ and by continuity

$$\sum_{m>k} \frac{s_{m-1}}{s_k}(x_m r)\varphi + g\varphi = \sum_{m>k} \frac{s_{m-1}}{s_k}(x_m r) + g.$$

We consider any $m > n, k$ and note that x_m is a torsion-free element of the R_n-module G_m and also $\varphi \in U(R_n)$, (hence G_m is invariant under application of φ, r) and $0 \neq x_m r\varphi, x_m\varphi \in G_m$. Restricting the support $[x^k r]$ of the last displayed equation to x_m shows that $\frac{s_{m-1}}{s_k}(x_m r) = \frac{s_{m-1}}{s_k}(x_m r)\varphi$. Hence $0 \neq (x_n r) = (x_n r)\varphi$ and passing to $h \in \mathfrak{p}G_m$ with $hq = x_n r$ we get $h\varphi = h$, hence $\varphi = 1$ by the U-property for (G_m, F_m). So φ acts as id_{G_m} for all $m \in \omega$ large enough. Thus φ is the identity on G, and the same holds for G' by continuity (G is dense in G' in the $\mathbb{S}$-topology). □

Next we prove the step lemma.

Step Lemma 4.4. *Let $P = \bigoplus_{(\alpha,i)\in I^*} \mathbb{Z}e_{\alpha,i}$ for some $I^* \subseteq \lambda \times \rho$ and let H be a subgroup of $\hat{B}$ with $P \subseteq_* H \subseteq_* \hat{B}$ which is $\aleph_1$-free and an R-module, where $R =^{\mathrm{df}} R^{\mathfrak{x}} = \bigcup_{\alpha\in I'} R_\alpha$ with $I' =^{\mathrm{df}} [I]_\lambda = \{\alpha < \lambda \mid \exists i < \rho, (\alpha, i) \in I^*\}$. Assume that $\mathfrak{x} = (H, \mathfrak{F}, \pi) \in \mathfrak{K}^*$ (thus $R^{\mathfrak{x}} = \mathbb{Z}\langle\mathfrak{F}\rangle$). Also suppose that there is a set $I = \{(\alpha_n, i_n) \mid n < \omega\} \subseteq [P] = I^*$ such that $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots (n < \omega)$ are discrete ordinals and*

(i) *$H = \bigcup_{n<\omega} G_{\alpha_n}$, $R = \bigcup_{n<\omega} R_{\alpha_n}$; where $R_{\alpha_n} = \mathbb{Z}\langle\mathfrak{F}_{\alpha_n}\rangle$, $\mathfrak{x}_n = (G_{\alpha_n}, \mathfrak{F}_{\alpha_n}, \pi_n) \in \mathfrak{K}^*$*

(ii) *$I_\lambda \cap [g]_\lambda$ is finite for all $g \in H$ $(I_\lambda = [I]_\lambda)$.*

If $\phi\colon P \to H$ is a homomorphism which is not multiplication by an element from R, then there exists an element $y \in \widehat{P}$ and $\mathfrak{x} \leq \mathfrak{x}' = (H', \mathfrak{F}', \pi') \in \mathfrak{K}^*$ such that $H \subseteq_* H' \subseteq_* \hat{B}$, $y \in H'$ and $y\phi \notin H'$.

Proof. By assumption H is an R-module and hence the $\mathbb{S}$-completion $\widehat{H}$ is an $\widehat{R}$-module. Let $\widehat{\mathbb{Z}}$ be the $\mathbb{S}$-completion of $\mathbb{Z}$ and let $x = \sum_{n\in\omega} s_n e_{\alpha_n,i_n} \in \widehat{P}$. We will use $y = x \in \widehat{P}$ or $y = x + b \in \widehat{P}$ for some $b \in \widehat{P}$ to construct two new groups $H \subseteq_* H_y$:

Let $H'_y = \langle H, yR\rangle_* \subseteq \hat{B}$. By Proposition 4.3 and Theorem 3.14 we obtain $\mathfrak{x}_y = (H_y, \mathfrak{F}_y, \pi_y) \in \mathfrak{K}^*$ such that $\mathfrak{x} \leq \mathfrak{x}_y$ and $y \in H_y$. Clearly $y\phi \in \widehat{H}$. Put $z = y\phi$

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and $y = x$. If $z \notin H_x$, then we choose $H' = H_x$ and have $\mathfrak{x}_x \in \mathfrak{K}^*$ with $\phi \notin \operatorname{End} H_x$. Also $R_x = \mathbb{Z}\langle \mathfrak{F}_x \rangle$.

If $z \in H_x$, then also $z \in H'_x$ by an easy limit argument and there exist integers k and n such that $s_k x\phi = gr_n + xr'_n$ for some $r_n, r'_n \in R$ and $g \in H$. It follows $x(s_k\phi - r'_n) = gr_n$ and $r_n, r'_n \in R_{n'}$ for some $n' \in \omega$. Since $(s_k\phi - r'_n) \neq 0$ there is $b' \in P$ such that $b =^{\mathrm{df}} b'(s_k\phi - r'_n) \neq 0$. Note that b' has finite support. Moreover, by the cotorsion-freeness of H there exists $\pi \in \widehat{\mathbb{Z}}$ such that $\pi b \notin H$. Let $y = x + \pi b'$. We claim that $\phi \notin \operatorname{End}(H_y)$. By way of contradiction assume that $s_l(x+\pi b')\phi = g^* r^*_m + (x+\pi b')r^{**}_m$ for some integer $l \geq k$ and elements $r^*_m, r^{**}_m \in R_{m'}$ for some $m' \in \omega$ and $g^* \in H$. Without loss of generality, we may assume $n = m$, hence $s_l(x + \pi b')\phi = g^* r^*_n + (x + \pi b')r^{**}_n$ and if $s = s_l/s_k$, then

$$\begin{aligned} s_l(\pi b')\phi = s_l(x + \pi b')\phi - s s_k x\phi = g^* r^*_n + (x + \pi b')r^{**}_n - s(gr_n + xr'_n) = \\ = (g^* r^*_n - sgr_n) + \pi b' r^{**}_n + x(r^{**}_n - sr'_n). \end{aligned}$$

Since $[\pi b'] = [b'], [\pi b'\phi] = [b'\phi]$ and $g^*, g \in H$ an easy support argument shows that $r^{**}_n = sr'_n$, hence $ss_k(\pi b')\phi = (g^* r^*_n - sgr_n) + s\pi b' r'_n$ and thus $s\pi(s_k b'\phi - b'r'_n) = (g^* r^*_n - sgr_n) \in H$. By purity we get $\pi(s_k b'\phi - b'r'_n) = \pi b \in H$ - a contradiction. □

Proof of Theorem 1.1. If $R = \operatorname{End} G$, then also $\operatorname{Aut} G = \pm F$, because $U(R) = \pm F$ and any $r \in R \setminus U(R)$ viewed as endomorphism of G is not bijective by Proposition 3.9. Hence it remains to show that $\operatorname{End} G \subseteq R$.

First we choose a new filtration of G and define

$$G^\alpha =^{\mathrm{df}} \{g \in G \mid \|g\| < \alpha\} \quad (\alpha < \lambda).$$

Lemma 4.5. *The new filtration on G has the following properties.*

(a) $G \cap \widehat{P_\beta} \subseteq G_{\beta+1}$ *for all* $\beta < \lambda$;
(b) $\{G^\alpha \mid \alpha < \lambda\}$ *is a λ-filtration of* G;
(c) *If $\alpha, \beta < \lambda$ are ordinals such that $\|\phi_\beta\| = \alpha$ then $G^\alpha \subseteq G_\beta$.*

Proof. Inspection of the definitions, or [**10, 9**]. □

Assume that $\phi \in \operatorname{End} G \setminus R$ and let $\phi' = \phi \restriction B$, hence $\phi' \notin R$. Let $I = \{(\alpha_n, i_n) \mid n < \omega\} \subseteq \lambda \times \rho$ such that $\alpha_0 < \alpha_1 < \cdots \alpha_n < \cdots$ is a sequence of discrete ordinals and $I_\lambda \cap [g]_\lambda$ is finite for all $g \in G$. Note that the existence of I can be easily arranged, e.g. let $E \subsetneq \lambda^o$, $\alpha \in \lambda^o \setminus E$, $i_n \in \rho$ $(n < \omega)$ arbitrary and $(\alpha_n)_{n<\omega}$ any discrete ladder on α.

By Lemma 4.4 there exist an element $y \in \hat{B}$, $(G, \mathfrak{F}, \pi) \leq (G', \mathfrak{F}', \pi') \in \mathfrak{K}^*$ such that $y \in G'$ and $y\phi' \notin G'$. (The last innocent sentence uses $\phi' \notin R$ and not just $\phi' \notin R_\beta$ for a suitable β, see our comment in the construction of G.) By the Black Box Theorem 4.2 the set

$$E' = \{\alpha \in E \mid \exists \beta < \lambda : \|\phi_\beta\| = \alpha, \phi_\beta \subseteq \phi', [y] \subseteq [\phi_\beta]\}$$

is stationary since $|[y]| \leq \aleph_0$. Note, $[y] \subseteq [\phi_\beta]$ implies that $y \in \widehat{P_\beta}$. Moreover, the set $C = \{\alpha < \lambda \mid G^\alpha\phi \subseteq G^\alpha\}$ is a cub in λ, hence $E' \cap C \neq \emptyset$. Let $\alpha \in E' \cap C$. Then $G^\alpha\phi' \subseteq G^\alpha$ and there exists an ordinal $\beta < \lambda$ such that $\|\phi_\beta\| = \alpha$, $\phi_\beta \subseteq \phi$ and $y \in \widehat{P_\beta}$. The first property implies that $G^\alpha \subseteq G_\beta$ by Lemma 4.5 and the latter

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properties imply that $\phi_\beta \notin R$. Moreover, $P_\beta \subseteq B$ with $\|P_\beta\| = \alpha$ and hence P_β, and also $P_\beta\phi$ are contained in $G^\alpha \subseteq G_\beta$.

Therefore $\phi_\beta \colon P_\beta \to G_\beta$ with $\phi_\beta \notin R_\beta$ and thus it follows form the construction that $y_\beta\phi_\beta \notin G_{\beta+1}$. On the other hand it follows from Lemma 4.5 that $y_\beta\phi_\beta = y_\beta\phi \in G \cap \widehat{P_\beta} \subseteq G_{\beta+1}$ - a contradiction. Thus $\operatorname{End} G = R$. □

We conclude this section with an immediate corollary of Theorem 1.1. In Section 2 we noted that $R = \mathbb{Z}F$ for some free group F has only trivial zero divisors. Therefore Theorem 1.1 has the following

Corollary 4.6. *For any cardinal $\lambda = \mu^+$ with $\mu^{\aleph_0} = \mu$ there is an $\aleph_1$-free, indecomposable abelian UT-group of cardinality λ.*

5. Discussions and applications

We want to discuss some consequences of our Main Theorem 1.1. Let $\operatorname{Mon} G$ be the monoid of all monomorphisms of an abelian group G. Also let be

$$\operatorname{Mon}_t G = \{\varphi \in \operatorname{Mon} G \mid G/\operatorname{Im}\varphi \text{ is torsion}\}.$$

We immediately note a

Proposition 5.1. *If G is an abelian group of type* 0, *then* $\operatorname{Mon}_t G$ *is a submonoid of* $\operatorname{Mon} G$.

Proof. If $\varphi, \psi \in \operatorname{Mon}_t G$, then $\varphi\psi \in \operatorname{Mon} G$. To see that $G/\operatorname{Im}(\varphi\psi)$ is torsion, consider any $x \in G$. There are $y \in G, 0 \neq n \in \omega$ with $nx = y\psi$; similarly $my = z\psi$, hence $mnx = m(y\psi) = z(\varphi\psi)$ and $\varphi\psi \in \operatorname{Mon}_t G$. □

We notice at many steps of the proof of Main Theorem 1.1 that UT (with respect to $\operatorname{Aut} G$) is a very strong property. If we replace $\operatorname{Aut} G$ by $\operatorname{Mon} G$ or $\operatorname{End} G$ in the definition of UT, this is even stronger: We either get nothing new or arrive on classical territory as shown next. We begin with an easy

Observation 5.2. *If G is an abelian group of type* 0 *and* $\operatorname{Mon} G$ *acts UT on* $\mathfrak{p}G$, *then* $\operatorname{Aut} G = \operatorname{Mon} G$.

Proof. Suppose $\varphi \in \operatorname{Mon} G$ and $x \in \mathfrak{p}G$, then $y =^{\mathrm{df}} x\varphi \in \mathfrak{p}G$ and there is $\psi \in \operatorname{Mon} G$ with $y\psi = x$, hence $x\varphi\psi = x, y\psi\varphi = y$ and $x\,\mathrm{id} = x, y\,\mathrm{id} = y$. Thus $\varphi\psi = \psi\varphi = \mathrm{id}$ follows by the U-property and $\varphi \in \operatorname{Aut} G$. □

Hence UT is the same for automorphisms and monomorphism. If we require even more, that $\operatorname{End} G$ acts transitive on G, then G is a vector space and the problem on the existence of UT-modules becomes trivial.

Definition 5.3. *We will say that G is E-transitive if for any pair $x, y \in \mathfrak{p}G$ there is $\sigma \in \operatorname{End} G$ such that $x\sigma = y$.*

Any $\sigma \in \operatorname{Aut} G$ induces a permutation of $\mathfrak{p}G$, but this does not hold for $\sigma \in \operatorname{End} G$. Our next results removes the set theoretic assumption $V = L$ from a main theorem in Dugas, Shelah [**5**] and strengthens the outcome. Recall from the

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introduction that A-transitive in the sense of [5] is the same as transitive (or just T) in this context.

Corollary 5.4. *For any cardinal $\lambda = \mu^+$ with $\mu^{\aleph_0} = \mu$ there is an $\aleph_1$-free abelian group of cardinality λ which is E-transitive, but not transitive.*

In view of Proposition 5.1 we will strengthen this corollary further and sketch a proof:

Theorem 5.5. *For any cardinal $\lambda = \mu^+$ with $\mu^{\aleph_0} = \mu$ there is an $\aleph_1$-free abelian group G of cardinality λ with* $\operatorname{Aut} G = \pm 1$, *which is transitive with respect to* $\operatorname{Mon}_t G$. *From* $\operatorname{Aut} G = \pm 1$ *follows immediately that G is not a T-group.*

Sketch of a proof. The proof is very similar to the proof given for $\operatorname{Aut} G$ but simpler because the U-property must go, see Observation 5.2. We must run through the paper once more, essentially replacing $\operatorname{Aut} G$ by $\operatorname{Mon}_t G$. First we will replace the definition $\operatorname{pAut} G$ by

$$\operatorname{pMon}_t G =^{\mathrm{df}} \{\varphi \mid \varphi\colon \operatorname{Dom}\varphi \to \operatorname{Im}\varphi \subseteq G, \varphi \text{ an isomorphism}, G/\operatorname{Dom}\varphi\ \aleph_1\text{-free}\}.$$

If $\mathfrak{F} \subseteq \operatorname{pMon}_t G$, then $\langle\mathfrak{F}\rangle$ is the submonoid of $\operatorname{pMon}_t G$ which is generated as a monoid by $\{\pm 1\} \cup \mathfrak{F}$, hence $\langle\mathfrak{F}\rangle \subseteq \operatorname{pMon}_t G$ by Proposition 5.1. The Definition 3.3 of an $\aleph_1$-free pairs $(G, (\varphi_t)_{t\in u})$ remains the same, except that pAut and automorphisms must be replaced by pMon_t and monomorphisms, respectively. Moreover condition (iii) is weaker (automatically) because $\langle\mathfrak{F}\rangle$ is not closed under weak inverses.

The U-property must be replaced by a very weak condition assuring that finally a free monoid will act on the group as monomorphisms which are not automorphisms. This is incorporated in the new definition for $\mathfrak{K}$ (compare Definition 3.5). We say that a quadruple $\mathfrak{x} = (G, \mathfrak{F}, \pi, h)$ belongs to $\mathfrak{K}$ if and only if the following holds.

(i) G is an $\aleph_1$-free abelian group.

(ii) $\mathfrak{F} = \{\varphi_t \mid t \in J\}$ is a family of symbols φ_t indexed by $J = J^{\mathfrak{x}}$, which generates a free monoid $\langle\mathfrak{F}\rangle$.

(iii) $\pi\colon \mathfrak{F} \to \operatorname{pMon}_t G$ is a map which naturally extends to $\langle\mathfrak{F}\rangle$ and satisfies the weak U-property: If $\varphi, \varphi' \in \langle\mathfrak{F}\rangle$, $D = \operatorname{Dom}\pi(\varphi) \cap \operatorname{Dom}\pi(\varphi') \neq 0$ and $x\varphi = x\varphi'$ for all $x \in \mathfrak{p}G \cap D$ then $\varphi = \varphi'$.

(iv) $h\colon \mathfrak{P}_{\aleph_0}(J) \to \operatorname{Im}(h)$ is a partial function from $\operatorname{Dom} h \subseteq \mathfrak{P}_{\aleph_0}(J)$ such that for $u \in \operatorname{Dom} h \subseteq G$ the following holds.

  (a) $h(u)$ is a countable subgroup of G and $(\operatorname{Dom}\varphi_t^{\mathfrak{x}} \cap h(u))\varphi_t^{\mathfrak{x}} \subseteq h(u)$ for all $t \in u$

  (b) If $\overline{G} = G/h(u)$ and $\overline{\varphi}_t^{\mathfrak{x}}$ denote the monomorphism induced by $\varphi_t^{\mathfrak{x}}$, then $(\overline{G}, (\overline{\varphi}_t^{\mathfrak{x}})_{t\in u})$ is $\aleph_1$-free.

The definition of an order on $\mathfrak{K}$ remains the same, except that condition $(iii)(d)$ must be removed. Following the arguments along Section 3 we note that Lemma 3.11 and a simple version of Lemma 3.12 are crucial for the weak U-property. We obtain a theorem parallel to Theorem 3.14: For each $\mathfrak{x} = (G, \mathfrak{F}, \pi, h) \in \mathfrak{K}$ we can find $\mathfrak{x}' = (G', \mathfrak{F}', \pi', h') \in \mathfrak{K}$ with $\mathfrak{x} \leq \mathfrak{x}'$ such that $\mathfrak{F}' \subset \operatorname{pMon}_t G'$ which freely

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generates $\langle\mathfrak{F}'\rangle$ as a submonoid, $\langle\mathfrak{F}'\rangle$ acts transitive on $\mathfrak{p}G'$, all members of $\langle\mathfrak{F}\rangle\setminus\{1\}$ are proper monomorphisms and $|G'|+|\mathfrak{F}'|=|G|+|\mathfrak{F}|$. As in Section 4 we modify G' and get a new G with endomorphism ring $R=\mathbb{Z}\langle\mathfrak{F}\rangle$. Recall that $\langle\mathfrak{F}\rangle$ is a free monoid generated by $\mathfrak{F}$, hence $U(R)=\pm 1$. Modification of Proposition 3.9 will show that any $\varphi\in R\setminus U(R)$ is not in $\mathrm{Mon}_t(G)$; thus $\mathrm{Mon}_t(G)=\langle\mathfrak{F}\rangle$ and $\mathrm{Aut}\, G=U(R)=\pm 1$. The group G cannot be transitive. □

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