I [Sh 345a]

BASIC: COFINALITIES OF SMALL REDUCED PRODUCTS

§0 Introduction

The theme of this Chapter is that we should look not at the power 2^λ, but at small products, say $\prod_{i<\kappa}\lambda_i$ with $\kappa<\lambda_i$; moreover, look at the cofinality of such products (so we assume each λ_i regular). Now $\prod_{i<\kappa}\lambda_i$ is naturally only partially ordered, and though its cofinality is well defined (Definition 1.1(1)(e)) it is clearer to look at $\prod_{i<\kappa}\lambda_i$ divided by an ultrafilter (see Definition 1.2(1)), and look at the set of cofinalities thus gotten named $\mathrm{pcf}(\{\lambda_i : i<\kappa\})$ (as $\kappa<\lambda_i=\mathrm{cf}(\lambda_i)$, repetitions do not matter, so we use simply any such set $\mathfrak{a}$).

The pcf first appeared in [Sh68], in an attempt to build Jonsson algebras in for example $\aleph_{\omega+1}$; they were obtained provided that for some non-principal ultrafilter D on ω, $\prod_{n<\omega}\aleph_n/D$ has cofinality $\aleph_{\omega+1}$; i.e. when $\aleph_{\omega+1}\in\mathrm{pcf}(\{\aleph_n : n<\omega\})$. It appears lightly in [Sh71] and [Sh111] (as some reduced products being λ-like), but heavily in the proof of for example $\aleph_\omega^{\aleph_0}<\aleph_{[2^{\aleph_0}]^+}$ in [Sh-b,XIII,§5,§6]. Some of the theorems proved here were proved there under the additional assumption $\min\mathfrak{a}>2^{|\mathfrak{a}|}$. It seems to me the theorem has attracted some attention, but the pcf was generally thought of as an artifact of the proof and not a meaningful important notion per se; this book is trying to prove the opposite. We further investigate pcf in [Sh282], probably the most interesting result is in §8 there; it says that, for example if $\aleph_{\omega_1}$ is strong limit, $2^{\aleph_{\omega_1}}>\aleph_{\omega_2}$, then also for some $\delta<\omega_2$ of cofinality $\aleph_0$, $\aleph_\delta^{\aleph_0}$ is quite high. We return to this in [Sh345] because of a desire to represent many regular cardinals as the (true) cofinality of the reduced product by the simplest ideal: the ideal of the bounded subsets of κ. Our view changes, deciding it is important to, for example, investigate what occurs below the continuum or even below $\min\{\lambda : 2^\lambda>2^{\aleph_0}\}$, where conventional cardinal arithmetic sees nothing.

The first section shows we can find ideals $J_{<\lambda}[\mathfrak{a}]$ on $\mathfrak{a}$, increasing continuous with λ, $J_{<\lambda}[\mathfrak{a}]\neq J_{<\lambda^+}[\mathfrak{a}]$ iff $\lambda\in\mathrm{pcf}(\mathfrak{a})$, and for an ultrafilter D on $\mathfrak{a}$ we have $\lambda=\mathrm{cf}\left(\prod\mathfrak{a}/D\right)$ iff $D\cap J_{<\lambda^+}[\mathfrak{a}]\neq\emptyset$ and $D\cap J_{<\lambda}[\mathfrak{a}]=\emptyset$, $\mathrm{pcf}(\mathfrak{a})$ has a last element and $\mathrm{pcf}(\mathfrak{a})$ is of cardinality $\leq 2^{|\mathfrak{a}|}$. All this plays a large role in the rest of the book.

In the second section we investigate when the description of $J_{<\lambda^+}[\mathfrak{a}]$ over $J_{<\lambda}[\mathfrak{a}]$ is simple: $J_{<\lambda^+}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda[\mathfrak{a}]$ for some $\mathfrak{b}_\lambda[\mathfrak{a}] \subseteq \mathfrak{a}$. Note: $\prod \mathfrak{b}_\lambda[\mathfrak{a}]/J_{<\lambda}[\mathfrak{a}]$ has true cofinality λ; i.e. there is an increasing cofinal sequence of this length.

Lastly, in the third section we investigate mainly when

$$\langle \mathfrak{b}_\lambda[\mathfrak{a}] : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$$

can be chosen "nicely".

The ideal $I[\lambda]$ used in §2 was introduced in [Sh108], concentrating on λ successor of strong limit singular (as under GCH the ideal is non-trivial only for such λ 's, which earlier confused many, including the author). This is re-represented in [Sh88a]. By [Sh-e,Ch.III,§6], [Sh351,§4] represented here in [Sh365], if $\lambda = \mu^+$, μ regular then $\{\delta < \lambda : \mathrm{cf}(\delta) < \mu\} \in I[\lambda]$. See more in [Sh420,§1]. Lemma 2.6A is a representation of Lemma 14 of [Sh282]; which improves earlier versions from [Sh68], [Sh71], [Sh111], [Sh-b,Ch.XIII,§5]. See more in [Sh355,§0].

§1 The basic properties of pcf($\mathfrak{a}$)

This is a central section.

After giving the standard definition of cofinality of a partial order and reduced products (of ordinals) (in Def 1.1) we present one of the book's main Definitions (1.2): for a set $\mathfrak{a}$ of regular cardinals, $\mathrm{pcf}(\mathfrak{a})$ is the set of the cardinals θ which has the form: θ is the cofinality of $\prod \mathfrak{a}/D$, D an ultrafilter on $\mathfrak{a}$. (Essentially $\mathfrak{a}$ is always, after 1.4, a set of regular cardinals $\geq |\mathfrak{a}|$, so we shall not always remember to state this).

A priori if for example $2^{\aleph_0} = \aleph_1$, $\mathfrak{a} = \{\aleph_n : 1 < n < \omega\}$, any function from the family of non-principal ultrafilters to the set of regular cardinals $\kappa \in (\aleph_\omega, \aleph_\omega^{\aleph_0}]$ may be realized as $D \mapsto \mathrm{cf}(\prod \mathfrak{a}/D)$. But the truth is not so chaotic. There is a natural sequence of ideals $\langle J_{<\lambda}[\mathfrak{a}] : \lambda\rangle$ increasing with λ such that $\mathrm{cf}(\prod \mathfrak{a}/D) = \min\{\lambda : D \cap J_{<\lambda^+}[\mathfrak{a}] \neq \emptyset\}$; (those natural ideals will be central, too, in this book). In particular, the cardinality of $\mathrm{pcf}(\mathfrak{a})$ is not the number of ultrafilters on $\mathfrak{a}$, but at most $2^{|\mathfrak{a}|}$. This, with more detailed information is done in 1.5, 1.6, 1.8. A consequence of those Lemmas is (1.9): $\mathrm{pcf}(\mathfrak{a})$ necessarily has a maximal element, $\max \mathrm{pcf}(\mathfrak{a})$, which will continue to play major role.

Lastly, 1.10, 1.11 tell us how cases of $\mathrm{cf}(\prod \mathfrak{a}/D)$ are related when we take a reduced product of reduced products, from which we conclude: if $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$ then $\mathrm{pcf}(\mathfrak{b}) \subseteq \mathrm{pcf}(\mathfrak{a})$ (we here assume $|\mathfrak{b}| < \min \mathfrak{b}$, $|\mathfrak{a}| < \min \mathfrak{a}$ of course, but not necessarily $|\mathfrak{b}| < \min \mathfrak{a}$).

Notation 1.0 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ denote sets of regular cardinals. I, J denote ideals (usually on some $\mathfrak{a}$), D a filter.

For a set A of ordinals with no last element,

$$J^{\rm bd}_A = \{B \subseteq A : \sup B < \sup A\},$$

i.e. the ideal of bounded subsets.

* * *

Definition 1.1 (1) For a partial order P

(a) P is λ-directed if: for every $A \subseteq P$, $|A| < \lambda$ there is $q \in P$ such that $\bigwedge_{p\in A} p \leq q$, and we say: q is an upper bound of A;

(b) P has true cofinality λ <u>if</u> there is $\langle p_i : i < \lambda\rangle$ cofinal in P, i.e.:
$\bigwedge_{i<j} p_i < p_j$
$\forall q \in P[\bigvee_i q \leq p_i]$
[and one writes $\mathrm{tcf}(P) = \lambda$ for the minimal λ]
(if P is linearly ordered it always has a true cofinality).

(c) P is endless if $\forall p \in P\, \exists q \in P\ \ < q]$ (so if P is endless, in (a), (b), (d) we can replace $\leq$ by $<$).

(d) $A \subseteq P$ is a cover if: $\forall p \in P\, \exists q \in A\ \ \leq q]$

(e) $\mathrm{cf}(P) = \min\{|A| : A \subseteq P$ is a cover$\}$.

(f) We say that , in P, p is a lub (least upper bound) of $A \subseteq P$ if:
(α) p is an upper bound of A (see (a))
(β) if p' is an upper bound of A then $p \leq p'$.

(2) If D is a filter on S, α_s (for $s \in S$) are ordinals, $f, g \in \prod_{s\in S} \alpha_s$, then: $f/D < g/D$, $f <_D g$ and $f < g \bmod D$ all mean

$$\{s \in S : f(s) < g(s)\} \in D.$$

Similarly for $\leq$, and we do not distinguish between a filter and the dual ideal in such notions. So if J is an ideal on $\mathfrak{a}$ and $f,\ g \in \prod \mathfrak{a}$, then

$$f < g \bmod J \text{ iff } \{\theta \in \mathfrak{a} : \neg f(\theta) < g(\theta)\} \in J.$$

Similarly if we replace the α_s's by partial orders.

(3) For $f, g : S \rightarrow$Ordinals, $f < g$ means $\bigwedge\limits_{s\in S} f(s) < g(s)$; similarly $f \leq g$.

(4) If I is an ideal on κ, $F \subseteq {}^\kappa\mathrm{Ord}$, we call $g \in {}^\kappa\mathrm{Ord}$ an $\leq_I$ $-$eub (exact upper bound) of F, or g an eub of F for I, if:

(α) g is an $\leq_I$-upper bound of F (in ${}^\kappa$Ord)

(β) if $h \in {}^\kappa\mathrm{Ord}$, $h <_I \max\{g, 1\}$ then for some $f \in F$,

$$h < \max\{f, 1\} \bmod I.$$

(γ) if $A \subseteq \kappa$, $A \neq \emptyset \bmod I$ and

$$[f \in F \Rightarrow f\restriction A =_I O_A \text{ i.e. } \{i \in A : f(i) \neq \emptyset\} \in I]$$

<u>then</u> $g\restriction A =_I O_A$.

Remark: Note that $g' = \max\{g, 1\}$ means $g'(i) = \max\{g(i), 1\}$; if there is $f \in F$, $\{i < \kappa : f(i) = 0\} \in I$ we can use g in clause (β) and omit clause (γ)).

Definition 1.2 (1) For a property Γ of ultrafilters (if Γ is the empty condition, we omit it):

$$\mathrm{pcf}_\Gamma(\mathfrak{a}) = \left\{\mathrm{tcf}(\prod \mathfrak{a}/D) : D \text{ is an ultrafilter on } \mathfrak{a} \text{ satisfying } \Gamma\right\}$$

(so $\mathfrak{a}$ is a set of regular cardinals; note: as D is an ultrafilter, $\prod \mathfrak{a}/D$ is linearly ordered).

(1A) More generally, for a property Γ of ideals on $\mathfrak{a}$ we let $\mathrm{pcf}_\Gamma(\mathfrak{a}) = \{\mathrm{tcf}(\prod \mathfrak{a}/J) : J$ is an ideal on $\mathfrak{a}$ satisfying Γ such that $\prod \mathfrak{a}/J$ has true cofinality$\}$.

(2) $J_{<\lambda}[\mathfrak{a}] = \{\mathfrak{b} \subseteq \mathfrak{a}$: for no ultrafilter D on $\mathfrak{a}$ to which $\mathfrak{b}$ belongs, is $\mathrm{tcf}(\prod \mathfrak{a}/D) \geq \lambda\}$.

(3) $J_{\leq\lambda}[\mathfrak{a}] = J_{<\lambda^+}[\mathfrak{a}]$.

(4) $\mathrm{pcf}_I(\mathfrak{a}) = \{\mathrm{tcf}(\prod \mathfrak{a}/D) : D$ an ultrafilter on $\mathfrak{a}$ disjoint to $I\}$.

Claim 1.3 (0) $(\prod \mathfrak{a}, <_J)$, $(\prod \mathfrak{a}, \leq_J)$ are endless.

(1) $\min(\mathrm{pcf}(\mathfrak{a})) = \min \mathfrak{a}$.

(2) If $\mathfrak{a} \subseteq \mathfrak{b}$ <u>then</u> $\mathrm{pcf}(\mathfrak{a}) \subseteq \mathrm{pcf}(\mathfrak{b})$; and for any $\mathfrak{b}, \mathfrak{c}$ we have:

$$\mathrm{pcf}(\mathfrak{c} \cup \mathfrak{b}) = \mathrm{pcf}(\mathfrak{c}) \cup \mathrm{pcf}(\mathfrak{b}) \text{ and}$$

$$x \in J_{<\lambda}[\mathfrak{b} \cup \mathfrak{c}] \Leftrightarrow x \subseteq \mathfrak{c} \cup \mathfrak{b} \;\&\; x \cap \mathfrak{c} \in J_{<\lambda}[\mathfrak{c}] \;\&\; x \cap \mathfrak{b} \in J_{<\lambda}[\mathfrak{b}].$$

(3) (i) if $\mathfrak{b} \subseteq \mathfrak{a}$, $\mathfrak{b}$ finite, <u>then</u>

$$\mathrm{pcf}(\mathfrak{b}) = \mathfrak{b} \text{ and } \mathrm{pcf}(\mathfrak{a}) \backslash \mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a} \backslash \mathfrak{b}) \subseteq \mathrm{pcf}(\mathfrak{a})$$

(ii) in addition if $\mathfrak{b} \subseteq \{\theta \in \mathfrak{a} : |\theta \cap \mathfrak{a}| < \aleph_0\}$; for example $\mathfrak{b} = \{\min(\mathfrak{a})\}$ <u>then</u> $\mathrm{pcf}(\mathfrak{a} \backslash \mathfrak{b}) = \mathrm{pcf}(\mathfrak{a}) \backslash \mathfrak{b}$.

(4) (i) If $\mathfrak{a}$ has no last element, J an ideal on $\mathfrak{a}$ and $J^{\mathrm{bd}}_{\mathfrak{a}} \subseteq J$ (equivalently: $\theta \in \mathfrak{a} \Rightarrow \mathfrak{a} \cap \theta^+ \in J$) <u>then</u>

$$(\prod \mathfrak{a}, <_J) \text{ is } (\sup \mathfrak{a})\text{-directed.}$$

(ii) If in (i), $\sup \mathfrak{a}$ is a singular cardinal, or at least $\mathfrak{a}$ is not a stationary subset of $\sup \mathfrak{a}$ or for some $\mathfrak{b} \in J$, $\mathfrak{a} \backslash \mathfrak{b}$ is not stationary in $\sup \mathfrak{a}$ <u>then</u> $(\prod \mathfrak{a}, <_J)$ is $(\sup \mathfrak{a})^+$-directed.

(iii) If D is an ultrafilter on $\mathfrak{a}$ such that for every $\theta \in \mathfrak{a}$ we have $(\mathfrak{a}\backslash\theta^+) \in D$, then $\mathrm{cf}(\prod \mathfrak{a}/D) \geq \sup \mathfrak{a}$ (and if equality holds, then $\sup \mathfrak{a}$ is a weakly inaccessible cardinal, D a weakly normal ultrafilter).

(5) If $\mathfrak{a}$ has no last element, then:

$$\text{there is } \lambda \in \mathrm{pcf}(\mathfrak{a}) \text{ such that } \sup \mathfrak{a} < \lambda$$

[also see the third possibility in (4)(ii)].

(6) If D is a filter on a set S and for $s \in S$, α_s is a limit ordinal then:

(i) $\mathrm{cf}(\prod_{s\in S} \alpha_s, <_D) = \mathrm{cf}(\prod_{s\in S} \mathrm{cf}(\alpha_s), <_D) = \mathrm{cf}(\prod_{s\in S}(\alpha_s, <)/D)$, and

(ii) $\mathrm{tcf}(\prod_{s\in S} \alpha_s, <_D) = \mathrm{tcf}(\prod_{s\in S} \mathrm{cf}(\alpha_s), <_D)$

$$= \mathrm{tcf}(\prod_{s\in S}(\alpha_s, <)/D).$$

In particular, if one of them is well defined, then so are the others. This is true even if we replace α_s by linear orders, or even partial orders with true cofinality.

(7) If D is an ultrafilter on a set S, λ_s a regular cardinal (for $s \in S$), then

$$\theta =: \mathrm{tcf}(\prod_{s\in S} \lambda_s, <_D) \text{ is well defined and}$$

$$|S| < \min\{\lambda_s : s \in S\} \text{ implies } \theta \in \mathrm{pcf}(\{\lambda_s : s \in S\}).$$

(8) If D is a filter on a set S, for $s \in S$, λ_s is a regular cardinal $> |S|$, $\mathfrak{a} = \{\lambda_s : s \in S\}$ and $E =: \{\mathfrak{b} : \mathfrak{b} \subseteq \mathfrak{a} \text{ and } \{s : \lambda_s \in \mathfrak{b}\} \in D\}$ then:

(i) E is a filter on $\mathfrak{a}$, and if D is an ultrafilter on S then E is an ultrafilter on $\mathfrak{a}$.

(ii) $\mathfrak{a}$ is a set of regular cardinals $> |\mathfrak{a}|$,

(iii) $F = \{f \in \prod_{s\in S} \lambda_s : \lambda_s = \lambda_t \Rightarrow f(s) = f(t)\}$ is a cover of $\prod\limits_{s\in S} \lambda_s$,

(iv) $\mathrm{cf}\left(\prod_{s\in S} \lambda_s/D\right) = \mathrm{cf}(\prod \mathfrak{a}/E)$ and

$$\mathrm{tcf}\left(\prod_{s\in S} \lambda_s/D\right) = \mathrm{tcf}(\prod \mathfrak{a}/E).$$

(9) Assume I is an ideal on κ, $F \subseteq {}^\kappa\mathrm{Ord}$, $g \in {}^\kappa\mathrm{Ord}$. If g is an eub of F for I then g is a $\leq_I$ – lub of F.

Proof: They are all very easy, but as this is the first claim in the book, we shall prove some.

(0) We shall show $(\prod \mathfrak{a}, <_J)$ is endless (assuming, of course, that J is a

proper ideal on $\mathfrak{a}$, $\mathfrak{a}$ is a set of regular (hence infinite) cardinals). Let $f \in \prod \mathfrak{a}$, then $g =: f+1$ (defined $(f+1)(\theta) = f(\theta)+1$) is in $\prod \mathfrak{a}$ too, as each $\theta \in \mathfrak{a}$ being an infinite cardinal is a limit ordinal, and $f < g \bmod J$.
(1) For the given $\mathfrak{a}$, let $\theta^* = \min \mathfrak{a}$. $D = \{\mathfrak{b} \subseteq \mathfrak{a} : \theta^* \in \mathfrak{b}\}$ is an ultrafilter on $\mathfrak{a}$ (such ultrafilters are "degenerated", usually called non-principal, still they are ultrafilters). Now $\prod \mathfrak{a}/D$ is isomorphic to $(\theta^*, <)$ hence has cofinality θ^*, so $\theta^* \in \mathrm{pcf}(\mathfrak{a})$ hence $\theta^* \geq \min(\mathrm{pcf}(\mathfrak{a}))$. On the other hand, $\prod \mathfrak{a}$ is θ^*-directed: (if $F \subseteq \prod \mathfrak{a}$, $|F| < \theta^*$ define $g \in \prod \mathfrak{a}$ by $g(\theta) = \bigcup_{f\in F} f(\theta)+1$), hence for every ultrafilter D on $\mathfrak{a}$, $\prod \mathfrak{a}/D$ is θ^*-directed hence $\mathrm{tcf}(\prod \mathfrak{a}/D) \geq \theta^*$. As this holds for every such D, clearly $[\theta \in \mathrm{pcf}(\mathfrak{a}) \Rightarrow \theta \geq \theta^*]$ hence $\min(\mathrm{pcf}(\mathfrak{a})) \geq \theta^*$.
9) Let us prove g is a $\leq_I$-lub of F in $({}^\kappa\mathrm{Ord}, \leq_I)$. As we can deal separately with $I+A$, $I+(\kappa\backslash A)$, where $A = \{i : g(i) = 0\}$, and the later case is trivial, we can assume $A = \emptyset$. So, assume g is not a lub, so there is an upper bound g' of F, but not $g \leq_I g'$. Define $g'' \in {}^\kappa\mathrm{Ord}$:

$$g''(i) = \begin{cases} 0 & \text{if } g(i) \leq g'(i) \\ g'(i) & \text{if } g'(i) < g(i) \end{cases}.$$

Clearly $g'' <_I g$. So, as g is an eub of F for I, there is $f \in F$ such that $g'' <_I \max\{f,1\}$, but $B =: \{i : g'(i) < g(i)\} \neq \emptyset \bmod I$ (as "not $g \leq_I g'$") so $g'{\restriction}B = g''{\restriction}B <_I \max\{f,1\}{\restriction}B$. But as g' is an $\leq_I$-upper bound of F we have: $f \leq_I g'$, so $f{\restriction}B <_I \max\{f,1\}{\restriction}B$ hence $f{\restriction}B =_I O_B$ hence $g'{\restriction}B =_I O_B$; as g' is a $\leq_I$-upper bound of F we know $[f' \in F \Rightarrow f'{\restriction}B =_I O_B]$, hence by (γ) of Definition 1.1(4) $g{\restriction}B =_I O_B$, contradicting the assumption $A = \emptyset$.
So we have proved clause (β) of Definition 1.1(1)(f); the other clause, (α), says "g is a $\leq_I$-upper bound of F", is obvious by clause (α) of Definition 1.1(4). $\square_{1.3}$

Claim 1.4

(1) $J_{<\lambda}[\mathfrak{a}]$ is an ideal (of $\mathcal{P}(\mathfrak{a})$)
(2) if $\lambda \leq \mu$, then $J_{<\lambda}[\mathfrak{a}] \subseteq J_{<\mu}[\mathfrak{a}]$
(3) if λ is singular, $J_{<\lambda}[\mathfrak{a}] = J_{<\lambda^+}[\mathfrak{a}] = J_{\leq\lambda}[\mathfrak{a}]$
(4) if $\lambda \notin \mathrm{pcf}(\mathfrak{a})$, then $J_{<\lambda}[\mathfrak{a}] = J_{\leq\lambda}[\mathfrak{a}]$.

Lemma 1.5 *If* $\min(\mathfrak{a}) \geq |\mathfrak{a}|$, λ *a cardinal* $\geq |\mathfrak{a}|$, *then* $(\prod \mathfrak{a}, <_{J_{<\lambda}[\mathfrak{a}]})$ *is* λ*-directed.*

Proof: By 1.3(3)(ii), without loss of generality $|\mathfrak{a}|, |\mathfrak{a}|^+ \notin \mathfrak{a}$ so $\min(\mathfrak{a}) > |\mathfrak{a}|^+$, and without loss of generality $\lambda \geq \min(\mathfrak{a})$, so $\lambda > |\mathfrak{a}|^+$. Note: if $f \in \prod \mathfrak{a}$ then $f < f+1 \in \prod \mathfrak{a}$, (i.e. $(\prod \mathfrak{a}, <_{J_\lambda[\mathfrak{a}]})$ is endless, where $f+1$ is defined by $(f+1)(\theta) = f(\theta)+1$). Let $F \subseteq \prod \mathfrak{a}$, $|F| < \lambda$, and we shall prove that for some $g \in \prod \mathfrak{a}$, $(\forall f \in F)\big(f \le g \mod J_{<\lambda}[\mathfrak{a}]\big)$. The proof is by induction on $|F|$. If $|F|$ is finite, this is trivial. Also if $|F| < \min(\mathfrak{a})$ it is easy: let $g \in \prod \mathfrak{a}$ be $g(\theta) = \sup\{f(\theta) : f \in F\}$. So, assume $|F| = \mu$, $\min(\mathfrak{a}) \le \mu < \lambda$, so let $F = \{f^0_i : i < \mu\}$. By the induction hypothesis we can choose by induction on $i < \mu$, $f^1_i \in \prod \mathfrak{a}$ such that:

(a) $f^0_i \le f^1_i \mod J_{<\lambda}[\mathfrak{a}]$

(b) for $j < i$ we have $f^1_j \le f^1_i \mod J_{<\lambda}[\mathfrak{a}]$.

If μ is singular, there is $C \subseteq \mu$ unbounded, $|C| = \mathrm{cf}(\mu) < \mu$, and by the induction hypothesis there is $g \in \prod \mathfrak{a}$ such that for $i \in C$, $f^1_i \le g \mod J_{<\lambda}[\mathfrak{a}]$. Now g is as required: $f^0_i \le f^1_i \le f^1_{\min(C\setminus i)} \le g \mod J_{<\lambda}[\mathfrak{a}]$. So, without loss of generality, μ is regular. Now we try to define by induction on $\alpha < |\mathfrak{a}|^+$, $g_\alpha, i_\alpha = i(\alpha) < \mu$, $\langle \mathfrak{b}^\alpha_i : i < \mu\rangle$ such that:

(i) $g_\alpha \in \prod \mathfrak{a}$

(ii) for $\beta < \alpha$ we have $g_\beta \le g_\alpha$

(iii) for $i < \mu$ let $\mathfrak{b}^\alpha_i =: \{\theta \in \mathfrak{a} : f^1_i(\theta) > g_\alpha(\theta)\}$

(iv) for each α, for every $i \in [i_{\alpha+1}, \mu)$, $\mathfrak{b}^\alpha_i \ne \mathfrak{b}^{\alpha+1}_i$.

We cannot carry out this definition: if we let $i(*) = \sup\{i_\alpha : \alpha < |\mathfrak{a}|^+\}$, then $i(*) < \mu$, since $\mu = \mathrm{cf}(\mu)$, $\mu \ge \min\ \mathfrak{a} > |\mathfrak{a}|^+$. We know that $\mathfrak{b}^\alpha_{i(*)} \ne \mathfrak{b}^{\alpha+1}_{i(*)}$ for $\alpha < |\mathfrak{a}|^+$ (by (iv)) and $\mathfrak{b}^\alpha_{i(*)} \subseteq \mathfrak{a}$ (by (iii)) and $[\alpha < \beta \Rightarrow \mathfrak{b}^\beta_{i(*)} \subseteq \mathfrak{b}^\alpha_{i(*)}]$ (by (ii)), together a contradiction.
Now, for $\alpha = 0$ let g_α be f^1_0 and $i_\alpha = 0$.
For α limit let $g_\alpha(\theta) = \bigcup\limits_{\beta<\alpha} g_\beta(\theta)$ (note: $g_\alpha \in \prod \mathfrak{a}$ as $\alpha < |\mathfrak{a}|^+ < \min\ \mathfrak{a}$ and $\mathfrak{a}$ is a set of regular cardinals) and let $i_\alpha = 0$.
For $\alpha = \beta+1$, suppose that g_β hence $\langle \mathfrak{b}^\beta_i : i < \mu\rangle$ are defined. If $\mathfrak{b}^\beta_i \in J_{<\lambda}[\mathfrak{a}]$ for unboundedly many $i < \mu$ then g_β is an upper bound for F and the proof is complete. So assume this fails, then there is a minimal $i(\alpha) < \mu$ such that $\mathfrak{b}^\beta_{i(\alpha)} \notin J_{<\lambda}[\mathfrak{a}]$. As $\mathfrak{b}^\beta_{i(\alpha)} \notin J_{<\lambda}[\mathfrak{a}]$, by Definition 1.2(2) for some ultrafilter D on $\mathfrak{a}$, $\mathfrak{b}^\beta_{i(\alpha)} \in D$ and $\mathrm{cf}(\prod \mathfrak{a}/D) \ge \lambda$. Hence $\{f^1_i/D : i < \mu\}$ has a bound h_α/D where $h_\alpha \in \prod \mathfrak{a}$. Let us define $g_\alpha \in \prod \mathfrak{a}$:

$$g_\alpha(\theta) = \max\{g_\beta(\theta), h_\alpha(\theta)\}.$$

Now (i), (ii) hold trivially and $\mathfrak{b}^\alpha_i$ is defined by (iii). Why does (iv) hold with $i_\alpha := i(\alpha)$? Suppose $i(\alpha) \le i < \mu$. As $f^1_{i(\alpha)} \le f^1_i \mod J_{<\lambda}[\mathfrak{a}]$ clearly $\mathfrak{b}^\beta_{i(\alpha)} \subseteq \mathfrak{b}^\beta_i \mod J_{<\lambda}[\mathfrak{a}]$. Moreover $J_{<\lambda}[\mathfrak{a}]$ is disjoint from D (by its choice) so $\mathfrak{b}^\beta_{i(\alpha)} \in D$ implies $\mathfrak{b}^\beta_i \in D$.
On the other hand, $\mathfrak{b}^\alpha_i$ is $\{\theta \in \mathfrak{a} : f^1_i(\theta) > g_\alpha(\theta)\}$ which is equal to

$$\{\theta \in \mathfrak{a} : f^1_i(\theta) > g_\beta(\theta), h_\alpha(\theta)\}$$

which does not belong to D (h_α was chosen such that $f^1_i \leq h_\alpha \bmod D$). We can conclude $\mathfrak{b}^\alpha_i \notin D$, whereas $\mathfrak{b}^\beta_i \in D$; so they are distinct.

Now we have said that we cannot carry out the definition for all $\alpha < |\mathfrak{a}|^+$, so we are stuck at some α; by the above, α is a successor, say $\alpha = \beta+1$, and g_β is as required: it bounds F.

$\square_{1.5}$

Lemma 1.6 *If* $\min \mathfrak{a} \geq |\mathfrak{a}|$, *$D$ is an ultrafilter on $\mathfrak{a}$ and*

$$\lambda = \mathrm{tcf}(\prod \mathfrak{a}, <_D),$$

then

for some $\mathfrak{b} \in D$, $(\prod \mathfrak{b}, <_{J_{<\lambda}[\mathfrak{a}]})$ has true cofinality λ.

(So $\mathfrak{b} \in J_{\leq\lambda}[\mathfrak{a}] \backslash J_{<\lambda}[\mathfrak{a}]$).

Proof: Again without loss of generality $\min \mathfrak{a} > |\mathfrak{a}|^+$; and we know $\lambda \geq \min \mathfrak{a}$.
Let $\langle f_i/D : i < \lambda\rangle$ be increasing unbounded in $\prod \mathfrak{a}/D$ (so $f_i \in \prod \mathfrak{a}$). By 1.5, without loss of generality $(\forall j < i)\,(f_j < f_i \bmod J_{<\lambda}[\mathfrak{a}])$.
Now 1.6 follows from 1.7 below: its hypothesis clearly holds. If $\bigwedge_{i<\lambda} \mathfrak{b}_i = \emptyset \bmod D$, (see (A) of 1.7) then (see (D) of 1.7) $J \cap D = \emptyset$ hence (see (D) of 1.7) g/D contradict the choice of $\langle f_i/D : i < \lambda\rangle$. So for some $i < \lambda$, $\mathfrak{b}_i \in D$; by (C) of 1.7 we get the desired conclusion.

$\square_{1.6}$

Lemma 1.7 *Suppose* $|\mathfrak{a}| < \min(\mathfrak{a})$, $\lambda > |\mathfrak{a}|^+$, $f_i \in \prod \mathfrak{a}$, $f_i < f_j \bmod J_{<\lambda}[\mathfrak{a}]$ *for $i < j < \lambda$, and there is no $g \in \prod \mathfrak{a}$ such that for every $i < \lambda$, $f_i < g \bmod J_{<\lambda}[\mathfrak{a}]$.*
Then there are $\mathfrak{b}_i$ $(i < \lambda)$ such that:

(A) $\mathfrak{b}_i \subseteq \mathfrak{a}$ *and for some* $i(*) < \lambda : i(*) \leq i < \lambda \Rightarrow \mathfrak{b}_i \notin J_{<\lambda}[\mathfrak{a}]$

(B) $i < j \Rightarrow \mathfrak{b}_i \subseteq \mathfrak{b}_j \bmod J_{<\lambda}[\mathfrak{a}]$ *(i.e.* $\mathfrak{b}_i \backslash \mathfrak{b}_j \in J_{<\lambda}[\mathfrak{a}]$*)*

(C) for each i, $\langle f_j {\restriction} \mathfrak{b}_i : j < \lambda\rangle$ is cofinal in $(\prod \mathfrak{b}_i, <_{J_{<\lambda}[\mathfrak{a}]})$ (better restrict yourselves to $i \geq i(*)$, see(A)); so necessarily $\mathfrak{b}_i \in J_{\leq\lambda}[\mathfrak{a}]$.

(D) for some $g \in \prod \mathfrak{a}$, $\bigwedge\limits_{i<\lambda} f_i \leq g \bmod J$ where $J = J_{<\lambda}[\mathfrak{a}] + \{\mathfrak{b}_i : i < \lambda\}$; in fact

(D)$^+$ for every $i < \lambda$, we have $f_i \leq g \bmod (J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_i)$

(E) if $g \leq g' \in \prod \mathfrak{a}$, then for arbitrarily large $i < \lambda$:

$$\bigwedge_{\theta\in\mathfrak{a}} [g(\theta) \geq f_i(\theta) \Leftrightarrow g'(\theta) \geq f_i(\theta)],$$

*(F) if δ is a limit ordinal $< \lambda$, f_δ is a $\leq_{J_{<\lambda}[\mathfrak{a}]}-$*lub *of $\{f_\alpha : \alpha < \delta\}$ then*

$$\mathfrak{b}_\delta \text{ is a lub of } \{\mathfrak{b}_\alpha : \alpha < \delta\} \text{ in } \mathcal{P}(\mathfrak{a})/J_{<\lambda}[\mathfrak{a}].$$

Proof of 1.7: Assume the lemma fails. We now define by induction on $\alpha < |\mathfrak{a}|^+$, g_α, $i(\alpha) < \lambda$, $\langle \mathfrak{b}_i^\alpha : i < \lambda\rangle$ such that:
(i) $g_\alpha \in \prod \mathfrak{a}$
(ii) for $\beta < \alpha$, $g_\beta \leq g_\alpha$
(iii) $\mathfrak{b}_i^\alpha =: \{\theta \in \mathfrak{a} : f_i(\theta) > g_\alpha(\theta)\}$
(iv) if $i(\alpha) \leq i < \lambda$ then $\mathfrak{b}_i^\alpha \neq \mathfrak{b}_i^{\alpha+1}$
For $\alpha = 0$ let $g_\alpha = f_0$.
For α limit let $g_\alpha(\theta) = \bigcup_{\beta<\alpha} g_\beta(\theta)$ (now $[\beta < \alpha \Rightarrow g_\beta \leq g_\alpha]$ holds trivially and $g_\alpha \in \prod \mathfrak{a}$ as min $\mathfrak{a} \geq |\mathfrak{a}|^+ > \alpha$).
For $\alpha = \beta + 1$, if $\{i < \lambda : \mathfrak{b}_i^\beta \in J_{<\lambda}[\mathfrak{a}]\}$ is unbounded in λ, then g_β is a bound for $\langle f_i : i < \lambda\rangle$ mod $J_{<\lambda}[\mathfrak{a}]$ contradicting an assumption. Clearly, $i < j < \lambda \Rightarrow \mathfrak{b}_i^\beta \subseteq \mathfrak{b}_j^\beta$ mod $J_{<\lambda}[\mathfrak{a}]$, hence $\{i < \lambda : \mathfrak{b}_i^\beta \in J_{<\lambda}[\mathfrak{a}]\}$ is an initial segment, so by the previous sentence there is $i(\alpha) < \lambda$ such that $\forall i \in [i(\alpha), \lambda)$, $\mathfrak{b}_i^\beta \notin J_{<\lambda}[\mathfrak{a}]$. If $\langle \mathfrak{b}_i^\beta : i < \lambda\rangle$ satisfies the desired conclusion, with $i(\alpha)$ for $i(*)$ in (A) and g_β for g in (D), (D)$^+$ and (E), we are done. Now, among the conditions in the conclusion of 1.7, (A) holds by the definition of $\mathfrak{b}_i^\beta$ and of $i(\alpha)$, (B) holds by $\mathfrak{b}_i^\beta$'s definition as $[i < j \Rightarrow f_i < f_j$ mod $J_{<\lambda}[\mathfrak{a}]]$, (D)$^+$ holds with $g = g_\beta$ by the choice of $\mathfrak{b}_i^\beta$. Lastly, if (E) fails, say for g', then it can serve as g_α. Now condition (F) follows immediately from (iii) (if (F) fails for δ, there is $\mathfrak{e} \subseteq \mathfrak{b}_\delta^\beta$, $\bigwedge_{i<\delta} \mathfrak{b}_i^\beta \subseteq \mathfrak{e} \bmod J_{<\lambda}[\mathfrak{a}]$ and $\mathfrak{b}_\delta^\beta \setminus \mathfrak{e} \notin J_{<\lambda}[\mathfrak{a}]$; now $(g{\restriction}(\mathfrak{a}\setminus\mathfrak{e})) \cup (f_\delta{\restriction}\mathfrak{e})$ contradicts "f_δ is a $\leq_{J_\lambda[\mathfrak{a}]}-$lub of $\{f_i : i < \delta\}$"). So only (C) (of 1.7) may fail, without loss of generality for $i = i(\alpha)$. I.e. $\langle f_j{\restriction}\mathfrak{b}_{i(\alpha)}^\beta : j < \lambda\rangle$ is not cofinal in $\left(\prod \mathfrak{b}_{i(\alpha)}^\beta, <_{J_{<\lambda}[\mathfrak{a}]}\right)$. As this sequence of functions is increasing *w.r.t.* $<_{J_{<\lambda}[\mathfrak{a}]}$, there is $h_\alpha \in \prod \mathfrak{b}_{i(\alpha)}^\beta$ such that for no $j < \lambda$, $h_\alpha \leq f_j{\restriction}\mathfrak{b}_{i(\alpha)}^\beta$ mod $J_{<\lambda}[\mathfrak{a}]$. Let $h'_\alpha = h_\alpha \cup 0_{(\mathfrak{a}\setminus\mathfrak{b}_{i(\alpha)}^\beta)}$, and $g_\alpha \in \prod \mathfrak{a}$ be defined by $g_\alpha(\theta) = \max\{g_\beta(\theta), h'_\alpha(\theta)\}$. Now define $\mathfrak{b}_i^\alpha$ by (iii), so (i), (ii), (iii) hold trivially, and we can check (iv). So we can define g_α, $i(\alpha)$ for $\alpha < |\mathfrak{a}|^+$, satisfying (i)-(iv). As in the proof of 1.5, this is impossible; so the lemma cannot fail. $\square_{1.7}$

Lemma 1.8 *Suppose* $|\mathfrak{a}| < \min(\mathfrak{a})$.
(1) For every $\mathfrak{b} \in J_{\leq\lambda}[\mathfrak{a}]\setminus J_{<\lambda}[\mathfrak{a}]$, we have:

$$\left(\prod \mathfrak{b}, <_{J_{<\lambda}[\mathfrak{a}]}\right) \text{ has true cofinality } \lambda$$

(hence λ is regular).

(2) If $0 < \alpha < \lambda$ and for $\beta < \alpha$, $\mathfrak{c}_\beta \in J_{\le\lambda}[\mathfrak{a}] \backslash J_{<\lambda}[\mathfrak{a}]$ then

$$(\exists \mathfrak{c} \in J_{\le\lambda}[\mathfrak{a}] \backslash J_{<\lambda}[\mathfrak{a}])(\forall \beta < \alpha)(\mathfrak{c}_\beta \subseteq \mathfrak{c} \bmod J_{<\lambda}[\mathfrak{a}]).$$

(3) If D is an ultrafilter on $\mathfrak{a}$, then $\mathrm{cf}(\prod \mathfrak{a}/D)$ is $\min\{\lambda : D \cap J_{\le\lambda}[\mathfrak{a}] \neq \emptyset\}$.

(4) (i) For λ limit $J_{<\lambda}[\mathfrak{a}] = \bigcup_{\mu<\lambda} J_{<\mu}[\mathfrak{a}]$, hence

(ii) for every λ, $J_{<\lambda}[\mathfrak{a}] = \bigcup_{\mu<\lambda} J_{\le\mu}[\mathfrak{a}]$.

(5) $|\mathrm{pcf}(\mathfrak{a})| \le 2^{|\mathfrak{a}|}$ and $[\lambda \in \mathrm{pcf}(\mathfrak{a}) \Leftrightarrow J_{<\lambda}[\mathfrak{a}] \neq J_{\le\lambda}[\mathfrak{a}]]$

Proof: (1) Let $J = \{\mathfrak{b} \subseteq \mathfrak{a} : \mathfrak{b} \in J_{<\lambda}[\mathfrak{a}]$ or $\mathfrak{b} \in J_{\le\lambda}[\mathfrak{a}] \backslash J_{<\lambda}[\mathfrak{a}]$ and $(\prod \mathfrak{b}, <_{J_{<\lambda}[\mathfrak{a}]})$ has true cofinality $\lambda\}$.

Clearly $J \subseteq J_{\le\lambda}[\mathfrak{a}]$; it is quite easy to check it is an ideal. Assume $J \neq J_{\le\lambda}[\mathfrak{a}]$ and we shall get a contradiction. Choose $\mathfrak{b} \in J_{\le\lambda}[\mathfrak{a}] \backslash J$; as J is an ideal, there is an ultrafilter D on $\mathfrak{a}$ such that: $D \cap J = \emptyset$ and $\mathfrak{b} \in D$. Now if $\mathrm{cf}(\prod \mathfrak{a}/D) \ge \lambda^+$, then $\mathfrak{b} \notin J_{\le\lambda}[\mathfrak{a}]$ (by the definition of $J_{\le\lambda}[\mathfrak{a}]$); contradiction. On the other hand, if $F \subseteq \prod \mathfrak{a}$, $|F| < \lambda$ there is $g \in \prod \mathfrak{a}$ such that $(\forall f \in F)(f < g \bmod J_{<\lambda}[\mathfrak{a}])$ so $(\forall f \in F)[f < g \bmod D]$ (as $J_{<\lambda}[\mathfrak{a}] \subseteq J$, $D \cap J = \emptyset$), and this says $\mathrm{cf}(\prod \mathfrak{a}/D) \ge \lambda$. By the last two sentences we know that $\mathrm{tcf}(\prod \mathfrak{a}/D)$ is λ. Now by 1.6 for some $\mathfrak{c} \in D$, $(\prod \mathfrak{c}, <_{J_{<\lambda}[\mathfrak{a}]})$ has true cofinality λ. Clearly, if $\mathfrak{c}' \subseteq \mathfrak{c}$, $\mathfrak{c}' \notin J_{<\lambda}[\mathfrak{a}]$ then also $(\prod \mathfrak{c}', <_{J_{<\lambda}[\mathfrak{a}]})$ has true cofinality λ, hence without loss of generality $\mathfrak{c} \subseteq \mathfrak{b}$; hence $\mathfrak{c} \in J_{\le\lambda}[\mathfrak{a}]$, hence, by the definition of J, we have $\mathfrak{c} \in J$. But this contradicts the choice of D as disjoint from J.

We have to conclude that $J = J_{\le\lambda}[\mathfrak{a}]$ so we have proved 1.8(1).

(2) For each $\beta < \alpha$ let $\langle f_j^\beta : j < \lambda\rangle$ exemplify that $(\prod \mathfrak{a}, <_{J_{<\lambda}[\mathfrak{a}]+(\mathfrak{a}\backslash\mathfrak{c}_\beta)})$ has true cofinality λ; so $f_j^\beta \in \prod \mathfrak{a}$ and

$$\left[j(1) < j(2) < \lambda \Rightarrow f_{j(1)}^\beta < f_{j(2)}^\beta \bmod (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a}\backslash\mathfrak{c}_\beta))\right]$$

and

$$(\forall g \in \prod \mathfrak{a})(\exists j < \lambda)\left[g < f_j^\beta \bmod (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a}\backslash\mathfrak{c}_\beta))\right].$$

By 1.5 we can define $f_j^* \in \prod \mathfrak{a}$ by induction on $j < \lambda$ such that:

(i) for $i < j$, $f_i^* < f_j^* \bmod J_{<\lambda}[\mathfrak{a}]$

(ii) for each $\beta < \alpha$ $f_j^\beta \le f_j^* \bmod J_{<\lambda}[\mathfrak{a}]$.

Let $\langle \mathfrak{b}_i : i < \lambda\rangle$ be as guaranteed by 1.7 (for $\langle f_j^* : j < \lambda\rangle$). Clearly for each $\beta < \alpha$, $\langle f_j^* : j < \lambda\rangle$ is $<_{J_{<\lambda}[\mathfrak{a}]+(\mathfrak{a}\backslash\mathfrak{c}_\beta)}$-increasing and cofinal. So for each $\beta < \alpha$ for some $i(\beta) < \lambda$

$$\mathfrak{c}_\beta \subseteq \mathfrak{b}_{i(\beta)} \bmod J_{<\lambda}[\mathfrak{a}].$$

[Why? If there is $\beta < \alpha$ such that $\neg\left(\bigvee_{i<\lambda} \mathfrak{c}_\beta \subseteq \mathfrak{b}_i \bmod J_{<\lambda}[\mathfrak{a}]\right)$, then (as $i < j < \lambda \Rightarrow \mathfrak{b}_i \subseteq \mathfrak{b}_j \bmod J_{<\lambda}[\mathfrak{a}]$) we have $\mathfrak{c}_\beta \notin J$, where J comes from

1.7(D). Choose now an ultrafilter D on $\mathfrak{a}$ such that $\mathfrak{c}_\beta \in D$ & $D \cap J = \emptyset$. Applying 1.7(D) yields a function $g \in \prod \mathfrak{a}$ such that $\bigwedge_{j<\lambda} f_j^* < g \bmod J$, so $\bigwedge_{j<\lambda} f_j^* < g \bmod D$. On the other hand, (by the choice of $\langle f_j^\beta : j < \lambda \rangle$) for some $j_0 < \lambda$, $g < f_{j_0}^\beta \le f_{j_0}^* \bmod J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \setminus \mathfrak{c}_\beta)$, so $g < f_{j_0}^* \bmod D$ (since $D \cap J_{<\lambda}[\mathfrak{a}] \subseteq D \cap J = \emptyset$ and $\mathfrak{c}_\beta \in D$)-a contradiction to the choice of g].

Let $i(*) = \sup_{\beta<\alpha} i(\beta)$. Now $i(*) < \lambda$ (as $\lambda = \mathrm{cf}(\lambda) > |\alpha|$) and $\mathfrak{c}_\beta \subseteq \mathfrak{b}_{i(*)} \bmod J_{<\lambda}[\mathfrak{a}]$ for each $\beta < \alpha$ (because $i_1 < i_2 \Rightarrow \mathfrak{b}_{i_1} \subseteq \mathfrak{b}_{i_2} \bmod J_{<\lambda}[\mathfrak{a}]$) and $\mathfrak{b}_{i(*)} \in J_{<\lambda^+}[\mathfrak{a}]$ (by the choice of $\langle \mathfrak{b}_i : i < \lambda \rangle$ in 1.7).

(3) Let $\lambda \in \mathrm{pcf}(\mathfrak{a})$ be minimal such that $D \cap J_{\le\lambda}[\mathfrak{a}] \ne \emptyset$ and choose $\mathfrak{b} \in D \cap J_{\le\lambda}[\mathfrak{a}]$. So $[\mu < \lambda \Rightarrow \mathfrak{b} \notin J_{\le\mu}[\mathfrak{a}]]$ (by the choice of λ) hence by 1.8(4)(ii) below, we have $\mathfrak{b} \notin J_{<\lambda}[\mathfrak{a}]$.
Similarly, by the choice of λ, $D \cap J_{<\lambda}[\mathfrak{a}] = \emptyset$. Now $\left(\prod \mathfrak{a}, <_{J_{<\lambda}[\mathfrak{a}]+(\mathfrak{a}\setminus\mathfrak{b})}\right)$ has true cofinality λ by 1.8(1). As $\mathfrak{b} \in D$ and $J_{<\lambda}[\mathfrak{a}] \cap D = \emptyset$, we have finished the proof.

(4) (i) Let $J =: \bigcup_{\mu<\lambda} J_{<\mu}[\mathfrak{a}]$. Now J is an ideal and $(\prod \mathfrak{a}, <_J)$ is λ-directed; i.e. if $\alpha^* < \lambda$ and $\{f_\alpha : \alpha < \alpha^*\} \subseteq \prod \mathfrak{a}$, then there exists $f \in \prod \mathfrak{a}$ such that

$$(\forall \alpha < \alpha^*)(f_\alpha < f \bmod J).$$

[Why? λ is a limit cardinal, hence there is μ^* such that $\alpha^* < \mu^* < \lambda$. By 1.5, there is $f \in \prod \mathfrak{a}$ such that $(\forall \alpha < \alpha^*)\left(f_\alpha < f \bmod J_{<\mu^*}[\mathfrak{a}]\right)$. Since $J_{<\mu^*}[\mathfrak{a}] \subseteq J$, it is immediate that

$$(\forall \alpha < \alpha^*)(f_\alpha < f \bmod J).]$$

Clearly $J = \bigcup\limits_{\mu<\lambda} J_{<\mu}[\mathfrak{a}] \subseteq J_{<\lambda}[\mathfrak{a}]$ by 1.4(2). On the other hand, let us suppose that there is $\mathfrak{b} \in \left(J_{<\lambda}[\mathfrak{a}] \setminus \bigcup\limits_{\mu<\lambda} J_{<\mu}[\mathfrak{a}]\right)$. Choose an ultrafilter D on $\mathfrak{a}$ such that $\mathfrak{b} \in D$ and $D \cap J = \emptyset$. Since $(\prod \mathfrak{a}, <_J)$ is λ-directed and $D \cap J = \emptyset$, one has $\mathrm{tcf}(\prod \mathfrak{a}/D) \ge \lambda$, but $\mathfrak{b} \in D \cap J_{<\lambda}[\mathfrak{a}]$, in contradiction to Definition 1.2(2).

(4)(ii) If λ limit — by part (i) and 1.4(2); if λ successor — by 1.4(2) and Definition 1.2(3).

(5) The second phrase is easy by 1.8(3) (and 1.4(4)). The first phrase follows as by the second phrase and 1.4(2) we know that: $\langle J_{<\lambda}(\mathfrak{a}) : \lambda \in \mathrm{pcf}(\mathfrak{a}) \rangle$ is a strictly increasing sequence of subsets of $\mathcal{P}(\mathfrak{a})$. $\square_{1.8}$

Conclusion 1.9 If $|\mathfrak{a}| < \min \mathfrak{a}$, then $\mathrm{pcf}(\mathfrak{a})$ has a last element.

Proof: This is the minimal λ such that $\mathfrak{a} \in J_{\le\lambda}[\mathfrak{a}]$.
[λ exists, since $\kappa =: |\prod \mathfrak{a}| \in \{\lambda : \mathfrak{a} \in J_{\le\lambda}[\mathfrak{a}]\} \ne \emptyset$]. $\square_{1.9}$

Claim 1.10 Suppose $\kappa < \min(\mathfrak{a})$, for $i < \kappa$, D_i is a filter on $\mathfrak{a}$, E a filter on κ and $D^* = \{\mathfrak{b} \subseteq \mathfrak{a} : \{i < \kappa : \mathfrak{b} \in D_i\} \in E\}$ (a filter on $\mathfrak{a}$). Let $\lambda_i =: \mathrm{tcf}(\prod \mathfrak{a}, <_{D_i})$ be well defined. Let

$$\lambda^* = \mathrm{tcf}(\prod \mathfrak{a}, <_{D^*}),\ \mu = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_E).$$

Then $\lambda^* = \mu$ (in particular, if one is well defined, then so is the other).

Proof: Let $\langle f^i_\alpha : \alpha < \lambda_i \rangle$ be an increasing cofinal sequence in $(\prod \mathfrak{a}, <_{D_i})$.

Define for $g \in \prod_{i<\kappa} \lambda_i$, $F(g) \in \prod \mathfrak{a}$ by:

$$F(g)(\theta) = \sup\{f^i_\beta(\theta) : i < \kappa, \beta = g(i)\} < \theta$$

(as $\kappa < \min \mathfrak{a}$).

Now for each $f \in \prod \mathfrak{a}$, define $G(f) \in \prod_{i<\kappa} \lambda_i$ by

$$G(f)(i) = \min\{\gamma < \lambda_i : f \le f^i_\gamma \bmod D_i\}$$

(it is well defined for $f \in \prod \mathfrak{a}$ by the choice of $\langle f^i_\gamma : \gamma < \lambda_i \rangle$).
Note that for $f^1, f^2 \in \prod \mathfrak{a}$:
$f^1 \le f^2 \bmod D^* \Rightarrow B(f^1, f^2) =: \{\theta \in \mathfrak{a} : f^1(\theta) \le f^2(\theta)\} \in D^*$
$\Rightarrow A(f^1, f^2) =: \{i < \kappa : B(f^1, f^2) \in D_i\} \in E$
$\Rightarrow \bigwedge_{i \in A(f^1,f^2)} G(f^1)(i) \le G(f^2)(i)$ where $A(f^1, f^2) \in E$
$\Rightarrow G(f^1) \le G(f^2) \bmod E$.
So G is a homomorphism from $(\prod \mathfrak{a}, \le_{D^*})$ into $(\prod_{i<\kappa} \lambda_i, \le_E)$. The range of G is a cover of $(\prod_{i<\kappa} \lambda_i, \le_E)$:

$$\text{if } g \in \prod_{i<\kappa} \lambda_i \text{ then } f^i_{g(i)} \le F(g)$$

(for every $i < \kappa$, see the definition of F) hence $g(i) \le [G(F(g))](i)$, hence $g \le G(F(g))$.
This finishes the proof. $\square_{1.10}$

Claim 1.11 In 1.10 if $|\mathfrak{a}|^+ < \min \mathfrak{a}$, we can weaken the hypothesis $\kappa < \min \mathfrak{a}$ to $\kappa < \min\{\lambda_i : i < \kappa\}$.

Proof: Similar to the proof of 1.10.

We define $G : \prod \mathfrak{a} \to \prod_{i<\kappa} \lambda_i$ exactly as previously and also the proof of $[f^1 \le f^2 \bmod D^* \Rightarrow G(f^1) \le G(f^2) \bmod E]$ does not change.

It is enough to prove that

$$\text{for } g \in \prod_{i<\kappa} \lambda_i, \text{ for some } f \in \prod \mathfrak{a}, g \le G(f) \bmod E.$$

By 1.5 $(\prod \mathfrak{a}, <_{J_{\le\kappa}[\mathfrak{a}]})$ is κ^+-directed, hence for some $f \in \prod \mathfrak{a}$:
$(*)_1$ for $i < \kappa$, $f^i_{g(i)} < f \bmod J_{\le\kappa}[\mathfrak{a}]$.
We assume $\kappa < \lambda_i$ hence $J_{\le\kappa}[\mathfrak{a}] \subseteq J_{<\lambda_i}[\mathfrak{a}]$ which is disjoint from D_i (use 1.8(3)), so together with $(*)_1$:
$(*)_2$ for $i < \kappa$, $f^i_{g(i)} < f \bmod D_i$.
So clearly $g < G(f)$ (more than required). $\square_{1.11}$

Conclusion 1.12 If $|\mathfrak{a}| < \min \mathfrak{a}$, $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$, $|\mathfrak{b}| < \min\ \mathfrak{b}$ then

$$\mathrm{pcf}(\mathfrak{b}) \subseteq \mathrm{pcf}(\mathfrak{a}).$$

§2 Normality of $\lambda \in \mathrm{pcf}(\mathfrak{a})$ for $\mathfrak{a}$

Having found those ideals $J_{<\lambda}[\mathfrak{a}]$ which are so central in this book, we would like to know more. As $J_{<\lambda}[\mathfrak{a}]$ is increasing continuous in λ the question is how $J_{<\lambda}[\mathfrak{a}]$, $J_{<\lambda^+}[\mathfrak{a}]$ are related.

The simplest relation is $J_{<\lambda^+}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}$ for some $\mathfrak{b} \subseteq \mathfrak{a}$, and then we call λ normal (for $\mathfrak{a}$) and denote $\mathfrak{b} = \mathfrak{b}_\lambda[\mathfrak{a}]$, though it is unique only modulo $J_{<\lambda}[\mathfrak{a}]$. We give sufficient conditions for this , use this in 2.8; give the necessary definitions in 2.3 and needed information in 2.4, 2.5, 2.6; 2.7 is the essential uniqueness of cofinal sequences in appropriate $\prod \mathfrak{a}/I$. Then in 2.11, we get a weaker result which does not need an extra assumption: there is $\mathfrak{b} \in J_{\le\lambda}[\mathfrak{a}]$ such that for every $\mathfrak{c} \in J_{\le\lambda}[\mathfrak{a}]$, $\mathfrak{c}\backslash\mathfrak{b}$ is included in a countable union of members of $J_{<\lambda}[\mathfrak{a}]$.

* * *

Definition 2.1 (1) We say $\lambda \in \mathrm{pcf}(\mathfrak{a})$ is normal (for $\mathfrak{a}$) <u>if</u> for some $\mathfrak{b} \subseteq \mathfrak{a}$, $J_{\le\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}$.

(2) We say $\lambda \in \mathrm{pcf}(\mathfrak{a})$ is semi-normal (for $\mathfrak{a}$) if there are $\mathfrak{b}_i$ for $i < \lambda$ such that:
(i) $i < j \Rightarrow \mathfrak{b}_i \subseteq \mathfrak{b}_j \bmod J_{<\lambda}[\mathfrak{a}]$
and
(ii) $J_{\le\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \{\mathfrak{b}_i : i < \lambda\}$.

(3) We say $\mathfrak{a}$ is normal if every $\lambda \in \mathrm{pcf}(\mathfrak{a})$ is normal for $\mathfrak{a}$.

Fact 2.2 Suppose $\min \mathfrak{a} > |\mathfrak{a}|$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$. Now:

(1) λ is semi-normal for $\mathfrak{a}$ iff for some $F = \{f_\alpha : \alpha < \lambda\} \subseteq \prod \mathfrak{a}$, for every ultrafilter D over $\mathfrak{a}$, F is unbounded in $(\prod \mathfrak{a}, <_D)$ whenever $\mathrm{tcf}(\prod \mathfrak{a}, <_D) = \lambda$.

(2) In 2.1(2) we can assume without loss of generality that either:
$\mathfrak{b}_i = \mathfrak{b}_0 \bmod J_{<\lambda}[\mathfrak{a}]$ (so λ is normal) or:
$\mathfrak{b}_i \neq \mathfrak{b}_j \bmod J_{<\lambda}[\mathfrak{a}]$ for $i < j < \lambda$ and in 2.2(1) above

$$[\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J_{<\lambda}[\mathfrak{a}]].$$

(3) Suppose $F = \langle f_\alpha : \alpha < \lambda \rangle$ is as in (1) and is $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing. Then λ is normal iff there is $g \in \prod_{\theta \in \mathfrak{a}}(\theta + 1)$ which is a $<_D$-least upper bound of F for every ultrafilter D such that $\mathrm{cf}(\prod \mathfrak{a}/D) = \lambda$; if the second clause holds then $\mathfrak{b} =: \{\theta \in \mathfrak{a} : g(\theta) = \theta\}$ generates $J_{\leq\lambda}[\mathfrak{a}]$.

Proof: Left to the reader. (Use 1.7, 1.8(3)).

We shall give some sufficient conditions for normality.

Definition 2.3 For given regular λ, $\theta < \mu < \lambda$, $S \subseteq \lambda$, $\sup S = \lambda$.

(1) We call $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ a continuity condition for (S, μ, θ) if: $A_\alpha \subseteq \alpha$, $|A_\alpha| < \mu$, $[\delta \in S \Rightarrow \mu > \mathrm{cf}(\delta) \geq \theta]$ and $[\beta \in A_\alpha \Rightarrow A_\beta = A_\alpha \cap \beta]$ and $[\delta \in S \Rightarrow \delta = \sup A_\delta]$.

(2) We say $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ obeys $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ if:
   (a) for $\beta \in A_\alpha$, $\bigwedge_{\theta \in \mathfrak{a}} f_\beta(\theta) < f_\alpha(\theta)$
   (b) if $\alpha \in S$ (is a limit ordinal) then $f_\alpha(\theta) = \sup_{\beta \in A_\alpha}(f_\beta(\theta) + 1)$ for every $\theta \in \mathfrak{a}$.

   If only (a) holds we say: $\bar{f}$ weakly obeys $\bar{A}$. Note: if $\bar{A}$ is a continuity condition, the $+1$ in clause (b) is redundant.

(3) If $\theta = \aleph_0$ we omit it, $(S, \mathfrak{a})$ stands for $(S, \min\ \mathfrak{a}, |\mathfrak{a}|^+)$, (λ, μ, θ) stands for "(S, μ, θ) for some stationary S".

(4) We add to "continuity condition" (in part (1)) the adjective "weak" if "$\beta \in A_\alpha \Rightarrow A_\beta = A_\alpha \cap \beta$" is replaced by

$$\text{“}\alpha \in S \ \&\ \beta \in A_\alpha \Rightarrow (\exists \gamma < \alpha)[A_\alpha \cap \beta \subseteq A_\gamma]\text{”}.$$

(5) $I[\lambda] = I^g[\lambda] =: \{S \subseteq \lambda :$ there is a sequence $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ such that: $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of α, and for every $\delta \in S$ for some unbounded $A \subseteq \delta :$ otp $A < \delta$ and $[\alpha \in A \Rightarrow A \cap \alpha \in \bigcup_{\beta < \delta} \mathcal{P}_\beta]\}$.

(6) $I^{\mathrm{wg}}_\mu[\lambda] = \big\{S \subseteq \lambda :$there is a sequence $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ such that $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of α, $[A \in \bigcup_\alpha \mathcal{P}_\alpha \Rightarrow |A| < \mu]$ and for every $\delta \in S$, for some unbounded $A \subseteq \delta :$
$\alpha \in A \Rightarrow (\exists B) \left[A \cap \alpha \subseteq B \in \bigcup_{\beta < \delta} \mathcal{P}_\beta\right]$
(so $|A| < \mu$)$\big\}$.

(7) $I^{\rm sg}_{\mu,\theta}[\lambda] = \{S \subseteq \lambda :$ there is a sequence $\langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$ such that $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of α each of power $< \mu$ and: for every $\delta \in S$, $\mathrm{cf}(\delta) < \mu$ and for some unbounded $A \subseteq \delta$: for every $\alpha \in A$, for some $i^* < \theta$ and $x_i \in \bigcup_{\beta<\delta} \mathcal{P}_\beta$ for $i < i^*$, $[A \cap \alpha \subseteq \bigcup_{i<i^*} x_i]\}$.

(8) Stationary members of $I[\lambda]$ are called good stationary sets; similarly, stationary members of $I^{\rm wg}_\mu[\lambda]$ are called weakly good stationary sets. Again $I^{\rm sg}_\mu[\lambda]$ stands for $I^{\rm sg}_{\mu,\aleph_0}[\lambda]$ (it is $I^{\rm wg}_\mu[\lambda]$).

Fact 2.4 (0) Let $S \subseteq \lambda$, $\lambda = \mathrm{cf}(\lambda) > \aleph_0$, $\lambda \geq \mu = \bigcup_{\delta\in S}(\mathrm{cf}(\delta))^+$ and $[\delta \in S \Rightarrow \mathrm{cf}(\delta) < \delta]$. Then $S \in I[\lambda]$ $[S \in I^{\rm wg}_\mu[\lambda]]$ iff for some club C of λ, there is a [weak] continuity condition $\bar{A}$ for (S, μ); moreover, $\delta \in S \Rightarrow \mathrm{otp}\, A_\delta = \mathrm{cf}(\delta)$.

(1) There is a [weak] continuity condition $\bar{A}$ for $(\lambda, \mathfrak{a})$ iff there is stationary S such that: $S \subseteq \{\delta < \lambda : |\mathfrak{a}| < \mathrm{cf}(\delta) < \min\, \mathfrak{a}\}$ is in $I[\lambda]$ [in $I^{\rm wg}_{\min\, \mathfrak{a}}[\lambda]$].

Remark 2.4A (1) The following will not be used and are included for the reader's amusement.

(2) If $\lambda = \mu^+$, $\mathrm{cf}(\mu) = \mu > \aleph_0$ then $\{\delta < \lambda : \mathrm{cf}(\delta) < \mu\}$ is in $I[\lambda]$ (in fact we prove this in [Sh365,2.14]).

(3) If $\lambda = \mu^+$, $\theta < \mathrm{cf}(\mu)$ then $\{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is the union of $\mathrm{cf}(\mu)$ stationary sets each of them from $I^{\rm sg}_\kappa[\lambda]$ for some $\kappa < \mu$ (not necessarily the same κ).

(4) If $\lambda = \mu^+$, μ singular $> \chi$ and $\chi \to (\theta)^2_{\mathrm{cf}(\mu)}$, then there are $\kappa < \mu$ and a stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ which is in $I^{\rm wg}_\kappa[\lambda]$.

(5) If $\lambda = \mu^+$, μ is singular strong limit then

$$\lambda \in I^{\rm sg}_{\mu,(\mathrm{cf}(\mu))^+}[\lambda] \text{ and } I^{\rm wg}_\mu[\lambda] = I[\lambda].$$

If λ is strongly inaccessible $> \aleph_0$ then $\lambda \in I[\lambda]$. If $\lambda > \theta$ are regular, $\lambda = \lambda^{<\theta}$ then there is a stationary $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ in $I^g[\lambda]$.

Fact 2.5 Suppose $\bar{A}$ is a weak continuity condition for $(S, \mathfrak{a})$, $f_\alpha \in \prod \mathfrak{a}$ for $\alpha < \lambda$, $\min \mathfrak{a} > |\mathfrak{a}|^+$, $\lambda = \mathrm{cf}(\lambda) > |\mathfrak{a}|$. Then

(1) we can find $\langle f'_\alpha : \alpha < \lambda\rangle$ obeying $\bar{A}$, $f'_\alpha \in \prod \mathfrak{a}$, such that:
   (i) for $\alpha \in \lambda\backslash S$ we have $f_\alpha \leq f'_\alpha$
   (ii) for every α, $f_\alpha \leq f'_{\alpha+1}$.

(2) Suppose $\langle f'_\alpha : \alpha < \lambda\rangle$ obeys $\bar{A}$ and satisfies (i). If $g_\alpha \in \prod \mathfrak{a}$, $\langle g_\alpha : \alpha < \lambda\rangle$ obeys $\bar{A}$ and $\bigwedge_\alpha g_\alpha \leq f_\alpha$ then $\bigwedge_\alpha g_\alpha \leq f'_\alpha$.

(3) We can add in (1)

(iii) if $\langle f''_\alpha : \alpha < \lambda \rangle$ obeys $\bar{A}$, $f''_\alpha \in \prod \mathfrak{a}$, and it satisfies (i) and (ii), <u>then</u> for every α, $f'_\alpha \leq f''_\alpha$.

Proof: Easy.

Lemma 2.6 *Suppose $f_\alpha \in \prod \mathfrak{a}$ for $\alpha < \lambda$, λ regular, $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ obeys some $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ which is a weak continuity condition for $(\lambda, \mathfrak{a})$, and $\bar{f}$ is $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing (so $\lambda \geq \min(\mathfrak{a}) > |\mathfrak{a}|^+$). Then*

(a) $\langle f_\alpha : \alpha < \lambda \rangle$ has a $<_{J_{<\lambda}[\mathfrak{a}]}$-exact upper bound $g \in \prod_{\theta \in \mathfrak{a}} (\theta + 1)$.

(b) $\mathfrak{b}_g \in J_{\leq\lambda}[\mathfrak{a}]$ where $\mathfrak{b}_g =: \{\theta \in \mathfrak{a} : g(\theta) = \theta\}$.

(c) Letting $\mu_\theta = \mathrm{cf}(g(\theta))$, we have that $\left(\prod_{\theta\in\mathfrak{a}} \mu_\theta, <_{J_{<\lambda}[\mathfrak{a}]}\right)$ has true cofinality λ and $\mu_\theta \leq \theta$.

Proof: First note (b), (c) follows from (a); $\mathfrak{b}_g \in J_{\leq\lambda}[\mathfrak{a}]$ as $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_{J_{<\lambda}[\mathfrak{a}]+(\mathfrak{a}\setminus\mathfrak{b}_g)}$-increasing and cofinal in $\prod \mathfrak{a}$, and similarly (c) holds (by 1.3(6)(ii)). So it suffices to prove:

Claim 2.6A Assume I an ideal on κ, $\lambda = \mathrm{cf}(\lambda) > \kappa^+$, $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \lambda$ is $\leq_I$-increasing and weakly obeys $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ which is a weak continuity condition for (S, μ, κ^+), S a stationary subset of λ (see Definition 2.3(1),(2),(3),(4)). <u>Then</u> $\langle f_\alpha : \alpha < \lambda \rangle$ has an $\leq_I$-exact upper bound.

Proof: We first prove the existence of a lub. We try to choose by induction on $\zeta < \kappa^+$, g_ζ and for ζ limit also α_ζ, $s_{\zeta,i} (i < \kappa)$, $f_{\zeta,\alpha} (\alpha < \lambda)$ such that:

(A) (a) $g_\zeta \in {}^\kappa\mathrm{Ord}$

(b) $f_\alpha \leq_I g_\zeta$ for $\alpha < \lambda$

(c) $\xi < \zeta \Rightarrow g_\zeta \leq_I g_\xi$

(d) $\xi < \zeta \Rightarrow \neg g_\zeta =_I g_\xi$

(by (c) it is enough: $\xi + 1 = \zeta \Rightarrow \neg g_\zeta =_I g_\xi$)

(B) if ζ is limit also

(e) $s_{\zeta,i} =: \{g_\xi(i) : \xi < \zeta\} \cup \{\sup_\alpha f_\alpha(i) + 1\}$

(f) $f_{\zeta,\alpha}(i) =: \min [s_{\zeta,i} \setminus f_\alpha(i)]$

(g) $g_\zeta = f_{\zeta,\alpha_\zeta}$ and $\alpha_\zeta \leq \alpha < \lambda \Rightarrow g_\zeta =_I f_{\zeta,\alpha}$.

We let $g_0 \in {}^\kappa\mathrm{Ord}$ be defined by $g_0(i) = \sup_{\alpha<\lambda} f_\alpha(i) + 1$. If $\zeta = \xi + 1$, and there is no g_ζ as required (relevant parts are (a) - (d)) then we are done and the conclusion holds. So assume ζ is a limit ordinal $< \kappa^+$, then define $s_{\zeta,i} (i < \kappa)$ as in (e) and $f_{\zeta,\alpha} (\alpha < \lambda)$ as in (f). If for some $\alpha_\zeta < \lambda$, $\left[\alpha_\zeta \leq \alpha < \lambda \Rightarrow f_{\zeta,\alpha_\zeta} =_I f_{\zeta,\alpha}\right]$ let $g_\zeta = f_{\zeta,\alpha_\zeta}$, and you can check that all conditions hold. If not, then for some club E of λ,

$$[\alpha < \beta \ \&\ \alpha \in E \ \&\ \beta \in E \Rightarrow \neg f_{\zeta,\alpha} =_I f_{\zeta,\beta}].$$

Note

$(*)$ $\alpha < \beta \Rightarrow f_{\zeta,\alpha} \le_I f_{\zeta,\beta}$.

Choose δ in S which is an accumulation point of E. So A_δ is defined, and for every $\beta \in A_\delta$ there is $\gamma_\beta < \delta$ such that $A_\delta \cap (\beta+1) \subseteq A_{\gamma_\beta}$ (see Definition 2.3(4)). Let $\{\beta_\epsilon : \epsilon < \mathrm{cf}(\delta)\} \subseteq A_\delta$ be unbounded such that:

(α) $\gamma_{\beta_\epsilon} < \beta_{\epsilon+1}$

(β) $\neg f_{\zeta,\gamma_{\beta_\epsilon}} =_I f_{\zeta,\beta_{\epsilon+1}}$ moreover $(\gamma_{\beta_\epsilon}, \beta_{\varepsilon+1}) \cap E \ne \emptyset$.

(why is this possible? (α) by 2.3(4) clause (β) holds as δ is an accumulation point of E).

Now for each $\epsilon < \mathrm{cf}(\delta)$ by clause (β) clearly

$$t_\epsilon =: \left\{i < \kappa : f_{\zeta,\gamma_{\beta_\epsilon}}(i) \ge f_{\zeta,\beta_{\epsilon+1}}(i)\right\} \text{ is } \ne \kappa \bmod I,$$

so there is $i(\epsilon) \in \kappa \backslash t_\epsilon$, hence for some $i(*) < \kappa$ the set

$$B = \{\epsilon < \mathrm{cf}(\delta) : i(\epsilon) = i(*)\}$$

is an unbounded subset of A_δ hence of δ (remember $\bar{A}$ is an (S, μ, κ^+)-weak continuity sequence and $\delta \in S$, so $\mathrm{cf}(\delta) > \kappa$). Now if $\epsilon(1) < \epsilon(2)$ are in B then $f_{\zeta,\beta_{\epsilon(1)+1}}(i(*)) \le f_{\zeta,\gamma_{\beta_{\epsilon(2)}}}(i(*))$ [as $\langle f_\alpha : \alpha < \lambda\rangle$ weakly obeys $\bar{A}$ and

$$\beta_{\epsilon(1)+1} \in A_\delta \cap (\beta_{\epsilon(1)+1} + 1) \subseteq A_\delta \cap (\beta_{\epsilon(2)} + 1) \subseteq A_{\gamma_{\beta(2)}}]$$

and $f_{\zeta,\gamma_{\beta_{\epsilon(2)}}}(i(*)) < f_{\zeta,\beta_{\epsilon(2)+1}}(i(*))$ [as $i(*) \notin t_{\epsilon(2)}$]; hence

$$f_{\zeta,\beta_{\epsilon(1)+1}}(i(*)) < f_{\zeta,\beta_{\epsilon(2)+1}}(i(*)).$$

So $\langle f_{\zeta,\beta_{\epsilon+1}}(i(*)) : \epsilon \in B\rangle$ is a strictly increasing sequence of ordinals, but by the definition of $f_{\zeta,\beta_{\epsilon+1}}$ they are all in $s_{\zeta,i(*)}$, but $s_{\zeta,i(*)}$ has cardinality $\le |\zeta| \le \kappa < \kappa^+$, contradiction.

So we can carry out the induction on $\zeta < \kappa^+$. Let $\alpha^* = \sup_{\zeta<\kappa^+} \alpha_\zeta < \lambda$. For each i, $s_{\zeta,i}$ increases with ζ, hence $f_{\zeta,\alpha^*}(i) = \min(s_{\zeta,i} \backslash f_{\alpha^*}(i))$ decreases with ζ hence is eventually constant; as this holds for each i, $\langle f_{\zeta,\alpha^*} : \zeta < \kappa^+\rangle$ is eventually constant; but $f_{\zeta,\alpha^*} = g_\zeta \bmod I$ for each ζ (by (g)) contradicting (d).

So $\bar{f}$ has a $\le_I -$lub g. If g is not an $\le_I -$eub, in Definition 1.1(4), parts (α), (γ) hold easily, so (β) fails, then for some $h \in {}^\kappa\mathrm{Ord}$, $h <_I \max\{g, 1\}$, but for no $\alpha < \lambda$, $h <_I \max\{f_\alpha, 1\}$. Let $\mathfrak{b}_\alpha = \{i < \alpha : h(i) \ge f_\alpha(i), 1\}$, so $\alpha < \beta \Rightarrow \mathfrak{b}_\alpha \supseteq \mathfrak{b}_\beta \bmod I$; but $\alpha < \lambda \Rightarrow \mathfrak{b}_\alpha \notin I$. If for some $\alpha^* < \lambda$, $[\alpha^* \le \alpha < \lambda \Rightarrow \mathfrak{b}_{\alpha^*} = \mathfrak{b}_\alpha \bmod I]$, then $g \restriction (\kappa \backslash \mathfrak{b}_{\alpha^*}) \cup h \restriction \mathfrak{b}_{\alpha^*}$ contradicts "g is a $\le_I -$lub of $\bar{f}$". So for some club E of λ we have

$$[\alpha < \beta \in E \Rightarrow \mathfrak{b}_\alpha \ne \mathfrak{b}_\beta \bmod I]$$

and we get a contradiction as above. $\square_{2.6A}$

$\square_{2.6}$

Claim 2.7 Suppose:

(a) $f_\alpha \in \prod \mathfrak{a}$ for $\alpha < \lambda$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$ and $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing.

(b) $\bar{f}$ obeys $\bar{A}$, a weak continuity condition for $(S, \mathfrak{a})$, $\lambda = \sup S$ (hence $\lambda \geq \min(\mathfrak{a}) > |\mathfrak{a}|^+$).

(c) J is an ideal on $\mathcal{P}(\mathfrak{a})$ extending $J_{<\lambda}[\mathfrak{a}]$, and $\langle f_\alpha / J : \alpha < \lambda \rangle$ is cofinal in $(\prod \mathfrak{a}, <_J)$ (for example $J = J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b})$, $\mathfrak{b} \in J_{\leq\lambda}[\mathfrak{a}] \backslash J_{<\lambda}[\mathfrak{a}]$).

(d) $\langle f'_\alpha : \alpha < \lambda \rangle$ satisfies (a), (b) above.

(e) $f_\alpha \leq f'_\alpha$ for $\alpha < \lambda$ (alternatively: $\langle f'_\alpha : \alpha < \lambda \rangle$ satisfies (c)).

Then $\{\delta < \lambda :$ if $\delta \in S$ then $f'_\delta = f_\delta \bmod J\}$ contains a club of λ.

Proof: Not hard.

Remark 2.7A Here (and even in 2.5 — 2.10) we can use $I^{\mathrm{sg}}_{\mu,\theta}[\lambda]$ provided that we restrict ourselves to θ-complete ideals.

Lemma 2.8 *Suppose* $\min \mathfrak{a} > |\mathfrak{a}|^+$, $\lambda = \mathrm{cf}(\lambda) \in \mathrm{pcf}(\mathfrak{a})$ *and there is a good stationary set* $\subseteq \{\delta < \lambda : |\mathfrak{a}| < \mathrm{cf}(\delta) < \min\ \mathfrak{a}\}$ *or at least a weakly good stationary set* $\subseteq \{\delta < \lambda : |\mathfrak{a}| < \mathrm{cf}(\delta) < \min\ \mathfrak{a}\}$. *Then* λ *is normal for* $\mathfrak{a}$.

Proof: Let $\bar{A}$ be a weak continuity condition for $(S, \mathfrak{a})$ for some S, a stationary subset of $\{\delta < \lambda : |\mathfrak{a}| < \mathrm{cf}(\delta) < \min\ \mathfrak{a}\}$. We assume λ is not normal for $\mathfrak{a}$ and eventually get a contradiction. By 2.2(1), 2.2(3), 2.5 and 2.6 λ is not semi-normal for $\mathfrak{a}$. Let us define by induction on $\zeta \leq |\mathfrak{a}|^+$, $\bar{f}^\zeta = \langle f^\zeta_\alpha : \alpha < \lambda \rangle$ and D_ζ such that:

(I) (i) $f^\zeta_\alpha \in \prod \mathfrak{a}$

(ii) $\alpha < \beta \Rightarrow f^\zeta_\alpha < f^\zeta_\beta \bmod J_{<\lambda}[\mathfrak{a}]$

(iii) $\bar{f}^\zeta$ obeys $\bar{A}$

(iv) for $\xi < \zeta \leq |\mathfrak{a}|^+$ and $\alpha < \lambda : f^\xi_\alpha \leq f^\zeta_\alpha$

(II) (i) D_ζ is an ultrafilter on $\mathfrak{a}$ such that: $\mathrm{cf}(\prod \mathfrak{a} / D_\zeta) = \lambda$

(ii) $\langle f^\zeta_\alpha / D_\zeta : \alpha < \lambda \rangle$ is not cofinal in $\prod \mathfrak{a} / D_\zeta$

(iii) $\langle f^{\zeta+1}_\alpha / D_\zeta : \alpha < \lambda \rangle$ is cofinal in $\prod \mathfrak{a} / D_\zeta$

(iv) $f^{\zeta+1}_0 / D_\zeta$ is above $\{f^\zeta_\alpha / D_\zeta : \alpha < \lambda\}$.

<u>For $\zeta = 0$:</u> no problem. [Use 2.5 and 1.5].

<u>For ζ limit:</u> Let $g^\zeta_\alpha \in \prod \mathfrak{a}$ be defined by $g^\zeta_\alpha(\theta) = \sup_{\xi<\zeta} f^\xi_\alpha(\theta)$, which belongs to $\prod \mathfrak{a}$ as $|\mathfrak{a}|^+ < \min(\mathfrak{a})$. Now use 2.5 (1)+(3) and get $\langle f^\zeta_\alpha : \alpha < \lambda \rangle$ obeying $\bar{A}$ as there, in particular such that $[\alpha \in \lambda \backslash S \Rightarrow g^\zeta_\alpha \leq f^\zeta_\alpha]$ and $[g^\zeta_\alpha \leq f^\zeta_{\alpha+1}]$. Use 2.2 to find an appropriate D_ζ (i.e. let $\mathfrak{b}_g$ be as in 2.2(3); if $J_{\leq\lambda}[\mathfrak{a}] \neq J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_g$ we can find an ultrafilter $D = D_\zeta$ on $\mathfrak{a}$, $D \cap J_{<\lambda}[\mathfrak{a}] = \emptyset$ $\mathfrak{b}_g \notin D$ and $D \cap J_{\leq\lambda}[\mathfrak{a}] \neq \emptyset$). Now $\langle f^\zeta_\alpha : \alpha < \lambda \rangle$ and D_ζ are as required.

<u>For $\zeta = \xi + 1$:</u> Let $\langle h^\zeta_\alpha : \alpha < \lambda \rangle$ be cofinal in $(\prod \mathfrak{a}, <_{D_\xi})$ and without loss of generality $f^\xi_\alpha \leq h^\zeta_0 \bmod D_\xi$. We get D_ζ and $\langle f^\zeta_\alpha : \alpha < \lambda \rangle$ by 2.2 and 2.5 for $\langle h^\zeta_\alpha : \alpha < \lambda \rangle$.

Now for each $\zeta < |\mathfrak{a}|^+$ we apply 2.7 for $\langle f_\alpha^{\zeta+1} : \alpha < \lambda \rangle$, $\langle f_\alpha^{|\mathfrak{a}|^+} : \alpha < \lambda \rangle$ and $J = \mathcal{P}(\mathfrak{a}) \backslash D_\zeta$ (which is a maximal ideal on $\mathfrak{a}$). We get a club C_ζ of λ such that:

(*) $\alpha \in S \cap C_\zeta \Rightarrow f_\alpha^{\zeta+1} = f_\alpha^{|\mathfrak{a}|^+} \bmod D_\zeta$.

So $\bigcap_{\zeta<|\mathfrak{a}|^+} C_\zeta$ is a club of λ since $|\mathfrak{a}|^+ < \lambda$, so we can choose $\alpha \in S \cap \bigcap_{\zeta<|\mathfrak{a}|^+} C_\zeta$. Let

$$\mathfrak{c}_\zeta = \{\theta \in \mathfrak{a} : f_\alpha^\zeta(\theta) = f_\alpha^{|\mathfrak{a}|^+}(\theta)\}.$$

By (*), $\mathfrak{c}_{\zeta+1} \in D_\zeta$; by (II)(iv) $\mathfrak{c}_\zeta \notin D_\zeta$, hence $\mathfrak{c}_\zeta \neq \mathfrak{c}_{\zeta+1}$.
On the other hand by (I)(iv), $\langle \mathfrak{c}_\zeta : \zeta < |\mathfrak{a}|^+ \rangle$ is $\subseteq$-increasing and by the previous sentence it is strictly $\subseteq$-increasing; contradiction. $\square_{2.8}$

Claim 2.9 Suppose $\min(\mathfrak{a}) > |\mathfrak{a}|^+$, $|\mathfrak{a}|^+ < \mu = \mathrm{cf}(\mu) < \lambda \in \mathrm{pcf}(\mathfrak{a})$. Then for some $\kappa_\theta = \mathrm{cf}(\kappa_\theta) < \theta$ (for $\theta \in \mathfrak{a}$) we have $\left(\prod_{\theta\in\mathfrak{a}} \kappa_\theta, <_{J_{<\lambda}[\mathfrak{a}]}\right)$ has true cofinality μ, provided that:

(*) μ has a weakly good stationary set

$$S \subseteq \{\delta < \mu : |\mathfrak{a}| < \mathrm{cf}(\delta) < \min\ \mathfrak{a}\}.$$

Proof: Easy, by 2.6A, 2.5.

Claim 2.10 Suppose the assumptions (a),(c),(d),(e) of 2.7 hold and
(b)′ $\bar{f}$ obeys $\bar{A}$, $\bar{A}$ a continuity condition for $(S, \kappa, \aleph_0)$(where $\lambda = \sup S$).
(f) J is κ-complete, $\kappa = \mathrm{cf}(\kappa) > \mathrm{cf}(\delta)$ for every $\delta \in S$.
Then for some club C of λ

$$\delta \in S \cap C \Rightarrow f'_\delta = f_\delta \bmod J.$$

Proof: Not hard. (See 2.7).

Lemma 2.11 *Suppose* $\min(\mathfrak{a}) > |\mathfrak{a}|^+$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$.
Then there is $\mathfrak{b}_\alpha \subseteq \mathfrak{a}$ *for* $\alpha < \lambda$ *such that* $\mathfrak{b} \in J_{\leq\lambda}[\mathfrak{a}]$ *and:*

(*) *for every* $\mathfrak{c} \in J_{\leq\lambda}[\mathfrak{a}]$ *there are* $\mathfrak{b}_n \in J_{<\lambda}[\mathfrak{a}]$ *for* $n < \omega$ *and* $\alpha < \lambda$ *such that* $\mathfrak{c} \subseteq \mathfrak{b}_\alpha \cup \bigcup_{n<\omega} \mathfrak{b}_n$.

Proof: Let

$$S = \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0 \text{ or } \delta \text{ is zero or } \delta \text{ is a successor ordinal}\}.$$

We can easily find a continuity condition $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$, for $(S, \aleph_1, \aleph_0)$ such that: for $\delta \in S$, A_δ is an unbounded subset of δ of order type ω, and for non-limit $\alpha \in S$, A_α is finite. Here is how one finds the continuity condition.

We prove by induction on $\alpha \leq \lambda$ the existence of a continuity condition $\bar{A}^\alpha = \langle A_i^\alpha : i \in \alpha \cap S \rangle$:

(1) $\alpha \leq \omega + 1$

Let $A_i^\alpha = i$ for $i < \alpha$.

(2) not (1) and $\alpha = \beta + \gamma$ where $\beta < \alpha$, $\gamma < \alpha$, $\mathrm{cf}(\beta) \neq \aleph_0, 1, 0$.

$$\text{Let } A_i^\alpha = \begin{cases} A_i^\beta & \underline{\text{if}}\ i \in \beta \cap S \\ \beta + A_j^\gamma & \underline{\text{if}}\ i \in \alpha \cap S \setminus \beta, \quad i - \beta = j \end{cases}$$

where $\beta + A = \{\beta + \zeta : \zeta \in A\}$.

(3) Not (1), (2) and $\alpha = \beta$, $\mathrm{cf}(\beta) = \aleph_0$ or $\alpha = \beta + 1$, $\mathrm{cf}(\beta) = \aleph_0$.

Let $\beta = \bigcup_{n<\omega} \alpha_n$, where $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots$, each α_{n+1} a successor ordinal, and let

(a) $A_\beta^\alpha = \{\alpha_n : n < \omega\}$ [if $\beta < \alpha$]

(b) $A_{\alpha_n}^\alpha = \{\alpha_m : m < n\}$

(c) if $\alpha_n < \gamma < \alpha_{n+1}$ let

$$A_\gamma^\alpha =: (\alpha_n + 1) + A_{\gamma-(\alpha_n+1)}^{\alpha_{n+1}-(\alpha_n+1)},$$

(4) Not 1), 2), 3), $\alpha > \mathrm{cf}(\alpha) > \aleph_0$.

Let $\kappa = \mathrm{cf}(\alpha)$. Let $\langle \alpha_i : i < \kappa \rangle$ be increasing continuous, $\bigcup_{i<\kappa} \alpha_i = \alpha$, $\alpha_0 = 0$, each α_{i+1} a successor ordinal.

We define by the induction hypothesis

$$A_\gamma^\alpha = (\alpha_i + 1) + A_{\gamma-(\alpha_i+1)}^{\alpha_{i+1}-(\alpha_i+1)}$$

for $\alpha_i < \gamma < \alpha_{i+1}$ and

$$A_{\alpha_i}^\alpha = \{\alpha_j : j \in A_i^\kappa\}.$$

(5) $\alpha = \mathrm{cf}(\alpha) > \aleph_0$.

Call $\alpha = \kappa$.

Choose $\langle \alpha_i : i < \kappa \rangle$ increasing continuous, $\bigcup_{i<\kappa} \alpha_i = \alpha$, $\alpha_0 = 0$, α_{i+1} successor of limit and $\alpha_{i+1} > (\omega + \omega) + (\alpha_i + \alpha_i) + \omega$. So

$$E_i = \{\delta + 1 : \delta \text{ limit}, \alpha_i < \delta + 1 < \alpha_{i+1}\} \text{ has power } \geq |\alpha_i|.$$

Let g_i be a function from E_i onto $\bigcup_{j<i} E_j$.

We define $h : \kappa \to \kappa$,

$$h(\alpha) = \begin{cases} \alpha + 1 & \alpha \text{ successor}, \\ \alpha & \text{otherwise}. \end{cases}$$

Choose A^α_γ as follows:

for $\alpha_i < \gamma < \alpha_{i+1}$, let $B^\alpha_\gamma = (\alpha_i + 1) + A^{\alpha_{i+1}-(\alpha_i+1)}_{\gamma-(\alpha_i+1)}$, and $A^\alpha_{h(\gamma)} = \{h(\zeta) : \zeta \in B^\alpha_\gamma\}$. So we have defined A^α_γ for $\gamma \in \bigcup_{i<\kappa}((\alpha_i, \alpha_{i+1})\backslash E_i) \cap S$.

For $\gamma \in E_i$ we define A^α_γ by induction on γ :

$i = 0$ $\quad A^\alpha_\gamma = \emptyset$

$i > 0$ $\quad A^\alpha_\gamma = \{g_i(\gamma)\} \cup A^\alpha_{g_i(\gamma)}$.

Lastly for $\gamma \in \{\alpha_i : i < \kappa\}$, if i is non-limit or $\mathrm{cf}(i) > \aleph_0$, let $A^\alpha_i = \emptyset$, otherwise $\mathrm{cf}(\alpha_i) = \aleph_0$ and $\mathrm{cf}(i) = \aleph_0$. So there are

$$\langle j_n : n < \omega\rangle : \; 0 = j_0 < j_1 \ldots$$

and

$$\bigcup_{n<\omega} j_n = i.$$

Choose inductively $\gamma^i_n \in E_{j_n}$, $h(\gamma^i_{n+1}) = \gamma^i_n$. So $A^\alpha_{\gamma^i_n} = \{\gamma^i_0, \ldots, \gamma^i_{n-1}\}$ and let $A^\alpha_{\alpha_i} =: \{\gamma^i_n : n < \omega\}$.

Now after this digression, we return to the proof of 2.11. The proof is the same as that of 2.8, using 2.10 instead of 2.7, applied to

$$J =: J^1_{<\lambda}[\mathfrak{a}] = \left\{\bigcup_n \mathfrak{b}_n : \mathfrak{b}_n \in J_{<\lambda}[\mathfrak{a}] \text{ for } n < \omega\right\}$$

which is an $\aleph_1$-complete ideal (we use J instead of $J_{<\lambda}[\mathfrak{a}]$). $\square_{2.11}$

Conclusion 2.12 Suppose $\min \mathfrak{a} > |\mathfrak{a}|^+$.

(1) We can find $\langle \mathfrak{b}_{\lambda,\alpha} : \alpha < \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ such that:

(i) $\mathfrak{b}_{\lambda,\alpha} \in J_{\leq\lambda}[\mathfrak{a}]\backslash J_{<\lambda}[\mathfrak{a}]$

(ii) every member of $J_{<\lambda}[\mathfrak{a}]$ is included in some $\bigcup_{n<\omega} \mathfrak{b}_{\lambda_n,\alpha_n}$, for some $\alpha_n < \lambda_n < \lambda$.

(2) If every $\lambda \in \mathrm{pcf}(\mathfrak{a})$ is normal for $\mathfrak{a}$, then we can replace (ii) above by

(ii)′ $J_{<\lambda}[\mathfrak{a}]$ is generated by $\{\mathfrak{b}_\mu : \mu \in \lambda \cap \mathrm{pcf}(\mathfrak{a})\}$.

§3 Getting better representation: generating sequences and cofinality systems

Assume for simplicity that our $\mathfrak{a}$'s are such that for every λ there is $\mathfrak{b}_\lambda = \mathfrak{b}_\lambda[\mathfrak{a}] \subseteq \mathfrak{a}$ such that $J_{\leq\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda$. Then $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ is called a generating sequence; it is a concise description of $\langle J_{<\lambda}[\mathfrak{a}] : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ as $J_{<\lambda}[\mathfrak{a}]$ is the ideal on $\mathfrak{a}$ generated by $\{\mathfrak{b}_\theta : \theta \in \lambda \cap \mathrm{pcf}(\mathfrak{a})\}$. An immediate useful property is the following "compactness" (3.2(5), 3.7 respectively):

$\otimes_1$ if $\mathfrak{b} \subseteq \mathfrak{a}$, then for some finite $\mathfrak{d} \subseteq \mathrm{pcf}(\mathfrak{b})$, $\mathfrak{b} \subseteq \bigcup_{\theta\in\mathfrak{d}} \mathfrak{b}_\theta$

$\otimes_2$ if $\lambda = \max\mathrm{pcf}(\mathfrak{a}), \sup(\lambda\cap\mathrm{pcf}(\mathfrak{a})$ singular, $\mathfrak{c}$ an unbounded subset of $\lambda\cap\mathrm{pcf}(\mathfrak{a})$ of cardinality $< \min\mathfrak{c}$ then λ is the true cofinality of $(\prod\mathfrak{c}, <_{J^{\mathrm{bd}}_{\mathfrak{c}}})$ (see 1.1(1)(*b*), and 1.0; see 1.10-1.12).

To apply this note

$\otimes_3$ if $\lambda = \max\mathrm{pcf}(\mathfrak{a})$ and μ is the maximal accumulation point of $\mathrm{pcf}(\mathfrak{a})$ (if $\mathfrak{a}$ is infinite it necessarily exists), *then* $\mathrm{pcf}(\mathfrak{a})\backslash\mu$ is finite and for some $\mathfrak{b} \in J_{<\lambda}[\mathfrak{a}]$, $\mu = \sup[\lambda\cap\mathrm{pcf}(\mathfrak{a}\backslash\mathfrak{b})]$ (and necessarily $\lambda = \max(\mathrm{pcf}(\mathfrak{a}\backslash\mathfrak{b}))$).

However, our terminology is somewhat misleading: $\mathfrak{b}_\theta[\mathfrak{a}]$ is not unique, it is defined only mod $J_{<\theta}[\mathfrak{a}]$. So it is natural to ask whether we can choose one (more exactly, a sequence) which is "nice". The suggested properties are: smooth if $\theta\in\mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda$, and closed if $\mathfrak{a}\cap\mathrm{pcf}(\mathfrak{b}_\lambda) = \mathfrak{b}_\lambda$. This is particularly interesting when $\mathfrak{a} = \mathrm{pcf}(\mathfrak{a})$ (so when $\min\mathfrak{a} > 2^{|\mathfrak{a}|}$, we know $|\mathrm{pcf}(\mathfrak{a})| < \min\ \mathfrak{a} = \min\ \mathrm{pcf}(\mathfrak{a})$, and by 1.12 $\mathrm{pcf}(\mathrm{pcf}(\mathfrak{a})) = \mathrm{pcf}(\mathfrak{a})$, so our theorem applies to $\mathfrak{a}' =: \mathrm{pcf}(\mathfrak{a})$).

Now we know that for each λ, we can find a $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing cofinal sequence of length λ in $\prod\mathfrak{b}_\lambda$ (or use $<_{J_{<\lambda}[\mathfrak{a}]+(\mathfrak{a}\backslash\mathfrak{b}_\lambda)}$ and $\prod\mathfrak{a}$, or other variant). For our end we want more specific properties; we define "nice", "continuous" such sequences (in Definition 3.3) then prove their existence (3.4(1)) and investigate what is the function $\theta\mapsto\sup(\theta\cap N)$ for appropriately closed elementary submodels N of large enough fragments $H(\chi)$ of the universe of sets (in 3.5) and use this to find smooth $\bar{\mathfrak{b}}$ (in 3.6) and under stronger conditions smooth and closed $\bar{\mathfrak{b}}$ (in 3.8).
In some possible representation this is central (as in my lectures on the subject in the Hebrew University, Spring 1989), deducing for example the localization theorem by formal manipulations of such generating sequences, but as the main conclusions have some restrictions, this will not be done here. On more see end of [AG 4.13].

* * *

We can replace systematically normal by semi-normal and $\mathfrak{b}_\lambda$ by $\langle\mathfrak{b}^\lambda_i : i<\lambda\rangle$ as in Definition 2.1, but avoid it to ease the reading.

Definition 3.1 (1) We say $\langle\mathfrak{b}_\lambda : \lambda\in\mathfrak{c}\rangle$ is a *generating sequence* for $\mathfrak{a}$ if:

(i) $\mathfrak{b}_\lambda \subseteq \mathfrak{a}$, $\mathfrak{c}\subseteq\mathrm{pcf}(\mathfrak{a})$

(ii) $J_{\le\lambda}[\mathfrak{a}] = (J_{<\lambda}[\mathfrak{a}]) + \mathfrak{b}_\lambda$.

(2) Let $J^{1,\kappa}_{<\lambda}[\mathfrak{a}]$ be the κ-complete ideal on $\mathcal{P}(\mathfrak{a})$ generated by $J_{<\lambda}[\mathfrak{a}]$.

(3) Let $\mathrm{pcf}^{1,\kappa}(\mathfrak{a}) = \{\lambda\in\mathrm{pcf}(\mathfrak{a}) : J^{1,\kappa}_{<\lambda}[\mathfrak{a}] \ne J^{1,\kappa}_{\le\lambda}[\mathfrak{a}]\}$. (See 3.1(6)).

(4) We say $\langle\mathfrak{b}_\lambda : \lambda\in\mathfrak{c}\rangle$ is a *weak generating sequence* for $\mathfrak{a}$ if:

(i) $\mathfrak{b}_\lambda\subseteq\mathfrak{a}$, $\mathfrak{b}_\lambda\notin J_{<\lambda}[\mathfrak{a}]$, $\mathfrak{b}_\lambda\in J_{\le\lambda}[\mathfrak{a}]$

(ii) $\mathfrak{c}\subseteq\mathrm{pcf}(\mathfrak{a})$.

(5) We say $\langle \mathfrak{b}_{\lambda,\alpha} : \alpha < \lambda \in \mathfrak{c}\rangle$ is a κ-*almost generating sequence* for $\mathfrak{a}$ if
(i) $\mathfrak{b}_{\lambda,\alpha} \subseteq \mathfrak{a}$, $\mathfrak{b}_{\lambda,\alpha} \in J_{<\lambda}[\mathfrak{a}]$, and $\alpha < \beta \Rightarrow \mathfrak{b}_{\lambda,\alpha} \subseteq \mathfrak{b}_{\lambda,\beta} \bmod J_{<\lambda}[\mathfrak{a}]$
(ii) $\mathfrak{c} \subseteq \mathrm{pcf}^{1,\kappa}(\mathfrak{a})$
(iii) $J^{1,\kappa}_{\leq\lambda}[\mathfrak{a}] = \left(J^{1,\kappa}_{<\lambda}[\mathfrak{a}]\right) + \{\mathfrak{b}_{\lambda,\alpha} : \alpha < \lambda\}$ whenever $\lambda \in \mathrm{pcf}^{1,\kappa}(\mathfrak{a})$.
If $\mathfrak{b}_{\lambda,\alpha} = \mathfrak{b}_\lambda$, we write $\langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$.

(6) In 2), 3), 5) if $\kappa = \aleph_1$, we omit it.

(7) We call $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ *smooth* if:
$\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda$.

(8) We call $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{a}\rangle$ closed if for each λ
$\mathfrak{b}_\lambda = \mathfrak{a} \cap \mathrm{pcf}(\mathfrak{b}_\lambda)$.

Fact 3.2 Let $|\mathfrak{a}|^+ < \min \mathfrak{a}$.

(1) $\lambda \in \mathrm{pcf}^1(\mathfrak{a})$ iff for some $\aleph_1$-complete ideal J on $\mathfrak{a}$, $\lambda = \mathrm{tcf}(\prod \mathfrak{a}, <_J)$. Similarly for $\lambda \in \mathrm{pcf}^{1,\kappa}(\mathfrak{a})$.

(2) There is an almost generating sequence $\langle \mathfrak{b}_{\lambda,\alpha} : \alpha < \lambda \in \mathrm{pcf}^1(\mathfrak{a})\rangle$ for $\mathfrak{a}$.

(3) There is a generating sequence $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ for $\mathfrak{a}$ if every $\lambda \in \mathrm{pcf}(\mathfrak{a})$ has a $(\lambda, \mathfrak{a})$-weakly continuity condition (see Definition 2.3(1)+(3)+(4)).

(4) An $\aleph_0$-almost generating sequence is a generating sequence.

(5) Suppose $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$, $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\mathfrak{a}$, and $\mathfrak{b} \subseteq \mathfrak{a}$, $\mathrm{pcf}(\mathfrak{b}) \subseteq \mathfrak{c}$ then for some finite $\mathfrak{d} \subseteq \mathfrak{c}$, $\mathfrak{b} \subseteq \bigcup_{\theta \in \mathfrak{d}} \mathfrak{b}_\theta$.

Proof: 1) If $\lambda \in \mathrm{pcf}^1(\mathfrak{a})$, i.e. $\lambda \in \mathrm{pcf}^{1,\aleph_1}(\mathfrak{a})$ (see 3.1(6)) this means $J^{1,\aleph_1}_{<\lambda}[\mathfrak{a}] \neq J^{1,\aleph_1}_{\leq\lambda}[\mathfrak{a}]$, i.e. $J^1_{<\lambda}[\mathfrak{a}] \neq J^1_{\leq\lambda}[\mathfrak{a}]$. So $J^1_{\leq\lambda}[\mathfrak{a}] \not\subseteq J^1_{<\lambda}[\mathfrak{a}]$ hence $J_{\leq\lambda}[\mathfrak{a}] \not\subseteq J^1_{<\lambda}[\mathfrak{a}]$. So choose $\mathfrak{b} \in J_{\leq\lambda}[\mathfrak{a}]$, $\mathfrak{b} \notin J^1_{<\lambda}[\mathfrak{a}]$, and let $J = J^1_{<\lambda}[\mathfrak{a}] + (\mathfrak{a}\backslash\mathfrak{b})$, obviously J is $\aleph_1$-complete. Apply 1.8(1).
The other direction is trivial too. (Use 1.8(3) and note that $J^1_{<\lambda}[\mathfrak{a}] \neq J^1_{\leq\lambda}[\mathfrak{a}]$ iff $J^1_{<\lambda}[\mathfrak{a}] \not\supseteq J_{\leq\lambda}[\mathfrak{a}]$).
2) By 2.11.
3) We can assume $\mathfrak{a}$ is infinite. Use 2.8.
4) Check.
5) If not, then $I = \{\mathfrak{b} \cap \bigcup_{\theta\in\mathfrak{d}} \mathfrak{b}_\theta : \mathfrak{d} \subseteq \mathfrak{c},\ \mathfrak{d}$ finite$\}$ is a family of subsets of $\mathfrak{b}$, closed under union, $\mathfrak{b} \notin I$, hence there is an ultrafilter D on $\mathfrak{b}$ disjoint from I. Let $\theta =: \mathrm{cf}(\prod \mathfrak{b}/D)$; necessarily $\theta \in \mathrm{pcf}(\mathfrak{b})$, hence $\theta \in \mathfrak{c}$. Let D' be the ultrafilter on $\mathfrak{a}$ which D generates, clearly $\theta = \mathrm{cf}(\prod \mathfrak{a}/D')$; by 1.8(3) we have $\mathfrak{b}_\theta \in D'$ hence $\mathfrak{b} \cap \mathfrak{b}_\theta \in D$, contradicting the choice of D. $\square_{3.2}$

Definition 3.3 (1) For a weak generating sequence $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ for $\mathfrak{a}$ we say
$\bar{f} = \langle\langle f_{\lambda,\alpha} : \alpha < \lambda\rangle : \lambda \in \mathfrak{c}\rangle$ is a cofinality sequence for $(\mathfrak{a}, \bar{\mathfrak{b}})$ if:
(i) $\langle f_{\lambda,\alpha} : \alpha < \lambda\rangle$ is strictly increasing mod $J_{<\lambda}[\mathfrak{a}]$ and cofinal in

$\left(\prod(\mathfrak{a}\cap\lambda^+), <_{J_{<\lambda}[\mathfrak{a}]+(\mathfrak{a}\setminus\mathfrak{b}_\lambda)}\right)$.

(ii) $f_{\lambda,0} = 0_{\mathfrak{a}\cap\lambda^+}$.

$(2)^a$ $\bar{f}$ is $\bar{S}-{}^a$continuous, where $\bar{S} = \langle S_\lambda : \lambda \in \mathfrak{c}\rangle$ and each S_λ is a stationary subset of λ if:

$(\mathrm{iii})^a$ $\delta \in S_\lambda \Rightarrow f_{\lambda,\delta}/J_{<\lambda}[\mathfrak{a}]$ is a $<_{J_{<\lambda}[\mathfrak{a}]}$ – lub of $\{f_{\lambda,\alpha}/J_{<\lambda}[\mathfrak{a}] : \alpha < \delta\}$.

We write μ instead of $\langle\{\delta < \lambda : \mathrm{cf}(\delta) = \mu\} : \lambda \in \mathfrak{c}\rangle$ and similarly $(> \mu)$. Instead of $(\geq \min \mathfrak{a})$ we write nothing. We add "weakly" if in (iii) we assume such lub exists. Let continuous mean ^{c}continuous (see below).

$(2)^b$ $\bar{f}$ is ^{b}continuous if:

$(\mathrm{iii})^b$ if $\delta < \lambda$, $|\mathfrak{a}| < \mathrm{cf}(\delta) < \min \mathfrak{a}$ then for $\theta \in \mathfrak{a}$

$$f_{\lambda,\delta}(\theta) = \min\Big\{ \bigcup_{\alpha\in C} f_{\lambda,\alpha}(\theta) : C \subseteq \delta \text{ is a club } \Big\}.$$

$(2)^c$ $\bar{f}$ is ^{c}continuous if:

$(\mathrm{iii})^c$ if $\delta < \lambda$, $|\mathfrak{a}| < \mathrm{cf}(\delta) < \min \mathfrak{a}$ then we first define

$$f^\epsilon_{\lambda,\delta} \in \prod(\mathfrak{a}\cap\lambda^+)$$

by induction on $\epsilon < \min \mathfrak{a}$, (and only later make our requirements on $f_{\lambda,\delta}$) :

(α) $\underline{\epsilon = 0}$, for $\theta \in \mathfrak{a}\cap\lambda^+$ we let
$f^\epsilon_{\lambda,\delta}(\theta) = \min\{\bigcup_{\alpha\in C} f_{\lambda,\alpha}(\theta) : C \subseteq \delta \text{ a club}\}$

(β) $\underline{\epsilon = \xi+1}$, for $\theta \in \mathfrak{a}\cap\lambda^+$ we let
$f^{\xi+1}_{\lambda,\delta}(\theta) = \sup\Big[\Big\{f^\xi_{\lambda,\delta}(\theta)\Big\} \cup \Big\{f_{\mu,f^\xi_{\lambda,\delta}(\mu)}(\theta) : \theta\leq\mu<\lambda, \mu\in\mathfrak{a}\Big\}\Big]$

(γ) $\underline{\epsilon \text{ limit}}$, for $\theta \in \mathfrak{a}\cap\lambda^+$ we let $f^\epsilon_{\lambda,\delta}(\theta) = \cup\{f^\zeta_{\lambda,\delta}(\theta) : \zeta < \epsilon\}$.

Lastly, $f_{\lambda,\delta} \in \prod(\mathfrak{a}\cap\lambda^+)$ is defined by:
for $\theta \in \mathfrak{a}\cap\lambda^+$, $f_{\lambda,\delta}(\theta) = \cup\{f^\epsilon_{\lambda,\delta}(\theta) : \epsilon < \min\ \mathfrak{a}\}$, except when it is equal to θ (possible only if $\theta = \min\ \mathfrak{a}$), and then $f_{\lambda,\delta}(\theta) = 0$[2].

$(2)^d$ For $\bar{\mathfrak{b}} = \langle\mathfrak{b}_\lambda : \lambda \in \mathfrak{a}\rangle$ a weak generating sequence, the notion "$\bar{f}$ is a ^{d}continuous cofinality sequence for $(\mathfrak{a}, \bar{\mathfrak{b}})$" is defined as in $(2)^c$, except that (β) is replaced by:

$(\beta)'$ $\underline{\epsilon = \xi+1}$ for $\theta \in \mathfrak{a}\cap\lambda^+$ we let

$$f^{\xi+1}_{\lambda,\delta}(\theta) = \sup\Big[\Big\{f^\xi_{\lambda,\delta}(\theta)\Big\} \cup \Big\{f_{\mu,f^\xi_{\lambda,\delta}(\mu)}(\theta) : \theta \leq \mu < \lambda, \mu \in \mathfrak{a}\Big\}$$
$$\cup \Big\{\min\{\gamma < \theta : f^\xi_{\lambda,\delta}\restriction(\mathfrak{a}\cap\theta^+) \leq f_{\theta,\gamma} \bmod J_{<\theta}[\mathfrak{a}] + (\mathfrak{a}\setminus\mathfrak{b}_\theta)\}\Big\}\Big].$$

[2]really $\langle f^\zeta_{\lambda,\delta} : \zeta < \min\ \mathfrak{a}\rangle$ is eventually constant, see later.

(3) $\bar{f}$ is ^{x}nice if it is ^{x}continuous and in addition:
   (iv) if $\alpha < \lambda$ and $\lambda \in \mathfrak{c}$, then
   $\theta \in \mathfrak{a} \cap \lambda^+ \ \& \ \sigma \in \mathfrak{a} \cap \theta^+ \Rightarrow f_{\theta, f_{\lambda,\alpha}(\theta)}(\sigma) \leq f_{\lambda,\alpha}(\sigma)$.

(4) Similar definitions for $\bar{\mathfrak{b}} = \langle \mathfrak{b}_{\lambda,\alpha} : \alpha < \lambda \in \mathfrak{c} \rangle$ as in Definition 3.1(5).

Remark 3.3A In [Sh345,7.3], another variant, in between ^{b}continuous and ^{c}continuous, is used.

Fact 3.4 Assume $|\mathfrak{a}| < \min \mathfrak{a}$.

(1) For every weak generating sequence $\bar{\mathfrak{b}}$ for $\mathfrak{a}$, ($\mathfrak{b} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c} \rangle$ or even $\langle \mathfrak{b}_{\lambda,\alpha} : \alpha < \lambda \in \mathfrak{c} \rangle$) some $\bar{f}$ is a ^{b}continuous cofinality sequence for $(\mathfrak{a}, \bar{\mathfrak{b}})$.

(2) If $\langle \langle f_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in \text{pcf}(\mathfrak{a}) \rangle$ is a cofinality sequence for $(\mathfrak{a}, \bar{\mathfrak{b}})$, $\bar{\mathfrak{b}}$ is a generating sequence for $\mathfrak{a}$ with domain $\text{pcf}(\mathfrak{a})$ then:

$(*)_2$ for every $g \in \prod \mathfrak{a}$ there are $n < \omega$, $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ from $\text{pcf}(\mathfrak{a})$ and $\alpha_\ell < \lambda_\ell$ for $\ell \leq n$ such that

$$g \leq \max\{f_{\lambda_\ell,\alpha_\ell} : \ell \leq n\}.$$

(3) In (2), if $\bar{\mathfrak{b}}$ is only a κ-almost generating sequence for $\mathfrak{a}$ (so its domain $\supseteq \text{pcf}^{1,\kappa}(\mathfrak{a})$) then:

$(*)_3$ for every $g \in \prod \mathfrak{a}$ there is a set $\mathfrak{b} \subseteq \text{pcf}^{1,\kappa}(\mathfrak{a})$ of cardinality $< \kappa$ and $\langle \alpha_\theta : \theta \in \mathfrak{b} \rangle$ such that $\alpha_\theta < \theta$ and

$$g < \sup\{f_{\lambda,\alpha_\lambda} : \lambda \in \mathfrak{b}\}$$

that is $(\forall \theta \in \mathfrak{a}) \bigvee_{\lambda \in \mathfrak{b}} g(\theta) < f_{\lambda,\alpha_\lambda}(\theta)$.

Proof: 1) We define $\langle f_{\lambda,\alpha} : \alpha < \lambda \rangle$ for each $\lambda \in \mathfrak{c}$. By 1.8(1) there is $\langle f^*_{\lambda,\alpha} : \alpha < \lambda \rangle$, $<_J$-increasing, where $J = (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)) \restriction (\mathfrak{a} \cap \lambda^+)$ and cofinal in $(\prod(\mathfrak{a} \cap \lambda^+), <_J)$. We now choose $f_{\lambda,\alpha}$ by induction on α such that:

(a) for $\alpha = 0$, $f_{\lambda,\alpha} = 0_{\mathfrak{a} \cap \lambda^+}$
(b) for α successor, $f^*_{\lambda,\alpha} \leq f_{\lambda,\alpha} \in \prod(\mathfrak{a} \cap \lambda^+)$
(c) for $\beta < \alpha$ $\ f_{\lambda,\beta} < f_{\lambda,\alpha}$ mod J
(d) if α is limit, $|\mathfrak{a}| < \text{cf}(\alpha) < \min \mathfrak{a}$, then (iii)^b of 3.3(2) holds.

The only problematic point is, why if $\alpha = \delta$, $|\mathfrak{a}| < \text{cf}(\delta) < \min \mathfrak{a}$, if we define $f_{\lambda,\delta}$ as required in (d), then it satisfies (c) and belongs to $\prod(\mathfrak{a} \cap \lambda^+)$. The latter holds as there is a closed unbounded $C \subseteq \delta$, with $\text{otp}(C) = \text{cf}(\delta) < \min \mathfrak{a}$, so $f_{\lambda,\alpha}(\theta) \leq \bigcup_{\beta \in C} f_{\lambda,\beta}(\theta) < \theta$ as $f_{\lambda,\beta}(\theta) < \theta$ and $\text{cf}(\theta) = \theta \geq \min\ \mathfrak{a} > |C|$.

For the first point (for $\beta < \alpha = \delta$, $f_{\lambda,\beta} < f_{\lambda,\delta}$ mod $J_{<\lambda}[\mathfrak{a}]$) for every $\theta \in \mathfrak{a} \cap \lambda^+$, for some club C_θ of δ we have

$(*)$ $f_{\lambda,\delta}(\theta) = \cup\{f_{\lambda,\beta}(\theta) : \beta \in C_\theta\}$.

We can find $\gamma \in \bigcap_{\theta \in \mathfrak{a} \cap \lambda^+} C_\theta$, $\gamma > \beta$; by the induction hypothesis $f_{\lambda,\beta} <_J f_{\lambda,\gamma}$, whereas by $(*)$ we have $f_{\lambda,\gamma} \le f_{\lambda,\delta}$. Together we finish.

(2) A particular case of part (3).

(3) Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$; for each $\lambda \in \mathfrak{c}$ we can find $\alpha = \alpha_\lambda < \lambda$ such that $g \restriction \mathfrak{b}_\lambda < f_{\lambda,\alpha} \restriction \mathfrak{b}_\lambda \mod J^{1,\kappa}_{<\lambda}$. Let $\mathfrak{b}^*_\lambda = \{\theta \in \mathfrak{b}_\lambda : g(\theta) < f_{\lambda,\alpha}(\theta)\}$, so $\mathfrak{b}^*_\lambda \subseteq \mathfrak{b}_\lambda$ and $\mathfrak{b}_\lambda \backslash \mathfrak{b}^*_\lambda \in J^{1,\kappa}_{<\lambda}$. If for some $\mathfrak{d} \subseteq \mathfrak{c}$, $|\mathfrak{d}| < \kappa$ and $\mathfrak{a} = \bigcup_{\lambda \in \mathfrak{d}} \mathfrak{b}^*_\lambda$, we are done; otherwise let J be the κ-complete filter on $\mathfrak{a}$ generated by $\{\mathfrak{b}^*_\lambda : \lambda \in \mathfrak{c}\}$, let μ be minimal in $\mathfrak{c}$ such that $J^{1,\kappa}_{\le\mu}[\mathfrak{a}] \not\subseteq J$.

Necessarily $\mu \in \mathrm{pcf}^{1,\kappa}(\mathfrak{a}) \subseteq \mathfrak{c}$, and choose $\mathfrak{d} \in J^{1,\kappa}_{<\mu^+}[\mathfrak{a}] \backslash J$ and even $\mathfrak{d} \in J_{<\mu^+}[\mathfrak{a}] \setminus J$; so $\mathfrak{d} \backslash \mathfrak{b}_\mu \in J^{1,\kappa}_{<\mu}[\mathfrak{a}] \subseteq J$ (by "$\bar{\mathfrak{b}}$ is a generating sequence" and the minimality of μ respectively) and $\mathfrak{b}_\mu \backslash \mathfrak{b}^*_\mu \in J^{1,\kappa}_{<\mu}[\mathfrak{a}] \subseteq J$ and $\mathfrak{b}^*_\mu \in J$, by the choice of $\mathfrak{b}_\mu$, see above. So $\mathfrak{d} \setminus \mathfrak{b}_\mu$, $\mathfrak{b}_\mu \setminus \mathfrak{b}^*_\mu$ and $\mathfrak{b}^*_\mu$ belong to J, hence $\mathfrak{d} \in J$, contradicting the choice of $\mathfrak{d}$.

$\square_{3.4}$

Claim 3.5 Suppose

(a) $|\mathfrak{a}|^+ < \min \mathfrak{a}$, $x \in \{b, c\}$

(b) $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta : \theta \in \mathfrak{a}\rangle$ is a weak generating sequence for $\mathfrak{a}$

(c) $\bar{f} = \langle < f_{\lambda,\alpha} : \alpha < \lambda >: \lambda \in \mathfrak{a}\rangle$ is a ^{x}continuous cofinality sequence for $(\mathfrak{a}, \bar{\mathfrak{b}})$ and if $x \in \{c\}$ then $f^\zeta_{\lambda,\delta}$ as in (iii)^x of 3.3(2^x).

(d) χ is large enough, $|\mathfrak{a}| < \sigma < \min \mathfrak{a}$, $\sigma = \mathrm{cf}(\sigma)$, $N_i \prec (H(\chi), \in, <^*_\chi)$ for $i \le \sigma$, $N_i \in N_{i+1}$, N_i increasing continuous in i, $\mathfrak{a} \in N_0$, $\bar{f} \in N_0$, $\mathfrak{a} \subseteq N_0$, $\|N_i\| < \min \mathfrak{a}$.

(e) Define $g_i \in \prod \mathfrak{a}$ by: $g_i(\theta) = \sup(N_i \cap \theta)$ (for $i \le \sigma$ and $\theta \in \mathfrak{a}$).

Then

(α) for $\lambda \in \mathfrak{a}$, $\delta \le \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$ we have $f_{\lambda, g_\delta(\lambda)} \le g_\delta \restriction (\mathfrak{a} \cap \lambda^+)$

(β) for $\lambda \in \mathfrak{a}$, $\delta \le \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$ we have $f_{\lambda, g_\delta(\lambda)} \restriction \mathfrak{b}_\lambda = g_\delta \restriction \mathfrak{b}_\lambda \mod J_{<\lambda}[\mathfrak{a}]$.

(γ) if $\bar{\mathfrak{b}}$ is a κ-almost generating sequence, $\delta \le \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$, $\mathfrak{a} = \mathrm{pcf}(\mathfrak{a}) = \mathrm{Dom}\, \bar{\mathfrak{b}}$, then for some $\mathfrak{d} \subseteq \mathfrak{a}$, $|\mathfrak{d}| < \kappa$ and $g_\delta = \max\{f_{\lambda, g_\delta(\lambda)} : \lambda \in \mathfrak{d}\}$.

(δ) if $x = c$, $\lambda \in \mathfrak{a}$ and $\langle \mathfrak{b}^\lambda_i : i < \lambda\rangle \in N_0$ is as in 1.7 for $\langle f_{\lambda,\alpha} : \alpha < \lambda\rangle$ and $\delta \le \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$ then $\mathfrak{d}_\lambda =: \{\theta \in \mathfrak{a} \cap \lambda^+ : f_{\lambda, g_\delta(\lambda)}(\theta) = g_\delta(\theta)\}$ satisfies $\mathfrak{d}_\lambda \in J_{\le\lambda}[\mathfrak{a}]$, moreover, for $i < g_\delta(\lambda)$:

$$\mathfrak{b}^\lambda_i \subseteq \mathfrak{d}_\lambda \subseteq \mathfrak{b}^\lambda_{g_\delta(\lambda)} \mod J_{<\lambda}[\mathfrak{a}].$$

[Hence if $J_{\le\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda$ then $\mathfrak{d}_\lambda = \mathfrak{b}_\lambda \mod J_{<\lambda}[\mathfrak{a}]$].

(ϵ) if $x = c$, $\delta \le \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$, $\lambda \in \mathfrak{a}$, $\gamma < \lambda$, $|\mathfrak{a}| < \mathrm{cf}(\gamma) < \min \mathfrak{a}$, $\gamma = g_\delta(\lambda)$ then for every large enough $\epsilon < \min \mathfrak{a}$, $f^\epsilon_{\lambda,\gamma} = f_{\lambda,\gamma}$ hence (iv) of 3.3(3) holds for $\bar{f}$ when $\min \mathfrak{a} > \mathrm{cf}(\alpha) > |\mathfrak{a}|$.

(ζ) if $\delta \leq \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$, $\lambda \in \mathfrak{a}$, then $f_{\lambda, g_\delta(\lambda)} \restriction \mathfrak{b}_\lambda$ is a $<_{J_{<\lambda}[\mathfrak{b}_\lambda]}$ − lub and even $<_{J_{<\lambda}[\mathfrak{b}_\lambda]}$-eub of $\{f_{\lambda,\alpha} \restriction \mathfrak{b}_\lambda : \alpha < g_\delta(\lambda)\}$.

Remark 3.5A (1) Using $J^{1,\kappa}_{<\lambda}[\mathfrak{a}]$, $(\lambda \in \mathrm{pcf}^{1,\kappa}(\mathfrak{a}))$ we have parallel results: if we restrict ourselves to $\mathrm{cf}(\delta) \in [\aleph_1, \kappa)$ the same continuity notion (i.e. as for $x = c$) is O.K. (i.e. in addition to $\mathrm{cf}(\delta) \in [|\mathfrak{a}|^+, \min \mathfrak{a})$).

(2) For $\mathrm{cf}(\delta) = \aleph_0$, we should have a preassigned unbounded $C_\delta \subseteq \delta$, otp $C_\delta = \omega$ for $\delta < \lambda$, $\mathrm{cf}(\delta) = \aleph_0$, and use $C \subseteq C_\delta$ in the definition of continuous.

(3) It follows that if $\lambda = \max \mathrm{pcf}(\mathfrak{a})$, $|\mathfrak{a}|^+ < \min \mathfrak{a}$ and $\langle f_{\lambda,\alpha} : \alpha < \lambda \rangle$ is a $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing cofinal in $\left(\prod \mathfrak{a}, <_{J_{<\lambda}[\mathfrak{a}]}\right)$, then
$S =: \{\delta < \lambda : |\mathfrak{a}| < \mathrm{cf}(\delta) < \min \mathfrak{a}$ and $\{f_{\lambda,\alpha} : \alpha < \delta\}$ has a $<_{J_{<\lambda}[\mathfrak{a}]}$-lub $\}$,
is not too small. *E.g.* no $S' \subseteq \{\delta < \lambda : |\mathfrak{a}| < \mathrm{cf}(\delta) < \min \ \mathfrak{a}\}$ disjoint to S is stationary and in $I[\lambda]$- see [Sh371,5.1A(6),(5)].

Proof: Note that if $i < j \leq \sigma$ then $g_i \in N_j$ so as $\mathfrak{a} \subseteq N_0$, clearly Rang $g_i \subseteq N_j$ hence $g_i < g_j$. As $\bar{f} \in N_i \prec N_j$ and $\mathfrak{a} \subseteq N_i \prec N_j$, for each $\theta \in \mathfrak{a}$ we have $g_i(\theta) \in N_j$ hence $f_{\theta, g_i(\theta)} \in N_j$ hence (as Dom $f_{\theta,g_i(\theta)} = \mathfrak{a} \cap \theta^+ \subseteq N_i \prec N_j$) we have Rang $f_{\theta,g_i(\theta)} \subseteq N_j$. By the definition of g_j this implies $f_{\theta,g_i(\theta)} \leq g_j \restriction (\mathfrak{a} \cap \theta^+)$.

Note that for $\theta \in \mathfrak{a}$, $\langle g_i(\theta) : i \leq \sigma \rangle$ is a strictly increasing continuous sequence of ordinals. So for limit $\delta \leq \sigma$,

$$\mathrm{cf}(g_\delta(\theta)) = \mathrm{cf}(\delta), \text{ and } C^\delta_\theta =: \{g_i(\theta) : i < \delta\} \text{ is a club of } g_\delta(\theta).$$

Now we shall prove:

Subclaim 3.5B If x, $\bar{f}$, $\mathfrak{a}$, $\bar{\mathfrak{b}}$, $\langle N_i : i \leq \sigma \rangle$ are as in 3.5, $\lambda \in \mathfrak{a}$, $\delta \leq \sigma$, $\mathrm{cf}(\delta) > |\mathfrak{a}|$ and γ belongs to the closure of $N_\delta \cap \lambda$, then:

(A) if $x \in \{b, c\}$ then $f_{\lambda,\gamma} \leq g_\delta \restriction (\mathfrak{a} \cap \lambda^+)$; moreover Rang $f_{\lambda,\gamma}$ is included in the closure of $N_\delta \cap \lambda$ (in the order topology)

(B) if $x = c$, $\|N_\delta\| \geq \mathrm{cf}(\gamma) > |\mathfrak{a}|$ and $\epsilon < \min \mathfrak{a}$ then $f^\epsilon_{\lambda,\gamma} \leq g_\delta \restriction (\mathfrak{a} \cap \lambda^+)$; moreover Rang $f^\epsilon_{\lambda,\gamma}$ is included in the closure of $N_\delta \cap \lambda$ (hence every sequence $\langle f^\epsilon_{\lambda,\gamma}(\alpha) : \epsilon < \min(\mathfrak{a}) \rangle$ is eventually constant).

This is enough for proving (α) (of 3.5), as $f_{\lambda, g_\delta(\lambda)} \leq g_\delta$ by (A) (because $g_\delta(\lambda)$ belongs to the closure of N_δ-by its definition); by the way, note as $g_\delta(\lambda) \notin N_\delta$, we have $\mathrm{cf}[g_\delta(\lambda)] \leq \|N_\delta\|$ (in fact is $\mathrm{cf}(\delta)$). It is also enough for (ϵ), as $\langle f^\epsilon_{\lambda,\gamma}(\theta) : \epsilon < \min \ \mathfrak{a} \rangle$ is non-decreasing whereas closure$(N_\delta \cap \lambda)$ has cardinality $\leq \|N_\delta\| < \min \mathfrak{a}$.

Proof of 3.5B: We prove, both parts, by induction on $\lambda \in \mathfrak{a}$.
Let λ be given. We first deal with (A); note that if $\gamma \in N_\delta$, then $f_{\lambda,\gamma} \in N_\delta$, hence (as $\mathfrak{a} \subseteq N_0 \subseteq N_\delta$) clearly Rang$(f_{\lambda,\gamma}) \subseteq N_\delta$, hence by the definition of g_δ, $f_{\lambda,\gamma} \leq g_\delta$ (and, of course, $N_\delta \cap \lambda \subseteq$closure$(N_\delta \cap \lambda)$), so (A) holds. On

the other hand if $\gamma \notin N_\delta$, as γ is in the closure of $N_\delta \cap \lambda$, and the closure of N_i is $\subseteq N_{i+1} \subseteq N_\delta$, necessarily $\langle \sup(N_i \cap \gamma) : i < \delta \rangle$ is strictly increasing hence $\mathrm{cf}(\gamma) = \mathrm{cf}(\delta)$ hence $\|N_\delta\| \geq \mathrm{cf}(\gamma) > |\mathfrak{a}|$, hence (B) will apply; now if we prove (B) then (A) follows when $x = c$, as by $(\mathrm{iii})^x$ of $3.3(2)^x$, for $\theta \in \mathfrak{a} \cap \lambda^+$:

$$f_{\lambda,\gamma}(\theta) = \cup\{f^\epsilon_{\lambda,\gamma}(\theta) : \epsilon < \min\ \mathfrak{a}\}$$

(if the latter is $< \theta$, but by (B) it is $\leq g_\delta(\theta) < \theta$ (remember $\|N_\delta\| \leq \sigma < \min(\mathfrak{a})$). When $x = b$, (A) amounts to (B) for $\epsilon = 0$.

So it suffices to prove (B) (for our λ).

We prove the statement in (B) by induction on ϵ.

First Case: $\epsilon = 0$

First we note that there is a strictly increasing continuous sequence $\langle \gamma_i : i < \mathrm{cf}(\gamma) \rangle$ of ordinals in $N_\delta \cap \lambda$ with limit γ.

[Why ? Subcase 1A: $\gamma \in N_\delta$. As $\mathrm{cf}(\gamma) \leq \|N_\delta\|$ (by assumption) it suffices to show $\|N_\delta\| \subseteq N_\delta$. For $i < \delta$, $N_i \in N_\delta$ and $N_i \prec N_\delta$ hence $\|N_i\| + 1 \subseteq N_\delta$; so the only case left is $\|N_\delta\| > \sup_{i<\delta} \|N_i\|$ so $\delta = \|N_\delta\|$. Now

$$\langle \min(\delta \setminus N_i) : i < \delta \rangle,$$

is a strictly increasing sequence of ordinals, so

$$N_\delta = \bigcup_{i<\delta} N_i \supseteq \bigcup_{i<\delta} (N_i \cap \delta) = \delta.$$

Subcase 1B: $\gamma \notin N_\delta$. Then necessarily $\mathrm{cf}(\gamma) = \mathrm{cf}(\delta)$, and

$$\langle \sup(N_i \cap \gamma) : i < \delta \rangle$$

is as required, as

$$\sup(N_i \cap \gamma) = \sup(N_i \cap (\min(N_i \cap \mathrm{Ord} \setminus \gamma)) \in N_{i+1} \subseteq N_\delta].$$

So by the definition of $f^0_{\lambda,\gamma}$, each $f^0_{\lambda,\gamma}(\theta)$ is $\cup\{f_{\lambda,\gamma_i}(\theta) : i \in C\}$ for some ($\equiv$ every small enough) club of $\mathrm{cf}(\gamma)$ and as $f_{\lambda,\gamma_i}(\theta) \in N_\delta$, it follows that $\mathrm{Rang}(f^0_{\lambda,\gamma})$ is a subset of the closure of $N_\delta \cap \lambda$.

Note that this also proves (α) for the case $x = b$.

Second case: $\epsilon = \xi + 1$

So,

$$f^\epsilon_{\lambda,\gamma}(\theta) = \sup\left[\{f^\xi_{\lambda,\gamma}(\theta)\} \cup \left\{f_{\mu, f^\xi_{\lambda,\gamma}(\mu)}(\theta) : \theta \leq \mu < \lambda, \mu \in \mathfrak{a}\right\}\right]$$

(see $(\mathrm{iii})^c$ of $3.3(2)^c$). As the closure of $N_\delta \cap \theta$ is closed, it is enough to show that every ordinal in the range of the sup is in this closure. Now $f^\xi_{\lambda,\gamma}(\theta)$

is, by the induction hypothesis on ϵ. As for $f_{\mu,f^{\xi}_{\lambda,\gamma}(\mu)}(\theta)$ we know (by the induction hypothesis on ϵ) that $f^{\xi}_{\lambda,\gamma}(\theta)$ is in the closure of $N_\delta \cap \theta$; by the induction hypothesis on λ, (part (A)) it follows that $f_{\mu,f^{\xi}_{\lambda,\gamma}(\mu)}(\theta)$ belongs to the closure, as required.

Third case: ϵ limit

As $f^{\zeta}_{\lambda,\gamma}(\theta) = \cup\{f^{\epsilon}_{\lambda,\gamma}(\theta) : \epsilon < \zeta\}$, as the closure of $N_\delta \cap \lambda$ is closed, and the induction hypothesis on ϵ, this is trivial.

So we have proved 3.5B, hence clauses (α), (ϵ) of 3.5. $\square_{3.5B}$

Continuation of the Proof of 3.5: for each $\lambda \in \mathfrak{a}$ and $i < j \le \sigma$, as $g_i \in (\prod \mathfrak{a}) \cap N_j$, for some $\alpha = \alpha(\lambda, i)$ we have:

$$\alpha \in \lambda \cap N_j \text{ and } \quad g_i < f_{\lambda,\alpha} \bmod (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)).$$

Now as $\alpha \in \lambda \cap N_j$, we have $\alpha < g_j(\lambda)$ so

$$f_{\lambda,\alpha} < f_{\lambda,g_j(\lambda)} \bmod \ (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda))$$

hence together

$$g_i < f_{\lambda,g_j(\lambda)} \bmod \ (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)).$$

So if $\delta \le \sigma$, $|\mathfrak{a}| < \mathrm{cf}(\delta)$, we have:

$$g_i < f_{\lambda,g_\delta(\lambda)} \bmod (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)) \text{ for each } i < \delta.$$

Let for $i \le \delta$,

$$\mathfrak{c}_i =: \{\theta \in \mathfrak{a} \cap \lambda^+ : g_i(\theta) > f_{\lambda,g_\delta(\lambda)}(\theta)\}.$$

Now as $[i < j \Rightarrow g_i \le g_j]$ we have $[i < j \Rightarrow \mathfrak{c}_i \subseteq \mathfrak{c}_j]$, so (as $\mathrm{cf}(\delta) > |\mathfrak{a}| = |\mathrm{Dom}\, g_i|$) clearly $\langle \mathfrak{c}_i : i < \delta \rangle$ is eventually constant; by the definition of the $\mathfrak{c}_j s$ and as $\langle g_j(\theta) : j \le \delta \rangle$ is continuously increasing clearly $\mathfrak{c}_\delta = \bigcup_{j<\delta} \mathfrak{c}_j$; hence we have $\mathfrak{c}_\delta = \mathfrak{c}_i$ for some $i < \delta$. But we have shown above that for $i < \delta$,

$\mathfrak{c}_i \in (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda))$ so $\mathfrak{c}_\delta \in (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda))$so

$\{\theta \in \mathfrak{a} \cap \lambda^+ : g_\delta(\theta) > f_{\lambda,g_\delta(\lambda)}(\theta)\} \in (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda))$so

$g_\delta \le f_{\lambda,g_\delta(\lambda)} \bmod (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda))$

As we have already proved clause (α) of 3.5:

$$g_\delta = f_{\lambda,g_\delta(\lambda)} \bmod (J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda))$$

i.e. we get clause (β).

Now clause (γ) is left to the reader (see 3.4(3)).

We try now to prove clause (δ) of 3.5. For a fixed λ let $g \in \prod \mathfrak{a}$ be as in 1.7$(D), (D)^+$; without loss of generality $g \in N_0$. Remember

$$\mathfrak{d}_\lambda =: \{\theta \in \mathfrak{a} \cap \lambda^+ : g_\delta(\theta) = f_{\lambda, g_\delta(\lambda)}(\theta)\}.$$

By the definition of $\mathfrak{d}_\lambda$ (and as $g < g_\delta$ because $g \in N_\delta$) we have:

$$\theta \in \mathfrak{d}_\lambda \Rightarrow g(\theta) < f_{\lambda, g_\delta(\lambda)}(\theta)$$

i.e. by 1.7$(D)^+$:

$(*)$ $\mathfrak{d}_\lambda \cap (\mathfrak{a} \backslash \mathfrak{b}^\lambda_{g_\delta(\lambda)}) \in J_{<\lambda}[\mathfrak{a}]$, i.e. $\mathfrak{d}_\lambda \subseteq \mathfrak{b}^\lambda_{g_\delta(\lambda)} \bmod J_{<\lambda}[\mathfrak{a}]$.

On the other hand by (E) of 1.7 (and 1.5) certainly for every $\alpha < \delta$, if $i \in \lambda \cap N_\alpha$, then the proof of (β) (of 3.5) holds also if we replace $\mathfrak{b}_\lambda$ by $\mathfrak{b}^\lambda_i$ hence

$$f_{\lambda, g_\delta(\lambda)} \restriction \mathfrak{b}^\lambda_i = g_\delta \restriction \mathfrak{b}^\lambda_i \bmod J_{<\lambda}[\mathfrak{a}],$$

hence $\mathfrak{b}^\lambda_i \subseteq \mathfrak{d}_\lambda \bmod J_{<\lambda}[\mathfrak{a}]$.

So we have finished proving clause (δ) of 3.5 by $(*)$ above.

Let us prove clause (ζ). By clause (β) it is enough to show that $g_\delta \restriction \mathfrak{b}_\lambda$ is a $<_{J_{<\lambda}[\mathfrak{b}_\lambda]}$-eubof $\{f_{\lambda,\alpha} : \alpha < g_\delta(\lambda)\}$. We first prove it is a $<_{J_{<\lambda}[\mathfrak{b}_\lambda]}$ $-$lub. For each $i < \sigma$ there is $\alpha(\lambda, i) \in \lambda \cap N_{i+1}$ such that $g_i \restriction \mathfrak{b}_\lambda < f_{\lambda,\alpha(\lambda,i)} \bmod J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)$, and as $\alpha(\lambda, i) \in \lambda \cap N_{i+1}$, clearly $f_{\lambda,\alpha(\lambda,i)} < g_{i+1}$. Let

$$\mathfrak{c}^*_i =: \{\theta \in \mathfrak{b}_\lambda : \text{ not "}g_i(\theta) < f_{\lambda,\alpha(\lambda,i)}(\theta) < g_{i+1}(\theta)\text{"}\},$$

so clearly $\mathfrak{c}^*_i \in J_{<\lambda}[\mathfrak{a}]$. As:

$$[\alpha < g_\delta(\lambda) \Rightarrow \text{ there is } \beta,\ \alpha < \beta \in N_\delta \cap \lambda],$$

$$[\alpha < \beta \in N_\delta \cap g_\delta(\lambda) \Rightarrow f_{\lambda,\alpha} < f_{\lambda,\beta} < g_\delta \bmod J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)],$$

we have that $g_\delta \restriction \mathfrak{b}_\lambda$ is *a* $<_{J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)}$-upper bound of $\{f_{\lambda,\alpha} \restriction \mathfrak{b}_\lambda : \alpha < g_\delta(\lambda)\}$. So suppose $g' \in \prod \mathfrak{a}$ is another $<_{J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)}$-upper bound of $\{f_{\lambda,\alpha} : \alpha < g_\delta(\lambda)\}$. Let for $i < \delta$,

$$\mathfrak{c}^{**}_i =: \mathfrak{c}^*_i \cup \{\theta \in \mathfrak{b}_\lambda : f_{\lambda,\alpha(\lambda,i)}(\theta) \geq g'(\theta)\}.$$

Clearly $\mathfrak{c}^{**}_i \in J_{<\lambda}[\mathfrak{a}]$. Define $g^* \in \prod(\mathfrak{a} \cap \lambda^+)$ by:

$$g^*(\theta) =: \sup\{f_{\lambda,\alpha(\lambda,i)}(\theta) : \theta \notin \mathfrak{c}^{**}_i, i < \delta\}.$$

So by the definition of $\mathfrak{c}^{**}_i$, $g^* \restriction \mathfrak{b}_\lambda \leq g'$ and as $\mathfrak{c}^{**}_i \in J_{<\lambda}[\mathfrak{a}]$ we have

$$f_{\lambda,\alpha(\lambda,i)} \leq g^* \bmod J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda).$$

So without loss of generality $g' = g^*$; let

$$\mathfrak{c}^+ = \{\theta \in \mathfrak{b}_\lambda : g^*(\theta) < g_\delta(\theta)\};$$

assume $\mathfrak{c}^+ \notin J_{<\lambda}[\mathfrak{a}]$ and we shall get a contradiction, thus finishing the proof of clause (ζ) hence of 3.5. For each $\theta \in \mathfrak{c}^+$, by the definition of $g_\delta(\theta)$ (and as $\langle g_j(\theta) : j \le \sigma\rangle$ is increasing continuous) there is $j_\theta < \delta$ such that $g^*(\theta) < g_{j_\theta}(\theta)$; let $j =: \sup\{j_\theta : \theta \in \mathfrak{c}^+\}$, so $j < \delta$ (as $\mathrm{cf}(\delta) > |\mathfrak{a}| \ge |\mathfrak{c}^+|$), and

$$\{\theta \in \mathfrak{b}_\lambda : g^*(\theta) < g_j(\theta)\} \notin J_{<\lambda}[\mathfrak{a}]$$

so

$$\neg[g^* > f_{\lambda,\alpha(\lambda,j)} \bmod J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\lambda)],$$

contradiction. We leave the proof of its being $<_{J_{<\lambda}[\mathfrak{b}_\lambda]}$ −eub to the reader. $\square_{3.5}$

Lemma 3.6 *Suppose* $|\mathfrak{a}| < \min \mathfrak{a}$, $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{a}\rangle$ *is a weak generating sequence for* $\mathfrak{a}$.
Then we can find $\bar{\mathfrak{b}}' = \langle \mathfrak{b}'_\lambda : \lambda \in \mathfrak{a}\rangle$, $\bar{f} = \langle\langle f_{\lambda,\alpha} : \alpha < \lambda\rangle : \lambda \in \mathfrak{a}\rangle$ *such that:*

(α) $\bar{\mathfrak{b}}'$ *is a smooth weak generating sequence for* $\mathfrak{a}$

(β) *for* $\lambda \in \mathfrak{a}$, $\mathfrak{b}_\lambda \subseteq \mathfrak{b}'_\lambda \bmod J_{<\lambda}[\mathfrak{a}]$

(γ) $\bar{f}$ *is a* ^{c}*nice cofinality system for* $(\mathfrak{a}, \bar{\mathfrak{b}}')$.

Proof: Let $\bar{f} = \langle\langle f^*_{\lambda,\alpha} : \alpha < \lambda\rangle : \lambda \in \mathfrak{a}\rangle$ be a ^{b}continuous cofinality system for $(\mathfrak{a}, \bar{\mathfrak{b}})$ (exist by 3.4(1)). We now define by induction on $\lambda \in \mathfrak{a}$ the sequence $\langle f_{\lambda,\alpha} : \alpha < \lambda\rangle$. We define $f_{\lambda,\alpha}$ by induction on α such that:

(1) $f^*_{\lambda,\alpha+1} \le f_{\lambda,\alpha+1} \in \prod(\mathfrak{a} \cap \lambda^+)$

(2) for $\beta < \alpha$ $f_{\lambda,\beta} \restriction \mathfrak{b}_\lambda < f_{\lambda,\alpha} \restriction \mathfrak{b}_\lambda \bmod J_{<\lambda}[\mathfrak{a}]$

(3) if $\alpha < \lambda$, $\mathrm{cf}(\alpha) \le |\mathfrak{a}|$ or $\mathrm{cf}(\alpha) \ge \min \mathfrak{a}$ we choose $f_{\lambda,\alpha}$ satisfying the relevant cases of (1) and (2) and, if possible:
$(*)$ $\theta \in \lambda \cap \mathfrak{a} \Rightarrow f_{\theta, f_{\lambda,\alpha}(\theta)} \le f_{\lambda,\alpha} \restriction (\mathfrak{a} \cap \theta^+)$.

(4) if $\alpha < \lambda$, $|\mathfrak{a}| < \mathrm{cf}(\alpha) < \min \mathfrak{a}$ then define:
$f^0_{\lambda,\alpha}(\theta) =: \min \left\{\bigcup_{\beta \in C} f_{\lambda,\beta}(\theta) : C \text{ a club of } \alpha\right\}$
$f^{\zeta+1}_{\lambda,\alpha}(\theta) =: \sup \left[\{f^\zeta_{\lambda,\alpha}(\theta)\} \cup \left\{f_{\mu, f^\zeta_{\lambda,\alpha}(\mu)}(\theta) : \theta \le \mu < \lambda, \mu \in \mathfrak{a}\right\}\right]$
$f^\zeta_{\lambda,\alpha}(\theta)$, for ζ limit is $\cup\{f^\epsilon_{\lambda,\alpha}(\theta) : \epsilon < \zeta\}$.

Let $f_{\lambda,\alpha}(\theta) = \cup\{f^\zeta_{\lambda,\alpha}(\theta) : \zeta < \min \mathfrak{a}\}$ except if it is θ, then $f_{\lambda,\alpha}(\theta) = 0$. There are no problems in this.
Clearly $\bar{f}$ is a ^{c}continuous cofinality system ((4) and (2) are compatible: proof as in 3.4).

Next choose χ large enough $\sigma =: |\mathfrak{a}|^+$ and $\langle N_i : i \le \sigma\rangle$ continuously increasing, $N_i \prec (H(\chi), \in, <^*_\chi)$, $\|N_i\| = |\mathfrak{a}|^+$, $|\mathfrak{a}|^+ \subseteq N_i$, $N_i \in N_{i+1}$ and $\{\bar{f}, \bar{\mathfrak{b}}, \mathfrak{a}\} \in N_0$.

Now apply 3.5(α), (β) for $\delta = \sigma$, $\lambda \in \mathfrak{a}$. We can now show that in (3) above, ($*$) was always possible: if not there is a minimal λ for which it fails and then a minimal α. So (λ, α) is definable from parameters which belongs to N_0, hence $(\lambda, \alpha) \in N_0$. Now $g_\sigma \restriction (\mathfrak{a} \cap \lambda^+)$ shows ($*$) is possible — by (α) of 3.5. Moreover ($*$) now holds also if $\alpha < \lambda$, $|\mathfrak{a}| < \mathrm{cf}(\alpha) < \min \mathfrak{a}$ by (ϵ) of 3.5 and Definition 3.3(2)^c. So $\bar{f}$ is ^{c}continuous and even ^{c}nice (check Definition 3.3(3)). Now let $\mathfrak{b}'_\lambda =: \{\theta \in \mathfrak{a} \cap \lambda^+ : g_\sigma(\theta) = f_{\lambda, g_\sigma(\lambda)}(\theta)\}$, they are as required: $\mathfrak{b}_\lambda \subseteq \mathfrak{b}'_\lambda$ mod $J_{<\lambda}[\mathfrak{a}]$ by clause (β) of 3.5 and smoothness (of $\langle \mathfrak{b}'_\lambda : \lambda \in \mathfrak{a}\rangle$) follows from ^{c}niceness because

$$\mathfrak{b}'_\lambda = \{\theta \in \mathfrak{a} \cap \lambda^+ : g_\sigma(\theta) \le f_{\lambda, g_\sigma(\lambda)}(\theta)\}$$

which holds again by clause (α) of 3.5.
Also $\mathfrak{b}'_\lambda \in J_{\le\lambda}[\mathfrak{a}]$, because $\left(\prod \mathfrak{a}, <_{J_{\le\lambda}[\mathfrak{a}]}\right)$ is λ^+-directed so there is $f \in (\prod \mathfrak{a}) \cap N_0$ such that:

$$\alpha < \lambda \Rightarrow f_{\lambda,\alpha} < f \bmod J_{\le\lambda}[\mathfrak{a}].$$

But $f \in N_0 \Rightarrow f < g_\sigma$.
This implies $f_{\lambda, g_\sigma(\lambda)} < g_\sigma$ mod $J_{\le\lambda}[\mathfrak{a}]$ hence $\mathfrak{b}'_\lambda \in J_{\le\lambda}[\mathfrak{a}]$.

So clause (α), (β) of 3.6 holds. As for clause (γ)- the proof above that it holds for $\bar{\mathfrak{b}}$ apply also for $\bar{\mathfrak{b}}'$. Use 3.5(δ) and 1.7 to see that $\langle f_{\lambda,\alpha}; \alpha < \lambda\rangle$ is still cofinal mod $J_{<\lambda}[\mathfrak{a}] + (\mathfrak{a} \setminus \mathfrak{b}'_\lambda)$. $\square_{3.6}$

Lemma 3.7 *If* $|\mathfrak{a}| < \min \mathfrak{a}$, $\lambda = \max[\mathrm{pcf}(\mathfrak{a})]$, $\sup[\lambda \cap \mathrm{pcf}(\mathfrak{a})]$ *is singular, then for every unbounded* $\mathfrak{c} \subseteq \lambda \cap \mathrm{pcf}(\mathfrak{a})$ *of power* $< \min\ \mathfrak{c}$,

$$\lambda = \mathrm{tcf}(\Pi \mathfrak{c}, <_{J^{\mathrm{bd}}_{\mathfrak{c}}}).$$

Proof: First $\max \mathrm{pcf}(\mathfrak{c}) \le \lambda$ as $\mathrm{pcf}(\mathfrak{c}) \subseteq \mathrm{pcf}(\mathfrak{a})$ by 1.12. If $\neg\left[\mathrm{tcf}\left(\prod \mathfrak{c}, <_{J^{\mathrm{bd}}_{\mathfrak{c}}}\right) = \lambda\right]$, then $J_{<\lambda}[\mathfrak{c}] \not\subseteq J^{\mathrm{bd}}_{\mathfrak{c}}$ (by 1.8(1)) so for some $\mathfrak{d} \subseteq \mathfrak{c}$; $\mathfrak{d} \notin J^{\mathrm{bd}}_{\mathfrak{c}}$ and $\theta =: \max \mathrm{pcf}(\mathfrak{d}) < \lambda$.
Now $\left(\prod \mathfrak{d}, <_{J^{\mathrm{bd}}_{\mathfrak{c}}}\right)$ is $\sup(\mathfrak{d})$-directed, so $\theta \ge \sup(\mathfrak{d})$; $\sup \mathfrak{d}$ is singular, so $\sup \mathfrak{d} < \theta < \lambda$. Now $\mathfrak{d} \subseteq \mathrm{pcf}(\mathfrak{a})$ and $|\mathfrak{d}| \le |\mathfrak{c}| < \min\ \mathfrak{c} \le \min\ \mathfrak{d}$, hence $\mathrm{pcf}(\mathfrak{d}) \subseteq \mathrm{pcf}(\mathfrak{a})$ by 1.12, but $\theta \in \mathrm{pcf}(\mathfrak{d})$ so $\theta \in \mathrm{pcf}(\mathfrak{a})$ and so

$$\theta \le \sup(\mathrm{pcf}(\mathfrak{a}) \cap \lambda) = \sup \mathfrak{c} = \sup \mathfrak{d} < \theta < \lambda$$

— contradiction. $\square_{3.7}$

Claim 3.8 (1) Suppose (with $\kappa = 2^{|\mathfrak{a}|}$) :

(a) $\kappa^+ < \min \mathfrak{a}$ and $x = d$

(b) $\langle \mathfrak{b}_\theta : \theta \in \mathfrak{a}\rangle$ is a weak generating sequence for $\mathfrak{a}$

(c) $\bar{f} = \langle\langle f_{\lambda,\alpha} : \lambda < \lambda\rangle : \lambda \in \mathfrak{a}\rangle$ is a ^{x}continuous cofinality sequence for $(\mathfrak{a}, \bar{\mathfrak{b}})$, and $f^\zeta_{\lambda,\delta}(|\mathfrak{a}| < \mathrm{cf}(\delta) < \min\ \mathfrak{a}, \zeta < \min \mathfrak{a})$ is as in (iii)^x

of 3.3(2)^x.

(d) χ large enough, $|\mathfrak{a}| < \mathrm{cf}(\sigma) = \sigma \leq \kappa^+$, $N_i \prec (H(\chi), \in, <^*_\chi)$ for $i \leq \sigma$, $N_i \in N_{i+1}$, $\kappa \subseteq N_i$, N_i increasing continuous in i, $\mathfrak{a} \in N_0$, $\bar{f} \in N_0$, $\langle N_i : i \leq j \rangle \in N_{j+1}$ and $\|N_i\| \leq |\kappa| + |i|$.

Then: (α)-(ζ) of 3.5 holds omitting "if $x = c$" in (δ), (ϵ).

(2) The parallel of 3.5B holds.

(3) If in 3.6 we add "$2^{|\mathfrak{a}|} < \min \mathfrak{a}$", we can strengthen in the conclusion:

$(\alpha)^+$ $\bar{\mathfrak{b}}'$ is a smooth closed generating sequence ("weak" disappears)

$(\gamma)^+$ $\bar{f}$ is a ^{d}nice cofinality system for $(\mathfrak{a}, \bar{\mathfrak{b}}')$.

(4) In (1) — (3) instead of "$\kappa = 2^{|\mathfrak{a}|} < \min \mathfrak{a}$" we can demand: $\kappa = |\mathfrak{a}|^+$ and "for every $\lambda \in \mathfrak{a}$, there is $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = |\mathfrak{a}|^+\}$ stationary and weakly good".

(5) In (1) — (3) instead of "$\kappa = 2^{|\mathfrak{a}|} < \min \mathfrak{a}$" we can add $|\mathfrak{a}| < \kappa < \min \mathfrak{a}$ and for every $\lambda \in \mathfrak{a}$ and $A \subseteq \lambda$ of power κ, E_A has $\leq \kappa$ equivalence classes, where:

for α, $\beta < \lambda$ we have: $\alpha \, E_A \beta$ iff
$\{\theta \in \mathfrak{a} : \min(A \backslash f_{\lambda,\alpha}(\theta)) \neq \min(A \backslash f_{\lambda,\beta}(\theta))\} \in J_{<\lambda}[\mathfrak{a}]$.

Proof: Clearly (1) — (3) are particular cases of (5) as E_A has $\leq |A|^{|\mathfrak{a}|} = \kappa^{|\mathfrak{a}|}$ classes. Also (4) is a particular case of (5) by the proofs in §2. Lastly, the proof of (5) is just like the proof of 3.5, 3.6, the only difference is that in the parallel to 3.5B we have one more thing to prove in the successor stage.

$\square_{3.8}$