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SPECTRA OF MONADIC SECOND ORDER SENTENCES

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Introduction

This continues [GuSh 536] and was announced there. For a monadic second order sentence ψ in the language with one unary functions and unary predicates, the spectra of the sentence (i.e., the set $\{||M|| : M$ a finite model of $\psi\}$ is (see [GuSh: 536]) periodic, but this fail badly when we allow, e.g. two unary functions. In the second section we characterize the family of finite structures which really behave like the unary function case.
In section one we assume that a monadic second order sentence satisfies: every model is not indecomposable, i.e., has a non trivial decoposition in a weak sense (see Definition 1.2). We conclude that the specra is not arbitrary, mainly - there are no big gaps in it (from some point on). This is of course considerably weaker conclusion than what we know for the languages with only a unary function (under a much weaker assumption).
Subsequently, (but before the release of this paper) Fischer and Makowsky [FiMw03] continues [GuSh 536] in a different direction, using counting monadic logic and dealing with width of graphs (and of models).
It seems that Definition 2.2 is a variant of "clique width of models"; see on this [FiMw03]. Clearly we can in §1 use operations like [2.2] instead of $M_1 \cup M_2$.

§1 Weakly decomposable

We can deal just with graphs just a this is traditional. The restriction to relational vocabulary.

1.1 Context. 1) Let τ be a finite relational vocabulary, i.e., a finite set of predicates this is for simplifying our statement.
2) Let $\mathfrak{K}^*_\tau$ be the class of τ-models and recall $\|M\|$ is the number of elements of $M \in \mathfrak{K}^*_\tau, R^M$ is the interpretation of $R \in \tau$.
3) Let $\mathfrak{K}$ denote a family of τ-models closed under isomorphisms.

1.2 Definition. 1) We say that $\mathfrak{K}$ is weakly k-decomposable <u>if</u>: for every m there is n such that

$\circledast_{k,m,n}$ if $M \in \mathfrak{K}, |M| \geq n$ <u>then</u> we can find submodels M_1, M_2 (for graphs-induced subgraphs G_1, G_2) such that

- (a) $M_1 \cup M_2 = M$ (i.e., $a \in M \Leftrightarrow a \in M_1 \cup a \in M_2$ and $R^{M_1} = F^{M_1} \cup R^{M_2}$ for any $R \in \tau$ (for graphs: G, G_1, G_2 let $G = G_1 \cup G_2$ mean that the set of nodes is the union of the set of nodes of G_1 and of G_2, and the set of edges of G is the union of the set of edges of G_1 and of G_2)
- (b) $|M_1 \cap M_2| \leq k$
- (c) $|M_\ell| \geq m$ for $\ell = 1, 2$.

2) For a monadic second order sentence ψ (in a vocabulary τ) we say that ψ is k-decomposable <u>if</u> $\mathfrak{K}^\tau_\psi$ is (see part (3)).
3) For a vocabulary τ (as in 1.1) and sentence ψ (in this vocabulary) let $\mathfrak{K}^\tau_\psi = \{M : M$ is a finite τ-model such that $M \models \psi\}$. We may suppress τ, when clear from the context.

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1.3 Claim. *Assume*

$(*)^{k^*}_\psi$ *ψ a monadic second order sentence, in the vocabulary $\tau, \mathfrak{K} = \mathfrak{K}^\tau_\psi$ is k^*-decomposable*

<u>then</u> $\mathrm{Sp}(\psi) = \{|M| : M \in \mathfrak{K}\}$ *satisfies for some n^*, that*

$\circledast$ *if $n_1 < n_2$ are successive members* $\mathrm{Sp}(\psi)$ *and $n^* < n_1$ then $n_2 < 2n_1$.*

Proof. Let ψ have quantifier depth $\leq d^*$.
Let $m^*_1 > k^*$ be large enough such that

$\boxdot_1$ if $M_1 \in \mathfrak{K}^*_\tau, \|M_1\| > k^*$ and $a_1, \dots, a_k \in M_1$ and $k \leq k^*$ <u>then</u> there is $M_2 \in \mathfrak{K}$ and $b_1, \dots, b_k \in M_2$ such that
$\mathrm{Th}^{d^*}(M_1, a_1, \dots, a_k) = \mathrm{Th}^{d^*}(M_2, b_1, \dots, b_k)$
and $k^* < |M_2| < m^*_1$.

Let m^*_2 be such that the statement $\circledast_{k^*, m^*_1, m^*_2}$ from Definition 1.2 holds (for $\mathfrak{K}$).

Now assume that $n_1 < n_2$ are successive members of $\mathrm{Sp}(\psi)$ and $n_1 > m^*_2$. Hence there is $M \in \mathfrak{K}$ with exactly n_2 members. So applying $\circledast_{k^*, m^*_1, m^*_2}$ to M we can find M_1, M_2 as in Definition 1.2 and let $\{a_1, \dots, a_k\}$ list $M_1 \cap M_2$; so $k \leq k^*$ and $|M_1|, |M_2| \geq m^*_1$. Without loss of generality $|M_1| \leq |M_2|$ still $|M_1| \geq m^*_1$.
By the choice of m^*_1 there is $(M'_1, b_1, \dots, b_k)$ such that $k^* < \|M'_1\| < m^*_1$ and

$$\mathrm{Th}^{d^*}(M'_1, b_1, \dots, b_k) = \mathrm{Th}^{d^*}(M_1, a_1, \dots, a_2).$$

Without loss of generality $\ell \in \{1, \dots, k\} \Rightarrow b_\ell = a_\ell$ and no member (for graphs - node) of M'_1 belongs to $M_2 \backslash \{a_1, \dots, a_2\}$. Let $M' = M'_1 \underset{\{a_1, \dots, a_k\}}{+} M_2$ be defined naturally (set of elements of M' = union of set of elements of M'_1 and set of elements of $M_2, R^M = R^{M'_1} \cup R^{M_2}$ for $R \in \tau$).

By the addition theorem $M' \models \psi$, i.e., $M' \in \mathfrak{K}$ and

$\boxdot$ $\frac{1}{2}\|M\| \leq \|M_2\| < \|M'\| = \|M'_1\| + \|M_2\| - k < m^*_1 + \|M_2\| - k \leq \|M_1\| + \|M_2\| - k < \|M\|$.

That is $n_2/2 < \|M'\| < n_2$ but $M' \in \mathfrak{K}$ so $\|G'\| \in \mathrm{Sp}(\mathfrak{K}), \|M'\| < n_2$ hence $\|M'\| \leq n_1$ so $n_2/2 < n_1$ so we are done. $\square_{1.3}$

1.4 Conclusion. If φ is a second order monadic sentence and $(*)^{k^*}_\varphi$ and α is a real > 0 <u>then</u> for every n large enough

$$n \in \mathrm{Sp}(\varphi) = (\exists m \in (\mathrm{Sp}(\varphi))[n < m < (1 + \alpha)n].$$

Proof. By Claim 1.3.

Let Ξ be the family of positive reals α such that

$\circledast_1$ for every monadic second order sentence ψ (for any vocabulary τ as in 1.1) such that $(*)^{k^*}_\psi$ holds, the conclusion of 1.3 holds
(no harm in varying k^*, too).

Note that allowing individual constants in τ is O.K. (either allow them or code them by unary predicates); for a vocabulary τ let τ^{+k} be $\tau + k$ individual constants.
Clearly $0 < \beta < \alpha$ & $\beta \in \Xi \Rightarrow \alpha \in \Xi$. By Claim 1.3 we have $1 \in \Xi$.
We shall now prove that

$\circledast_2$ if $\alpha \in \Xi \Rightarrow \alpha/2 \in \Xi$.

This clearly suffices. Before proving $\circledast_2$ note that given ($\tau, \mathfrak{K}$ and ψ), let d be above the quantifier depth of ψ. For $k \leq k^*$ let

$$\begin{aligned}\mathfrak{K}'_k = \{(M', a_1, \ldots, a_k) :&\text{for some } M \in \mathfrak{K} \text{ and } M_1, M_2 \text{ as in 1.2}\\ &\text{with } |M_1 \cap M_2| \leq k^* \text{ we have } M' = M\\ &\text{and } \{a_1, \ldots, a_k\} \text{ lists } M_1 \cap M_2\}.\end{aligned}$$

This is a class of τ^{+k} models. Let $\{\mathrm{Th}^d(M', a_1, \ldots, a_k) : (M', a_1, \ldots, a_k) \in \mathfrak{K}'_k\}$ be listed as $\mathbf{t}_1, \ldots, \mathbf{t}_m$ and for $\ell \in \{1, \ldots, m\}$ let $\mathfrak{K}'_{k,\ell} = \{(M', a_1, \ldots, a_k) \in \mathfrak{K}'_k : \mathrm{Th}^d(M', a_1, \ldots, a_k) = \mathbf{t}_\ell\}$.
It is not hard to see

$\circledast_3$ for some monadic second order sentence ψ_ℓ of quantifier depth $d, \mathfrak{K}'_{k,\ell}$ is the class of models of ψ_ℓ and $(*)^{k^*}_{\psi_\ell}$ holds.

Let us prove $\circledast_2$ so $\alpha, \tau, \mathfrak{K}, \psi, d$ are given and let $\mathfrak{K}'_k, \mathfrak{K}'_{k,\ell}$ be as above. Now for any $M \in \mathfrak{K}$ by the proof of 1.3, as we are assuming that $\alpha \in \Xi$ we can choose M'_1 such that

$$\frac{\|M_1\|}{1+\alpha} < \|M'_1\| < \|M_1\|$$

hence

$$\begin{aligned}\|M'\| &= \|M_2\| + \|M'_1\| - k > \|M_2\| + \frac{\|M_1\|}{1+\alpha} - k\\ &= \frac{1}{(1+\alpha)}(\|M_2\| + \alpha\|M_2\| + \|M_1\| - k - \alpha k)\\ &= \frac{1}{1+\alpha}(\|M\| + \alpha\|M_2\| - \alpha k)\\ &\geq \frac{1}{1+\alpha}(\|M\| + \alpha\|M\|/2) - \frac{\alpha k}{1+\alpha} = \frac{1+\alpha/2}{1+\alpha}|M| - \frac{\alpha k}{1+\alpha}\end{aligned}$$

so we conclude: if conclusion 1.4 holds for $\alpha > 0$ it holds for $\alpha/2$ because

$$\beta = \frac{1+\alpha}{1+\alpha/2} - 1 = \frac{\alpha/2}{1+\alpha/2} < \alpha/2.$$

So we can prove by induction on i that it holds for $\alpha \geq \frac{1}{2^i}$. $\square_{1.4}$

§2 WHAT THE METHOD OF [GuSh 536] GIVES

2.1 Discussion: The result above is interesting but leave us unsatisfied. For trees we get essentially sharp results. Here the spectra is not characterized. We know that it is quite restricted but, e.g. is it almost periodic?

The problem is that we do not see here a parallel to the operations generating the class.

We may consider such classes:

2.2 Definition. Let τ and k^* be fixed and let $\mathfrak{K}_{k^*}$ be the minimal family of $(M, a_1, \ldots, a_k)$, M a finite τ-model $k \leq k^*, a_\ell \in M$ such that

(a) K_{k^*} includes those with $\leq k^*$ elements

(b) if $(M_\ell, a_1^\ell, \ldots, a_{k_\ell}^\ell) \in \mathfrak{K}_{k^*}$ for $\ell = 1, 2$ and $x \in M_1 \wedge x \in M_2 \Rightarrow x \in \{a_1^1, \ldots, a_{k_1}^1\} \cap \{a_1^2, \ldots, a_{k_2}^2\}$ then $(M, b_1, \ldots, b_k) \in \mathfrak{K}_{k^*}$ when:

$\circledast$ (i) x an element of $M \Rightarrow x$ an element node of M_1 or of M_2

(ii) x an element of $M_\ell, x \notin \{a_1^\ell, \ldots, a_{k_2}^\ell\} \Rightarrow x$ an element of M

(iii) $\{b_1, \ldots, b_k\} \subseteq \{a_2^1, \ldots, a_{k_1}^1\} \cup \{a_1^2 \ldots a_{k_2}^2\}$

(iv) if $R \in \tau$ is k-place predicate, and $y_1, \ldots, y_k \in M, z_1, \ldots, z_k \in M$ then $\langle y_1, \ldots, y_k \rangle \in R^M \equiv \langle z_1, \ldots, z_k \rangle \in R^M$ when:

($\boxdot$) $(z_i = z_j) \equiv (y_i = y_j), (z_i = a_\ell^1) \equiv (y_i = a_\ell^1)$ $(z_i = a_\ell^2) \equiv (y_i = a_\ell^2), (z_i \in M_\ell) \equiv (y_i \in M_\ell)$ and letting $w_\ell = \{i : y_i \in G_\ell\}$ the quantifier free type of $\langle y_i : i \in w_\ell \rangle$ in M_ℓ is equal to the quantifier free type of $\langle z_i : i \in w_\ell \rangle$ in M_ℓ.

2.3 Claim. *We can prove for K_{k^*} what we have proved for trees; including almost periodically of the spectrum.*

2.4 Question: Is the class K_{k^*} known? Interesting?

REFERENCES

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