Archive for Mathematical Logic (2020) 59:879–892
https://doi.org/10.1007/s00153-020-00721-y

Mathematical Logic

A version of κ-Miller forcing

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Received: 12 December 2018 / Accepted: 14 February 2020 / Published online: 20 February 2020

Abstract
We consider a version of κ-Miller forcing on an uncountable cardinal κ. We show that under $2^{<\kappa} = \kappa$ this forcing collapses 2^κ to ω and adds a κ-Cohen real. The same holds under the weaker assumptions that $\mathrm{cf}(\kappa) > \omega$, $2^{2^{<\kappa}} = 2^\kappa$, and forcing with $([\kappa]^\kappa, \subseteq)$ collapses 2^κ to ω.

Keyword Forcing with higher perfect trees

Mathematics Subject Classification Primary 03E05; Secondary 03E04 · 03E15

1 Introduction

Many of the tree forcings on the classical Baire space have various analogues for higher cardinals. Here we are concerned with Miller forcing [4]. In the classical case, a Miller condition is a superperfect subtree of $\omega^{<\omega}$. The subtree is ordered by the end-extension relation on $\omega^{<\omega}$. The forcing order is simply $\subseteq$. A tree is superperfect if each node has an extension that has infinitely many immediate tree successors. Such a node is called a splitting node. We can assume that each node has just one direct successor or infinitely many.

For a κ-version of Miller forcing, superperfectness and splitting are usually interpreted as follows: Above each node $t \in p \subseteq \kappa^{<\kappa}$ there is a node splitting node s. The common interpretation of "s is a splitting node of p" is:

This research, No. 1154 on the second author's list, was partially supported by European Research Council Grant 338821.

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$$\{\alpha \in \kappa \,:\, s^\frown\langle\alpha\rangle \in p\} \text{ contains a club subset of } \kappa.$$

In order to gain $(<\kappa)$-closure of the notion of forcing, in addition to the club version of superperfectness one usually requires for conditions that (see, e.g., [2, Section 5.2]) limits of length less than κ of splitting nodes be splitting nodes as well.

In this paper we investigate a version of κ-Miller forcing where the conditions on superperfectness and $(<\kappa)$-closure of splitting nodes are kept and the definition of "s is a splitting node of p" is weakened to

$$|\{\alpha \,:\, s^\frown\langle\alpha\rangle \in p\}| = \kappa.$$

We show: If $\mathrm{cf}(\kappa) > \omega$, $\mathrm{cf}(\kappa) = \kappa$ or $\mathrm{cf}(\kappa) < 2^{\mathrm{cf}(\kappa)} \le \kappa$, $2^{2^{<\kappa}} = 2^\kappa$, and there is a κ-mad family of size 2^κ, then this variant of Miller forcing is related to the forcing $([\kappa]^\kappa, \subseteq)$ and collapses 2^κ to ω. In particular, if $\omega < \kappa^{<\kappa} = \kappa$, then our four premises are fulfilled. Thus we provide some mathematical justification of the customary choice of higher Miller forcing.

Throughout the paper we let κ be an uncountable cardinal. We do not make the general assumption that $2^{<\kappa} = \kappa$, nor do we assume that κ is regular.

We denote forcing orders in the form $(\mathbb{P}, \le_{\mathbb{P}})$ and let $q \le_{\mathbb{P}} p$ mean that q is *stronger* than p.

If $\mathrm{dom}(t)$, i are ordinals, we write $t^\frown\langle i\rangle$ for the concatenation of t with the singleton function $\{(0, i)\}$, i.e., $t^\frown\langle i\rangle = t \cup \{(\mathrm{dom}(t), i)\}$. For cardinals κ, λ, we write ${}^{<\lambda}\kappa$ for the set of functions $f\colon \alpha \to \kappa$ for some $\alpha < \lambda$. For $s, t \in \kappa^{<\lambda}$ we write $s \trianglelefteq t$ if $s = t \restriction \mathrm{dom}(s)$, and the corresponding strict order is written as $\triangleleft$. The domain α of f is also called the length of f. The set of subsets of κ of size κ is denoted by $[\kappa]^\kappa$.

Definition 1.1 (1) $\mathbb{Q}^1_\kappa$ is the forcing $([\kappa]^\kappa, \subseteq)$.

(2) $\mathbb{Q}^2_\kappa$ is the following version of κ-Miller forcing: Conditions are trees $T \subseteq {}^{\kappa>}\kappa$ that are κ *superperfect*: for each $s \in T$ there is $s \trianglelefteq t$ such that t is a κ-splitting node of T. A node $t \in T$ is called a κ-*splitting node* if

$$\mathrm{set}_p(t) = \{\alpha < \kappa \,:\, t^\frown\langle\alpha\rangle \in T\}$$

has size κ. The set of splitting nodes of T is denoted by $\mathrm{spl}(T)$.
We furthermore require for $p \in \mathbb{Q}^2_\kappa$ that the limit of an $\triangleleft$-increasing sequence of length less than κ of κ-splitting nodes is a κ-splitting node if it has length less than κ.
For $p, q \in \mathbb{Q}^2_\kappa$ we write $q \le_{\mathbb{Q}^2_\kappa} p$ if $q \subseteq p$. So subtrees are stronger conditions.

(3) For $p \in \mathbb{Q}^2_\kappa$ and $\eta \in p$ we let $\mathrm{suc}_p(s) = \{t \in {}^{\kappa>}\kappa \,:\, (\exists\alpha < \kappa)(t = s^\frown\langle\alpha\rangle \in p)\}$.

(4) Let $s \in p \in \mathbb{Q}^2_\kappa$. We let $p^{\langle s\rangle} = \{t \in p \,:\, t \trianglelefteq s \vee s \trianglelefteq t\}$.

(5) For $a, b \subseteq \kappa$ we write $a \subseteq^*_\kappa b$ if $|a \setminus b| < \kappa$.

Each of the two forcing orders $\mathbb{P}$ has a weakest element, denoted by $1_{\mathbb{P}}$. Namely, $\mathbb{Q}^1_\kappa$ has as a weakest element $1_{\mathbb{Q}^1_\kappa} = \kappa$, and $\mathbb{Q}^2_\kappa$ has as a weakest element the full tree ${}^{\kappa>}\kappa$. We write $\mathbb{P} \Vdash \varphi$ if the weakest condition forces φ.

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2 Results about $\mathbb{Q}^1_\kappa$

In this section we consider $\mathbb{Q}^1_\kappa$. The purpose is to provide standardised $\mathbb{Q}^1_\kappa$-names for collapses. Later these particular $\mathbb{Q}^1_\kappa$-names shall be translated to $\mathbb{Q}^2_\kappa$-names.

Definition 2.1 A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called a κ-almost disjoint family if for $A \neq B \in \mathcal{A}$, $|A \cap B| < \kappa$.

Observation 2.2 *If $2^{<\kappa} = \kappa$, there is a κ-ad family $\mathcal{A} \subseteq [\kappa]^\kappa$ of size 2^κ.*

Proof We let $f : {}^{\kappa>}2 \to \kappa$ be an injection. We assign to each branch b of ${}^{\kappa>}2$ a set $a_b = \{f(s) : s \in b\}$. The resulting family $\{a_b : b \text{ branch of } {}^{\kappa>}2\}$ is κ-ad. □

Observation 2.3 *If $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω, then there is a κ-ad family of size 2^κ.*

Proof $\mathbb{Q}^1_\kappa$ cannot have the 2^κ-c.c. Hence there is an antichain of size 2^κ. Since $p \perp_{\mathbb{Q}^1_\kappa} q$ means $|p \cap q| < \kappa$, the antichain is a κ-ad family. □

We will apply the following result for $\chi = 2^\kappa$.

Theorem 2.4 [5, Theorem 0.5] *Suppose that there is an antichain in $\mathbb{Q}^1_\kappa$ of size χ. Then the following holds.*

(1) *Forcing with $\mathbb{Q}^1_\kappa$ collapses χ to $\aleph_0$ if $\aleph_0 < \mathrm{cf}(\kappa) = \kappa$ or if $\aleph_0 < \mathrm{cf}(\kappa) < 2^{\mathrm{cf}(\kappa)} \leq \kappa$.*
(2) *Forcing with $\mathbb{Q}^1_\kappa$ collapses χ to $\aleph_1$ in the case of $\aleph_0 = \mathrm{cf}(\kappa) < \kappa$.*

Now we start defining tree structures from $\mathbb{Q}^1_\kappa$-names for collapsing functions. Those trees will later be used to define dense suborders $Q_\mathcal{T}$ of $\mathbb{Q}^2_\kappa$. The idea of $Q_\mathcal{T}$ is that the sets $\mathrm{set}_p(t)$, $t \in \mathrm{spl}(p)$, for $p \in Q_\mathcal{T}$ will be sufficiently strong $\mathbb{Q}^1_\kappa$ conditions.

Lemma 2.5 *Suppose that $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω. Then there is a $\mathbb{Q}^1_\kappa$-name $\underset{\sim}{\tau} : \aleph_0 \to 2^\kappa$ for a surjection, and there is a labelled tree $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho_\eta) : \eta \in {}^{\omega>}(2^\kappa)\rangle$ with the following properties*

(a) *$a_{\langle\rangle} = \kappa$ and for any $\eta \in {}^{\omega>}(2^\kappa)$, $a_\eta \in [\kappa]^\kappa$.*
(b) *$\eta_1 \lhd \eta_2$ implies $a_{\eta_1} \supseteq a_{\eta_2}$.*
(c) *$n_\eta \in [\mathrm{dom}(\eta) + 1, \omega)$.*
(d) *If $a \in [\kappa]^\kappa$ then there is some $\eta \in {}^{\omega>}(2^\kappa)$ such that $a \supseteq a_\eta$.*
(e) *If $\eta^\frown\langle\beta\rangle \in T$ then $a_{\eta^\frown\langle\beta\rangle}$ forces $\underset{\sim}{\tau} \restriction n_\eta = \varrho_{\eta^\frown\langle\beta\rangle}$ for some $\varrho_{\eta^\frown\langle\beta\rangle} \in {}^{n_\eta}(2^\kappa)$, such that the $\varrho_{\eta^\frown\langle\beta\rangle}$, $\beta \in 2^\kappa$, are pairwise different. Hence for any $\eta \in {}^{\omega>}(2^\kappa)$, the family $\{a_{\eta^\frown\langle\alpha\rangle} : \alpha < 2^\kappa\}$ is a κ-ad family in $[a_\eta]^\kappa$.*

Proof Let $\underset{\sim}{\tau}$ be a $\mathbb{Q}^1_\kappa$-name such that $\mathbb{Q}^1_\kappa \Vdash \underset{\sim}{\tau} : \aleph_0 \to 2^\kappa$ is onto. For $\alpha < 2^\kappa$ let AP_α be the set of objects $\bar{m}$ satisfying

$(*)_1$ (1.1) $\bar{m} = (T, \bar{a}, \bar{n}, \bar{\varrho}) = (T_{\bar{m}}, \bar{a}_{\bar{m}}, \bar{n}_{\bar{m}}, \bar{\varrho}_{\bar{m}})$.
(1.2) T is a subtree of $({}^{\omega>}(2^\kappa), \lhd)$ of cardinality $\leq |\alpha| + \kappa$ and $\langle\rangle \in T$.
(1.3) $\bar{a} = \langle a_\eta : \eta \in T\rangle$ fulfils $\eta \lhd \nu \to a_\nu \subseteq a_\eta$ and $a_{\langle\rangle} = \kappa$ and $a_\eta \in [\kappa]^\kappa$.
(1.4) $\bar{n} = \langle n_\eta : \eta \in T\rangle$ fulfils $\mathrm{dom}(\varrho_{\eta^\frown\langle\beta\rangle}) = n_\eta > \mathrm{dom}(\eta)$ for any $\eta^\frown\langle\beta\rangle \in T$.

(1.5) If $\eta\hat{}\langle\beta\rangle \in T$, then $a_{\eta\hat{}\langle\beta\rangle}$ forces a value to $\underset{\sim}{\tau} \restriction n_\eta$, called $\varrho_{\eta\hat{}\langle\beta\rangle}$, and for $\beta \neq \gamma$ we have $\varrho_{\eta\hat{}\langle\beta\rangle} \neq \varrho_{\eta\hat{}\langle\gamma\rangle}$. Hence for any $\eta\hat{}\langle\beta\rangle, \eta\hat{}\langle\gamma\rangle \in T_{\bar{m}}$, $\beta \neq \gamma$ implies $a_{\eta\hat{}\langle\beta\rangle} \cap a_{\eta\hat{}\langle\gamma\rangle} \in [\kappa]^{<\kappa}$.
(1.6) For $\eta \in T_{\bar{m}}$, we let

$$\mathrm{Pos}(a_\eta, n_\eta) = \{\varrho \in {}^{n_\eta}(2^\kappa) \,:\, a_\eta \not\Vdash_{\mathbb{Q}^1_\kappa} \underset{\sim}{\tau} \restriction n_\eta \neq \varrho\},$$

and require that the latter has cardinality 2^κ.

In the next items we state some properties of AP_α that are derived from $(*)_1$.

$(*)_2$ $AP = \bigcup\{AP_\alpha \,:\, \alpha < 2^\kappa\}$ is ordered naturally by $\leq_{AP}$, which means end extension.
$(*)_3$ (a) AP_α is not empty and increasing in α.
(b) For infinite α, AP_α is closed under unions of increasing sequences of length $< |\alpha|^+$.
$(*)_4$ Let $\gamma < 2^\kappa$. If $\bar{m} \in AP_\gamma$ and $\eta \in T_{\bar{m}}$ and $\eta\hat{}\langle\alpha\rangle \notin T_{\bar{m}}$ then there is $\bar{m}' \in AP_\gamma$ such that $\bar{m} \leq_{AP} \bar{m}'$ and $T_{\bar{m}'} = T_{\bar{m}} \cup \{\eta\hat{}\langle\alpha\rangle\}$.
Proof: For $\eta \in T_{\bar{m}}$,

$$\mathcal{U} = \mathrm{Pos}(a_\eta, n_\eta) = \{\varrho \in {}^{n_\eta}(2^\kappa) \,:\, a_\eta \not\Vdash_{\mathbb{Q}^1_\kappa} \underset{\sim}{\tau} \restriction n_\eta \neq \varrho\} \text{ has size } 2^\kappa,$$

whereas

$$\Lambda_\eta = \{\varrho_{\eta\hat{}\langle\beta\rangle} \restriction n_\eta \,:\, \beta \in 2^\kappa \wedge \eta\hat{}\langle\beta\rangle \in T_{\bar{m}}\}$$

is of size $\leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Hence we can choose $\varrho_* \in \mathcal{U} \setminus \Lambda_\eta$ and $b_* \in [a_\eta]^\kappa$ such that $b_* \Vdash_{\mathbb{Q}^1_\kappa} \varrho_* = \underset{\sim}{\tau} \restriction n_\eta$. We let $\varrho_{\eta\hat{}\langle\alpha\rangle} = \varrho_*$. Since b_* forces a value of $\tau \restriction n_\eta$ that is incompatible with the one forced by $a_{\eta\hat{}\langle\beta\rangle}$ for any $\eta\hat{}\langle\beta\rangle \in T_{\bar{m}}$, the set b_* is κ-almost disjoint from $a_{\eta\hat{}\langle\beta\rangle}$ for any $\eta\hat{}\langle\beta\rangle \in T_{\bar{m}}$. We take $b_* = a_{\bar{m}',\eta\hat{}\langle\alpha\rangle} \subseteq a_{\bar{m},\eta}$.

Since $\mathrm{cf}(2^\kappa) > \aleph_0$ and since

$$|\{\mathrm{range}(\varrho) \,:\, \varrho \in {}^{\omega>}(2^\kappa) \wedge b_* \not\Vdash_{\mathbb{Q}^1_\kappa} \underset{\sim}{\tau} \restriction n \neq \varrho\}| = 2^\kappa,$$

there is an n such that

$$\mathrm{Pos}(b_*, n) = \{\varrho \in {}^{n}(2^\kappa) \,:\, b_* \not\Vdash_{\mathbb{Q}^1_\kappa} \underset{\sim}{\tau} \restriction n \neq \varrho\}$$

has cardinality 2^κ. We take the minimal one and let it be $n_{\eta\hat{}\langle\alpha\rangle}$.

$(*)_5$ If $\bar{m} \in AP_\alpha$ and $a \in [\kappa]^\kappa$ then there is some $\bar{m}' \geq \bar{m}$, such that there is $\eta \in T_{\bar{m}'}$ with $a_{\bar{m}',\eta} \subseteq a$.
Let

$$\mathcal{U}_a = \{\varrho \in {}^{\omega>}(2^\kappa) \,:\, a \not\Vdash_{\mathbb{Q}^1_\kappa} \varrho \ntriangleleft \underset{\sim}{\tau}\},$$

i.e.

$$\mathcal{U}_a = \{\varrho \in {}^{\omega>}(2^\kappa) : (\exists b \geq_{\mathbb{Q}^1_\kappa} a)(b \Vdash_{\mathbb{Q}^1_\kappa} \varrho \lhd \underset{\sim}{\tau})\}.$$

This set has cardinality 2^κ because $\mathbb{Q}^1_\kappa \Vdash \underset{\sim}{\tau} : \omega \to 2^\kappa$ is onto. We take n minimal such that

$$\mathcal{U}_{a,n} = \{\varrho \in {}^{n}(2^\kappa) : (\exists b \geq_{\mathbb{Q}^1_\kappa} a)(b \Vdash_{\mathbb{Q}^1_\kappa} \varrho \lhd \underset{\sim}{\tau})\}$$

has size 2^κ. We let

$$\text{set}^+_n(\bar{m}) = \{\varrho_\eta : \eta \in T_{\bar{m}}, \text{dom}(\varrho_\eta) \geq n\}.$$

Clearly $|\text{set}^+_n(\bar{m})| \leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Thus we can take $\varrho_a \in \mathcal{U}_{a,n}$ that is incompatible with every element of $\text{set}^+_n(\bar{m})$. We take some $b_a \in [a]^\kappa$ such that $b_a \Vdash_{\mathbb{Q}^1_\kappa} \varrho_a \trianglelefteq \underset{\sim}{\tau}$. The set

$$\Lambda_a = \{\eta \in T_{\bar{m}} : b_a \subseteq^*_\kappa a_\eta\}$$

is $\lhd$-linearly ordered by $(*)_1$ clauses 1.3 and 1.5 and $\langle\rangle \in \Lambda_a$. Since b_a does not pin down $\underset{\sim}{\tau}$, Λ_a has a $\lhd$-maximal member η_a. Now we take $\alpha_* = \min\{\beta : \eta_a{}^\frown\langle\beta\rangle \notin T_{\bar{m}}\}$. For any $\eta_a{}^\frown\langle\beta\rangle \in T_{\bar{m}}$ we have $\varrho_{\eta_a{}^\frown\langle\beta\rangle}$ and ϱ_a are incompatible, and hence $a_{\eta_a{}^\frown\langle\beta\rangle} \cap b_a \in [\kappa]^{<\kappa}$. Now we choose $b^1_a \in [b_a]^\kappa$ and ϱ^*_a such that $b^1_a \Vdash_{\mathbb{Q}^1_\kappa} \varrho^*_a \lhd \underset{\sim}{\tau}$ and $\text{dom}(\varrho^*_a) \geq n_{\bar{m},\eta_a} > \text{dom}(\eta_a)$.
We let

$$T_{\bar{m}'} = T_{\bar{m}} \cup \{\eta_a{}^\frown\langle\alpha_*\rangle\},$$
$$a_{\eta_a{}^\frown\langle\alpha_*\rangle} = b^1_a,$$

We let $n_{\eta_a{}^\frown\langle\alpha_*\rangle}$ be the minimal n such that $|\text{Pos}(b^1_a, n)| \geq 2^\kappa$. So $(*)_5$ holds.

Now we are ready to construct $\mathcal{T}$ as in the statement of the lemma. We do this by recursion on $\alpha \leq 2^\kappa$. First we enumerate $[\kappa]^\kappa$ as $\langle c_\alpha : \alpha < 2^\kappa\rangle$, and we enumerate ${}^{\omega>}(2^\kappa)$ as $\langle \eta_\alpha : \alpha < 2^\kappa\rangle$ such that $\eta_\alpha \lhd \eta_\beta$ implies $\alpha < \beta$. We choose an increasing sequence $\bar{m}_\alpha$ by induction on $\alpha < 2^\kappa$. We start with the tree $\{\langle\rangle\}$, $a_{\langle\rangle} = \kappa$, $\varrho_{\langle\rangle} = \emptyset$, $n_{\langle\rangle}$ be minimal such that $|\text{Pos}(\kappa, n)| = 2^\kappa$. In the odd successor steps we take $\bar{m}_{2\alpha+1} \geq_{AP} \bar{m}_\alpha$ so that $a_\eta \subseteq c_\alpha$ for some $\eta \in T_{2\alpha+1}$. This is done according to $(*)_5$. In the even successor steps we take $\bar{m}_{2\alpha+2} \geq_{AP} \bar{m}_{2\alpha+1}$ such that $\eta_\alpha \in T_{2\alpha+2}$. Since all initial segments of η_α appeared among the η_β, $\beta < \alpha$, $\bar{m}_{2\alpha+2}$ is found according to $(*)_4$. In the limit steps we take unions. Then $\mathcal{T}$ that is given by the last three components of $\bar{m}_{2^\kappa}$ has properties (a) to (e). □

Since $\tau = \underset{\sim}{\tau}[G]$ is not in $\mathbf{V}$, for any $\mathcal{T}$ as in Lemma 2.5, for any $f \in {}^\omega(2^\kappa) \cap \mathbf{V}$, the branch $\langle(a_{f\restriction m}, n_{f\restriction m}, \varrho_{f\restriction m}) : m \in \omega\rangle$ of $\mathcal{T}$ has a no $\subseteq^*_\kappa$-lower bound for its first coordinate.

3 Transfer to $\mathbb{Q}^2_\kappa$

In this section we use the tree $\mathcal{T}$ from Lemma 2.5 for finding $\mathbb{Q}^2_\kappa$-names. First we establish a dense subforcing $Q_\mathcal{T}$ of $\mathbb{Q}^2_\kappa$. Then we construct $Q_\mathcal{T}$-names that are based on a $\mathbb{Q}^1_\kappa$-name of a collapse and on the equation $2^{2^{<\kappa}} = 2^\kappa$.

Definition 3.1 Let μ, λ be cardinals. For $\nu, \nu' \in {}^{\lambda>}\mu$ we write $\nu \perp \nu'$ if $\nu \ntrianglelefteq \nu'$ and $\nu' \ntrianglelefteq \nu$.

Typical pairs (λ, μ) are $(\omega, 2^\kappa)$ and (κ, κ).

An important tool for the analysis of $\mathbb{Q}^2_\kappa$ is the following particular kind of fusion sequence $\langle p_\alpha : \alpha < \kappa^{<\kappa}\rangle$ in $\mathbb{Q}^2_\kappa$. Since we do not suppose $\kappa^{<\kappa} = \kappa$, a fusion sequence can be longer than κ. An important property is that for each $\nu \in {}^{\kappa>}\kappa$ there is at most one $\alpha < \kappa^{<\kappa}$ such that $\mathrm{set}_{p_\alpha}(\nu) \supsetneq \mathrm{set}_{p_{\alpha+1}}(\nu)$.

Lemma 3.2 *Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa}\rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that*

$$\nu_\alpha \triangleleft \nu_\beta \rightarrow \alpha < \beta. \tag{3.1}$$

Let $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha < \kappa^{<\kappa}\rangle$ be a sequence such that for any $\alpha \leq \lambda$ the following holds:

(a) $p_0 \in \mathbb{Q}^2_\kappa$.
(b1) *If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \in \mathrm{spl}(p_\beta)$, then*

$$c_\beta \in [\mathrm{suc}_{p_\beta}(\nu_\beta)]^\kappa \text{ and}$$
$$p_\alpha = p_\beta(\nu_\beta, c_\beta) := \bigcup\{p_\beta^{\langle \nu_\beta{}^\frown\langle i\rangle\rangle} : i \in c_\beta\} \cup \bigcup\{p_\beta^{\langle\eta\rangle} : \eta \ntrianglelefteq \nu_\beta \wedge \nu_\beta \ntrianglelefteq \eta\}$$

(b2) *If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \notin \mathrm{spl}(p_\beta)$ then $p_\alpha = p_\beta$.*
(c) $p_\alpha = \bigcap\{p_\beta : \beta < \alpha\}$ *for limit $\alpha \leq \kappa^{<\kappa}$.*

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}^2_\kappa$ and $\forall \beta < \lambda$, $p_\lambda \leq_{\mathbb{Q}^2_\kappa} p_\beta$.

Proof We go by induction on λ. The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \leq \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}^2_\kappa$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \trianglerighteq t$ that is a splitting node in p_λ.

We fix the smallest α such that $\nu_\alpha \trianglerighteq_{p_0} t$ is a splitting node in p_0. Then in p_0 there are no splitting nodes in $\{s : t \trianglelefteq s \triangleleft \nu_\alpha\}$. Hence $\nu_\alpha \in \mathrm{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle \nu^i : i < \gamma\rangle$ be an $\triangleleft$-increasing sequence of splitting nodes of p_λ with union $\nu \in \kappa^{<\kappa}$. Then ν is a splitting node of each p_α, $\alpha < \lambda$, and also in p_λ since $\langle \mathrm{set}_{p_\alpha}(\nu) : \alpha < \lambda\rangle$ has at most two entries and their intersection has size κ. □

We use yet another, richer type of fusion sequence.

Definition 3.3 Let $p \in \mathbb{Q}^2_\kappa$ and let $\nu \in \mathrm{spl}(p)$.

(1) We say η is *the shortest splitting node above ν in p* and write $\eta = \mathrm{sucspl}_p(\nu)$ if η is the shortest splitting node in p such that $\eta \supseteq \nu$. Equality is allowed and occurs if ν is a splitting node.
(2) We say $F \subseteq p$ is *the front of next splitting nodes above ν in p*, if

$$F = \{\eta' \in \mathrm{spl}(p) \,:\, \exists(\eta \in \mathrm{suc}_p(\nu))(\eta' = \mathrm{sucspl}_p(\eta))\}.$$

Lemma 3.4 *Let $\langle \nu_\alpha \,:\, \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that*

$$\nu_\alpha \lhd \nu_\beta \to \alpha < \beta. \tag{3.2}$$

Let $\langle p_\alpha, \nu_\alpha, c_\alpha, F_\alpha \,:\, \alpha < \kappa^{<\kappa} \rangle$ be a sequence such that for any $\alpha \le \lambda$ the following holds:

(a) $p_0 \in \mathbb{Q}^2_\kappa$.
(b1) *If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \in sp(p_\beta)$, then $c_\beta \in [\mathrm{suc}_{p_\beta}(\nu_\beta)]^\kappa$, F_β contains for each $i \in c_\beta$ exactly one $\eta \in \mathrm{spl}(p_\beta^{\langle \nu_\beta{}^\frown \langle i \rangle \rangle})$, and*

$$\begin{aligned} p_\alpha = p_\beta(\nu_\beta, c_\beta, F_\beta) := & \bigcup \{p_\beta^{\langle \eta \rangle} \,:\, i \in c_\beta, \eta \in F_\beta\} \\ & \cup \bigcup \{p_\beta^{\langle \eta \rangle} \,:\, \eta \not\trianglelefteq \nu_\beta \wedge \nu_\beta \not\trianglelefteq \eta\}. \end{aligned}$$

Note that this implies that F_β is the front of next splitting nodes of p_α above ν_β.
(b2) *If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \notin \mathrm{spl}(p_\beta)$ then $p_\alpha = p_\beta$.*
(c) $p_\alpha = \bigcap \{p_\beta \,:\, \beta < \alpha\}$ *for limit $\alpha \le \kappa^{<\kappa}$.*

Then for any $\lambda \le \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}^2_\kappa$ and $\forall \beta < \lambda$, $p_\lambda \le_{\mathbb{Q}^2_\kappa} p_\beta$.

Proof We go by induction on λ. The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \le \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}^2_\kappa$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \trianglerighteq t$ that is a splitting node in p_λ.

We fix the smallest α such that $\nu_\alpha \trianglerighteq_{p_0} t$ is a splitting node in p_0. Then in p_0 there are no splitting nodes in $\{s \,:\, t \trianglelefteq s \lhd \nu_\alpha\}$. Hence $\nu_\alpha \in \mathrm{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle \nu^i \,:\, i < \gamma \rangle$ be an $\lhd$-increasing sequence of splitting nodes of p_λ with union $\nu \in \kappa^{<\kappa}$. Then ν is a splitting node of each p_α, $\alpha < \lambda$, and also in p_λ since $\langle \mathrm{set}_{p_\alpha}(\nu) \,:\, \alpha < \lambda \rangle$ has at most two entries and their intersection has size κ. □

In the special case $F_\beta = \{\nu_\beta{}^\frown \langle j \rangle \,:\, j \in c_\beta\}$, the construction of Lemma 3.4 coincides with the simpler construction from Lemma 3.2.

Definition 3.5 We assume $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω. Let $\underset{\sim}{\tau}$ and $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho) \,:\, \eta \in {}^{\omega>}(2^\kappa) \rangle$ be as in Lemma 2.5. Now let $Q_{\mathcal{T}}$ be the set of κ-Miller trees p such that for every $\nu \in \mathrm{spl}(p)$ there is $\eta_{p,\nu} = \eta_\nu \in {}^{\omega>}(2^\kappa)$ such that

$$\mathrm{set}_p(\nu) = \{\varepsilon \in \kappa \,:\, \nu^\frown \langle \varepsilon \rangle \in p\} = a_{\eta_\nu}. \tag{3.3}$$

By the properties of $\mathcal{T}$, the node $\eta_{p,\nu}$ is unique.

Lemma 3.6 *Assume that $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω, let $\mathcal{T}$ be chosen as in Lemma 2.5, and let $Q_{\mathcal{T}}$ be defined from $\mathcal{T}$ as above. Then $Q_{\mathcal{T}}$ is dense in $\mathbb{Q}^2_\kappa$.*

Proof Let $p_0 = T \in \mathbb{Q}^2_\kappa$. Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa}\rangle$ be an injective enumeration of $\kappa^{<\kappa}$ with property (3.1). We now define fusion sequence $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha \le \kappa^\kappa\rangle$ according to the pattern in Lemma 3.2 in order to find $p_{\kappa^{<\kappa}} \ge T$ such that $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$.

Suppose that p_α and ν_α are given. If ν_α is not in p_α or is not a splitting node in p_α, then we let $p_{\alpha+1} = p_\alpha$. If $\nu_\alpha \in \mathrm{spl}(p_\alpha)$, then according to Lemma 2.5 clause (d) there is $\eta \in {}^{\omega>}(2^\kappa)$ such that $\mathrm{suc}_{p_\alpha}(\nu_\alpha) \supseteq a_\eta$. We choose such an η of minimal length and call it $\eta(\alpha)$.

Then we strengthen p_α to

$$\begin{aligned} p_{\alpha+1} = & \bigcup\{p_\alpha^{\langle \nu'\rangle} : \nu' = \nu_\alpha{}^\frown\langle i\rangle \wedge i \in a_{\eta(\alpha)}\}\cup \\ & \bigcup\{p_\alpha^{\langle \eta\rangle} : \eta \not\trianglelefteq \nu_\alpha \wedge \nu_\alpha \not\trianglelefteq \eta\}. \end{aligned} \tag{3.4}$$

Now we have that

$$\eta_{p_{\alpha+1},\nu_\alpha} = \eta(\alpha), c_\alpha = a_{\eta(\alpha)}.$$

For limit ordinals $\lambda \le \kappa^{<\kappa}$, we let $p_\lambda = \bigcap\{p_\beta : \beta < \lambda\}$. Since the sequence $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha \le \kappa^{<\kappa}\rangle$ matches the pattern in Lemma 3.2, we have $p_{\kappa^{<\kappa}} \in \mathbb{Q}^2_\kappa$. By construction, for any $\alpha < \kappa^{<\kappa}$ for any $\delta \in [\alpha+1, \kappa^{<\kappa})$, $\nu_\alpha \in \mathrm{spl}(p_\delta)$ implies

$$\mathrm{set}_{p_{\alpha+1}}(\nu_\alpha) = \mathrm{set}_{p_\delta}(\nu_\alpha) = a_{\eta(\alpha)}.$$

Hence the condition $p = p_{\kappa^{<\kappa}}$ fulfils Equation (3.3) in its splitting node ν_α with witness $\eta_{p,\nu_\alpha} = \eta(\alpha)$. Since all nodes are enumerated, we have $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$. □

We use only the inclusion $\mathrm{set}_p(\nu) \subseteq a_{\eta_\nu}$ from Definition 3.5.

Definition 3.7 We assume that $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω and the $\mathcal{T}$ is as in Lemma 2.5. For $T \in Q_{\mathcal{T}}$ and a splitting node ν of T we set $\varrho_{T,\nu} := \varrho_{\eta_{T,\nu}} \in {}^{\omega>}(2^\kappa)$. Recall $\eta_{T,\nu}$ is defined in Def. 3.5, and ϱ is a component of $\mathcal{T}$.

For $p \in Q_{\mathcal{T}}$, the relation $\nu \trianglelefteq \nu' \in p$ does neither imply $\eta_\nu \trianglelefteq \eta_{\nu'}$ nor $\varrho_\nu \trianglelefteq \varrho_{\nu'}$. However, $\eta_\nu \triangleleft \eta_{\nu'}$ implies $a_{\eta_\nu} \supset a_{\eta_{\nu'}}$ and $\varrho_\nu \triangleleft \varrho_{\nu'}$.

Observation 3.8 *We assume that $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω. Let $p_1, p_2 \in Q_{\mathcal{T}}$. If $p_1 \le_{\mathbb{Q}^2_\kappa} p_2$ then for $\nu \in \mathrm{spl}(p_2)$ we have $\nu \in \mathrm{spl}(p_1)$ and $\varrho_{p_1,\nu} \trianglelefteq \varrho_{p_2,\nu}$.*

We introduce dense sets:

Definition 3.9 We assume that $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω. Let $n \in \omega$.

$$D_n = \big\{p \in Q_{\mathcal{T}} : (\forall \nu \in \mathrm{spl}(p))(\mathrm{dom}(\varrho_{p,\nu}) > n)\big\}.$$

D_n is open dense in $Q_{\mathcal{T}}$ and the intersection of the D_n is empty.

Recall, by Lemma 3.6 we can work with the dense subforcing $Q_{\mathcal{T}}$ of $\mathbb{Q}^2_\kappa$. The following technical lemma is the next step of a transformation of a $\mathbb{Q}^1_\kappa$-name of a surjection from ω onto 2^κ into a $Q_{\mathcal{T}}$-name of such a surjection. The coordinate $\bar{\gamma}_\alpha$ and the clauses (d), (e), (f) are used for a counting argument in the induction steps. Later, only the coordinates p_α, n_α, and clauses (a), (b), (c) and Remark 3.11 will be used.

Lemma 3.10 *We assume that $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω, $\mathrm{cf}(\kappa) > \omega$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa\rangle$ enumerate $\mathbb{Q}^2_\kappa$ such that each Miller tree appears 2^κ times. There is $\langle (p_\alpha, n_\alpha, \bar{\gamma}_\alpha) : \alpha < 2^\kappa\rangle$ such that*

- (a) $n_\alpha < \omega$,
- (b) $p_\alpha \in D_{n_\alpha} \subseteq Q_{\mathcal{T}}$ *and* $p_\alpha \geq T_\alpha$.
- (c) *If* $\beta < \alpha$ *and* $n_\beta \geq n_\alpha$ *then* $p_\beta \perp p_\alpha$.
- (d) $\bar{\gamma}_\alpha = \langle \gamma_{\alpha,\nu} : \nu \in \mathrm{spl}(p_\alpha)\rangle$.
- (e) $(\forall \nu \in \mathrm{spl}(p_\alpha))(a_{\eta_{p_\alpha,\nu}} \Vdash_{\mathbb{Q}^1_\kappa} \gamma_{\alpha,\nu} \in \mathrm{range}(\varrho_{p_\alpha,\nu}))$.
- (f) $\gamma_{\alpha,\nu} \in 2^\kappa \setminus W_{<\alpha,\nu}$ *with*

$$W_{<\alpha,\nu} = \bigcup\{\mathrm{range}(\varrho_{p_\beta,\nu}) : \beta < \alpha, \nu \in \mathrm{spl}(p_\beta)\}.$$

Proof Assume that $\langle (p_\beta, n_\beta, \bar{\gamma}_\beta) : \beta < \alpha\rangle$ has been defined and we are to define $(p_\alpha, n_\alpha, \bar{\gamma}_\alpha)$. Note that the p_β need not be increasing in strength.

$(\oplus)_1$ The choice of the a_η in Lemma 2.5 and the choice $Q_{\mathcal{T}}$ and of $\eta_{p_\beta,\nu}$ for $\nu \in \mathrm{spl}(p_\beta)$, $\beta < \alpha$, imply that the set $W_{<\alpha,\nu}$ is well defined and of cardinality $\leq |\alpha| + \aleph_0 < 2^\kappa$. Hence we can choose $\gamma_{\alpha,\nu} \in 2^\kappa \setminus W_{<\alpha,\nu}$.

$(\oplus)_2$ With the fusion Lemma 3.2 we choose $q_\alpha \geq T_\alpha$, $q_\alpha \in Q_{\mathcal{T}}$, such that

$$(\forall \nu \in \mathrm{spl}(q_\alpha))(a_{\eta_{q_\alpha,\nu}} \Vdash_{\mathbb{Q}^1_\kappa} \gamma_{\alpha,\nu} \in \mathrm{range}(\varrho_{q_\alpha,\nu})).$$

$(\oplus)_3$ Let $q \in \mathbb{Q}^2_\kappa$. For $n \in \omega$ and $\nu \in \mathrm{spl}(q)$ we let

$$\mathcal{U}_{\alpha,\nu,n}(q) = \{\beta < \alpha : n_\beta = n, \nu \in \mathrm{spl}(p_\beta) \wedge |\,\mathrm{set}_q(\nu) \cap \mathrm{set}_{p_\beta}(\nu)| = \kappa\}.$$
$$\mathcal{U}_{\alpha,\nu}(q) = \bigcup\{\mathcal{U}_{\alpha,\nu,n}(q) : n \in \omega\}.$$

$(\oplus)_4$ (a) If $n \in \omega$ and $\nu \in \mathrm{spl}(q_\alpha)$ then

$$\beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha) \to \varrho_{p_\beta,\nu} \trianglelefteq \varrho_{q_\alpha,\nu}.$$

This is seen as follows. We let $a = \mathrm{set}_{p_\beta}(\nu) \cap \mathrm{set}_{q_\alpha}(\nu)$. Since $\beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha)$, $a \in [\kappa]^\kappa$. Clearly $a \Vdash_{\mathbb{Q}^1_\kappa} \underaccent{\tilde}{\tau} \rhd \varrho_{p_\beta,\nu}, \varrho_{q_\alpha,\nu}$. So either $\varrho_{p_\beta,\nu} \lhd \varrho_{q_\alpha,\nu}$ or $\varrho_{p_\beta,\nu} \trianglerighteq \varrho_{q_\alpha,\nu}$. However, since $\gamma_{\alpha,\nu} \in \mathrm{range}(\varrho_{q_\alpha,\nu}) \setminus W_{<\alpha,\nu}$, only $\varrho_{q_\alpha,\nu} \rhd \varrho_{p_\beta,\nu}$ is possible.
(b) So for $\nu \in \mathrm{spl}(q_\alpha)$, the set $\{\varrho_{p_\beta,\nu} : \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha)\}$ has at most $\mathrm{dom}(\varrho_{q_\alpha,\nu})$ elements.
(c) The assigment $\beta \mapsto \varrho_{p_\beta,\nu}$ is is defined between $\mathcal{U}_{\alpha,\nu}(q_\alpha)$ and $\{\varrho_{p_\beta,\nu} : \beta \in$

$\mathcal{U}_{\alpha,\nu}(q_\alpha)\}$. According to properties (e) and (f) in the induction hypothesis, the assigment is injective, and hence
$|\mathcal{U}_{\alpha,\nu}(q_\alpha)| \le \mathrm{dom}(\varrho_{q_\alpha,\nu})$.
(d) We state for further use that $\mathcal{U}_{\alpha,\nu}(q_\alpha)$ is finite and for any $q \le q_\alpha, \mathcal{U}_{\alpha,\nu}(q) \subseteq \mathcal{U}_{\alpha,\nu}(q_\alpha)$.

$(\oplus)_5$ We look at the cone above q_α and show:

$$\begin{aligned}&(\forall q \ge q_\alpha)(\forall \nu \in \mathrm{spl}(q))(\exists r_{\alpha,\nu} \le_{\mathbb{Q}^2_\kappa} q)\\ &(\exists c \in [\mathrm{set}_q(\nu)]^\kappa)(\exists F \subseteq \{\eta \in \mathrm{spl}(q) \,:\, \eta \rhd \nu\})\\ &\big(r_{\alpha,\nu} = q(\nu, c, F) \wedge (\forall \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha))(r^{\langle\nu\rangle}_{\alpha,\nu} \perp p^{\langle\nu\rangle}_\beta \vee p^{\langle\nu\rangle}_\beta \ge r^{\langle\nu\rangle}_{\alpha,\nu})\big).\end{aligned} \tag{3.5}$$

How do we find $r_{\alpha,\nu} = r_{\alpha,\nu}(q)$? Given $q \le_{\mathbb{Q}^2_\kappa} q_\alpha$, $\nu \in \mathrm{spl}(q)$ we enumerate $\mathcal{U}_{\alpha,\nu}(q_\alpha)$ as $\beta_0, \dots, \beta_{k-1}$. We let $r_0 = q$ and by induction on $i \le k$ we define r_i, increasing in strength, with $\nu \in \mathrm{spl}(r_i)$ and $c_i = \mathrm{set}_{r_i}(\nu)$. Thus the c_i are $\subseteq$-decreasing sets of size κ. Given r_i, we distinguish cases:

First case: $\beta_i \notin \mathcal{U}_{\alpha,\nu}(r_i)$. Then there is $c_{i+1} \in [\mathrm{set}_{r_i}(\nu)]^\kappa$, $c_{i+1} \cap \mathrm{set}_{p_{\beta_i}}(\nu) = \emptyset$. We let $r_{i+1} = r_i(\nu, c_{i+1})$ and thus have $r^{\langle\nu\rangle}_{i+1} \perp p_{\beta_i}$.

Second case: $\beta_i \in \mathcal{U}_{\alpha,\nu}(r_i)$. We let

$$c_i = \{j \in \mathrm{set}_{r_i}(\nu) \,:\, r_i^{\langle\nu\hat{}\langle j\rangle\rangle} \le p_{\beta_i}^{\langle\nu\hat{}\langle j\rangle\rangle}\} \cup \{j \in \mathrm{set}_{r_i}(\nu) \,:\, r_i^{\langle\nu\hat{}\langle j\rangle\rangle} \not\le p_{\beta_i}^{\langle\nu\hat{}\langle j\rangle\rangle}\}.$$

If $c_{i,1} = \{j \in \mathrm{set}_{r_i}(\nu) \,:\, r_i^{\langle\nu\hat{}\langle j\rangle\rangle} \le p_{\beta_i}^{\langle\nu\hat{}\langle j\rangle\rangle}\}$ has size κ, then we let $c_{i+1} = c_{1,i}$ and $r_{i+1} = r_i(\nu, c_{i+1})$ and thus get $r^{\langle\nu\rangle}_{i+1} \ge p_{\beta_i}$.

If $|c_{i,1}| < \kappa$, then $c_{i,2} = \{j \in \mathrm{set}_{r_i}(\nu) \,:\, r_i^{\langle\nu\hat{}\langle j\rangle\rangle} \not\le p_{\beta_i}^{\langle\nu\hat{}\langle j\rangle\rangle}\}$ has size κ, and we let $c_{i+1} = c_{i,2}$. For $j \in c_{i+1}$, $r_i^{\langle\nu\hat{}\langle j\rangle\rangle} \not\le p_{\beta_i}^{\langle\nu\hat{}\langle j\rangle\rangle}$. Thus we can find a node in the $r_i^{\langle\nu\hat{}\langle j\rangle\rangle} \setminus p_{\beta_i}^{\langle\nu\hat{}\langle j\rangle\rangle}$ and above this node we find a splitting node of r_i. We take this latter splitting node into r_{i+1} as the direct successor splitting node to $\nu\hat{}\langle j\rangle$. Doing so for every $j \in c_{i+1}$ we get $F_{\nu,i}$, a front strictly above ν in $r_{i+1} = r_i(\nu, c_{i+1}, F_{\nu,i})$. Again we get $r^{\langle\nu\rangle}_{i+1} \perp p_{\beta_i}$.

In the end we let $r_{\alpha,\nu} = r_k$. There is a front F that contains for each $j \in c_k$ the shortest splitting node of r_k above $\nu\hat{}\langle j\rangle$. Thus we have $r_k = r_{\alpha,\nu} = q(\nu, c_k, F)$ and $r_{\alpha,\nu}$ fulfils (3.5).

$(\oplus)_6$ Now we use $(\oplus)_5$ iteratively along all $\nu \in \kappa^{<\kappa}$ to find a fusion sequence $\langle r_{\alpha,\nu}, \nu, c_\nu, F_\nu \,:\, \nu < \kappa^{<\kappa}\rangle$ with starting point $q_\alpha = r_{0,\nu_0}$. In this sequence, $r_{\alpha,\nu}$ is chosen as $r_{\alpha,\nu}(q)$ in $\oplus_5$ for $q = \bigcap_{\beta<\alpha} r_\beta$, if $\nu \in \mathrm{spl}(q)$. If $\nu \notin \mathrm{spl}(q)$, then $r_{\alpha,\nu} = q$. Then we apply the fusion Lemma 3.4 and get an lower bound r_α of $r_{\alpha,\nu}$, $\nu \in {}^{\kappa>}\kappa$. Note $r^{\langle\nu\rangle}_\alpha \perp p_\beta$ iff $r^{\langle\nu\rangle}_\alpha \perp p^{\langle\nu\rangle}_\beta$ and $r^{\langle\nu\rangle}_\alpha \le p_\beta$ iff $r^{\langle\nu\rangle}_\alpha \le p^{\langle\nu\rangle}_\beta$. Hence $r_\alpha \ge q_\alpha$ and

$$(\forall \nu \in \mathrm{spl}(r_\alpha))(\forall \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha))(r^{\langle\nu\rangle}_\alpha \perp p_\beta \vee p_\beta \ge r^{\langle\nu\rangle}_\alpha).$$

($\oplus$)$_7$ Finally we choose n_α and p_α. There are k and ν such that $n < \omega$ and $\nu \in \mathrm{spl}(r_\alpha)$ such that $p_\alpha = r_\alpha^{\langle \nu \rangle}$ fulfils

$$(\forall \beta < \alpha)(n_\beta \geq k \to p_\alpha \perp p_\beta).$$

Proof of existence. By induction on $k \in \omega$ we try to find $\langle \nu_k, \beta_k : k \in \omega \rangle$ such that
(a) $\nu_k \in \mathrm{spl}(r_\alpha)$,
(b) $\nu_k \lhd \nu_m$ for $k < m$,
(c) $\beta_k < \alpha$ and $n_{\beta_k} \geq k$ and $r_\alpha^{\langle \nu_k \rangle} \leq p_{\beta_k}$.
If we succeed, then $\nu_* = \bigcup \{\nu_k : k \in \omega\} = \nu^* \in \mathrm{spl}(r_\alpha)$ by Definition 1.1 (2). Here we use that $\mathrm{cf}(\kappa) > \omega$. Hence

$$r_\alpha^{\langle \nu^* \rangle} \in Q_{\mathcal{T}} \cap \bigcap \{D_k : k < \omega\} \text{ and}$$
$$a_{\eta_{r_\alpha, \nu^*}} \text{ determines in } \Vdash_{\mathbb{Q}^1_\kappa} \text{ for any } k < \omega \text{ the value of } \underset{\sim}{\tau} \restriction k.$$

This is a contradiction.

So there is a smallest k such that ν_k cannot be defined. We let $n_\alpha = k$. We let p_α be a strengthening of $r_\alpha^{\langle \nu_{k-1} \rangle}$ such that $p_\alpha \in D_{n_\alpha}$. For finding such a strengthening we again invoke the fusion Lemma 3.2.

We show that $p_\alpha \perp p_\beta$ for $\beta < \alpha$ with $n_\beta \geq k$. Otherwise, having arrived at $r_\alpha^{\langle \nu_{k-1} \rangle}$ we find some β_k, α such that $n_{\beta_k} \geq k$ and $r_\alpha^{\langle \nu_{k-1} \rangle}$ is compatible with p_{β_k}. Then we can prolong ν_{k-1} to a splitting node $\nu_k \in \mathrm{spl}(p_{\beta_k}) \cap \mathrm{spl}(r_\alpha)$. By the choice of r_α the latter implies that $r_\alpha^{\langle \nu_k \rangle} \leq p_{\beta_k}$. However, now we would have found ν_k, β_k as required in contradiction to the choice of k. □

Remark 3.11 Conditions (a) to (c) of Lemma 3.10 yield: For any $k < \omega$,

$$\{p_\alpha : n_\alpha \geq k\} \text{ is dense in } \mathbb{Q}^2_\kappa.$$

Proof Let k and p be given. There is α_0 such that $T_{\alpha_0} \in D_0$ and $T_{\alpha_0} \leq_{\mathbb{Q}^2_\kappa} p$. Then $p_{\alpha_0} \leq T_{\alpha_0}$ and $n_{\alpha_0} \geq 0$. Then there is $\alpha_1 > \alpha_0$ such that $T_{\alpha_1} \leq_{\mathbb{Q}^2_\kappa} p_{\alpha_0}$. Then $p_{\alpha_1} \leq T_{\alpha_1}$ and hence by condition (c), $n_{\alpha_1} > n_{\alpha_0} \geq 0$. We can can repeat the argument $k - 1$ times. □

Now we drop the component $\bar{\gamma}_\alpha$ from a sequence $\langle p_\alpha, n_\alpha, \bar{\gamma}_\alpha : \alpha < 2^\kappa \rangle$ given by Lemma 3.10. Then we get a sequence with properties (a), (b), and a weakening (c) with the property stated in the remark. This sequence, combined with $2^{(2^{<\kappa})} = 2^\kappa$, allows to define a $\mathbb{Q}^2_\kappa$-name of a collapse.

Lemma 3.12 *We assume that $\mathbb{Q}^1_\kappa$ collapses 2^κ to ω, $\mathrm{cf}(\kappa) > \omega$ and $2^{(2^{<\kappa})} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate all Miller trees that such each tree appears 2^κ times. Assume that $\langle (p_\alpha, n_\alpha) : \alpha < 2^\kappa \rangle$ are such that*

(a) *$n_\alpha < \omega$,*

(b) $p_\alpha \in D_{n_\alpha} \subseteq Q_{\mathcal{T}}$ *and* $p_\alpha \geq T_\alpha$,
(c) *if* $\beta < \alpha$ *and* $n_\beta = n_\alpha$ *then* $p_\beta \perp p_\alpha$,
(d) *for any* $k \in \omega$, $\{p_\alpha : n_\alpha \geq k\}$ *is dense in* $\mathbb{Q}^2_\kappa$.

Then there is a $\mathbb{Q}^2_\kappa$*-name* $\underset{\sim}{\tau}'$ *for a surjection of* ω *onto* 2^κ.

Proof Let G be a $\mathbb{Q}^2_\kappa$-generic filter over $\mathbf{V}$. We define $\underset{\sim}{\tau}(n)$, a $\mathbb{Q}^2_\kappa$-name by $\underset{\sim}{\tau}(n)[G] = \alpha$ if $p_\alpha \in G$ and $n_\alpha = n$. The name $\underset{\sim}{\tau}$ is a name of a function by (c). By (d), the domain of $\underset{\sim}{\tau}$ is forced to be infinite. For any $p \in \mathbb{Q}^2_\kappa$ we let $U_p = \{\alpha : T_\alpha = p\}$. U_p is of size 2^κ, in particular for $\alpha \in 2^\kappa$ we have $|U_{p_\alpha}| = 2^\kappa$ and U_{p_α} contains the antichain $\{p_{\delta(\alpha,i)} : i < 2^\kappa\}$. Hence there is $f : 2^\kappa \to 2^\kappa$ in $\mathbf{V}[G]$ such that for any $\gamma, \alpha \in 2^\kappa$ there is $\beta = \delta(\alpha, \gamma) \in U_{p_\alpha}$ for some function $\delta : 2^\kappa \times 2^\kappa \to 2^\kappa$ with $f(\beta) = \gamma$ forced by $p_{\delta(\alpha,\gamma)}$. We let

$$\underset{\sim}{\tau}' = f(\underset{\sim}{\tau}) = \{\langle (n_\alpha, \gamma), p_{\delta(\alpha,\gamma)} \rangle : \alpha, \gamma \in 2^\kappa\}.$$

Next we show

$$\mathbb{Q}^2_\kappa \Vdash \text{range}(\underset{\sim}{\tau}') = 2^\kappa.$$

Suppose $p \in Q_{\mathcal{T}}$ and $\gamma < 2^\kappa$ are given. By construction the sequence $\{p_\beta : \beta < 2^\kappa\}$ is dense. Let $p \leq p_\alpha$. Then there is $\beta = \delta(\alpha, \gamma) \in U_{p_\alpha}$ $p_{\delta(\alpha,\gamma)} \leq p_\gamma$, with $f(\alpha) = \gamma$. However, $\delta(\alpha, \gamma) = \beta \in U_{p_\alpha}$ means $T_\beta = p_\alpha \geq p_\beta$ by construction. By the definition of $\underset{\sim}{\tau}$, $p_\beta \Vdash \underset{\sim}{\tau}(n_\alpha) = \alpha$, so $p_\beta \Vdash (f(\underset{\sim}{\tau}))(n_\alpha) = \gamma$. □

So we can sum up:

Theorem 3.13 *We assume that* $\mathbb{Q}^1_\kappa$ *collapses* 2^κ *to* ω *and* $\text{cf}(\kappa) > \omega$ *and* $2^{(\kappa^{<\kappa})} = 2^\kappa$. *Then the forcing with* $\mathbb{Q}^2_\kappa$ *collapses* 2^κ *to* $\aleph_0$.

4 *κ*-Cohen reals and the Levy collapse

Many κ-tree forcings add a κ-Cohen real, sometimes even if their ω-version does not add a Cohen real. Also our forcing $\mathbb{Q}^2_\kappa$ is of this kind. Classical Miller forcing preserves P-points and hence does not add a Cohen real. In this section we show that under the above conditions, $\mathbb{Q}^\kappa_2$ add a κ-Cohen real and is equivalent to the Levy collapse of 2^κ to $\aleph_0$.

Lemma 4.1 *If* $\mathbb{Q}^2_\kappa$ *collapses* 2^κ *to* $\aleph_0$, $\text{cf}(\kappa) > \aleph_0$, *and* $2^{2^{<\kappa}} = 2^\kappa$, *then* $\mathbb{Q}^2_\kappa$ *adds a* κ*-Cohen real.*

Proof Let G be $\mathbb{Q}^2_\kappa$-generic over $\mathbf{V}$. Let $f : \omega \to 2^{<\kappa}$ be a function in $\mathbf{V}[G]$, such that $(\forall \eta \in 2^{<\kappa})(\exists^\infty k f(k) = \eta)$. Such a function exists since $2^{<\kappa} \leq 2^\kappa$.

Since $2^{2^{<\kappa}} = 2^\kappa$, we can enumerate all antichains in $\mathbb{C}(\kappa)$ in $\alpha_* \leq 2^\kappa$ many steps. In $\mathbf{V}[G]$, α_* is countable. We list it as $\langle \alpha_n : n < \omega \rangle$. Now we choose $\eta_n \in \mathbb{C}(\kappa)^{\mathbf{V}}$ by induction on n in $\mathbf{V}[G]$: $\eta_0 = \emptyset$. Given η_n we choose k_n such that $f(k_n) = \eta_n$ and then we choose $\eta_{n+1} \trianglerighteq \eta_n$, such that $\eta_{n+1} \in I_{\alpha_n}$. Then $\{\eta : (\exists n < \omega)(\eta \trianglelefteq f(k_n))\}$ is a $\mathbb{C}(\kappa)$-generic filter over $\mathbf{V}$ and it exists in $V[G]$, since it is definable from $\{f(k_n) : n < \omega\}$. □

Two forcings $\mathbb{P}_1, \mathbb{P}_2$ are said to be equivalent if their regular open algebras $\mathrm{RO}(\mathbb{P}_i)$ coincide (for a definition of the regular open algebra of a poset, see, e.g., [3, Corollary 14.12]). Some forcings are characterised up to equivalence just by their size and their collapsing behaviour.

Lemma 4.2 [3, Lemma 26.7]. *Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that Q collapses λ onto $\aleph_0$, i.e.,*

$$1_Q \Vdash_Q |\check{\lambda}| = \aleph_0.$$

Then $\mathrm{RO}(Q) = \mathrm{Levy}(\aleph_0, \lambda)$.

Lemma 4.3 *If $\mathbb{Q}^1_\kappa$ collapses 2^κ to $\aleph_0$, then $\mathbb{Q}^1_\kappa$ is equivalent of* $\mathrm{Levy}(\aleph_0, 2^\kappa)$.

Proof $\mathbb{Q}^1_\kappa$ has size 2^κ. Hence Lemma 4.2 yields $\mathrm{RO}(\mathbb{Q}^1_\kappa) = \mathrm{Levy}(\aleph_0, 2^\kappa)$. □

Definition 4.4 A Boolean algebra is (θ, λ)-*nowhere distributive* if there are antichains $\bar{p}^\varepsilon = \langle p^\varepsilon_\alpha : \alpha < \alpha_\varepsilon \rangle$ of $\mathbb{P}$ for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta$

$$|\{\alpha < \alpha_\varepsilon : p \not\perp p^\varepsilon_\alpha\}| \geq \lambda.$$

Definition 4.5 Let B be a Boolean algebra. We write $B^+ = B \setminus \{0\}$. A subset $D \subseteq B^+$ is called *dense* if $(\forall b \in B^+)(\exists d \in D)(d \leq b)$. The *density* of a Boolean algebra B is the least size of a dense subset of B. A Boolean algebra B has uniform density if for every $a \in B^+$, $B \restriction a$ has the same density. The *density* of a forcing order $(\mathbb{P}, \leq_\mathbb{P})$ is the density of the regular open algebra $\mathrm{RO}(\mathbb{P})$.

Lemma 4.6 [1, Theorem 1.15] *Let $\theta < \lambda$ be regular cardinals.*

(1) *Suppose that $\mathbb{P}$ has the following properties* (a) *to* (c).

  (a) *$\mathbb{P}$ is a (θ, λ)-nowhere distributive forcing notion,*
  (b) *$\mathbb{P}$ has density λ,*
  (c) *in case $\theta > \aleph_0$, $\mathbb{P}$ has a θ-complete dense subset S. The latter means:* $(\forall B \in [S]^{<\theta})(\exists s \in S)(\forall b \in B)(b \leq_\mathbb{P} s)$.

  Then $\mathbb{P}$ *is equivalent to* $\mathrm{Levy}(\theta, \lambda)$.

(2) *Under* (a) *and* (b) *$\mathbb{P}$ collapses λ to θ (and may or may not collapse λ to $\aleph_0$).*

Proposition 4.7 *If there is a κ-mad family of size 2^κ the forcing $\mathbb{Q}^1_\kappa$ is $(\aleph_0, 2^\kappa)$-nowhere distributive.*

Proof Lemma 2.5 gives $\mathcal{T}$ such that $\bar{p}^n = \{a_\eta : \eta \in {}^n(2^\kappa)\}$, $n \in \omega$, witnesses $(\aleph_0, 2^\kappa)$-nowhere distributivity. □

By Lemma 4.2 and Theorem 3.13 we get:

Proposition 4.8 *If $\mathbb{Q}^1_\kappa$ collapses 2^κ to $\aleph_0$,* $\mathrm{cf}(\kappa) > \aleph_0$ *and $2^{(\kappa^{<\kappa})} = 2^\kappa$ then $\mathbb{Q}^2_\kappa$ is equivalent to* $\mathrm{Levy}(\aleph_0, 2^\kappa)$.

Acknowledgements Open Access funding provided by Projekt DEAL. We thank Marlene Koelbing for pointing out a gap in an earlier version.

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