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Specializing trees and answer to a question of Williams

Mohammad Golshani*
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P. O. Box 19395-5746, Tehran, Iran
golshani.m@gmail.com

Saharon Shelah
Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem 91904, Israel
Department of Mathematics, Rutgers University
New Brunswick, NJ 08854, USA
shelah@math.huji.ac.il

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We show that if $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$, then any nontrivial $\aleph_1$-closed forcing notion of size $\leq 2^{\aleph_0}$ is forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$, the Cohen forcing for adding a new Cohen subset of ω_1. We also produce, relative to the existence of suitable large cardinals, a model of ZFC in which $2^{\aleph_0} = \aleph_2$ and all $\aleph_1$-closed forcing notion of size $\leq 2^{\aleph_0}$ collapse $\aleph_2$, and hence are forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$. These results answer a question of Scott Williams from 1978. We also extend a result of Todorcevic and Foreman–Magidor–Shelah by showing that it is consistent that every partial order which adds a new subset of $\aleph_2$, collapses $\aleph_2$ or $\aleph_3$.

Keywords: Tree specialization; Arosiszajn trees; collapsing cardinals; supercompact cardinals.

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1. Introduction

For an infinite cardinal κ, let $\mathrm{Add}(\kappa, 1)$ denote the Cohen forcing for adding a new Cohen subset of κ; thus conditions in $\mathrm{Add}(\kappa, 1)$ are partial functions $p : \kappa \to \{0, 1\}$ of size less than κ, ordered by reverse inclusion. The forcing is $\mathrm{cf}(\kappa)$-closed and satisfies $(2^{<\kappa})^+$-c.c., in particular, if κ is regular and $2^{<\kappa} = \kappa$, then it preserves all cardinals.

*Corresponding author.

It is well known that if the continuum hypothesis (CH), holds, then any $\aleph_1$-closed forcing notion of size continuum is forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$. In [19] (see also [20]), Williams asked if the converse is also true, i.e. if CH follows from the assumption "any $\aleph_1$-closed forcing notion of size continuum is forcing equivalent to the Cohen forcing $\mathrm{Add}(\aleph_1, 1)$". We will show that $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$ is sufficient to conclude that all $\aleph_1$-closed forcing notions of size continuum are forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$. Since $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$ is consistent with $\neg$CH, this gives a negative answer to Williams question.

Theorem 1.1. *Assume* $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$. *Then any nontrivial* $\aleph_1$*-closed forcing notion of size* $\leq 2^{\aleph_0}$ *is forcing equivalent to* $\mathrm{Add}(\aleph_1, 1)$.

Remark 1.2. (1) We can replace $\aleph_1, 2^{\aleph_0}$ by $\kappa = \mu^+, 2^\mu$, respectively, with $\mathrm{cf}(2^\mu) = \kappa$; or by $\kappa, 2^\mu$, respectively, if κ is weakly inaccessible, $\mu < \kappa$, $2^\mu = 2^{<\kappa}$ and $\mathrm{cf}(2^\mu) = \kappa$.

(2) If $2^{\aleph_0} = 2^{\aleph_1}$, then $\mathrm{Add}(\aleph_2, 1)$ is $\aleph_1$-closed of size continuum, but it is not forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$.

On the other hand, it is not difficult to prove the consistency of "$2^{\aleph_0} = \aleph_2$ and there exists a nontrivial $\aleph_1$-closed (but not $\aleph_2$-closed) forcing notion of size $\aleph_2$ which preserves all cardinals" (see [9]). So, it is natural to ask if we can have the same result as in Theorem 1.1 with $2^{\aleph_0}$ being regular. We show that this is indeed the case, if we assume the existence of large cardinals.

Recall that an uncountable cardinal κ is *supercompact* if for every cardinal $\lambda > \kappa$ there exists a nontrivial elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$. It is 2-*Mahlo* if $\{\mu < \kappa : \mu \text{ is a Mahlo cardinal}\}$ is stationary in κ.

Theorem 1.3. *Assume* κ *is a supercompact cardinal and* $\lambda > \kappa$ *is a* 2*-Mahlo cardinal. Then there is a generic extension of the universe in which the following hold*:

(a) $2^{\aleph_0} = \kappa = \aleph_2$;
(b) $2^{\aleph_1} = \lambda = \aleph_3$;
(c) *Any* $\aleph_1$*-closed forcing notion of size* $\leq \aleph_2$ *collapses* $\aleph_2$ *into* $\aleph_1$, *in particular it is forcing equivalent to* $\mathrm{Add}(\aleph_1, 1)$.

Following [4], let *Todorcevic's maximality principle* be the assertion: "every partial order which adds a fresh subset of $\aleph_1$, collapses $\aleph_1$ or $\aleph_2$", where by a fresh subset of a cardinal κ we mean a subset of κ which is not in the ground model but all of its proper initial segments are in the ground model.

In [16], Todorcevic showed that if $2^{\aleph_0} = \aleph_2$ and every $\aleph_1$-tree of size $\aleph_1$ is special, then Todorcevic's maximality principle holds.

By results of Baumgartner [2] and Todorcevic [15], "$2^{\aleph_0} = \aleph_2 +$ every $\aleph_1$-tree of size $\aleph_1$ is special" is consistent, and hence Todorcevic's maximality principle is consistent as well. On the other hand, Foreman–Magidor–Shelah [8] showed that PFA implies the same conclusion. In [18], Viale and Weiss introduced the principle guessing model principle (GMP) and showed that it follows from PFA. Cox and Krueger [3], introduced the stronger principle indestructible GMP (IGMP) and showed that PFA implies IGMP which in turn implies Todorcevic's maximality principle. On the other hand, in [4], they showed that Todorcevic's maximality principle does not follow from GMP.

We extends the above result of Todorcevic to higher cardinals, and prove the following theorem.

Theorem 1.4. *Assume κ is a supercompact cardinal and $\lambda > \kappa$ is a 2-Mahlo cardinal. Then there is a generic extension of the universe in which the following hold*:

(a) $2^{\aleph_0} = \aleph_1$;
(b) $\kappa = \aleph_2$;
(c) $2^{\aleph_1} = \lambda = \aleph_3$;
(d) *Every partial order which adds a fresh subset of $\aleph_2$, collapses $\aleph_2$ or $\aleph_3$.*

Remark 1.5. In Theorems 1.3 and 1.4, we can replace the cardinals $\aleph_0, \aleph_1$ and $\aleph_2$ by the cardinals η, η^+ and η^{++}, respectively, where η is a regular cardinal less than κ.

The above result is related to Foreman's maximality principle [7], which asserts that any nontrivial forcing notion either adds a new real or collapses some cardinals.

This paper is organized as follows. In Sec. 2, we prove Theorem 1.1. Sections 3 and 4 are devoted to some preliminary results which are then used in Sec. 5 for the proof of Theorem 1.3. In Sec. 6, we prove Theorem 1.4.

To avoid trivialities, by a forcing notion we always mean a nontrivial separative forcing notion. We use $\simeq$ for the equivalence of forcing notions, so

$$\mathbb{P} \simeq \mathbb{Q} \Leftrightarrow RO(\mathbb{P}) \text{ is isomorphic to } RO(\mathbb{Q}),$$

where $RO(\mathbb{P})$ denotes the Boolean completion of $\mathbb{P}$. Also $\mathbb{P} \lessdot \mathbb{Q}$ means that $\mathbb{P}$ is a regular sub-forcing of $\mathbb{Q}$.

2. A Negative Answer to Williams Question When the Continuum is Singular

In this section, we prove Theorem 1.1. In [9], it is shown that if $\mathbb{Q}$ is any $\aleph_1$-closed forcing notion[a] of size $\leq 2^{\aleph_0}$ and if λ is the least cardinal such that forcing with $\mathbb{Q}$ adds a fresh λ-sequence of ordinals, then forcing with $\mathbb{Q}$ collapses $2^{\aleph_0}$ into λ, and hence, if in addition $\lambda = \aleph_1$, then $\mathbb{Q} \simeq \mathrm{Add}(\aleph_1, 1)$. Thus to prove Theorem 1.1, it suffices to show that if $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$, then any $\aleph_1$-closed forcing notion $\mathbb{Q}$ of size at

[a]In fact being $\omega + 1$-strategically closed is sufficient.

most $2^{\aleph_0}$ adds a fresh set of ordinals of size $\aleph_1$. We give a direct proof of this fact which is of its own interest, and avoids the use of the results of [9].

If $2^{\aleph_0} = \aleph_1$, then the result is known to hold, so assume that $\aleph_1 < 2^{\aleph_0}$ and $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$. Let $\mathbb{Q}$ be a nontrivial $\aleph_1$-closed forcing notion of size $\leq 2^{\aleph_0}$. We are going to show that $\mathbb{Q}$ is forcing equivalent to $\mathrm{Add}(\aleph_1, 1)$.

Natation 2.1. For a forcing notion $\mathbb{P}$ and a condition $p \in \mathbb{P}$, let $\mathbb{P} \downarrow p$ denote the set of all conditions in $\mathbb{P}$ which extend p; i.e. $\mathbb{P} \downarrow p = \{q \in \mathbb{P} : q \leq_{\mathbb{P}} p\}$.

Let $\langle \mathbb{Q}_i : i < \omega_1 \rangle$ be a $\subseteq$-increasing and continuous sequence of subsets of $\mathbb{Q}$ such that $\mathbb{Q}_0 = \emptyset$, for all $i < \omega_1, |\mathbb{Q}_i| < 2^{\aleph_0}$, and $\mathbb{Q} = \bigcup_{i<\omega_1} \mathbb{Q}_i$.

Lemma 2.2. *For every $i < \omega_1$ and every $p \in \mathbb{Q}$, there exists $q \leq_{\mathbb{Q}} p$ such that there is no $r \in \mathbb{Q}_i$ with $r \leq q$.*

Proof. Let A be a maximal antichain in $\mathbb{Q}$ below p of size $2^{\aleph_0}$, which exists as $\mathbb{Q}$ is nontrivial and $\aleph_1$-closed. As $|\mathbb{Q}_i| < 2^{\aleph_0}$, we can find $q \in A$ such that $(\mathbb{Q} \downarrow q) \cap \mathbb{Q}_i = \emptyset$. Then q is as required. $\square$

We now define by induction on $i < \omega_1$ a sequence $\bar{p}_i$ such that:

(1) $\bar{p}_i = \langle \bar{p}_i(\eta) : \eta \in ({}^{i+1}\ (2^{\aleph_0})) \rangle$ is a maximal antichain in $\mathbb{Q}$;
(2) if $j < i$ and $\eta \in ({}^{i+1}(2^{\aleph_0}))$, then $\bar{p}_i(\eta) \leq_{\mathbb{Q}} \bar{p}_j(\eta \restriction (j+1))$;
(3) if $\eta \in ({}^{i+1}(2^{\aleph_0}))$, then there is no member of $\mathbb{Q}_i$ which is below $\bar{p}_i(\eta)$.

***i* = 0:** Let $\bar{p}_0 = \langle \bar{p}_0(\eta) : \eta \in ({}^{1}(2^{\aleph_0})) \rangle$ be any maximal antichain in $\mathbb{Q}$. Note that clauses (2) and (3) above are vacuous as $\mathbb{Q}_0$ is empty.

***i* > 0:** For every $\nu \in ({}^{i}(2^{\aleph_0}))$ set $\bar{p}^1_{i,\nu} = \langle \bar{p}_j(\nu \restriction (j+1)) : j < i \rangle$. Then, by the induction hypothesis, $\bar{p}^1_{i,\nu}$ is a countable decreasing sequence of conditions in $\mathbb{Q}$, and so the set

$$\mathbb{P}^2_{i,\nu} = \{q \in \mathbb{Q} : j < i \Rightarrow q \leq \bar{p}_j(\nu \restriction (j+1))\}$$

is nonempty. Let

$$\mathbb{P}^3_{i,\nu} = \{q \in \mathbb{P}^2_{i,\nu} : \forall\, z \in \mathbb{Q}_i[z \not\leq_{\mathbb{Q}} q \text{ and moreover } z \nVdash \text{“}q \in \dot{G}_{\mathbb{Q}}\text{”}]\}.$$

$\mathbb{P}^3_{i,\nu}$ is easily seen to be a dense subset of $\mathbb{P}^2_{i,\nu}$, hence, we can find a maximal antichain, say $\bar{\mathbb{P}}_{i,\nu} = \{p_{i,\nu}(\alpha) : \alpha < 2^{\aleph_0}\}$, in it. For $\eta \in ({}^{i+1}(2^{\aleph_0}))$ set $\bar{p}_i(\eta) = p_{i,\eta \restriction i}(\eta(i))$. Then it is easily seen that $\bar{p}_i = \langle \bar{p}_i(\eta) : \eta \in ({}^{i+1}(2^{\aleph_0})) \rangle$ is as required.

Let $\underset{\sim}{\nu} \in V^{\mathbb{Q}}$ be the $\mathbb{Q}$-name

$$\underset{\sim}{\nu} = \{\langle (\check{j}, \check{\eta}(\check{j})), \bar{p}_i(\eta) \rangle : j \leq i < \omega_1 \text{ and } \eta \in ({}^{i+1}(2^{\aleph_0}))\}.$$

Claim 2.3. (a) *For every $i < \omega_1$ and $\eta \in ({}^{i+1}(2^{\aleph_0})), \bar{p}_i(\eta) \Vdash$“$\eta = \underset{\sim}{\nu} \restriction i+1$”.*
(b) $1_{\mathbb{Q}} \Vdash$“$\underset{\sim}{\nu} \in ({}^{\omega_1}(2^{\aleph_0}))$”.

Proof. (a) is clear from the definition of $\underset{\sim}{\nu}$.

(b) Let G be $\mathbb{Q}$-generic over V. Then for each $i < \omega_1$ we can find a unique $\eta_i \in ({}^{i+1}(2^{\aleph_0}))$ such that $\bar{p}_i(\eta_i) \in G$. If $j < i$, then $\bar{p}_i(\eta_i) \leq_{\mathbb{Q}} \bar{p}_j(\eta_i \restriction (j+1))$, and so $\eta_j = \eta_i \restriction j+1$. It then immediately follows that

$$\underset{\sim}{\nu}[G] = \{(i, \eta_i(j)) : j \leq i < \omega_1\} = \bigcup_{i<\omega_1} \{\eta \in ({}^{i+1}(2^{\aleph_0})) : \bar{p}_i(\eta) \in G\}$$

is a function from ω_1 into $2^{\aleph_0}$. □

We now define a $\mathbb{Q}$-name $\underset{\sim}{\tau}$ for a function from ω_1 into 2 as follows: let $\langle \rho_\alpha : \alpha < 2^{\aleph_0}\rangle$ be an enumeration of ${}^{\omega}2$ with no repetitions. Then let $\underset{\sim}{\tau}$ be such that

$$\Vdash_{\mathbb{Q}} \underset{\sim}{\tau}(\omega \cdot i + n) = \rho_{\underset{\sim}{\nu}(i)}(n).$$

Lemma 2.4. $\Vdash_{\mathbb{Q}}$"$\underset{\sim}{\tau} \in ({}^{\omega_1}2)$ *and* $\underset{\sim}{\tau} \notin \check{V}$".

Proof. Let $q_1 \in \mathbb{Q}$. Then for some $i < \omega_1, q_1 \in \mathbb{Q}_i$. Since $\langle \bar{p}_i(\eta) : \eta \in ({}^{i+1}(2^{\aleph_0}))\rangle$ is a maximal antichain, we can find $\eta \in ({}^{i+1}(2^{\aleph_0}))$ such that q_1 is compatible with $\bar{p}_i(\eta)$. But $q_1 \nVdash$"$\bar{p}_i(\eta) \in \dot{G}_{\mathbb{Q}}$", so there is $q_2 \leq q_1$ such that q_2 is incompatible with $\bar{p}_i(\eta)$. But again as $\langle \bar{p}_i(\eta) : \eta \in ({}^{i+1}(2^{\aleph_0}))\rangle$ is a maximal antichain, there exists $\rho \in ({}^{i+1}(2^{\aleph_0}))$ such that q_2 and $\bar{p}_i(\rho)$ are compatible. Let $q_3 \leq q_2, \bar{p}_i(\rho)$, and let $j \leq i$ be maximal such that $\eta \restriction j = \rho \restriction j$ and $\eta \restriction (j+1) \neq \rho \restriction (j+1)$. Then $q_3, \bar{p}_i(\eta)$ are compatible with q_1, but they force contradictory information about $\underset{\sim}{\tau} \restriction [\omega \cdot j, \omega \cdot j + \omega)$. The result follows immediately. □

Lemma 2.5. *There exists a dense subset $\mathbb{Q}'$ of $\mathbb{Q}$ which is the union of $\aleph_1$-many maximal antichains $\langle I_i^* : i < \omega_1\rangle$ of $\mathbb{Q}$.*

Proof. For any $p \in \mathbb{Q}$, by the previous lemma, p does not force any value for $\underset{\sim}{\tau}$, hence there are ordinal $i < \omega_1$ and conditions $p_0, p_1 \leq p$ such that $p_l \Vdash$"$\underset{\sim}{\tau}(i) = l$", $l = 0, 1$. Hence, we can define by recursion a sequence

$$\langle\langle q_{p,\eta}, i_{p,\eta}, \sigma_{p,\eta}\rangle : \eta \in ({}^{<\omega}2)\rangle$$

such that:

(1) $q_{p,\langle\rangle} = p$;
(2) $\nu \lhd \eta \Rightarrow q_{p,\eta} \leq q_{p,\nu}$;
(3) $i_{p,\eta}$ is the least ordinal i less than ω_1 such that $q_{p,\eta}$ does not decide $\underset{\sim}{\tau}(i)$;
(4) $q_{p,\eta} \Vdash$"$\forall j < i_{p,\eta}, \underset{\sim}{\tau}(i_{p,\eta \restriction j}) = \sigma_{p,\eta}(j)$".

It is evident that if $\nu \lhd \eta$, then $i_{p,\nu} < i_{p,\eta}$.

Claim 2.6. *For any $p \in \mathbb{Q}$, there exists a perfect subtree T_p of ${}^{\omega}2$ such that for some limit ordinal δ_p and every $\rho \in \mathrm{Lim}(T_p), \bigcup_n i_{p,\rho \restriction n} = \delta_p$, where $\mathrm{Lim}(T_p)$ is the set of all branches through T_p.*

Proof. For any $\eta \in ({}^{<\omega}2)$ set

$$\delta_{p,\eta} = \sup\{i_{p,\nu} : \eta \lhd \nu \in ({}^{<\omega}2)\}.$$

For some η_*, the ordinal δ_{p,η_*} is minimal. δ_{p,η_*} is a limit ordinal of cofinality $\aleph_0$, so let $\langle \eta_{p,m} : m < \omega \rangle$ be an increasing sequence cofinal in δ_{p,η_*} such that $\eta_{p,0} = \mathrm{lh}(\eta_*)$. We define $h_m : {}^m 2 \to {}^{\eta_{p,m}}2$, by induction on $m < \omega$, such that:

(1) h_m is $1-1$;
(2) $h_0(\langle\rangle) = \eta_*$;
(3) if $n < m$ and $\eta \in ({}^m 2)$, then $h_n(\eta \restriction n) \lhd h_m(\eta)$;
(4) if $\eta \in ({}^m 2)$, then $i_{p,h_m(\eta)} > \eta_{p,m}$.

Then $T_p = \{h_m(\eta) : m < \omega \text{ and } \eta \in ({}^m 2)\}$ and $\delta_p = \delta_{p,\eta_*}$ are as required. $\square$

For each limit ordinal $\delta < \omega_1$ set

$$I^1_\delta = \{p \in \mathbb{Q} : \delta_p = \delta\}.$$

Then clearly $\mathbb{Q} = \bigcup\{I^1_\delta : \delta \text{ is a limit ordinal less than } \omega_1\}$.

Claim 2.7. *Let δ be a countable limit ordinal. Then there exists an antichain $\bar{q}^\delta = \langle q^\delta_p : p \in I^1_\delta \rangle$ such that for each $p \in I^1_\delta, q^\delta_p \leq p$.*

Proof. Let $\langle p_\alpha : \alpha < \alpha_\delta \leq 2^{\aleph_0} \rangle$ enumerate I^1_δ. We choose, by induction on α, a pair $\langle r_\alpha, v_\alpha \rangle$ such that:

(1) $r_\alpha \leq p_\alpha$ and $v_\alpha \in ({}^\delta 2)$;
(2) $r_\alpha \Vdash$ "$\underset{\sim}{\tau} \restriction \delta_{p_\alpha} = v_\alpha$;
(3) $\alpha \neq \beta \Rightarrow v_\alpha \neq v_\beta$.

Suppose $\alpha < \alpha_\delta$ and we have defined $\langle r_\beta, v_\beta \rangle$ for all $\beta < \alpha$ as above. We define $\langle r_\alpha, v_\alpha \rangle$.

For every $\rho \in \mathrm{Lim}\ (T_{p_\alpha})$, the sequence $\langle q_{p,\rho\restriction n} : n < \omega \rangle$ is a decreasing chain of conditions in $\mathbb{Q}$, and hence there is a condition $q^*_{\rho,\alpha}$ which extends all of them. We may further suppose that it forces a value $v_{\rho,\alpha}$ for $\underset{\sim}{\tau} \restriction \delta$, where $\delta = \delta_{p_\alpha}$. Also note that by the choice of $\langle q_{\rho,\alpha} : \rho \in ({}^{<\omega}2)\rangle$, for $\rho_1 \neq \rho_2$ in $\mathrm{Lim}\ (T_{p_\alpha})$, we have $v_{\rho_1,\alpha} \neq v_{\rho_2,\alpha}$. Now $\{v_\beta : \beta < \alpha\} \subseteq {}^\delta 2$, hence for some $\rho = \rho_\alpha \in \mathrm{Lim}\ (p_\alpha)$, we have that $v_{\rho,\alpha} \notin \{v_\beta : \beta < \alpha\}$. Let $r_\alpha = q_{\rho_\alpha,\alpha}$ and $v_\alpha = v_{\rho_\alpha,\alpha}$. $\square$

Now, for each limit ordinal $\delta < \omega_1$ let I_δ be a maximal antichain of $\mathbb{Q}$, such that $I_\delta \supseteq \{q_{p,\delta} : p \in I^1_\delta\}$, and let $\mathbb{Q}' = \bigcup\{I_\delta : \delta \text{ is a countable limit ordinal}\}$. Then $\mathbb{Q}'$ is as required which completes the proof of Lemma 2.5.

As each I^*_i is a maximal antichain in $\mathbb{Q}'$ and hence also in $\mathbb{Q}$, it can easily seen that there are $\bar{p}^*_i, i < \omega_1$, such that:

(1) $\bar{p}^*_i = \langle p^*_\eta : \eta \in ({}^{i+1}(2^{\aleph_0}))\rangle$ is a maximal antichain of $\mathbb{Q}'$ (and hence of $\mathbb{Q}$);

(2) if $j < i$ and $\eta \in ({}^{i+1}(2^{\aleph_0}))$, then $p^*_\eta \leq p^*_{\eta \restriction (j+1)}$;
(3) if $i = j + 1$ and $\eta \in ({}^{i+1}(2^{\aleph_0}))$, then p^*_η is stronger than some condition in I^*_i.

Let

$$\mathbb{Q}'' = \{p^*_\eta : \exists i < \omega_1, \eta \in ({}^{i+1}(2^{\aleph_0}))\}.$$

Lemma 2.8. *$\mathbb{Q}''$ is a dense subset of $\mathbb{Q}$.*

Proof. Let $p \in \mathbb{Q}$. By Lemma 2.5, we can find some $i < \omega_1$ and some $p_1 \in I^*_i \subseteq \mathbb{Q}'$ such that $p_1 \leq p$. By above (3), each $p^*_\eta, \eta \in ({}^{i+1}(2^{\aleph_0}))$, is stronger than some condition in I^*_i. If there is no η with $p^*_\eta \leq p_1$, then we contradict with above (1). The result follows immediately. $\square$

Finally note that the map

$$\eta \mapsto p^*_\eta$$

defines an isomorphism between a dense subset of $\mathrm{Col}(\aleph_1, 2^{\aleph_0})$ and $\mathbb{Q}''$. It follows that

$$\mathbb{Q} \simeq \mathbb{Q}'' \simeq \mathrm{Col}(\aleph_1, 2^{\aleph_0}) \simeq \mathrm{Add}(\aleph_1, 1).$$

The theorem follows.

3. A Note on $\aleph_1$-Closed Forcing Notions of Size Continuum

In this section, we present a result about $\aleph_1$-closed forcing notions of size continuum which will be used in Sec. 5 for the proof of Theorem 1.3.

Assume that $2^{\aleph_0} = \aleph_2$ and that $\mathbb{R}$ is an $\aleph_1$-closed forcing notions of size continuum which does not collapse $\aleph_2$. It then follows from [9] that the forcing notion $\mathbb{R}$ does not add a fresh sequence of ordinals of size $\aleph_1$ and hence it is $\aleph_2$-distributive. The following result is proved in [1, Theorem 2.1].

Lemma 3.1. *There exists a sequence $\langle T_\alpha : \alpha < \aleph_2\rangle$ of subsets of $\mathbb{R}$ such that:*

(1) *Each T_α is a maximal antichain in $\mathbb{R}$.*
(2) *If $T = \bigcup\{T_\alpha : \alpha < \aleph_2\}$, then $(T, \geq_{\mathbb{R}})$ is a tree of height $\aleph_2$, where T_α is the αth level of T.*
(3) *Each $t \in T$ has $\aleph_2$-many immediate successors.*
(4) *T is dense in $\mathbb{R}$.*

We denote the above tree T by $T(\mathbb{R})$, and call it a base tree of $\mathbb{R}$. Note that by clause (4), $\mathbb{R} \simeq T(\mathbb{R})$.

4. Specializing $\aleph_2$-Trees Which Have Few Branches

In this section, we consider trees of size and height κ which have $\leq \kappa$-many branches, and define a suitable forcing notion for specializing them. As we allow our trees to have cofinal branches, we need a slightly different definition of the concept of a special tree than the usual ones.

Definition 4.1. Let $\kappa = \varrho^+$, where ϱ is a regular cardinal.

(1) A κ-tree is a tree of height and size κ (so we allow the levels of the tree to have size κ).
(2) Let T be a κ-tree [2]. T is *special* if there exists a function $F : T \to \varrho$ such that for all $x, y, z \in T$ if $x \leq_T y, z$ and $F(x) = F(y) = F(z)$, then either $y \leq_T z$ or $z \leq_T y$.

By [2, Theorem 8.1], a κ-special tree has at most κ-many cofinal branches.

Let $\kappa = \varrho^+$, where ϱ is a regular cardinal. Let also $\theta > \kappa$ be large enough regular and let $\prec$ be a well ordering of $H(\theta)$. Let Λ denote the set of all κ-trees T with at most κ-many cofinal branches, such that for all $t \in T, \mathrm{Suc}_T(t)$, the set of successors of t in T, has size κ.

We define a map $\star : \Lambda \to \Lambda$, where to each $T \in \Lambda$, assigns a subtree $T^\star = \star(T)$ of T, such that $T^\star$ is dense in T and it has no cofinal branches. Thus let $T \in \Lambda$. Let $\langle b_\alpha : \alpha < \kappa \rangle$ be the $\prec$-least enumeration of the cofinal branches through T, and for each $\alpha < \kappa$ set

$$s_\alpha = \text{the } \leq_T \text{-least element of } b_\alpha \Big\backslash \bigcup_{\beta<\alpha} b_\beta.$$

Finally, set

$$T^\star = \{t \in T : \neg(\exists \alpha,\ s_\alpha <_T t \in b_\alpha)\}.$$

Lemma 4.2. (a) *$T^\star$ has no cofinal branches.*

(b) *$(T^\star, \geq_T)$ is dense in $(T, \geq_T)$ (when considered as forcing notions), in particular $(T^\star, \geq_T) \simeq (T, \geq_T)$.*

Proof. (a) Assume not, and let b be a branch through $T^\star$. then for some $\alpha, b \subseteq b_\alpha$, and then clearly $b \cap (T \backslash T^\star) \neq \emptyset$, which is a contradiction.

(b) Let $t \in T$. If $t \in T^\star$, then we are done; so assume that $t \notin T^\star$. Then for some $\alpha < \kappa, s_\alpha <_T t \in b_\alpha$. Let $t' \in \mathrm{Suc}_T(t) \backslash \bigcup_{\beta \leq \alpha} b_\beta$. Then $t' \in T^\star$ and $t' \geq_T t$. □

Lemma 4.3. *Assume there exists $F : T^\star \to \varrho$ such that if $F(x) = F(y)$, then x and y are incomparable in T. Then there exists $F' : T \to \varrho$ such that $F' \supseteq F$ and F' specializes T.*

Proof. Define $G : (T \backslash T^\star) \to \varrho$ as follows: Let $t \in (T \backslash T^\star)$. Then for some $\alpha < \kappa$, $s_\alpha <_T t \in b_\alpha$. Set $G(t) = F(s_\alpha)$. It is now easily seen that $F' = F \cup G$ is as required. □

Thus, in order to define a forcing notion which specializes T, it suffices to define a forcing notion which adds a function $F : T^\star \to \varrho$ as in Lemma 4.3.

Definition 4.4. The forcing notion $\mathbb{Q}(T^\star)$, for specializing $T^\star$, is defined as follows:

(a) A condition in $\mathbb{Q}(T^\star)$ is a partial function $f : T^\star \to \varrho$ such that:

(1) $\mathrm{dom}(f)$ has size $< \varrho$;
(2) if $x <_T y$ and $x, y \in \mathrm{dom}(f)$ then $f(x) \neq f(y)$.

(b) $f \leq_{\mathbb{Q}(T^\star)} g$ if and only if $f \supseteq g$.

It is clear that the forcing notion $\mathbb{Q}(T^\star)$ is ϱ-directed closed. But in general, there is no guarantee that the forcing $\mathbb{Q}(T^\star)$ satisfies the κ-c.c., or preserves all cardinals, even if we assume GCH (see [5, 14]).

Lemma 4.5. *Forcing with $\mathbb{Q}(T^\star) * T$ collapses κ into ϱ.*

Proof. By Lemma 4.2(b), $\mathbb{Q}(T^\star) * T \simeq \mathbb{Q}(T^\star) * T^\star$. Let G be $\mathbb{Q}(T^\star)$-generic over V and H be $T^\star$-generic over $V[G]$. Let also $F = \bigcup\{f : f \in G\}$. Then $F : T^\star \to \varrho$ and for all $x <_T y$ in T^* we have $F(x) \neq F(y)$. Let $b \in V[G * H]$ be a cofinal branch in $T^\star$. Then $F \restriction b : b \to \varrho$ is an injection, and $|b| = \kappa$. Hence, κ is collapsed into ϱ. $\square$

Given an infinite cardinal κ, let $\mathrm{Add}(\aleph_0, \kappa)$ denote the Cohen forcing for adding κ-many new Cohen reals; thus conditions are finite partial functions $p : \kappa \times \omega \to \{0, 1\}$ ordered by reverse inclusion. The forcing is c.c.c., and hence it preserves all cardinals and cofinalities.

For our purpose in Sec. 5, we will work with $\mathrm{Add}(\aleph_0, \kappa)$-names of trees, and we now modify the above results to cover this case.

Assume $\kappa = \varrho^+$, where ϱ is a regular cardinal and let $\underset{\sim}{T}$ be an $\mathrm{Add}(\aleph_0, \kappa)$-name for a κ-tree which is forced to have $\leq \kappa$-many cofinal branches. Let $\underset{\sim}{T}^\star$ be an $\mathrm{Add}(\aleph_0, \kappa)$-name such that it is forced by $\mathrm{Add}(\aleph_0, \kappa)$ that "$\underset{\sim}{T}^\star$ is the subtree of $\underset{\sim}{T}$ defined using the function $\star$". We assume, without loss of generality, that it is forced by $\mathrm{Add}(\aleph_0, \kappa)$ that "the set of nodes of $\underset{\sim}{T}^\star$ is $\kappa \times \kappa$ and for each $\alpha < \kappa$, the αth level of $\underset{\sim}{T}^\star$ is $\{\alpha\} \times \kappa$". We now define $\mathbb{Q}_A(\underset{\sim}{T}^\star) \in V$ as follows.

Definition 4.6. (a) A condition in $\mathbb{Q}_A(\underset{\sim}{T}^\star)$ is a partial function $f : \kappa \times \kappa \to \varrho$ such that:

(1) $\mathrm{dom}(f)$ is a subset of $\kappa \times \kappa$ of size $< \varrho$.
(2) If $x, y \in \mathrm{dom}(f)$ and $f(x) = f(y)$, then $\Vdash_{\mathrm{Add}(\aleph_0,\kappa)}$ "x and y are incompatible in the tree ordering, $x \perp y$".

(b) $f \leq_{\mathbb{Q}_A(\underset{\sim}{T}^\star)} g$ if and only if $f \supseteq g$.

Note that we defined the forcing notion $\mathbb{Q}_A(\underset{\sim}{T}^\star)$ in V and not in the generic extension by $\mathrm{Add}(\aleph_0, \kappa)$.

Lemma 4.7. (a) $\mathbb{Q}_A(\underset{\sim}{T}^\star)$ *is ϱ-directed closed.*

(b) *Let G be $\mathbb{Q}_A(\underset{\sim}{T}^\star)$-generic over V. Then in $V[G]$, there exists a function $F : \kappa \times \kappa \to \varrho$, such that for any H which is $\mathrm{Add}(\aleph_0, \kappa)$-generic over $V[G]$, F is a specializing function for $\underset{\sim}{T}^\star[H]$.*

The next lemma can be proved as in Lemma 4.5.

Lemma 4.8. *Let $\underset{\sim}{T}$ be an $\mathrm{Add}(\aleph_0, \kappa)$-name for a κ-tree which has $\leq \kappa$-many cofinal branches. Then*

$$\Vdash_{\mathbb{Q}_A(\underset{\sim}{T}^\star) * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“forcing with } \underset{\sim}{T} \text{ collapses } \kappa\text{”}.$$

Proof. By Lemma 4.2(b),

$$(\mathbb{Q}_A(\underset{\sim}{T}^\star) * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)) * \underset{\sim}{T} \simeq (\mathbb{Q}_A(\underset{\sim}{T}^\star) * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)) * \underset{\sim}{T}^\star,$$

and hence it suffices to show that

$$\Vdash_{\mathbb{Q}_A(\underset{\sim}{T}^\star) * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“forcing with } \underset{\sim}{T}^\star \text{ collapses } \kappa\text{”}.$$

Let $(G_1 * G_2) * H$ be $(\mathbb{Q}_A(\underset{\sim}{T}^\star) * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)) * \underset{\sim}{T}^\star$-generic over V and $F = \bigcup\{f : f \in G_1\}$. By Lemma 4.7, $F : \kappa \times \kappa \to \varrho$, and if $T^\star = \underset{\sim}{T}^\star[G_2]$, then for $x <_{T^\star} y$, $F(x) \neq F(y)$.

Let $b \in V[(G_1 * G_2) * H]$ be a cofinal branch of $T^\star$. Then $F \restriction b : b \to \varrho$ is an injection, and hence κ is collapsed into ϱ. $\square$

5. A Negative Answer to Williams Question When the Continuum is Regular

In this section, we prove Theorem 1.3. In Sec. 5.1, we define the main forcing construction $\mathbb{P}$ and prove some of its basic properties. In Sec. 5.2, we show that forcing with $\mathbb{P}$ preserves κ. Then in Sec. 5.3 more properties of the forcing notion $\mathbb{P}$ are proved and finally in Sec. 5.4 we complete the proof of Theorem 1.3.

5.1. *The main forcing construction and its basic properties*

In this section, we define the main forcing notion that will be used in the proof of Theorem 1.3. Let κ be a supercompact cardinal, and let $\lambda > \kappa$ be a 2-Mahlo cardinal. By [12], we may assume that κ is Laver indestructible, i.e. the supercompactness of κ is preserved under κ-directed closed forcing notions, and that GCH holds at and above κ.

Let $\Phi : \lambda \to H(\lambda)$ be such that for each $x \in H(\lambda)$, $\Phi^{-1}(x) \cap \{\beta+2 : \beta \text{ is Mahlo}\}$ is unbounded in λ. Such a Φ exists as $|H(\lambda)| = 2^{<\lambda} = \lambda$ and λ is a 2-Mahlo cardinal.

We define, by induction on $\alpha \leq \lambda$, an iteration

$$\mathbb{P} = \langle\langle \mathbb{P}_\alpha : \alpha \leq \lambda\rangle, \langle \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda\rangle\rangle$$

of forcing notions of length λ. Suppose $\alpha \leq \lambda$ and we have defined $\mathbb{P}_\beta$, for all $\beta < \alpha$. We define $\mathbb{P}_\alpha$ as follows.

Definition 5.1. A condition p is in $\mathbb{P}_\alpha$, if and only if p is a function with domain α such that for every $\beta < \alpha, \Vdash_\beta$"$p(\beta) \in \underset{\sim}{\mathbb{Q}}_\beta$", where:

(1) $\mathrm{supp}(p)$ has size less than κ, where $\mathrm{supp}(p)$ denotes the support of p.
(2) $\{\beta \in \mathrm{supp}(p) : \beta \equiv 0 \pmod 3$ or $\beta \equiv 2 \pmod 3\}$ has size less than $\aleph_1$.
(3) If $\beta < \kappa$ and $\beta \equiv 0 \pmod 3$ or $\beta \equiv 2 \pmod 3$, then $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\aleph_1, \aleph_2 + |\beta|)$".
(4) If $\beta \geq \kappa$, $\beta \equiv 0 \pmod 3$ and β is inaccessible, then $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Add}}(\aleph_1, \kappa)$".
(5) If $\beta \geq \kappa$, $\beta \equiv 1 \pmod 3$ and $\beta - 1$ is inaccessible, then $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\kappa, 2^{|\mathbb{P}_\beta|}) = \underset{\sim}{\mathrm{Col}}(\kappa, 2^{|\beta|})$" (as $|\mathbb{P}_\beta| = |\beta|$).
(6) If $\beta \geq \kappa$, $\beta \equiv 2 \pmod 3$, $\beta - 2$ is inaccessible and if $\Vdash_\beta$ "$\kappa = \aleph_2$" and $\Phi(\beta)$ is a $\mathbb{P}_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name for a κ-tree which has $\leq \kappa$-many cofinal branches, then $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathbb{Q}}_A(\Phi(\beta)^\star)$".[b]
(7) Otherwise, $\Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is the trivial forcing notion".

Also, set $\mathbb{P} = \mathbb{P}_\lambda$.

The next lemma gives some basic properties of the forcing notion $\mathbb{P}$.

Lemma 5.2. (a) $\mathbb{P}$ *is* $\aleph_1$*-directed closed, and hence it preserves CH.*
(b) *If* $\mu \in (\kappa, \lambda]$ *is Mahlo, then* $\mathbb{P}_\mu$ *satisfies the* μ-c.c.
(c) $\mathbb{P}_\lambda$ *collapses all cardinals in* $(\aleph_1, \kappa)$ *into* $\aleph_1$, *so, if* κ *is not collapsed, then* $\Vdash_\mathbb{P}$ "$\kappa = \aleph_2$".
(d) *In* $V^\mathbb{P}, \lambda$ *is preserved, but all* $\mu \in (\kappa, \lambda)$ *are collapsed, so, if* κ *is not collapsed, then* $\Vdash_\mathbb{P}$ "$\lambda = \kappa^+ = \aleph_3$".
(e) $\Vdash_\mathbb{P}$ "$2^{\aleph_1} = \lambda$".

Proof. (a) is clear as all forcing notions considered in the iteration are $\aleph_1$-directed closed and the support of the iteration is at least countable.

(b) Assume $A \subseteq \mathbb{P}_\mu$ is a maximal antichain of size μ and let $\langle p^\xi : \xi < \mu\rangle$ be an enumeration of A. Define $F : \mu \to \mu$ by $F(\xi) =$ the least η such that $\mathrm{supp}(p^\xi) \restriction \xi \subseteq \eta$. F is a regressive function on the stationary set $X = \{\xi < \mu : \xi \text{ is inaccessible}\}$, and hence F is constant on some stationary subset Y of X. Let η be the resulting fixed value. So, for all $\xi \in Y, \mathrm{supp}(p^\xi) \restriction \xi \subseteq \eta$. We may further suppose that if $\xi_1 < \xi_2$ are in Y, then $\mathrm{supp}(p^{\xi_1}) \subseteq \xi_2$.

As $\mathbb{P}_\eta$ has size less than μ, there are $\xi_1 < \xi_2$ in Y such that $p^{\xi_1} \restriction \eta$ is compatible with $p^{\xi_2} \restriction \eta$. But then in fact p^{ξ_1} is compatible with p^{ξ_2} and we get a contradiction.

(c), (e) and the fact that forcing with $\mathbb{P}_\lambda$ collapses all cardinals in $(\aleph_1, \kappa)$ into $\aleph_1$ are clear and the rest of (d) follows from (b). The lemma follows. $\square$

[b] $\Phi(\beta)^\star$ is defined from $\Phi(\beta)$ as in Sec. 4 using a fixed well ordering of a large initial segment of the universe.

5.2. *Preservation of κ*

In this section, we show that forcing with $\mathbb{P}$ preserves κ. Let

$$G = \langle \langle G_\alpha : \alpha \le \lambda \rangle, \langle H_\alpha : \alpha < \lambda \rangle \rangle$$

be $\mathbb{P}$-generic over V, i.e. each G_α is $\mathbb{P}_\alpha$-generic over V and H_α is $\underset{\sim}{\mathbb{Q}}_\alpha[G_\alpha]$-generic over $V[G_\alpha]$.

For each $\alpha \le \lambda$, we define the forcing notion $\mathbb{P}^U_\alpha \in V$, the $\mathbb{P}^U_\alpha$-name $\underset{\sim}{\mathbb{P}}^C_\alpha$ for a forcing notion, in such a way that:

(a) There are projections $\chi_\alpha : \mathbb{P}_\alpha \to \mathbb{P}^U_\alpha$ and $\pi_\alpha : \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha \to \mathbb{P}_\alpha$.

(b) If $G^U_\alpha = \chi_\alpha[G_\alpha]$, for $\alpha \le \lambda$, then there exists a function $\Psi \in V[G^U_\lambda]$ such that for each ordinal $\alpha = \beta + 2 > \kappa$, where β is inaccessible, $\Psi(\alpha) \in V[G^U_\alpha]$ is a $\mathbb{P}^C_\alpha * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name such that

$$\Phi(\alpha)[G^U_\alpha * H] = \Psi(\alpha)[H],$$

for any H which is $\mathbb{P}^C_\alpha * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-generic over $V[G^U_\alpha]$. Further, if $\Phi(\alpha)$ is a $\mathbb{P}_\alpha * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name for a κ-tree with $\le \kappa$-many cofinal branches, then in $V[G^U_\lambda]$, $\Psi(\alpha)$ is a $\mathbb{P}^C_\alpha * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name for a κ-tree with $\le \kappa$-many cofinal branches.

(c) $\mathbb{P}^U_\alpha$ is κ-directed closed.

(d) For every $\gamma \in [\alpha, \lambda]$, $\mathbb{P}^U_\gamma \Vdash$ "$\underset{\sim}{\mathbb{P}}^C_\alpha$ is $\aleph_1$-directed closed".[c]

Let us first define the forcing notions $\mathbb{P}^U_\alpha$ and the corresponding projections χ_α. Let

$$U = \{\beta < \lambda : \beta \equiv 1 \pmod 3 \text{ and } \beta - 1 \text{ is inaccessible}\}.$$

For $\beta \in U$, let $\mathbb{Q}^U_\beta$ be the term forcing, whose conditions are $\mathbb{P}_\beta$-names $\underset{\sim}{p}$ such that $\Vdash_{\mathbb{P}_\beta}$"$\underset{\sim}{p} \in \underset{\sim}{\mathrm{Col}}(\kappa, 2^{|\beta|})$", ordered by $\underset{\sim}{p} \le_{\mathbb{Q}^U_\beta} \underset{\sim}{q}$ if and only if $\Vdash_{\mathbb{P}_\beta}$"$\underset{\sim}{p} \le_{\underset{\sim}{\mathrm{Col}}(\kappa, 2^{|\beta|})} \underset{\sim}{q}$". Then set $\mathbb{P}^U_\alpha$ be the $< \kappa$-support product of the forcing notions $\mathbb{Q}^U_\beta$, where $\beta \in U \cap \alpha$. Then there is a natural projection $\chi_\alpha : \mathbb{P}_\alpha \to \mathbb{P}^U_\alpha$.

We now define, by induction on $\alpha \le \lambda$, the $\mathbb{P}^U_\alpha$-name $\underset{\sim}{\mathbb{P}}^C_\alpha$ and the corresponding projection $\pi_\alpha : \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha \to \mathbb{P}_\alpha$. We also inductively verify (d) along the way.

(1) $\alpha \le \kappa$: Let $\mathbb{P}^C_\alpha$ to be $\mathbb{P}_\alpha / \chi^{-1}_\alpha[\dot{G}^U_\alpha]$. It is then clear that $\mathbb{P}^C_\alpha \simeq \mathbb{P}_\alpha$ and there exist a forcing isomorphism $\pi_\alpha : \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha \simeq \mathbb{P}_\alpha$. It is also clear that clause (d) holds.

[c]It is also possible to define the forcing notions $\mathbb{P}^U_\alpha \in V$ and $\mathbb{P}^C_\alpha \in V[G^U_\alpha]$ directly, by setting

$$\mathbb{P}^U_\alpha = \{p \in \mathbb{P}_\alpha : \mathrm{supp}(p) \subseteq \{\beta < \lambda : \beta \equiv 1 \pmod 3\}\}$$

and

$$\mathbb{P}^C_\alpha = \{p \in \mathbb{P}_\alpha : \mathrm{supp}(p) \subseteq \{\beta < \lambda : \beta \equiv 0 \pmod 3 \text{ or } \beta \equiv 2 \pmod 3\} \text{ and } p \text{ is compatible with } G^U_\alpha\},$$

where $G^U_\alpha = G_\alpha \cap \mathbb{P}^U_\alpha$. We will give a more explicit definition for $\mathbb{P}^C_\alpha$, that will be useful in the proof of Lemma 5.3.

(2) $\alpha=\beta+1>\kappa$, $\beta\equiv 0 \pmod 3$ *and β is inaccessible*: Then set

$$\Vdash_{\mathbb{P}^U_\alpha} \text{“}\underset{\sim}{\mathbb{P}}^C_\alpha \simeq \underset{\sim}{\mathbb{P}}^C_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_1,\kappa)\text{”}.$$

By the induction, there is a projection $\pi_\beta : \mathbb{P}^U_\beta * \underset{\sim}{\mathbb{P}}^C_\beta \to \mathbb{P}_\beta$. Since $\mathbb{P}^U_\alpha = \mathbb{P}^U_\beta$, and since by clause (d) we can regard each $\mathbb{P}^U_\beta * \underset{\sim}{\mathbb{P}}^C_\beta$-name for an element of $\mathrm{Add}(\aleph_1,\kappa)$ as a $\mathbb{P}_\beta$-name, this induces the projection $\pi_\alpha : \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha \to \mathbb{P}_\alpha$, which is defined by $\pi_\alpha(p,(\underset{\sim}{q},\underset{\sim}{r})) = (\pi_\beta(p,\underset{\sim}{q}),\underset{\sim}{r})$. It is also clear that clause (d) holds in this case.

(3) $\alpha=\beta+1>\kappa$, $\beta\equiv 1 \pmod 3$ *and $\beta-1$ is inaccessible*: Let $\Vdash_{\mathbb{P}^U_\beta} \text{“}\underset{\sim}{\mathbb{P}}^C_\alpha = \underset{\sim}{\mathbb{P}}^C_\beta\text{”}$[d]. In this case $\mathbb{P}^U_\alpha = \mathbb{P}^U_\beta \times \mathbb{Q}^U_\beta$, and by the induction hypothesis, there is a projection $\pi_\beta : \mathbb{P}^U_\beta * \underset{\sim}{\mathbb{P}}^C_\beta \to \mathbb{P}_\beta$. This induces the projection $\pi_\alpha : \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha \to \mathbb{P}_\alpha$, which is defined by $\pi_\alpha((p,\underset{\sim}{q}),\underset{\sim}{r}) = (\pi_\beta(p,\underset{\sim}{r}),\underset{\sim}{q})$. Again clause (d) is easily verified.

(4) $\alpha=\beta+1>\kappa$, $\beta\geq\kappa$, $\beta\equiv 2 \pmod 3$ *and $\beta-2$ is inaccessible*: Then we may assume that $\Phi(\beta)$ is a $\mathbb{P}_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)$-name for a κ-tree which has $\leq\kappa$-many cofinal branches, as otherwise the forcing at stage β is the trivial forcing and the result follows from the induction hypothesis. By the induction hypothesis, we have projections $\chi_\beta,\ \pi_\beta$, and so there exists $\Psi(\beta)\in V[G^U_\beta]$ as in clause (b) above. Set

$$\Vdash_{\mathbb{P}^U_\alpha} \text{“}\underset{\sim}{\mathbb{P}}^C_\alpha \simeq \underset{\sim}{\mathbb{P}}^C_\beta * \underset{\sim}{\mathbb{Q}}_A(\Psi(\beta)^\star)\text{”}.$$

By the choice of $\Psi(\beta)$, it is clear that we have a natural projection $\pi_\alpha : \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha \to \mathbb{P}_\alpha$, which extends π_β. Clause (d) is easily verified in this case as well.

(4) $\alpha=\beta+1$ *is not as above*: Then set

$$\Vdash_{\mathbb{P}^U_\alpha} \text{“}\underset{\sim}{\mathbb{P}}^C_\alpha = \underset{\sim}{\mathbb{P}}^C_\beta\text{”}.$$

Set also $\pi_\alpha=\pi_\beta$. It is also clear that clause (d) holds.

(6) *α is a limit ordinal*: Then set

$$\Vdash_{\mathbb{P}^U_\alpha} \text{“}\underset{\sim}{\mathbb{P}}^C_\alpha \quad \text{is the countable support iteration of } \langle \underset{\sim}{\mathbb{P}}^C_\beta : \beta<\alpha\rangle\text{”}.$$

Note that π_α can be defined in a uniform way from π_β's, $\beta<\alpha$. To be more precise, let $\pi_\alpha : \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha \to \mathbb{P}_\alpha$ be defined by

$$\pi_\alpha(p,\langle \underset{\sim}{q}_\beta : \beta<\alpha\rangle) = \langle \pi_\beta(p,\underset{\sim}{q}_\beta) : \beta<\alpha\rangle.$$

It is evident that $\pi_\alpha(1_{\mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha}) = 1_{\mathbb{P}_\alpha}$ and that π_α is order preserving. Now suppose $\langle p,\langle \underset{\sim}{q}_\beta : \beta<\alpha\rangle\rangle \in \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha$, $r\in\mathbb{P}_\alpha$ and suppose that $r\leq \pi_\alpha(p,\langle \underset{\sim}{q}_\beta : \beta<\alpha\rangle) = \langle \pi_\beta(p,\underset{\sim}{q}_\beta) : \beta<\alpha\rangle$. By 5.1(2), the set

$$S=\{\beta\in\mathrm{supp}(r) : \beta\equiv 0 \pmod 3 \text{ or } \beta\equiv 2 \pmod 3\}$$

[d]Note that we can consider $\underset{\sim}{\mathbb{P}}^C_\beta$ as a $\mathbb{P}^U_\alpha$-name as well.

is at most countable. By induction on $\beta \in S$, we can find $(\bar{p}_\beta, \underset{\sim}{\bar{q}}_\beta) \in \mathbb{P}^U_\beta * \underset{\sim}{\mathbb{P}}^C_\beta$ such that:

(i) $\pi_\beta(\bar{p}_\beta, \underset{\sim}{\bar{q}}_\beta) \leq r \restriction \beta$;

(ii) if $\beta_0 < \beta_1$ are in S, then $\bar{p}_{\beta_1} \leq \bar{p}_{\beta_0}$.

Each $\bar{p}_\beta$, for $\beta \in S$, is in $\mathbb{P}^U_\alpha$, and since it is κ-closed, we can find $\bar{p} \in \mathbb{P}^U_\alpha$ which extends all $\bar{p}_\beta$'s, $\beta \in S$. Then $(\bar{p}, \langle \underset{\sim}{\bar{q}}_\beta : \beta < \alpha \rangle) \in \mathbb{P}^U_\alpha * \underset{\sim}{\mathbb{P}}^C_\alpha$, where for $\beta \in \alpha \backslash S$, $\Vdash_{\mathbb{P}^U_\beta}$ "$\underset{\sim}{\bar{q}}_\beta = 1_{\underset{\sim}{\mathbb{P}}^C_\beta}$", and it satisfies

$$\pi_\alpha(\bar{p}, \langle \underset{\sim}{\bar{q}}_\beta : \beta \in S \rangle) = \langle \pi_\beta(\bar{p}, \underset{\sim}{\bar{q}}_\beta) : \beta < \alpha \rangle \leq r.$$

Clause (d) follows easily from the induction hypothesis and the fact that $\mathbb{P}^U_\beta$'s are κ-directed closed and hence κ-distributive.

Lemma 5.3. *For every $\alpha \leq \lambda$, $\mathbb{P}^U_\lambda \Vdash$ "$\underset{\sim}{\mathbb{P}}^C_\alpha$ is κ-c.c.".*

Proof. Let G^U_λ be $\mathbb{P}^U_\lambda$-generic over V and for each $\beta < \lambda$ let $G^U_\beta = G^U_\lambda \cap \mathbb{P}^U_\beta$. Then G^U_β is $\mathbb{P}^U_\beta$-generic over V. It follows that for any ordinal $\alpha = \beta + 1 > \kappa$, where $\beta \equiv 2 \pmod 3$ and $\beta - 2$ is inaccessible, we have

$$V[G^U_\lambda] \models \text{“}\mathbb{P}^C_{\alpha+1} \simeq \mathbb{P}^C_\alpha * \underset{\sim}{\mathbb{Q}}_A(\Psi(\alpha)^\star)\text{”}.$$

As $\mathbb{P}^U_\lambda$ is κ-directed closed and κ is assumed to be Laver indestructible, κ remains supercompact, and hence weakly compact, in $V[G^U_\lambda]$.

Working in $V[G^U_\lambda]$, let $\mathcal{F}$ be the weakly compact filter on κ, i.e. the filter on κ generated by the sets $\{\lambda < \kappa : (V_\lambda, \in, B \cap V_\lambda) \models \psi\}$, where $B \subseteq V_\kappa$ and ψ is a Π^1_1 sentence for the structure $(V_\kappa, \in, B)$. Let also $\mathcal{S}$ be the collection of $\mathcal{F}$-positive sets, i.e. $\mathcal{S} = \{X \subseteq \kappa : \forall B \in \mathcal{F}, X \cap B \neq \emptyset\}$.

The proof of the next claim is as in the proof of [10, Lemma 2.13], where the forcing notions $\mathbb{P}^2_\alpha$ and $\mathbb{P}^1_\alpha$ there, are replaced with $\mathbb{P}^C_\alpha$ and $\mathrm{Add}(\aleph_0, \kappa)$.[e]

Claim 5.4. *Work in $V[G^U_\lambda]$. Let $\alpha \leq \lambda$. For any sequence $\langle q_i : i < \kappa \rangle$ of conditions in $\mathbb{P}^C_\alpha$, there exist a set $X \in \mathcal{S}$ and two sequences $\langle q^1_i : i \in X \rangle$ and $\langle q^2_i : i \in X \rangle$ of conditions in $\mathbb{P}^C_\alpha$, such that:*

- *For all $i \in X$, $q^1_i, q^2_i \leq q_i$.*
- *For all $i < j$ in X, q^1_i is compatible with q^2_j, and this is witnessed by a condition q, such that for every $\xi < \alpha$, $q \restriction \xi \Vdash$ "$q(\xi) = q^1_i(\xi) \cup q^2_j(\xi)$".*

By Claim 5.4, for every $\alpha \leq \lambda$,

$$V[G^U_\lambda] \models \text{“}\mathbb{P}^C_\alpha \text{ is } \kappa\text{-c.c.”},$$

and the lemma follows. $\square$

The following is immediate.

[e]In [10], only forcing notions for specializing $\mathbb{P}^1_\alpha$-names for trees are considered, while in our forcing, we also consider the Cohen forcing $\mathrm{Add}(\aleph_1, \kappa)$, but this does not produce any problems, as the forcing $\mathrm{Add}(\aleph_1, \kappa)$ is well-behaved and is κ-c.c.

Lemma 5.5. $V^{\mathbb{P}} \models$ "$CH + \kappa = \aleph_2 + \lambda = \aleph_3 = 2^{\aleph_1}$" .

5.3. *More on the forcing notion* $\mathbb{P}$

In this section, we prove a few more properties of the forcing notions $\mathbb{P}$ that will be used in the proof of Theorem 1.3.

Lemma 5.6. *Assume that* $\mu \in (\kappa, \lambda)$ *is a Mahlo cardinal. Let* $\underset{\sim}{T}$ *be a* $\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$*-name of a* κ*-tree. Then*

$$\Vdash_{\mathbb{P}_{\mu+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“}\underset{\sim}{T} \textit{ has } \leq \kappa\textit{-many } \kappa\textit{-branches”}.$$

Proof. Let G_μ be $\mathbb{P}_\mu$-generic over V and $V_1 = V[G_\mu]$. In V_1, $\underset{\sim}{T}$ can be considered as an $\mathrm{Add}(\aleph_0, \kappa)$-name. Note that in V_1, $\kappa = \aleph_2$, $\mu = \aleph_3$ and $2^{\aleph_1} = \aleph_3$. Further, we have

$$\mathbb{P}_{\mu+2}/\mathbb{P}_\mu \simeq \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} = \mathrm{Add}(\aleph_1, \kappa) * \underset{\sim}{\mathrm{Col}}(\kappa, 2^\mu),$$

and

$$\Vdash_{\mathbb{P}_{\mu+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“}|\{b \in V_1 : b \text{ is a branch of } \underset{\sim}{T}\}| \leq |(2^\kappa)^{V_1}| = \kappa\text{”}.$$

So, it suffices to show that forcing with $\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$ adds no new cofinal branches. Assume by contradiction that $\underset{\sim}{\eta}$ is a $\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name which is forced to be a new κ-branch of $\underset{\sim}{T}$. The next claim follows easily from the assumption that $\underset{\sim}{\eta}$ is forced to be a new branch.

Claim 5.7. *For every* $\langle p^0, p^1, p^2 \rangle \in \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$, *there are conditions* $\langle q_i^0, q_i^1, q_i^2 \rangle$, *for* $i = 0, 1$, $\delta < \kappa$ *and* x_0, x_1 *such that*:

(a) $\langle q_0^0, q_0^1, q_0^2 \rangle, \langle q_1^0, q_1^1, q_1^2 \rangle \leq \langle p^0, p^1, p^2 \rangle$;
(b) $x_0 \neq x_1$;
(c) $\Vdash$ "$x_0, x_1 \in \underset{\sim}{T}_\delta$, *the* δ*th level of* $\underset{\sim}{T}$";
(d) $\langle q_i^0, q_i^1, q_i^2 \rangle \Vdash$ "$x_i \in \underset{\sim}{\eta}$" $(i = 0, 1)$. □

In fact, as the forcing notions $\mathrm{Add}(\aleph_1, \kappa)$ and $\mathrm{Add}(\aleph_0, \kappa)$ are κ-c.c. and $\mathrm{Col}(\kappa, 2^\mu)$ is forced to be κ-closed, we can show that the conditions $\langle q_0^0, q_0^1, q_0^2 \rangle$ and $\langle q_1^0, q_1^1, q_1^2 \rangle$ in the claim can be chosen so that $q_0^0 = q_1^0 = p^0$ and $q_0^2 = q_1^2 = p^2$ (see [11] for similar arguments).

Let us assume that the empty condition forces $\underset{\sim}{\eta}$ is a new branch. By repeated application of Claim 5.7, we can build a sequence $\langle \underset{\sim}{q}_\nu^1 : \nu \in ({}^{<\omega_1}2) \rangle$ of $\mathbb{Q}_\mu$-names of elements of $\underset{\sim}{\mathbb{Q}}_{\mu+1}$, an increasing continuous sequence $\langle \delta_i : i < \omega_1 \rangle$ of ordinals less than κ and a sequence $\langle x_\nu : \nu \in ({}^{<\omega_1}2) \rangle$ such that:

(1) $\nu_1 \trianglelefteq \nu_2 \Rightarrow \Vdash_{\mathbb{Q}_\mu}$ "$\underset{\sim}{q}_{\nu_2}^1 \leq \underset{\sim}{q}_{\nu_1}^1$";
(2) $\langle \emptyset, \underset{\sim}{q}_\nu^1, \emptyset \rangle \Vdash$"$x_\nu \in \underset{\sim}{T}_{\delta_i}$" where $i = \mathrm{lh}(\nu)$;
(3) $x_{\nu^\frown\langle 0 \rangle} \neq x_{\nu^\frown\langle 1 \rangle}$;

(4) $\langle\emptyset, \underset{\sim}{q}^1_\nu, \emptyset\rangle \Vdash$ "$x_\nu \in \underset{\sim}{\eta}$";
(5) $\nu_1 \trianglelefteq \nu_2 \Rightarrow \langle\emptyset, \underset{\sim}{q}^1_{\nu_2}, \emptyset\rangle \Vdash$ "$x_{\nu_1} <_{\underset{\sim}{T}} x_{\nu_2}$".

For some $\xi < \kappa$, $\langle \underset{\sim}{q}^1_\nu : \nu \in ({}^{<\omega_1}2)\rangle$ is in fact an $\mathrm{Add}(\aleph_1, \xi)$-name. Now, we have

$$\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa) \simeq \mathrm{Add}(\aleph_1, \xi) * \underset{\sim}{\mathrm{Add}}(\aleph_1, [\xi, \kappa)) * \underset{\sim}{\mathbb{Q}}_{\mu+1} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa),$$

and in the generic extension $V^{\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_1, \xi)}$, we have an interpretation q^1_ν of the name $\underset{\sim}{q}^1_\nu$, where $\nu \in ({}^{<\omega_1}2)$.

Work in $V^{\mathbb{P}_\mu * \underset{\sim}{\mathbb{Q}}_\mu}$. For each $\tau \in ({}^{\omega_1}2)$, let $q^1_\tau \le q^1_{\tau\restriction i}, i < \omega_1$ and let $\delta = \sup\{\delta_i : i < \omega_1\} < \kappa$. By extending q^1_τ if necessary, we can assume that for some x_τ,

$$\langle\emptyset, q^1_\tau, \emptyset\rangle \Vdash \text{“}x_\tau \in \underset{\sim}{T}_\delta \cap \underset{\sim}{\eta}\text{”}.$$

But then for all $\tau_1 \ne \tau_2$ in ${}^{\omega_1}2$ we have $x_{\tau_1} \ne x_{\tau_2}$, and so

$$\Vdash_{\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“the } \delta\text{th level of the tree has at least } 2^{\aleph_1} = \mu = \aleph_3\text{-many nodes”}.$$

But $\Vdash_{\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)}$ "$|\underset{\sim}{T}_\delta| \le \kappa < \mu$", and we get a contradiction. □

The next lemma follows from Lemma 5.6 and the fact that $\mathbb{P}_\mu * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa) \lessdot \mathbb{P}_{\mu+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$.

Lemma 5.8. *With the same hypotheses as in Lemma* 5.6, *we have the following*: *In* $V^{\mathbb{P}_{\mu+2}}$, $\underset{\sim}{T}$ *is isomorphic to some* $\underset{\sim}{T}'$, *which is an* $\mathrm{Add}(\aleph_0, \kappa)$*-name of a* κ*-tree with* $\le \kappa$*-many cofinal branches.*

5.4. *Completing the proof of Theorem* 1.3

Finally, in this section, we complete the proof of Theorem 1.3. We first prove the following lemma.

Lemma 5.9. $\Vdash_{\mathbb{P} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)}$ "*Any* $\aleph_1$*-closed forcing notion of size* $\le \kappa$ *collapses* κ".

Proof. Let $G * H$ be $\mathbb{P} * \mathrm{Add}(\aleph_0, \kappa)$-generic over V and assume $\mathbb{R} \in V[G * H]$ is an $\aleph_1$-closed forcing notion of size $\le \kappa = \aleph_2$.

Assume towards a contradiction that forcing with $\mathbb{R}$ over $V[G * H]$ does not collapse $\aleph_2$. It then follows from [10] that $\mathbb{R}$ is $\aleph_2$-distributive, and hence by Lemma 3.1, there exists a κ-tree $T = T(\mathbb{R})$, the base tree of $\mathbb{R}$, which is dense in $\mathbb{R}$.

Let $\underset{\sim}{T}$ be a $\mathbb{P} * \mathrm{Add}(\aleph_0, \kappa)$-name for T. By Lemma 5.2, we may assume that $\underset{\sim}{T} \in H(\lambda)$, and hence there exists some Mahlo cardinal $\beta \in (\kappa, \lambda)$ such that $\underset{\sim}{T}$ is a $\mathbb{P}_\beta * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name. By Lemma 5.6, $\underset{\sim}{T}$ is isomorphic to some $\underset{\sim}{T}' \in H(\lambda)$ which is a $\mathbb{P}_{\beta+2} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$-name for a κ-tree which has $\le \kappa$-many cofinal branches. On the other hand $\{\beta + 2 : \beta \in (\kappa, \lambda)$ is a Mahlo cardinal and $\Phi(\beta + 2) = \underset{\sim}{T}'\}$ in unbounded in λ, and hence we can choose β as above such that $\underset{\sim}{T}' = \Phi(\beta + 2)$.

Then $\mathbb{P}_{\beta+3} \simeq \mathbb{P}_{\beta+2} * \underset{\sim}{\mathbb{Q}}_A(\Phi(\beta+2)^\star)$, and by Lemma 4.8,

$$\Vdash_{\mathbb{P}_{\beta+3} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“Forcing with } \underset{\sim}{T}' \text{ collapses } \kappa \text{ into } \aleph_1\text{”}.$$

As $\mathbb{P}_{\beta+3} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa) \lessdot \mathbb{P} * \underset{\sim}{\mathrm{Add}}(\aleph_0, \kappa)$,

$$\Vdash_{\mathbb{P} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“Forcing with } \underset{\sim}{T}' \text{ collapses } \kappa \text{ into } \aleph_1\text{”}.$$

This implies

$$\Vdash_{\mathbb{P} * \underset{\sim}{\mathrm{Add}}(\aleph_0,\kappa)} \text{“Forcing with } \underset{\sim}{\mathbb{R}} \text{ collapses } \kappa \text{ into } \aleph_1\text{”}.$$

The lemma follows. □

Now let $G * H$ be $\mathbb{P} * \mathrm{Add}(\aleph_0, \kappa)$-generic over V. Let also $\mathbb{R} \in V[G * H]$ be an $\aleph_1$-closed forcing notion of size $\leq \kappa = \aleph_2$. By Lemma 5.9,

$$V[G * H] \models \text{“}\mathbb{R} \simeq \mathrm{Col}(\aleph_1, \kappa) \simeq \mathrm{Add}(\aleph_1, 1)\text{”}.$$

6. Consistency, Every Forcing Which Adds a Fresh Subset of $\aleph_2$ Collapses a Cardinal

In this section, we prove Theorem 1.4. Thus assume that GCH holds and $\lambda > \kappa$ are such that κ is supercompact and Laver indestructible, and λ is a 2-Mahlo cardinal. Let also $\Phi : \lambda \to H(\lambda)$ be such that for each $x \in H(\lambda), \Phi^{-1}(x) \cap \{\beta + 2 : \beta$ is Mahlo$\}$ is unbounded in λ. The forcing notion we define is very similar the forcing notion of Sec. 5.

Definition 6.1. Let

$$\langle \langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle, \langle \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle \rangle$$

be an iteration of forcing notions such that for each $\alpha \leq \lambda, p \in \mathbb{P}_\alpha$ if and only if p is a function with domain α such that:

(1) p has support of size less than κ.
(2) $\{\beta \in \mathrm{supp}(p) : \beta \equiv 0 \pmod 3$ or $\beta \equiv 2 \pmod 3\}$ has size less than $\aleph_1$.
(3) If $\beta < \kappa$ and $\beta \equiv 0 \pmod 3$ or $\beta \equiv 2 \pmod 3$, then $\Vdash_\beta$ “$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\aleph_1, \aleph_2 + |\beta|)$”.
(4) If $\beta \geq \kappa$, $\beta \equiv 0 \pmod 3$ and β is inaccessible, then $\Vdash_\beta$ “$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Add}}(\aleph_1, \kappa)$”.
(5) If $\beta \geq \kappa$, $\beta \equiv 1 \pmod 3$ and $\beta - 1$ is inaccessible, then $\Vdash_\beta$ “$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathrm{Col}}(\kappa, 2^{|\mathbb{P}_\beta|})$”.
(6) If $\beta \geq \kappa$, $\beta \equiv 2 \pmod 3$ $\beta - 2$ is inaccessible, and $\Phi(\beta)$ is a $\mathbb{P}_\beta$-name for κ-tree with $\leq \kappa$-many cofinal branches, then $\Vdash_\beta$ “$\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathbb{Q}}_A(\Phi(\beta)^\star)$”.
(7) Otherwise, $\Vdash_\beta$ “$\underset{\sim}{\mathbb{Q}}_\beta$ is the trivial forcing notion”.

Finally set $\mathbb{P} = \mathbb{P}_\lambda$.

The next lemma can be proved as in Sec. 5.

Lemma 6.2. *Let G be $\mathbb{P}$-generic over V. Then the following hold in $V[G]$:*

(a) $2^{\aleph_0} = \aleph_1 < \kappa = \aleph_2 < 2^{\aleph_1} = \lambda = \aleph_3$,
(b) *Every tree of size and height $\aleph_2$ is specialized.*

Thus (a)–(c) of Theorem 1.4 are satisfied. Let's prove Theorem 1.4(d). The proof is similar to Todorcevic's proof in [16], and we present it here for completeness.

Work in $V[G]$. Let $\mathbb{P}$ be any forcing notion, and suppose that forcing with $\mathbb{P}$ adds a fresh subset of $\aleph_2$ without collapsing it. We show that forcing with $\mathbb{P}$ collapses $\aleph_3$. Let $\mathbb{B} = RO(\mathbb{P})$. Let also $\underset{\sim}{\tau}$ be a name for a fresh subset of $\aleph_2$ such that

$$\|(\underset{\sim}{\tau} \subseteq \aleph_2) \wedge (\underset{\sim}{\tau} \notin \check{V}) \wedge (\forall \alpha < \aleph_2, \underset{\sim}{\tau} \cap \alpha \in \check{V})\|_{\mathbb{B}} = 1.$$

For $\alpha < \aleph_2$, set $a_{\alpha,0} = \|\alpha \in \underset{\sim}{\tau}\|_{\mathbb{B}}$ and $a_{\alpha,1} = \|\alpha \notin \underset{\sim}{\tau}\|_{\mathbb{B}}$. Let $T_0 = \{1_{\mathbb{B}}\}$, and for $0 < \alpha < \aleph_2$ set

$$T_\alpha = \Big\{\bigwedge\{a_{\beta,f(\beta)} : \beta < \alpha\} : f \in {}^{\alpha}2, \bigwedge\{a_{\beta,f(\beta)} : \beta < \alpha\} \neq 0_{\mathbb{B}}\Big\}.$$

By the assumption on $\underset{\sim}{\tau}$, each T_α is a partition of $1_{\mathbb{B}}$, for $\beta < \alpha, T_\alpha$ refines T_β and so $T = \bigcup\{T_\alpha : \alpha < \aleph_2\}$ is a tree of height $\aleph_2$, whose αth level is T_α. Also, clearly $|T| = 2^{\aleph_1} = \aleph_3$.

Claim 6.3. *For every $0_{\mathbb{B}} \neq b \in \mathbb{B}$, there exists $\alpha < \aleph_2$ such that*

$$|\{a \in T_\alpha : a \wedge b \neq 0_{\mathbb{B}}\}| > \aleph_2.$$

Proof. Suppose not. So, we can find $0_{\mathbb{B}} \neq b \in \mathbb{B}$ such that for each $\alpha < \aleph_2$, $|\{a \in T_\alpha : a \wedge b \neq 0_{\mathbb{B}}\}| \leq \aleph_2$. Define a new tree $T^* = \bigcup\{T^*_\alpha : \alpha < \aleph_2\}$, where for each α,

$$T^*_\alpha = \{a \wedge b : a \in T_\alpha, a \wedge b > 0_{\mathbb{B}}\}.$$

Then T^* is an $\aleph_2$-tree of size $\aleph_2$ and hence it is specialized. But then

$$\|\dot{G}_{\mathbb{B}} \cap T^* \text{ is a new cofinal branch of } T^*\|_{\mathbb{B}} \geq b,$$

where $\dot{G}_{\mathbb{B}}$ is the canonical name for a generic ultrafilter over $\mathbb{B}$. This is impossible as T^* is specialized and forcing with $\mathbb{B}$ preserves $\aleph_2$. $\square$

For each $\alpha < \aleph_2$, let $\langle a_\alpha(\xi) : \xi < \lambda_\alpha \leq \aleph_3\rangle$ be an enumeration of T_α, and let $\underset{\sim}{f}$ be a name for a function from $\aleph_2$ into $\aleph_3$ defined by

$$\|\underset{\sim}{f}(\alpha) = \xi\|_{\mathbb{B}} = \begin{cases} a_\alpha(\xi) & \text{if } \xi < \lambda_\alpha, \\ 0_{\mathbb{B}} & \text{Otherwise.} \end{cases}$$

Claim 6.4. $\|\text{range}(\underset{\sim}{f})$ *is unbounded in* $\aleph_3\|_{\mathbb{B}} = 1_{\mathbb{B}}$.

Proof. Assume not. Then for some $\delta < \aleph_3$, $b = \|\mathrm{range}(\underset{\sim}{f}) \subseteq \delta\|_{\mathbb{B}} > 0_{\mathbb{B}}$. By Claim 6.3, we can find $\alpha < \aleph_2$ such that $|\{a \in T_\alpha : a \wedge b \neq 0_{\mathbb{B}}\}| = \aleph_3$, so $\lambda_\alpha = \aleph_3$. Pick some $\xi > \delta$ so that $a_\alpha(\xi) \wedge b \neq 0_{\mathbb{B}}$. This implies

$$\|\underset{\sim}{f}(\alpha) = \xi\|_{\mathbb{B}} \wedge \|\mathrm{range}(\underset{\sim}{f}) \subseteq \delta\|_{\mathbb{B}} \neq 0_{\mathbb{B}},$$

and we get a contradiction (as $\xi > \delta$). □

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