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SOME RESULTS ON POLISH GROUPS

A b s t r a c t. We prove that no quantifier-free formula in the language of group theory can define the $\aleph_1$-half graph in a Polish group, thus generalising some results from [6]. We then pose some questions on the space of groups of automorphisms of a given Borel complete class, and observe that this space must contain at least one uncountable group. Finally, we prove some results on the structure of the group of automorphisms of a locally finite group: firstly, we prove that it is not the case that every group of automorphisms of a graph of power λ is the group of automorphism of a locally finite group of power λ; secondly, we conjecture that the group of automorphisms of a locally finite group of power λ has a locally finite subgroup of power λ, and reduce the problem to a problem on p-groups, thus settling the conjecture in the case $\lambda = \aleph_0$.

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1. Introduction

We collect some results (of different nature) on the theory of Polish groups.

Section 2. Definable $\aleph_1$-Half Graphs in Polish Groups.

By the $\aleph_1$-half graph $\Gamma(\aleph_1)$ we mean the graph on vertex set $\{a_\alpha : \alpha < \aleph_1\} \cup \{b_\beta : \beta < \aleph_1\}$ with edge relation $a_\alpha E_\Gamma b_\beta$ if and only if $\alpha < \beta$. In the process of characterization of the graph products of cyclic groups embeddable in a Polish group [6], we observed that the commutation relation $x^{-1}y^{-1}xy = e$ can never define the $\aleph_1$-half graph in a Polish group G. Here we generalize this to:

Theorem 1.1. *No quantifier-free formula $\varphi(\bar{x}, \bar{y})$ in the language of group theory can define the $\aleph_1$-half graph in a Polish group G.*

We actually prove a stronger result of independent interest, i.e. that Polish groups do not admit "polarized $\aleph_1$-partitions", see Theorem 2.1 for the detailed statement of this result. Finally, we would like to mention that Theorem 1.1 can be considered as a form of model-theoretic stability for Polish groups, in this direction see also [9].

Section 3. The Space of Automorphism Groups of a Borel Complete Class.

By a Borel complete class we mean a Borel class $\mathbf{K}$ of structures with domain ω in a fixed language L such that the isomorphism relation on $\mathbf{K}$ is as complicated as possible (equivalently, the countable graph isomorphism relation is reducible to it – cf. Definition 3.2). We wonder here: how complex can $Aut(\mathbf{K}) = \{Aut(A) : A \in \mathbf{K}\}$ be for a given Borel complete class? Can $Aut(\mathbf{K})$ contain only one isomorphism type, resp. finitely many, resp. countably many (cf. Problem 3.5)? In this direction:

Proposition 1.2. *Let $\mathbf{K}$ be a Borel class of L-structures with domain $\aleph_0$ such that for every $G \in Aut(\mathbf{K})$ we have that $|G| \leqslant \aleph_0$. Then the isomorphism relation on $\mathbf{K}$* is *Borel, and so in particular $\mathbf{K}$ is* not *Borel complete (cf. Definition 3.2).*

On questions affine to this topic see also the interesting recent work [5].

Section 4. Group of Automorphisms of Locally Finite Groups.

Notation 1.3. *(1) We denote by $\mathbf{K}_{\mathrm{lf}}$ the class of locally finite groups.*

(2) We denote by $\mathbf{K}_{\mathrm{gf}}$ the class of graphs.

(3) For $\mathbf{K}$ *a class of* L*-structures and* λ *an infinite cardinal, we let:*

(3.1) $\mathbf{K}^\lambda = \{M \in \mathbf{K} : \text{the domain of } M \text{ is } \lambda\}$*;*

(3.2) $Aut(\mathbf{K}) = \{Aut(M) : M \in \mathbf{K}\}$*.*

Fact 1.4 ([4])**.** *The class* $\mathbf{K}^{\aleph_0}_{\mathrm{lf}}$ *is Borel complete (cf. Definition 3.2).*

In this section we deal with the following problem:

Problem 1.5. *Characterize* $Aut(\mathbf{K}^\lambda_{\mathrm{lf}})$*, for* $\lambda \geqslant \aleph_0$*.*

In this direction we first prove:

Theorem 1.6. *Let* $\lambda \geqslant \aleph_0$*, then:*

$$Aut(\mathbf{K}^\lambda_{\mathrm{gf}}) \neq Aut(\mathbf{K}^\lambda_{\mathrm{lf}}).$$

The proof of Theorem 1.6 leads to the following conjecture:

Conjecture 1.7. *If* $G \in \mathbf{K}^\lambda_{\mathrm{lf}}$*, then* $Aut(G)$ *has a locally finite subgroup of power* λ*.*

In the case of $\aleph_0$ we prove that this is indeed the case:

Proposition 1.8. *If* $G \in \mathbf{K}^{\aleph_0}_{\mathrm{lf}}$*, then* $Aut(G)$ *has a locally finite infinite subgroup.*

On the other hand, we do not settle here Conjecture 1.7 in general, but we prove:

Lemma 1.9. *To prove Conjecture 1.7 it suffices to prove Conjecture 1.10, where:*

Conjecture 1.10. *If* $\lambda > \aleph_0$*,* $G \in \mathbf{K}^\lambda_{\mathrm{lf}}$ *is an abelian* p*-group,* $H \leqslant G$ *and* $|H| < \lambda$*, then* $Aut_H(G)$ *has a locally finite subgroup of power* λ*.*

Finally, we would like to mention that in [7] we give a close analysis of the group of automorphisms of Philip Hall's universal locally finite group.

2. Definable $\aleph_1$-Half Graphs in Polish Groups

Theorem 2.1. *Let G be a Polish group, and for $\ell < \ell(*) < \omega$ let:*

(i) $\bar{\mathbf{g}}^\ell = (\bar{g}^\ell_\alpha : \alpha \in A_\ell)$;

(ii) $A_\ell \in [\omega_1]^{\aleph_1}$;

(iii) $\bar{g}^\ell_\alpha \in G^{n(\ell)}$;

(iv) Δ *be a finite set of q.f. formulas of the form* $\varphi(\bar{x}^0_{n(0)}, ..., \bar{x}^{\ell(*)-1}_{n(\ell(*)-1)})$ *in the language of group theory such that* $lg(\bar{x}^\ell_{n(\ell)}) = n(\ell)$.

Then there are $B_\ell \in [A_\ell]^{\aleph_1}$, *for* $\ell < \ell(*)$, *and truth value* $t \in \{0, 1\}$ *such that if* $\alpha(\ell) \in B_\ell$, *for* $\ell < \ell(*)$, *then:*

$$G \models \varphi^t(\bar{g}^0_{\alpha(0)}, ..., \bar{x}^{\ell(*)-1}_{\alpha(\ell(*)-1)}).$$

Proof. First of all notice that it suffices to prove the claim for:

$$\Delta = \{\sigma(\bar{x}^0_{n(0)}, ..., \bar{x}^{\ell(*)-1}_{n(\ell(*)-1)}) = e\},$$

and $\sigma(\bar{x}^0_{n(0)}, ..., \bar{x}^{\ell(*)-1}_{n(\ell(*)-1)})$ a term in the language of group theory $L = \{e, \cdot, ()^{-1}\}$.
[Why? First of all, without loss of generality, we can assume that each $\varphi \in \Delta$ is a Boolean combination of formulas of the form $\sigma(\bar{x}^0_{n(0)}, ..., \bar{x}^{\ell(*)-1}_{n(\ell(*)-1)}) = e$. So let $(\sigma_i(\bar{x}^0_{n(0)}, ..., \bar{x}^{\ell(*)-1}_{n(\ell(*)-1)}) = e : i < i(*) < \omega)$ list them. Now choose $(B_{i,\ell} : \ell < \ell(*))$ by induction on $i \leqslant i*$ such that:

(a) $B_{i,\ell} \in [\omega_1]^{\aleph_1}$;

(b) $B_{0,\ell} = A_\ell$;

(c) $B_{i+1,\ell} \subseteq B_{i,\ell}$;

(d) $(B_{i+1,\ell} : \ell < \ell(*))$ satisfies the desired conclusion for:

$$\Delta = \{\sigma_i(\bar{x}^0_{n(0)}, ..., \bar{x}^{\ell(*)-1}_{n(\ell(*)-1)}) = e\}.$$

Then $(B_{i(*),\ell} : \ell < \ell(*))$ is as wanted.]
Let (G, d) witness the Polishness of G. For $\ell < \ell(*)$ and $\alpha, \beta \in A_\ell$, let $d(\bar{g}^\ell_\alpha, \bar{g}^\ell_\beta) = max\{d(g^{\ell,i}_\alpha, g^{\ell,i}_\beta) : i < n(\ell)\}$, and:

$$\mathcal{U}_\ell = \{\alpha < \omega_1 : \text{ for some } \varepsilon \in (0,1)_{\mathbb{R}}$$

$$\text{the set } \{\beta \in A_\ell : d(\bar{g}^\ell_\alpha, \bar{g}^\ell_\beta) < \varepsilon\} \text{ is countable}\}.$$

Since (G, d) is separable, for every $\ell < \ell(*)$, the set $\mathcal{U}_\ell$ is countable, and so we can find $\alpha(*) < \omega_1$ such that $\bigcup_{\ell<\ell(*)} \mathcal{U}_\ell \subseteq \alpha(*)$. Now, if $B_\ell = A_\ell - \alpha(*)$ is such that for every $\alpha(\ell) \in B_\ell$ we have that $G \models \sigma(\bar{g}^0_{\alpha(0)}, ..., \bar{g}^{\ell(*)-1}_{\alpha(\ell(*)-1)}) = e$, then we are done. So suppose that this is not the case, then we can find $\alpha_\ell \in A_\ell - \alpha(*)$ such that:

$$\varepsilon = d(\sigma(\bar{g}^0_{\alpha(0)}, ..., \bar{g}^{\ell(*)-1}_{\alpha(\ell(*)-1)}), e) \neq 0.$$

As G is Polish, there is $\xi \in (0,1)_{\mathbb{R}}$ such that:

$$\text{if } \bar{a}_\ell \in G^{n(\ell)} \text{ and } d(\bar{a}_\ell, \bar{g}^\ell_{\alpha(\ell)}), \text{ then } d(\sigma(\bar{a}^0_0, ..., \bar{a}^{\ell(*)-1}_{\ell(*)-1}), e) > \varepsilon/2. \quad (1)$$

Now, for $\ell < \ell(*)$, let $B_\ell = \{\alpha \in A_\ell : d(\bar{g}^\ell_\alpha, \bar{g}^\ell_{\alpha_\ell}) < \xi\}$. Then $B_\ell \subseteq A_\ell$ and, as $\alpha(*) \leqslant \alpha_\ell$, clearly $|B_\ell| = \alpha_1$. Hence, by (1), $(B_\ell : \ell < \ell(*))$ is as required. □

Theorem 1.1. *No quantifier-free formula $\varphi(\bar{x}, \bar{y})$ in the language of group theory can define the $\aleph_1$-half graph in a Polish group G.*

Proof. Immediate from Theorem 2.1. □

3. The Space of Automorphism Groups of a Borel Complete Class

For an overview (and careful explanation) of the descriptive set theoretic notions occurring in this section cf. e.g. [8, Chapter 11].

Notation 3.1. *We denote by $\mathbf{K}^{\aleph_0}_{\mathrm{gf}}$ the standard Borel space of graphs with domain $\aleph_0$.*

Definition 3.2. Let $\mathbf{K}$ be a Borel class of L-structures with domain $\aleph_0$. We say that $\mathbf{K}$ is *Borel complete* if there exists a Borel map $f : \mathbf{K}_{\mathrm{gf}}^{\aleph_0} \to \mathbf{K}$ such that for every $A, B \in \mathbf{K}_0$ we have $A \cong B$ if and only if $f(A) \cong f(B)$.

Notation 3.3. *Let G and H be topological groups.*

(1) *We write $G \cong H$ to mean that G and H are isomorphic as abstract groups.*

(2) *We write $G \cong_t H$ to mean that G and H are isomorphic as topological groups.*

Notation 3.4. *(As in Notation 1.3) Given a class $\mathbf{K}$ of L-structures, we let:*

$$Aut(\mathbf{K}) = \{Aut(A) : A \in \mathbf{K}\}.$$

Proposition 1.2. *Let $\mathbf{K}$ be a Borel class of L-structures with domain $\aleph_0$ such that for every $G \in Aut(\mathbf{K})$ we have that $|G| \leqslant \aleph_0$. Then the isomorphism relation on $\mathbf{K}$* is *Borel, and so in particular $\mathbf{K}$ is* not *Borel complete.*

Proof. We show that for any such class $\mathbf{K}$ the isomorphism relation $\cong$ on $\mathbf{K}$ is Borel. Notice that for $A, B \in \mathbf{K}$ we have that $A \cong B$ if and only if there are countably many $f \in S_\infty := \{f : \omega \to \omega : f \text{ is a bijection}\}$ such that $f : A \cong B$. Thus, the relation $\cong$ on $\mathbf{K}$ is the projection of a Borel relation R:

$$(\mathbf{K} \times \mathbf{K}) \times S_\infty \supseteq R = \{(A, B, f) : f : A \cong B\}$$

with countable sections $R_{(A,B)}$ (for $(A, B) \in \mathbf{K} \times \mathbf{K}$). Hence, by [3, Lemma 18.12], the relation $\cong$ on $\mathbf{K}$ is Borel, and so we are done. □

We are interested in the following open problem:

Problem 3.5. *Let $\mathbf{K}$ be a Borel complete class.*

(1) *Can $Aut(\mathbf{K})/\cong$ have size 1? Can $Aut(\mathbf{K})/\cong_t$ have size 1?*

(2) *Can $Aut(\mathbf{K})/\cong$ be finite? Can $Aut(\mathbf{K})/\cong_t$ be finite?*

(3) *Can $Aut(\mathbf{K})/\cong$ be countable? Can $Aut(\mathbf{K})/\cong_t$ be countable?*

4. Groups of Automorphisms of Locally Finite Groups

Definition 4.1. (1) We denote by P the set of prime numbers.

(2) For $p \in P$, we denote by $G^*_{(p,\infty)}$ the divisible abelian p-group of rank 1.

(3) For $p \in P$ and $\ell < \omega$ we denote by $G^*_{(p,\ell)}$ the finite cyclic group of order p^ℓ.

(4) For $p \in P$, we let $S_p = \{(p,\ell) : \ell < \omega\}$ and $S_p^+ = S_p \cup \{(p,\infty)\}$.

(5) For $s \in S_p^+$ and λ a cardinal, we let $G^*_{s,\lambda}$ be the direct sum of λ copies of G^*_s.

(6) For $p \in P$, we denote by J_p the group of p-adic integers.

(7) We say that an abelian group G is bounded if there exists $n < \omega$ such that for every $g \in G$ we have $ng = 0$.

(8) We say that G is unbounded if it is not bounded.

(9) We say that G is torsion if every element of G has finite order.

Fact 4.2 ([1][Theorem 17.2])**.** *Let G be a bounded abelian group. Then G is a direct sum of cyclic groups.*

Fact 4.3. *If an abelian p-group G is bounded, then there exists $n < \omega$ such that:*

$$G = \bigoplus_{\ell < n} G^*_{(p,\ell),\lambda_\ell}.$$

Proof. This is a consequence of Fact 4.2. □

Fact 4.4 ([1][Theorem 8.4])**.** *Let G be a torsion abelian group. Then:*

$$G = \bigoplus_{p \in P} G_p,$$

with G_p a p-group, for every $p \in P$.

Remark 4.5 ([2][pg. 250])**.** Let G be an abelian group and suppose that $G = \bigoplus_{i \in I} G_i$, then $\bigoplus_{i \in I} Aut(G_i)$ can be embedded into $Aut(G)$.

Fact 4.6 ([2, Theorem 115.1]). *Let G be an unbounded abelian p-group, then there exists an embedding $f : J_p \to Aut(G)$.*

Lemma 4.7. *Let $G \in \mathbf{K}^{\lambda}_{\rm lf}$ (cf. Definition 1.3). Then $Aut(G)$ has a non-trivial locally finite subgroup.*

Proof. We distinguish three cases:

(i) G is not abelian.

(ii) G is abelian and not bounded.

(iii) G is abelian and bounded.

If (i), then $G/Cent(G) \in \mathbf{K}_{\rm lf}$ is non-trivial and it can be embedded into $Aut(G)$, and so we are done. If (ii), then we are done by Facts 4.4 and 4.6, and Remark 4.5. Finally, if (iii), then by Facts 4.3 and 4.4, there exists a direct summand G_p of G such that $G_p = \bigoplus_{\ell < n} G^*_{(p,\ell),\lambda_\ell}$ and for some $0 < \ell(*) < n < \omega$ we have $\lambda_{\ell(*)} \geqslant \aleph_0$. Let $G_{p^\ell}(\alpha)$ be the α-th copy of $G^*_{(p,\ell),\lambda_\ell}$. If $p > 2$, then consider:

$$\{\pi \in Aut(G_p) : \pi \text{ maps } G_{p^\ell}(\alpha) \text{ onto itself, for every } \alpha < \lambda_\ell \text{ and } \ell < n\}.$$

If $p = 2$, then consider:

$$\{\pi \in Aut(G_p) : \pi \text{ maps } G_{p^{\ell(*)}}(2\alpha) \oplus G_{p^{\ell(*)}}(2\alpha + 1) \text{ onto itself,}$$

$$\text{for every } \alpha < \lambda_{\ell(*)}\}.$$

Hence, by Remark 4.5, also (iii) is taken care of. □

The following facts are folklore:

Fact 4.8. *(1) If M is a linear order, then $Aut(M)$ has no element of finite order.*

(2) For every infinite structure M there exists a graph Γ_M of the same cardinality of M such that $Aut(\Gamma_M) \cong Aut(M)$.

Theorem 1.6. *Let $\lambda \geqslant \aleph_0$, then:*

$$Aut(\mathbf{K}^{\lambda}_{\rm gf}) \neq Aut(\mathbf{K}^{\lambda}_{\rm lf}).$$

Proof. By Lemma 4.7 and Fact 4.8. □

We devote the rest of the section to the proof of Proposition 1.8 and Lemma 1.9.

Notation 4.9. *Given a group G and $H \leqslant G$ we let:*

$$Aut_H(G) = \{\pi \in Aut(G) : \pi \restriction H = id_H\}.$$

Fact 4.10. *If $G \in \mathbf{K}^{\lambda}_{\mathrm{lf}}$ and $|G/Cent(G)| < \lambda$, then there is $H \leqslant G$ such that $|H| < \lambda$ and $G = \langle H \cup Cent(G)\rangle_G$.*

Lemma 4.11. *If (A) then (B), where:*

(A) *(a)* $G \in \mathbf{K}^{\lambda}_{\mathrm{lf}}$;
- *(b)* $H \leqslant G$ *and* $|H| < \lambda$;
- *(c)* $G^* = Cent(G)$ *and* $G = \langle H \cup G^*\rangle_G$;
- *(d)* $G^* = \bigoplus_{p \in P} G_p$ *and* $H_p = H \cap G_p$;

(B) *(a)* *if* $\pi \in Aut_H(G)$, *then* $\pi(p) := \pi \restriction G_p \in Aut_{H_p}(G_p)$;
- *(b)* *the mapping* $\pi \mapsto (\pi(p) : p \in P)$ *from* $Aut_H(G)$ *into* $\prod_{p \in P} Aut_{H_p}(G_p)$ *is an embedding;*
- *(c)* *the embedding in (b) is onto.*

Proof. The non-trivial part is item (B)(c). To this extent, let $\pi_p \in Aut_{H_p}(G_p)$, for $p \in P$. It suffices to find $\pi \in Aut_H(G)$ such that $\pi(p) = \pi_p$, for every $p \in P$. We define π as follows. For $p_1 < \cdots < p_n \in P$ an initial segment of P with the induced order, $y_{p_\ell} \in G_{p_\ell}$ and $y \in H$ we let:

$$\pi(yy_{p_1} \cdots y_{p_n}) = y\pi_{p_1}(y_{p_1}) \cdots \pi_{p_n}(y_{p_n}).$$

Now, by (A), every $g \in G$ has at least one representation of the form $g = yy_{p_1} \cdots y_{p_n}$, and so, for every $g \in G$, $\pi(g)$ has at least one definition. We are then left to show that the choice of representation $g = yy_{p_1} \cdots y_{p_n}$ does not matter. To this extent, let $g \in G$ and suppose that:

$$yy_{p_1} \cdots y_{p_m} = g = y'y'_{p_1} \cdots y'_{p_k},$$

By adding occurrences of e in the representations we can assume without loss of generality that:

$$yy_{p_1} \cdots y_{p_n} = g = y'y'_{p_1} \cdots y'_{p_n}.$$

Notice now that for $1 \leqslant \ell \leqslant n$ we have:

(a) $y'_{p_\ell} \in y_{p_\ell} H_{p_\ell}$, say $y'_{p_\ell} = z_{p_\ell} y_{p_\ell}$ with $z_{p_\ell} \in H_{p_\ell}$;

(b) $y' y'_{p_1} \cdots y'_{p_n} = y'(z_{p_1} y_{p_1}) \cdots (z_{p_n} y_{p_n}) = (y'(z_{p_1} \cdots z_{p_n})) y_{p_1} \cdots y_{p_n}$;

(c) $y y_{p_1} \cdots y_{p_n} = (y'(z_{p_1} \cdots z_{p_n})) y_{p_1} \cdots y_{p_n}$;

(d) $y = y'(z_{p_1} \cdots z_{p_n})$.

Hence, we have:

$$\begin{aligned}
\pi(y y_{p_1} \cdots y_{p_n}) &= y \pi_{p_1}(y_{p_1}) \cdots \pi_{p_n}(y_{p_n}) \\
&= (y'(z_{p_1} \cdots z_{p_n})) \pi_{p_1}(y_{p_1}), ..., \pi_{p_n}(y_{p_n}) \\
&= y' z_{p_1} \pi_{p_1}(y_{p_1}) \cdots z_{p_n} \pi_{p_n}(y_{p_n}) \\
&= y' \pi_{p_1}(z_{p_1} y_{p_1}) \cdots \pi_{p_n}(z_{p_n} y_{p_n}) \\
&= y' \pi_{p_1}(y'_{p_1}) \cdots \pi_{p_n}(y'_{p_n}) \\
&= \pi(y' y'_{p_1} \cdots y'_{p_n}).
\end{aligned}$$

□

Proposition 1.8. *If $G \in \mathbf{K}_{\mathrm{lf}}^{\aleph_0}$, then $Aut(G)$ has a locally finite infinite subgroup.*

Proof. Let $G \in \mathbf{K}_{\mathrm{lf}}^{\aleph_0}$. If $G/Cent(G)$ is infinite, then we are done, since we can embed $G/Cent(G)$ into $Aut(G)$. So suppose that $G/Cent(G)$ is finite and let G^*, G_p and H_p be as in Lemma 4.11. If for some $p \in P$ we have that G_p is infinite use Lemma 4.11 and Fact 4.6, unless G_p is bounded, in which case use Lemma 4.11 and Fact 4.3. If for every $p \in P$ we have that G_p is finite, then the set:

$$P^* = \{p \in P : G_p \neq H_p \text{ and } [G_p : H_p] > 2\}$$

is infinite, and so for every $p \in P^*$ we have that $Aut_{H_p}(G_p)$ is non-trivial. Hence, considering $\prod_{p \in P^*} Aut_{H_p}(G_p)$ and using Lemma 4.11 we are done. □

Lemma 1.9. *To prove Conjecture 1.7 it suffices to prove Conjecture 1.10, where:*

Conjecture 1.10. *If $\lambda > \aleph_0$, $G \in \mathbf{K}_{\mathrm{lf}}^{\lambda}$ is an abelian p-group, $H \leqslant G$ and $|H| < \lambda$, then $Aut_H(G)$ has a locally finite subgroup of power λ.*

Proof of Lemma 1.9. By Fact 4.10 and Lemma 4.11. □

References

[1] L. Fuchs, Infinite Abelian Groups – Vol. I, Pure and Applied Mathematics, Vol. 36, Academic Press, New York-London 1970.

[2] L. Fuchs, Infinite Abelian Groups – Vol. II, Pure and Applied Mathematics. Vol. 36-II, Academic Press, New York-London, 1973.

[3] A.S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.

[4] A.H. Mekler, Stability of Nilpotent Groups of Class 2 and Prime Exponent, *J. Symbolic Logic* **46:4** (1981), 781–788.

[5] A.S. Kechris, A.Nies, K. Tent, The Complexity of Topological Group Isomorphism, *J. Symbolic Logic* **83:3** (2018), 1190–1203.

[6] G. Paolini, S. Shelah, Groups Metrics for Graph Products of Cyclic Groups, *Topology Appl.* **232** (2017), 281–287.

[7] G. Paolini and S. Shelah, The Automorphism Group of Hall's Universal Group, *Proc. Amer. Math. Soc.* **146** (2018), 1439–1445.

[8] Su Gao, Invariant Descriptive Set Theory, Pure and Applied Mathematics (Boca Raton), 293. CRC Press, Boca Raton, FL, 2009.

[9] S. Shelah, Beginning of Stability Theory for Polish Spaces, *Israel J. Math.* **214:2** (2016), 507–537.

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