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MUTUAL STATIONARITY AND SINGULAR JONSSON CARDINALS

S. SHELAH

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram,
The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
Department of Mathematics, Hill Center – Busch Campus Rutgers,
The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA
e-mail: shelah@math.huji.ac.il

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Abstract. We prove that if the sequence $\langle k_n : 1 \leq n < \omega\rangle$ contains a so-called gap then the sequence $\langle S^{\aleph_n}_{\aleph_{k_n}} : 1 \leq n < \omega\rangle$ of stationary sets is not mutually stationary, provided that $k_n < n$ for every $n \in \omega$. We also prove a sufficient condition for being singular Jonsson cardinals.

0. Introduction

Mutual stationarity appeared first in a seminal paper of Foreman and Magidor [2]. The combinatorial motivation can be described as follows. If $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ then club subsets of κ and stationary subsets of κ are extremely important concepts with a very rich structure. But if $\mu > \mathrm{cf}(\mu) = \aleph_0$ then a straightforward generalization of these concepts is almost meaningless. For example, one can easily define two disjoint clubs of μ. Mutual stationarity is an attempt to capture the parallel of stationarity at regular cardinals while dealing with singular cardinals with countable cofinality. Earlier work on the case where each S_n is $S^{\lambda(n)}_{\kappa(n)}$ is Liu–Shelah [4]. Lately, Ben Neria has worked on this.

A particular case which attracted some attention is $\mu = \aleph_\omega$. One reason is the possible connection between mutual stationarity at $\mu > \mathrm{cf}(\mu) = \aleph_0$ and the possible Jonssonicity of μ, a long standing open problem in set theory. The main result of this paper is that one can prove the existence of a sequence of non-mutual stationary subsets of the $\aleph_n$'s.

The author thanks Alice Leonhardt for the beautiful typing. References like [5, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. This is work 1158 in the author's list.

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Other results are focused on singular Jonsson cardinals and, in particular, the preservation of Jonssonicity under forcing extensions. The accepted wisdom was always that, to prove the consistency of $\aleph_\omega$ is a Jonsson cardinal we should force $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ and that for any given $M_* \in \mathscr{M}_{\aleph_\omega}$, 0.2(2). we should find an elementary sub-model M of M_* such that for every $n < \omega$ we have $\mathrm{cf}(\sup(M \cap \omega_{n+1})) = \aleph_n = \|M \cap \omega_{n+1}\|$; naturally starting with the natural large cardinal. We have thought that it is more natural to try to force that M satisfies $\mathrm{cf}(\sup(M \cap \omega_n)) = \aleph_{h(n)}$ where h is a function from ω to ω satisfying $h(n+1) \leq h(n)+1$, for some increasing sequence $0 = n_0 < n_1 \dots$ we have $h \restriction [n_i + 1, n_i]$ is non-decreasing from its domain onto $[0, n_i]$.

The paper answers a question which arose during a lecture of Ben-Neria in the Hebrew University, Fall 2017. In fact, it had been asked by Foreman [3].

We thank Ben-Neria and Shimoni for their help.

Notation 0.1. For regular $\kappa < \lambda$ let $S^\lambda_\kappa = S[\lambda, \kappa]$ be the set $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ and let $S[\lambda, \geq \kappa]$ be the set $\{\delta < \lambda : \mathrm{cf}(\delta) \geq \kappa\}$.

Convention 0.2. 1) Let μ be a cardinal. We follow the convention by which an *algebra* M_* on $\mathscr{H}(\mu)$ is a model in a countable language, which expands $(\mathscr{H}(\mu), \in)$. By Skolemizing, we may always assume that M_* has definable Skolem functions, whose induced closure function is denoted by $F_{M_*} \colon [\mathscr{H}(\mu)]^{<\aleph_0} \to [\mathscr{H}(\mu)]^{\aleph_0}$ and let $\mathscr{M}_\mu = \mathscr{M}(\mu)$ be the set of such models M_*.

2) A *sub-algebra* M of M_* is an elementary substructure $M \prec M_*$. Therefore, for every $X \subseteq \mathscr{H}(\mu)$, the Skolem-hull of X,

$$\mathrm{Sk}^{M_*}(X) = F_{M_*}``[X]^{<\aleph_0}$$

is a sub-algebra of M_* of cardinality $|X| + \aleph_0$.

3) For every cardinal $\kappa \in M$ we define $\chi_M(\kappa) = \sup(M \cap \kappa)$.

Definition 0.3. We say μ is Jonsson <u>when</u> for every algebra M_* on $\mathscr{H}(\mu)$ there exists a sub-algebra $M \prec M_*$ such that $|M \cap \mu| = \mu$ and $M \cap \mu \neq \mu$.

Definition 0.4. 1) Let $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle \in M$ be a sequence of cardinals, and $\vec{S} = \langle S_n : n < \omega \rangle$ be a sequence of sets, $S_n \subseteq \kappa_n = \sup(S_n)$. We say that $\vec{S}$ is mutually stationary if for every algebra M_* on $\mathscr{H}(\mu)$, $\mu = \bigcup_n \kappa_n$, there exists a sub-algebra $M \prec M_*$ such that $\chi_M(\kappa_n) \in S_n$ for every $n < \omega$.

2) Similarly for an increasing sequence $\langle \kappa_\alpha : \alpha < \alpha(*) \rangle$ of regular cardinals and sequence $\langle S_\alpha : \alpha < \alpha(*) \rangle$ when with S_α a sequence with S_α an unbounded subset of κ_α

DEFINITION 0.5. An infinite cardinal λ is $I1$ iff there is an elementary embedding $\mathbf{j} : \mathbf{V}_{\lambda+1} \to \mathbf{V}_{\lambda+1}$. It is easy to see that if λ is $I1$ then, e.g. λ is an ω-limit of measurable cardinals and hence Jonsson.

1. Mutual stationarity at the $\aleph_n$'s

In this section we shall give a positive answer to Question 4.3 from [3].

Before stating the main theorem of this section we define the following concept:

DEFINITION 1.1. Let $\langle k_n : n_0 \leq n < \omega \rangle$ be a sequence of integers satisfying $k_n < n$. We say that the sequence contains a gap if there are $k < \omega$ and $n < \omega$, so that $0 < k < k_n$ but $k \neq k_m$ for all $m < n$.

THEOREM 1.2. *There exists a sequence $\langle S_n : 1 \leq n < \omega \rangle$ of stationary sets $S_n \subseteq \omega_n$ which is not mutually stationary. Moreover, each S_n can be taken to be $S^n_{k_n}$ for some $k_n < n$.*

PROOF. Let $\langle k_n : 1 \leq n < \omega \rangle$ be a sequence of integers $k_n < n$, which contains a gap. We prove that the sequence of stationary sets $\langle S_n : 1 \leq n < \omega \rangle$, $S_n = S^n_{k_n}$, is not mutually stationary.

By the definition of mutual stationarity we should find $M_* \in \mathscr{M}_{\aleph_\omega}$ such that for every $M \prec M_*$ there exists $n \in \omega$ such that $\chi_M(\aleph_n) \notin S^n_{k_n}$. Hence by stipulating $M_* \in \mathscr{M}(\aleph_\omega)$, it suffices to show that there are no sub-algebras $M \prec M_*$ for which $n < \omega \Rightarrow \mathrm{cf}(\chi_M(\aleph_n)) = \aleph_{k_n}$. Suppose otherwise. Let $\mu = \aleph_\omega$ and fix a counter example $M \prec M_*$ and $k < \omega, n' < n < \omega$, for which $k_{n'} < k < k_n$ and $k \neq k_m$ for all $m < n$. Also define $u = \{m \leq n : \mathrm{cf}(\chi_M(\aleph_m)) > \aleph_k\}$.

For each $m \leq n$, let $A_m \subseteq (\chi_M(\aleph_m) \setminus \omega_{m-1}) \cap M$ be a cofinal subset of $\chi_M(\aleph_m)$ of minimal order type $\mathrm{otp}(A_m) = \aleph_{k_m}$. Also choose $B \subseteq A_n$ of cardinality $|B| = \aleph_k$; possible because $|A_n| = \aleph_{k_n}$, $k_n > k$. To establish a contradiction, we shall show that under the above assumptions, it is possible to construct a sub-algebra $N_0 \prec M$ of cardinality $|N_0| \leq \aleph_{k-1}$, which contains B.

To this end, we shall define by a decreasing induction on $i = n, n-1, \ldots, 1, 0$, a sequence of sub-algebras N_i of M together with a sequence of ordinals $\alpha_i \in A_i (\subseteq M)$ when $i \in u$ and $B \subseteq N_i$. If $0 < i \leq n$ is such that α_j have been defined for every $j \in u \setminus i$, then we define

$$N_i = \mathrm{Sk}^M\Big((M \cap \omega_{i-1}) \cup \{\alpha_j : j \in u \setminus i\} \cup \Big(\bigcup_{j \in (n \setminus (u \cup i))} A_j\Big)\Big).$$

If $i = 0$ then let $N_0 = \mathrm{Sk}^M\big(\{\alpha_j : j \in u\} \cup \big(\bigcup_{j \in n \setminus u} A_j\big)\big)$.

It is clear from this definition of the sub-algebras N_i, that $N_n \subseteq M$ and they form a decreasing sequence $N_n \supseteq N_{n-1} \supseteq \ldots \supseteq N_0$, that $\|N_i\| < \aleph_{\max\{i,k\}}$ and that $\|N_0\| < \aleph_k$.

The reason for $\|N_0\| < \aleph_k$ is that $j \in n \backslash u \Rightarrow \neg(|A_j| > \aleph_k)$ and then necessarily $|A_j| < \aleph_k$ since $k \neq k_m$ for every $m < n$. Note that this is the only point in which we use the fact that our sequence contains a gap. The key is therefore to choose the ordinals α_j for $j \in u$, so that $B \subseteq N_i$ for every i; this will give the desired contradiction because $\aleph_k = |B| < \aleph_k$.

Case I: $i = n$. Note that $B \subseteq A_n$ has cardinality $|B| = \aleph_k < \aleph_{k_n} = \mathrm{cf}(\chi_M(\aleph_n))$ and is therefore bounded by some $\alpha_n \in A_n$. Since $\alpha_n \cap M \subseteq N_n = \mathrm{Sk}^M((M \cap \omega_{n-1}) \cup \{\alpha_n\})$ we conclude that $B \subseteq N_n$.

Next, let $i < n$ and suppose that $\{\alpha_j : j \in u \setminus (i+1)\}$ have been defined so that $B \subseteq N_{i+1}$.

Case II: $i \notin u$ and $i < n$. The inductive assumption is that $B \subseteq N_{i+1}$, and we shall show that in this case $N_i = N_{i+1}$ which is stronger than what we have to prove. Since A_i is cofinal in $\chi_M(\aleph_i)$, and $A_i \subseteq N_i$ we have that $M \cap \omega_i \subseteq \mathrm{Sk}^M((M \cap \omega_{i-1}) \cup A_i)$. As $i \notin u, \{\alpha_j : j \in u \setminus i\} = \{\alpha_j : j \in u \setminus (i+1)\}$, and it follows at once that

$$N_i = \mathrm{Sk}^M\left((M \cap \omega_{i-1}) \cup \{\alpha_j : j \in u \setminus i\} \cup \left(\bigcup_{j \in (n \setminus (u \cup i))} A_j\right)\right)$$

$$= \mathrm{Sk}^M\left((M \cap \omega_i) \cup \{\alpha_j : j \in u \setminus i+1\} \cup \left(\bigcup_{j \in n \setminus (u \cup (i+1))} A_j\right)\right) = N_{i+1}.$$

In particular, $B \subseteq N_i$.

Case III: $i \in u$ and $i < n$. For each $\alpha \in A_i$, consider the sub-algebra

$$N_{i,\alpha} = \mathrm{Sk}^M\left((M \cap \omega_{i-1}) \cup \{\alpha_j : j \in u \setminus (i+1)\} \cup \left(\bigcup_{j \in (n \setminus (u \cup i))} A_j\right) \cup \{\alpha\}\right).$$

It is clear from our definition of N_{i+1} that $\langle N_{i,\alpha} : \alpha \in A_i \rangle$ is a $\subseteq$-increasing sequence of sub-algebras of M and even of N_{i+1}, which cover N_{i+1}. As $\mathrm{otp}(A_i) \geq \aleph_{k+1} > |B|$, there must exist some $\alpha \in A_i$ such that $B \subseteq N_{i,\alpha}$. We define α_i to be the minimal such $\alpha \in A_i$. It is clear from the definitions that $N_i = N_{i,\alpha_i}$, and thus, $B \subseteq N_i$ as required. $\square_{1.2}$

Assume that $\lambda = \bigcup_{n \in \omega} \kappa_n$, where $\langle \kappa_n : n \in \omega \rangle$ is an increasing sequence of measurable cardinals; see [6, (3a), p. 506] or [1, Theorem 5.2], the statement of 1.2 fails at λ. The reason is that in this case, λ is a fixed point of the $\aleph$-function, as can be deduced from the proof. Therefore, we can phrase the following:

CLAIM 1.3. *Let* $\lambda = \aleph_\delta > \mathrm{cf}(\lambda) = \aleph_0$ *and suppose that* $\delta < \aleph_\delta = \lambda$.

1) *Assume* $\delta = \alpha + \omega$ *and* $\langle n_i = n(i) : i < \omega \rangle$ *is increasing sequence of non-zero natural numbers and* $\kappa_i = \aleph_{\alpha+n(i)}$.

Then there exists a sequence $\langle S_i : i \in \omega \rangle$ *such that*:

(a) $S_i \subseteq \kappa_i$ *is stationary for every* $i < \omega$

(b) $\langle S_i : i < \omega \rangle$ *is not mutual stationary.*

2) *In part* (1), *if* $n_i + 1 < n_{i+1}$ *then we can choose* $S_j = S[\kappa_j \geq \kappa_{n_i+1}]$ *when* $j \in [n_i, n_{i+1})$ *and* $S_j = \kappa_j$ *when* $j < \omega$, $j \notin [n_i, n_{i+1})$.

3) *In part* 1), *if clause* (A) *below holds then we can choose the* S_i*-s as in clause* (B), *where*

(A) (a) $i_1 = i(1) < i_2 = i(2) < \omega$

(b) *for* $i \in [i_1, i_2]$ *we have* $w_i \subseteq [i(1), i]$

(c) *there is no* f *satisfying*: *its domain is* $[i_1, i_2]$ *and when* $f(i)$ *is well defined then it belongs to* w_i *and is equal to* $n_{i(1)} - 1$ *or it belongs to* $[i(1), i)$; *and its range include* $[i(1), f(i_2))$.

(B) (a) *if* $i \in [i_1, i_2]$ *then* $S_i = \big\{ \beta < \kappa_i : \mathrm{cf}(\beta) = \aleph_{\alpha+i(1)}$ *or* $\mathrm{cf}(\beta) \in \{\aleph_{\alpha+n} : n \in w_i\} \big\}$

(b) *if* $i < \omega, i \notin [i_1, i_2]$ *then* $S_i = \kappa_i$.

4) *The following is impossible*: $M_* \in \mathscr{M}_\lambda$ *and* $|\delta| < \aleph_{i(*)}$, $j(*) < i(*) < \delta$ *and* $i \in [j(*)+1, i(*)) \Rightarrow \chi_M(\aleph_{i+1}) \neq \aleph_{j(*)+1}$ *and* $\chi_{M_*}(\aleph_i) > \aleph_{j(*)+1}$.

PROOF. 1) If $n_i = i + 1$, the proof is exactly as the proof of 2.1, or note that forcing by the Levy collapse of $\aleph_{\alpha+n(0)-1}$ to $\aleph_0$. If not as above, this reduce to Theorem 1.2 and anyhow then we can apply part (2).

2) A special case of part (3).

3) Consider a model $M_* \in \mathscr{M}_\lambda$ and let M be an elementary sub-model of M_* and let f be the function with domain $[i(1), i(2))$ defined by: if $\chi_M(\aleph_{\alpha+n})$ $\leq \aleph_{\alpha+i(1)-1}$ then $f(n) = i(1) - 1$ and otherwise $\chi_M(\aleph_n) = \aleph_{\alpha+f(n)}$. Now continue as in the proof of 2.1 with $n(i(1)) - 1, i(2), f(m)$ here playing the role of $0, n, k_m$.

4) We choose A_i an unbounded subset of $\aleph_i \cap M$ of order-type $\chi_M(\aleph_i)$, so is of cardinality $\aleph_i$ when $\aleph_i \cap M$ is unbounded in $\aleph_i$. Let B be a subset of $A_{n(i(2))}$ of cardinality $\aleph_{j(*)+1}$, let $C = \bigcup\{A_i : i < \delta, \chi_M(\aleph_i) \leq \aleph_{\alpha+j(*)}\} \cap \omega_{j(*)}$ so C is a subset of M of cardinality at most $\aleph_{j(*)}$. Now by induction on $k < \omega$ choose B_k, N_k, M_k such that:

(a) B_k is a subset of M of cardinality $\leq \aleph_{j(*)}$

(b) M_k is the Skolem hull of $\cup\{B_m : m < k\} \cup C$

(c) N_k is the Skolem hull of $M_k \cup B$

(d) if $i < \delta, i > i(*)$ and A_i has cardinality $> \aleph_{j(*)}$ then B_k contains a member of A_i which is above $N_k \cap \aleph_{i+1}$

Now we can prove that $N = \cup\{N_k : k < \omega\} = \cup\{M_k : k < \omega\}$, as in [7, Ch.XIII], [8, Ch.VII]. $\square_{1.3}$

2. On singular Jonsson cardinals

Suppose that μ is a strong limit singular cardinal. The purpose of this section is to provide sufficient conditions for μ being Jonsson.

Let M_* be an algebra on $\mathscr{H}(\mu)$. As before, let F_{M_*} be the function induced by a definable collection of Skolem functions from M_*. For every cardinal $\lambda < \mu$, we define $M_* \restriction \mathscr{H}(\lambda)$ to be the algebra on $\mathscr{H}(\lambda)$ generated by $F^{\lambda}_{M_*} : [\mathscr{H}(\lambda)]^{<\omega} \to \mathscr{H}(\lambda)$, where

$$F^{\lambda}_{M_*}(v_0, \dots, v_{m-1}) = \begin{cases} F_{M_*}(v_0, \dots, v_{m-1}) & \text{if } F_{M_*}(v_0, \dots, v_{m-1}) \in \mathscr{H}(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for every $x \subseteq \lambda$, $F^{\lambda}_{M_*}$“$[x]^{<\omega} = F_{M_*}$“$[x]^{<\omega} \cap \mathscr{H}(\lambda)$. In particular, for every sub-algebra M' of $M_* \restriction \mathscr{H}(\lambda)$, there exists a sub-algebra M of M_* so that $M' = M \cap \mathscr{H}(\lambda)$.

Claim 2.1. *Suppose that there exist three increasing sequences $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$, $\vec{\mu} = \langle \mu_n : n < \omega \rangle$, $\vec{\lambda} = \langle \lambda_n : n < \omega \rangle$, all cofinal in μ. The following conditions guarantee that μ is Jonsson*:

(1) $\kappa_n < \mu_n < \lambda_n < \kappa_{n+1}$ *for every* $n < \omega$

(2) $2^{\mu_n} \leq \lambda_n$ *for every* $n < \omega$

(3) *for every algebra M_* on $\mathscr{H}(\mu)$, there exists a sequence $\vec{M} = \langle M_n : n < \omega \rangle$ so that $\{\kappa_k, \mu_k, \lambda_k : k < \omega\} \cap \lambda_n \subseteq M_n, \kappa_0 \nsubseteq M_0$, and for every $n < \omega$*:

- (a) *M_n is a sub-algebra of the algebra $M_* \restriction \mathscr{H}(\lambda_n)$,*
- (b) $|M_n \cap \mu_n| \geq \kappa_n$,
- (c) $M_{n+1} \cap \lambda_n \subseteq M_n$.

Proof. Fix an algebra $\mathscr{M}_*$ on $\mathscr{H}(\mu)$. Denote for each $n < \omega$, $M_n \cap \mu_n$ by A_n, and define $M = \mathrm{Sk}^{M_*}\big(\bigcup_n A_n\big) = F_{M_*}$“$[\bigcup_n A_n]^{<\aleph_0}$. Clearly M is elementary in M_* and has cardinality $|M| = \sum_n |A_n| = \sum_n \kappa_n = \mu$. To show that M is nontrivial, we verify that $M \cap \kappa_0 = M_0 \cap \kappa_0$. In particular, $\kappa_0 \nsubseteq M$ by our assumptions.

Clearly $M_0 \cap \kappa_0 \subseteq M \cap \kappa_0$, so let us prove that $M_0 \cap \kappa_0 \supseteq M \cap \kappa_0$.

Fix some $\tau \in M \cap \kappa_0$. By the definition of M, there is $m < \omega$ and finite sequences $a_i \in {}^{\omega>}(A_i), i = 0, \dots, m$, so that $\tau = F_{M_*}(a_0, a_1, \dots, a_m)$. We proceed to show by downward induction on $i = m, m-1, \dots 1$, that for every i, there exists a function $f_i : {}^{\omega>}(\mu_{i-1}) \to \kappa_0$ in M_{i-1}, such that $\tau = f_i(a_0, \dots, a_{i-1})$.

Starting from $i = m$, define $f_m \colon [\mu_{m-1}]^{<\omega} \to \kappa_0$ by

$$f_m(v_0, \dots, v_{m-1}) = \begin{cases} F_{M_*}(v_0, \dots, v_{m-1}, a_m) & \text{if } F_{M_*}(v_0, \dots, v_{m-1}, a_m) < \kappa_0 \\ 0 & \text{otherwise.} \end{cases}$$

$f_m \in M_m$ since $a_m, \kappa_0, \mu_{m-1} \in M_m$. Moreover, the fact that $2^{\mu_{m-1}} \leq \lambda_{m-1}$ and $M_m \cap \lambda_{m-1} \subseteq M_{m-1}$, implies that $f_m \in M_{m-1}$.

Next, suppose that $f_{i+1} \in M_i, f_{i+1} \colon {}^{\omega>}(\mu_i) \to \kappa_0$ has been defined. Let $f_i \colon {}^{\omega>}(\mu_{i-1}) \to \kappa_0$, by $f_i(v_0, \ldots, v_{i-1}) = f_{i+1}(v_0, \ldots, v_{i-1}, a_i)$. f_i belongs to M_i since f_{i+1}, a_i do. It further belongs to M_{i-1} since $2^{\mu_{i-1}} \leq \lambda_{i-1}$ and $M_i \cap \lambda_{i-1} \subseteq M_{i-1}$.

Finally, for $i = 1$, we have that $f_1 \colon [\mu_0]^{<\omega} \to \kappa_0$ belongs to M_0. Since both $f_1, a_0 \in M_0, \tau = f_1(a_0)$ belongs to $M_0 \cap \kappa_0$. $\Box_{2.1}$

EXAMPLE 2.2. Suppose that $\mathbf{j} \colon \mathbf{V}_{\tau+1} \to \mathbf{V}_{\tau+1}$ is an elementary embedding. Let $\tau_0 = \text{cp}(\mathbf{j})$ and $\tau_{k+1} = \mathbf{j}(\tau_k)$ for every $k < \omega$. In particular $\tau = \bigcup_k \tau_k$. It is not difficult to see if $(\kappa_n, \mu_n, \lambda_n)$ $n < \omega$, are chosen among the cardinals τ_k, $k < \omega$, so that for every $n < \omega$, $\kappa_n = \tau_{t_n}$ implies $\mu_n = \tau_{t_n+1}$ and $\lambda_n = 2^{\mu_n}$, then the resulting sequences $\vec{\kappa}$, $\vec{\mu}$, $\vec{\lambda}$ satisfy the conditions of Claim 2.1. This is an immediate consequence of the elementarity of the embedding $\mathbf{j}$ and the fact that for every algebra M_* on μ, the sequence $\langle \mathbf{j}''(M_* \restriction \mathscr{H}(\lambda_n)) : n < \omega \rangle$ satisfies the desired condition with respect to $\mathbf{j}(M_*)$.

The proof of Claim 2.1 naturally generalizes to cases where μ is singular of an arbitrary cofinalily. We state the relevant result.

CLAIM 2.3. *Suppose that μ is a singular limit of sequences $\vec{\kappa} = \langle \kappa_i : i < \text{cf}(\mu) \rangle$, $\vec{\mu} = \langle \mu_i : i < \text{cf}(\mu) \rangle$, $\vec{\lambda} = \langle \lambda_i : i < \text{cf}(\mu) \rangle$, which satisfy the following conditions*:

(1) $\kappa_i < \mu_i < \lambda_i$ *and* $2^{\mu_i} \leq \lambda_i$ *for every* $i < \text{cf}(\mu)$

(2) $\lambda_i < \kappa_j$ *whenever* $i < j < \text{cf}(\mu)$

(3) *For every algebra M_* on $\mathscr{H}(\mu)$ there is a sequence $\langle M_i : i < \text{cf}(\mu) \rangle$ of sub-algebras of M_*, such that $\kappa_0 \nsubseteq M_0$, and the following holds for each $i < \text{cf}(\mu)$*:

(a) $|M_i \cap \mu_i| \geq \kappa_i$,

(b) $M_j \cap \lambda_i \subseteq M_i$ *for every* $j > i$.

(c) $M_0 \cap \kappa_0 \neq \kappa_0$.

Then μ is Jonsson.

2.1. Speculating on preserving the Jonsson property in generic extensions. Building on the result of the first claim, we proceed to describe conditions under which μ remains Jonsson after collapsing certain cardinals below μ. Recall that $\mu = \cup\{\lambda_n : n < \omega\}$.

Suppose that $\mathbb{P} = \langle \mathbb{P}_n, \mathbb{Q}_n : n < \omega \rangle$ is a full-support iteration of posets $\mathbb{Q}_n$, so that for each $n < \omega, \mathbb{Q}_n$ collapses certain cardinals in the interval $(\lambda_{n-1}, \lambda_n)$. For completeness, we set $\lambda_{-1} = \aleph_0$. Suppose also that for each $n < \omega$, $\mathbb{P}_n$ satisfies the λ_{n-1}-c.c, and that $\mathbb{P}/\mathbb{P}_n$ is λ^+_{n-1}-closed. We naturally assume $\mathbb{P}_{n+1} \subseteq \mathscr{H}(\lambda_n)$ hence every antichain A of $\mathbb{P}_{n+1}$ belongs to $\mathscr{H}(\lambda_n)$.

Let $\vec{M} = \langle M_n : n < \omega \rangle$ be a sequence of sub-algebras satisfying the conditions of Claim 2.1, where for each $n < \omega, \mathbb{P}_{n+1}$ is definable over M_n (i.e., M_n is elementary in an expansion of $(\mathscr{H}(\lambda_n), \in, \mathbb{P}_{n+1})$).

DEFINITION 2.4. We say that a condition $\vec{p^*} = \langle p^*_n : n < \omega \rangle$ of $\mathbb{P}$ has property (*) if for every algebra M_* on $\mathscr{H}(\mu)$, and every condition $\vec{p} = \langle p_n : n < \omega \rangle$ which extends $\vec{p^*}$, there exists a sequence $\vec{M} = \langle M_n : n < \omega \rangle$ of sub-algebras $M_n \prec M_* \restriction \mathscr{H}(\lambda_n)$, as in the statement of Claim 2.1 above, and an extension $\vec{q}$ of $\vec{p}$, so that for each $n < \omega$, and a $\mathbb{P}_{n+1}$-name $\underset{\sim}{\sigma} \in M_n$ of an ordinal below λ_{n-1}, there exists a $\mathbb{P}_n$-name $\underset{\sim}{\sigma}^* \in M_n$, such that $\vec{q} \Vdash \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^*$.

REMARK 2.5. Let $\vec{p^*}$ be a condition of $\mathbb{P}$ and suppose that for every $\vec{p} \geq \vec{p^*}$ and an algebra M_* on $\mathscr{H}(\mu)$, there exist a sequence $\vec{M}$ as in the statement of Claim 2.1, and an extension $\vec{q} \geq \vec{p}$, so that for every $n < \omega$, $\vec{q} \restriction n$ forces that q_n is $M_n[\underset{\sim}{\mathbf{G}}_n]$-generic for $\mathbb{Q}_n$ (here, $\underset{\sim}{\mathbf{G}}_n$ is the canonical name for a $\mathbb{P}_n$-generic filter). Then $\vec{p^*}$ satisfies property $(*)$. For this, note that for every $\mathbb{P}_{n+1}$-name $\underset{\sim}{\sigma} \in M_n$ of an ordinal there is a $\mathbb{P}_n$-name of a dense set D in $\mathbb{Q}_n$, so that each $r \in D$ forces that $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^*$ for some $\mathbb{P}_n$-name $\underset{\sim}{\sigma}^*$. Therefore, if $\underset{\sim}{\sigma} \in M_n$ and q_n is forced to be $M_n[\underset{\sim}{\mathbf{G}}_n]$ generic, by $\vec{q} \restriction n$, then q_n is forced to belong to $D \cap M_n[\underset{\sim}{G}_n]$, and thus, to force that $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^*$ for some $\underset{\sim}{\sigma}^* \in M_{n-1}[\underset{\sim}{\mathbf{G}}_n]$.

EXAMPLE 2.6. Suppose $\mathbf{j} \colon \mathbf{V}_{\tau+1} \to \mathbf{V}_{\tau+1}$ is an elementary embedding, as in Example 2.2 above and let $\mathbb{P} = \langle \mathbb{P}_n, \mathbb{Q}_n : n < \omega \rangle$ be a full support iteration, so that for each $n < \omega, \mathbb{Q}_n$ is an Easton support iteration of $\mathrm{Coll}(\alpha^+, \alpha^{+13})$ where α ranges over all strongly inaccessible cardinal in $[\lambda_{n-1}, \lambda_n)$. Define $\vec{p^*} = \langle p^*_n : n < \omega \rangle$ as follows.

For each $n < \omega$, $p^*_n = \langle p^*_n(\alpha) : \alpha \in [\lambda_{n-1}, \lambda_{n+1}) \text{ inaccessible} \rangle$, is defined by:

(1) p^*_0 is the empty condition of $\mathbb{Q}_0$,

(2) for every $n \geq 1, \vec{p^*} \restriction n \,\hat{}\, p^*_n \restriction \alpha \Vdash_{\mathbb{P}_n * (\mathbb{Q}_n \restriction \alpha)} p^*_n(\alpha) = \emptyset$ if $\alpha \notin \mathbf{j}'' \lambda_{n-1}$, and

(3) $\vec{p^*} \restriction n \,\hat{}\, p^*_n \restriction \mathbf{j}(\beta) \Vdash_{\mathbb{P}_n * (\mathbb{Q}_n \restriction \mathbf{j}(\beta))} p^*_n(j(\beta)) = \mathbf{j}'' \underset{\sim}{g}_{n-1}(\beta)$ where $g_{n-1}(\beta)$ is the canonical name of the $\mathbb{Q}_n(\beta)$-generic collapse function $g_{n-1}(\beta) : \beta^+ \to \beta^{+13}$.

It is straightforward to verify that:

(i) $\vec{p^*}$ can be identified with a condition in $\mathbf{j}(\mathbb{P})$ (i.e., by a simple renaming of its indices) which extends $\mathbf{j}(\vec{p^*})$

(ii) for every algebra $\mathscr{A}$ on either $\mathscr{H}(\mu)$ or $\mathscr{H}(\tau_n)$, $\vec{p^*}$ is $M = \mathbf{j}'' \mathscr{A}$-generic.

It follows that $\mathbf{j}(\vec{p^*})$ has the $(*)$-property-witness $\vec{p^*}$ with respect to $\mathbf{j}(M_*)$, for every algebra M_* on $\mathscr{H}(\mu)$. Therefore $\vec{p^*}$ satisfies $(*)$.

As shown below, the existence of a condition $\vec{p^*}$ which satisfies $(*)$ guarantees that μ remains Jonsson after forcing with $\mathbb{P}$ above $\vec{p^*}$. We note that this example is somewhat superfluous, as here, the condition $\vec{p^*}$ guarantees that the I1 embedding $\mathbf{j}$ in V, extends to the $\mathbb{P}$-generic extension.

Claim 2.7. *Extending the conditions of Claim 2.1 above, if $\mathbf{G} \subseteq \mathbb{P}$ is a generic filter containing a condition $\vec{p^*} \in \mathbb{P}$ which satisfies $(*)$ from Definition 2.4, <u>then</u> μ remains Jonsson in $V[\mathbf{G}]$.*

Proof. Let M_* be an algebra on $\mathscr{H}(\mu)^{V[\mathbf{G}]}$. Fix a $\mathbb{P}$-name $\underset{\sim}{M}_*$ and $\vec{p} \in \mathbf{G}$ which extends $\vec{p^*}$, and forces $\underset{\sim}{M}_*$ is an algebra on $\mathscr{H}(\mu)$. Recall that our assumptions on $\mathbb{P}$ include that for every $n < \omega$, $\mathbb{P}/\mathbb{P}_{n+1}$ is λ_n^+-closed. We may therefore assume that $\vec{p}$ reduces $\underset{\sim}{M}_* \restriction \mathscr{H}(\lambda_n)$ to a $\mathbb{P}_{n+1}$-name, for each $n < \omega$. Since $\vec{p^*}$ satisfies property $(*)$, there exists a sequence $\langle N_n : n < \omega \rangle$ satisfying the conditions of Claim 2.1, so that each N_n is a sub-algebra of $\langle \mathscr{H}(\lambda_n), \in, \mathbb{P}_n, F_{\underset{\sim}{M}_* \restriction \mathscr{H}(\lambda_n)} \rangle$, and a condition $\vec{q} \in \mathbf{G}$, which forces that $N_n[\mathbf{G}] \cap \lambda_{n-1} \subseteq (N_n \cap \mathscr{H}(\lambda_{n-1}))[\mathbf{G} \restriction n]$. Recall that our assumptions on $\vec{N}$ in the statement of Claim 2.1 guarantee that $N_n \cap \mathscr{H}(\lambda_{n-1}) \subseteq N_{n-1}$. It follows that $N_n[\mathbf{G}] \cap \lambda_{n-1} \subseteq N_{n-1}[\mathbf{G} \restriction n] = N_{n-1}[\mathbf{G}]$. Clearly, $N_n[\mathbf{G}]$ is a sub-algebra of $M_* \restriction \mathscr{H}(\lambda_n)^{\mathbf{V[G]}}$ and $|N_n[\mathbf{G}] \cap \mu_n| \geq \kappa_n$ for every $n < \omega$.

We conclude that the sequence $\vec{M} = \langle M_n : n < \omega \rangle$ of sub-algebras, $M_n = N_n[\mathbf{G}] \prec M_* \restriction \mathscr{H}(\lambda_n)^{\mathbf{V[G]}}$, satisfies the conditions of Claim 2.1. We may therefore apply the claim in $V[\mathbf{G}]$ and conclude that μ is Jonsson. $\square_{2.7}$

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