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QUITE FREE COMPLICATED ABELIAN GROUPS, PCF AND BLACK BOXES

BY

SAHARON SHELAH*

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram
The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
and
Department of Mathematics, Hill Center—Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA
e-mail: shelah@math.huji.ac.il
URL: http://shelah.logic.at

ABSTRACT

We would like to build Abelian groups (or R-modules) which on the one hand are quite free, say $\aleph_{\omega+1}$-free, and on the other hand are complicated in a suitable sense. We choose as our test problem one having no non-trivial homomorphism to $\mathbb{Z}$ (known classically for $\aleph_1$-free, recently for $\aleph_n$-free). We succeed to prove the existence of even $\aleph_{\omega_1 \cdot n}$-free ones. This requires building n-dimensional black boxes, which are quite free. This combinatorics is of self interest and we believe will be useful also for other purposes. On the other hand, modulo suitable large cardinals, we prove that it is consistent that every $\aleph_{\omega_1 \cdot \omega}$-free Abelian group has non-trivial homomorphisms to $\mathbb{Z}$.

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0. Introduction

0(A). Abelian groups. We would like to determine the supremum of all λ for which we can prove TDC_λ, so, dually, the minimal λ such that consistently we have NTDC_λ which means the failure of TDC_λ, the trivial dual conjecture for λ, where:

(TDC_λ) there is a λ-free Abelian group G such that

$$\mathrm{Hom}(G, \mathbb{Z}) = 0.$$

This seems the weakest algebraic statement of this kind; it is consistent that the number is ∞, as if $\mathbf{V} = \mathbf{L}$ then TDC_λ holds for every λ (see, e.g., [GT12]). On the one hand by Magidor–Shelah [MS94], for

$$\lambda = \min\{\lambda : \lambda \text{ is a fixed point, that is } \lambda = \aleph_\lambda\},$$

NTDC_λ is consistent, as more is proved there: consistently "λ-free $\Rightarrow$ free". On the other hand, for a long time we have known the following for $\lambda = \aleph_1$, and recently by [She07] we know that for $\lambda = \aleph_n$ there are examples using the n-BB (n-dimensional black boxes) introduced there (for every n). Subsequently, those were used for more complicated algebraic relatives in Göbel–Shelah [GS09], Göbel–Shelah–Strüngman [GSS13] and Göbel–Herden–Shelah [GHS]. In [She13b] we have several close approximations to proving in ZFC the existence for $\aleph_\omega$, that is $\mathrm{TDC}_{\aleph_\omega}$ using 1-black boxes.

Here we finally fully prove that $\mathrm{TDC}_{\aleph_\omega}$ holds and much more; $\lambda = \aleph_{\omega_1 \cdot \omega}$ is the first cardinal for which TDC_λ cannot be proved in ZFC. The existence proof for $\lambda' < \lambda$ is a major result here, relying on the existence proof of quite free n-black boxes (in §1) which use results on pcf (see [She13a]). For complementary consistency results we start with the universe forced in [MS94] and then we force with a c.c.c. forcing notion making "$\mathrm{MA} + 2^{\aleph_0}$ large" but we have to work to show the desired result.

Of course, we can get better results (μ^+-free) when $\mu \in \mathbf{C}_\theta$ (see Definition 0.2) is so-called 1-solvable or $2^\mu = 2^{<\Upsilon} < 2^\Upsilon$ and $\Upsilon < 2^\mu$.

Note a point which complicates our work relative to previous ones: the amount of freeness (i.e., the κ such that we demand κ-free) and the cardinality of the structure are markedly different. In [She13b] this point is manifested when we construct say G of cardinality λ which is μ^+-free, where $\mu \in \mathbf{C}_{\aleph_0}$ or $\mu \in \mathbf{C}_{\aleph_1}$ and $\lambda = 2^\mu$ or $\min\{\lambda : 2^\lambda > 2^\mu\}$. The "distance" is even larger in [She07].

An interesting point here is that for many non-structure problems we naturally end up with two incomparable proofs. One is when we have a μ^+-free $\mathscr{F} \subseteq {}^{\partial}\mu$ of cardinality λ, λ as above. In this case the amount of freeness is large. In the other, we use the black box from Theorem 1.25. But we may like to use more sophisticated black boxes, say start with $\lambda_\ell, \mu_\ell(\ell \leq \mathbf{k})$, a black box $\mathbf{x}$ as in Theorem 1.25 and combine it with [She05]. The quotients $G/G_{\delta+1}, \delta$ a limit ordinal, are close to being $\lambda_{\mathbf{k}}^+$-free, replacing free by direct sums of small subgroups.

Recall from [Sheb, §3]: if we are given BB approximating models with universe, e.g., κ_2 by "guesses of cardinality κ_1", and usually models $\kappa_2 = \kappa_2^{\kappa_1}$, then we can construct models of cardinality κ_2 quite freely except the "corrections" toward avoiding, e.g., undesirable endomorphisms, i.e., for each approximation of such endomorphisms given by the BB is seen as a "task" how to avoid that in the end there will be an endomorphism extending the one given by the approximation. The "price" is that we make the construction not free, but between the various approximations there is little interaction. This will hopefully help in a planned continuation of [Shed] to use $\partial > \aleph_0$ and here to try to sort out the complicated cases like $\mathrm{End}(G) \cong R$. Maybe we can get a neater proof.

In [She75b], [She74] we suggested that combinatorial proofs from [She78, Ch. VIII], [She90, Ch. VIII], should be useful for proving the existence of many non-isomorphic structures, as well as rigid and indecomposable ones. The most successful case were black boxes applied to Abelian groups and modules first applied in [She84a], [She84b], that is:

(A) For separable Abelian p-groups G, proving the existence of those of cardinality $\lambda = \lambda^{\aleph_0}$ with only so called small endomorphisms ([She84a]).

(B) Let R be a ring whose additive group R^+ is cotorsion-free, i.e., R^+ is reduced and has no subgroups isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or to the p-adic integers. For $\lambda = \lambda^{\aleph_0} > |R|$ there is an abelian group G of cardinality λ whose endomorphism ring is isomorphic to R and as an R-module it is $\aleph_1$-free ([She84b, Th. 0.1, p. 40]).

We can relax the demands on R^+ and may require that G extends a suitable group G_0 such that R is realized on $\mathrm{End}(G)$ modulo a suitable ideal of "small" endomorphisms.

(C) Let R be a ring whose additive group is the completion of a direct sum of copies of the p-adic integers. If $\lambda^{\aleph_0} \geq |R|$, then there exists a separable Abelian p-group G with so-called basic subgroup of cardinality λ and $R = \mathrm{End}(G)/\mathrm{End}_s(G)$. As usual we get $\mathrm{End}(G) = \mathrm{End}_s(G) \oplus R$ ([She84b, Th. 0.2, p. 41]).

For previous history of those algebraic problems see [EM02], [GT12]. Quite many works using black boxes follow, starting with Corner–Göbel [CG85]; see again [EM02], [GT12]. On black boxes in set theory with weak versions of choice see [She16, §3A], with no choice see [She16, §3B] and for **k**-dimensional hopefully see [She16].

For further applications of those black boxes continuing the present work, mainly representation of a ring R and the endomorphism ring of a quite free Abelian group, see [Shed].

Discussion 0.1: (1) Note that usually, the known constructions were either for a λ-free R-module of cardinality λ using a non-reflecting $S \subseteq S^\lambda_{\aleph_0}$ with diamond or $\aleph_1$-free of some cardinality λ (mainly $\lambda = (\mu^{\aleph_0})^+$ but also in some other cases) many times using a black box (see [Sheb]) or "the elevator" (see [GT12]). In the former we use induction on $\alpha < \lambda$ and each α has kind of a "one task".

That is, using black boxes in the nice versions, we have for each $\delta \in S$ a perfect set of pairwise isomorphic tasks.

To deal with getting an $\aleph_n$-free Abelian group G with $\mathrm{Hom}(G, \mathbb{Z}) = 0$, the n-dimensional black boxes actually constructed and used in [She07] were products of black boxes from [Sheb]; each black box separately is only $\aleph_1$-free but the product of k gives $\aleph_k$-freeness. Here things are more complicated.

(2) Here cardinality and freeness differ.

(3) Note that the versions of freeness of BB in [She13b] and here are not the same.

0(B). Notation. Recall (on pp see [She94] but the reader can just use 0.3 below)

Definition 0.2: Let $\mathbf{C} = \{\mu : \mu \text{ strong limit singular and } \mathrm{pp}(\mu) =^+ 2^\mu\}$,

$$\mathbf{C}_\kappa = \{\mu \in \mathbf{C} : \mathrm{cf}(\mu) = \kappa\}.$$

Claim 0.3:

(a) *$\mu \in \mathbf{C}$ if μ is strong limit singular of uncountable cofinality,*
(b) *if $\mu = \beth_\delta > \mathrm{cf}(\mu)$ and $\delta = \omega_1$ or just $\mathrm{cf}(\delta) > \aleph_0$, then $\mu \in \mathbf{C}_{\mathrm{cf}(\mu)}$ and for a club (= a closed unbounded subset) of $\alpha < \delta$ we have $\beth_\alpha \in \mathbf{C}$.*

Proof. Clause (a) holds by [She94, Ch. II, §2] and clause (b) by [She94, Ch. IX, §5]. $\blacksquare_{0.3}$

Explanation 0.4: (1) A reader, particularly with algebraic background, may wonder how the ideals defined in Definition 0.5 below are used in the algebraic construction. For an ideal J on a set S we may try to find an Abelian group G_1 extending the free Abelian group

$$G_0 = \oplus\{\mathbb{Z}x_s : s \in S\}$$

such that the quotient

$$G_1/\oplus\{\mathbb{Z}_s : s \in S_1\}$$

is free for every $S_1 \in J$. In particular, we would like to have some $h_0 \in \mathrm{Hom}(G_0, \mathbb{Z})$ which cannot be extended to a homomorphism from G_1 to $\mathbb{Z}$. Copies of such tuples (S, J, G_1, G_0, h_0) are used as "the building block" in the constructions, so finding such examples is crucial; we find some, see in §2; more in [Shed].

(2) Concerning Observation 0.6, note that the product $J_1 \times J_2$ is not symmetric (even up to isomorphisms) because if, e.g., $\partial < \kappa$ are regular, then

$$J_\partial \times J_\kappa = \{A \subseteq \partial \times \kappa : \text{ for some } i < \partial, j < \kappa \text{ we have } A \subseteq (i \times \kappa) \cup (j \times \partial)\};$$

but $J_\kappa \times J_\partial$ has no such representation.

Definition 0.5: (1) For a set S of ordinals with no last member let J^{bd}_S be the ideal consisting of the bounded subsets of S.

(2) If J_ℓ is an ideal on S_ℓ for $\ell = 1, 2$, then $J_1 \times J_2$ is the ideal on $S_1 \times S_2$ consisting of the $S \subseteq S_1 \times S_2$ such that

$$\{s_1 \in S_1 : \{s_2 \in S_2 : (s_1, s_2) \in S\} \notin J_2\}$$

belongs to J_1.

(3) If δ_1, δ_2 are limit ordinals, J_ℓ is an ideal on δ_ℓ and $\delta_1 \cdot \delta_2 = \delta_3$, then $J_1 * J_2$ is the following ideal on δ_3: it consists of

$$\{\{\delta_1 \cdot i + j : i < \delta_2, j < \delta_1 \text{ and } (j, i) \in A\} : A \in J_1 \times J_2\}.$$

(4) If δ_1, δ_2 are limit ordinals, J_ℓ is an ideal on δ_ℓ for $\ell = 1, 2$ and $\delta_1 \cdot \delta_2 = \delta_3$, then $J_2 \odot J_1$ is the following ideal on δ_3: it consists of

$$\{\{\delta_1 \cdot i + j : i < \delta_2, j < \delta_1 \text{ and } (i, j) \in A\} : A \in J_2 \times J_1\}.$$

Observation 0.6: *If $\partial > \kappa$ are regular cardinals, then $J^{\mathrm{bd}}_\partial \times J^{\mathrm{bd}}_\kappa$ is isomorphic to $J^{\mathrm{bd}}_\partial * J^{\mathrm{bd}}_\kappa$ which includes $J^{\mathrm{bd}}_\partial \odot J^{\mathrm{bd}}_\kappa$, which is isomorphic to $J^{\mathrm{bd}}_\kappa \times J^{\mathrm{bd}}_\partial$.*

Proof. Should be clear but we elaborate the first equivalence.

Why is $J' = J^{\rm bd}_\partial \times J^{\rm bd}_\kappa$ isomorphic to $J'' = J^{\rm bd}_\partial * J^{\rm bd}_\kappa$?

Note that J' is an ideal on $\partial \times \kappa$ and J'' is an ideal on $\partial \cdot \kappa$. We define a function $\pi : \partial \times \kappa \to \partial \cdot \kappa$ by

(*) $\pi((i,j)) = \partial \cdot j + i$, so π is a one-to-one function from $\partial \times \kappa$ onto $\partial \cdot \kappa$ by the rules of ordinal division.

It suffices to prove that for any $A \subseteq \partial \times \kappa$, $A \in J' \Leftrightarrow \pi''(A) \in J''$; so fix $A \subseteq \partial \times \kappa$ and below we have $\bullet_i \Leftrightarrow \bullet_{i+1}$, hence $\bullet_1 \Leftrightarrow \bullet_4$, which suffices, when:

- $\bullet_1$ $A \in J'$,
- $\bullet_2$ $\{s_1 \in \partial : \{s_2 \in \kappa : (s_1, s_2) \in A\} \notin J^{\rm bd}_\kappa\} \in J^{\rm bd}_\partial$,
- $\bullet_3$ $\{i < \partial : \{j < \kappa : \partial j + i \in \pi''(A)\} \notin J^{\rm bd}_\kappa\} \in J^{\rm bd}_\partial$,
- $\bullet_4$ $\pi''(A) \in J''$.

That is, $\bullet_1 \leftrightarrow \bullet_2$ by the definition of J' and $\bullet_2 \leftrightarrow \bullet_3$ by the choice of π and $\bullet_3 \leftrightarrow \bullet_4$ by the definition of J''. $\blacksquare_{0.6}$

Definition 0.7: (1) We say $\mathscr{F} \subseteq {}^S X$ is **(θ, J)-free** when[1] J is an ideal on S and for every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \theta$ there is a sequence $\langle w_\eta : \eta \in \mathscr{F}' \rangle$ such that: $\eta \in \mathscr{F}' \Rightarrow w_\eta \in J$ and if $\eta_1 \neq \eta_2 \in \mathscr{F}'$ and $s \in S \backslash (w_{\eta_1} \cup w_{\eta_2})$ then $\eta_1(s) \neq \eta_2(s)$.

(2) We say $\mathscr{F} \subseteq {}^S X$ is **$[\theta, J]$-free** when J is an ideal on S and for every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \theta$ there is a list $\langle \eta_\alpha : \alpha < \alpha_* \rangle$ of $\mathscr{F}'$ such that: if $\alpha < \alpha_*$ then the set $w_\alpha := \{s \in S : \eta_\alpha(s) \in \{\eta_\beta(s) : \beta < \alpha\}\}$ belongs to J.

(3) Let θ-free or (θ)-free mean (θ, J)-free when $S \subseteq {\rm Ord}$, $J = J^{\rm bd}_S$.

(4) We say μ is **1-solvable** when μ is singular strong limit and there is a μ^+-free family $\mathscr{F} \subseteq {}^{{\rm cf}(\mu)}\mu$ of cardinality 2^μ.

(5) We say μ is **$(\theta, 1)$-solvable** when above we weaken "μ^+-free" to "θ-free".

(6) We say $\mathscr{F} \subseteq {}^S X$ is **weakly ordinary** when each $\eta \in \mathscr{F}$ is a one-to-one function. We say $\mathscr{F} \subseteq {}^\gamma{\rm Ord}$ is **ordinary** when each $\eta \in \mathscr{F}$ is an increasing function.

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[1] E.g., in [She94], this version is used. Sometimes we even demand

$$\alpha < \alpha_* \Rightarrow \{s \in S : \eta_\alpha(s) \in \{\eta_\beta(t) : \beta < \alpha, t \in I\}\} \in J.$$

But in the main case "J is a θ-complete filter on θ", the versions in 0.7(1),(2) are equivalent; see 1.16.

Claim 0.8: *Assume $\theta > \partial$ and ∂ is regular, J is an ideal on ∂ extending $[\partial]^{<\partial}$ and $\mathscr{F} \subseteq {}^{\partial}\mathrm{Ord}$ and*[2] *$\eta \neq \nu \in \mathscr{F} \Rightarrow |\{i < \partial : \eta(i) \in \mathrm{Rang}(\nu)\}| < \partial$.*

(1) *We have $\mathscr{F}$ is (θ, J)-free iff $\mathscr{F}$ is $[\theta, J]$-free.*

(2) *If every $\eta \in \mathscr{F}$ is one-to-one, then we can add in Definition 0.7(2), $\eta_\alpha(s) \notin \{\eta_\beta(t) : \beta < \alpha, t \in S\}$.*

Remark 0.9: (1) We may consider only the case $i \neq j \Rightarrow \eta(i) \neq \nu(j)$ in 0.7(1), 1.2(6), 1.11(1).

(2) Compare with [She94], [She13b].

(3) Because of 0.8 the difference between (θ, J)-free and $[\theta, J]$-free is not serious. For **k**-c.p. **x** see Definition 1.5; there we use only the latter version so do not write $[\theta, J]$.

Proof. (1) It is enough to prove for every $\mathscr{F} \subseteq {}^{\partial}\mathrm{Ord}$ of cardinality $< \theta$ that $\mathscr{F}$ is (θ, J)-free iff $\mathscr{F}$ is $[\theta, J]$-free.

First, if $\mathscr{F}$ is $[\theta, J]$-free, then there is a sequence $\langle \eta_\alpha : \alpha < \alpha_* \rangle$ enumerating $\mathscr{F}$ as in Definition 0.7(2), i.e., $\alpha < \alpha_* \Rightarrow w^1_\alpha := \{i < \partial : \eta_\alpha(i) \in \{\eta_\beta(i) : \beta < \alpha\}\} \in J$. Define w_η by $\eta = \eta_\alpha \Rightarrow w_\eta = w^1_\alpha$; easily $\langle w_\eta : \eta \in \mathscr{F} \rangle$ is as required in Definition 0.7(1).

Second, if $\mathscr{F}$ is (θ, J)-free, then there is $\langle w_\eta : \eta \in \mathscr{F} \rangle$ which is as required in Definition 0.7(1).

Let $\langle \eta^1_\alpha : \alpha < \alpha_* \rangle$ list $\mathscr{F}$ and by induction on n for each α we define $u_{\alpha,n}$ as follows:

$(*)^1_\alpha$ (a) $u_{\alpha,0} = \{\alpha\}$,

(b) $u_{\alpha,n+1} = u_{\alpha,n} \cup \{\beta < \alpha_* :$ for some $i \in \partial \backslash w_\beta$ we have $\eta_\beta(i) \in \{\eta_\gamma(i) : \gamma \in u_{\alpha,n}\}\}$.

Now

$(*)^2_\alpha$ $|u_{\alpha,n}| \leq \partial$ and $u_{\alpha,n} \subseteq \alpha_*$.

Why? Trivially $u_{\alpha,n} \subseteq \alpha_*$. Also $|u_{\alpha,0}| = 1 \leq \partial$, and if $|u_{\alpha,n}| \leq \partial$ then

$$\begin{aligned} |u_{\alpha,n+1}| \leq & |u_{\alpha,n}| + \sum_{i<\partial} \sum_{\gamma \in u_{\alpha,n}} |\{\beta < \alpha_* : \beta \text{ satisfies } i \notin w_\beta \wedge \eta_\beta(i) = \eta_\gamma(i)\}| \\ = & |u_{\alpha,n}| + \sum_{i<\partial} \sum_{\gamma \in u_{\alpha,n}} 1 \\ \leq & \partial + \partial \cdot \partial \cdot 1 = \partial. \end{aligned}$$

[2] We can replace "$< \partial$" by "$\in J'$" when $J' \subseteq J$ is a ∂-complete ideal.

We define u_α by induction on $\alpha < \alpha_*$ as follows:

$$u_\alpha = \bigcup_n u_{\alpha,n} \backslash \bigcup_{\beta<\alpha} u_\beta,$$

so $\langle u_\alpha : \alpha < \alpha_* \rangle$ is a partition of α_* to sets each of cardinality $\leq \partial$, so we can let $\langle \beta_{\partial\alpha+i} : i < i_\alpha \leq \partial \rangle$ list u_α. Let

$$\mathscr{U} = \{\partial\alpha + i : \alpha < \alpha_*, i < i_\alpha \text{ and } \beta_{\partial\alpha+i} \notin \cup\{u_\gamma : \gamma < \alpha\}\},$$

so $\{\beta_\gamma : \gamma \in \mathscr{U}\}$ lists α_* with no repetitions and easily $\langle \eta_{\beta_\zeta} : \zeta \in \mathscr{U} \rangle$ is a list as required in Definition 0.7(2). That is, let

$$\beta = \beta_{\partial\alpha+i} = \beta(\gamma_\alpha + i), \quad i < i_\alpha.$$

So $\{i < \partial : \eta_\beta(j) \in \{\eta_\gamma(j) : \gamma \in \mathscr{U} \cap \beta\}\}$ is the union of the following sets: $w^2_\beta := \{j < \partial : \eta_\beta(j) \in \{\eta_\gamma(j) : \gamma \in \mathscr{U} \cap \beta_{\partial,\alpha}\}\}$ and $w^2_{\beta,\iota} = \{j < \partial : \eta_\beta(j) = \eta_{\partial\alpha+\iota}(j)\}$ for $\iota < i$. Now each of those sets belong to J. [Why? w^2_β by the choice of the $u_{\gamma,n}$'s and the u_γ's; $w^2_{\beta,\iota}$ as it is included in $w_{\eta_{\beta(\partial\alpha+i)}}$.] So if J is a ∂-complete ideal we are done, and if not, by part of the assumption of the claim, $\iota < i \Rightarrow |w^2_{\beta,\iota}| < \partial$, so recalling ∂ is regular, $\bigcup_{\iota<i} w^2_{\beta,\iota}$ has cardinality $< \partial$ hence belongs to J, so as J is an ideal we are done.

Pedantically

$$\langle \eta'_\gamma : \gamma < \mathrm{otp}(\mathscr{U}) \rangle$$

is such a list when we define η'_γ for $\gamma < \mathrm{otp}(\mathscr{U})$ by $\eta'_{\mathrm{otp}(\zeta \cap \mathscr{U})} = \eta_{\beta_\zeta}$.

(2) Similarly to the "Second" in the proof that 0.7(1) holds, except that $(*)^1_\alpha$(b) is:

(b)′ $u_{\alpha,n+1} = u_{\alpha,n} \cup \{\beta < \alpha_* : \text{for some } i \in \partial \backslash w_\beta,$

$$\eta_\beta(i) = \{\eta_\gamma(j) : \gamma \in u_{\alpha,n}, j < \partial\}\}. \qquad \blacksquare_{0.8}$$

Question 0.10: (1) If μ is strong limit $\aleph_0 = \mathrm{cf}(\mu) < \mu$ (but not necessarily $\mu \in \mathbf{C}$), can we get the freeness results of [She13a]?

(2) In the cases we have, can we strengthen the χ-BB by having $F : \Lambda_{\mathbf{x}} \to \chi$ and demand $\eta_m(i) \in F(\bar\eta \upharpoonright (m, < i))$?

(2A) Is this preserved by products?

1. Black boxes

We generalize the $\mathbf{k}$-dimensional black box from [She07], where we deal with the special case when $\ell < \mathbf{k} \Rightarrow \partial_\ell = \aleph_0$ because this seems natural for Abelian groups; the black boxes earlier to [She07] were for $\mathbf{k} = 1$.

But here, for Abelian groups the most interesting cases are when

$$\{\partial_\ell : \ell < \mathbf{k}\} \subseteq \{\aleph_0, \aleph_1\}.$$

In the cases we prove existence, the $\mathbf{k}$-dimensional black box is the product of black boxes, i.e., those for $\mathbf{k} = 1$.

The main result is Theorem 1.25 telling us that there are $\mathbf{k}$-dimensional black boxes which are quite free.

The central notion here is combinatorial parameters, those objects ($\mathbf{x}$) consisting of the relevant finitely many cardinals ($\langle \partial_\ell : \ell < \mathbf{k}\rangle$) and sets ($\langle S_\ell : \ell < \mathbf{k}\rangle$) and a family ($\Lambda$) of sequences $\langle \eta_\ell : \ell < \mathbf{k}\rangle$ with η_ℓ a sequence of length ∂_ℓ of members of S_ℓ. Such objects are used in the construction of Abelian groups G. The point is that, on the one hand, the relevant (algebraic) freeness of the Abelian group G is deduced from (set theoretic) freeness of $\mathbf{x}$, i.e., of Λ, and on the other hand, e.g., $\mathrm{Hom}(G, \mathbb{Z}) = 0$ is deduced by using the $\mathbf{x}$ having a black box (which is used in the construction). See more in 1.4.

Convention 1.1: (1) $\bar\partial$ will denote a sequence $\langle \partial_\ell : \ell < \mathbf{k}\rangle$ of regular cardinals or just limit ordinals of length $\mathbf{k} \geq 1$ and then $\partial(\ell) = \partial_\ell$, but note that $k = \mathbf{k} - 1$ was used in [She07]; a major case is $\bar\partial$ is constant, i.e., $\bigwedge_\ell \partial_\ell = \partial$ for some ∂.

(2) Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ denote combinatorial parameters; see Definition 1.5 below.

Notation 1.2: (0) Here $\bar S = \langle S_\ell : \ell < \mathbf{k}\rangle$ and $\bar\partial = \langle \partial_\ell = \partial(\ell) : \ell < \mathbf{k}\rangle$.

(1) Let $\bar S^{[\bar\partial]} = \prod_{\ell<\mathbf{k}} {}^{\partial(\ell)}(S_\ell)$ and $\bar S^{[\bar\partial,u]} = \prod_{\ell\in u} {}^{\partial(\ell)} S_\ell$ for $u \subseteq \{0, \ldots, \mathbf{k}-1\}$, and if each S_ℓ is a set of ordinals let $\bar S^{<\bar\partial>} = \{\bar\eta \in \bar S^{[\bar\partial]}$: each η_ℓ is increasing$\}$ and similarly $\bar S^{<\bar\partial,u>}$.

(2) If $\bar\eta \in \bar S^{[\bar\partial]}, m < \mathbf{k}$ and $i < \partial_m$, then[3] $\bar\eta \upharpoonleft (m, i) = \bar\eta \upharpoonleft_{\mathbf{x}} (m, i)$ is $\langle \eta'_\ell : \ell < \mathbf{k}\rangle$ where η'_ℓ is η_ℓ when $\ell < \mathbf{k} \wedge \ell \neq m$ and is $\eta_\ell \restriction \{i\}$ if $\ell = m$; this is close to but not the same as in [She07].[4] Also for $w \subseteq \partial_m, \bar\eta \upharpoonleft (m, = w)$ is defined

[3] It is sometimes natural to replace "$i < \partial_\ell$" by "i a subset of ∂_ℓ from some family $\mathscr{P}_\ell$ and $\eta'_\ell = \eta_\ell \restriction i$ when $\ell = m$", say using $J^{\rm bd}_{\aleph_1} * J^{\rm bd}_{\aleph_1}$ as in [She13b]. In [She07] this version was used.

[4] But if we use a tree like $\Lambda \subseteq \bar S^{[\bar\partial]}$, see 1.2(6), the difference is small; what we use there is called here $\bar\eta \upharpoonleft (m, = i)$.

as $\langle \eta'_\ell : \ell < \mathbf{k} \rangle$ where $\eta'_\ell = \eta_\ell$ if $\ell < \mathbf{k} \wedge \ell \neq m$ and $\eta'_\ell = \eta_\ell{\restriction}w$ if $\ell = m$. Let $\bar{\eta} \upharpoonleft (m) = \langle \eta_\ell : \ell \neq m, \ell < \mathbf{k} \rangle$.

(3) If $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}, m < \mathbf{k}$ and $i < \partial_m$, then $\Lambda \upharpoonleft_{\mathbf{x}} (m,i) = \{\bar{\eta} \upharpoonleft (m,i) : \bar{\eta} \in \Lambda\}$; we define similarly $\Lambda \upharpoonleft_{\mathbf{x}} (\eta, = w)$.

(4) If $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}$, $m < \mathbf{k}$ and $i \leq \partial_m$, then $\Lambda \upharpoonleft_{\mathbf{x}} (m, < i) = \cup\{\Lambda \upharpoonleft (m, i_1) : i_1 < i\}$.

(5) $\Lambda_{\mathbf{x}, \in u} = \cup\{\Lambda_{\mathbf{x}} \upharpoonleft (m,i) : m \in u, i < \partial_m\}$ for $u \subseteq \{0, \ldots, \mathbf{k} - 1\}$. We may write "$< m$" instead of "$\in m$" when "$u = \{0, \ldots, m-1\}$" and let $\Lambda_{\mathbf{x},m} = \Lambda_{\mathbf{x}, \in \{m\}}$.

(6) We say $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}$ is tree-like when $\bar{\eta}, \bar{\nu} \in \Lambda, \bar{\eta} \upharpoonleft (m,i) = \bar{\nu} \upharpoonleft (m,j)$ implies $\eta_m{\restriction}i = \nu_m{\restriction}j$, so in particular it implies $i = j$.

(7) We say $\Lambda \subseteq \bar{S}^{<\bar{\partial}>}$ is normal when: if $\bar{\eta}, \bar{\nu} \in \Lambda, m < \mathbf{k}, i, j < \partial_m$ and $\eta_m(i) = \nu_m(j)$, then $i = j$ (hence each ν_m is one-to-one; this follows from being tree-like).

We now define in Definition 1.3 the standard $\mathbf{x}$, as it is more transparent than the general case (in 1.5), but we will not use it as the ZFC-existence results are not standard; see explanation after Definition 1.3. The main difference is that in the general (i.e., not necessarily standard) version, we have the extra parameter J_ℓ, ideal on ∂_ℓ.

Definition 1.3: (1) We say $\mathbf{x}$ is a **standard $\bar{\partial}$-c.p.** (combinatorial $\bar{\partial}$-parameter) when

$$\mathbf{x} = (\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda) = (\mathbf{k}_{\mathbf{x}}, \bar{\partial}_{\mathbf{x}}, \bar{S}_{\mathbf{x}}, \Lambda_{\mathbf{x}})$$

and it satisfies:

(a) $\mathbf{k} \in \{1, 2, \ldots\}$ and let $k = k_{\mathbf{x}} = \mathbf{k} - 1$ (this is to fit the notation in [She07]),

(b) $\bar{\partial} = \langle \partial_\ell : \ell < \mathbf{k} \rangle$ is a sequence of regular cardinals, so $\partial_\ell = \partial_{\mathbf{x},\ell}$,

(c) $\bar{S} = \langle S_\ell : \ell < \mathbf{k} \rangle, S_\ell$ a set of ordinals, so $S_\ell = S_{\mathbf{x},\ell}$,

(d) $\Lambda \subseteq \bar{S}^{[\bar{\partial}]} = \prod_{\ell < \mathbf{k}} {}^{\partial(\ell)}(S_\ell)$, see 1.2(1).

(2) If $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial$, we may write ∂ instead of $\bar{\partial}$ in $(\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda)$ and may say combinatorial $(\partial, \mathbf{k})$-parameter. If $\ell < \mathbf{k} \Rightarrow \partial_\ell = \aleph_0$, we may omit $\bar{\partial}$ and write "$\mathbf{x}$ is a combinatorial $\mathbf{k}$-parameter". If $\ell < \mathbf{k} \Rightarrow S_\ell = S$, we may write S instead of $\bar{S}$. Also we may write $\mathbf{k}(\mathbf{x})$ for $\mathbf{k}_{\mathbf{x}}$.

(3) We say $\mathbf{x}$ (or Λ) is **ordinary** when (each S_ℓ is a set of ordinals and) $\bar{\eta} \in \Lambda \Rightarrow$ each η_ℓ is increasing. We say $\mathbf{x}$ (or Λ) is **weakly ordinary** when $\bar{\eta} \in \Lambda \wedge m < \ell g(\bar{\eta}) \Rightarrow \eta_m$ is one-to-one. We say $\mathbf{x}$ is **disjoint** when $\langle S_{\mathbf{x},m} : m < \mathbf{k} \rangle$

is a sequence of pairwise disjoint sets. We say $\mathbf{x}$ is **ordinarily full** when it is ordinary and

$$\Lambda_{\mathbf{x}} = \{\langle \eta_\ell : \ell < \mathbf{k}\rangle : \eta_\ell \in {}^{\partial(\ell)}(S_\ell) \text{ is increasing for } \ell < \mathbf{k}\}.$$

Similarly for **weakly ordinary**.

(4) We say $\mathbf{y}$ is a permutation of $\mathbf{x}$ when for some permutation π of $\{0, \dots, \mathbf{k}-1\}$ we have $m < k \Rightarrow \partial_{\mathbf{x},m} = \partial_{\mathbf{y},\pi(m)}$ and $m < k \Rightarrow S_{\mathbf{x},m} = S_{\mathbf{y},\pi(m)}$ and

$$\Lambda_{\mathbf{y}} = \{\langle \eta_{\pi(m)} : m < \mathbf{k}\rangle : \langle \eta_m : m < \mathbf{k}\rangle \in \Lambda_{\mathbf{x}}\}.$$

(5) We say $\bar{\pi}$ is an isomorphism from $\mathbf{x}$ onto $\mathbf{y}$ when:

(a) $\mathbf{k}_{\mathbf{y}} = \mathbf{k}_{\mathbf{x}}$ call it $\mathbf{k}$,
(b) $\bar{\pi} = \langle \pi_m : m \le \mathbf{k}\rangle$,
(c) $\pi_{\mathbf{k}}$ is a permutation of $\{0, \dots, \mathbf{k}-1\}$,
(d) $\partial_{\mathbf{x},m} = \partial_{\mathbf{y},\pi_{\mathbf{k}}(m)}$ for $m < \mathbf{k}$,
(e) π_m is a one-to-one function from $S_{\mathbf{x},m}$ onto $S_{\mathbf{y},\pi_{\mathbf{k}}(m)}$ for $m < \mathbf{k}$,
(f) $\langle \nu_m : m < \mathbf{k}\rangle \in \Lambda_{\mathbf{y}}$ iff for some $\langle \eta_m : m < \mathbf{k}\rangle \in \Lambda_{\mathbf{x}}$ we have $\nu_{\pi_{\mathbf{k}}(m)} = \langle \pi_m(\eta_m(i)) : i < \partial_{\mathbf{x},m}\rangle$.

Discussion 1.4: It may be helpful to the reader to indicate how such $\mathbf{x}$ helps to construct, e.g., Abelian groups; for simplicity each ∂_ℓ is $\aleph_0$ (this suffices for constructing an $\aleph_{\omega\cdot n}$-free G, which already is new).

First, let $\langle x_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}} \upharpoonright (m,i)$ for some m and $i\rangle$ freely generate an Abelian group G_0 and for such $\bar{\eta} \in \Lambda_{\mathbf{x}}$ we add elements like

$$y_{\bar{\eta},n} = \Sigma\Big\{\Big(\frac{i!}{n!}\Big)(x_{\bar{\eta}\upharpoonright(m,i)} + a_{\bar{\eta},m} x_{\nu_{\bar{\eta}}}) : m < \mathbf{k}_{\mathbf{x}},\ i \text{ finite } \ge n\Big\}$$

for some $\nu_{\bar{\eta}} \in \Lambda_{\mathbf{x}_1,<\mathbf{k}}$, $n < \omega$ and $a_{\bar{\eta},m} \in \mathbb{Z}$ getting $G_1 \supseteq G_0$. Now, on the one hand, we like G_1 to be θ-free and, on the other hand, we like it, e.g., to have no non-zero homomorphism into $\mathbb{Z}$. For the second task, we need a BB (black box) property, that is, for each possible $\nu_{\bar{\eta}}$ to have, for each $\bar{\eta} \in \Lambda$, a homomorphism $h_{\bar{\eta}}$ from $\Sigma\{\mathbb{Z}x_{\bar{\eta}\upharpoonright(m,i)} : m < \mathbf{k}, i \text{ finite}\} \oplus \mathbb{Z}x_{\nu_{\bar{\eta}}}$ into $\mathbb{Z}$ such that $\{h_{\bar{\eta}} : \bar{\eta} \in \Lambda\}$ is dense (or see Definition 1.7(1), called $\bar{\alpha}_{\bar{\eta}}$ there) and choose the $a_{\bar{\eta},n}$'s to "defeat $h_{\bar{\eta}}$", i.e., to ensure no $h \in \mathrm{Hom}(G_1, \mathbb{Z})$ extends $h_{\bar{\eta}}$.

Concerning the first task, we like to ensure $\mathbf{x}$ is θ-free, meaning that for any $\Lambda \subseteq \Lambda_{\mathbf{x}}$ of cardinality $< \theta$ we can list its members as $\langle \bar{\eta}_\alpha : \alpha < \alpha_*\rangle$ such that for every α for some m, i we have $j \ge i \Rightarrow \bar{\eta}_\alpha \upharpoonright (m,j) \notin \{\bar{\eta}_\beta \upharpoonright (m,j) : \beta < \alpha\}$; see Definition 1.7(3).

In the existence proofs the novel main point is getting enough freeness relying on the pcf theory, i.e., in §1 we prove the existence of suitable c.p. $\mathbf{x}$.

Definition 1.5: (1) We say $\mathbf{x}$ is a $\bar{\partial}$**-c.p.** (combinatorial $\bar{\partial}$-parameter) when

$$\mathbf{x} = (\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda, \bar{J}) = (\mathbf{k}_\mathbf{x}, \bar{\partial}_\mathbf{x}, \bar{S}_\mathbf{x}, \Lambda_\mathbf{x}, \bar{J}_\mathbf{x})$$

and they satisfy (in the standard case $J_m = \{w \subseteq \partial_\ell : w \text{ is bounded}\}$):

(a) $\bar{\partial} = \langle \partial_m : m < \mathbf{k} \rangle$, a sequence of limit ordinals,
(b) $\bar{J} = \langle J_m : m < \mathbf{k} \rangle$,
(c) J_m is an ideal on ∂_m ,
(d) $\bar{S} = \langle S_m : m < \mathbf{k} \rangle, S_m$ a set of ordinals unless stated otherwise,
(e) $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}$.

(2) We adopt the conventions and definitions in 1.3(2)–(5).

Convention 1.6: (1) If $\mathbf{x}$ is clear from the context, we may write $\mathbf{k}$ for $\mathbf{k}(\mathbf{x})$, k for $k(\mathbf{x})$ and $S, \Lambda, \bar{J}$ instead of $\mathbf{k}_\mathbf{x}, k_\mathbf{x}, \bar{S}_\mathbf{x}, \Lambda_\mathbf{x}, \bar{J}_\mathbf{x}$ respectively.

(2) If not said otherwise $\mathbf{x}$ is weakly ordinary; see 1.3(3).

Definition 1.7: Assume $\mathbf{x}$ is a $\bar{\partial}$-c.p.

(1) We say $\mathbf{x}$ has $(\bar{\chi}, \mathbf{k}, 1)$**-Black Box** or $\bar{\chi}$**-pre-black box** when some $\bar{\alpha}$ is a $(\bar{\chi}, \mathbf{k}, 1)$-black box for $\mathbf{x}$ or $(\mathbf{x}, \bar{\chi})$-pre-black box, which means:

(a) $\bar{\chi} = \langle \chi_m : m < \mathbf{k}_\mathbf{x} \rangle$ is a sequence of cardinals,
(b) $\bar{\alpha} = \langle \bar{\alpha}_{\bar{\eta}} : \bar{\eta} \in \Lambda_\mathbf{x} \rangle$,
(c) $\bar{\alpha}_{\bar{\eta}} = \langle \alpha_{\bar{\eta},m,i} : m < \mathbf{k}_\mathbf{x}, i < \partial_m \rangle$ and $\alpha_{\bar{\eta},m,i} < \chi_m$,
(d) if $h_m : \Lambda_{\mathbf{x},m} \to \chi_m$ for $m < \mathbf{k}_\mathbf{x}$ recalling 1.2(5), then for some $\bar{\eta} \in \Lambda_\mathbf{x}$ we have: $m < \mathbf{k}_\mathbf{x} \wedge i < \partial_m \Rightarrow h_m(\bar{\eta} \upharpoonright \langle m, i \rangle) = \alpha_{\bar{\eta},m,i}$.

(2) For $\Lambda \subseteq \Lambda_\mathbf{x}$ we define $\mathbf{x}{\restriction}\Lambda$ naturally as $(\mathbf{k}_\mathbf{x}, \bar{\partial}_\mathbf{x}, \bar{S}_\mathbf{x}, \Lambda, \bar{J})$.

(3) We may write $\bar{\alpha}$ as $\mathbf{b}$, a function with domain

$$\{(\bar{\eta}, m, i) : \bar{\eta} \in \Lambda_\mathbf{x}, m < \mathbf{k}, i < \partial_m\}$$

such that

$$\mathbf{b}_{\bar{\eta}}(m, i) = \mathbf{b}(\bar{\eta}, m, i) = \alpha_{\bar{\eta},m,i}.$$

We may replace $\bar{\chi}$ by χ if $\bar{\chi} = \langle \chi : \ell < \mathbf{k}_\mathbf{x} \rangle$ or by $\bar{C} = \langle C_\ell : \ell < \mathbf{k} \rangle$ when $|C_\ell| = \chi_\ell$, and we demand $\mathrm{Rang}(h_\ell) \subseteq C_\ell$. We may replace $\mathbf{x}$ by $\Lambda = \Lambda_\mathbf{x}$ (so say $\bar{\alpha}$ is a $(\Lambda, \bar{\chi})$-pre-black box).

(4) Omitting the "pre" in part (1) means that there is a partition

$$\bar{\Lambda} = \langle \Lambda_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$$

of $\Lambda_{\mathbf{x}}$ such that each $\mathbf{x} \restriction \Lambda_\alpha$ has a $\bar{\chi}$-pre-black box and some $\langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$ witnesses it, which means that:

- (a) $\{\bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}}|\} = \Lambda_{\mathbf{x}}$,
- (b) letting μ be maximal such that $(\forall \ell < \mathbf{k}) 2^{<\mu} \leq \chi_\ell$ we have
$$\alpha < \beta < \alpha + \mu \Rightarrow \bar{\nu}_\alpha = \bar{\nu}_\beta,$$
- (c) if $\alpha \leq \beta < |\Lambda_{\mathbf{x}}|$, $(\alpha, \beta) \neq (0,0)$ and $\bar{\eta} \in \Lambda_\beta$ then $\nu_{\alpha,\mathbf{k}-1} < \eta_{\mathbf{k}-1} \bmod J_{\mathbf{x},\mathbf{k}-1}$.

(5) We may write BB instead of black box.

(6) We say $\mathbf{x}$ essentially has a $\bar{\chi}$-black box when some $(\bar{\Lambda}, \mathbf{n})$ witnesses it, which means:[5]

- (a) $\bar{\Lambda} = \langle \Lambda_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$ is a sequence of pairwise disjoint subsets of $\Lambda_{\mathbf{x}}$,
- (b) $\mathbf{x} \restriction \Lambda_\alpha$ has a $\bar{\chi}$-pre-black box,
- (c) $\mathbf{n} = \langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$,
- (d) if $\bar{\nu} \in \Lambda_{\mathbf{x}}$ then $\bar{\nu} \in \{\bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}}|\}$,
- (e) if $\mu = \sup\{\mu : 2^\mu < \min\{|S_{\mathbf{x},\ell}| : \ell < \mathbf{k}_{\mathbf{x}}\}$, then $\alpha < \beta < \alpha + \mu \Rightarrow \bar{\nu}_\alpha = \bar{\nu}_\beta$ and $\alpha \leq \beta < \lambda \wedge \bar{\eta} \in \Lambda_{\mathbf{x}_\alpha} \Rightarrow \nu_{\alpha,\mathbf{k}-1} <_{J_{\mathbf{x},\ell}} \eta_{\mathbf{k}-1}$ (we can use a variant of this), but this suffices presently.

We shall use freely

Observation 1.8: *If* (A) *then* (B)*:*

- (A) $\mathbf{x}$ *is a* $\bar{\partial}$-c.p. *and* $(\bar{\Lambda}, \mathbf{n})$ *witness* $\mathbf{x}$ *essentially have a* $\bar{\chi}$*-black box,*
- (B) *there is* $\mathbf{y} = \mathbf{x} \restriction \Lambda$ *for some* $\Lambda \subseteq \Lambda_{\mathbf{x}}$ *which has a* $\bar{\chi}$*-black box.*

Proof. We choose $\Omega_n \subseteq \Lambda_{\mathbf{x}}$ by induction on n by:

$(*)$ (a) if $n = 0$ then $\Omega_0 = \Lambda_0 \cup \{\bar{\nu}_0\}$
 (b) if $n = m+1$ then $\Omega_n = \cup\{\Lambda_\alpha : \alpha < \lambda = |\Lambda_{\mathbf{x}}| \text{ and } \bar{\nu}_\alpha \in \Omega_m\} \cup \Omega_m$.

Now $\mathbf{x} \restriction \bigcup_n \Omega_n$ is as required. $\blacksquare_{1.8}$

Observation 1.9: (1) *In Definition 1.7(4) we may use* $\Lambda_{\mathbf{x}}$ *as the index set of* $\bar{\Lambda}$ *instead of* $|\Lambda_{\mathbf{x}}|$.

(2) *If* $\mathbf{x}$ *is a* $\bar{\partial}$*-c.p.,* $\bar{\chi} = \langle \chi_\ell : \ell < \mathbf{k}_{\mathbf{x}} \rangle$ *and* $|\Lambda_{\mathbf{x}}| = \max\{\chi_\ell : \ell < \mathbf{k}_{\mathbf{x}}\}$*, then* $\mathbf{x}$ *has a* $\bar{\chi}$*-black box iff* $\mathbf{x}$ *has a* $\bar{\chi}$*-pre-black box.*

[5] See the proof of 2.10(2).

Remark 1.10: Concerning the variants below our aim is to have "$\mathbf{x}$ is (θ)-free", but to get it we use the other versions.

Definition 1.11: (1) For $\Lambda_* \subseteq \bar{S}^{[\bar{\partial}]}$, we say "$\mathbf{x}$ is (θ, u)-free over Λ_*" when $\mathbf{x}$ is weakly ordinary,[6] $u \subseteq \{0, \ldots, \mathbf{k}_\mathbf{x} - 1\}$ and for every $\Lambda \subseteq \Lambda_\mathbf{x} \backslash \Lambda_*$ of cardinality $< \theta$ there is a list $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ of Λ such that: for every α for some $m \in u$ and $w \in J_{\mathbf{x},m}$ we have[7]

$$\bar{\nu} \in \{\bar{\eta}_\beta : \beta < \alpha\} \cup \Lambda_* \wedge \bar{\nu} \upharpoonright (m) = \bar{\eta}_\alpha \upharpoonright (m) \wedge j < \partial_{\mathbf{x},m} \wedge i \in \partial_{\mathbf{x},m} \backslash w \Rightarrow \nu_m(j) \neq \eta_{\alpha,m}(i).$$

(2) If $\theta > |\Lambda_\mathbf{x}|$ we may (in part (1)) write (∞, u)-free or u-free; we may omit "over Λ_*" when $\Lambda_* = \emptyset$.

(3) If $u = \{0, \ldots, \mathbf{k} - 1\}$ we may omit it.

(4) Suppose we are given cardinals $\theta_1 \leq \theta_2$, combinatorial $\bar{\partial}$-parameter $\mathbf{x}$, Λ_* (usually $\subseteq \Lambda_\mathbf{x}$) and $u \subseteq \{0, \ldots, \mathbf{k}_\mathbf{x} - 1\}$ and k.

We say $\mathbf{x}$ is $(\theta_2, \theta_1, u, k)$-free over Λ_* when:

(a) $\theta_2 \geq \theta_1 \geq 1$,

(b) $1 \leq k \leq \mathbf{k}_\mathbf{x}$, if $k = 1$ we may omit it,

(c) $u \subseteq \{0, \ldots, \mathbf{k}_\mathbf{x} - 1\}$ has $\geq k$ members

(d) for every $\Lambda \subseteq \Lambda_\mathbf{x} \backslash \Lambda_*$ of cardinality $< \theta_2$ there is a witness $(\bar{\Lambda}, g, \bar{h})$ which means:

(α) $\bar{\Lambda} = \langle \Lambda_\gamma : \gamma < \gamma(*) \rangle$ is a partition of Λ to sets each of cardinality $< \theta_1$, so $\gamma(*)$ is an ordinal $< \theta_2$.

(β) $g : \gamma(*) \to [u]^k$; when $k = 1$ we usually use $g' : \gamma(*) \to u$ where $g(\gamma) = \{g'(\gamma)\}$ for $\gamma < \gamma(*)$ or even use $g'' : \Lambda \to [u]^1$ where $g''(\bar{\eta}) = g'(\gamma)$ when $\bar{\eta} \in \Lambda_\gamma$. Occasionally (when the meaning of $\bar{\eta}_\beta$ is clear) we may write $g(\bar{\eta}_\beta)$ or $g'(\bar{\eta}_\beta)$ instead of $g(\beta)$ and $g'(\beta)$ (so we consider $\Lambda_\mathbf{x}$ as the domain of g, g' instead of $\gamma(*)$).

(γ) $\bar{\eta}, \bar{\nu} \in \Lambda_\gamma \wedge m \in (\mathbf{k}_\mathbf{x} \backslash g(\gamma)) \Rightarrow \eta_m = \nu_m$.

(δ) $\bar{h} = \langle h_m : m \in u \rangle$.

(ε) $h_m : \Lambda \to J_m$; really just

$$h_m {\restriction} \{\bar{\eta} \in \Lambda : \text{ if } \gamma < \gamma(*) \text{ and } \bar{\eta} \in \Lambda_\gamma \text{ then } m \in g(\gamma)\}$$

matters. Here again, we may write $h_m(\beta)$ instead of $h_m(\bar{\eta}_\beta)$

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[6] so if $k_\mathbf{x}$ is 1, then "$\mathbf{x}$ is $(\theta, \{0\})$-free" has a closer meaning to "$\{\eta : \langle \eta \rangle \in \Lambda_\mathbf{x}\}$ is $[\theta, J_{\mathbf{x},0}]$-free" than to $(\theta, J_{\mathbf{x},0})$-free; see Definition 0.8.

[7] If $\Lambda_\mathbf{x}$ is normal, we can restrict ourselves to $i = j$ and this is the usual case.

(ζ) if $\bar{\eta} \in \Lambda_\beta$ and $m \in g(\beta)$ and $\bar{\nu} \in \cup\{\Lambda_\alpha : \alpha < \beta\} \cup \Lambda_*$ and $\bar{\nu} \restriction (m, = \emptyset) = \bar{\eta} \restriction (m, = \emptyset)$, then $i \in \partial_m \backslash h_m(\bar{\eta}) \Rightarrow \eta_m(i) \neq \nu_m(i)$.

(5) In (4), if $\theta_2 > |\Lambda_{\mathbf{x}}|$ we may write (∞, θ_1, u, k)-free; we may omit Λ_* if $\Lambda_* = \emptyset$ and if $k = 1$ we may omit k.

(6) We say $\mathbf{x}$ is (θ, u)-free over Λ_* respecting $\bar{\Lambda}$ (so we may write $\mathbf{k}$ instead of $u = \{\ell : \ell < \mathbf{k}\}$ and θ-free instead of $(\theta, \{\ell : \ell < \mathbf{k}\})$) when $\bar{\Lambda} = \langle \Lambda_{\bar{\nu}} : \bar{\nu} \in \Lambda_{\mathbf{x}} \rangle, \Lambda_{\bar{\nu}} \subseteq \Lambda_{\mathbf{x}}$, and for every $\Lambda \subseteq \Lambda_{\mathbf{x}} \backslash \Lambda_*$ of cardinality $< \theta$ there is a list $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ of Λ such that:

- $\bullet_1$ if $\bar{\eta}_\alpha \in \Lambda_{\bar{\nu}}$ so $\bar{\nu} \in \Lambda_{\mathbf{x}}$, then $\bar{\nu} \in \{\bar{\eta}_\beta : \beta < \alpha\} \cup \Lambda_*$,
- $\bullet_2$ for every $\alpha < \alpha_*$, for some $m \in u$ and $w \in J_{\mathbf{x},m}$, we have

$$\bar{\nu} \in \{\bar{\eta}_\beta : \beta < \alpha\} \cup \Lambda_* \cap \bar{\nu} \restriction (m) = \bar{\eta}_\alpha \restriction m \wedge j \in \partial_{\mathbf{x},m} \backslash w \wedge i < \partial_{\mathbf{x},m} \Rightarrow \nu_m(i) \neq \eta_m(j).$$

(7) For $\mathbf{x}, \theta_1, \theta_2, \Lambda_*, u$ as in Definition 1.11(4) and a sequence $\bar{\Lambda}^* = \langle \Lambda^*_{\bar{\rho}} : \bar{\rho} \in \Lambda_{\mathbf{x}} \rangle$ of subsets of $\Lambda_{\mathbf{x}}$, we say $\mathbf{x}$ is $(\theta_2, \theta_1, u, k)$-free over Λ_* respecting $\bar{\Lambda}^*$, when clauses (a)–(d) of Definition 1.11(4) hold and we add to clause (d)

(η) if $\bar{\eta} \in \Lambda_\alpha$ and $\bar{\eta} \in \Lambda^*_{\bar{\rho}}$ then $\bar{\rho} \in \cup\{\Lambda_\beta : \beta < \alpha\} \cup \Lambda_*$.

Claim 1.12: *Assume $\mathbf{x}$ is a $\bar{\partial}$-c.p. and $u \subseteq \{0, \dots, \mathbf{k}_{\mathbf{x}} - 1\}$ is not empty.*

(1) *$\mathbf{x}$ is $(\theta_2, 2, u, 1)$-free over Λ_* iff $\mathbf{x}$ is (θ_2, u)-free over $\Lambda_*, \theta_2 \geq 2$.*
(2) *If $\partial > \max\{\partial_\ell : \ell < \mathbf{k}_{\mathbf{x}}\}, \mathbf{x}$ is (θ, ∂, u)-free over Λ_* and for each $\ell \in u, \mathbf{x}$ is $(\partial, 2, \{\ell\})$-free, then $\mathbf{x}$ is $(\theta, 2, u)$-free over Λ_* (equivalently (θ, u) free over Λ_*).*

Proof. Should be clear but we elaborate.

(1) It is enough to deal with the case $|\Lambda_{\mathbf{x}} \backslash \Lambda_*| < \theta_2$. First, assume $\theta_2 \geq 2$ and $\mathbf{x}$ is (θ_2, u)-free over Λ_*, let $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ listing $\Lambda_{\mathbf{x}} \backslash \Lambda_*$ be as in Definition 1.11(1). Let $\Lambda_\alpha = \{\bar{\eta}_\alpha\}$ for $\alpha < \alpha_*$ and define $g' : \alpha_* \to u$ by $g'(\alpha) =$ the minimal $m \in u$ such that for some $w \in J_m$ the condition in Definition 1.11(1) holds. By the assumption that $\mathbf{x}$ is (θ_2, u)-free over Λ_*, g' is well defined. Let $g : \alpha_* \to [u]^1$ be $g(\alpha) = \{g'(\alpha)\}$. Also we define $h_m : \alpha_* \to J_m$ for $m \in u$ such that: if $\alpha < \alpha_*$ and $m = g'(\alpha)$ then $h_m(\alpha)$ is any $w \in J_m$ such that the condition in Definition 1.11(1) holds. Now clearly in Definition 1.11(4), clause (a) holds (letting $\theta_1 = 2$ as $\theta_2 \geq 2 = \theta_1$), clause (b) holds as $k = 1 \in [1, \mathbf{k}_{\mathbf{x}}]$ and clause (c) is obvious. We shall check clauses (d)(α)–(ζ) hence finishing proving the "if" implication.

Let $\gamma(*) = \alpha_*$ and $\bar\Lambda = \langle \Lambda_\alpha : \alpha < \alpha_* \rangle$. This definition takes care of (d)(α) and the above definition of g, g' ensures (d)(β). Clause (d)(γ) is immediate since each Λ_α is a singleton. Clauses (d)(δ), (d)(ε) follow from the definition of the h_m's. Finally, clause (d)(ζ) follows from Definition 1.11(1).

Second, assume $\mathbf{x}$ is $(\theta_2, 2, u, 1)$-free and let $(\bar\Lambda, g, \bar h)$ witness this so $\theta_1 = 2$; note that $\theta_2 \geq 2$, since $\theta_1 = 2$ and $\theta_2 \geq \theta_1$ by Definition 1.11(4)(a). So $\bar\Lambda = \langle \Lambda_\alpha : \alpha < \alpha_* \rangle$ and $\bar h = \langle h_m : m \in u \rangle$ and $g : \alpha_* \to [u]^1$, so for some function $g' : \alpha_* \to u$ we have $\alpha < \alpha_* \Rightarrow g(\alpha) = \{g'(\alpha)\}$. As $|\Lambda_\alpha| < \theta_2 = 2$ we have $|\Lambda_\alpha| \leq 1$; without loss of generality $\bigwedge_\alpha \Lambda_\alpha \neq \emptyset$, hence there is a unique $\bar\eta_\alpha \in \Lambda_{\mathbf{x}} \backslash \Lambda_*$ such that $\Lambda_\alpha = \{\bar\eta_\alpha\}$. So $\langle \bar\eta_\alpha : \alpha < \alpha_* \rangle$ lists $\Lambda_{\mathbf{x}} \backslash \Lambda_*$, and it suffices to check that for every $\alpha < \alpha_*$ the condition in Definition 1.11(1) holds. We choose $m = g'(\alpha)$ so $m \in u$ and we choose $w = h_m(\alpha)$ so $w \in J_m$ indeed, and the condition there holds for m, w by clause (d)(ζ) of Definition 1.11(4) as $\Lambda_\alpha = \{\bar\eta_\alpha\}, \beta < \alpha \Rightarrow \Lambda_\beta = \{\bar\eta_\beta\}$.

(2) As $\mathbf{x}$ is (θ, ∂, u)-free over Λ_* there is a triple $(\bar\Lambda^*, g^*, \bar h^*)$ witnessing it, as in Definition 1.11(4), and let $\bar\Lambda^* = \langle \Lambda^*_\alpha : \alpha < \alpha_* \rangle$ and $\bar h^* = \langle h^*_m : m \in u \rangle$. For each $\ell \in u$ and $\alpha < \alpha_*$ we know that $\mathbf{x}$ is $(\partial, 2, \{\ell\})$-free and Λ^*_α is a subset of $\Lambda_{\mathbf{x}} \backslash \Lambda_*$ of cardinality $< \partial$, hence there is a triple $(\bar\Lambda_\alpha, g_\alpha, \bar h_\alpha)$ witnessing it. Let $\bar\Lambda_\alpha = \langle \Lambda_{\alpha,\beta} : \beta < \beta_\alpha \rangle$ and so $|\Lambda_{\alpha,\beta}| < 2$ and without loss of generality $\Lambda_{\alpha,\beta} \neq \emptyset$, so let $\Lambda_{\alpha,\beta} = \{\bar\eta_{\alpha,\beta}\}$ and (as $k = 1$, see end of 1.11(5)) $g_\alpha(\beta) = \{g'_\alpha(\beta)\}$, where $g'_\alpha : \beta_\alpha \to u$ and let $\bar h_\alpha = \langle h_{\alpha,m} : m \in u \rangle$.

Let $\gamma_\alpha = \Sigma\{\beta_{\alpha_1} : \alpha_1 < \alpha\}$ for $\alpha < \alpha_*$, so clearly $\langle \gamma_\alpha : \alpha \leq \alpha_* \rangle$ is increasing continuous and $\gamma_0 = 0$ and let $\gamma_* = \gamma_{\alpha_*}$; we define $\bar\eta_\gamma$ for $\gamma < \gamma_*$ by: if $\gamma = \gamma_\alpha + \beta$, $\beta < \beta_\alpha$, then we let $\bar\eta_\gamma = \bar\eta_{\alpha,\beta}$. Also let $g' : \gamma_* \to [u]^1$ be defined by $g' {\restriction} [\gamma_\alpha, \gamma_{\alpha+1})$ is constantly $\{g^*(\alpha)\}$, let $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ where $\Lambda_\gamma = \{\bar\eta_\gamma\}$, and let $\bar h = \langle h_m : m \in u \rangle, h_m : \gamma_* \to J_m$ be $h_m(\gamma_\alpha + \beta) = h_{\alpha,m}(\beta)$ if $\alpha < \alpha_*, \beta < \beta_\alpha$. So it is enough to check that $(\bar\Lambda, g', \bar h)$ witnesses $\Lambda_{\mathbf{x}}$ is $(\theta, 2, u)$-free over Λ_*, e.g., why clause (ζ) of Definition 1.11(d) holds.

Let $\bar\eta \in \Lambda_\gamma, m \in g'(\gamma)$ and $\bar\nu \in \cup\{\Lambda_\alpha : \alpha < \gamma\} \cup \Lambda_*$. So $\bar\eta = \bar\eta_\gamma$ and one of the following cases occurs, letting $\gamma = \gamma_\alpha + \beta, \beta < \beta_\alpha$.

Case 1: $\bar\nu \in \cup\{\Lambda^*_{\alpha'} : \alpha' < \alpha\} \cup \Lambda_*$.

Use "$(\bar\Lambda^*, g^*, \bar h^*)$ witness $\Lambda_{\mathbf{x}}$ is (θ, ∂, u)-free over Λ_*".

Case 2: $\bar\nu \in \Lambda^*_\alpha$.

Use "$(\bar\Lambda_\alpha, g_\alpha, \bar h_\alpha)$ witness Λ_α is $(\partial, 2, \{\ell\})$-free" for $\ell = g^*(\alpha)$. $\blacksquare_{1.12}$

Definition 1.13: We say $(\mathbf{x}, \bar{\Lambda})$ witness $\mathrm{BB}^3_{\mathbf{k}}(\lambda, \Theta, \bar{\chi}, \bar{\partial})$ when:

(a) $\mathbf{x}$ is a $\bar{\partial}$-c.p. with $|\Lambda_{\mathbf{x}}| = \lambda$ and $\mathbf{k} = \mathbf{k}_{\mathbf{x}}$, i.e., $= \ell g(\bar{\partial})$,

(b) $\bar{\Lambda} = \langle \Lambda_{\bar{\nu}} : \bar{\nu} \in \Lambda_{\mathbf{x}} \rangle$ is a sequence of pairwise disjoint subsets of $\Lambda_{\mathbf{x}}$,[8]

(c) $\mathbf{x} \restriction \Lambda_{\bar{\nu}}$ has $\bar{\chi}$-pre-black box for every $\bar{\nu} \in \Lambda_{\mathbf{x}}$

(d) Θ is a collection of cardinals and pairs of cardinals,

(e) if $\theta \in \Theta$, then $\mathbf{x}$ is $(\theta, \mathbf{k})$-free respecting $\bar{\Lambda}$, see 1.11(6), which means that in the list $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ in Definition 1.11(1), we have

$$(\alpha > 0) \wedge \bar{\eta}_\alpha \in \Lambda_{\bar{\nu}} \Rightarrow \bar{\nu} \in \{\bar{\eta}_\beta : \beta < \alpha\},$$

(f) if $(\theta_2, \theta_1) \in \Theta$ then $\mathbf{x}$ is $(\theta_2, \theta_1, \mathbf{k}, 1)$-free respecting $\bar{\Lambda}$, see 1.11(7).

Remark 1.14: Note that in Definition 1.13 necessarily we have

$$\Sigma\{\chi_\ell : \ell < \mathbf{k}\} \leq |\Lambda_{\mathbf{x}}|.$$

Clearly

Claim 1.15: *Assume μ is strong limit $> \mathrm{cf}(\mu) = \partial, \mathscr{F} \subseteq {}^{\partial}\mu$ has cardinality $\lambda = 2^\mu$ and $\mathscr{F}$ is θ-free (i.e., $(\theta, J^{\mathrm{bd}}_{\partial})$-free); moreover, $[\theta, J^{\mathrm{bd}}_{\partial}]$-free and weakly ordinary, see 0.7(1),(2),(6).*

Then there is a $\langle \partial \rangle$-c.p. $\mathbf{x}$ with $\Lambda_{\mathbf{x}} = \mathscr{F}$ which is θ-free and has the λ-BB (i.e., $(\langle \lambda \rangle, 1, 1)$-BB).

Proof. The point is that the set of functions from ${}^{\partial>}\mu$ to λ has cardinality $\lambda = |\mathscr{F}|$, see more in [She13b, 2.2=Ld.6]. $\blacksquare_{1.15}$

Claim 1.16: (1) *Assume $\mathbf{x}$ is a $\mathbf{k}$-c.p.,*

$$\theta_2 \geq \theta_1 = \mathrm{cf}(\theta_1) > \max\{\partial_{\mathbf{x},\ell} : \ell < \mathbf{k}_{\mathbf{x}}\}$$

and

$$u \subseteq \{0, \ldots, \mathbf{k}_{\mathbf{x}} - 1\}, |u| = k \geq 1.$$

The following conditions (A),(B),(C) on $\mathbf{x}, \theta_2, \theta_1, u, k$ are equivalent:

(A) *$\mathbf{x}$ is $(\theta_2, \theta_1, u, k)$-free over Λ_*,*

(B) *as in Definition 1.11(4) omitting clause (d)(γ), in this case we call $(\bar{\Lambda}, g, \bar{h})$ an almost witness,*

(C) *for every $\Lambda \subseteq \Lambda_{\mathbf{x}} \backslash \Lambda_*$ of cardinality $< \theta_2$ there is a weak witness $(g, \bar{h})$ which means: clauses (δ), (ε) of 1.11(4)(d) and*

[8] In [She07] we use $\Lambda_{\mathbf{x},<k}$ as index set which, if $k = 1$, may have smaller cardinality; so far not a significant difference.

$(\beta)'$ $g : \Lambda \to [u]^k$,

$(\zeta)'$ *if* $\bar\eta_1 \in \Lambda$ *and* $m \in u$ *then for all but* $< \theta_1$ *of the sequences* $\bar\eta_2 \in \Lambda$ *we have*

- *if* $\bar\eta_1 \neq \bar\eta_2$, $\bar\eta_1 \upharpoonright (m, = \emptyset) = \bar\eta_2 \upharpoonright (m, = \emptyset)$ *and* $m \in g(\bar\eta_1) \cap g(\bar\eta_2)$ *and* $i \in \partial_m \backslash (h_m(\bar\eta_1) \cup h_m(\bar\eta_2))$, *then*

$$\eta_{1,m}(i) \neq \eta_{2,m}(i),$$

$(\eta)'$ *if* $\bar\eta_1 \in \Lambda$ *and* $\bar\eta_2 \in \Lambda_*$, *then* $\bullet$ *of* $(\zeta)'$ *holds demanding only* $m \in g(\bar\eta_1)$.

(2) *If in addition* $\mathbf{x}$ *is normal (see 1.2(7)) we can add:*

(D) *like* (C) *but we replace* $\bullet$ *inside* $(\zeta)'$ *(and similarly in* $(\eta)'$*) by*

- *if* $\bar\eta_1 \neq \bar\eta_2 \in \Lambda$, $\bar\eta_1 \upharpoonright (m, = \emptyset) = \bar\eta_2 {\restriction} (m, \emptyset)$ *and* $m \in g(\bar\eta_1) \cap g(\bar\eta_2)$ *and* $i, j \in \partial_m \backslash (h_m(\bar\eta_1) \cup h_m(\bar\eta_2))$, *then*

$$\eta_{1,m}(i) \neq \eta_{2,m}(j).$$

(3) *If in addition* $\Lambda_* \subseteq \Lambda_{\mathbf{x}}$ *and each* $J_{\mathbf{x},\ell}$ *is* σ*-complete, then*

$$\{\Lambda : \Lambda \subseteq \Lambda_{\mathbf{x}} \backslash \Lambda_* \text{ is } (\theta_2, \theta_1, u, k)\text{-free over } \Lambda_*\}$$

is a σ*-complete ideal on* $\Lambda_{\mathbf{x}} \backslash \Lambda_*$.

Proof. (1) (A)$\Rightarrow$(B):

Obvious by the formulation of (B).

(B)$\Rightarrow$(C):

Let $\Lambda \subseteq \Lambda_{\mathbf{x}} \backslash \Lambda_*$ have cardinality $< \theta_2$; by clause (B) we can choose $(\bar\Lambda, g, \bar h)$, an almost witness (for Λ). As $|u| = k$, necessarily g is constantly u, so let $g' : \gamma(*) \to [u]^k$ be constantly u, hence it is enough to prove that $(g', \bar h)$ is a weak witness; clearly clause $(\beta)'$ of (C) holds. So by the phrasing of (B) and (C) it is enough to prove clauses $(\zeta)', (\eta)'$ of (C). But clause $(\eta)'$ follows from clause (ζ) of (B), i.e., (d)(ζ) of Definition 1.11(4). Now for clause $(\zeta)'$, let $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma(*) \rangle$ and assume $\bar\eta_\iota \in \Lambda_{\beta_\iota}$ for $\iota = 1, 2$ and $\beta_1 \neq \beta_2 < \gamma(*)$ and $m \in u$ and it suffices to prove $\bullet$ of $(\zeta)'$. Clearly $m \in g'(\beta_1) \cap g'(\beta_2)$. So assuming

$$\bar\eta_1 \upharpoonright (m, = \emptyset) = \bar\eta_2 \upharpoonright (m, = \emptyset)$$

and $i \in \partial_m \backslash (h_m(\bar\eta_1) \cup h_m(\bar\eta_2))$ we should prove that $\eta_{1,m}(i) \neq \eta_{2,n}(i)$. By the symmetry without loss of generality $\beta_1 < \beta_2$ and we apply clause (ζ) of (B) with $\bar\eta_1, \bar\eta_2, \beta_1, \beta_2, m$ here standing for $\bar\nu, \bar\eta, \beta, m$ there and get $\eta_{1,m}(i) \neq \eta_{2,m}(i)$ as promised.

(C)⇒(A):

So assume that $\Lambda \subseteq \Lambda_{\mathbf{x}} \backslash \Lambda_*$ has cardinality $< \theta_2$ and let $(g, \bar{h})$ be a weak witness for it (actually we have no further use of $|\Lambda| < \theta_2$); again necessarily g is constantly u. So for $m \in u, i < \partial_m$ and every $\bar{\eta} \in \Lambda$ let

$$\Omega^1_{i,m,\bar{\eta}} = \{\bar{\nu} \in \Lambda : \bar{\nu} \upharpoonright (m, = \emptyset) = \bar{\eta} \upharpoonright (m, = \emptyset) \\ \text{and } i \in \partial_m \backslash h_m(\bar{\nu}) \text{ and } \bar{\nu}_m(i) = \bar{\eta}_m(i)\}.$$

By the choice of $(g, \bar{h})$ and the definition of $\Omega^1_{i,m,\bar{\eta}}$ we have:

- $\bullet_1$ if $\bar{\nu}, \bar{\rho} \in \Omega^1_{i,m,\bar{\eta}}$, then $\bar{\nu} \upharpoonright (m, = \emptyset) = \bar{\rho} \upharpoonright (m, = \emptyset)$ and $\bar{\eta}_m(i) = \bar{\nu}_m(i)$ and $i \in \partial_m \backslash (h_m(\bar{\nu}) \cup h_m(\bar{\rho}))$,

hence applying clause $(\zeta)'$ of (C) to any $\bar{\eta}_1 \in \Omega_{i,m,\bar{\eta}}$ we have

- $\bullet_2$ $\Omega^1_{i,m,\bar{\eta}}$ has $< \theta_1$ members.

Let

$$\Omega^1_{\bar{\eta}} = \cup\{\Omega^1_{i,m,\bar{\eta}} : m \in u, i < \partial_m\} \cup \{\bar{\eta}\},$$

so recalling the claim assumption $\theta_1 = \mathrm{cf}(\theta_1) > \sum_m \partial_m$ clearly

- $\bullet_3$ if $\bar{\eta} \in \Lambda$ then $\Omega^1_{\bar{\eta}}$ has cardinality $< \theta_1$.

By transitivity of equality

- $\bullet_4$ if $\bar{\nu} \in \Omega^1_{\bar{\eta}}$ then $m < \mathbf{k} \wedge m \notin u \Rightarrow \bar{\nu}_m = \bar{\eta}_m$.

For $\bar{\eta} \in \Lambda$ let $\Omega^2_{\bar{\eta}}$ be the minimal subset Ω of Λ such that $\bar{\eta} \in \Omega$ and

$$\bar{\nu} \in \Omega \Rightarrow \Omega^1_{\bar{\nu}} \subseteq \Omega,$$

so recalling θ_1 is regular necessarily $|\Omega^2_{\bar{\eta}}| < \theta_1$.

Let $\langle \bar{\eta}^*_\gamma : \gamma < \gamma(*) \rangle$ list Λ. We now choose Λ^1_γ for $\gamma < \gamma(*)$ by

$$\Lambda^1_\gamma = \cup\{\Omega^2_{\bar{\eta}^*_\beta} : \beta \leq \gamma\}$$

so $\{\bar{\eta}^*_\gamma\} \subseteq \Lambda^1_\gamma \subseteq \Lambda$, so clearly $\cup\{\Lambda^1_\gamma : \gamma < \gamma(*)\} = \Lambda$.

Lastly, let $\Lambda^2_\gamma = \Lambda^1_\gamma \backslash \cup \{\Lambda^1_\beta : \beta < \gamma\}$, so obviously $\bar{\Lambda}^2 = \langle \Lambda^2_\gamma : \gamma < \gamma(*) \rangle$ is a partition of Λ. Let $g_* : \gamma(*) \to [u]^k$ be constantly u and $\bar{h} = \langle h_m : m \in u \rangle$, and we shall show that the triple $(\bar{\Lambda}^2, g_*, \bar{h})$ is as required in 1.11(4)(d).

Now clauses (α)–(ε) hold by our choices noting that by $\bullet_4$ we have: if $\bar{\eta}, \bar{\nu} \in \Lambda^2_\gamma$ and $m < \mathbf{k}, m \notin u$ then $\bar{\eta}_m = \bar{\nu}_m$. As for clause (ζ) let $\bar{\eta} \in \Lambda_\beta, m \in g(\beta), \alpha < \beta$ and $\bar{\nu} \in \Lambda^2_\alpha, \bar{\nu} \upharpoonright (m, = \emptyset) = \bar{\eta} \upharpoonright (m, = \emptyset)$ and $i \in \partial_m \backslash h_m(\bar{\eta})$ and we should prove that $\bar{\nu}_m(i) \neq \bar{\eta}_m(i)$. But otherwise $\bar{\eta} \in \Omega^1_{\bar{\nu}} \subseteq \Omega^2_{\bar{\eta}^*_\alpha} \subseteq \cup\{\Lambda^2_{\alpha_1} : \alpha_1 \leq \alpha\}$, contradiction.

(2) Similarly.

(3) By part (1), as we can use as definition clause (C) of (1), assume that $\Lambda = \bigcup_{i<i(*)} \Lambda_i \subseteq \Lambda_{\mathbf{x}} \backslash \Lambda_*$ and $i(*) < \sigma$ and $h_{i,m} : \Lambda \to J_m$ and $(g_i, \bar{h}_i)$ weakly witnesses Λ_i. As $|u| = k$ necessarily $g_0 := \bigcup_i g_i$ is the constant function from Λ into $\{u\}$ and let $h_m : \Lambda \to \mathscr{P}(\partial_m)$ be

$$h_m(\bar{\eta}) = \cup\{h_{i,m}(\bar{\eta}) : i < i(*) \text{ and } \bar{\eta} \in \Lambda_i\}.$$

Now h_m is into J_m as J_m is a σ-complete ideal and $i(*) < \sigma$. Lastly, clearly $(g_0, \langle h_m : m \in u\rangle)$ is a weak witness for Λ so we are done. $\blacksquare_{1.16}$

Remark 1.17: Why the demand $|u| = k$ in the claim?

Our problem is: in (A) we promise that the function g gives (for a fixed one γ) for all $\bar{\eta} \in \Lambda_\gamma$ the same u whereas in clause (C) this is not the case, in fact, not well defined. It is natural then to divide Λ_γ to $\leq 2^{\mathbf{k}}$ cases according to the value of g, but then it is not clear that clause (ζ) of (A) holds. To avoid this we assume $|u| = k$. Maybe 1.16(3) helps but this is not crucial.

Definition 1.18: If $\ell g(\bar{\partial}_\iota) = \mathbf{k}_\iota$ and $\mathbf{x}_\iota$ is a combinatorial $\bar{\partial}_\iota$-parameter for $\iota = 1, 2, 3$ then we say $\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3$ when:

(a) $\bar{\partial}_3 = \bar{\partial}_1{}^\frown\bar{\partial}_2$ hence $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$,
(b) $\bar{J}_{\mathbf{x}_3} = \bar{J}_{\mathbf{x}_1}{}^\frown\bar{J}_{\mathbf{x}_2}$,
(c) $\bar{S}_{\mathbf{x}_3}$ is $\bar{S}_{\mathbf{x}_1}{}^\frown\bar{S}_{\mathbf{x}_2}$, that is
  - $S_{\mathbf{x}_1,\ell}$ if $\ell < \mathbf{k}_1$,
  - $S_{\mathbf{x}_2,\ell-\mathbf{k}_1}$ if $\ell \geq \mathbf{k}_1$,

(d) $\Lambda_{\mathbf{x}_3}$ is the set of $\bar{\eta} \in \prod_{\ell<\mathbf{k}_3} {}^{\partial_3(\ell)}(S_{\mathbf{x}_3,\ell})$ such that for some $\bar{\nu} \in \Lambda_{\mathbf{x}_1}$ and $\bar{\rho} \in \Lambda_{\mathbf{x}_2}$ we have:
  - if $\ell < \mathbf{k}_1$, then $\eta_\ell = \nu_\ell$,
  - if $\ell \geq \mathbf{k}_1$, then $\eta_\ell = \rho_{\ell-\mathbf{k}_1}$.

Explanation 1.19: What is the role of the next claim? We shall prove for $(\partial, J) = (\aleph_0, J^{\mathrm{bd}}_\omega)$ and $(\aleph_1, J^{\mathrm{bd}}_{\aleph_1} \times J^{\mathrm{bd}}_{\aleph_0})$, that for many strong limit singular μ, there is a $1-$c.p. $\mathbf{x}$ such that $(\partial_{\mathbf{x},0}, J_{\mathbf{x},0}) = (\partial, J)$ and $\mathbf{x}$ has 2^μ-BB and $\mathbf{x}$ is quite free. But we do not know how to get one which is even just $\aleph_{\omega+1}$-free. But such freeness is needed in §2! However, using long enough finite products we can get enough freeness. More fully, first by 1.20, the product gives a combinational parameter of the expected length (the sum) and weak ordinariness, ordinariness and normality are preserved.

Second, by 1.21 the products have the appropriate (pre-)black box if each product has one.

Third, in 1.21–1.24 we get that if each $\mathbf{x}_\ell$ satisfies enough cases of (θ_2, θ_1, u)-freeness conditions then their product satisfies more.

Fourth, in Theorem 1.25 we prove the existence of $\mathbf{x}_\ell\,(\ell < \mathbf{k})$ as required relying on [She13a].

Lastly, in Conclusion 1.27 we get the desired conclusion used in §2.

CLAIM 1.20: (1) *If* $\mathbf{x}_\iota$ *is a combinatorial* $\bar{\partial}_\iota$*-parameter for* $\iota = 1, 2$, *then there is one and only one combinatorial parameter* $\mathbf{x}_3$ *such that* $\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3$.

(2) *The product in Definition 1.18 is associative.*

(3) *If* $\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3$, *then* $\mathbf{x}_2 \times \mathbf{x}_1$ *is a permutation of* $\mathbf{x}_3$*; see Definition 1.3(4).*

(4) *If in Definition 1.18,* $\mathbf{x}_1, \mathbf{x}_2$ *are (weakly) ordinary and/or normal, see 1.3(3), 1.2(7), then so is* $\mathbf{x}_3$.

Proof. Straightforward. $\blacksquare_{1.20}$

CLAIM 1.21: (1) $\mathbf{x}_3$ *has* $\bar{\chi}_3$*-pre-black box when:*

(a) $\mathbf{x}_\iota$ *is a combinatorial* $\bar{\partial}_\iota$*-parameter for* $\iota = 1, 2, 3$,

(b) $\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3$,

(c) $\mathbf{x}_\iota$ *has* $\bar{\chi}_\iota$*-pre-black box for* $\iota = 1, 2$,

(d) $\bar{\chi}_3 = \bar{\chi}_1{}^\frown\bar{\chi}_2$,

(e) *if* $\ell < \ell g(\bar{\partial}_2)$ *then* $\chi_{2,\ell} = (\chi_{2,\ell})^{|\Lambda_{\mathbf{x}_1}|}$.

(2) *Moreover,* $\mathbf{x}_3$ *has a* $\bar{\chi}_3$*-black box when in addition*

(c)$^+$ $\mathbf{x}_2$ *has a* $\bar{\chi}_2$*-black box and* $\chi_{2,n} = (\chi_{2,n})^{|\Lambda_{\mathbf{x}_1}|}$.

Proof. (1) For each $m < \mathbf{k}_{\mathbf{x}_2}$ let $\bar{F}^m = \langle F^m_\alpha : \alpha < \chi_{2,m}\rangle$ list

$$\{F : F \text{ a function from } \Lambda_{\mathbf{x}_1} \text{ into } \chi_{2,m}\}.$$

By clause (e) of the assumption, such sequence exists. Let $\bar{\alpha}^1$ be a $\bar{\chi}_1$-pre-black box for $\mathbf{x}_1$ and let $\bar{\alpha}^2$ be a $\bar{\chi}_2$-pre-black box for $\mathbf{x}_2$; they exist by clause (c) of the assumption.

Lastly, we define $\bar{\alpha} = \langle \bar{\alpha}_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}_3}\rangle$ where

$$\bar{\alpha}_{\bar{\eta}} = \langle \alpha_{\bar{\eta},m,i} : m < \mathbf{k}_{\mathbf{x}_3}, i < \partial_m\rangle$$

as follows: for $\bar{\eta} \in \Lambda_{\mathbf{x}_3}, m < \mathbf{k}_{\mathbf{x}_3}$ and $i < \partial_{\mathbf{x}_3,m}$ we let:

- if $m < \mathbf{k}_{\mathbf{x}_1}$ then $\alpha_{\bar{\eta},m,i} = \alpha^1_{\bar{\eta}\restriction\mathbf{k}(\mathbf{x}_1),m,i}$,
- if $m = \mathbf{k}_{\mathbf{x}_1} + \ell$ and $\ell < \mathbf{k}_{\mathbf{x}_2}$ then $\alpha_{\bar{\eta},m,i} = F^m_{\alpha^2_{\bar{\nu},\ell,i}}(\bar{\eta}\restriction\mathbf{k}_{\mathbf{x}_1})$, where $\bar{\nu} = \bar{\eta}\restriction[\mathbf{k}_{\mathbf{x}_1}, \mathbf{k}_{\mathbf{x}_3})$, i.e., $\bar{\nu} = \langle \eta_{\mathbf{k}(\mathbf{x}_1)+n} : n < \mathbf{k}_{\mathbf{x}_2}\rangle$.

Clearly $\bar{\alpha}$ is of the right form, but is it really a $\bar{\chi}_3$-pre-black box? So assume $h_m : \Lambda_{\mathbf{x}_3,m} \to \chi_{3,m}$ for $m < \mathbf{k}_{\mathbf{x}_3}$ and we should find $\bar{\eta} \in \Lambda_{\mathbf{x}_3}$ as in Definition 1.7(1). Now first we define $h^2_m : \Lambda_{\mathbf{x}_2,m} \to \chi_{2,m}$ for $m < \mathbf{k}_{\mathbf{x}_2}$ as follows: $h^2_m(\bar{\nu})$ is the unique $\alpha < \chi_{2,m}$ such that:

$$\bar{\rho} \in \Lambda_{\mathbf{x}_1} \Rightarrow h_{\mathbf{k}(\mathbf{x}_1)+m}(\bar{\rho}\hat{\ }\bar{\nu}) = F^m_\alpha(\bar{\rho}),$$

possible by the choice of $\bar{F}^m$ above. As $\bar{\alpha}^2$ is a $\bar{\chi}_2$-pre-black box, clearly there is $\bar{\nu} \in \Lambda_{\mathbf{x}_2}$ such that

$$m < \mathbf{k}_{\mathbf{x}_2} \wedge i < \partial_{\mathbf{x}_2,m} \Rightarrow h^2_m(\bar{\nu} \upharpoonright (m,i)) = \alpha^2_{\bar{\nu},m,i}.$$

Fix a sequence $\bar{\nu} \in \Lambda_{\mathbf{x}_2}$ as in the former paragraph. Now for $m < \mathbf{k}_{\mathbf{x}_1}$ we define $h^1_m : \Lambda_{\mathbf{x}_1,m} \to \chi_{1,m}$ by $h^1_m(\bar{\rho}) = h_m(\bar{\rho}\hat{\ }\bar{\nu})$ for $\bar{\rho} \in \Lambda_{\mathbf{x}_1,m}$, it is well defined by our assumptions on h_m, it has domain $\Lambda_{\mathbf{x}_1,m}$ and, as $\bar{\nu} \in \Lambda_{\mathbf{x}_2}$, clearly $\bar{\rho}\hat{\ }\bar{\nu} \in \Lambda_{\mathbf{x}_3,m}$ by the definition of $\mathbf{x}_3$. As $\bar{\alpha}^1$ is a $\bar{\chi}_1$-pre-black box for $\mathbf{x}_1$ there is $\bar{\rho} \in \Lambda_{\mathbf{x}_1}$ such that $m < \mathbf{k}_{\mathbf{x}_1} \wedge i < \partial_{\mathbf{x}_1,m} \Rightarrow h^1_m(\bar{\rho}) = \alpha^1_{\bar{\rho},m,i}$. We shall show that

$$\bar{\eta} := \bar{\rho}\hat{\ }\bar{\nu}$$

is as required.

First, $\bar{\eta} \in \Lambda_{\mathbf{x}_3}$ because $\mathbf{x}_3 = \mathbf{x}_1 \times \mathbf{x}_2, \bar{\rho} \in \Lambda_{\mathbf{x}_1}$ and $\bar{\nu} \in \Lambda_{\mathbf{x}_2}$.

Second, if $m < \mathbf{k}_{\mathbf{x}_1} \wedge i < \partial_{\mathbf{x}_3,m} = \partial_{\mathbf{x}_1,m}$ then

(∗) (a) $h_m(\bar{\eta} \upharpoonright (m,i)) = h^1_m(\bar{\rho} \upharpoonright (m,i))$ by the choices of $\bar{\eta}$ and h^1_m,
(b) $h^1_m(\bar{\rho} \upharpoonright (m,i)) = \alpha^1_{\bar{\rho},m,i}$ by the choice of $\bar{\rho}$,
(c) $\alpha^1_{\bar{\rho},m,i} = \alpha_{\bar{\eta},m,i}$ by the choice of $\alpha_{\bar{\eta},m,i}$, so together
(d) $h_m(\bar{\eta} \upharpoonright (m,i)) = \alpha_{\bar{\eta},m,i}$.

Third, if $m \in [\mathbf{k}_{\mathbf{x}_1}, \mathbf{k}_{\mathbf{x}_3}) \wedge i < \partial_{\mathbf{x}_3,n}$, then $m = \mathbf{k}_{\mathbf{x}_1} + \ell$, $\ell < \mathbf{k}_{\mathbf{x}_2}$ for some ℓ and use the choices of $\alpha_{\bar{\eta},m,i}$ and of $\bar{\nu}$.

(2) We have to deal with the black box case. So recalling Definition 1.7(4) we are assuming:

(∗) (a) $\bar{\Lambda}^2 = \langle \Lambda^2_\gamma : \gamma < |\Lambda_{\mathbf{x}_2}|\rangle$ is a partition of $\Lambda_{\mathbf{x}_2}$,
(b) if $\gamma < |\Lambda_{\mathbf{x}_2}|$, then $\mathbf{x}_2{\restriction}\Lambda^2_\gamma$ has a $\bar{\chi}_2$-pre-black box.

Now repeating the proof above, note:

(c) $\langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}_2}|\rangle$ list $\Lambda_{\mathbf{x}}$ as required in Definition 1.7(4).

We can choose $\bar{\alpha}^2$ such that not only is it a $\bar{\chi}_2$-pre-black box, but also

$$\bar{\alpha}^2{\restriction}\Lambda^2_\gamma = \langle \bar{\alpha}^2_{\bar{\nu}} : \bar{\nu} \in \Lambda^2_\gamma\rangle$$

is a $\bar{\chi}_2$-pre-black box for each $\gamma < |\Lambda_{\mathbf{x}_2}|$.

Having defined $\bar{\alpha} = \langle \bar{\alpha}_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}} \rangle$ note that:

(*) $|\Lambda_{\mathbf{x}_1}| \leq \chi_{\mathbf{x}_2,0}$ (by clause (e) of the claim) and $\chi_{\mathbf{x}_2,0} \leq |\Lambda_{\mathbf{x}_2}|$ (by 1.14) and $|\Lambda_{\mathbf{x}_2}|$ is infinite (otherwise the $\bar{\chi}_2$-black box fails), hence

$$|\Lambda_{\mathbf{x}_3}| = |\Lambda_{\mathbf{x}_2}| \times |\Lambda_{\mathbf{x}_2}| = |\Lambda_{\mathbf{x}_2}|,$$

(*) letting $\Lambda_\gamma = \Lambda_{\mathbf{x}_1} \times \Lambda^2_\gamma$ the sequence $\langle \Lambda_\gamma : \gamma < |\Lambda_{\mathbf{x}_3}| \rangle$ is a partition of $\Lambda_{\mathbf{x}_3}$.

Mainly we need to prove that: if $\gamma < |\Lambda_{\mathbf{x}_3}|$ then $\bar{\alpha} {\restriction} \Lambda_\gamma$ is a $\bar{\chi}$-pre-black box. This proof is exactly as in the proof of the first part.

Lastly, we choose $\langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}_3}| \rangle$ as required. Toward this let

$$\mu_\iota = \max\{\mu : (\forall \ell < \mathbf{k}_\iota)(2^{<\mu} \leq \chi_{\iota,\ell})\};$$

note that necessarily $|\Lambda_{\mathbf{x}_2}| = |\Lambda_{\mathbf{x}_3}|$ and $\mu_1 \leq |\Lambda_{\mathbf{x}_1}| < \mu_2$. Now choose $\langle \bar{\nu}^1_\alpha : \alpha < |\Lambda_{\mathbf{x}_1}| \rangle$ such that

$$\{\bar{\nu}^1_\alpha : \alpha < |\Lambda_{\mathbf{x}_1}|\} = \Lambda_{\mathbf{x}_1}$$

and

$$\alpha \leq \beta < \alpha + \mu_1 \Rightarrow \bar{\nu}^1_\alpha = \bar{\nu}^1_\beta.$$

To finish, define $\langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}_3}| \rangle$ by:

- if $\gamma = |\Lambda_{\mathbf{x}_1}| \cdot \alpha + \beta$ and $\beta < |\Lambda_{\mathbf{x}_1}|$, then $\bar{\nu}_\gamma = \bar{\nu}^1_\beta {}^\frown \bar{\nu}^2_\gamma$.

Recalling $\gamma_1 < \gamma_2 < \gamma_1 + \mu_2 \Rightarrow \bar{\nu}^2_{\gamma_1} = \bar{\nu}^2_{\gamma_2}$ we are easily done. $\blacksquare_{1.21}$

The following Definition is somewhat similar to [She07], with different notation than earlier.

Definition 1.22: Let $\mathbf{x} = (\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda, \bar{J})$ be disjoint for notational transparency (see 1.3(3)).

(0) For $u \subseteq \{0, \ldots, \mathbf{k} - 1\}$ let $u^\perp = \{\ell < \mathbf{k} : \ell \notin u\}$.

(1) For $\mathscr{U} \subseteq \bigcup_{\ell < \mathbf{k}} {}^{\partial(\ell)}(S_{\mathbf{x},\ell})$ let

$$\Lambda_{\mathscr{U}} = \Lambda_{\mathbf{x},\mathscr{U}} = \Lambda_{\mathbf{x}}(\mathscr{U}) = \{\bar{\eta} \in \Lambda_{\mathbf{x}} : \eta_\ell \in \mathscr{U} \text{ for every } \ell < \mathbf{k}\}.$$

(2) For $\mathscr{U} \subseteq \bigcup_{\ell < \mathbf{k}} {}^{\partial(\ell)}(S_{\mathbf{x},\ell})$ and $u \subseteq \{0, \ldots, \mathbf{k} - 1\}$ let:

(a) $\mathrm{add}_{\mathbf{x}}(u) = \{\mathbf{u} : \mathbf{u} \subseteq \bigcup_{\ell \in u} {}^{\partial(\ell)}(S_{\mathbf{x},\ell})$ satisfies $|\mathbf{u} \cap {}^{\partial(\ell)}(S_{\mathbf{x},\ell})| = 1$ for $\ell \in u\}$,

note that $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u) \Rightarrow |\mathbf{u}| = |u|$,

(b) for $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u)$ let

$$\begin{aligned}\Lambda_{\mathscr{U},\mathbf{u}} = \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u}) := \{\bar\eta \in \Lambda_{\mathbf{x}} : & \text{ for some } m \in u \text{ we have} \\ & \ell < \mathbf{k} \wedge \ell \neq m \Rightarrow \eta_\ell \in (\mathscr{U} \cup \mathbf{u}) \cap {}^{\partial(\ell)}(S_{\mathbf{x},\ell}) \\ & \text{and } \ell < \mathbf{k} \wedge \ell = m \Rightarrow \eta_\ell \in \mathscr{U} \cap {}^{\partial(\ell)}(S_{\mathbf{x},\ell})\},\end{aligned}$$

(c) $\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u}) := \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u}) \backslash \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$; this set is interesting, i.e., non-empty only when $\mathscr{U} \cap \mathbf{u} = \emptyset$ and then it is equal to

$$\{\bar\eta \in \Lambda_{\mathbf{x}} : \text{if } \ell \in u \text{ then } \eta_\ell \in \mathbf{u} \text{ and if } \ell \in \mathbf{k}\backslash u \text{ then } \eta_\ell \in \mathscr{U}\}.$$

(3) For non-empty $u \subseteq \{0,\dots,\mathbf{k}-1\}$ we say $\mathbf{x}$ is θ-(u,k)-free when: if $\mathscr{U} \subseteq \bigcup_{\ell<\mathbf{k}} {}^{\partial(\ell)}(S_{\mathbf{x},\ell})$ has cardinality $< \theta$ and $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$ is disjoint to $\mathscr{U}$ then $\Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})$ is $(\infty,2,u,k)$-free over $\Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ recalling 1.11(4),(5).

(3A) If $\theta > |\Lambda_{\mathbf{x}}|$ we may write ∞ instead of θ in part (3).

(4) For non-empty $u \subseteq \{0,\dots,\mathbf{k}-1\}$ we say $\mathbf{x}$ is (θ_2,θ_1)-(u,k)-free when: if $\mathscr{U} \subseteq \bigcup_{\ell<\mathbf{k}} {}^{\partial(\ell)}(S_{\mathbf{x},\ell})$ and $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$ is disjoint to $\mathscr{U}$ then $\Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})$ is (θ_2,θ_1,u,k)-free over $\Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ recalling 1.11(4).

Observation 1.23: (1) *In Definition 1.22(3), the conclusion is equivalent to "$\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u}) = \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})\backslash\Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ is $(\infty,2,u,k)$-free".*

(2) *Similarly in 1.22(4); that is, assume $u \subseteq \{0,\dots,\mathbf{k}-1\}$, $\mathscr{U} \subseteq \bigcup_{\ell<\mathbf{k}} {}^{\partial(\ell)}(S_{\mathbf{x},\ell})$ and $\mathbf{u} \in \mathrm{add}(u^\perp)$ is disjoint to $\mathscr{U}$, then: $\Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})$ is (θ_2,θ_1,u,k)-free over $\Lambda_{\mathbf{x}}(\mathscr{U} < \mathbf{u})$ iff $\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u}) = \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})\backslash\Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ is (θ_2,θ_1,u,k)-free..*

(3) *If $\mathbf{x}$ is θ-(u,k)-free, then $\mathbf{x}$ is (θ,u,k)-free; see Definition 1.22(3), 1.11(4), (5) respectively.*

(4) *If $\mathbf{x}$ is (θ_2,θ_1)-(u,k)-free, then $\mathbf{x}$ is (θ_2,θ_1,u,k)-free; see Definition 1.22(4), 1.11(4) respectively.*

Proof. (1) If $\bar\eta \in \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})\backslash\Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ and $\bar\nu \in \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ as $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$ it follows that $(\exists m \in u^\perp)[\eta_m \neq \nu_m]$.

(2) Similarly.

(3), (4) Straightforward. $\blacksquare_{1.23}$

The gain in the following theorem is that taking products of combinatorial parameters, we gain new cases of freeness.

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The Freeness Theorem 1.24: *If $\boxplus$ below holds, then $\mathbf{x}$ is $(\theta_{\mathbf{m}}, \theta_0^+)$-$(u,1)$-free. If, in addition, every $\mathbf{x}_\ell$ is θ_0^+-free, then $\mathbf{x}$ is $(\theta_{\mathbf{m}}, u)$-free:*

$\boxplus$ (a) $\mathbf{x}_\ell$ *is a combinatorial* $\langle \partial_\ell \rangle$*-parameter for* $\ell < \mathbf{k}$,
(b) $\mathbf{x} = \mathbf{x}_0 \times \cdots \times \mathbf{x}_{\mathbf{k}-1}$,
(c) $u \subseteq \{0, \ldots, \mathbf{k}-1\}$ *and* $\mathbf{m} = |u| > 0$, *hence* $\mathbf{m} \leq \mathbf{k}$,
(d) $\theta_0 < \theta_1 < \cdots < \theta_{\mathbf{m}}$ *are regular except possibly* θ_0,
(e) $\partial_{\mathbf{x}_\ell} \leq \theta_0$ *for* $\ell < \mathbf{k}$,
(f) $\mathbf{x}_k$ *is* $(\theta_{m+1}, \theta_m^+)$*-free when* $k \in u \wedge m < \mathbf{m}$.

Proof. Without loss of generality $\mathbf{x}$ is disjoint, i.e., the sets $S_\ell := S_{\mathbf{x},\ell}$ are pairwise disjoint for $\ell < \mathbf{k}$. We prove the claim by induction on $\mathbf{m}$ (so fix $\mathbf{k}$ but we vary u and the θ_m's). So let $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$ and $\mathscr{U} \subseteq \bigcup_{\ell<\mathbf{k}} {}^{\partial_\ell}(S_\ell)$ has cardinality $< \theta_{\mathbf{m}}$ and we shall prove that $\Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$ is $(\infty, \theta_0^+, u, 1)$-free. Clearly this suffices for the first phrase and the second follows recalling 1.12(2), 1.23(2).

Case 1: $\mathbf{m} = 1$

So $|u| = 1$ and let $u = \{\ell\}$, hence $\bar\eta \mapsto \eta_\ell$ is a one-to-one function from $\Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$ onto

$$\mathscr{U}_\ell := \mathscr{U} \cap \Lambda_{\mathbf{x}_\ell}.$$

We know that $\mathbf{x}_\ell$ is (θ_1, θ_0^+)-free and $|\mathscr{U}_\ell| < \theta_1$, hence there is a partition $\langle \mathscr{U}_{\ell,\alpha} : \alpha < \alpha(*) \rangle$ of $\mathscr{U}_\ell$ to sets each of cardinality $\leq \theta_0$, $\alpha(*) \leq |\mathscr{U}_\ell| < \theta_1$ and $h_\ell : \mathscr{U}_\ell \to J_{\mathbf{x}_\ell}$ such that

$$\alpha < \beta < \alpha(*) \wedge \eta \in \mathscr{U}_{\ell,\alpha} \wedge \nu \in \mathscr{U}_{\ell,\beta} \wedge \partial_\ell > i \notin h_\ell(\nu) \Rightarrow \eta(i) \neq \nu(i).$$

For $\alpha < \alpha(*)$ let

$$\Lambda_\alpha = \{\bar\eta \in \Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u}) : \eta_\ell \in \mathscr{U}_{\ell,\alpha}\};$$

clearly $\langle \Lambda_\alpha : \alpha < \alpha(*) \rangle$ is a partition of $\Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$ to sets each of cardinality $\leq \theta_0$. Let the function g from $\alpha(*)$ to $[u]^1 = \{\{\ell\}\}$ be defined by $g(\alpha) = \{\ell\}$; clearly the partition $\langle \Lambda_\alpha : \alpha < \alpha(*) \rangle$ and the functions g, h_ℓ witness that $\Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$ is $(\theta_{\mathbf{m}}, \theta_0^+)$-free, as required.

Case 2: $\mathbf{m} > 1$

Let $m = \mathbf{m} - 1$; as $\mathbf{m} > 1$, clearly m is ≥ 1. So for $k \in u$ the c.p. $\mathbf{x}_k$ is $(|\mathscr{U}|^+, \theta_m^+)$-free and let $\mathscr{U}_k = \mathscr{U} \cap {}^{\partial(k)}(S_k) \subseteq \Lambda_{\mathbf{x}_k}$, and by the induction hypothesis, without loss of generality $|\mathscr{U}| \geq ! \theta_m$. Hence as in earlier cases (see 1.16(1)(C)) we can find a function $h_k^* : \mathscr{U}_k \to J_{\mathbf{x}_k}$ such that in the directed graph $(\mathscr{U}_k, R_k)$

each node has out-degree $\le \theta_m$, that is, $(\forall \eta \in \mathscr{U}_k)(\exists^{\le \theta_m} \nu \in \mathscr{U}_k)[\eta R_k \nu]$, where

$(*)_1$ $R_k = R_{k,h_k} = \{(\eta,\nu) : \eta,\nu \in \mathscr{U}_k$ and for some $i < \partial_k$ we have $i \notin h^*_k(\nu)$, $\eta(i) = \nu(i)\}$,

$(*)_2$ let Λ_* be $\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u}) = \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u}) \backslash \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$,

and let

$(*)_3$ $R_* = \{(\bar\eta,\bar\nu) : \bar\eta,\bar\nu \in \Lambda_*$ and for some $k \in u$ we have $\eta_k R_k \nu_k$ and $\ell < \mathbf{k} \wedge \ell \ne k \Rightarrow \eta_\ell = \nu_\ell\}$.

Clearly

$(*)_4$ (Λ_*, R_*) is a directed graph with each node having out-degree $\le \theta_m$.

Let $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma(*)\rangle$ be such that:

$(*)_5$ (a) $\bar\Lambda$ is a partition of Λ_*,

(b) Λ_γ has cardinality $\le \theta_m$,

(c) if $\bar\eta \in \Lambda_\beta, \bar\nu \in \Lambda_\gamma$ and $\beta < \gamma < \gamma(*)$, then $\neg(\bar\eta R_* \bar\nu)$; that is

- if $\ell \in u$ and $\bar\eta \restriction (\ell, < 0) = \bar\nu \restriction (\ell, < 0)$, then $\neg(\eta_\ell R_\ell \nu_\ell)$.

[Why? Let $\langle \bar\eta_\alpha : \alpha < |\Lambda_*|\rangle$ list Λ_* with no repetition. For $\alpha < |\Lambda_*|$ we define $u_{\alpha,n} \in [|\Lambda_*|]^{\le \theta_m}$ by induction on n, increasing with n by

$$u_{\alpha,0} = \{\alpha\}, u_{\alpha,n+1} = \{\beta : \text{ for some } \gamma \in u_{\alpha,n} \text{ we have } \bar\eta_\gamma R_* \bar\eta_\beta \text{ or } \beta = \gamma\}.$$

So $u_\alpha = \bigcup_n u_{\alpha,n} \in [|\Lambda_*|]^{\le \theta_m}$, $\alpha \in u_\alpha$ and $[\bar\eta_\beta R_* \bar\eta_\gamma \wedge \beta \in u_\alpha \Rightarrow \gamma \in u_\alpha]$. Let $\Lambda_\alpha = \{\bar\eta_\gamma : \gamma \in u_\alpha$ but $(\forall \beta < \alpha)(\gamma \notin u_\beta)\}$, now check that $\bar\Lambda = \langle \Lambda_\alpha : \alpha < |\Lambda_*|\rangle$ is as required.]

$(*)_6$ It is enough to prove for each $\gamma < \gamma(*)$ that Λ_γ is $(\infty, \theta_0^+, u, 1)$-free.

[Why? It is enough to prove Λ_* is $(\infty, \theta_0^+, u, 1)$-free.

By the assumption of $(*)_6$ for each $\gamma < \gamma(*)$ let $\bar\Lambda_\gamma, g_\gamma, \bar h_\gamma$ witness that Λ_γ is (∞, θ_0^+, u)-free, that is (recall Definition 1.11(4); for $k = 1$, see 1.11(5)):

- $\bar\Lambda_\gamma = \langle \Lambda_{\gamma,\varepsilon} : \varepsilon < \varepsilon_\gamma\rangle$ is a partition of Λ_γ
- $\Lambda_{\gamma,\varepsilon}$ has cardinality $\le \theta_0$,
- $g_\gamma : \varepsilon_\gamma \to u$,
- if $\bar\eta, \bar\nu \in \Lambda_{\gamma,\varepsilon}$ and $k \in u \subseteq \mathbf{k}, k \ne g_\gamma(\varepsilon)$ then $\eta_k = \nu_k$,
- $\bar h_\gamma = \langle h_{\gamma,k} : k \in u\rangle$,
- $h_{\gamma,m}$ is a function from Λ_γ into J_m,
- if $\bar\eta \in \Lambda_{\gamma,\varepsilon}$ and $\bar\nu \in \cup\{\Lambda_{\gamma,\xi} : \xi < \varepsilon\}, m = g_\gamma(\varepsilon), \bar\nu \restriction (m) = \bar\eta \restriction (m)$ and $i \in \partial_k \backslash h_{\gamma,k}(\bar\eta)$ then $\eta_k(i) \ne \nu_k(i)$.

Let

- $\zeta_\gamma = \sum_{\beta<\gamma} \varepsilon_\beta$ for $\gamma \le \gamma(*)$,
- $\Lambda'_\varepsilon = \Lambda_{\gamma,\varepsilon-\zeta_\gamma}$ when $\varepsilon \in [\zeta_\gamma, \zeta_{\gamma+1}]$,
- g is the function with domain $\zeta_{\gamma(*)}$,
- $g(\varepsilon) = g_\gamma(\varepsilon - \zeta_\gamma)$ when $\varepsilon \in [\zeta_\gamma, \zeta_{\gamma+1})$ and $\gamma < \gamma(*)$,
- h_k is the function with domain Λ_* defined by: if $\bar\eta \in \Lambda_\zeta, \zeta = \zeta_\gamma + \varepsilon$ and $\varepsilon < \varepsilon_\gamma$ then $h_k(\bar\eta) = h_{\gamma,k}(\bar\eta) \cup h^*_k(\bar\eta)$.

Now check Definition 1.11(4).]

Fix $\gamma < \gamma(*)$ and we shall prove for it the condition from $(*)_6$. If $|\Lambda_\gamma| < \theta_m$ the desired statement follows from the induction hypothesis, so assume $|\Lambda_\gamma| = \theta_m$. Let $\langle \eta_{\gamma,\alpha} : \alpha < \theta_m\rangle$ list $\{\nu_k : \bar\nu \in \Lambda_\gamma$ and $k \in u\}$.

For $\beta < \theta_m$ let $\mathscr{U}_{\gamma,\beta} = \{\eta_{\gamma,\alpha} : \alpha < \beta\}$ and let $k(\beta)$ be the unique $k \in u$ such that $\eta_{\gamma,\beta} \in {}^{\partial_k}(S_k)$. Clearly $|\mathscr{U}_{\gamma,\beta}| < \theta_m$. Also $\langle \mathscr{U}_{\gamma,\beta} : \beta < \theta_m\rangle$ is $\subseteq$-increasing continuous with union $\cup\{\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) : \beta < \theta_m\} = \Lambda_\gamma$.

By induction on $\beta < \theta_m$ we choose $\langle \bar\Lambda_\beta, g_\beta, \bar h^\beta\rangle$ such that

$(*)_7$ (a) $\bar\Lambda_\beta = \langle \Lambda_{\gamma,\varepsilon} : \varepsilon < \varepsilon_\beta\rangle$ is a partition of $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u})$ so

$$\alpha < \beta \Rightarrow \bar\Lambda_\alpha \triangleleft \bar\Lambda_\beta,$$

(b) each $\Lambda_{\gamma,\varepsilon}$ has cardinality $\le \theta_0$,
(c) $g_\beta : \varepsilon_\beta \to u$ such that $\alpha < \beta \Rightarrow g_\alpha \subseteq g_\beta$,
(d) $\bar h^\beta = \langle h^\beta_k : k \in u\rangle$,
(e) $h_{\beta,k} : \Lambda_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) \to J_k$ and $\alpha < \beta \Rightarrow h^\alpha_k \subseteq h^\beta_k$,
(f) if $\varepsilon < \varepsilon_\beta, \bar\eta \in \Lambda_{\gamma,\varepsilon}, g_\beta(\bar\eta) = k$ so $k \in u$ and $\bar\nu \in \cup\{\Lambda_{\gamma,\zeta} : \zeta < \varepsilon\}$ and $\bar\nu \upharpoonright (k, < 0) = \bar\eta \upharpoonright (k, < 0)$, then

$$i \in \partial_{\mathbf{x},k} \backslash h_{\beta,k}(\bar\eta) \Rightarrow \nu_k(i) \ne \eta_k(i).$$

For $\beta = 0$, $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) = \emptyset$ so this is obvious. For β limit take unions.

Lastly, for $\beta = \beta_* + 1$, it is enough to show that $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u})$ is (∞, θ_0^+, u)-free over $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta_*}, \mathbf{u})$. Now

$$\mathscr{U}_{\gamma,\beta} \backslash \mathscr{U}_{\gamma,\beta_*} = \{\eta_{\gamma,\beta_*}\}, \quad \eta_{\gamma,\beta_*} \in {}^{\partial_{k(\beta_*)}}(S_{k(\beta_*)}),$$

hence $\eta_{\gamma,\beta_*} \in \mathscr{U}$. So let $u_{\gamma,\beta} = u \backslash \{k(\beta_*)\}$, $\mathbf{u}_{\gamma,\beta} = \mathbf{u} \cup \{\eta_{\gamma,\beta_*}\}$, so $\mathbf{u}_{\gamma,\beta} \in \mathrm{add}_{\mathbf{x}}(u^\perp_{\gamma,\beta})$, $u_{\gamma,\beta} \subseteq \{0, \dots, \mathbf{k}_{\mathbf{x}} - 1\}$ has m members because $|\mathbf{u}| = \mathbf{m} = m + 1$. Recall

$$\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta}) = \Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) \backslash \Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta_*}, \mathbf{u})$$

and by the induction hypothesis on m we know $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta})$ is $(\infty, \theta_0^+, u_{\gamma,\beta})$-free so there is a witness $(\bar{\Lambda}^*_{\gamma,\beta}, g^*_{\gamma,\beta}, \bar{h}^*_{\gamma,\beta})$, i.e., it is as in 1.11(4)(d) for $k = 1$, in particular:

$(*)_8$ $\bar{\Lambda}^*_{\gamma,\beta} = \langle \Lambda^*_{\gamma,\beta,\zeta} : \zeta < \zeta_{\gamma,\beta}\rangle$ is a partition of $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta})$.

We define

$(*)_9$
- $\varepsilon_\beta = \varepsilon_{\beta_*} + \zeta_{\gamma,\beta}$,
- $\Lambda_{\varepsilon_{\beta_*}+\zeta} = \Lambda^*_{\gamma,\beta,\zeta}$ for $\zeta < \zeta_{\gamma,\beta}$,
- $g_\beta(\varepsilon_{\beta_*} + \zeta) = g^*_{\gamma,\beta}(\zeta)$ for $\zeta < \zeta_{\gamma,\beta}$, i.e., g_β is the function with domain ε_β extending g_{β_*} and defined on $[\varepsilon_{\beta_*}, \varepsilon_\beta)$ as above,
- $h_{\beta,k}$ is a function with domain $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta,\mathbf{u}}) = \cup\{\Lambda_\varepsilon : \varepsilon < \varepsilon_\beta\}$ extending $h_{\gamma,\beta_*,k}$,
- $h_{\beta,k}(\bar{\eta}) = h^*_{\gamma,\beta,k}(\bar{\eta})$ if $\bar{\eta} \in \Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta})$.

Now check. Notice that if $\xi < \varepsilon_{\beta_*} \le \varepsilon < \varepsilon_\beta$ and $\bar{\nu} \in \Lambda_{\gamma,\xi}$ and $\bar{\eta} \in \Lambda_{\gamma,\varepsilon} = \Lambda^*_{\gamma,\beta,\varepsilon-\varepsilon_{\beta_*}}$ and $m = g_\beta(\varepsilon) = g^*_{\beta,\gamma}(\varepsilon - \varepsilon_{\beta_*})$, then $m \ne k(\beta_*)$ and $\eta_{k(\beta_*)} \ne \nu_{k(\beta_*)}$, so no problem arises and the rest should be clear. $\blacksquare_{1.24}$

In what follows we assume $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial$ to simplify; anyhow we have not sorted out what occurred to (B)(d) when $\bar{\partial}$ is not constant

Theorem 1.25: *If* (A) *then* (B) *where:*

(A) (a) $\bar{\partial} = \langle \partial_\ell : \ell < \mathbf{k}\rangle$ *such that* $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial = \mathrm{cf}(\partial)$,
 (b) $\mu_\ell \in \mathbf{C}_{\partial_\ell}$ *for* $\ell < \mathbf{k}$, *see 0.2, 0.3,*
 (c) $\mu_\ell < \mu_{\ell+1}$ *for* $\ell < \mathbf{k}$,
 (d) $\chi_\ell = 2^{\mu_\ell}$,
 (e) *for some regular* σ, $\bigwedge_\ell \sigma < \partial_\ell$ *and* $J_\ell = J^{\mathrm{bd}}_\sigma \odot J^{\mathrm{bd}}_{\partial_\ell}$ *for* $\ell < \mathbf{k}$;

(B) *there is* $\mathbf{x}$ *such that:*
 (a) $\mathbf{x}$ *is a combinatorial* $\bar{\partial}$*-parameter of cardinality* $\le \chi_{\mathbf{k}-1}$ *with* $J_{\mathbf{x},\ell} = J_\ell$,
 (b) $\mathbf{x}$ *has a* $\bar{\chi}$*-black box*
 (c) $\mathbf{x}$ *is* (θ_*, θ^+)*-free when* $n(*) \ge 1, \theta = \mathrm{cf}(\theta) \ge \partial, \theta_* = \theta^{+(\partial\cdot n(*))} < \mu_0$ *and* $3n(*) + 4 < \mathbf{k}$,
 (d) $\mathbf{x}$ *is* θ_{**}*-free when* $\theta_{**} = \partial^{+(\partial\cdot n(*)+\partial)} < \mu_0, 3n(*)+4 < \mathbf{k}, n(*) \ge 1$.

Remark 1.26: Note that the proof is somewhat easier when $\theta^{+\partial(n(*)+1)} < \mu_0$ and the loss is minor.

Proof. For each $\ell < \mathbf{k}$ we can choose $\mathbf{x}_\ell$ such that:

$\oplus$ (a) $\mathbf{x}_\ell$ is a combinatorial $\langle \partial_\ell \rangle$-parameter,
(b) $\mathbf{x}_\ell$ is $(\theta^{+\partial+1}, \theta^{+4})$-free when $\partial \le \theta < \mu_0$ and $J_{\mathbf{x},e} = J_e$,
(c) $\mathbf{x}_\ell$ has a χ_ℓ-pre-black box, moreover,
(c)$^+$ $\mathbf{x}_\ell$ has a χ_ℓ-black box,
(d) $\Lambda_{\mathbf{x}_\ell}$ has cardinality χ_ℓ,
(e) $\mathbf{x}_\ell$ is ∂^+-free.

Why? By [She13a, 0.4,0.5,0.6=y19,y22,y40], when we weaken clause $\oplus(c)^+$ to $\mathbf{x}_\ell$ has a χ_ℓ-pre-black box; anyhow we elaborate (also when $\partial = \aleph_0$ we have to say a little more) so let $\ell < \mathbf{k}$ and $\mu = \mu_\ell, \lambda = \chi_\ell$.

First, assume that there is a $(\mu^+, J^{\mathrm{bd}}_\partial)$-free subset $\mathscr{F}$ of ${}^\partial(\mu)$ of cardinality $\lambda = 2^\mu$. We define $\mathbf{x}_\ell$ by $\Lambda_{\mathbf{x}_\ell} = \{\langle \eta \rangle : \eta \in \mathscr{F}\}, J_{\mathbf{x}_\ell} = J^{\mathrm{bd}}_\partial$.

Now $\mathbf{x}_\ell$ has a λ-black box (by [She13b, §3]); easy as the number of functions from ${}^{\partial>}(\mu)$ to λ is $\lambda^\mu = \lambda$.

Note also that $\mathbf{x}_\ell$ is tree-like; this is enough for $\oplus$(a), (b), (c), (d), (e). Without loss of generality there is a list $\langle \eta_\alpha : \alpha < \lambda \rangle$ of the elements of $\mathscr{F}$ such that $\alpha < \beta \Rightarrow \eta_\alpha <_{J^{\mathrm{bd}}_\partial} \eta_\beta$ (see the proof of [She13b, 3.10=L1f.28]. Let $\langle \mathscr{U}_\alpha : \alpha < \lambda \rangle$ be a sequence of pairwise disjoint subsets of λ each of cardinality λ such that

$$\min(\mathscr{U}_\alpha) > \mu^\omega \cdot \alpha$$

and let $\mathscr{F}_\alpha = \{\eta_\beta : \beta \in \mathscr{U}_\alpha\}$ and $\nu_\alpha = \eta_\beta$ when $\alpha \in [\mu \cdot \beta, \mu \cdot \beta + \mu)$ and $\mathscr{F}_* = \bigcup_\alpha \mathscr{F}_\alpha$. Now we choose

$$\Lambda_{\mathbf{x}_\ell} = \{\langle \eta \rangle : \eta \in \mathscr{F}_*\}, \quad \Lambda^*_\alpha = \{\langle \eta \rangle : \eta \in \mathscr{F}_\alpha\}$$

so $\langle \nu_\alpha : \alpha < \lambda \rangle$ witness $\mathbf{x}_\ell$ has a $\bar{\chi}$-black box.

Second, assume that there is no $\mathscr{F}$ as above; it follows that $\lambda = 2^\mu$ is regular (see [She13a, 0.4] or [She13b, §3] using the "no hole claim"). Note that if there is a $\langle \partial \rangle$-c.p. $\mathbf{x}$ which is (θ_2, θ_1)-free, $\Lambda_{\mathbf{x}} \subseteq {}^\partial\mu$ pedantically $\Lambda_{\mathbf{x}} \subseteq \{\langle \eta \rangle : \eta \in {}^\partial\mu\}$ and $|\Lambda_{\mathbf{x}}| = 2^\mu$, J and ideal on ∂, $J \supseteq J_{\mathbf{x}} = J^{\mathrm{bd}}_\partial$, then there is such $\mathbf{y}$ with $J_{\mathbf{y}} = J$ and as above both have the λ-BB.

Now as $\lambda = \mathrm{cf}(\lambda) = 2^\mu, \mu \in \mathbf{C}_\partial$, there is a sequence $\langle \lambda_i : i < \partial \rangle$ of regular cardinals $< \mu$ and (see [She96, 6.5]) the ∂-complete ideal $J = J^{\mathrm{bd}}_\partial$ such that

$$\chi = \mathrm{tcf}\bigg(\prod_{i<\partial} \lambda_i, <_J\bigg),$$

so let $\langle \eta_\alpha : \alpha < \chi \rangle$ be $<_J$-increasing cofinal in $(\prod_{i<\partial} \lambda_i, <_J)$. By [She13a, 0.1=L41] there is $S \in \check{I}_{\theta^+}[\lambda]$ such that: if $\delta < \lambda \wedge \mathrm{cf}(\delta) \geq \theta^{+4}$ then

$$\{\delta_1 < \delta : \mathrm{cf}(\delta_1) = \theta^{+3} \text{ and } S \cap \delta_1 \text{ is a stationary subset of } \delta_1\}$$

is stationary in δ; note that there $\mathrm{cf}(\delta) = \theta^{+4}$, but the general case of $\mathrm{cf}(\delta) \geq \theta^{+4}$ follows.

Recall $\lambda = \mathrm{cf}(\lambda), S \subseteq \lambda, \sup(S) = \lambda$ and we recall some things from [She13a]; $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$-increasing, J an ideal on $\partial, f_\alpha : \partial \to \mathrm{Ord}$ and $u_\alpha \subseteq \alpha$ for $\alpha < \lambda$, we say $\bar{f}$ obeys the sequence of sets $\bar{u} = \langle u_\alpha : \alpha < \lambda \rangle$ when for every βu_α we have $\bigwedge_{\gamma<\partial} f_\beta(\gamma) < f_\alpha(\gamma)$ and if $\alpha \in S$ is a limit ordinal then $f_\alpha(\gamma) = \sup_{\beta \in u_\alpha}(f_\beta(\gamma) + 1)$ for every $\gamma < \partial$.

For $\theta = \mathrm{cf}(\theta) < \lambda$, we say $\bar{u}$ as above is a witness for $S \in \check{I}_\theta[\lambda]$ when:

- $\alpha \in S \Rightarrow \mathrm{cf}(\alpha) = \theta$,
- $\alpha < \lambda \Rightarrow |u_\alpha| < \theta$,
- $\alpha \in u_\beta \Rightarrow u_\alpha = u_\beta \cap \alpha$,
- there is a club E of λ such that if $\delta \in S \cap E$ then u_α is an unbounded subset of α of order type θ.

We say $\bar{f}$ is good in a limit ordinal $\delta < \lambda$ when there are $u \subseteq \delta = \sup(\delta)$ and $\bar{w} = \langle w_\alpha : \alpha \in u \rangle \in {}^u J$ such that

$$\alpha \in u \wedge \beta \in u \wedge \alpha < \beta \wedge i \in \partial \backslash (w_\alpha \cup w_\beta) \Rightarrow f_\alpha(i) < f_\beta(i).$$

So without loss of generality $\bar{f}$ obeys a witness for $S \in \check{I}_{\partial^+}[\lambda]$, hence is good in δ when: $\delta \in S$ or $(S \cap \delta)$ is a stationary subset of δ and $\mathrm{cf}(\delta) \in (\theta, \theta^{+\partial})$. Let $S_\bullet \in \check{I}_\sigma[\lambda]$ be stationary such that $\delta \in S_\bullet \Rightarrow \mu^\omega | \delta$ and for $\delta \in S_\bullet$ we let $\rho_\delta \in^\sigma \delta$ be increasing with limit δ. Now let $f'_\delta \in^\sigma \delta$ be such that $i < \partial \wedge j < \sigma \Rightarrow f'_\delta(\sigma i + j) = \mu f_\delta(i) + \rho_\delta(j)$. Hence $\{f'_\alpha : \alpha < \lambda\}$ is $(\theta^{+\partial+1}, \theta^{+4})$-free for every $\theta \geq \partial$, see [She13a, 0.4=Ly19]; in more detail, in [She13a, 0.4=Ly19] we conclude (A) or (B), now (A) there is covered by "first" whereas if (B) there holds see [She13a, 0.6(e)=Ly40(e)].

Together we are done except for $\oplus$(c), (c)$^+$ which is proved by [She13a, 0.6(g), (g')].

So we have finished proving $\oplus$.

Let $\mathbf{x} = \mathbf{y}_\mathbf{k}$, where for $m \in \{1, \ldots, \mathbf{k}\}$ we let $\mathbf{y}_m = \mathbf{x}_0 \times \mathbf{x}_1 \times \cdots \times \mathbf{x}_{m-1}$ and we shall show it is as required.

Clause (B)(a), which says "$\mathbf{x}$ is a combinatorial $\bar{\partial}$-parameter of cardinality $\chi_{\mathbf{k}-1}$", holds by 1.20(1), i.e., we can prove "$\mathbf{y}_m$ is a $\langle \partial_\ell : \ell < m \rangle$-c.p. of cardinality χ_{m-1}" by induction on $m = 1, \ldots, \mathbf{k}$.

Clause (B)(b), which says "$\mathbf{x}$ has a $\bar{\chi}$-BB" holds by 1.21, that is, again by induction on $m = 1, \dots, \mathbf{k}$ we can prove that $\mathbf{y}_m$ has the $\langle \chi_\ell : \ell < m \rangle$-BB.

We now shall prove:

Clause (B)(c): we deduce it from 1.24+$\oplus$(b). We are given $\theta, n(*)$ as there. Let $\langle \theta_m : m \leq m(*) \rangle$ be defined by: $m(*) = 3n(*) + 4$, $\theta_\iota := \theta^{+\iota}$ for $\iota = 0, 1, 2, 3$ and $\theta_{3+3m+\iota} := (\theta_{3+3m})^{+(\partial+\iota)}$ for $\iota = 1, 2, 3$ when $m < n(*)$ and

$$\theta_{m(*)} := \theta^{+\partial+1}_{3n(*)+4} < \mu_0;$$

the "$\leq \mu_0$" holds by the assumption of clause (B)(c). Note that if $\theta_{m+1} = \theta^+_m$ then "$\mathbf{x}_\ell$ being $(\theta_{m+1}, \theta^+_m)$-free" is trivial.

To apply Theorem 1.24 with $\mathbf{x}_\ell$ as in $\oplus$ above, $\mathbf{x}$ as above, $\mathbf{m} = m(*)$, $u = \{0, \dots, \mathbf{k} - 1\}$ has $\mathbf{m}$ members and θ_ℓ for $\ell \leq \mathbf{m}$ as above, we have to verify clauses (a)–(f) of $\boxplus$ of 1.24.

Now clause (a) stating $\mathbf{x}_\ell$ is a combinatorial $\langle \partial_\ell \rangle$-parameter holds by $\oplus$(a).

Now clause (b) stating $\mathbf{x} = \mathbf{x}_0 \times \cdots \mathbf{x}_{\mathbf{k}-1}$ holds by the choice of $\mathbf{x}$ above.

Clause (c) stating "$u \subseteq \{0, \dots, \mathbf{k} - 1\}$ and $\mathbf{m} = |u| > 0$" holds by the choice of u and the assumption on $m(*)$.

Clause (d) stating "$\theta_0 < \cdots < \theta_{\mathbf{m}}$" holds by the choice of the θ_ℓ's above. Notice that each $\theta_\ell (\ell > 0)$ is a successor and hence regular.

Clause (e) stating "$\partial_{\mathbf{x}_\ell} \leq \theta_0$ for $\ell < \mathbf{k}$" holds because $\theta_0 = \theta \geq \partial = \partial_\ell$ for $\ell < \mathbf{k}$.

Clause (f) stating "$\mathbf{x}_\ell$ is $(\theta_{m+1}, \theta^+_m)$-free", "when $\ell \in u, m < \mathbf{m}$", holds; we check this by cases.

Case 1: $\theta_{m+1} = \theta^+_m, (f)$ holds trivially.

Case 2: $m = 3, (\theta_{m+1}, \theta^+_m) = (\theta^{+(\partial+1)}, \theta^{+4})$ holds by clause (b) of $\oplus$.

Case 3: $m = 3n + 3$ where $n < n(*)$ so $(\theta_{m+1}, \theta^+_m) = (\theta^{\partial \cdot (n+1)+1}, \theta^{\partial \cdot n+4})$.

By clause (b) of $\oplus$ above applied to $\theta = \partial^{+\partial \cdot n}$.

So all clauses of $\boxplus$ of Theorem 1.24 hold, hence its conclusion which says $\mathbf{x}$ is $(\theta_{\mathbf{m}}, \theta^+_0)$-free but $\theta_{\mathbf{m}} = \theta_*$ and $\theta_0 = \partial$, so we are done proving clause (c) of 1.25(B).

Clause (B)(d) says that "$\mathbf{x}$ is θ_* free" assuming $\theta_* = \partial^{+(\partial \cdot n(*)+\partial)} < \mu_0$, $3m + 4 < \mathbf{k}$ and $\bigwedge_{\ell < \mathbf{k}} \partial_\ell = \partial$. We will deduce it from clause (B)(c) by applying it choosing $\theta'_* = \theta^{+\partial \cdot n(*)+4}$, $\theta = \partial$ and $m(*) = m$. The assumptions in clause (c) hold: $\theta = \partial^+$ so $\theta \geq \partial$ and θ'_* is as θ_* is there and $\theta'_* < \mu_0$ by an assumption of clause (d) which also says $3n(*) + 4 < \mathbf{k}$.

So the conclusion of clause (c) holds, i.e., $\mathbf{x}$ is (θ'_*, θ)-free. But $\theta_* \leq \theta'_*$ so $\mathbf{x}$ is (θ_*, ∂^+)-free. Also each $\mathbf{x}_\ell$ is ∂^+-free by $\boxplus(e)$ hence by 1.12 the last two statements imply $\mathbf{x}$ is θ_*-free. $\blacksquare_{1.25}$

Conclusion 1.27: (1) *If $\sigma < \partial$ are regular and $\chi \geq \partial$ and $n \geq 1$, then there is an $\aleph_{\partial \cdot n}$-free, m-c.p.* $\mathbf{x}$ *for some m which has the χ-BB and $|\Lambda_{\mathbf{x}}| < \beth_{\partial \cdot \omega}(\chi)$ and $J_{\mathbf{x},m} = J^{\rm bd}_\partial \odot J^{\rm bd}_\sigma$.*

(2) *If $\sigma = \partial$ is regular and $\chi \geq \partial$ and $n \geq 1$, then there is an $\aleph_{\partial \cdot n}$-free m-c.p. (for some m),* $\mathbf{x}$ *has the χ-BB which is not free (really follows) and $\Lambda_{\mathbf{x}}$ is not even the union of $\leq \chi$ free subsets and* $\mathbf{x}$ *has cardinality $< \beth_{\partial \cdot \omega}(\chi) + \beth_{\omega_1}(\chi)$ and $J_{\mathbf{x},m} = J^{\rm bd}_\partial$.*

(3) *If $m = 3n+5$, $\sigma = {\rm cf}(\sigma) < \partial = {\rm cf}(\partial) < \chi < \mu_0 < \cdots < \mu_{m-1}$ and $\mu_\ell \in \mathbf{C}_\partial$ for $\ell < m$, $\lambda_\ell = {\rm cf}(2^{\mu_\ell})$, $S_\ell \subseteq \{\delta < \lambda_\ell : {\rm cf}(\delta) = \sigma\}$ stationary from $\check{I}_\sigma[\lambda_\ell]$ and $J = J^{\rm bd}_\partial \times J^{\rm bd}_\sigma$, then we have* (A) *or* (B)*, where:*

(A) *for some ℓ*

- (a) *there is an $\mathscr{F} \subseteq {}^\partial(\mu_\ell)$ of cardinality 2^{μ_ℓ} which is μ^+_ℓ-free, i.e., is $(\mu^+_\ell, J^{\rm bd}_\partial)$-free, see Definition 0.7(1), hence (μ^+_ℓ, J)-free,*
- (b) *hence letting* $\mathbf{x}$ *be the 1.-c.p. such that $\Lambda_{\mathbf{x}} = \{\langle \eta \rangle : \eta \in \mathscr{F}\}$, it is a 2^{μ_ℓ}-BB for* $\mathbf{x}$ *which is μ^+_ℓ-free and $J_{\mathbf{x}} = J$;*

(B) *we can choose $\mathbf{x} = \mathbf{x}_0 \times \cdots \times \mathbf{x}_{m-1}, \mathbf{x}_\ell$ as a 1-c.p., $\Lambda_{\mathbf{x}_\ell} = \{\eta_{\ell,\delta} : \delta \in S_\ell\}$, $\lim_{J_{\mathbf{x}_\ell}}(\eta_{\ell,\delta}) = \delta$, moreover $\eta_{\ell,\delta}$ is increasing with limit δ and $J_{\mathbf{x}_\ell} = J_\partial \odot J_\sigma$ and $\mathbf{x}_\ell$ has the χ-BB if $\chi < \mu_\ell$.*

(4) *Given $n, m, \sigma < \partial < \chi$ as in part (3), we can find μ_ℓ (and λ_ℓ, S_ℓ) as there such that:*

- (a) *if $\partial > \aleph_0$ then $\mu_\ell = \beth_{\partial \cdot (1+\ell)}(\chi)$, we'll have "$\mathbf{x}$ is θ_*-free" we need $\chi \geq \theta$,*
- (b) *if $\partial = \aleph_0$ for some club E of ω_1 and $\mu_\ell \in \{\beth_\delta(\chi) : \delta \in E\}$ are O.K.*

Proof. (1) Let $\mathbf{k} = 3n+5$ and for $\ell < \mathbf{k}$ we let $\partial_\ell = \partial$, $\mu_\ell = \beth_{\partial \cdot (1+\ell)}(\partial^{+(\partial \cdot n+1)} + \chi)$ and $\chi_\ell = 2^{\mu_\ell}$. So each μ_ℓ is strong limit of cofinality $\partial = {\rm cf}(\partial) > \sigma \geq \aleph_0$; recalling 0.3 we have $\mu_\ell \in \mathbf{C}_{\partial_\ell}$, i.e., clause (A)(b) of Theorem 1.25 holds.

Clauses (A)(a),(c),(d),(e) of 1.25 are obvious, hence there is $\mathbf{x}$ as in clause (B) of 1.25, in particular it is $\partial^{+(\partial \cdot n+1)}$-free. Also $\partial^{+(\partial \cdot n+1)} < \beth_{\partial \cdot \omega}(\chi)$, hence also $\mu_\ell = \beth_{\partial \cdot (n+2)}(\partial^{+(\partial n+1)} + \chi)$ is $< \beth_{\partial \cdot \omega}(\chi)$, hence $|\Lambda_{\mathbf{x}}| \leq 2^{\mu_{\mathbf{k}-1}} < \beth_{\partial \cdot \omega}(\chi)$, so we are done.

(2) If $\partial > \aleph_0$, the proof of part (1) holds and $|\Lambda_{\mathbf{x}}| < \beth_{\partial \cdot \omega}(\chi)$. If $\partial = \aleph_0$, we know (see [She94]) that there is a club E of ω_1 consisting of limit ordinals such that $\delta \in E \Rightarrow \beth_\delta(\chi) \in \mathbf{C}_\partial$. We define $\mathbf{k}, \partial_\ell$ as above and for $\ell < \mathbf{k}$ let δ_ℓ be the ℓ-th member of E and let $\mu_\ell = \beth_{\delta_\ell}(\chi)$, and we continue the proof in [Shee], and anyhow not used.

(3) This is straightforward by [She13b] but we elaborate to some extent. First assume that for some $\ell < \mathbf{k}$ clause (A)(a) of 1.27(3) holds, so $\mathbf{x}$ from (A)(b) is a well defined 1-c.p. and is μ_ℓ^+-free, and letting $\chi = 2^{\mu_\ell}$ there is a χ-BB for $\mathbf{x}$ because the number of $h : {}^{\partial>}(\mu_\ell) \to \chi$ is $\leq \chi^{\mu_\ell} = \chi$, and diagonalizing we can choose a χ-pre-BB for $\mathbf{x}$ (see 1.15). To get a χ-BB we work as in the proof of 1.21(2).

So assume there is no such ℓ. Then for each ℓ, we know that $\lambda_\ell = 2^{\mu_\ell}$ is regular (see [She13b, 3.10(3)=L1f.28, p. 39]). By the proof of $\oplus$ in the beginning of the proof of 1.25, there is $\mathbf{x}_{\ell,1}$ as there, so as $\Lambda_{\mathbf{x}_{\ell,1}} \subseteq {}^{\partial}(\mu_\ell)$. By [She13b, 3.6=L1f.21], we know that $\alpha < \lambda_\ell \Rightarrow |\alpha|^\sigma < \lambda_\ell$, hence obviously there is a stationary set $S_\ell \subseteq \check{I}_\sigma[\lambda_\ell]$ (in fact, $\{\delta < \lambda_\ell : \mathrm{cf}(\delta) = \sigma\}$ belongs to $\check{I}_\sigma[\lambda_\ell]$), see [She93a, Claim 2.14]) and without loss of generality $\delta \in S_\ell \Rightarrow \mu_\ell^\omega | \delta$.

Hence we can find $\bar{\nu} = \langle \nu_\delta : \delta \in S_\ell \rangle$ such that:

- $\nu_\delta \in {}^\sigma\delta$ is increasing with limit δ,
- $\nu_{\delta_1}(i_1) = \nu_{\delta_2}(i_2) \Rightarrow \ell_1 = \ell_2 \wedge \nu_{\delta_1} \restriction i_1 = \nu_{\delta_2} \restriction i_2$,
- $\nu_\delta(i)$ is divisible by μ_ℓ.

Let $\langle \rho_\delta : \delta \in S_\ell \rangle$ list $\Lambda_{\mathbf{x}_{\ell,1}}$ and for $\delta \in S_\ell$ let $\eta_\delta \in {}^\partial\delta$ be: if $i < \partial, j < \sigma$ then

$$\eta_\delta(\sigma i + j) = \nu_\delta(j) + \rho_\delta(i).$$

We define $\mathbf{x}_\ell$ by $\Lambda_{\mathbf{x}_\ell} = \{\eta_\delta : \delta \in S_\ell\}$, $J_{\mathbf{x}_\ell} = J_\sigma \circledast J^\delta_\partial$, etc. Now

(∗) $\mathbf{x}_\ell$ is a $\langle \partial \rangle$-pre-BB of cardinality χ_ℓ, with the freeness properties from 1.25.

What about χ-pre-BB? By [She13b, §3] this holds whenever $\chi < \mu_\ell$, which is enough for applying. To get χ-BB let $\langle \delta(\zeta) : \zeta < \lambda \rangle$ list S_ℓ in increasing order and let $\langle S_\alpha : \alpha < \lambda_\ell \rangle$ be a sequence of pairwise disjoint stationary subsets of S_ℓ such that $\min(S_\alpha) > \delta(\alpha)$. Let $\nu_\xi = \eta_{\delta(\zeta)}$ when $\zeta \cdot \mu \leq \xi < \zeta \cdot \mu + \mu$.

We define $\Lambda_\alpha = \Lambda^\ell_\alpha = \{\eta_\delta : \delta \in S_\alpha\}$ so for each α there is a χ_ℓ-pre-BB for Λ_α and we continue as in the proof of 1.25. We now continue as in part (1) within the proof of 1.25.

(4) By the proofs above this should be clear. $\blacksquare_{1.27}$

Discussion 1.28: (1) The following statement appears in [She13a, 0.4=Ly19]. If $\sigma = \mathrm{cf}(\sigma) < \kappa = \mathrm{cf}(\kappa)$ and $\mu \in \mathbf{C}_\kappa$, then at least one of the following holds:

(A) there exists a μ^+-free $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality $\lambda = 2^\mu$,
(B) $\lambda = 2^\mu$ is regular and there is a $(\lambda, \mu, \sigma, \kappa) - 5$-solution.

If (A) holds, then we get more than promised (i.e., μ_ℓ^+-freeness). Hence we may assume, without loss of generality, that (B) holds. We shall return to this point (and then recall the definition of 5-solution).

(2) We can vary the definition of the BB, using values in χ or using models.

(3) We can use just a product of two combinatorial parameters but with any $\mathbf{k_x}$. At present this makes no real difference.

Discussion 1.29: Assume $\mathbf{x}$ is a combinatorial $\bar\partial$-parameter, $\bar\partial = \bar\partial_{\mathbf{x}}$ and $\bar\partial' = \langle \partial'_\ell : \ell < \mathbf{k_x}\rangle$ is a sequence of limit ordinals such that

$$\ell < \mathbf{k} \Rightarrow \mathrm{cf}(\partial'_\ell) = \partial_\ell.$$

It follows that there is $\mathbf{y}$ such that:

(∗) (a) $\mathbf{y}$ is a combinatorial $\bar\partial'$-parameter,

(b) $\mathbf{S}_{\mathbf{y},\ell} = \{\partial'_\ell\alpha + i : \alpha \in S_{\mathbf{x},\ell} \text{ and } i < \partial'_\ell\}$,

(c) $\Lambda_{\mathbf{y}} = \{g(\bar\eta) : \bar\eta \in \Lambda\}$, where

(d) $g : \bar S^{[\bar\partial]}_{\mathbf{x}} \to \bar S^{[\bar\partial']}_{\mathbf{y}}$ is defined as follows: for each $\ell < \mathbf{k}$ for some increasing continuous sequence $\langle \varepsilon_{\ell,i} : i \le \partial_\ell\rangle$ of ordinals with $\varepsilon_{\ell,0} = 0$, $\varepsilon_{\ell,\partial_\ell} = \partial'_\ell$ we have $g(\bar\eta) = \bar\nu$ iff $\bar\eta = \langle \eta_\ell : \ell < \mathbf{k}\rangle$, $\bar\nu = \langle \nu_\ell : \ell < k\rangle$ and

$$\varepsilon_{\ell,i} \le \varepsilon < \varepsilon_{\ell,i+1} \Rightarrow \nu_\ell(\varepsilon) = \partial'_\ell \cdot \eta_\ell(i) + \varepsilon$$

(of course, we could have "economical"),

(e) if $\mathbf{x}$ has $\bar\chi$ – BB and $\chi_\ell = \chi_\ell^{\partial'_\ell}$ for $\ell < \mathbf{k}$ then $\mathbf{y}$ has $\bar\chi$ – BB.

Definition 1.30: We say a $\mathbf{k}$-c.p. $\mathbf{x}$ is (θ,σ)**-well orderable** $(\bar\chi, \mathbf{k}, 1)$-BB when there is a witness $\bar\Lambda$ which means:

(a) $\bar\Lambda = \langle \Lambda_\alpha : \alpha < \delta\rangle$,

(b) $\bar\Lambda$ is increasing continuous,

(c) $\mathrm{cf}(\delta) \ge \sigma$ and δ is divisible by θ,

(d) if $\alpha < \delta$ then $\mathbf{x}{\restriction}(\Lambda_{\alpha+1}\backslash\Lambda_\alpha)$ has a $\bar\chi$-pre-black box

(e) if $\alpha < \delta, \bar\eta \in \Lambda_{\alpha+1}\backslash\Lambda_\alpha$ and $m < \mathbf{k}$ then the following set belongs to $J_{\mathbf{x},m}$:

- $\{i < \partial_{\mathbf{x},m}$: for some $\bar\nu \in \Lambda_\alpha$ we have $\bar\eta \upharpoonright (m,i) = \bar\nu \upharpoonright (m,i)\}$.

Claim 1.31: (1) *In Theorem 1.25, for any $\theta = \mathrm{cf}(\theta) \le \chi_{\mathbf{k}-1}$ Clause* (B)(b) *can be strengthened to:* $\mathbf{x}$ *has a θ-well orderable $\bar\chi$-black box.*

(2) *Parallel to Conclusion 1.27.*

2. Building Abelian groups and modules with small dual

For transparency we restrict ourselves to hereditary rings.

Convention 2.1: (1) All rings R are hereditary, i.e., if M is a free R-module then any pure sub-module N of M is free.

(2) An alternative is to interpret "G is a θ-free ring" by demanding $\mathrm{cf}(\theta) > \aleph_0$ and in the game of choosing $A_n \in [G]^{<\theta}$ increasing with n, the even player can guarantee the sub-module $\langle\bigcup_n A_n\rangle_G$ of G is free.

We shall try to use a $\bar\partial$-BB to construct Abelian groups and modules. In 2.2 we present a quite clear case: the case $\bigwedge_\ell \partial_\ell = \aleph_0$, the ring is $\mathbb{Z}$ (and the equations are simple). Note that the addition of z (in 2.2(1)(b), 2.4(1)(a)) is natural when we are trying to prove $h \in \mathrm{Hom}(G, \mathbb{Z}) \Rightarrow h(z) = 0$ which is central in this section, but is not natural for treating some other questions. When dealing with TDC_λ we may restrict ourselves to G simply derived from $\mathbf{x}$, see 2.2(3), so can ignore 2.2(1A),(2).

Definition 2.2: Let $\mathbf{x}$ be a tree-like[9] (see Definition 1.2(1)) combinatorial $\bar\partial$-parameter and let $\mathbf{k} = \mathbf{k}_\mathbf{x}$.

(1) If $k < \mathbf{k}_\mathbf{x} \Rightarrow \partial_\ell = \aleph_0$, then we say an Abelian group G is **derived from x** when

- (a) G is generated by $X \cup Y$ where:
  - (α) $X = \{x_{\bar\eta\restriction(m,n)} : \bar\eta \in \Lambda_\mathbf{x}, m < \mathbf{k}_\mathbf{x} \text{ and } n \in \mathbb{N}\} \cup \{z\}$,
  - (β) $Y = \{y_{\bar\eta,n} : \bar\eta \in \Lambda_\mathbf{x} \text{ and } n \in \mathbb{N}\}$;
- (b) moreover, generated freely except the following set of equations:

$$\Xi_\mathbf{x} = \{(n+1)y_{\bar\eta,n+1} = y_{\bar\eta,n} - \Sigma\{x_{\bar\eta\restriction(m,n)} : m < \mathbf{k}\} - a_{\bar\eta,n} z_{\bar\eta} : \bar\eta \in \Lambda_\mathbf{x} \text{ and } n \in \mathbb{N}\},$$

where

- $\bullet_1$ $z_{\bar\eta} \in \Sigma\{\mathbb{Z}x_{\bar\eta\restriction(m,n)} : \bar\eta \in \Lambda_\mathbf{x}, m < \mathbf{k}, n \in \mathbb{N}\} \oplus \mathbb{Z}z$,
- $\bullet_2$ $a_{\bar\eta,n} \in \mathbb{Z}$.

(1A) We say the Abelian group G is **canonically** derived from **x** when above we omit the $z_{\bar\eta}$'s equivalently $a_{\bar\eta,n} = 0$. If we omit z we say strictly derived.

[9] In [She07] this was not necessary, as the definition of $\eta \restriction (m, n)$ there is $\eta\restriction(m,<n+1)$ here.

(2) We say the derivation of G in part (1) is **well orderable** (or "G or $\langle z_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$ universally respect $\mathbf{x}$") when we replace $\bullet_1$ above by:

- $\bullet_1'$ there is a list $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle$ of $\Lambda_{\mathbf{x}}$ such that

$$z_{\bar\eta_\alpha} \in \Sigma\{\mathbb{Z}x_{\bar\eta_\beta \restriction (m,n)} : \beta < \alpha, m < \mathbf{k}\} \oplus \mathbb{Z}z$$

for every $\alpha < \alpha(*)$; such a sequence is called a witness.

(3) We add **simply** (derived from $\mathbf{x}$) when $z_{\bar\eta} = z$ for every $\bar\eta$.

Remark 2.3: (1) We can replace $(n+1)y_{\bar\eta,n+1}$ by $k_{\bar\eta,n}y_{\bar\eta,n+1}$ with $k_{\bar\eta,n}\in\{2,3,\ldots\}$.

(2) By combining Abelian groups, the "simply derived" is enough for cases of the TDC_λ. Instead of "simply derived" we may restrict $\langle z_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$ more than in 2.2(2).

A more general case than 2.2 is:

Definition 2.4: (1) We say an R-module G is **derived from** a combinatorial $\bar\partial$-parameter $\mathbf{x}$ when (R is a ring and):

(a) G_* is an R-module freely generated by

$$X_* = \{x_{\bar\eta\restriction(m,i)} : m < \mathbf{k}_{\mathbf{x}}, i < \partial_m \text{ and } \bar\eta \in \Lambda_{\mathbf{x}}\} \cup \{z\},$$

(b) the R-module G is generated by $\cup\{G_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\} \cup X_*$, also $G_* \subseteq G$,

(c) G/G_* is the direct sum of $\langle (G_{\bar\eta} + G_*)/G_* : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$,

(d) $Z_{\bar\eta} \subseteq X_* \subseteq G_*$ for $\bar\eta \in \Lambda_{\mathbf{x}}$; if $Z_{\bar\eta} = \{z_{\bar\eta}\}$ we may write $z_{\bar\eta}$ instead of $Z_{\bar\eta}$,

(e) if $\bar\eta \in \Lambda_{\mathbf{x}}$, then the R-submodule $G_{\bar\eta} \cap G_*$ of G is generated (not only included in the submodule generated) by

$$\{x_{\bar\eta\restriction(m,i)} : m < \mathbf{k}_{\mathbf{x}} \text{ and } i < \partial_{\mathbf{x},m}\} \cup Z_{\bar\eta} \subseteq X_*.$$

(1A) We say $\mathfrak{x}$ is an R**-construction** or $(R,\mathbf{x})$-construction when it consists of $\mathbf{x}, R, G_*, G, \langle x_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x},<\mathbf{k}}\rangle, \langle G_{\bar\eta}, Z_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$ as above and we shall write $\Lambda_{\mathfrak{x}} = \Lambda_{\mathbf{x}}, G_*^{\mathfrak{x}} = G_*, G_{\mathfrak{x}} = G, G_{\mathfrak{x},\bar\eta} = G_{\bar\eta}$, etc. (so in 2.2(1) we have a $\mathbb{Z}$-construction with $G_{\bar\eta}/(G_{\bar\eta} \cap G_*)$ being isomorphic to $(\mathbb{Q},+)$). We may say $\mathfrak{x}$ is for $\mathbf{x}$ but we may write G rather than $G_{\mathfrak{x}}$, etc. when $\mathfrak{x}$ is clear from the context.

(1B) For an R-construction $\mathfrak{x}$ we say: "universally respecting $\mathbf{x}$" or "$\mathfrak{x}$ is well orderable" when we can find $\bar\Lambda$ which $\mathfrak{x}$ obeys meaning:

(f) (α) $\bar\Lambda = \langle \Lambda_\alpha : \alpha \leq \alpha_*\rangle$ is increasing continuous,

(β) $\Lambda_{\alpha_*} = \Lambda_{\mathbf{x}}$ and $\Lambda_0 = \emptyset$,

(γ) if $\bar\eta \in \Lambda_{\alpha+1}\backslash\Lambda_\alpha$ and $m < \mathbf{k}$ then

$$\{i < \partial_m : (\exists \bar\nu \in \Lambda_\alpha)(\bar\eta \restriction (m,i) = \bar\nu \restriction (m,i)\} \in J_{\mathbf{x},m},$$

(δ) if $\bar\eta \in \Lambda_{\alpha+1}\backslash\Lambda_\alpha$ then $Z_{\bar\eta} \subseteq \langle\{G_{\bar\nu} : \bar\nu \in \Lambda_\alpha\} \cup \{z\}\rangle_G$.

(1C) We may say "G is derived from $\mathbf{x}$" and $\mathfrak{x}$ is derived from $\mathbf{x}$.

(1D) We add "simple" or "simply derived" when $z_{\bar\eta} = z$, hence $Z_{\bar\eta} = \{z\}$ for every $\bar\eta \in \Lambda$.

(1E) We say $\mathfrak{x}$ is almost simple if $|Z_{\bar\eta} \backslash \{z\}| \leq 1$.

(2) Above we say $\mathfrak{x}$ is a locally free derivation or locally free or G in part (1) is freely derived when in addition:

(g) if $\bar\eta \in \Lambda_{\mathbf{x}}, m < \mathbf{k}$ and $w \in J_{\mathbf{x},m}$, then $(G_{\bar\eta}/G_{\bar\eta,m,w})$ is a free R-module where $G_{\bar\eta,m,w}$ is the R-submodule of G generated by

$$\{x_{\bar\eta\restriction(m_1,i_1)} : m_1 < k, i_1 < \partial_{m_1} \text{ and } m_1 = m \Rightarrow i_1 \in w\} \cup Z_{\bar\eta}$$

so $G_{\bar\eta} = G^{\perp}_{\bar\eta,m,w} \oplus G_{\bar\eta,m,w}$ for some R-submodule $G^{\perp}_{\bar\eta,m,w}$ and let $\mathfrak{x}$ determine it.

(3) Above we say $\mathfrak{x}$ is $(< \theta)$-locally free or $\mathfrak{x}$ is a free $(< \theta)$-derivation when[10] in addition to part (1):

(g)$^+$ like (g) but the quotient $G_{\bar\eta}/G_{\bar\eta,m,w}$ is θ-free,

(h) $\mathbf{x}$ is θ-free.

(4) We say $\mathfrak{x}$ is a canonical R-construction or canonical $(R, \mathbf{x})$-construction when $\bar\eta \in \Lambda_{\mathbf{x}} \Rightarrow Z_{\bar\eta} = \emptyset$. We say canonically* when we omit z and we write $G^-_{\mathfrak{x}}$.

(5) We say $\mathfrak{x}$ or just $(\mathbf{x}, \bar Z)$ where $\bar Z = \langle Z_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$ is θ-well orderable when for every $\Lambda \subseteq \Lambda_{\mathbf{x}}$ of cardinality $< \theta$ there is $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle, \Lambda' \supseteq \Lambda$ witnessing which means:

(a) $\bar\eta_\alpha \in \Lambda_{\mathbf{x}}$ with no repetitions,

(b) if $\bar\eta \in \Lambda$ then

- $\bar\eta = \bar\eta_\alpha$ for some α,
- $Z_{\bar\eta} \subseteq \{\bar\eta_\beta : \beta < \alpha\}$,
- for some $m_* < \mathbf{k}$ and $w \in J_{\mathbf{x},m}$ we have

$$i \in \partial_{m_*} \backslash w \Rightarrow \eta \restriction (m_*, i) \notin \{\bar\eta_\beta \restriction (m_i, j) : m < \mathbf{k}, j < \partial_m\}.$$

Remark 2.5: In Definition 2.4, we may like in $G_{\bar\eta}$ to have more elements from G_*. This can be accomplished by replacing $x_{\bar\nu}, \bar\nu \in \Lambda_{\mathbf{x},<\mathbf{k}}$ by $x_{\bar\nu,t}$ for $t \in T_{m,i}$ when $\bar\nu = \bar\eta \restriction (m, i), \bar\eta \in \Lambda_{\mathbf{x}}$.

However, we can just as well replace ∂_ℓ by $\partial'_\ell = \gamma\partial_\ell$ for some non-zero ordinal γ (and J_ℓ by $J'_\ell = \{w \subseteq \partial'_\ell$: for some $u \in J_\ell$ we have $w \subseteq \{\gamma i + \beta : \beta < \gamma$ and $i \in u\}\}$).

[10] So "$\mathfrak{x}$ is locally free" does not imply "$\mathfrak{x}$ is θ-free" because of clause (h).

Claim 2.6: *Assume $\mathfrak{x}$ is a simple R-construction (see 2.4(1A),(1D)) which is a ($< \theta$)-locally-free (see 2.4(2) respectively) and $G = G_{\mathfrak{x}}$ so it is derived from $\mathbf{x}$ and $\mathbf{x}$ is θ-free.*

1. *G is a θ-free R-module.*
2. *If in addition $(R, +)$, that is R as an additive (so Abelian) group, is free, then $(G, +), G$ as an Abelian group, is θ-free.*
3. *In part (2) it suffices that $(R, +)$ is a θ-free Abelian group.*
4. *In (1),(2),(3) we can replace "derived" by "($< \theta$)-derived".*
5. *Instead, assuming "$\mathfrak{x}$ is simply derived" we can demand "$\mathfrak{x}$ is well orderable and almost simple"; see Definition 2.4(1B),(1E).*

Proof. (1) Let $X \subseteq G$ have cardinality $< \theta$. By the Definition 2.4(1) there are $\Lambda \subseteq \Lambda_{\mathbf{x}}$ of cardinality $< \theta$ and $\Lambda_* \subseteq \Lambda_{\mathbf{x},<\mathbf{k}}$ of cardinality $< \theta$ such that $X \subseteq \langle \{x_{\bar\eta} : \bar\eta \in \Lambda_*\} \cup \{G_{\bar\eta} : \bar\eta \in \Lambda\}\rangle_G$, recalling $\{z\} = Z_{\bar\eta} \subseteq G_{\bar\eta}$ for every $\bar\eta \in \Lambda_{\mathbf{x}}$ so, without loss of generality $X = \{x_{\bar\eta} : \bar\eta \in \Lambda_*\} \cup \{Y_{\bar\eta} : \bar\eta \in \Lambda\}$ where $Y_{\bar\eta} \subseteq G_{\bar\eta}, |Y_{\bar\eta}| < \theta$ for $\bar\eta \in \Lambda$ and $[m < \mathbf{k} \wedge i < \partial_m \Rightarrow \bar\eta \upharpoonright (m, i) \in \Lambda_*]$.

As $\mathbf{x}$ is θ-free we can find the following objects:

(a) $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle$ list Λ,
(b) $m_\alpha < \mathbf{k}_{\mathbf{x}}$ and $w_\alpha \in J_{\mathbf{x},m_\alpha}$ for $\alpha < \alpha_*$,
(c) if $\alpha < \beta$ and $i \in \partial_{\mathbf{x},m_\beta} \backslash w_\beta$ then $\eta_{\beta,m}(i) \neq \eta_{\alpha,m}(i)$.

For $\alpha \leq \alpha(*)$, let

$$G_\alpha = \langle \cup\{G_{\bar\eta_\beta} : \beta < \alpha\}\rangle_G \quad \text{and} \quad G_{\alpha(*)+1} = \langle G_{\alpha(*)} \cup \{x_{\bar\eta} : \bar\eta \in \Lambda_*\}\rangle_G.$$

So $\langle G_\alpha : \alpha \leq \alpha(*) + 1\rangle$ is an increasing continuous sequence of sub-modules, $G_0 = 0$ and $G_{\alpha(*)+1}$ includes X. Also $G_{\alpha(*)+1}/G_{\alpha(*)}$ is free by their choice above.

Lastly, if $\alpha < \alpha(*)$ then $G_{\alpha+1}/G_\alpha$ is a θ-free R-module because it is isomorphic to $G_{\bar\eta_\alpha}/G_\alpha = G_{\bar\eta_\alpha}/G_{\bar\eta_\alpha,m_\alpha,w_\alpha}$ which is θ-free by Definition 2.4(3)(g)$^+$.

So clearly we are done.

(2), (3) Follow.

(4) Similarly.

(5) Let X, Λ and Λ_* be as in the proof of part (1) and let $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle$ list Λ.

Let $\bar\Lambda$ witness the well orderability of $\mathfrak{x}$. Then (recalling Definition 2.4(1B)) there is a function h such that:

(d) $h : \alpha_* \to \ell g(\bar\Lambda)$,
(e) if $\alpha < \alpha_*$ then $\bar\eta_\alpha \in \Lambda_{h(\alpha)+1} \backslash \Lambda_{h(\alpha)}$.

Let

(f) $Z_{\bar{\eta}_\alpha} \backslash \{z\} \subseteq \{\nu_\alpha\} \subseteq \{\bar{\eta}_\beta \restriction (m, i) : \mathbf{n} < \mathbf{k}, i < \partial_m \text{ and } \beta < \alpha\} \subseteq \Lambda_*$.

Also without loss of generality, as in §1,

(g) h is non-decreasing.

Now as Λ is θ-free, as in §1, looking carefully at 2.4(1B), without loss of generality $|\Lambda_{\alpha+1} \backslash \Lambda_\alpha| \leq 1$, so without loss of generality

(g)′ h is the identity.

The rest is as before. $\blacksquare_{2.6}$

Definition 2.7: (1) An Abelian group H is (θ_2, θ_1)-**1-free** when: if $X \subseteq H$, $|X| < \theta_2$ then we can find a $\bar{G}$ such that:

- $\bar{G} = \langle G_\alpha : \alpha < \alpha(*)\rangle$ is a sequence of subgroups of G,
- $G := \sum_{\alpha<\alpha(*)} G_\alpha \subseteq H$, both G and H include X,
- G_α is generated by a set of $< \theta_1$ members,
- $G = \bigoplus_{\alpha<\alpha(*)} G_\alpha$.

(2) Similarly for R-modules when each $G_{\bar{\eta}} (\bar{\eta} \in \Lambda_{\mathbf{x}})$ has cardinality $< \theta_n$.

Claim 2.8: (1) *If* $\mathbf{x}$ *is a* $\mathbf{k}$*-c.p.,* (θ_2, θ_1)*-free (see 1.11) and* $\mathfrak{x}$ *is a canonical* $(R, \mathbf{x})$*-construction which is locally free simply derived from* $\mathbf{x}$*, then* G *is* (θ_2, θ_1)*-1-free.*

(2) *Similarly for modules.*

Proof. (1) By (2) using $R = \mathbb{Z}$.

(2) Let $G = G_{\mathfrak{x}}$ such that $|\Lambda|, |\Lambda_*| < \theta_2$ and let $X \subseteq G$ be of cardinality $< \theta_2$. Choose Λ, Λ_* as in the proof of 2.6(1). As we are assuming "$\mathbf{x}$ is (θ_2, θ_1)-free" and $\Lambda \subseteq \Lambda_{\mathbf{x}}$ has cardinality $< \theta_2$, there is a sequence $\langle \bar{\Lambda}, g, \bar{h}\rangle$ witnessing it, see 1.11(4)(d), such that $\bar{\Lambda} = \langle \Lambda_\gamma : \gamma < \gamma(*)\rangle$ and $\Lambda = \bigcup_\gamma \Lambda_\gamma$. We define the sequence $\langle G_\gamma : \gamma \leq \gamma(*) + 1\rangle$ as follows.

For $\gamma < \gamma(*)$ let G_γ be the submodule of $G_{\mathbf{x}}$ generated by

$$\cup\{G^{\perp}_{\bar{\eta}, g(\gamma), h_\gamma(\bar{\eta})} : \bar{\eta} \in \Lambda_\gamma\}.$$

We may assume that $G_0 = \{0\}$. For $\gamma = \gamma(*)$ let G_γ be the submodule of $G_{\mathbf{x}}$ generated by

$$\{x_{\bar{\nu}} : \bar{\nu} \in \Lambda_* \text{ but for no } \gamma < \gamma(*),\ \bar{\eta} \in \Lambda_\gamma,\ i \in \partial_{\mathbf{x}, g(\gamma)} \backslash h_\gamma(\bar{\eta}) \\ \text{do we have } \bar{\nu} = \bar{\eta} \restriction (g(\gamma), i)\}.$$

Finally, for $\gamma = \gamma(*) + 1$ let G_γ be $\sum_{\beta \leq \gamma(*)} G_\beta$.

For every $\gamma \le \gamma(*)+1$ let $G_{<\gamma}$ be the submodule generated by $\cup\{G_\alpha : \alpha < \gamma\}$. Notice that the sequence $\langle G_{<\gamma} : \gamma \le \gamma(*)+1\rangle$ is increasing and continuous. It suffices to prove that $G_\gamma \cap G_{<\gamma} = \{0\}$. If not, then for some n and pairwise distinct $\bar\eta_0, \dots, \bar\eta_{n-1} \in \Lambda_\gamma$,

$$\left(\sum_{\ell<n} G^{\perp}_{\bar\eta_\ell, g(\gamma), h_\gamma(\bar\eta_\ell)}\right) \cap G_{<\gamma} \neq \{0\},$$

see 2.4(2).

If $0 \neq x \in (\sum_{\ell\le n} G^{\perp}_{\bar\eta_\ell, g(\gamma), h_\gamma(\bar\eta_\ell)}) \cap G_{<\gamma}$ then there are $x_\ell \in G^{\perp}_{\bar\eta_0, g(\gamma), h_\gamma(\bar\eta_\ell)}$ for $\ell < n$ such that $x = \sum_{\ell<n} x_\ell$. Recalling

$$\text{“}G_{\mathfrak{x}}/G_* = \oplus\{G_{\bar\eta}/(G_{\bar\eta} \cap G_*) : \bar\eta \in \Lambda_{\mathbf{x}}\}\text{”}$$

necessarily $x \in G_*$; moreover, recalling 2.4(1)(c) for each $\ell < n$ we have $x_\ell \in G_*$ so $x_\ell \in G^{\perp}_{\bar\eta, g(\gamma), h_\gamma(\bar\eta_\gamma)} \cap G_*$ which is

$$\subseteq \oplus\{x_{\eta_\ell \restriction (m,i)} : m = g(\gamma) \text{ and } i \in \partial_{\mathbf{x},m} \backslash h_\gamma(\bar\eta_\ell)\} \oplus Rz,$$

see 2.4(2).

Hence

$$x = \sum_{\ell<n} x_\ell \in H_1 := \oplus\left\{x_{\bar\eta_\ell \restriction (m,i)} : \ell < n, m = g(\gamma), i \in \bigcup_{\ell_1<n} h_\gamma(\eta_{\ell_1})\right\}.$$

By the choice of $(\bar\Lambda, g, \bar h)$,

$$H_2 := G_{<\gamma} \cap G_* \subseteq \oplus\{Rx_{\bar\nu} : \text{for some } \alpha<\gamma,\ \bar\eta\in\Lambda_\alpha,\ m=g(\alpha),\ i<\partial_{\mathbf{x},g(\alpha)},\ i\notin h_\alpha(\bar\eta), \\ \bar\nu = \bar\eta \restriction (m,i)\}.$$

Hence $x \in H_1 \cap H_2 = \{0\}$, contradiction. $\blacksquare_{2.8}$

Claim 2.9: *Assume* $\mathbf{x}$ *is an* $(\aleph_0, \mathbf{k})$*-c.p. with* $(\aleph_0, \mathbf{k})$*-BB.*

(1) *There is canonical* $\mathbb{Z}$*-construction* $\mathfrak{x}$ *such that:*

(a) $G = G_{\mathfrak{x}}$ *so* G *is an Abelian group of cardinality* $|\Lambda_{\mathbf{x}}|$,

(b) G *is not Whitehead,*

(c) G *is* θ*-free if* $\mathbf{x}$ *is* θ*-free,*

(d) G *is* (θ_2, θ_1)*-1-free if* $\mathbf{x}$ *is* (θ_2, θ_1)*-free, see 2.7(1),*

(e) G *has a* $\mathbb{Z}$*-adic dense subgroup of cardinality* $|\Lambda_{\mathbf{x},<\mathbf{k}}|$.

(2) *We can add:*

$(\mathrm{b})^+$ $\mathrm{Hom}(G, \mathbb{Z}) = 0$.

Remark 2.10: Recall that "$\mathbf{b}$ is a $(\chi, \mathbf{k})$-BB" means $\mathbf{b}$ is a function with range $\subseteq \chi$, see Definition 1.7.

Proof. (1) Let $G_0 = \oplus\{\mathbb{Z}x_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x},<\mathbf{k}}\} \oplus \mathbb{Z}z$ and G_1 be the $\mathbb{Z}$-adic closure of G_0 so G_1 is a complete metric space under the $\mathbb{Z}$-adic metric.

For $\bar\eta \in \Lambda_{\mathbf{x}}$ and $\bar a \in {}^\omega\mathbb{Z}$ and $n(*)$, in G_1 we let

$$y_{\bar a,\bar\eta,n(*)} = \sum_{n \geq n(*)} (n!/n(*)!)\Big(\sum_{m<\mathbf{k}} x_{\bar\eta\restriction(m,n)} - \sum_{m<\mathbf{k}} \mathbf{b}(\eta,m,n)z + a_n z\Big).$$

Let $\{b_i : i < \omega\}$ list the elements of $\mathbb{Z}$ and let $\bar{\mathbf{c}} = \langle \mathbf{c}_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$ be an $(\aleph_0, \mathbf{k})$-BB with $\mathbf{c}_{\bar\eta}$ a function from $\{\bar\eta \restriction (m,n) : m < \mathbf{k}, n < \omega\}$ to $\mathbb{Z}$. Now for each $\bar\eta \in \Lambda_{\mathbf{x}}$ let

$$G^0_{\bar\eta} = \Sigma\{\mathbb{Z}x_{\bar\eta\restriction(m,n)} : m < \mathbf{k}, n < \omega\} \oplus \mathbb{Z}z$$

and $h_{\bar\eta} \in \mathrm{Hom}(G^0_{\bar\eta}, \mathbb{Z}, z)$ be such that $h_{\bar\eta}(z) = z, h_{\bar\eta}(x_{\bar\eta\restriction(m,n)}) = b_{\mathbf{c}_{\bar\eta}(\bar\eta\restriction(m,n))}z$.

$(*)_1$ We can choose $\bar a = \bar a[\bar\eta] \in {}^\omega\mathbb{Z}$ such that there is no extension $h^1 \in \mathrm{Hom}(G^1_{\bar a,\bar\eta}, \mathbb{Z})$ of $h_{\bar\eta}$ where $G^1_{\bar a,\bar\eta} = \langle G^0_{\bar\eta} \cup \{y_{\bar a,\bar\eta,n} : n < \omega\}\rangle_{G_1}$.

[Why? Well known but we elaborate. It suffices to prove that

$$\mathscr{A} = \{\bar a \in {}^\omega 2 : h_{\bar\eta} \text{ has an extension in } \mathrm{Hom}(G^1_{\bar a,\bar\eta}, \mathbb{Z}) \text{ and } a_0 = 0 = a_1\}$$

is a countable subset of ${}^\omega\mathbb{Z}$; we could have allowed $\bar a \in {}^\omega\mathbb{Z}$ but this seems more transparent to restrict ourselves. For $\bar a \in \mathscr{A}$ let $h_{\bar a,\bar\eta}$ be an extension witnessing it.

Now

- For each $b \in \mathbb{Z}$ the set

  $$\mathscr{A}_b = \{\bar a \in \mathscr{A} \subseteq {}^\omega 2 : h_{\bar a,\bar\eta}(y_{\bar\eta,0}) = b \text{ and so } a_0 = a_1 = 0\}$$

  has at most one member.

[Why? Toward contradiction assume $\bar a_1 \neq \bar a_2 \in \mathscr{A}_b$ and let n be minimal such that $a_{1,n} \neq a_{2,n}$; now $n = 0, 1$ is impossible as $\bar a_1, \bar a_2 \in \mathscr{A}_b$, so $n \geq 2$. Now prove by induction on $\ell < n$ that $h_{\bar a_1,\bar\eta}(y_{\bar\eta,\ell}) = h_{\bar a_2,\eta}(y_{\bar\eta,\ell})$; for $\ell = 0, 1$ use $\bar a_1, \bar a_2 \in \mathscr{A}_b$ and for $\ell = j+1$ recall $\ell y_{\eta,\ell} = y_{\eta,j} - (\sum_{m<\mathbf{k}} x_{\bar\eta\restriction(m,j)} + a_{\iota,j}z)$ for $\iota = 1, 2$; apply $h_{\bar a_\iota,\bar\eta}$ and use the induction hypothesis. Now on this equation for $\ell = n$, $\iota = 1, 2$ apply $h_{\bar a_\iota,\bar\eta}$ and then substracting we get $a_{1,n} - a_{2,n}$ is divisible by ℓ and $\ell \geq 2$ but $a_{1,n} - a_{2,n} \in \{1, -1\}$, contradiction.]

So clearly there is $\bar a \in {}^\omega 2 \setminus \cup\{\mathscr{A}_b : b \in \mathbb{Z}\}$ such that $a_0 = 0 = a_1$; it is as required. So $(*)_1$ holds indeed.]

Lastly,

$(*)_2$ let $G_1 = \langle G_0 \cup \{y_{\bar{a}[\bar{\eta}],\bar{\eta},n} : n < \omega\} : \bar{\eta} \in \Lambda_{\mathbf{x}}\}\rangle_{G_1}$.

Now G_1 witnesses that $G_2 = G_1/\mathbb{Z}z$ is not a Whitehead group. [Why? Let $G_2 = G_1/\mathbb{Z}z$ and let h_* be the canonical homomorphism from G_1 onto $G_1/\mathbb{Z}z$, i.e., $h_*(x) = x + \mathbb{Z}z$ for $x \in G_1$. Toward contradiction assume G_2 is a Whitehead group; this means that there is a homomorphism g_* from G_2 into G_1 inverting h_*, that is, $y \in G_2 \Rightarrow h_*(g_*(y)) = y$.

As $\mathrm{Ker}(g_*) = \mathbb{Z}z$ clearly $x \in G_1 \Rightarrow g_*(h_*(x)) - x \in \mathbb{Z}z$, so let h_ℓ be the unique function from $\Lambda_{\mathbf{x},<\mathbf{k}}$ into $\mathbb{Z}$ defined by $h_0(\bar{\nu}) = b$ iff $\bar{\nu} \in \Lambda_{\mathbf{x},<\mathbf{k}}, k \in \mathbb{Z}$ and $g_*(h_*(x_{\bar{\nu}})) - x_{\bar{\nu}} = bz$. By the choice of $\mathbf{b}$ there is $\bar{\eta} \in \Lambda_{\mathbf{x}}$ such that $m < \mathbf{k} \wedge n < \omega \Rightarrow \mathbf{k}(\bar{\eta}, m, n) = h_\bullet(\bar{\eta} \upharpoonright (m,n))$. So $x \mapsto x - g_*(h_*(x))$ defines a homomorphism from $G_{\bar{a}(\bar{\eta}),\bar{\eta}}$ onto $\mathbb{Z}z$ mapping z to itself and mapping $x_{\bar{\eta}\upharpoonright(m,n)}$ to $b_{\mathbf{c}_\eta(\bar{\eta}\upharpoonright(m,n))}z$, contradicting the choice of $\bar{a}(\eta)$. So $G_2 = G_1/\mathbb{Z}z$ is Whitehead indeed.

Now clearly for some canonical* $\mathbb{Z}$-construction $\mathfrak{x}, G_{\mathfrak{x}} = G^-_{\mathfrak{x}} \oplus \mathbb{Z}z$, and easily $G_2 \cong G^-_{\mathfrak{x}}$ and G_2 is a direct summand of $G_{\mathfrak{x}}$ so (by the well known group theory) also $G_{\mathfrak{x}}$ is not a Whitehead group. The cardinality and freeness demands are obvious.]

(2) For transparency we ignore the "Whitehead". Recall we assume $\mathbf{x}$ has the $\aleph_0$-black box not just the $\aleph_0$-pre-black box (see 1.7(1),(4)).

Let $\langle \Lambda_\alpha : \alpha < |\Lambda_{\mathbf{x}}|\rangle, \langle \bar{\nu}_\alpha : \alpha < \alpha_*\rangle$ be as in Definition 1.7(4). Let $\langle h_{\bar{\eta}} : \bar{\eta} \in \Lambda_\alpha\rangle$ be an $\aleph_0$-BB. We choose $(\mathbb{Z}, \mathbf{x})$-construction $\mathfrak{x}$ by choosing $(z_{\bar{\eta}}, \bar{a}_{\bar{\eta}})$ for $\bar{\eta} \in \Lambda_\alpha$ by induction on α such that:

- $\bullet_1$ $z_{\bar{\eta}} = z_0 = z$ if $\alpha = 0$ (alternatively, omit z),
- $\bullet_2$ $z_{\bar{\eta}} = z_\alpha = x_{\bar{\nu}_\alpha}$ if $\bar{\eta} \in \Lambda_{1+\alpha}$,
- $\bullet_3$ $\bar{a}_{\bar{\eta}}$ for $\bar{\eta} \in \Lambda_\alpha$ is chosen such that: there is no homomorphism h from $G_{\bar{\eta}}$ into $\mathbb{Z}$ such that $(h(x_{\bar{\eta}\upharpoonright(m,i)}), h(z_\alpha))$ is coded by $h_{\bar{\eta}}(\bar{\eta} \upharpoonright (m,i))$.

So if $h \in \mathrm{Hom}(G_{\mathfrak{x}}, \mathbb{Z})$ then $\alpha < \alpha_* \Rightarrow h(z_\alpha) = 0$ but $\oplus \sum_\alpha \mathbb{Z}z_\alpha = G_0$, so $h{\restriction}G_0$ is zero but $G_{\mathfrak{x}}/G_0$ is divisible hence h is zero.

Alternatively omitting "$G = G_{\mathfrak{x}}$", this follows easily by repeated amalgamation of the G constructed in part (1) over pure subgroups isomorphic to $\mathbb{Z}$; see the proof of 2.12(3) or, e.g., [She16, §3]. $\blacksquare_{2.9}$

* * *

Now Claim 2.9 as stated is enough when we use §1 to get $\aleph_{\omega \cdot n}$-free $\mathbf{x}$ with χ-BB (see 1.27(1),(2)) but not for $\aleph_{\omega_1 \cdot n}$-free, because there we need for $\partial = \aleph_1$, $J = J^{\rm bd}_\kappa \odot J^{\rm bd}_\sigma$, $\sigma < \kappa$ regular, in particular $(\sigma, \kappa) = (\aleph_0, \aleph_1)$. So we better use the construction from Definition 2.4 rather than 2.2. Also we prefer to have general R-modules and we formalize the relevant property of $R, \bar\partial, \bar J, \theta$. We use ${}_R R$ to denote R as a left R-module.

Definition 2.11: (1) We say that $(\bar\partial, \bar J)$ does θ**-fit** R or the triple $(\bar\partial, \bar J, \theta)$**-fit** R (but if $\bar\partial = \bar\partial_{\mathbf{x}}, \bar J = \bar J_{\mathbf{x}}$ then we may write $\mathbf{x}$ instead of $(\bar\partial, \bar J)$) when:

(A) (a) R is a ring,
- (b) $\mathbf{k}$ is a natural number ≥ 1,
- (c) $\bar\partial = \langle \partial_\ell : \ell < \mathbf{k} \rangle$,
- (d) ∂_ℓ is a regular cardinal,
- (e) $\bar J = \langle J_\ell : \ell < \mathbf{k} \rangle$,
- (f) J_ℓ is an ideal on ∂_ℓ.

(B) If $G_0 = \oplus\{Rx_{m,i} : m < \mathbf{k}, i < \partial_m\} \oplus Rz$ and $h \in \mathrm{Hom}(G_0, {}_R R)$ and $h(z) \neq 0$, then there is G_1 such that
- $(*)$ (α) G_1 is an R-module extending G_0,
  - (β) G_1 has cardinality $< \theta$,
  - (γ) there is no homomorphism from G_1 to ${}_R R$ (i.e., R as a left R-module) extending h.

(1A) We replace "fit" by "weakly fit" when in clause (B) we further demand on $h, h(x_{m,2i}) = h(x_{m,2i+1})$.

(2) We say $(\bar\partial, \bar J)$ freely θ-fits R or $(\bar\partial, \bar J, \theta)$-fit R (but if $\bar\partial = \bar\partial_{\mathbf{x}}, \bar J = \bar J_{\mathbf{x}}$ we may replace $(\bar\partial, \bar J)$ by $\mathbf{x}$) when:

(A) (a)–(f) as above,

(B) as above adding
- (δ) if $m_* < \mathbf{k} \wedge w \in J_{m_*}$, then G_1 is free over

$$\oplus\{Rx_{m,i} : m < \mathbf{k}, i < \partial_m \text{ and } m = m_* \Rightarrow i \in w\} \oplus Rz.$$

(3) In part (1) above and also parts (4)–(6) below we may write $(\partial, J, \mathbf{k})$ instead of $(\langle \partial_\ell : \ell < \mathbf{k} \rangle, \langle J_\ell : \ell < \mathbf{k} \rangle)$ when $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial \wedge J_\ell = J$ so we may write $(\partial, J, \mathbf{k}, \theta)$. Also we may write J if $m < \mathbf{k} \Rightarrow J_m = J$ and omit $\bar J$ when $\ell < \mathbf{k} \Rightarrow J_\ell = J^{\rm bd}_{\partial_\ell}$.

(4) We may above replace "J_ℓ is an ideal on ∂_ℓ" by $J_\ell \subseteq \mathscr{P}(\partial_\ell)$.

(5) We may omit θ when $\theta = |R|^+ + \max\{\partial_m^+ : m < \mathbf{k}\}$.

(6) We replace fit by $\mathbb{I}$-fit when:

(a) $\mathbb{I}$ is a set of ideals of R closed under intersection of two including $I_0 = \{0_R\}$,

(b) replace Rz by $(R/I)z$, $I \in \mathbb{I}$; the default value of $\mathbb{I}$ is

$$\{\{a : ab = 0\} : b \in R\},$$

(c) in $(B)(*)$, if $x \in G_1 \backslash \{0\}$ then $\operatorname{ann}(x, G_1) = \{a \in R : ax = 0\} \in \mathbb{I}$.

Claim 2.12: (1) *Assume* $\mathbf{x}$ *is a* $\mathbf{k}$*-c.p.,* R *is a ring,* $\mathbf{x}$ *does* θ*-fit* R*,* $\chi^+ \geq \theta + |R|^+$ *and* $\mathbf{x}$ *has* $(\chi, \mathbf{k}, 1)$*-BB.*

There is $\mathfrak{x}$ *such that:*

(a) $\mathfrak{x}$ *is an* $(R, \mathbf{x})$*-construction,*

(b) $G = G_{\mathfrak{x}}$ *is an* R*-module of cardinality* $|\Lambda_{\mathbf{x}}|$*,*

(c) *there is no* $h \in \operatorname{Hom}(G, {}_R R)$ *such that* $h(z) \neq 0$*,*

(d) $\mathfrak{x}$ *is simple, that is,* $z_{\bar{\eta}} = z$ *for* $\bar{\eta} \in \Lambda_{\mathbf{x}}$*.*

(2) *If in addition* $\mathbf{x}$ *freely* θ*-fits* R*, then we can add:*

(e) G *is* σ*-free if* $\mathbf{x}$ *is* σ*-free (holds always for* $\sigma = \min(\bar{\partial}_{\mathbf{x}})$*),*

(f) G *is* (θ_2, θ_1)*-1-free if* $\mathbf{x}$ *is* (θ_2, θ_1)*-free.*

(3) *In (2) we can add:*

(g) $\operatorname{Hom}(G, {}_R R) = 0$.

(4) *We can use above "weakly fit".*

Proof. Let $G_* = \oplus\{Rx_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x},<\mathbf{k}}\} \oplus Rz$. See more in [Shee].

(1) Let $\{(a^1_\varepsilon, a^2_\varepsilon) : \varepsilon < \chi\}$ list, possibly with repetitions, the members of $R \times (R \backslash \{0_R\})$ and let $\mathbf{b}$ be a $(\chi, \mathbf{k}, 1)$-BB for $\mathbf{x}$ and let $\mathbf{b}', \mathbf{b}''$ be defined such that: $\varepsilon = \mathbf{b}_{\bar{\eta}}(m, i)$ implies $\mathbf{b}'_{\bar{\eta}}(m, i) = a^1_\varepsilon, \mathbf{b}''_{\bar{\eta}}(m, i) := a^2_\varepsilon$.

For $\bar{\eta} \in \Lambda_{\mathbf{x}}$ let $G^0_{\bar{\eta}} = \Sigma\{Rx_{\bar{\eta}\restriction(m,i)} : m < \mathbf{k}, i < \partial_m\} \oplus Rz \subseteq G_*$ and let $h_{\bar{\eta}}$ be the unique homomorphism from $G^0_{\bar{\eta}}$ into ${}_R R$ satisfying $h_{\bar{\eta}}(x_{\bar{\eta}\restriction(m,i)}) = \mathbf{b}'_{\bar{\eta}}(m, i)$ and $h_{\bar{\eta}}(z) = \mathbf{b}''_{\bar{\eta}}(0, 0)$ and let $G^1_{\bar{\eta}}$ be an R-module extending $G^0_{\bar{\eta}}$ such that $(G^1_{\bar{\eta}}, G^0_{\bar{\eta}}, h_{\bar{\eta}})$ here are like (G_1, G_0, h) in Definition 2.11(1)(B)(*), so in particular there is no homomorphism from $G^1_{\bar{\eta}}$ into ${}_R R$ extending $h_{\bar{\eta}}$. Without loss of generality $G^1_{\bar{\eta}} \cap G_0 = G^0_{\bar{\eta}}$ and $\langle G^1_{\bar{\eta}} \backslash G^0_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}}\rangle$ is a sequence of pairwise disjoint sets. Let G be the R-module generated by $\cup\{G^1_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}}\} \cup G_0$ extending each $G^1_{\bar{\eta}}$ and G_*, freely except this. Clearly we have defined an R-construction $\mathfrak{x}$ with $\mathbf{x}_{\mathfrak{x}} = \mathbf{x}$, $G_{\mathfrak{x}} = G$, $z_{\mathfrak{x},\bar{\eta}} = \{z\}$ and clauses (a),(b),(d) of the desired conclusion hold. To prove clause (c) toward contradiction assume that $h \in \operatorname{Hom}(G, {}_R R)$ satisfies $h(z) \neq 0$. Let $g : \Lambda_{\mathbf{x},<\mathbf{k}} \to \chi$ be defined by

$$g(\bar{\nu}) = \min\{\varepsilon < \chi : (h(x_{\bar{\nu}}), h(z)) = (a^1_\varepsilon, a^2_\varepsilon)\}.$$

Clearly the function is well defined, hence as $\mathbf{x}$ has $(\chi, \mathbf{k}, 1)$-BB, that is by the choice of $\mathbf{b}$ there is $\bar\eta \in \Lambda_{\mathbf{x}}$ such that $m < \mathbf{k} \wedge i < \partial_m \Rightarrow g(\bar\eta \restriction (m,i)) = \mathbf{b}_{\bar\eta}(m,i)$. We get easy contradiction.

What about the cardinality $|G|$? Note that $|G^1_{\bar\eta}| < \theta$ and $\theta \le \chi^+$.

(2) In the proof of part (1), choosing $G^1_{\bar\eta}$ we add the parallel of clause $(*)(\delta)$ of 2.11(B). Now clause (e) of 2.12(2) holds by 2.6(1) and clause (f) by 2.8(2).

(3) Let G be as constructed in part (1), and let

$$Y = \{y \in G : G/Ry \text{ is } \aleph_1\text{-free or even } \min(\bar\partial)^+\text{-free}\}$$

(recall 2.6 + freeness of $\mathbf{x}$).

So by part (2) the set Y generates G, let $\langle G_\varrho, h_\rho : \varrho \in {}^{\omega>}Y\rangle$ be such that G_ϱ is an R-module, h_ϱ is an isomorphism from G onto G_ϱ, without loss of generality $0_{G_\varrho} = 0$ for every ϱ and $G_{\varrho_1} \cap G_{\varrho_2} = \{0\}$ for $\varrho_1 \neq \varrho_2$.

Let $H_1 = \oplus\{G_\varrho : \varrho \in {}^{\omega>}Y\}$ and let H_0 be the R-submodule of H_1 generated by

$$X = \{h_{\varrho\char94\langle y\rangle}(z) - h_\varrho(y) : \varrho \in {}^{\omega>}Y \text{ and } y \in Y\}.$$

Let $H = H_1/H_0$ and we shall prove it is as required (on G), the main point is proving $\mathrm{Hom}(H, {}_RR)$. That is, toward contradiction $f_0 \in \mathrm{Hom}(H, {}_RR)$ is not zero and $f_1 \in \mathrm{Hom}(H_1, {}_RR)$ is defined by $f_1(x) = h(x + H_0)$, so also f_1 is not zero but $x \in X \Rightarrow f_1(x) = 0$. By the choice of H_1, there is $\varrho \in {}^{\omega>}Y$ such that $f_1 \restriction G_\varrho$ is not zero. But recall that G is generated by Y, hence G_ϱ is generated by $\{f_1, h_\varrho(y) : y \in Y\}$, hence for some $n \ge 1$ and $y_0, \dots, y_{n-1} \in Y$ and $b_0, \dots, b_{n-1} \in R \backslash \{0_R\}$ we have $f_1(h_\rho(\sum_{\ell<n} b_\ell, y_\ell)) \in R\backslash\{0\}$ hence for some $\ell < n$,

$$0 \neq f_1(h_\varrho(b_\ell y_\ell)) = f_1(b_\ell h_\varrho(y_\ell)).$$

So letting $y = h_\varrho(y_\ell)$ we have $y \in G_\varrho$ and for some $b \in R\backslash\{0\}$,

$$c = f_1(bh_\varrho(y_\ell)) = f_2(by).$$

As said above about f_1 we have

$$f_1(y) = f_1(h_\varrho(y_\ell)) = f_1(h_{\varrho\char94\langle y_\ell\rangle}(z))$$

so $f_1(h\varrho\char94\langle y_\ell\rangle(z)) = b \in R\backslash\{0\}$. So $h_{\varrho\char94\langle y_\ell\rangle} \circ f_1 \in \mathrm{Hom}(G, {}_RR)$ maps z into $b \in R\backslash\{0\}$, contradiction.

(4) Similarly, but replacing $x_{\bar\eta}(\bar\eta \in \Lambda_{\mathbf{x},<\mathbf{k}})$ by $x_{\bar\eta,\zeta}(\zeta < |R|^+)$, but we elaborate.

Let $\langle(\bar\alpha_\varepsilon, a_\varepsilon) : \varepsilon < \chi\rangle$ list, possibly with repetitions, the members of

$$\{(\bar\alpha, a_\varepsilon) : \bar\alpha = (\alpha_0, \alpha_1) \text{ such that } \alpha_0 < \alpha_1 < \chi \text{ and } a \in R\backslash\{0_R\}\},$$

and $\mathbf{b}$ be a $(\chi,\mathbf{k},1)$-BB for $\mathbf{x}$ and let $\mathbf{b}^\iota$ for $\iota=0,1,2$ be the functions with the same domain as $\mathbf{b}$ (writing $\mathbf{b}^\iota_{\bar\eta}(m,i)$ or $\mathbf{b}_{\eta,\iota}(m,i)$ for $\mathbf{b}^\iota(\bar\eta,m,i)$) such that $\varepsilon=\mathbf{b}_{\bar\eta}(m,i)$ implies

$$(\alpha_{\varepsilon.0},\alpha_{\varepsilon,1},a_\varepsilon)=(\mathbf{b}^0_{\bar\eta}(m,i),\mathbf{b}^1_{\bar\eta}(m,i),\mathbf{b}^2_{\bar\eta}(m,i)).$$

Let $G_0=\oplus\{Rx_{\bar\eta,\varepsilon}:\bar\eta\in\Lambda_{\mathbf{x},<\mathbf{k}}$ and $\varepsilon<\chi\}$ and

$(*)_1$ for $\bar\eta\in\Lambda_{\mathbf{x}}$ let

(a) $G^0_{\bar\eta}=\Sigma\{Rx_{\bar\eta\restriction(m,i),\varepsilon}:m<\mathbf{k},i<\partial_m$ and $\varepsilon<\chi\}$,

(b) $G^{0,0}_{\bar\eta}=\Sigma\{R(x_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,1}(m,i)}-x_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,0}(m,i)}):m<\mathbf{k},i<\partial_m\}\oplus Rz$,

(c) $G^{0,1}_{\bar\eta}=G^{0,0}\oplus Rz$,

(d) $h_{\bar\eta}$ be the homomorphism from $G^{0,0}_{\bar\eta}$ into R such that:

- $h_{\bar\eta}\restriction G^{0,0}$ is constantly zero,
- $h_{\bar\eta}(z)$ is $\mathbf{b}_{\bar\eta,2}(0,0)\in R\backslash\{0\}$,

(e) $\mathbf{h}_{\bar\eta}$ be the isomorphism from $G_0=\oplus\{Rx_{\bar\eta\restriction(m,i)}:m<\mathbf{k},i<\partial_m\}\oplus R_\eta$ onto $G^{0,1}_{\bar\eta}$ such that

$$\mathbf{h}_{\bar\eta}(z)=z,\mathbf{h}_{\bar\eta}(x_{m,i})=(x_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,1}(m,i)}-x_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,0}(m,i)}),$$

(f) $G^\bullet_{\bar\eta,1}$ be an R-module extending the R-module $G^\bullet_{\bar\eta}$ such that the triple $(G^\bullet_0,G_{\bar\eta,1},h^\bullet_{\bar\eta}\circ\mathbf{h}_{\bar\eta})$ is as in 2.11(1)(B)$(*)$,

(g) $\mathbf{h}^+_{\bar\eta},G^1_{\bar\eta}$ be such that $G^1_{\bar\eta}$ is an R-module extending $G^0_{\bar\eta}$ and $h^+_{\bar\eta}$ is an isomorphism from $G^\bullet_{\bar\eta,1}$ onto $G^1_{\bar\eta}$ extending $\mathbf{h}_{\bar\eta}$.

Lastly, let

$(*)$ without loss of generality $G^1_{\bar\eta}\cap G_0=G^{0,0}_{\bar\eta}$, $\langle G^1_{\bar\eta}\backslash G^{0,0}_{\bar\eta}:\bar\eta\in\Lambda_{\mathbf{x}}\rangle$ are pairwise disjoint and G^*_1 is an R-module extending G_0 and $G^1_{\bar\eta}$ for $\bar\eta\in\Lambda_{\mathbf{x}}$ and generated by their union freely (except the equations implicit in "extending" above).

Note:

$(*)$ if $h\in\mathrm{Hom}(G,{}_RR)$ satisfies $h(z)\neq 0_R$ then we define a function $\mathbf{c}:\Lambda_{\mathbf{x},<\mathbf{k}}\to\chi$ as follows: $\mathbf{c}$ is the minimal $\varepsilon<\chi$ such that:

- $h(x_{\bar\eta,\alpha_{\varepsilon,0}})=h(x_{\bar\eta,\alpha_{\varepsilon,1}})$,
- $h(z)=a_\varepsilon$.

The rest should be clear. $\blacksquare_{2.12}$

Remark 2.13: We can use a 2^χ-BB $\mathbf{b}$ and then let $\mathbf{c}(\bar\eta)$ code

$$(h\restriction\{x_{\bar\eta,\varepsilon}:\varepsilon<\chi\},h(z)).$$

Remark 2.14: (1) There is an alternative to the proof of 2.12(3), assume that $\mathbf{x}$ has $\aleph_0$-well orderable $(\chi, \mathbf{k}, 1)$-BB $\bar{\alpha}$ as witnessed by $\bar{\Lambda}$, see Definition 1.30. We then can find a $(R, \mathbf{x})$-construction obeying $\bar{\Lambda}$, see 2.4(1B).

(2) It may suffice for us to prove in 2.12 that $\mathfrak{x}$ is simple and Rz is not a direct summand of the R-module $G_{\mathfrak{x}}$. For this we can weaken the demand in Definition 2.11(1)(B) demanding $h(z) = 1_R$.

CLAIM 2.15: (1) *Let $\partial = \aleph_0, J = J^{\rm bd}_\partial$ and $\mathbf{k} = 1$; then $(\partial, J, \mathbf{k}, \theta)$ freely fit R when:*

$\oplus_1$ (a) *R is an infinite ring,*

(b) *if $d \in R\backslash\{0\}$ and $\bar{d} \in {}^\omega R$, then we can find $a^\iota_n \in R$ for $\iota = 1, 2, 3$ and $n < \omega$ such that the following set Γ of equations cannot be solved in R:*

$$\Gamma = \{a_n x_{n+2} = x_n + d_n + b_n d : n < \omega\}.$$

(2) *For $\partial, J, \mathbf{k}$ as above, $(\partial, J, \mathbf{k}, \theta)$ freely weakly fit R when:*

$\oplus_2$ (a) *as above,*

(b) *for every $d \in R\backslash\{0\}$ letting $\bigwedge_n d_n = 0_R$, the demand in $\oplus$ above holds, i.e., there are $a_n, b_n \in R$ for $n < \omega$ such that the following set Γ of equations is not solved in R:*

- $\Gamma = \{a_n x_{n+1} = x_n + b_n d : n < \omega\}$.

(3) *If R is an infinite ring, then $\oplus_1$ holds when:*

$\oplus_3$ (a) *as above,*

(b) *$(R, +)$ is $\aleph_1$-free or at least $\cap\{nR : n \geq 2\} = \{0\}$.*

Proof. (1) We should check all the clauses in Definition 2.11. First, Clause (A) is obvious: R is a ring by $\oplus_1$(a), $\mathbf{k} = 1 > 0$ by assumption, of course, letting $\bar{\partial} = \langle \partial_0 \rangle$, $\partial_0 = \partial$, ∂ is regular being $\aleph_0$ and $\bar{J} = \langle J_0 \rangle$, $J_0 = J$ is $J^{\rm bd}_\partial = J^{\rm bd}_{\aleph_0}$ so an ideal on ∂.

Second, toward proving Clause (B), assume

$$G_0 = \oplus\{Rx_{m,i} : m < \mathbf{k} = 1 \text{ so } m = 0 \text{ and } i < \partial\} \oplus \mathbb{Z}z,$$

$h_0 \in \operatorname{Hom}(G_0, {}_RR)$ and $d := h_0(z) \neq 0_R$ and let $d - n = h(x_{0,n})$. We should find G_1 satisfying $(*)$ there. Let

$$\langle (a^\bullet_n, b_n) : n < k \rangle$$

be as guaranteed by $\oplus_1$(b) of the claim for d, $\langle d_n : n < \omega \rangle$ from above.

For each $i < \partial$ let $G_n^* = G_0 \oplus (Ry_n)$ be an R-module; clearly there is an embedding $g_n : G_n^* \to G_{n+1}^*$ such that $g_n{\restriction}G_0 = \mathrm{id}_{G_0}$ and

$$g_n(y_n) = a_n^\bullet y_{n+1} + x_{0,n} + b_n z$$

where the $a_n, b_n \in R$ are from $\oplus_1$(b) for our h.

Renaming without loss of generality $G_n^* \subseteq G_{n+1}^*$ and g_n is the identity on G_n^*. Lastly, let

$$G_1 = \bigcup_n G_n^*$$

and it suffices to prove that $(*)$ of Definition 2.11 is satisfied. Clearly G_1 is an R-module extending G_0, i.e., $(*)(\alpha)$ holds. Also

$$|G_1| \le \aleph_0 + |G_0| = \aleph_0 + \aleph_0 \cdot |R| = |R| < |R|^+ = \theta,$$

recalling R is an infinite ring, so also $(*)(\beta)$ holds.

Lastly, to prove $(*)(\gamma)$, toward contradiction assume $h_2 \in \mathrm{Hom}(G_1, {}_RR)$ extends h. Let $c_n := h_2(y_n) \in R$. Now

$(*)$ (a) $\bar{c} = \langle c_n : n < \omega \rangle \in {}^\omega\mathbb{R}$,

(b) $$\begin{aligned} a_n c_{n+1} =& a_n h_1(y_{n+1}) = h_2(a_\eta^1 y_{n+1}) \\ =& h_2(y_n + x_{0,n} + b_n z) = h_2(y_n) + h_1(x_{0,n}) + b_n h_2(z) \\ =& c_n + d_n + b_n d. \end{aligned}$$

So $\bar{c}$ solves (in R) the set of equations

$$\Gamma = \{a_n z_{n+1} = z_n + d_n + b_n d : n < \omega\},$$

contradicting the choice of $\langle (a_n, h_n) : n < \omega \rangle$.

We still have to justify the "freely", i.e., clause (δ) of 2.11(2). So let $m_* < \mathbf{k}$, i.e., $m_* = 0$ and $w \in J_0 = J_\partial^{\mathrm{bd}}$ so w is finite and let

$$G_0 = \oplus\{Rx_{0,i} : i \in w\},$$

let n_* be such that $\sup(w) < n_*$ and we easily finish by noting:

$(*)$ the sequence $\langle y_n : n > n_* \rangle\hat{}\langle x_{0,m} : m \le n^* \rangle\hat{}\langle z \rangle$ generates G_1.

[Why? Freely, it generates G_1 because $x_{0,m} = a_n y_{m+2} - b_m y_m$ for $m > n_*$, use $y_n = a_n y_{n+1} - x_{0,n} - b_n z$ by downward induction on $n \le n_*$; translating the equations they become trivial.]

(2) Similarly but we choose g_n such that

$$g_n(y_n) = a_n y_{n+1} + (x_{0,2n} - x_{0,2n+1}) + b_n z_n.$$

(3) Choose $b_n = 1_R, a_n : n! \cdot 1_R$. $\blacksquare_{2.15}$

CLAIM 2.16: (1) *The quadruple* $(\partial, J, \mathbf{k}, \theta)$ *freely fit* $\mathbb{Z}$ *when:*

(a) $\theta = \aleph_2, \partial = \aleph_1$ *and* $\mathbf{k} > 0$

(b) $J = J^{\rm bd}_{\aleph_1} \times J^{\rm bd}_{\aleph_0}$, *but pedantically use the isomorphic copy*

$$J_{\aleph_1 * \aleph_0} = \{A : \text{for some } n_\alpha < \omega \text{ for } \alpha < \omega_1, i_* < \omega_1$$
$$\text{we have } A \subseteq \{\omega \cdot i + n : i < i_* \text{ or } n < n_\alpha\}\};$$

better, it is also O.K. to use $J = J^{\rm bd}_{\aleph_1} \odot J^{\rm bd}_{\aleph_0}$.

(2) *The quadruple* $(\aleph_1, J, \mathbf{k}, \theta)$ *freely fits* R *when:*

(a),(b) *as above,*

(c) $\theta = \aleph_2$,

(d) *given* $b_{\alpha,n} \in R$ *for* $\alpha < \omega_1, n < \omega$ *and* $t \in R \backslash \{0_R\}$, *there are pairwise distinct* $\rho_\alpha \in {}^\omega 2$ *for* $\alpha < \omega_1$ *and* $a_{\alpha,n}, d_{\alpha,n} \in R$ *such that the following set of equations is not solvable in* R*:*

- $d_{\alpha,n+1} y^1_{\alpha,n+1} = y^1_{\alpha,n} - y^2_{\rho_\alpha \restriction n} - b_{\alpha,n} - a_{\alpha,n} t.$

(3) *Similarly for "weakly" fit*

Remark 2.17: (1) Probably we can use $\bar\partial = \langle \partial_\ell : \ell < \mathbf{k} \rangle$ with $\partial_\ell \in \{\aleph_0, \aleph_1\}$ but there is no real need so far.

(2) This is essentially [She80, §4] and [She13b, 4.10(C)=L5e.28].

Proof. (1) Proving clause (A) of 2.11(1) and clause (B)(δ) of 2.11(2) is easy as in 2.15, so we concentrate on 2.11(1)(B).

So let G_0, h be as in 2.11(1)(B). Choose p_n by induction on n as follows: $p_0 = 2, p_{n+1}$ the first prime $> p_n + n$ such that

$$p_{n+1}!/(c_{n+1} - n) > \sqrt{p_{n+1!}},$$

where we let

$$c_n = \prod_{m<n} (p_m!).$$

Now observe that:

⊞ for $n \geq 100$ there is $C_n \subseteq \{0, 1, \ldots, (p_n!) - 1\}$ such that: if $b \in \mathbb{Z}$ and $t \in \mathbb{Z}$ satisfies $0 < |t| < n$, then for some $a_0, a_1 \in \mathbb{Z}$ we have

- $b + c_n a_0 t \in \cup\{i + (p_{n+1}! - 1)\mathbb{Z} : i \in C_n\}$,
- $b + c_n a_1 t \notin \cup\{i + (p_{n+1}!)\mathbb{Z} : i \in C_n\}$.

[Why? It suffices to consider $b \in \{0, \ldots, p_n! - 1\}$, $t \in \{\ell, -\ell : \ell \leq n, \ell \neq 0\}$ and let $A_{b,t} = \{b + c_n a t : a \in \mathbb{Z}\} \cap \{0, \ldots, p_{n+1}! - 1\}$. Clearly

$$|A_{b,t}| = (p_n!)/(c_n \cdot |t|) > \sqrt{p_n!}.$$

The family $\{A_{b,t} : b \in \{0, \ldots, p_{n+1}! - 1\}, t \in \{\ell, -\ell : \ell \le n, \ell \ne= 0\}\}$ has at most $2n(p_n!)$ members. Easily the number of $C \subseteq \{0, \ldots, p_n! - 1\}$ such that $(C \supseteq A_{b,t}) \vee (C \cap A_{b,t}) = \emptyset$ for some pair (b, t) as above is[11] $< 2^{\sqrt{p}_{n+1}!}$, hence there is C_n as required.]

Let $\Omega \subseteq {}^{\omega}2$ be of cardinality $\aleph_1$ and $\langle \rho_\alpha : \alpha < \omega_1 \rangle$ list Ω without repetitions.

Let G be generated by

$$\{x_{m,\alpha} : \alpha < \aleph_1, m < \mathbf{k}\} \cup \{y^1_{\rho,n} : \rho \in \Omega \text{ and } n < \omega\} \cup \{y^2_{\varrho} : \varrho \in {}^{\omega>}2\} \cup \{z\}$$

freely except the equations:

$(*)^1_{\alpha,n}$ $p_n! y^1_{\alpha,n+1} = y^1_{\alpha,n} - y^2_{\rho_\alpha \restriction n} - \sum_{m<\mathbf{k}} x_{m,\omega\cdot\alpha+n} - a_{\alpha,n} z$

where $a_{\alpha,n} \in \mathbb{Z}$ are chosen below; let $\bar{a} = \langle a_{\alpha,n} : \alpha < \omega_1, n < \omega \rangle$, so really $G = G_{\bar{a}}$ and let $\bar{a}_{\alpha,<n} = \langle a_{\alpha,n_1} : n_1 < n \rangle$.

Note that in G

$(*)^2_{\alpha,n}$ $y^1_{\alpha,0} = c_n y^1_{\alpha,n} + \sum_{n_1<n} c_{n_1}(y^2_{\rho_\alpha \restriction n_1} + \sum_{m<\mathbf{k}} x_{m,\omega\cdot\alpha+n} + a_{\alpha,n_1} z)$.

Define

$(*)^3_{\alpha,n}$ $b_{\alpha,n} = \sum_{n_1 \le n} h(\sum_{m<\mathbf{k}} c_{n_1} x_{m,\omega\cdot\alpha+n}) \in \mathbb{Z}$.

Recall G_0, h are as in 2.11(1)(B). Let $n_* = |h(z)|$ so $n_* > 0$. We choose $a_{\alpha,n} \in \mathbb{Z}$ by induction on n such that: if $n > |h(z)|$ then

$(*)^5_{\alpha,n}$ $\rho_\alpha(n) = 1$ iff $(b_{\alpha,n} + \sum_{n_1 \le n} c_{n_1} a_{\alpha,n_1} h(z))$ is equal to some $a \in C_n$ modulo $< p_n!$.

[Why possible? Arriving at n, the sum on the right side is

$$b_{\alpha,n} + \sum_{n_1<n} c_{n_1} a_{\alpha,n_1} h(z)) + c_n a_{\alpha,n} h(z) \in \mathbb{Z},$$

with the first two summands being already determined, i.e., they are computable from $\bar{a}_{\alpha,<n}$ and $|h(z)| \le n$, applying $\boxplus$ with

$$(n, h(z), b_{\alpha,n} + \Sigma\{c_{n_1} a_{\alpha,n_1} h(z) : n_1 < n\})$$

here standing for (n, t, b) there, so we get there a_0, a_1 and let $a_{\alpha,n}$ be a_0 if $\rho_\alpha(n) = 0$ and a_1 if $\rho_\alpha(n) = 1$. So for every $n, a_{\alpha,n}$ is as required and can be chosen.]

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[11] In other words, for each b, ta above a random $C \subseteq \{0, \ldots, p_{n+1}! - 1\}$ has probability $\le \frac{2}{2^{|A_{b,t}|}} \le \frac{2}{2^{\sqrt{p}_n!}}$ to include $A_{b,t}$ or to be disjoint to it. So the probability that this occurs for some pair (b, t) is $\le 2 \cdot |\{A_{b,t} : b, t \text{ is as above}\}|/2^{\sqrt{p}_n!} \le 4n(p_n!)/2^{\sqrt{p_n!}}$ which is $\ll 1$.

Having chosen $\bar{a} = \langle a_{\alpha,m} : \alpha < \omega_1, m < \omega\rangle$, the Abelian group $G = G_{\bar{a}}$ is chosen. Hence we just have to prove that G is as required in clause (B) of 2.11(1),(2). First, for 2.11(1)(B)

- $\odot$ toward contradiction assume that $f \in \mathrm{Hom}(G,\mathbb{Z})$ extends h and $n_* = |f(z)|$ is > 0,

hence (for every α, n applying f_n to the equation in $(*)^2_{\alpha,n}$):

$$(*)^6_{\alpha,n}\ \ f(y^1_{\alpha,0}) = c_n! f(y^1_{\alpha,n}) + \sum_{n_1<n} c_{n_1} f(y^2_{\rho_\alpha\restriction n_1}) + \sum_{n_1<n}\sum_{m<\mathbf{k}} c_{n_1} f(x_{m,\omega\cdot\alpha+n_1}) + \sum_{n_1<n} c_{n_1} a_{\alpha,n} f(z).$$

So recalling $|h(z)| = n_*$ for some $\rho_* \in {}^{n_*+100}2$ and $a_* \in \mathbb{Z}$ we have $|S| = \aleph_1$, where

$$S = \{\alpha < \aleph_1 : f(y^1_{\alpha,0}) \equiv a_* \text{ and } \rho_\alpha\restriction(n_*+1)) = \rho_*\}.$$

So choose $\alpha < \beta$ from S and let

$$n = \min\{\ell : \rho_\alpha(\ell) \neq \rho_\beta(\ell)\};$$

clearly we have $n > n_*$, hence $n \geq n_* + 1 \geq 2$, and subtracting the equations $(*)^6_{\alpha,n+1}, (*)^6_{\beta,n+1}$, in the left side we get a multiple of c_{n+1}, so a number divisible by $p_n!$, and in the right side we get the sum of the following four differences:

- $\odot_1$ $f(y^1_{\alpha,0}) - (f(y^1_{\beta,0})$ which is zero by the choice of S and the demand $\alpha,\beta \in S$,
- $\odot_2$ $\sum_{n_1\leq n} c_{n_1} f(y^2_{\rho_\alpha\restriction n_1}) - \sum_{n_1\leq n} c_{n_1} f(y^2_{\rho_\beta\restriction n_1})$ which is zero as
$$n_1 \leq n \Rightarrow \rho_\alpha\restriction n_1 = \rho_\beta\restriction n_1,$$
- $\odot_3$ $\sum_{n_1\leq n}\sum_{m<\mathbf{k}} c_{n_1} f(x_{m,\omega\cdot\alpha+n}) - \sum_{n_1\leq n}\sum_{m<\mathbf{k}} c_{n_1}\ f(x_{m,\omega\cdot\beta+n})$ which, recalling $(*)^3_{\alpha,n} + (*)^3_{\beta,n}$, is equal to $b_{\alpha,n} - b_{\beta,n}$ by the choice of $b_{\alpha,n}, b_{\beta,n}$ as f, h agree on G_0,
- $\odot_4$ $\sum_{n_1\leq n} c_{n_1} a_{\alpha,n_1} f(z) - \sum_{n_1\leq n} c_{n_1} a_{\beta,n_1} f(z)$.

Hence (recalling $f(z) = h(z)$)

- $\boxdot$ $(b_{\alpha,n} + \sum_{n_1\leq n} c_{n_1} a_{\alpha,n_1} f_\alpha(z)) - (b_{\beta,n} + \sum_{n_1\leq n} c_{n_1} a_{\beta,n} f(z))$ is divisible by $p_n!$ in $\mathbb{Z}$.

But by the choice of $a_{\alpha,n}$, i.e., by $(*)^5_{\alpha,n}$ we know that

$$\left(b_{\alpha,n} + \sum_{n_1\leq n} c_{n_1} a_{\alpha,n} f(z)\right)$$

is equal modulo $p_n!$ to some $i \in C_n$ iff $\rho_\alpha(n) = 1$. Similarly for β, but $\rho_\alpha(n) \neq \rho_\beta(n)$, contradiction to $\boxdot$. So indeed, $\odot$ leads to contradiction. This means that the demand in 2.11(1)(B) is satisfied. Second, recall that we need to verify the "freely fit". This means that

$\circledast_1$ for $\bar{a}$ as above and $w \in J$, the Abelian group $G_{\bar{a}}/\oplus\{\mathbb{Z}x_\alpha : \alpha \in w\}$ is free,

$\circledast_2$ $G_{\bar{a}}$ is free.

[Why? Easy.]

Hence

$\circledast_3$ without loss of generality $w = \{w\alpha + n : \alpha < \alpha_* \text{ or } \alpha < \omega_1 \restriction n < n^*_\alpha\}$ for some $\alpha_* < w_1$ and $n^*_\alpha < \omega$ for $\alpha < \omega_1$.

Now

$\circledast_4$ letting

$$G_* = \oplus\{\mathbb{Z}y^2_\varrho : \varrho \in {}^{\omega>}2\} \oplus \bigoplus\{\mathbb{Z}X_\alpha : \alpha < \omega\alpha_*\},$$
$$B_\omega = \oplus\{\mathbb{Z}X_\alpha : \alpha \in \omega, \alpha \geq \omega\alpha_*\}$$

we have

(a) $G_\omega + G_* = G_\omega \oplus G_*$. [Why? Check.]

(b) It suffices to prove $G_{\bar{a}}/(G_\omega \oplus G_*)$ is free. [Why? By (a).]

(c) $G_{\bar{a}}/(G_\omega \oplus G_*)$ is the direct such of

$$H'_\alpha := \langle H_\alpha + (G_\omega \oplus G_*\rangle / G_\omega \oplus G_* : \alpha \in [\omega\alpha_*, \omega_1]\rangle$$

where H_α is the subgroup of $G_{\bar{a}}$ generated by

$$\{X_{\omega\alpha+n} : n < \omega\} \cup \{y^1_{\alpha,n} : n < \omega\} \cup \{y^2_{\rho_\alpha \restriction n} : n < \omega\}.$$

[Why? Check.]

(d) it suffices to prove each H'_α is a free Abelian group. [Why? By (c).]

(e) H'_α is isomorphic to

$$H_\alpha/\oplus(\cup\{\mathbb{Z}X_{\omega\alpha,n} : n < n_\alpha\} \cup \{\mathbb{Z}y^2_{\rho_\alpha\restriction n} : n < \omega\}).$$

[Why? Check]

(f) H'_α is indeed free. [Why? By the same proof as in 2.15.]

So $(\partial, J, \mathbf{k}, \theta)$ freely fits $\mathbb{Z}$ indeed.

(2) We can fix $G_0 = \oplus\{RX_{m,i} : m < \mathbf{k}, i < \partial_m\} \oplus Rz$, $h \in \mathrm{Hom}(G_0, {}_RR)$ such that $h(z) \neq 0$. Let $\Omega, \langle\rho_\alpha : \alpha < \omega_1\rangle$ be as in the proof of part (1).

We are given $b_{\alpha,n} = h(x_{m,\omega\alpha+n}(\alpha < \aleph_1, n \in \mathbb{N})$ and $t = h(z)$ from R. We shall choose $\langle (a_{\alpha,n}, d_{\alpha,m}) : \alpha < \omega_1, n < \omega\rangle$ and will let G be the R-module generated by

$$\{x_{m,\alpha} : \alpha < \aleph_1, m < \mathbf{k}\} \cup \{y^1_{\alpha,n} : \alpha < \aleph_1 \text{ and } n < \omega\} \cup \{y^2_\varrho : \varrho \in {}^{\omega>}2\} \cup \{z\}$$

freely except the equations

$(*)_{\alpha,n}$ $d_{\alpha,n} y^1_{\alpha,n+1} = y^1_{\alpha,n} + y^2_{\rho_\alpha\restriction n} + \sum_{m<\mathbf{k}} x_{m,\omega\cdot\alpha+n} - a_{\alpha,n} z.$

Hence

$$(*)'_{\alpha,n}\ \ y^1_{\alpha,0} = \Big(\prod_{\ell<n} d_{\alpha,\ell}\Big) y^1_{\alpha,n} + \sum_{n_1<n} \Big(\prod_{\ell=n_1}^{n-1} d_{\alpha,\ell}\Big) y^2_{\rho_\alpha\restriction\ell} + \sum_{n_1<n}\sum_{m<\mathbf{k}} \Big(\prod_{\ell=n_1}^{n-1} d_{\alpha,\ell}\Big) x_{m,\omega\cdot\alpha+n} + \sum_{n_1<n}\Big(\prod_{\ell=n_1}^{n-1} d_\ell\Big) a_{\alpha,n_1} z.$$

Now continue as in the proof of part (1). $\blacksquare_{2.16}$

We now can put things together

THEOREM 2.18: (1) *For every $k \geq 1$ there is an $\aleph_{\omega_1\cdot k}$-free Abelian group G which is not Whitehead and even* $\mathrm{Hom}(G,\mathbb{Z}) = 0$.

(2) *If the ring R satisfies the demands in clause (c) part (2) from 2.16, then for every k there is an $\aleph_{\omega_1\cdot k}$-free R-module such that*

$$\mathrm{Hom}(G, {}_RR) = 0 \quad \textit{and} \quad \mathrm{Ext}(G, {}_RR) \neq 0.$$

Proof. (1) Given k we use 1.27 to find a c.p. $\mathbf{x}$ which is $\aleph_{\omega_1\cdot k}$-free and has χ-BB where $\chi = |R| + \aleph_1$ and $J = J^{\mathrm{bd}}_{\aleph_1} \odot J^{\mathrm{bd}}_{\aleph_0}$. Now apply 2.16(1) so $(\aleph_1, J, k, \aleph_1)$ fits $\mathbb{Z}$ and by 2.12(1),(2) we get the desired conclusion.

(2) Similarly, but now we use 2.16(2) rather than 2.16(1). $\blacksquare_{2.18}$

3. Forcing

The main result of the former section is the existence in ZFC of $\aleph_{\omega_1\cdot n}$-free Abelian groups G (for every $n \in \omega$) such that $\mathrm{Hom}(G,\mathbb{Z}) = 0$. The purpose of this section is to show that this result is best possible in the sense of freeness amount. Assuming the existence of $\aleph_0$-many supercompact cardinals in the ground model, we shall force the following statement. For every $\aleph_{\omega_1\cdot\omega}$-free non-trivial Abelian group $G, \mathrm{Hom}(G,\mathbb{Z}) \neq 0$.

This section is divided into two subsections. In §3(A), like §1 is combinatorial, we describe a general framework for dealing with freeness of R-modules (this continues [She85], [She96] and [Shec]; but we have to work more).

In §3(B) we rely on forcing, we focus on $R = \mathbb{Z}$ (hence R-modules are simply Abelian groups), and we prove the main consistency result in Theorem 3.9 which relies on Magidor–Shelah [MS94]. The proof is based on the context of §1(A), with double meaning.

3(A). Freeness classes.

Context 3.1: (1) R is a ring with no zero divisors and is hereditary (see 2.1(1A)).

(2) $\mathbf{K}$ is the class of R-rings closed under isomorphisms.

(3) $\mathbf{K}_*$ will denote a class $\subseteq \mathbf{K}$.

Definition 3.2: (0) $\mathbf{K}^w = \{M \in \mathbf{K} : M$ a Whitehead module that is, $\mathrm{Ext}(M, {}_R R) = 0$

equivalently, if $N_1 \subseteq N_2$ are R-modules,

$N_2/N_1 \cong M$ and $h_1 \in \mathrm{Hom}(N_1, {}_R R)$,

then there is $h_2 \in \mathrm{Hom}(N_2, {}_R R)$ extending $h_1\}$.

(1) We say $\mathbf{K}_*$ is a **λ-freeness class** inside $\mathbf{K}$ when:

(a) $\mathbf{K}_* \subseteq \mathbf{K}_{<\lambda}$ where for any cardinality θ we let $\mathbf{K}_{<\theta} := \{M \in \mathbf{K} : \|M\| < \theta\}$,

(b) $\mathbf{K}_*$ is closed under isomorphisms

(c) for simplicity $\lambda > |R|$.

(1A) We say $\mathbf{K}_*$ is **hereditary** when $\mathbf{K}_*$ is closed under pure submodules, i.e., $M \subseteq_{\mathrm{pr}} N \in \mathbf{K}_* \Rightarrow M \in \mathbf{K}_*$. We may in (1) omit $\mathbf{k}$ when clear from the context.

(2) We say $M \in \mathbf{K}$ is **$\mathbf{K}_*$-free** when there is $\bar{M}$ such that $\bar{M} = \langle M_\alpha : \alpha \leq \alpha_* \rangle$ is purely increasing continuous, M_0 is the zero module and

$$\alpha < \alpha_* \Rightarrow M_{\alpha+1}/M_\alpha \in \mathbf{K}_* \quad \text{and} \quad M_{\alpha_*} = M.$$

(2A) $M \in \mathbf{K}$ is **$(\lambda, \mathbf{K}_*)$-free** when every $M' \subseteq_{\mathrm{pr}} M$ of cardinality $< \lambda$ is $\mathbf{K}_*$-free.

(3) $\mathbf{K}^*_{<\theta} = \mathbf{K}_* \cap \mathbf{K}_{<\theta}$ for any cardinal θ.

(4) The class $\mathbf{K}_*$ is called a **(λ, κ)-freeness class** when: $\mathbf{K}_*$ is a λ-freeness class, $\mathbf{K}_*$ is hereditary and if $M \in \mathbf{K}_{<\lambda} \backslash \mathbf{K}_*$, then there is $N \subseteq_{\mathrm{pr}} M$ from $\mathbf{K}_{<\kappa} \backslash \mathbf{K}_*$.

The main example here is:

Claim 3.3: *Assume* $R = \mathbb{Z}, \lambda \geq \aleph_1$ *and* $\mathbf{K} =$ *the class of* R*-modules, and let*

$$\mathbf{K}_{whu} = \mathbf{K}_* = \{M \in \mathbf{K}_{<\lambda} : M \text{ is a Whitehead module, equivalently satisfies the condition inside 3.2(0)}\}$$

and $\mathbf{K}_{\mathrm{fr}} = \{M \in \mathbf{K}_{<\aleph_1} : M \text{ free}\}$.

(0) $\mathbf{K}_{\mathrm{fr}}$ *is a hereditary* $\aleph_1$*-freeness class.*
(1) *If* $\lambda > \aleph_2$ *and* $\mathrm{MA}_{<\lambda}$*, then* $\mathbf{K}_*$ *is a hereditary* $(\lambda, \aleph_2)$*-freeness class.*
(2) *If* $M \in \mathbf{K}$ *is* $\mathbf{K}_*$*-free, then* M *is a Whitehead group.*
(3) *If* $M_1 \subseteq_{\mathrm{pr}} M_2$ *and* M_2/M_1 *is* $\mathbf{K}_*$*-free and* $h_1 \in \mathrm{Hom}(M_1, {}_RR)$*, then there is* $h_2 \in \mathrm{Hom}(M_2, {}_RR)$ *extending* h_1*.*
(4) *If*

$$\mathbf{K}_{**} = \{M \in K_{<\lambda} : \text{for every c.c.c. forcing } \mathbb{P}_1 \text{ for some c.c.c. forcing notion } \mathbb{P}_2 \text{ satisfying } \mathbb{P}_1 \lessdot \mathbb{P}_2 \text{ we have } \Vdash_{\mathbb{P}_2} \text{“}M \text{ is a Whitehead group”}\}$$

then $\mathbf{K}_{**}$ *is* $(\lambda, \aleph_2)$*-freeness class.*

Proof. (0) Obvious as $\mathbb{Z}$ is countable.

(1) The first property in 3.2(4) holds trivially by the choice of $\mathbf{K}_*$. As for the second property it is well known that $\mathbf{K}_*$ is a hereditary class; see [Fuc73]. The third property in 3.2(4) follows from the full characterization of being Whitehead for Abelian group G of cardinality $< \lambda$ when $\mathrm{MA}_{<\lambda}$ holds (not just proving “strongly $\aleph_1$-free is enough”); in particular G is Whitehead if every subgroup of cardinality $\leq \aleph_1$ is Whitehead; see [EM02].

(2) Follows by (3).

(3) Without loss of generality let $M = M_2/M_1$ and $\pi \in \mathrm{Hom}(M_2, M)$ be onto with kernel M_1. Let $\langle M'_\alpha : \alpha \leq \alpha_* \rangle$ be as in 3.2(4) for M and let $N_\alpha = \pi^{-1}(M'_\alpha)$, so $\langle N_\alpha : \alpha \leq \alpha_* \rangle$ is purely increasing continuous, $N_0^* = M_1, N_{\alpha_*} = M_2$ and $N_{\alpha+1}/N_\alpha \in \mathbf{K}_*$.

Given $h_1 \in \mathrm{Hom}(M_1, {}_RR)$ by induction on α we choose $f_\alpha \in \mathrm{Hom}(N_\alpha, {}_RR)$ increasing continuous with α. For $\alpha = 0$ let $f_\alpha = h_1$, for α limit let

$$f_\alpha = \cup\{f_\beta : \beta < \alpha\}$$

and for $\alpha = \beta + 1$ use $N_\alpha/N_\beta \cong M'_\alpha/M'_\beta \in \mathbf{K}_*$ and the choice of $\mathbf{K}_*$.

Lastly, $h_2 = f_{\alpha_*}$ is as required.

(4) Easy. $\blacksquare_{3.3}$

Now on those freeness contexts see [She75a] or better [Shec] and history there. Note that we shall in §3(B) use 3.7(B)(c), and for this we need witnesses **s** from those references. Recall (see [Shec])

Definition 3.4: (1) We say **c** is a **pre-1-freeness** context when **c** consists of:

(a) $\mathscr{U}$ is a fixed set (we shall deal with subsets of it) or $\mathfrak{U}$ is an algebra with universe $\mathscr{U}$ (maybe with empty set of functions); let $c\ell_{\mathbf{c}}(A)$ be the closure of the set $A \subseteq \mathscr{U}$ in the algebra $\mathfrak{U}$; but we may sometime say $\mathscr{U}$ instead of $\mathfrak{U}$.

(b) $\mathscr{F}$ a family of pairs of subsets of $\mathscr{U}$; we may write "A/B is free" or "A is free over B" for (A, B) in $\mathscr{F}$.

(c) χ, μ will be fixed cardinals such that

$$|\tau(\mathfrak{U})| \le \chi < \mu \le \infty \quad \text{and} \quad (A, B) \in \mathscr{F} \Rightarrow |A| + |B| < \mu,$$

but if $\mu = \infty$ (equivalently, $\mu > |\mathscr{U}|$) we may omit it.

(2) We say "for the χ-majority of $X \subseteq A, P(X)$" (for a property P) when there is an algebra $\mathfrak{B}$ with universe A and χ functions, such that any $X \subseteq A$ closed under those functions satisfies P. We can replace $X \subseteq A$ by $X \in \mathscr{P}(A)$ or $X \in \mathscr{P}_{<\lambda}(A)$: alternatively we may say $\{X \subseteq A : P(A)\}$ is a χ**-majority**.

(3) We say **c** is a **freeness context** when in addition to (a),(b),(c) of part (1) it satisfies the following (adding a superscript $^+$ to an axiom means that whenever "$A/B \in \mathscr{F}$" or its negation appears in the assumption, then we demand B to be free over $\emptyset$. Of course, $\mathscr{F}_{\mathbf{c}} = \mathbf{F}, \chi_{\mathbf{c}} = \chi$, etc.):

Ax II_μ:

(a) A/B is free iff $A \cup B/B$ is free.

$(\mathrm{b})_\mu$ A/B is free when $|B| < \mu$ and $A \subseteq B$.

Ax III [2-transitivity]: If A/B and B/C are free and $C \subseteq B \subseteq A$, then A/C is free.

Ax $\mathrm{IV}_{\lambda,\mu}$ [continuous transitivity]: If $A_i (i < \lambda)$ is increasing, for $i \le \gamma < \lambda$ we have $A_\gamma / \bigcup_{j<i} A_j \cup B$ is free, $\lambda < \mu$ and $|\bigcup_{i<\lambda} A_i| < \mu$, then $\bigcup_{i<\lambda} A_i/B$ is free.

Let $\mathrm{Ax}(\mathrm{IV}_{<\lambda,\mu})$ mean $\theta < \lambda \Rightarrow \mathrm{Ax}\,(\mathrm{IV}_{\theta,\mu})$; and Ax IV_μ will mean Ax $\mathrm{IV}_{<\mu,\mu}$ and IV means IV_∞.

Ax VI: If A is free over $B \cup C$, then for the $\chi_{\mathbf{c}}$-majority of $X \subseteq A \cup B \cup C$ the pair $A \cap X/(B \cap X) \cup C$ is free.

Ax VII: If A is free over B, then for the $\chi_{\mathbf{c}}$-majority of $X \subseteq A \cup B$ the pair $A/(A \cap X) \cup B$ is free.

(4) We say $\mathbf{c}$ is a **freeness$^+$ context** when in addition

Ax I**: If A/B is free and $A^* \subseteq A$, then A^*/B is free.

(5) We say $\mathbf{c}$ is a (λ, κ)**-freeness context** when: in addition $\chi_{\mathbf{c}} \leq \kappa$, Ax I** and if A/B is not $\mathbf{c}$-free and $|A| < \lambda$ then for some $A' \subseteq A$ of cardinality $< \kappa$, A'/B is not $\mathbf{c}$-free

Definition 3.5: For a λ-freeness class $\mathbf{K}_*$ and R-module G and $\chi \geq |R| + \aleph_0$ (if equal, then χ may be omitted) we define what we call a **pre-freeness context** $\mathbf{c} = \mathbf{c}_G = \mathbf{c}_{\mathbf{K}_*, G, \chi}$ (this is proved in 3.6) as the quadruple

$$(\mathscr{U}, \mathfrak{A}, \mathscr{F}, \chi) = (\mathscr{U}_{\mathbf{x}}, \mathfrak{A}_{\mathbf{c}}, \mathscr{F}_{\mathbf{c}}, \chi_{\mathbf{c}})$$

where:

(a) $\mathscr{U} = G$ as a set and $\mathfrak{A}$ is an expansion of G by $F_a^{\mathfrak{A}}(a \in R)$ such that: if $G \models ax = y$ and $y' = F_a(y)$ then $G \models ay' = y$, if $y \notin aG$ then $F_a(y) = 0$,

(b) $\mathscr{F} = \{A/B : B, A \subseteq \mathscr{U}$ and $\langle A \cup B \rangle_{\mathfrak{A}} / \langle B \rangle_{\mathfrak{A}}$ is $\mathbf{K}_*$-free$\}$, we may say A/B is $\mathbf{c}$-free so A/B stands for the formal quotient, so pedantically is just the pair (A, B), where $\langle B \rangle_G$ is the minimal pure[12] sub-module of G which includes B,

(c) $\chi_{\mathbf{c}} = \chi$ so $\geq |R| + \aleph_0$ (and $\mu_{\mathbf{c}} = \infty$).

Fact 3.6: Assume $\mathbf{K}_*$ is a hereditary λ-freeness class and $\chi = |R| + \aleph_0$.

(1) Being $\mathbf{K}_*$-free has compactness in singular cardinals $> \lambda$.

(2) For any R-module $G_*, \mathbf{c} = \mathbf{c}_{\mathbf{K}_*, G_*, \chi}$ defined in 3.5 above is a freeness context and satisfies Ax I**.

(3) If $\mathbf{K}_*$ is moreover a (λ, κ)-freeness class (see 3.2(4)), then $\mathbf{c}$ is a (λ, κ)-freeness context (see 3.4(5)).

Proof. (1) By part (2) and [Shec], see history there.

(2) Check.

(3) Easy. $\blacksquare_{3.6}$

[12] Our modules are torsion free, i.e., $a \in R \wedge x \in G \Rightarrow (ax = 0 \Leftrightarrow (a = 0_{\mathbb{R}} \vee x = 0_G))$ holds when $R = \mathbb{Z}$; this is no problem. Otherwise, recall we have expanded G to an algebra $\mathfrak{A}$ such that $A = c\ell_{\mathscr{U}}(A) \Rightarrow A \subseteq_{\mathrm{pr}} G$.

CLAIM 3.7: *If* (A) *then* (B), *where:*

(A) (a) $\mathbf{K}_*$ *is a* (λ,κ)*-freeness class; see Definition 3.2(4),*
(b) $G \in \mathbf{K}$ *is* $(\mathbf{K}_*,\lambda)$*-free not* $\mathbf{K}_*$*-free, see Definition 3.2(2A); fix such* G *of minimal cardinality called* μ,
(c) $\mathbf{c} = \mathbf{c}_{\mathbf{K}_*,G,\kappa}$; *see Definition 3.5(1).*

(B) *There is a witness* $\mathbf{s}$ *for* G *in the context* $\mathbf{c}$ *(see* [She85, §2] *and better* [She96, §3]*) such that:*
(a) $B^{\mathbf{s}}_{<>} = \emptyset, B^{\mathbf{s}}_{<>^+} \subseteq G$ *so* $\lambda(<>, S_{\mathbf{s}}) \leq \|M\|$,
(b) *if* $\eta \notin \operatorname{fin}(S_{\mathbf{s}})$ *then* $\lambda_{\mathbf{s},\eta} \geq \lambda$,
(c) *if* $\eta\hat{\ }\langle\delta\rangle \in S_{\mathbf{s}}$ *then* $\operatorname{cf}(\delta) \notin [\kappa,\lambda)$,
(d) *if* $\eta \in \operatorname{fin}(S_{\mathbf{s}})$ *then* $B_{\mathbf{s},\eta^+} \backslash B_{\mathbf{s},\eta}$ *has cardinality* $< \kappa$.

Proof. By 3.6 we can apply 3.8 below. $\blacksquare_{3.7}$

CLAIM 3.8: *If* (A) *then* (B), *where:*

(A) (a) $\mathbf{c}$ *is a freeness context satisfying Ax* I^{**}
(b) $\mathbf{c}$ *is* (λ,κ)*-freeness context,*
(c) A/B *is* λ*-free not free pair, and with* $|A|$ *minimal.*

(B) *There is a witness* $\mathbf{s}$ *such that:*
(a) $B^{\mathbf{s}}_{<>} = B$ *and* $B^{\mathbf{s}}_{<>^+} \subseteq A$ *so* $\lambda(<>, S_{\mathbf{s}}) \leq |A|$ *but* $\geq \lambda$,
(b) *if* $\eta \notin \operatorname{fin}(S_{\mathbf{s}})$, *then* $\lambda_{\mathbf{s},\eta} \geq \lambda$,
(c) *if* $\eta\hat{\ }\langle\delta\rangle \in S_{\mathbf{s}}$, *then* $\operatorname{cf}(\delta) \notin [\kappa,\lambda)$,
(d) *if* $\eta \in \operatorname{fin}(S_{\mathbf{s}})$, *then* $B_{\mathbf{s},\eta^+} \backslash B_{\mathbf{s},\eta}$ *has cardinality* $< \kappa$.

Proof. Now (see [She96, §3] or better yet see [Shea, 4.5=Ld15]), there is a disjoint witness $\mathbf{s}$ for A/B being non-$\mathbf{c}$-free. So without loss of generality ($n = n(\mathbf{s})$ is well defined and) for some

$$\bar{\lambda}^* = \langle \lambda^*_\ell : \ell < n \rangle, \quad \bar{\kappa}^* = \langle \kappa^*_\ell : \ell < n \rangle$$

we have:

$(*)_1$ (a) for each $\ell < n$ one of the following holds:
(α) λ_ℓ is a regular cardinal and $\eta \in S_{\mathbf{s},\ell} \Rightarrow \lambda(\eta, S_{\mathbf{s}}) = \lambda^*_\ell$,
(β) $\lambda_\ell = *$ and $\eta \in S_{\mathbf{s},\ell} \Rightarrow \lambda(\eta, S_{\mathbf{s}})$ is (possibly weakly) inaccessible and $\ell > 0$,
(b) for each $\ell < n$, either κ_ℓ is a regular cardinal and

$$\eta \in S_{\mathbf{s},\ell} \wedge \delta \in W(\eta, S_{\mathbf{s}}) \Rightarrow \operatorname{cf}(\delta) = \kappa_\ell$$

or $\kappa_\ell = *$ and $\lambda_{\ell+1}$ is $*$.

See more there; naturally without loss of generality

$(*)_2$ $\mathbf{s}$ is minimal which means that (fixing A and B):

(a) $n = n(\mathbf{s})$ is minimal,

(b) under (a), $\bar{\lambda}^*$ is minimal under the lexicographical order,

(c) under (a) + (b), $\bar{\kappa}^*$ is minimal under the lexicopgrahical order.

Now:

$(*)_3$ If $\eta \in \mathrm{ini}(S_{\mathbf{s}})$ then $\lambda(\eta, S_{\mathbf{s}}) \geq \lambda$.

[Why? Otherwise choose a counterexample η with $\lambda(\eta, S_{\mathbf{s}})$ minimal so by the definition of a witness as $\chi_{\mathbf{c}} \leq \kappa$ we have $B^{\mathbf{s}}_{\eta^+}/B^{\mathbf{s}}_{\leq\eta}$ is not free (for $\mathbf{c}$), $B^{\mathbf{s}}_{\eta^+}\backslash B^{\mathbf{s}}_{\eta}$ has cardinality $\lambda(\eta, S_{\mathbf{s}})$ so $< \lambda$. Recalling "$\mathbf{c}$ is a (λ, κ)-freeness context", see Definition 3.4(5) and Fact 3.6(3), there is $C_\eta \subseteq B^{\mathbf{s}}_{\eta^+}$ of cardinality $\leq \kappa$ such that $C_\eta/B^{\mathbf{s}}_{\leq\eta}$ is not $\mathbf{c}$-free. So (it follows by minimality of $\mathbf{s}$) we get contradiction, so $\lambda(\eta, S_{\mathbf{s}}) \geq \lambda$ as promised in $(*)_3$.]

$(*)_4$ If $\eta\hat{\ }\langle\delta\rangle \in S_{\mathbf{s}}$ then $\mathrm{cf}(\delta) \notin [\kappa, \lambda)$.

[Why? As in the proof in [She85, She96] for each $\eta \in S_{\mathbf{s}}$ satisfying $\mathrm{cf}(\delta) \geq \kappa$ by the minimality, $\mathrm{cf}(\delta) \in \{\lambda(\nu, S_{\mathbf{s}}) : \nu \in S_{\mathbf{s}}$ satisfies $\eta \triangleleft \nu\}$, so $(*)_4$ follows by $(*)_3$.]

So we are done. $\blacksquare_{3.8}$

3(B). The main independence result. Below, it is reasonable to assume that the ring R is $\mathbb{Z}$ and we assume this is the nice version. Note that we prove that a non-Whitehead group has a non-free subgroup of small cardinality, not necessarily a non-Whitehead one. This is connected to the black boxes here having cardinality (much) bigger than the amount of freedom. For simplicity, presently we deal with freeness only in hereditary cases.

Recall that μ is supercompact iff for every ∂ there exists an elementary embedding $j : \mathbf{V} \to M$ such that M is a transitive class satisfying ${}^{\partial}M \subseteq M$ and ∂ is the critical cardinal.

Theorem 3.9: *If in $\mathbf{V}$ there are $\aleph_0$-many supercompact cardinals, then in some forcing extension we have for $\mu_* = \aleph_{\omega_1 \cdot \omega}$:*

$\oplus_{\mu_*}$ (a) *if G is a non-trivial μ_*-free Abelian group, then $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$,*

(b) *if $G \subseteq H$ are Abelian groups and H/G is μ_*-free and $h \in \mathrm{Hom}(G, \mathbb{Z})$, then h can be extended to a homomorphism from H to $\mathbb{Z}$ (this is an equivalent definition of "H/G is Whitehead", the reader may use it here as a definition).*

This will be proved below. As usual in such a proof, we collapse a large cardinal into quite small ones, so they cannot be really large but some remnant of their early largeness remains and is enough for our purpose. This is the rationale of Definition 3.10 below.

Definition 3.10: Let $\Pr_{\lambda_*,\mu_*,\kappa_*}$ mean[13]

(A) (a) $\lambda_* > \mu_* > \kappa_*$,
(b) λ_*, κ_* are regular uncountable cardinals
(c) μ_* is a limit cardinal;

(B) if (a) then (b), where
(a) (α) λ is a regular cardinal $\geq \lambda_*$,
(β) $\chi > \lambda$ and $\mu < \mu_*$ and $x \in \mathscr{H}(\chi)$,
(γ) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) < \kappa_*\}$ is a stationary subset of λ,
(δ) $u_\alpha \in [\alpha]^{\leq \mu}$ for $\alpha \in S$;
(b) there are a regular $\lambda' \in (\mu + \kappa_*, \mu_*)$ and an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \lambda' \rangle$ of ordinals $< \lambda$ such that the set

$$\{\varepsilon < \lambda' : \alpha_\varepsilon \in S \text{ and } u_{\alpha_\varepsilon} \subseteq \{\alpha_\zeta : \zeta < \varepsilon\}\}$$

is a stationary subset of λ'.

On the strong hypothesis above, see [She93b], it is a sufficient condition for the SCH, that is,

$$\partial = \mathrm{cf}(\mu) \wedge 2^\partial < \mu \Rightarrow \mu^\partial = \mu^+.$$

Definition 3.11: We say the universe $\mathbf{V}$ satisfies the strong hypothesis above λ when: if $\xi < \mathrm{cf}(\xi) + \lambda + \mu$ then $\mathrm{cf}([\chi]^{<\mu_1}, \subseteq) \geq \chi^+$.

THEOREM 3.12: (1) *Assume in $\mathbf{V}_0$ there are infinitely many supercompact cardinals $> \theta$ and $\theta = \mathrm{cf}(\theta) \in [\aleph_1, \aleph_{\omega_1})$. Then for some forcing notion $\mathbb{Q}$ not adding new subsets to $\theta, \mathbf{V}_1 = \mathbf{V}_0^{\mathbb{Q}}$ satisfies $\Pr_{\lambda_*,\mu_*,\kappa_*}$ where $\lambda_* = \mathrm{cf}(\lambda_*) = \mu_*^+$, $\mu_* = \aleph_{\theta\cdot\omega}$ and $\kappa_* = \theta^+$.*

(1A) *We can (by preliminary forcing) assume that the universe $\mathbf{V}_1$ above satisfies also G.C.H. above θ (we use just "above μ_*") and $\diamondsuit^*_\lambda$ holds for every regular uncountable λ above μ_*.*

(2) *If $\Pr_{\lambda_*,\mu_*,\kappa_*}$ holds in $\mathbf{V}$ and the c.c.c. forcing $\mathbb{P}$ has cardinality λ_*, then in $\mathbf{V}^{\mathbb{P}}$ still $\Pr_{\lambda_*,\mu_*,\kappa_*}$ holds.*

(3) *Part (1) holds for any freeness$^+$ context (see Definition 3.4(3),(4)).*

13 We may allow $\lambda_* = \mu_*$ here and in 3.13, but then we have to say somewhat more.

Proof. (1), (1A) Similarly to [MS94, §4, Th. 1, p. 807]. As there, let $\langle\kappa_n : n < \omega\rangle$ be an increasing sequence of supercompact cardinals. Without loss of generality G.C.H. holds above $\mu = \sum_n \kappa_n$ (called κ there) and $\diamondsuit^*_\chi$ holds for every $\chi = \mathrm{cf}(\chi) > \mu$. Also for each n, the supercompactness of κ_n is preserved by forcing notions which are κ_n-directed closed.

We proceed as there but now in the interval (κ_{n-1}, κ_n), the set of cardinals we do not collapse has order type $\theta + 2$.

(2), (3) Easy. $\blacksquare_{3.12}$

Proof of 3.9. Let $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{Q}}$ be as in 3.12(1)(1A) with $\theta = \aleph_1$, so $\kappa_* = \aleph_2$, $\mu_* = \aleph_{\omega_1\cdot\omega}$, $\lambda_* = \mu_*^+$, and in $\mathbf{V}_1$ let $\mathbb{P}$ be a c.c.c. forcing notion of cardinality λ_* such that $\Vdash_{\mathbb{P}}$ "$\mathrm{MA} + 2^{\aleph_0} = \lambda_*$". The result follows from Theorem 3.13 below. Clause (d) there holds because

$$\mathbf{V} = \mathbf{V}_1^{\mathbb{P}};$$

see 3.12(2). $\blacksquare_{3.9}$

THEOREM 3.13: *The statement $\oplus_{\mu_*}$ from 3.9 holds when $\mathbf{V}$ satisfies:*

(a) *the statement* $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$ *from Definition 3.10,*
(b) $\lambda_* = \lambda_*^{<\lambda_*} > \mu_*$,
(c) $\kappa_* = \aleph_2$,
(d) $\mathrm{MA} + 2^{\aleph_0} = \lambda_*$ *and* $\mathbf{V}$ *satisfies the strong hypothesis above* λ_**; see 3.11 or* [She93b].

Proof. We rely on 3.1–3.8. The first clause (b) of $\oplus_{\mu_*}$ implies clause (a); why? because if H is a μ_*-free Abelian group, let $x \in H\backslash\{0_H\}$ and without loss of generality x is not divisible by any $n \in \{2, 3, \ldots\}$, hence $K := \mathbb{Z}x$ is a pure subgroup of H; let h be an isomorphism from K onto $\mathbb{Z}$. As H is μ_*-free easily also H/K is μ_*-free, hence by $\oplus_{\mu_*}$(b) there is a homomorphism h^+ from H to $\mathbb{Z}$ extending h so $h^+(x) \neq 0_{\mathbb{Z}}$, hence $h^+ \in \mathrm{Hom}(H, \mathbb{Z})$ is non-zero, as required.

So it suffices to prove clause (b) of $\oplus_{\mu_*}$.

Let $R = \mathbb{Z}$ and let $\mathbf{K}, \mathbf{K}_*$ be as in Claim 3.3 for λ_* so $\mathbf{K}_*$ is a hereditary $(\mu_*, \aleph_2)$-freeness class (see Definition 3.2(1),(1A),(4)) by 3.3(1). So toward contradiction assume $G \in \mathbf{K}$ is a counterexample of minimal cardinality called λ so G is μ_*-free. To get a contradiction and finish the proof it suffices to assume $G_1 \subseteq_{\mathrm{pr}} G_2, G_2/G_1 \cong G$ and $h_1 \in \mathrm{Hom}(G_1, \mathbb{Z})$ and prove that there is $h_2 \in \mathrm{Hom}(G_2, \mathbb{Z})$ extending h_1. If G is $\mathbf{K}_*$-free (see Definition 3.2(2)) then by 3.3(3) a homomorphism h_2 as required exists.

Hence without loss of generality G is not $\mathbf{K}_*$-free and let $\mathbf{c} = \mathbf{c}_{\mathbf{K}_*,G,\theta}$, see Definition 3.5, so by 3.6(3), $\mathbf{c}$ is a (λ_*, κ_*)-freeness context and by 3.7(2),(3) (with λ_*, κ_* here standing for λ, κ there) there is a witness $\mathbf{s}$ as there. By 3.3(1) we have $\lambda(\langle\rangle, S_{\mathbf{s}}) \geq \lambda_*$.

Let $\mathbf{c}_1 = \mathbf{c}_{\mathbf{K}_{\mathrm{fr}},G,\theta}$; it is a $(\lambda, \aleph_1)$-freeness context. (Why? By 3.6 with K_{fr} (see 3.3) playing the role of K_*.)

Let $S_1 = W(<>, S_{\mathbf{s}})$, so for each $\delta \in S_1, B^{\mathbf{s}}_{<\delta+1>}/B^{\mathbf{s}}_{<\delta>}$ is not free for $\mathbf{c}$ so cannot be μ_*-free for $\mathbf{c}_1$ (as we have chosen a counter-example of minimal cardinality). Hence there is $A_\delta \subseteq B^{\mathbf{s}}_{<\delta+1>}$ of cardinality $< \mu_*$ such that $A_\delta/B^{\mathbf{s}}_{<\delta>}$ is not free for $\mathbf{c}_1$.

Let $B'_\delta \subseteq B^{\mathbf{s}}_{<\delta>}$ be of cardinality $\leq |A_\delta| + \kappa_*$ such that $B'_\delta \subseteq B' \subseteq B^{\mathbf{s}}_{<\delta>} \Rightarrow A_\delta/B'$ is not free for $\mathbf{c}_1$; it exists by properties of Abelian groups as $B^{\mathbf{s}}_{\langle\delta\rangle} \subseteq B^{\mathbf{s}}_{\langle\delta+1\rangle}$ are free (for $\mathbf{c}_1$) and $A_\delta/B^{\mathbf{s}}_{<\delta>}$ not free for $\mathbf{c}_1$.

So for some $\mu < \mu_*$ the set $S_2 = \{\delta \in S_1 : |A_\delta \cup B'_\delta| + \kappa_* = \mu\}$ is a stationary subset of $\lambda(\langle\rangle, S_{\mathbf{s}})$. Let h be a one-to-one function from $\lambda(\langle\rangle, S_{\mathbf{s}})$ onto $B^{\mathbf{s}}_{<\lambda>}$ and let $C := \{\delta < \lambda(\langle\rangle, S_{\mathbf{s}}) : h \text{ maps } \delta \text{ onto } B^{\mathbf{s}}_{\langle\delta\rangle}\}$; it is a club of $\lambda(\langle\rangle, S_{\mathbf{s}})$ hence $S_3 := S_2 \cap C$ is a stationary subset of $\lambda(\langle\rangle, S_{\mathbf{s}})$. Also for $\delta \in S_3$ let $u_\delta = \{\alpha < \delta : h(\alpha) \in B'_\delta\}$.

By clause (B)(c) of 3.7, i.e., the choice of $\mathbf{s}$, without loss of generality one of the following occurs:

(a) $\delta \in S_3 \Rightarrow \mathrm{cf}(\delta) = \kappa_1$ for some regular $\kappa_1 < \kappa_*$,
(b) every $\delta \in S_3$ has cofinality $\geq \lambda_*$.

Case 1: $\kappa_1 < \kappa_*$ is as in clause (a)

Just use $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$ for $\lambda, S_3, \langle u_\delta : \delta \in S_3\rangle$ to prove G is not a μ_*-free, a contradiction.

Case 2: Clause (b) above holds

For $\delta \in S_3$ clearly $|u_\delta| = |A_\delta \cup B'_\delta| + \kappa_* = \mu < \mu_* \leq \lambda_* \leq \mathrm{cf}(\delta)$ hence there is $\gamma_\delta < \delta$ such that $u_\delta \subseteq \gamma_\delta$, hence for some $\gamma_* < \lambda$ the set $S_4 = \{\delta \in S_3 : u_\delta \subseteq \gamma_*\}$ is stationary.

Subcase 2A: $\mathrm{cf}([\gamma_*]^{\leq \mu_*}, \subseteq)$ is $< \lambda(\langle\rangle, S_{\mathbf{s}})$.

So for some $u_* \in [\gamma_*]^{\leq \mu}$ the set $S_5 = \{\delta < \lambda : u_\delta \subseteq u_*\}$ is a stationary subset of λ. Let $S_6 \subseteq S_5$ be of cardinality μ^+ and let

$$A^* = \cup\{A_\delta : \delta \in S_6\} \cup \{h(\alpha) : \alpha \in u_*\}.$$

Clearly $A^* \subseteq G$ is of cardinality $< \mu$ and $A^*/\emptyset$ is not free for $\mathbf{c}_1$.

So G has a non-free subgroup of cardinality $< \mu_*$, contradiction to the assumption "$G = G_2/G_1$ is μ_*-free".

Subcase 2B: $\mathrm{cf}([\gamma_*]^{\leq \mu}, \subseteq) \geq \lambda(\langle\rangle, S_{\mathbf{s}})$.

Note that because $\mathbf{V}$ satisfies the strong hypothesis (see [She93b]), necessarily for some cardinal ∂ of cofinality $< \kappa_*$ we have $\lambda(\langle\rangle, S_{\mathbf{s}}) = \partial^+$.

In any case clearly for every $\alpha \in [\gamma_*, \lambda)$, letting $\beta_\alpha = \min(S_4 \backslash \alpha)$, the pair $A_{\beta_\alpha}/B_{<\alpha>}$ is not $\mathbf{c}_1$-free. So renaming without loss of generality

$$\alpha \geq \gamma_* \wedge \mathrm{cf}(\alpha) = \aleph_0 \Rightarrow \langle \alpha \rangle \in S$$

and we continue as in Case 1, so this works also in Subcase 2A. $\blacksquare_{3.13}$

References

[CG85] A. L. S. Corner and R. Göbel, *Prescribing endomorphism algebras, a unified treatment*, Proceedings of the London Mathematical Society **50** (1985), 447–479.

[EM02] P. C. Eklof and A. Mekler, *Almost Free Modules: Set Theoretic Methods*, North-Holland Mathematical Library, Vol. 65, North-Holland, Amsterdam, 2002.

[Fuc73] L. Fuchs, *Infinite Abelian Groups. Vols. I, II*, Pure and Applied Mathematics, Vols. 36, 36-II, Academic Press, New York–London, 1970, 1973.

[GHS] R. Göbel, D. Herden and S. Shelah, *Prescribing endomorphism rings of $\aleph_n$-free modules*, Journal of the European Mathematical Society **16** (2014), 1775–1816.

[GS09] R. Göbel and S. Shelah, *$\aleph_n$-free modules with trivial dual*, Results in Mathematics **54** (2009), 53–64.

[GSS13] R. Göbel S. Shelah and L. Struengmann, *$\aleph_n$-free Modules over complete discrete valuation domains with small dual*, Glasgow Mathematical Journal **55** (2013), 369–380.

[GT12] R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules. Vols. 1, 2*, de Gruyter Expositions in Mathematics, Vol. 41, Walter de Gruyter, Berlin, 2012.

[MS94] M. Magidor and S. Shelah, *When does almost free imply free? (For groups, transversal etc.)*, Journal of the American Mathematical Society **7** (1994), 769–830.

[Shea] S. Shelah, *Compactness spectrum.*

[Sheb] S. Shelah, *Black boxes*, `https://arxiv.org/abs/0812.0656`

[Shec] S. Shelah, *Compactness in singular cardinals revisited*, Sarajevo Journal of Mathematics, to appear, `https://arxiv.org/abs/1401.3175`.

[Shed] S. Shelah, *Quite free Abelian groups with prescribed endomorphism ring.*

[Shee] S. Shelah, *More on black boxes*, in preparation.

[She74] S. Shelah, *Why there are many nonisomorphic models for unsuperstable theories*, in *Proceedings of the International Congress of Mathematicians (Vancouver, BC, 1974), Vol. 1*, Canadian Mathematical Congress, Montreal, QC, 1975, pp. 259–263.

[She75a] S. Shelah, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel Journal of Mathematics **21** (1975), 319–349.

[She75b] S. Shelah, *The lazy model-theoretician's guide to stability*, Logique et Analyse **18** (1975), 241–308.

[She78] S. Shelah, *Classification Theory and the Number of Nonisomorphic Models*, Studies in Logic and the Foundations of Mathematics, Vol. 92, North-Holland, Amsterdam–New York, 1978.

[She96] S. Shelah. *Further cardianl arithmetic*, Israel Journal of Mathematics **95** (1996), 61–114.

[She80] S. Shelah, *Whitehead groups may not be free, even assuming CH. II*, Israel Journal of Mathematics **35** (1980), 257–285.

[She84a] S. Shelah, *A combinatorial principle and endomorphism rings. I. On p-groups*, Israel Journal of Mathematics **49** (1984), 239–257.

[She84b] S. Shelah, *A combinatorial theorem and endomorphism rings of abelian groups. II*, in *Abelian Groups and Modules (Udine, 1984)*, CISM Courses and Lectures, Vol. 287, Springer, Vienna, 1984, pp. 37–86.

[She85] S. Shelah, *Incompactness in regular cardinals*, Notre Dame Journal of Formal Logic **26** (1985), 195–228.

[She90] S. Shelah, *Classification Theory and the Number of Nonisomorphic Models*, Studies in Logic and the Foundations of Mathematics, Vol. 92, North-Holland, Amsterdam, 1990.

[She93a] S. Shelah, *Advances in cardinal arithmetic*, in *Finite and Infinite Combinatorics in Sets and Logic*, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, Vol. 411, Kluwer Academic Publishers, Doedrecht, 1993, pp. 355–383.

[She93b] S. Shelah, *More on Cardinal Arithmetic*, Archive for Mathematical Logic **32** (1993), 399–428,

[She94] S. Shelah, *Cardinal Arithmetic*, Oxford Logic Guides, Vol. 29, Clarendon Press, Oxford University Press, New York, 1994.

[She96] S. Shelah, *If there is an exactly λ-free abelian group then there is an exactly λ-separable one*, Journal of Symbolic Logic **61** (1996), 1261–1278.

[She05] S. Shelah, *Middle diamond*, Archive for Mathematical Logic **44** (2005), 527–560.

[She07] S. Shelah, *$\aleph_n$-free abelain group with no non-zero homomorphism to $\mathbb{Z}$*, Cubo **9** (2007), 59–79.

[She08] S. Shelah, *Theories with EF-equivalent non-isomorphic models*, Tbilisi Mathematical Journal **1** (2008), 133–164.

[She13a] S. Shelah, *Non-reflection of the bad set for $\check{I}_\theta[\lambda]$ and pcf*, Acta Mathematica Hungarica **141** (2013), 11–35.

[She13b] S. Shelah, *PCF and abelian groups*, Forum Mathematicum **25** (2013), 967–1038.

[She16] S. Shelah, *ZF + DC + AX$_4$*, Archive for Mathematical Logic **55** (2016), 239–294.