DENSITY OF INDECOMPOSABLE LOCALLY FINITE GROUPS SH1181

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Dedicated to my friend, Laszlo Fuchs

Abstract. We prove that for locally finite group there is an extension of the same cardinality which is indecomposable for almost all regular cardinals smaller than its cardinality, noting that a group G is called θ-indecomposable when for every increasing sequence $\langle G_i : i < \theta \rangle$ of groups with union G there is $i < \theta$ such that $G = G_i$

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We thank Alice Leonhardt for the beautiful typing. First typed February 18, 2016 as part of [Sheb]. In References [She17, 0.22=Lz19] means [She17, 0.22] has label z19 there, L stands for label; so will help if [She17] will change. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. References like [Sheb, 1.3=La11] means we cite from [Sheb], Definition 1.3 which has label La18, this to help if [Sheb] will be revised. This is publication number 1181 in Saharon Shelah's list.

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§ 0. Introduction

We are interested here in the class $\mathbf{K}_{\rm lf}$ of locally finite groups; the subject naturally use finite group theory and infinte combinatorics, see the book Kegel-Wehfritz [KW73].

Wehrfritz asked about the categoricity of the class $\mathbf{K}_{\rm exlf}$ of exlf (existentially closed, locally finite, see 0.2) groups in any $\lambda > \aleph_0$. This was answered by Macintyre-Shelah [MS76] which proved that in every $\lambda > \aleph_0$ there are 2^λ non-isomorphic members of $\mathbf{K}^{\rm exlf}_\lambda$. This was disappointing in some sense: in $\aleph_0$ the class is categorical, so the question was perhaps motivated by the hope that also general structures in the class can be understood to some extent.

A natural and frequent question on a class of structures is the existence of rigid members, i.e. ones with no non-trivial automorphism. Now any exlf group $G \in \mathbf{K}_{\rm exlf}$ has non-trivial automorphisms - the inner automorphisms (recalling it has a trivial center). So the natural question is about complete members where a group is called complete <u>iff</u> it has no non-inner automorphism.

Concerning the existence of a complete, existentially closed locally finite group of cardinality λ: Hickin [Hic78] proved one exists in $\aleph_1$ (and more, e.g. he finds a family of $2^{\aleph_1}$ such groups pairwise far apart, i.e. no uncountable group is embeddable in two of them). Thomas [Tho86] assumed G.C.H. and built one in every successor cardinal (and more, e.g. it has no Abelian or just solvable subgroup of the same cardinality). Related are Macintyre [Mac76], Giorgetta-Shelah [GS84], Shelah-Zigler [SZ79], which investigate the so called $\mathbf{K}_{G_*}$. Recall that we assume that G_* is a countable existentially closed group and K_{G_*} is the class of groups such that every finitely generated subgroup is embeddable into G_*.

On the existence and non-existence of universal members see Grossberg-Shelah [GS83].

The paper [ST97] investigate the group of permutation of the natural numbers, and ask: what can be the set of regular cardinals θ such that the group is θ-indecomposable (called there $\theta \in {\rm CF}(G)$); the result is that essentially there are some so called pcf restriction (on pcf see [She94]) and those essentially are all the restrictions.

Lately has finally appeared [She17] which connect to stability theory, in particular though the class $K_{\rm exlf}$ is very unstable it has many definable complete quantifier free type. One application was to use this to to build canonical extensions of a locally finite group which are existentially closed and of the same cardinality. Another was to build so called complete extension in λ for $G \in \mathbf{K}^{\rm exlf}_\lambda$ for many cardinals λ.

Here we deal more specifically with the density of so called θ-idecomposable extensions of the same cardinality, simultaneously for almost all relevant regular cardinals θ, essentially best possible. Observe that for a regular cardinal θ, a group G of cardinality λ is trivially θ-indecomposable if $\theta > \lambda$ and is not so if $\theta = \lambda$ or just θ is equal to the cofinality of λ. Those are almost the only restrictions. The problematic case is $\theta \neq {\rm cf}(\mu) < \mu, \mu^+ = \lambda$ and more, see 1.5, 1.7

We prove that essentially for every locally finite group G there is a locally finite group H extending G of the same cardinality which is κ-indecomposable for every regular $\kappa \neq {\rm cf}(|G|)$ and sometimes $\kappa \neq {\rm cf}(\mu)$ when ${\rm cf}(\mu) < \mu, \mu^+ = \lambda$).

In addition of being of self interest, this helps in [Shea], in proving that: for μ strong limit singular of cofinality $\aleph_0$, there is a universal locally finite group

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of cardinality μ iff there is a canonical such group. The results apply to many other classes (in general for so-called abstract elementary classes) which has enough indecomposable members.

The result here also help in [Sheb], in proving results of the form "any locally finite group of cardinality $\lambda > \aleph_0$ can be extended to a complete one of the same cardinality (not just its successor as in earlier proofs)".

The current work and [Shea] were original part of [Sheb] but were separated by requests. In 2019, the existence of θ-indecomposable in λ (see 1.5) were considerably improved after Corson-Shelah [CS20] deal with indecomposable groups (while we are dealing with locally finite groups). The improvement was that earlier it was for many rather than all cardinals;. The aim of [CS20] was to prove the existence of strongly bounded groups

It is fitting that this work is dedicated to Laszlo: he has been the father of model Abelian group theory and much more; his book [Fuc73] made me in 1973 start to work in group theory (in particular, on Whitehead problem (in [She74], [She75] and the old better versions of the general compactness theorem in [She19]).

We thank the referee for helping to make the paper more reader friendly and Mark Poor for pointing out a problem.

The following started in Todorcevic [Tod87] and is used in the proof of 1.5.

Claim 0.1. *1)* $\mu^+ \to [\mu^+]^2_{\lambda^+}$ *except possibly when* $\lambda = \mu^+, \mu$ *singular limit of (possibly weakly) inaccessibles.*
2) If $\lambda > \aleph_0$ *is regular, then* $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \aleph_0)$.
3) $\aleph_1 \nrightarrow (\aleph_1; \aleph_1)^2_{\aleph_1}$.

Proof. 1) By Todorcevic [Tod87] and [She88, 3.1,3.3(3)].
2) By [Shear, Ch.IV], see history and the definition of Pr_1 there.
3) By Moore [Moo06]. $\square_{0.1}$

Definition 0.2. 1) Let $\mathbf{K}_{\mathrm{lf}}$ be the class of locally finite groups
2) Let $\mathbf{K}^{\mathrm{lf}}_\lambda$ be the class of $G \in \mathbf{K}_{\mathrm{lf}}$ which are of cardinality λ
3) For a group G and a set A of elements of G let $\mathrm{sb}(A, G)$ be the subgroup of G generated by A
4) K_{exlf}, the class of locally finite existentially closed groups, is the class of locally finite groups G, such that for every finite groups $H_1 \subseteq H_2$ and embedding f_1 of H_1 into G there is an embedding f_2 of H_2 into G extending f_1.
5) Let $K^{\mathrm{exlf}}_\lambda$ be the class of $G \in K_{\mathrm{exlf}}$ of cardinality λ.

Convention 0.3. 1) $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$ denote an a.e.c., see [She09]. with $K_{\mathfrak{k}}$ being a class of structures and $\leq_{\mathfrak{k}}$ a partial order on it (the reader can ignore this or use $\leq_{\mathfrak{k}}$ being a sub-structure)
2) A major case here is $\mathfrak{k}$ being a universal class (see below).

where

Definition 0.4. 1) We say $\mathbf{K}$ is a universal class when:

(a) for some vocabulary $\tau, \mathbf{K}$ is a class of τ-models;
(b) $\mathbf{K}$ is closed under isomorphisms;
(c) for a τ-model $M, M \in \mathbf{K}$ iff every finitely generated submodel of M belongs to $\mathbf{K}$.

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The following result from [She17] is quoted in this work but only superficially, however in application this is important.

Theorem 0.5. *Let $\mathfrak{S}$ be as in* [She17] *and λ be any cardinal $\geq |\mathfrak{S}|$.*
1) For every $G \in K^{\rm lf}_{\leq\lambda}$ there is $H_G \in \mathbf{K}^{\rm exlf}_{\lambda}$ which is λ-full over G (hence over any $G' \subseteq G$; see Definition [She17, 1.15=La33]*) and $\mathfrak{S}$-constructible over it (see* [She17, 1.19=La37]*).*
2) If $H \in \mathbf{K}^{\rm lf}_{<\lambda}$ is λ-full over $G (\in \mathbf{K}^{\rm lf}_{\leq\lambda})$ then H_G from above can be embedded into H over G, see [She17, 1.23(4)=La41(4)].

Notation 0.6. 1) Let G, H, K denote groups, usually locally finite
2) Let δ denote a limit ordinal; k, ℓ, m, n natural numbers; $i, j, \alpha, \beta, \gamma$ ordinals and $\lambda, \mu, \kappa, \theta$ cardinals

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§ 1. Indecomposability

Here we show the density of indecomposable locally finite groups, moreover for any $\lambda > \aleph_0$ and locally finite group G of cardinality λ there is an extension H of the same cardinality which is θ-indecomposable for almost all regular cardinals θ, noting that for $\theta > \lambda$ this trivially holds and for $\theta = \mathrm{cf}(\lambda)$ it trivially fail. The only additional exclusion is that for λ a successor a of singular, we may exclude the singular's cofinality. This is proved in 1.5(3)(b); before this in 1.4 we show how to use a colouring $\mathbf{c} : [\lambda]^2 \to \lambda$ to build a group extension. Lastly in 1.7 we justify the excluded cardinal.

Definition 1.1. 1) We say M is θ-decomposable or $\theta \in \mathrm{CF}(M)$ <u>when</u>: θ is regular and if $\langle M_i : i < \theta\rangle$ is $\subseteq$-increasing with union M, <u>then</u> $M = M_i$ for some i.
2) We say M is Θ-indecomposable <u>when</u> it is θ-indecomposable for every $\theta \in \Theta$.
3) We say M is $(\neq \theta)$-indecomposable <u>when</u>: θ is regular and if $\sigma = \mathrm{cf}(\sigma) \neq \theta$ then M is σ-indecomposable.
4) We say $\mathbf{c} : [\lambda]^2 \to S$ is θ-indecomposable <u>when</u>: if $\langle u_i : i < \theta\rangle$ is $\subseteq$-increasing with union λ then $S = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u_i\}$ for some $i < \theta$; similarly for the other variants.
5) If we replace $\subseteq$ by $\leq_{\mathfrak{k}}$ where $\mathfrak{k}$ is an a.e.c., <u>then</u> we write "$\theta - \mathfrak{k}$-indecomposable" or $\theta \in \mathrm{CF}_{\mathfrak{k}}(M)$.

Note that group G may be indecomposable as a group or as a semi-group; the default choice is semi-group; but note that for locally finite groups the two are the same.

Definition 1.2. We say G is θ-indecomposable inside G^+ <u>when</u> the following hold:

(a) $\theta = \mathrm{cf}(\theta)$;
(b) $G \subseteq G^+$;
(c) if $\langle G_i : i \leq \theta\rangle$ is $\subseteq$-increasing continuous and $G \subseteq G_\theta = G^+$ then for some $i < \theta$ we have $G \subseteq G_i$.

The point of the definition of indecomposable is the following observation, 1.3.

Using cases of indecomposability, see 1.5, help elsewhere to prove density of complete members of $\mathbf{K}^{\mathrm{lf}}_\lambda$ and improve characterization of the existence of universal members in e.g. cardinality $\beth_\omega$.

Below recall that δ is here a limit ordinal.

Observation 1.3. *1) Assume $\langle M_i : i < \delta\rangle$ is $\leq_{\mathfrak{k}}$-increasing with union M, each M_{i+1} is $\theta - \mathfrak{k}$-indecomposable or just each M_{2i+1} is $\theta - \mathfrak{k}$-indecomposable in M_{2i+2}. If $\mathrm{cf}(\delta) \neq \theta$, <u>then</u> M is $\theta - \mathfrak{k}$-indecomposable.*
2) If for $\ell = 1, 2$ the sequence $\langle M^\ell_i : i < \theta\rangle$ is $\leq_{\mathfrak{k}}$-increasing and $\bigcup_i M^1_i = M = \bigcup_i M^2_i$ and each M^1_i is $\theta - \mathfrak{k}$-indecomposable or just M^1_{2i+1} is θ-indecomposable inside M^1_{2i+2} for $i < \theta$, <u>then</u> $\bigwedge_{i<\theta} \bigvee_{j<\theta} M^1_i \leq_{\mathfrak{k}} M^2_j$.
3) If for $\ell = 1, 2$ the sequence $\langle M^\ell_i : i \leq \delta\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and each M^ℓ_{i+1} is $\theta - \mathfrak{k}$-indecomposable or just M^ℓ_{2i+1} is θ-indecomposable in M^ℓ_{2i+2} for $i < \delta$ and $M^1_\delta = M^2_\delta$ and $\theta = \mathrm{cf}(\delta) > \aleph_0$, <u>then</u> $\{i < \delta : M^1_i = M^2_i\}$ is a club of δ.
4) If M is a Jonsson algebra of cardinality λ, <u>then</u> M is $(\neq \mathrm{cf}(\lambda))$-indecomposable.

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5) Assume J is a directed partial order, $\langle M_s : s \in J\rangle$ is $\subseteq$-increasing and $J_* := \{s \in J : M_s$ is $\theta - \mathfrak{k}$-indecomposable$\}$ is cofinal in J. Then $\bigcup_{s\in J} M_s$ is $\theta - \mathfrak{k}$-indecomposable provided that:

(∗) *if $\bigcup_{i<\theta} J_i \subseteq J$ is cofinal in J and $\langle J_i : i < \theta\rangle$ is $\subseteq$-increasing, then for some i, J_i is cofinal in J or at least $\bigcup_{s\in J_i} M_s = \bigcup_{s\in J} M_s$.*

6) Assume G is a model (e.g. a group), $\alpha_* < \theta = \mathrm{cf}(\theta), G_\alpha \subseteq G \subseteq H$ for $\alpha < \alpha_*$ and $\cup\{G_\alpha : \alpha < \alpha_*\}$ generate G. If each G_α is θ-indecomposable inside H then G is θ-indecomposable inside H.
7) G is θ-indecomposable iff G is θ-indecomposable inside G.
8) If $G_1 \subseteq G_2 \subseteq H_2 \subseteq H_1$ and G_2 is θ-indecomposable inside H_2 then G_1 is θ-indecomposable inside H_1.

Proof. Should be clear but we elaborate, e.g.:
5) Toward contradiction let $\langle N_i : i < \theta\rangle$ be $\subseteq$-increasing with union $\bigcup_{s\in J} M_s$. For each $s \in J_*$ there is $i(s) < \theta$ such that $N_{i(s)} \supseteq M_s$. Let $J_j = \{i(s) : s \in J_*$ and $i(s) \leq j\}$ for $i < \theta$. Clearly $\langle J_i : i < \theta\rangle$ is as required in the assumption of (∗), hence for some $i < \theta$ we have $\bigcup_{s\in J} M_s = \bigcup_{s\in J_i} M_s$, so necessarily $N_i \supseteq \bigcup_{s\in J} M_s\}$, and thus equality holds. $\square_{1.3}$

We turn to $\mathbf{K}_{\mathrm{lf}}$.

Proposition 1.4. *1) Assume I is a linear order and $\mathbf{c} : [I]^2 \to \mathscr{U}$ is θ-indecomposable (hence onto $\mathscr{U}$, see Definition 1.1(4)) $G_1 \in \mathbf{K}_{\mathrm{lf}}$ and $a_i \in G_1 (i \in \mathscr{U})$ are[1] pairwise commuting and each of order 2 (or 1).*
Then there are $G_1, \bar b$ such that:

(a) *$G_2 \in \mathbf{K}_{\mathrm{lf}}$ extends G_1;*
(b) *G_2 is generated by $G_1 \cup \bar b$ where $\bar b = \langle b_s : s \in I\rangle$;*
(c) *b_s has order 2 for $s \in I$;*
(d) *if $s_1 \neq s_2$ are from I then $a_{\mathbf{c}\{s_1,s_2\}} \in \mathrm{sb}(\{b_{s_1}, b_{s_2}\})$ and moreover $a_{\mathbf{c}\{s_1,s_2\}} = [b_{s_1}, b_{s_2}]$*
(e) *$G_1 \subseteq G_2$, moreover $G_1 \subseteq_{\mathfrak{S}} G_2$, for $\mathfrak{S} = \Omega[\mathbf{K}_{\mathrm{lf}}]$ (used only in* [Sheb]*, we can use much smaller $\mathfrak{S}$, see* [She17, Def. 0.9=La14, 1.4=La18, Claim 1.16=La34]*;)*
(f) *$\mathrm{sb}(\{a_i : i \in \mathscr{U}\}, G_1)$ (the subgroup of G_1 generated by $\{a_i : i \in \mathscr{U}\}$) is θ-indecomposable inside G_2; see Definition 1.2. ,*

2) Assume $G_1 \in \mathbf{K}_{\mathrm{lf}}$ and I a linear order which is the disjoint union of $\langle I_\alpha : \alpha < \alpha_*\rangle, u_\alpha \subseteq \mathrm{Ord}$ and $\mathbf{c}_\alpha : [I_\alpha]^2 \to u_\alpha$ is θ_α-indecomposable for $\alpha < \alpha_*, \langle u_\alpha : \alpha < \alpha_*\rangle$ is a sequence of pairwise disjoint sets with union $\mathscr{U}$ and $0 \notin \mathscr{U}$ and $a_\varepsilon \in G_1$ for $\varepsilon \in \mathscr{U}$ and a_ε, a_ζ commute for $\varepsilon, \zeta \in u_\alpha, \alpha < \alpha_*$ and each a_ε has order 2 (or 1), and we let $a_0 = e$.
Let $\mathbf{c} : [I]^2 \to \mathscr{U} \cup \{0\}$ extend each $\mathbf{c}_\alpha$ and be zero otherwise.
Then there are $G_2, \bar b, \bar d$ such that:

[1] The demand "the a_i's commute in G_1" is used in the proof of $(*)_8$, and the demand "a_{β_i} has order 2 (or 1)" is used in the proof of $(*)_7$.

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(a)-(c) *as above*

(d)' *if* $\varepsilon \neq \zeta \in u_\alpha$ *then* $a_{\mathbf{c}\{,\varepsilon,\zeta\}} = d_\alpha a'_{\mathbf{c}\{\varepsilon,\zeta\}} d_\alpha^{-1} = d_\alpha [b_\varepsilon, b_\zeta] d_\alpha^{-1}$

(e) *as above*

(f)' *if* $\alpha < \alpha_*$ *then* $\mathrm{sb}(\{a_\varepsilon : \varepsilon \in u_\alpha\}, G_2)$ *is* θ_α*-indecomposable inside* G_2.

(g) $\bar{d} = \langle d_\alpha : \alpha < \alpha_* \rangle$ *is a sequence if pairwise commuting and distinct elements of order 2*

(h) *if* $\varepsilon \in u_\alpha, \zeta \in u_\beta$, *and* $\alpha \neq \beta$ *the* b_ε, b_ζ *commute*

3) In parts (1), (2)

(a) *The cardinality of* G_2 *is* $|G_1| + |I|$ *(or both are finite)*

(b) *If we omit the assumption "*$\mathbf{c}$ *is* θ*-indecomposable" then still clauses (a)-(e) of part (1) holds.*

(c) *Moreover, in part (1), if* σ *is a regular cardinal and* $\mathbf{c}$ *is* σ*-indecomposable then* $\mathrm{sb}(\{a_i : i \in \mathscr{U}\}, G_1)$ *is* σ*-indecomposable in* G_2.

(d) *Moreover, in part (2), if* $\alpha < \alpha_*$ *and* $\mathbf{c}_\alpha$ *is a* σ*-indecomposable function, then* $\mathrm{sb}(\{a_s : s \in I_\alpha\}, G_1)$ *is* σ*-indecomposable in* G_2 .

Proof. 1) Let

$(*)_1$ $\mathscr{X} = \{(u, a) : u \subseteq I$ is finite and $a \in G_1\}$.

We shall choose below members $h_{c,}, h_s \in \mathrm{Sym}(\mathscr{X})$ for $c \in G_1, s \in I$.
First,

$(*)_2$ for $c \in G_1$ we choose $h_c \in \mathrm{Sym}(\mathscr{X})$ as follows: for $u \in [I]^{<\aleph_0}$ and $a \in G_1$ let $h_c(u, a)$ be
- (u, ac^{-1})

Now clearly,

$(*)_3$ (a) indeed $h_c \in \mathrm{Sym}(\mathscr{X})$ for $c \in G_1$
(b) the mapping $c \mapsto h_c$ is an embedding of G_1 into $\mathrm{Sym}(\mathscr{X})$.
(c) so without loss of generality this embedding is the identity

Next

$(*)_4$ for $t \in I$ we define $h_t : \mathscr{X} \to \mathscr{X}$ by defining $h_t(u, a)$ by induction on $|u|$ for $(u, a) \in \mathscr{X}$ as follows:
(a) if $u = \emptyset$ then $h_t(u, a) = (\{t\}, a)$
(b) if $u = \{s\}$ then $h_t(u, a)$ is defined as follows:
(α) if $t <_I s$ then $h_t(u, a) = (\{t, s\}, a)$
(β) if $t = s$ then $h_t(u, a) = (\emptyset, a)$
(γ) if $s <_I t$ then $h_t(u, a) = (\{s, t\}, d)$ where :
- we have $d = aa_{\mathbf{c}\{s,t\}})$

(c) if $s_1 < \ldots < s_n$ list $u \in [I]^n$ and $k \in \{0, \ldots, n\}$ and $s \in (s_k, s_{k+1})_I$ where we stipulate $s_0 = -\infty, s_{n+1} = +\infty$ then $h_t(u, a)$ is equal to:
$\bullet_1$ $(u \cup \{t\}, aa_{\mathbf{c}\{s_1,t\}} \ldots a_{\mathbf{c}\{s_k,t\}})$

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(d) if $s_1 < \ldots < s_n$ list $u \in [I]^n$ and $k \in \{0, \ldots, n-1\}$ and $t = s_{k+1}$ then $h_t(u, a)$ is equal to[2]
- $(u \backslash \{t\}, aa^{-1}_{\mathbf{c}\{s_k,t\}}, \ldots, a^{-1}_{\mathbf{c}\{s_2,t\}} a_{\mathbf{c}^{-1}\{s_1,t\}})$

Note that

$(*)_5$ (a) $(*)_4(b)(\alpha)$ is the same as $(*)_4(c)$ for $n = 1, k = 0$
(b) $(*)_4(b)(\beta)$ is the same as $(*)_4(d)$ for $n = 1, k = 0$
(c) $(*)_4(b)(\gamma)$ is the same as $(*)_4(c)$ for $n = 1, k = 1$
(d) $(*)_4(a)$ is the same as $(*)_4(c)$ for $n = 0, k = 0$.

$(*)_6$ (a) indeed h_a, h_s are permutations of $\mathscr{X}$
(b) let G_2 be the subgroup of $\mathrm{Sym}(\mathscr{X})$ generated by $Y = \{h_a, h_s : a \in G_1, s \in I\}$
(c) the group G_2 is locally finite

[Why? Clause (a), just check and clause (b) is a definition. For clause (c), let Z be a finite subset of Y, without loss of generality for some finite subgroup H of G_1 and finite subset J of I the set Z is included in the set $\{h_a, h_s : a \in H, s \in J\}$. Without loss of generality $\{\mathbf{c}\{s,t\} : s \neq t \in J\} \subseteq H$. It suffice to prove that for every pair $(u, a) \in \mathscr{X}$ the closure of $\{(u, a)\}$ under $\{h_d, h_s : d \in H, s \in J\}$ is not just finite but has at most $2^{|J|} \times |H|$ elements. Now this closure is obviously included in the set $\{((u \setminus v) \cup w, c) : v = J \cap u, w \subseteq J \setminus u, c \in (aH)\}$ which satisfies the inequality.]

Now clearly:

$(*)_7$ if $t \in I$ then $h_t \in \mathrm{Sym}(\mathscr{X})$ has order 2

[It is enough to prove $h_t(h_t(u, a)) = (u, a)$. We divide to cases according to "by which clause of $(*)_4$ is $h_t(u, a)$ defined".

If the definition is by $(*)_4(a)$ then $h_t(\emptyset, a) = (\{t\}, a)$ and by $(*)_4(b)(\beta)$

$$h_t h_t(\emptyset, a) = h_t(\{t\}, a) = (\emptyset, a).$$

If the definition is by $(*)_4(b)(\beta)$, the proof is similar.

If the definition is by $(*)_4(b)(\gamma)$ then recalling $(*)_4(d)$

$$h_t(h_t(u, a)) = h_t(h_t(\{s\}, a)) = h_t(\{s, t\}, aa_{\mathbf{c}\{s,t\}}) = (\{s\}, aa_{\mathbf{c}\{s,t\}} a^{-1}_{\mathbf{c}\{s,t\}}). = (u, a)$$

If the definition is by $(*)(b)(\alpha)$, the proof is similar.
If the definition is by $(*)_4(c)$, then recall $(*)_4(d)$ and compute similarly to the two previous cases, recalling $\langle a_{\mathbf{c}\{s,t\}} : s \in I\rangle$ are pairwise commuting of order 2 (or 1).

If the definition is by $(*)_4(d)$ - this is just like the last case.

So $(*)_7$ holds indeed]

$(*)_8$ if $s \neq t \in I$ then $[h_s, h_t] = h_{a_i}$ in G_2 where $i = \mathbf{c}\{s, t\}$

[2]The a_s^{-1} and inverting the order are more natural but immaterial as long as we are assuming the "of order 2" and "pairwise commuting, but those are now used in fewer points.

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[Why? We have to check by cases; here we use "the a_i's are pairwise commuting in G_1 for $i \in \mathscr{U}$" Without loss of generality $s <_I t$, we shall now checked four representative cases (the point is that for (u, c), the members of $u \setminus \{s, t\}$ have little influence).

First

$(*)_{8.1}$ how is $(\emptyset, c)$ mapped?
- (a) $h_s^{-1} h_t^{-1} h_s h_t(\emptyset, c) =$ by $(*)_4(a)$
- (b) $h_s^{-1} h_t^{-1} h_s(\{t\}, c) =$ by $(*)_4(b)(\alpha)$
- (c) $h_s^{-1} h_t^{-1}(\{s, t\}, c) =$ by $(*)_4(b)(\gamma)$
- (d) $h_s^{-1}(\{s\}, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(a)$
- (e) $(\emptyset, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_2$
- (f) $h_{\mathbf{c}\{s,t\}}(\emptyset, c)$

Second

$(*)_{8.2}$ how is $(\{s\}, c)$ mapped?
- (a) $h_s^{-1} h_t^{-1} h_s h_t(\{s\}, c) =$ by $(*)_4(b)(\gamma)$
- (b) $h_s^{-1} h_t^{-1} h_s(\{s, t\}, ca_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(d)$ with $(s_1, s_2) = (s, t), k = 0$
- (c) $h_s^{-1} h_t^{-1}(\{t\}, ca_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(b)(\beta)$
- (d) $h_s^{-1}(\emptyset, ca_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(a)$
- (e) $(\{s\}, ca_{\mathbf{c}\{s,t\}}) =$ by "every a_i has order 2"
- (f) $(\{s\}, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_2$
- (g) $h_{\mathbf{c}\{s,t\}}(\{s\}, c)$

Third

$(*)_{8.3}$ how is $(\{t\}, c)$ mapped?
- (a) $h_s^{-1} h_t^{-1} h_s h_t(\{t\}, c) =$ by $(*)_4(b)(\beta)$
- (b) $h_s^{-1} h_t^{-1} h_s(\emptyset, c) =$ by $(*)_4(a)$
- (c) $h_s^{-1} h_t^{-1}(\{s\}, c) =$ by $(*)_4(d)$ with $(s_1, s_2) = (s, t), k = 1$
- (d) $h_s^{-1}(\{s, t\}, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(d)$ with $(s_1, s_2) = (s, t), k = 0$
- (e) $(\{t\}, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_2$
- (f) $h_{\mathbf{c}\{s,t\}}(\{t\}, c)$

Fourth and lastly

$(*)_{8.4}$ how is $(\{s, t\}, c)$ mapped?
- (a) $h_s^{-1} h_t^{-1} h_s h_t(\{s, t\}, c) =$ by $(*)_4(d)$ with $(s_1.s_2) = (s, t), k = 1$
- (b) $h_s^{-1} h_t^{-1} h_s(\{s\}, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(b)(\beta)$
- (c) $h_s^{-1} h_t^{-1}(\emptyset, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(b)(\beta)$
- (d) $h_s^{-1}(\{t\}, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_4(c)$ with $(s_1, s_2) = (s, t), k = 0$
- (e) $(\{s, t\}, ca^{-1}_{\mathbf{c}\{s,t\}}) =$ by $(*)_2$
- (f) $h_{\mathbf{c}\{s,t\}}(\{s, t\}, c)$

]

$(*)_9$ $\mathrm{sb}(\{a_i : i \in S\}, G_1)$ is θ-indecomposable inside G_2.

[Why? Because the function $\mathbf{c}$ is θ-indecomposable by an assumption of the proposition and $(*)_8$.]

Together we are done proving part (1).

2) First

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$(*)_{11}$ we can find a pair $(G_2, \bar{d})$ such that (this G_2 is not the final one):

(a) $G_2 \in \mathbf{K}^{\rm lf}_\lambda$

(b) $\bar{d} = \langle d_\alpha : \alpha < \alpha_* \rangle$

(c) $\bar{d}$ is a sequence of members of G_2, pairwise commuting each of order 2, and letting d_u be the product $\langle d_\alpha : \alpha \in u \rangle$ for finite $u \subseteq \alpha_*$ we have $d_u = e$ iff $u = \emptyset$

(d) the group G_2 extend G_1 and is generated by $G_1 \cup \langle d_\alpha : \alpha < \alpha_* \rangle$

(e) the sequence $\langle d_u^{-1} G_1 d_u : u \in [\alpha_*]^{<\aleph_0} \rangle$ is a sequence of pairwise commuting subgroups, with the intersection of any two being $\{e\}$

(f) (follows) $G_1 \le_{\mathfrak{S}} G_2$, (see clause (e) of 1.4(1)))

[Why? Let $\mathscr{X} = [\alpha_*]^{<\aleph_0} \times G_1$. For $c \in G_1$ we define the permutation h_c of $\mathscr{X}$ by: $h_c(u, s) = (u, ac^{-1})$ if $u = \emptyset$ and $h_c(u, a) = (u, a)$ otherwise. Next for $\alpha < \alpha_*$ we define h_α, a permutation of $\mathscr{X}$ by: $h_\alpha((u, a)) = (u \Delta \{\alpha\}, a)$ where Δ is the symmetric difference.

Easy to check.]

Now let $a'_i = d_\alpha^{-1} a_i d_\alpha$ for $i \in u_\alpha$; so clearly they are pairwise commuting, each of order 2 (or 1). So we can apply part (1) with $G_2, \langle a'_i : i \in \mathscr{U} \rangle, \mathbf{c} : [I]^2 \to \mathscr{U} \cup \{0\}$ here standing for $G_1, \langle a_i : i \in \mathscr{U} \rangle, \mathbf{c} : [I]^2 \to \mathscr{U}$ there. We get $G_3, \langle b^2_s : s \in I \rangle$.

Let $\bar{b} = \bar{b}^2$ and we shall show that the triple $(G_2, \bar{b}, \bar{d})$ is as require, this suffice.

Clauses (a)-(c), (e) are obvious. As for clause (f), fix $\alpha < \alpha_*$, and let $\langle G_{2,i} : i < \theta \rangle$ be an increasing sequence of subgroups of G_2 with union G_2. Recalling $\mathbf{c}_\alpha = \mathbf{c} \restriction [I_\alpha]^2$, as in the proof of part (1) for some $i < \theta_\alpha$ the set $\{a'_s : s \in I_\alpha\}$ is included in $G_{2,i}$. Without loss of generality $d_\alpha \in G_{2,i}$ hence for every $s \in I_\alpha$ we have $a_\alpha = d_\alpha a'_s d_\alpha^{-1} \in G_{2,i}$ so we are done.

For clasue (h) consider $\varepsilon \in I_\alpha, \zeta \in I_\beta, \alpha < \beta$, so $\mathbf{c}\{\varepsilon, \zeta\} = 0$ hence $a'_{\{\varepsilon,\zeta\}} = a'_0 = d_0^{-1} a_0 d_0 = d_0^{-1} e d_0 = e$ hence by clause (d) of the first partwe have $[b_\varepsilon, b_\sigma] = e$ which means that they are commuting.

For clause (d)', let $\varepsilon \neq \zeta \in u_\alpha$ so $a'_\alpha = d_\alpha^{-1} a_\alpha d_\alpha$ and $[b_\varepsilon, b_\zeta] = a'_\alpha$. Together clause (d)' holds

Lastly clause (g) holds by $(*)_{11}$.

3) By the proofs of parts (1) and (2). $\square_{1.4}$

Our main result is 1.5, in particular part (3).

Theorem 1.5. *1) If $G_1 \in \mathbf{K}^{\rm lf}_{\le\lambda}$ <u>then</u> for some $G_2 \in \mathbf{K}^{\rm lf}_\lambda$ extending G_1 and $a^\ell_\alpha \in G_2$ for $\ell \in \{0, 1, 2\}, \alpha < \lambda$ we have:*

$\oplus$ (a) $\mathrm{sb}(\{a^\ell_\alpha : \ell \in \{0,1,2\}, \alpha < \lambda\}, G_2)$ *includes* G_1

(b) *if $\ell \in \{0, 1, 2\}$ then $\langle a^\ell_\alpha : \alpha < \lambda \rangle$ is a sequence of pairwise distinct commuting elements of order 2 of G_2*

(c) *G_2 is generated by $\{a^\ell_\alpha : \alpha < \lambda, \ell \in \{0, 1, 2\}\}$.*

(d) *$G_1 \le_{\mathfrak{S}} G_2$, like clause (e) of 1.4(1)*

2) If $\lambda \ge \mu$ and $\mathbf{c} : [\lambda]^2 \to \mu$ is θ-indecomposable and $G_1 \in \mathbf{K}^{\rm lf}_{\le\mu}$ <u>then</u> there is $G_2 \in \mathbf{K}^{\rm lf}_\lambda$ extending G_1 such that G_1 is θ-indecomposable inside G_2 and $G_1 \le_{\mathfrak{S}} G_2$, like clause (e) of 1.4(1).

3) If $\lambda \ge \aleph_1$ and we let $\Theta = \Theta_\lambda = \{\mathrm{cf}(\lambda)\}$ except that $\Theta = \Theta_\lambda = \{\mathrm{cf}(\lambda), \partial\}$ when $(c)_{\lambda,\partial}$ below holds, <u>then</u> (a),(b) holds

(a) *some $\mathbf{c} : [\lambda]^2 \to \lambda$ is θ-indecomposable for every $\theta = \mathrm{cf}(\theta) \notin \Theta$*

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(b) *for every* $G_1 \in \mathbf{K}^{\rm lf}_{\le\lambda}$ *there is an extension* $G_2 \in \mathbf{K}^{\rm exlf}_\lambda$ *which is* θ*-indecomposable for every regular* $\theta \notin \Theta$ *(and* $G_1 \le_{\mathfrak{S}} G_2$*, see clause (e) of 1.4(1))*

$(c)_{\lambda,\partial}$ *for some* $\mu, \lambda = \mu^+, \mu > \partial = {\rm cf}(\mu)$ *and* $\mu = \sup\{\theta < \mu : \theta$ *is a regular Jonsson cardinal*$\}$.

Remark 1.6. 1) 1) Note that given $\lambda \ge \aleph_1$ the demand $(c)_{\lambda.\partial}$ determine ∂ and implies $\lambda > \aleph_\omega$
2) We intend to sharpen $(c)_{\lambda,\partial}$ in [Sheb]

Proof. 1) Without loss of generality the group G_1 is generated by its set of elements of order 2 (see [KW73] or [She17], but for clause (d) of 1.4(1) only the later). Let $\bar a = \langle a_i : i < \lambda\rangle$ list the elements of G_1 of order 2, possibly with repetitions.

Let $\alpha_* = \lambda, I = \lambda \times \{1,2\}$ lexicographically ordered, $I_\alpha = \{\alpha\} \times \{1,2\}$, $a'_{1+\alpha} = a_\alpha, u_\alpha = \{1+\alpha\}, \mathscr{U} = \{1+\alpha : \alpha < \alpha_*\}$, $\mathbf{c}_\alpha\{(\alpha,1),(\alpha,2)\} = 1+\alpha$ and apply 1.4(2) getting G_2 and $\langle b_s : s \in I\rangle$ and $\langle d_\alpha : \alpha < \lambda\rangle$. Letting $a^\ell_\alpha = b_{(\alpha,\ell)}$ for $\alpha < \lambda, \ell \in \{1,2\}$ and $a^0_\alpha = d_\alpha$ we are done.

In particular:

Clause (a) of $\oplus$

It holds by clause (d)' of 1.4(2)

Clause (b) of $\oplus$

We split the proof by cases. First if $\ell = 1,2$ then $\langle a^\ell_\alpha : \alpha < \lambda\rangle = \langle b_{(\alpha,\ell} : \alpha < \lambda\rangle$ is a sequence of pairwise commuting elements of order 2 (or 1) by clause (h) of 1.4(2).

Second, if $\ell = 0$ then $\langle a^\ell_\alpha : \alpha < \lambda\rangle = \langle d_\alpha : \alpha < \lambda\rangle$ is a sequence of pairwise commuting elements of order 2 (or 1) by clause (g) of 1.4(2)

Clause (c) of $\oplus$

By our choices.

Clause (d) of $\oplus$

By 1.4(2((e).

2) Let $G'_0 = G_1$, by part (1) with μ here for λ there is $G'_1 \in \mathbf{K}^{\rm lf}_\mu$ $\le_{\mathfrak{S}}$-extending G'_1 with $\langle a^\ell_\alpha : \ell \in \{0,1,2\}, i < \mu\rangle$ as there. Next choose $G'_2 \in \mathbf{K}^{\rm lf}_\lambda$ $\le_{\mathfrak{S}}$-extending G'_1.

Now the pair $(G'_2, \langle a^1_i : i < \mu\rangle)$ satisfies the assumptions in 1.4(1) hence there is $G'_3 \in \mathbf{K}^{\rm lf}_\lambda$ $\le_{\mathfrak{S}}$-extending G'_2 such that $H_1 = {\rm sb}(\{a^1_i : i < \mu\}, G'_2)$ is θ-indecomposable in G'_3. Similarly there is $G'_4 \in \mathbf{K}^{\rm lf}_\lambda$ $\le_{\mathfrak{S}}$-extending G'_3 such that $H_2 = {\rm sb}(\{a^2_i : i < \mu\}, G'_2)$ is θ-indecomposable inside G'_4 and $H_0 = {\rm sb}(\{a^0_i : i < \mu\}, G'_2)$ is θ-indecomposable inside G'_4. Now $H = {\rm sb}(H_0 \cup H_1 \cup H_2, G'_2)$ include G'_1 and recalling the previous sentences, by 1.3(6), it is θ-indecomposable inside G'_4 but $G_1 = G'_1 \subseteq H$ hence by 1.3(8) also G_1 is θ-indecomposable inside G'_4, so letting $G_2 = G'_4$ we are done.
3) For proving it:

$(*)_1$ it suffices to prove clause (a).

Why? So we are given $G_1 \in \mathbf{K}^{\rm lf}_{\le\lambda}$. Let $\Theta' = \{\theta \le \lambda : \theta = {\rm cf}(\theta)\} \backslash \Theta$ and $\sigma = {\rm cf}(\lambda)$ so it is a regular cardinal $\le \lambda$. Let $\partial = |\Theta'|$ so it is a cardinal $\le \lambda$ and let $\langle \theta_\varepsilon : \varepsilon < \partial\rangle$ list Θ'. We choose $G_{2,i}$ by induction on $i \le \partial\sigma$ ($\partial\sigma$ is ordinal product) such that:

$(*)_{1.1}$ (a) $G_{2,i} \in \mathbf{K}^{\rm exlf}_\lambda$
(b) $\langle G_{2,j} : j \le i\rangle$ is increasing continuous
(c) $G_{2,0}$ extends G_1

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(d) if $i = \delta j + \varepsilon, \varepsilon < \partial$ then $G_{2,i}$ is θ_ε-indecomposable inside $G_{2,i+1}$
(d) $G_i \le_{\mathfrak{G}} G_{i+1}$ see clause (e) of 1.4(1))

We can carry the induction, e.g. for $i = \partial j + \varepsilon + 1$ by 1.5(2), well for having $G_i \in \mathbf{K}^{\rm exlf}_\lambda$ we use 0.5, (recalling 1.3(8). By 1.3, $G_2 := G_{2,\partial\sigma}$ is as required.

We shall now prove clause (a) by induction on λ.

Case 1: $\lambda = \partial^+, \partial$ regular

Recall 0.1(1).

Case 2: λ a limit cardinal and $\lambda > \theta$

Let $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of regular cardinals with limit λ, now let:

$(*)_2$ (a) $\mathbf{c}_{i+1} : [\lambda_i^{++}]^2 \to \lambda_i^{++}$
(b) $\langle \mathbf{c}_j : j \le i \rangle$ is $\subseteq$-increasing
(c) $\mathbf{c}_i$ is θ-indecomposable, for every regular $\theta \ne \lambda_i^{++}$.

Arriving to i use Case 1 knowing that $\mathbf{c}_i \restriction [\bigcup\limits_{j<i} \lambda_j^{++}]^2$ does not matter.

Now $\mathbf{c} = \cup\{\mathbf{c}_i : i < \mathrm{cf}(\lambda)\}$ is as required by 1.3(8), and 1.3(5).

Case 3: $\lambda = \mu^+, \mu > \kappa = \mathrm{cf}(\mu) \ne \theta$ and $\mu > \theta$

Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence of cardinals $> \theta$ with limit μ, each a successor of regular.

Let $\mathbf{c}_i : [\lambda_i]^2 \to \lambda_i$ witness $\lambda_i \nrightarrow [\lambda_i]^2_{\lambda_i}$.

Let $\lambda_{<i} = \cup\{\lambda_j : j < i\}$.

For $\varepsilon < \lambda$ let f_ε be a one-to-one function from $\mu(1+\varepsilon)$ onto μ. Now define $\mathbf{c} : [\lambda]^2 \to \lambda$ such that:

$(*)_3$ (a) if $\alpha \ne \beta$ belongs to the interval $[\mu(1+\varepsilon) + \lambda_{<i}, \mu(1+\varepsilon) + \lambda_i)$ then $\mathbf{c}\{\alpha, \beta\} = f_\varepsilon^{-1}\big(\mathbf{c}_i\{\alpha - \mu(1+\varepsilon), \beta - \mu(i+\varepsilon)\}\big)$.
(b) if not then $\mathbf{c}\{\alpha, \beta\} = 0$.

Then

$(*)_4$ it suffices to prove $\mathbf{c}$ witness the desired conclusion.

So let θ be a regular cardinal not from Θ , without loss of generality $\theta < \lambda$; hence $\theta < \mu$ so for some $i(*) < \kappa$ we have $\theta < \lambda_{i(*)}$.

$(*)_5$ let $h : \lambda \to \theta$ and we should prove that for some $\varepsilon < \theta, \{\mathbf{c}\{\alpha, \beta\} : h(\alpha), h(\beta) < \varepsilon\}$ is equal to λ.

Now for each $\gamma < \lambda$ and $i < \kappa$, we define a function $h_{\gamma,i} : \lambda_i \to \theta$ by:

$(*)_6$ $h_{\gamma,i}(\alpha) = h((1+\gamma)\mu + \alpha)$ for $\alpha < \lambda_i$.

By the choice of $\mathbf{c}_i$:

$(*)_7$ for $\gamma < \lambda, i < \kappa$ there is $\varepsilon_{\gamma,i} < \theta$ such that the set $\{\mathbf{c}_i(\{\alpha, \beta\} : \alpha, \beta < \lambda$ and $h_{\gamma,i}(\alpha), h_{\gamma,i}(\beta) < \varepsilon_{\gamma,i}\}$ is equal to λ_i.

[Why $\varepsilon_{\gamma,i}$ exists? By the choice of $\mathbf{c}_i$.]

$(*)_8$ for each $\gamma < \lambda$ there is $\varepsilon_\gamma < \theta$ such that $\kappa = \sup\{i < \kappa : \varepsilon_{\gamma,i} \le \varepsilon_\gamma\}$.

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[Why? Because κ, θ are regular cardinals and $\kappa \neq \theta$.]

$(*)_9$ there is $\varepsilon < \theta$ such that $\lambda = \sup\{\gamma < \lambda : \varepsilon_\gamma \leq \varepsilon\}$.

[Why ε exists? Because λ is a regular cardinal $> \theta$.]

Now by the choices of the f_γ's and of $\mathbf{c}$ we can finish.

Case 4: $\lambda = \mu^+, \mu > \kappa = \mathrm{cf}(\mu) = \theta$ but μ not a limit of Jonsson cardinals.

Let $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta, \delta$ divisible by μ for transparency$\}$ and let $\bar{C}$ be such that:

$\boxplus_1$ (a) $\bar{C} = \langle C_\delta : \delta \in S \rangle$
- (b) (α) C_δ is a club of δ
  - (β) C_δ is of order type κ if $\kappa > \aleph_0$ and μ if $\kappa = \aleph_0$
  - (γ) $0 \in C_\delta$
  - (δ) each $\alpha \in C_\delta \backslash \{0\}$ is a limit ordinal
- (c) if E is a club λ then for some $\delta \in S \cap E$ we have:
  - for every $\sigma < \mu$ we have $\mu = \sup\{\alpha \in \mathrm{nacc}(C_\delta) : \mathrm{cf}(\alpha) > \sigma$ and $\alpha \in C$; moreover, $\alpha = \sup(E \cap \alpha)\}$

[Why such $\bar{C}$ exists? See [She94, Ch.III,§1].

$\boxplus_2$ choose
- (a) $\bar{e} = \langle e_\alpha : \alpha < \lambda \rangle, e_\alpha$ a club of α of order type $\mathrm{cf}(\alpha)$
- (b) $\mathbf{c}_\partial : [\partial]^{<\aleph_0} \to \partial$ witness $\partial \nrightarrow [\partial]^{<\aleph_0}_\partial$ for ∂ a regular non-Jonsson cardinal from (∂_*, μ) for some $\partial_* \in [\theta, \mu]$
- (c) $\bar{f} = \langle f_\alpha : \alpha \in [\mu, \lambda) \rangle, f_\alpha$ is a function from μ onto α.

Now a major point is the choice of $\mathbf{c} : [\lambda]^2 \to \lambda$:

$\boxplus_3$ we choose $\mathbf{c} : [\lambda]^2 \to \lambda$ such that if (A) then (B) where:
- (A) (a) $\delta_2 \in S$ and $\delta_1 \in S \cap \delta_2$
  - (b) $\beta = \min\{\beta : \delta_1 < \beta \in C_{\delta_2}\}$ so necessarily $\beta \in \mathrm{nacc}(C_2)$; recalling $\mathrm{nacc}(C) = \{\alpha \in C : \alpha > \sup(C \cap \alpha)\}$
  - (c) $\mathrm{cf}(\beta) > \partial_*$
  - (d) $u = \{\gamma \in e_\beta$: for some $\alpha \in C_{\delta_1}, \gamma = \mathrm{suc}_{e_\beta}(\alpha)\}$; recalling $\mathrm{suc}_e(\alpha) = \min\{\beta \in e : \beta > \alpha\}$
  - (e) $\mathrm{otp}(u)$ is $\zeta + n, \zeta$ is zero or a limit ordinal
  - (f) $\gamma_0 < \ldots < \gamma_{n-1}$ list the last n members of u
  - (g) $\partial = \mathrm{cf}(\beta)$
- (B) $\mathbf{c}(\{\delta_1, \delta_2\}) = f_{\delta_2}(\mathbf{c}_\partial(\{\mathrm{otp}(e_\beta \cap \gamma_\ell) : \ell < n\}))$.

Now

$\boxplus_4$ there is indeed $\mathbf{c}$ as in $\boxplus_3$.

[Why? The point is proving that for any $\delta_1 < \delta_2$ from S, at most one case of (A) of $\boxplus_3$ holds, i.e. there is at most one sequence pair$(\beta, \langle \gamma_\ell : \ell < n \rangle)$ as there. But this is obvious from the way $\boxplus_3(A)$ is stated.]

So it suffices to prove:

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$\boxplus_5$ $\mathbf{c}$ is θ-indecomposable, moreover it witnesses $\lambda \nrightarrow [\lambda]^2_\lambda$

$\boxplus_6$ let $h : \lambda \to \theta$ and it suffices to prove $(\exists \zeta < \theta)[\lambda = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta < \lambda$ and $h(\alpha), h(\beta) < \zeta\}]$.

Let

$\boxplus_{6.1}$ (a) let $\chi = [2^\lambda]^+ : <^*_\chi$ a well ordering of $\mathscr{H}(\chi)$
(b) $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$ is $\prec$-increasing continuous
(c) $M_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ and M_α has cardinality $\leq \mu$ for $\alpha < \lambda$
(d) $\mathbf{c}, \bar{e}, \bar{C}$ and h belong to M_0 hence to M_α for $\alpha < \lambda$
(e) $\bar{M} \restriction (\alpha + 1) \in M_{\alpha+1}$.

Next

$\boxplus_{6.2}$ (a) let $E_1 = \{\alpha < \lambda : M_\alpha \cap \lambda = \alpha\}$
(b) let $E_2 = \{\delta \in E_2 : \mathrm{otp}(E_1 \cap \delta) = \delta\}$.

Now

$\boxplus_7$ there is δ_2 such that:
(a) $\delta_2 \in E_2 \cap S$
(b) for every $\sigma < \mu$ we have:
$\delta_2 = \sup(A_\sigma)$ where $A_\sigma = \{\alpha \in \mathrm{nacc}(C_{\delta_2}) : \alpha \in E_2$ and $\mathrm{cf}(\alpha) > \sigma\}$.

The rest is as in [She03]. $\square_{1.5}$

Can we eliminate the exceptional θ in 1.5(3)(b)? By the following claim we cannot, at least as long as the following famous open problem is unresolved (it is whether every successor of singular cardinality a Jonsson algebra.

Claim 1.7. *1) If $\lambda = \mu^+, \mu$ singular and λ is a Jonsson cardinal, then every $G \in \mathbf{K}^{\mathrm{lf}}_\lambda$ is $\mathrm{cf}(\mu)$-decomposable.*
2) Moreover this holds for every model M with universe λ and vocabulary of cardinality $< \mu$.

Proof. Easy and it will not be used; in short let M be a model with countable vocabulary and universe λ coding enough set theory. By the assumption on λ there is a proper elementary submodel N of M of cardinality λ. For $\alpha < \mu$ let $N_\alpha$be the Skolem hull of $N \cup \alpha$ inside M. We know that each N_α is not equal to M, is non decreasing with α and the union of $\langle N_\alpha : \alpha < \mu \rangle$ is equal to M. $\square_{1.7}$

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