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RANDOM GRAPH: STRONGER LOGIC BUT WITH THE ZERO ONE LAW
SH1077

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Abstract. We like to find a logic really stronger than first order for the random graph with edge probability $\frac{1}{2}$ but satisfies the 0-1 law. This means that on the one hand it satisfies the 0-1 law, e.g. for the random graph $\mathscr{G}_{n,1/2}$ and on the other hand there is a formula $\varphi(x)$ such that for no first order $\psi(x)$ do we have: for every random enough $\mathscr{G}_{n,1/2}$ the formulas $\varphi(x), \psi(x)$ are equivalent in it. We do it adding a quantifier on graphs $\mathbb{Q}_{\mathbf{t}}$, i.e. have a class of finite graphs closed under isomorphisms and being able to say that if $(\varphi_0(x,\bar{c}), \varphi_1(x_0,x_1,\bar{c}))$, a pair of formulas with parameters defining a graph in $\mathscr{G}_{n,1/2}$, then we can form a formula $\psi(\bar{y})$ such $\psi(\bar{c})$ says that the graph belongs $K_{\bar{\mathbf{t}}}$. Presently we do it for random enough $\bar{\mathbf{t}}$.

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Annotated Content

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§ 0. Introduction

Our aim is to find a logic $\mathscr{L}$ stronger than first order such that: for $p \in (0,1)_{\mathbb{R}}$, the p-random graph $\mathscr{G} = \mathscr{G}_{n,p}$ (i.e. with edge probability p) satisfies the 0-1-law but some formula $\varphi(x) \in \mathbb{L}(\text{graphs})$ defines in random enough graph $\mathscr{G}_{n,p}$ a set of nodes not definable by any first order logic formula (of course, small enough compared to n, even with parameters).

The logic is gotten from first order $\mathbb{L}$ by adding a (Lindström) quantifier $\mathbf{Q}_{\bar{\mathbf{t}}} = \mathbf{Q}_{\mathbf{K}_{\bar{\mathbf{t}}}}$ gotten from a "random enough" $\bar{\mathbf{t}} \in {}^{\mathbb{N}}\{0,1\}$; on quantifiers see [Be85]. We may wonder, can we replace $\mathbf{Q}$ by a "reasonably defined quantifier"? We may from the proof see what we need from $\mathbf{K}$, the class defining the quantifier $\mathbf{Q}_{\mathbf{K}}$, i.e. a class of (finite) graphs closed under isomorphisms. Excluding some graphs which we call low, the membership in $\mathbf{K}$ should be random enough in the sense that if we consider only random enough $\mathscr{G}_{n,p}$, the non-trivial $\mathbb{L}(\mathbf{Q}_{\mathbf{K}})$-formulas with parameters will define graphs which are not low and are pairwise non-isomorphic except in trivial cases. So we just need a definition satisfying this; we hope to try to do it in a work in preparation.

How does the randomness of $\bar{\mathbf{t}}$ help us to get the zero-one law? The idea is that for the quantifier $\mathbf{Q}_{\bar{t}}$ (see §2) used here, if we expand $\mathscr{G}_{n,p}$ by finitely many relations definable by formulas from $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$, we get a random structure with more relations essentially with constant probabilities, i.e. is interpretable in a suitable $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$, see §1, it look like $\mathscr{G}_{p,n}$ (but with some relations of suitable kinds as we sort out), with, e.g. $\bar{p} = \langle p_n : n < \omega \rangle$ with p_n going slowly to zero.

That is, fixing formulas $\varphi_\ell(\bar{x}_\ell) \in \mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$ starting with $\mathbf{Q}_{\bar{\mathbf{t}}}, \ell < k$ with no obvious connections we decide a priory that for a random enough $\mathscr{G}_{n,\bar{p}}$ the structure $([n], R_\ell^{\mathscr{G}_{n,p}})_{\ell<k} = (\varphi(\mathscr{G}_{p.n}), \dots, \varphi_\ell(\mathscr{G}_{p,n}), \dots)_\ell$ for suitable formulas $\varphi(x), \varphi_\ell(\bar{x}_\ell)$, will look like $\mathscr{M}$ above.

The decision is the simplest one: look as if truth values of $R_\ell^{\mathscr{G}_{n,p}}(\bar{a})$ were drawn independently, with probability p_n. This is an over simplification! We need a more involved such drawing, reflecting the original $\bar{\varphi}_\ell$ to some extent, see below.

We may replace $\mathscr{M}_{\mathbf{s},\bar{p},n}$ by using (for some irrational $\alpha \in (0,1)$) $\bar{p}_n = (p, p_n)$, such that $p_n = \frac{1}{n^\alpha}$, except the original drawing of the graphs as in [SS88]. We can also analyze $\mathscr{G}n, rn^\alpha$ and use several pairs (r,α) in the analysis (as long as the sets of α's is linearly independent over the rationals). We hope later to show that for some such version there is a more natural definable $\mathbf{Q}_{\mathbf{K}}$ which imitate its behavior.

So in the proof we have two questions to address: first fixing $G = ([n], R_\ell)_{\ell<k}$, drawing the quantifiers, how $([n], R_\ell^G, \dots)$ look like. Second, we need to consider all the G's on $[n]$. For the first stage the main problems are: two definably derived graphs which are isomorphic.

We do some kind of elimination of quantifiers: essentially if $\mathscr{M}_n$ is a τ-structure (τ relational and finite) drawn randomly according to the sequence $\langle p_{\tau,R} : R \in \tau \rangle$ of fixed probabilities, applying $\mathbf{Q}_{\bar{\mathbf{t}}}$ to some finitely many schemes $\langle \mathbf{s}_0, \dots, \mathbf{s}_k \rangle$ of interpreting graphs, define a random $\mathscr{M}'_n$ for τ'-structures by expanding $\mathscr{M}_n$ by $R_\ell = \{\bar{c} : \ell g(\bar{c}) = \ell g(\bar{z}_{\mathbf{s}_\ell})$ and the graph $H_{\mathbf{s}_\ell,\bar{c}}$ interpreted by $\mathbf{s}_\ell$ for the parameter $\bar{c}$ is in the class $\mathbf{Q}_{\bar{\mathbf{t}}}\}$.

Our use of vocabulary and structure deviates a little from the standard, but fits with the use in graph theory and is natural here. In graph theory the edge relation R is assume to be symmetric and irrefelxive. So we use (say k_t-place predicate)

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R_t such that it is always irreflexive (fails for k_t-tuples with a repetition) and K_t-invariant for some group K_t of permutation of $\{0,\dots,k_t-1\}$, i.e. if $\langle a_\ell:\ell<k_t\rangle$ satisfies it then so does $\langle \bar a_{\pi(\ell)}:\ell<k_t\rangle$ for every $\pi\in K_t$. This is natural because when the pair of formulas $\bar\varphi(\bar c)$ defines a graph $H=H_{M,\bar\varphi,\bar c}$ in the structure M (e.g. a graph) and we like to draw a truth value for "$H\in\mathbf{K}_{\bar{\mathbf{t}}}$", a group of permutation of $\ell g(\bar c)$ is dictated by $\bar\varphi$.

Why the random auxiliary structures are better defined in a different way? Recall the truth value of "$H\in\mathbf{K}_{\bar{\mathbf{t}}}$" is chosen randomly, but if H is definable in the graph G, say is $H_{G,\bar\varphi,\bar c,\bar{\mathbf{t}}}$ then the probability of "$H\in\mathbf{K}_{\mathbf{t}}$" depends on H, and in natural cases, on $|H|$, the number of nodes of H. But if $\mathscr{M}=([n],\dots,R^0_\ell,\dots)$ is random, the standard way to make the probability of $\bar c\in R^G_\ell$ naturally depend on n and in many cases $n\neq|H|$.

We could have allowed using the quantifiers only on graphs H definable in $\mathscr{G}_{n,q}$ with set of nodes $[n]$ but this seems to me quite undesirable, restricting our logic too much. We restrict ourselves to the class of graphs - twice, we consider $\mathscr{G}_{n,q}$ and the quantifier $\mathbf{Q}_{\bar{\mathbf{t}}}$ is on graphs. But in both cases this is not really needed.

We thank Simi Haber for raising again the problem and for some stimulating discussions and Noga Alon for asking during a lecture in the Noga-fest, January 2011, why we ignore the weak graph; a reasonable interpretation is: why we do not draw a truth value for "G is green" for G a empty graph. One problem is that the sentence ψ saying "the graph with all nodes (is $[n]$) and no edges" the probability that $\mathscr{G}_{p,n}$ satisfies it is always zero or one and in non-trial cases is not eventually constant; see more in §3

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§ 1. Identifying the low graphs

We like to add a quantifier $\mathbf{Q}$ on finite graphs, which give a property of finite graphs respecting isomorphism (i.e. a subset closed under automormorphisms). The aim is that for e.g. for the random graph $\mathscr{G}_{n,p}$, the 0-1 law holds for $\mathbb{L}(\mathbf{Q})$ but there is an $\mathbb{L}(\mathbf{Q})$-formula $\varphi(x)$ such that for no first order $\psi(x)$ are $\varphi(x), \psi(x)$ equivalently in $\mathscr{G}_{n,p}$.

More specifically, we better make the quantifier trivial on too simple graphs, then we intend that for any fix finite set of formulas from $\mathbb{L}(\mathbf{Q})$, for random enough $G_{n,p}$ the structure $(G, \varphi^G(-))_{\varphi\in\Delta}$ is a random structure excluding the "problematic" graphs.

§ 1(A). **Interpretation.**

Convention 1.1. 1) $\mathbf{h} : \mathbb{N} \to (0, \frac{1}{2})_{\mathbb{R}}$ goes to zero slowly enough, e.g. $\mathbf{h}(n) = 1/\log_2\log_2(n)$ for $n > 16$ and $= 1$ if $n \le 16$; slowly enough actually means:

- (a) $\alpha \in (0,1)_{\mathbb{R}} \Rightarrow \infty = \lim\langle \mathbf{h}(n) n^\alpha : n < \omega\rangle$
- (b) $0 = \lim\langle \mathbf{h}(n) : n < \omega\rangle$
- (c) $n^{\mathbf{h}(n)}$ is non-decreasing (for simplicity).

2) $\mathbf{g} : \mathbb{N} \to \mathbb{R}_{\ge 0}$ be $\mathbf{g}(n) = n^{\mathbf{h}(n)}$ hence $\mathbf{g}(1+n) \ge 1$ and so $\mathbf{g}$ go to infinity slowly enough.

Notation 1.2. 1) Let $[n] = \{1, \dots, n\}$ or $\{0, \dots, n-1\}$ if you prefer (serve as the universe of the n-th random graph).
2) Let τ denote a vocabulary (e.g. $\tau = \tau_{\rm gr}$ is the vocabulary of graphs; see Definition 1.3 below). Let $\mathbb{L}$ be first order logic so $\mathbb{L}(\tau)$ is the set of first order formulas in the vocabulary τ, but below we may write $\mathbb{L}(\mathbf{s})$ instead of $\mathbb{L}(\tau_{\mathbf{s}})$.
3) A τ-model M is defined as usual.
4) For a formula $\varphi = \varphi(\bar{x}, \bar{y})$, model M and $\bar{b} \in {}^{\ell g(\bar{y})}M$ let $\varphi(M, \bar{b}) = \{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}, \bar{b}]\}$.

The following is a central definition, explicating the restriction to what is definable.

Definition 1.3. 1) For a finite set I we say $\mathbf{s}$ is an I-kind or an I-kind sequence (of a vocabulary) and write $I_{\mathbf{s}} = I$ <u>when</u>:

- (a) $\mathbf{s} = \langle (k_t, K_t) : t \in I\rangle = \langle (k_{\mathbf{s},t}, K_{\mathbf{s},t}) : t \in I\rangle$
- (b) $k_t \in \mathbb{N}$
- (c) K_t is a group of permutations of $\{0, \dots, k_t - 1\}$.

1A) Let $\mathbf{s}_{\rm gr} = \mathbf{s}({\rm gr})$ be defined by (gr stands for graphs) $I_{\mathbf{s}} = \{s_{\rm gr}\}, s_{\rm gr}$ fix, e.g. $0, k_{\mathbf{s},s_0} = 2, K_{\mathbf{s},s_0} = {\rm sym}(2)$, the group of permutations of $\{0,1\}$.
2) For $\mathbf{s}$ an I-kind sequence we define:

- (a) the $\mathbf{s}$-vocabulary $\tau_{\mathbf{s}}$ is $\{R_t : t \in I\}, R_t$ a $k_{\mathbf{s},t}$-place predicate
- (b) an $\mathbf{s}$-structure is $M = (|M|, R_t^M)_{t\in I}$ such that (so the universe $|M|$ of M may be empty):
  - (α) R_t^M is a k_t-place relation on $|M|$

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(β) R^M_t is K_t-invariant, i.e. if $\langle a_\ell : \ell < k_t\rangle \in R^M_t \wedge \bar b \in \bar a/E_{K_t} \Rightarrow \bar b \in R^M_t$ where $\bar a/E_{K_t} = \{\langle a_{\pi(\ell)} : \ell < n_t\rangle : \pi \in K_t\}$; let $E_{\mathbf{s},t} = E_{K_t}$

(γ) R^M_t is irreflexive, i.e. $\bar a \in R^M_t \Rightarrow \bar a$ with no repetitions.

(c) $\mathbf{M}_{\mathbf{s}} = \cup\{\mathbf{M}_{s,n} : n \in \mathbb{N}\}$ where $\mathbf{M}_{s,m} = \{M : M$ an $\mathbf{s}$-structure with set of elements $[n]\}$.

3) For an I-kind $\mathbf{s}$ let $\mathbf{P}^1_{\mathbf{s}}$ be the set of $\bar p = \langle p_{t,n} : t \in I, n \in \mathbb{N}\rangle, p_{t,n} \in (0,1)_{\mathbb{R}}$, so $p_{t,n} \notin \{0,1\}$. We define the $(\mathbf{s},\bar p)$-random structure on $[n], \mathscr{M} = \mathscr{M}_{\mathbf{s},\bar p,n}$ as follows (see more in part (5),(6): for $t \in I$ and $\bar a \in {}^{k_t}([n])$ with no repetitions we draw a truth value for $\bar a \in R^{\mathscr{M}}_t$ with probability $p_{t,n}$, but demanding we have the same result for $\bar a', \bar a''$ when they are $E_{\mathbf{s},t}$-equivalent and independent otherwise.
3A) Let $\mathbf{P}^0_{\mathbf{s}}$ for $\mathbf{s}$ as above be the set of $\bar p \in \mathbf{P}^1_{\mathbf{s}}$ such that $t \in I_{\mathbf{s}} \wedge n \in \mathbb{N} \Rightarrow p_{t,n} = p_{t,0}$, so we may write p_t instead of $p_{t,0}$. If $\mathbf{s} = \mathbf{s}_{\rm gr}$, we may write gr instead $\mathbf{s}$.
4) Let $\mathbf{P}^2_{\mathbf{s}}$ be the set of $\bar p \in \mathbf{P}^1_{\mathbf{s}}$ such that for some $\bar q \in \mathbf{P}^0_{\mathbf{s}}$ and partition $\bar I = (I_0, I_1)$ of I, we have $p_{t,n}$ is q_t if $t \in I_0$ and is $q_0/\mathbf{g}(n)$ if $t \in I_1$; we denote $\bar p$ by $\bar p_{\bar q,I_0} = \bar p_{\bar q,\bar I} = \bar p[\bar q,\bar I]$.
4A) We may write p instead of $\langle p_t : t \in I_{\rm gr}\rangle$ when $p_{s_{\rm gr}} = p$.
5) For $\bar p \in \mathbf{P}^1_{\mathbf{s}}$ let $\mu_{\mathbf{s},\bar p,n}$ be the distribution (= probability space) on $\mathbf{M}_{\mathbf{s},n}$ corresponding to drawing the truth value of $R_t(\bar a)$ really of $\langle \mathbf{R}_t(\bar a') : \bar a' \in \bar a/E_{\mathbf{s},t}\rangle$ for a sequence $\bar a$ with no repetitions of length $k_{\mathbf{s},t}$ with probability $p_{t,n}$, independently of the other choices.
6) Let $\mathscr{M}_{\mathbf{s},\bar p,n}$ be the random variable for the finite probability space $(\mathbf{M}_{\mathbf{s},n}, \mu_{\mathbf{s},\bar p,n})$. If $\mathbf{s} = \mathbf{s}_{{\rm gr},q}$ let $\mathscr{G}_{q,n} = \mathscr{G}_{{\rm gr},q,n} = \mathscr{M}_{\mathbf{s}_{\rm gr},q,n}$ and $\mu_{{\rm gr},q,n} = \mu_{\mathbf{s}_{\rm gr},\bar p_{{\rm gr},q},n}$.

Recall

Fact 1.4. 1) $\mathbf{P}^0_{\mathbf{s}} \subseteq \mathbf{P}^2_{\mathbf{s}} \subseteq \mathbf{P}^1_{\mathbf{s}}$.
2) For every $\bar p \in \mathbf{P}^0_{\mathbf{s}}$ or $\bar p \in \mathbf{P}^2_{\mathbf{s}}, \mathscr{M}_{\mathbf{s},\bar p,n}$ satisfies the 0-1 law for first order logic and the limit theory $T_{\mathbf{s},\bar p}$ has elimination of quantifiers, really is $T_{\mathbf{s}}$, i.e. does not depend on $\bar p$ and $\mathbf{g}$ and $\mathbf{h}$ (as long as they are as in 1.1(2)).
3) $\mathbf{M}_{\mathbf{s}_{\rm gr},n}$ is the set of graphs with set of nodes $[n]$.

Proof. Should be clear. $\square_{1.4}$

Remark 1.5. 1) We first concentrate on one application of the quantifier.
2) We are interested in interpreting graphs. We give the most general case. Note that we intend the quantifier to be a property of graphs. So we have to think of an interpretation of a graph. In such general interpretations using quantifier free formulas the elements may be only: a set of elements definable by a formula $\varphi(x,\bar a), \bar a$ is a sequence of parameters or more generally such a set of k-tuples, maybe modulo suitable E_K, or even a finite union of such. For each pair of the nodes (fixing from where in the union they come) we define when it is an edge by a quantifier free formula. So below $\bar z$ are parameters, $\mathbf{i}(\bar\varphi)$ number of "kinds of elements", ways to define a node; $\varphi_{0,i}$ restrict the i-th kind, $\varphi_2(\bar z)$ describes the relevant parameters, $\varphi_{1,i,j}$ describes the edges between a node of the i-th kind and a node of the j-th kind.
3) Generally in interpretations we allow the set of elements to be e.g. the set of equivalence classes of an equivalence relation defined say by $\varphi(\bar x', \bar x'', \bar a)$, where $\lg(\bar x') = \lg(\bar x'')$ but in our case those will always be degenerated, see 1.10.

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Definition 1.6. 1) For $\mathbf{s}$ an I-kind, we say $\bar{\varphi}$ is a $\mathbf{s}$-scheme (of a graph interpretation in $\mathbf{s}$-structures) <u>when</u> it consists of:

- (a) $\langle \varphi_{0,i}(\bar{x}_i, \bar{z}), \varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z}), \varphi_2(\bar{z}) : i, j < \mathbf{i}(\bar{\varphi}) \rangle$ such that:
  - $\bullet_1$ $\ell g(\bar{x}'_j) = \ell g(\bar{x}_j)$, it is possibly zero
  - $\bullet_2$ $\langle \bar{x}_i, \bar{x}'_i : i < \mathbf{i}(\varphi) \rangle$ are pairwise disjoint, each with no repetitions
  - $\bullet_3$ $\mathbf{i}(\bar{\varphi})$ is a non-zero natural number; if we allow $\mathbf{i}(\bar{\varphi}) = 0$ then we have to allow the empty graph.
- (b) $\varphi_{0,i}, \varphi_{1,i,j}, \varphi_2$ are formulas in the vocabulary $\tau_{\mathbf{s}}$, in this section they always are quantifier free formulas in $\mathbb{L}(\tau_{\mathbf{s}})$, note that possibly $\varphi_{1,i=\varphi_{1,j}}$ though $i \neq j$.
- (c) $K_i = K_{\bar{\varphi},i}$ is a group of permutations of $\{0, \dots, \ell g(\bar{x}_i) - 1\}$, not related to $K_{\mathbf{s},t}(t \in I)$!
- (d) $\varphi_{0,i}(\bar{x}_i, \bar{z})$ is invariant under permuting $\bar{x}_i$ by any $\pi \in K_i$; that is if $\pi \in K_i$; $\bar{x}'_i = \langle x_{i,\pi(f)} : \ell < \ell g(\bar{x}_i) \rangle$ then $\varphi_2(\bar{z}) \vdash_{\mathbf{s}} (\forall x_0 \dots x_\ell \dots)(\varphi_{1,i}(\dots x_\ell \dots; \bar{z}) \equiv \varphi_{1,i}(\dots x_{\pi(\ell)}, \dots, \bar{z}))$ where $\vdash_{\mathbf{s}}$ means implication in every $\mathbf{s}$-structure
- (e) $\varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z})$ is invariant under permuting $\bar{x}_i, \bar{x}'_j$ by $\pi \in K_i, \varkappa \in K_j$ respectively, and $\vdash \varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z}) \equiv \varphi_{1,j,i}(\bar{x}'_j, \bar{x}_i, \bar{z})$ and $\vdash \neg\varphi_{1,i,i}(\bar{x}_i, \bar{x}_i, \bar{z})$
- (f) if M is a $\tau_{\mathbf{s}}$-structure and $G \models \varphi_{0,i}[\bar{a}, \bar{c}]$, so $\ell g(\bar{c}) = \ell g(\bar{z})$ <u>then</u> $\bar{a}^\frown \bar{c}$ is with no repetitions.

So if we have $\bar{\varphi} = \bar{\varphi}^\iota$ then $\varphi^\iota_{0,i} = \varphi_{0,i}$, etc. and we may write $\bar{z}_{\bar{\varphi}}, \bar{x}_{\bar{\varphi},1,i}, \bar{x}'_{\bar{\varphi},1,1}$.
2) If $\mathbf{s}$ and $\bar{\varphi}$ are as above, M is an $\mathbf{s}$-structure and $\bar{c} \in \varphi_2(M)$, i.e. $\bar{c} \in {}^{\ell g(\bar{z})}M$ satisfies $M \models \varphi_2[\bar{c}]$ <u>then</u> $H = H_{\bar{\varphi},\mathscr{M},\bar{c}}$ is the follwoing graph:

- (α) the set of nodes is $\{(i, \bar{a}/E_{K_i}) : M \models \varphi_{0,i}[\bar{a}, \bar{c}]$ for some $i < \mathbf{i}(\bar{\varphi})$ and $\bar{a} \in {}^{\ell g(\bar{x}_i)}M\}$, see 1.3(2)(b)$(\beta)$
- (β) $\{(i, \bar{a}/E_{K_i}), (j, \bar{b}/E_{K_j})\}$ is an edge iff $M \models \varphi_{1,i,j}[\bar{a}, \bar{b}, \bar{c}]$.

3) Let $k_{\mathbf{s}}(\bar{\varphi}) = \max(\{\ell g(\bar{x}_i) : i < \mathbf{i}(\bar{\varphi})\} \cup \{\ell g(\bar{z})\})$ and let $k_{\mathbf{s},i}(\bar{\varphi}) = \ell g(\bar{x}_i), k^*_{\mathbf{s}}(\bar{\varphi}) = \max\{\ell g(\bar{x}_i) : i < \mathbf{i}(\bar{\varphi})\}$.
4) We say $\varphi_2(\bar{z})$ is complete <u>when</u> for any $\mathbf{s}$-structure M, if $\bar{a}_1, \bar{a}_2 \in \varphi_2(M)$ then $\bar{a}_1, \bar{a}_2$ realizes the same quantifier free type in M.
5) We say $\bar{\varphi}$ is complete <u>when</u> $\varphi_2(\bar{z})$ and each $\varphi_{0,i}(\bar{x}_i, \bar{z})$ is (not contradictory and is) complete (see (4)) and $\varphi_{0,i}(\bar{x}_i, \bar{z}) \vdash \varphi_2(\bar{z})$. If not said otherwise, we assume $\bar{\varphi}$ is complete.

Observation 1.7. *1) In Definition 1.6(2), $H_{\bar{\varphi},M,\bar{c}}$ is indeed a graph (possibly empty) and is finite <u>when</u> M is finite $\tau_{\mathbf{s}}$-structure.*
2) For each $\bar{\varphi}$ as in 1.6(1), for each $i < \mathbf{i}(\bar{\varphi})$ one of the following holds:

- (α) *for some $k, \varphi_2(\bar{z}) \vdash (\exists^{!k} \bar{x}_i)(\varphi_i(\bar{x}_i, \bar{z}))$*
- (β) *for every k for some $\mathbf{s}$-structures M, in M we have $\varphi_2(\bar{z}) \vdash (\exists^{\geq k} \bar{x})\varphi_{0,i}(\bar{x}_i, \bar{z})$.*

Proof. Read Definition 1.6(1). $\square_{1.7}$

Observation 1.8. *1) Let $\mathbf{s}$ be an I-kind and $\bar{\varphi}$ is a complete $\mathbf{s}$-scheme.*
The following are equivalent:

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(a) *for every* $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ *and random enough* $\mathscr{M} = \mathscr{M}_{\mathbf{s},n}$ *we have* $\varphi_2(\mathscr{M}) \neq \emptyset$

(b) *for some* $\bar{p} \in \mathbf{P}^2_{\mathbf{s}} \cup \mathbf{P}^0_s$ *we have* $0 < \limsup_n \operatorname{Prob}(\varphi_2(\mathscr{M}_{\mathbf{s},p,n}) \neq \emptyset)$.

2) For any sentence $\psi \in \mathbb{L}(\tau_{\mathbf{s}})$, *similarly replacing* $\varphi_2(\mathscr{M}) \neq \emptyset$ *by "for some* $\bar{c}, H_{\bar{\varphi},\mathscr{M},\bar{c}} \models \psi$".

Proof. Easy. $\square_{1.8}$

Definition 1.9. 1) We call an $\mathbf{s}$-scheme $\bar{\varphi}$ trivial <u>when</u> for each $i < \mathbf{i}(\varphi)$ we have $\ell g(\bar{x}_i) = 0$.
2) We call an $\mathbf{s}$-scheme $\bar{\varphi}$ degenerated <u>when</u> the conditions of 1.8 fail; as long as $\bar{\varphi}$ is complete this does not occur as $\mathbf{i}(\bar{\varphi}) \neq 0$, Def 1.6(1)$ab_3$.

3) We say the $\mathbf{s}$-scheme $\bar{\varphi}$ is 1-weak <u>when</u> at least one of the following holds:

(a) $\mathbf{s}$ is degenerated <u>or</u> $\mathbf{s}$ is trivial, i.e. $\ell g(\bar{x}_i) = 0$ for every $i < \mathbf{i}(\varphi)$ <u>or</u>

(b) for some truth value $\mathbf{t}$ and $i_1, i_2 < \mathbf{i}(\varphi)$ satisfying $\ell g(\bar{x}_{i_1}), \ell g(\bar{x}_{i_2}) \geq 1$ and $v_1 \subsetneqq \ell g(\bar{x}_{i_1}), v_2 \subsetneqq \ell g(\bar{x}_{i_2})$ we have
- for some (equivalently any) $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$, for random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$, for some $\bar{c} \in \varphi_2(\mathscr{M})$ and $\bar{a}^*_\ell \in \varphi_{1,i_\ell}(\mathscr{M}, \bar{c})$ for $\ell = 1, 2$ we have
- if $\bar{a}_\ell \in \varphi_{1,i_\ell}(\mathscr{M}, \bar{c})$ and $\bar{a}_\ell{\restriction}v_\ell = \bar{a}^*_\ell{\restriction}v_\ell$ for $\ell = 1, 2$ and $\operatorname{rang}(\bar{a}_1) \cap \operatorname{rang}(a_2) \subseteq \operatorname{rang}(\bar{a}^*_1{\restriction}v_1) \cap \operatorname{rang}(\bar{a}^*_2{\restriction}v_2)$ <u>then</u> $\mathscr{M} \models \varphi_{i_1,i_2}[\bar{a}_1, \bar{a}_2, \bar{c}]^{\mathrm{if}(\mathbf{t})}$.

4) We say the $\mathbf{s}$-scheme $\bar{\varphi}$ is 2-weak <u>when</u> at least one of the following holds:

(a) it is degenerated <u>or</u> trivial, i.e. as in (a) of part (3)

(b) for some $i < \mathbf{i}(\varphi), \ell g(\bar{x}_i) \geq 2$

(c) for some $i_1, i_2 < i(\varphi)$ with $\ell g(\bar{x}_{i_2}) = 1 = \ell g(\bar{x}_{i_2})$ and $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ and random enough $\mathscr{M} = \mathscr{M}_{s,\bar{p},n}$ and $\bar{c} \in \varphi_2(\mathscr{M})$ there is $\mathbf{t} \in \{0, 1\}$ such that for every $a_1 \in \varphi_{1,i_1}(\mathscr{M}), a_2 \in \varphi_{1,i_2}(\mathscr{M})$ we have $a_1 \neq a_2 \Rightarrow H_{\bar{\varphi},\mathscr{M},\bar{c}} \models$ "$a_1 R a_2$ iff $\mathbf{t} = 1$".

5) We say the $\mathbf{s}$-scheme is 3-weak <u>when</u> it is 1-weak or 2-weak.

Claim 1.10. *1) For any* k, *if* $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ *is random enough for* k *and* $\bar{c} \in {}^{k\geq}M$, *and there is an interpretation using as parameter* $\bar{c} \in {}^{k\geq}\mathscr{M}$ *of a graph* H *in* $\mathscr{M}$ *using* $(\leq k)$*-tuples (in the widest sense - the elements can be equivalence classes of suitable definable equivalence relations on set of tuples satisfying a formula) by formulas of length* $\leq k$ <u>*then*</u> *there is a complete* $\mathbf{s}$*-scheme* $\bar{\varphi}$ *such that* $H = H_{\bar{\varphi},\mathscr{M},\bar{c}}$ *and* $k(\bar{\varphi}) \leq k$.
1A) For any interpretation by first order formulas with parameter $\bar{z}$

(*) *there is an* $\mathbf{s}$*-scheme interpretation equivalent to it, and we can compute it,*

(*) *moreover we can compute a finite sequence* $\langle \bar{\varphi}_i : i < i_* \rangle$, *each* $\bar{\varphi}_i$ *is complete, all with the same parameter* $\bar{c}$.

2) In fact $\bar{\varphi}$ *depends just on the interpretation and the quantifier free type of* $\bar{c}$ *in* $\mathscr{M}$, *not on* $\mathscr{M}$ *(and even* n*).*
3) Given $\mathbf{s}$ *and* k *there only finitely many scheme* $\bar{\varphi}$ *as above.*

Proof. Obvious. $\square_{1.10}$

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Definition 1.11. Let $\mathbf{s}, \bar{\varphi}$ be as above, $\bar{\varphi}$ is complete, see 1.6(5).
We say $(\mathbf{s}, \bar{\varphi})$ is reduced when: for every $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ and random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ and $\bar{c} \in {}^{\ell g(\bar{z}_{\bar{\varphi}})}\mathscr{M}$ satisfying $\varphi_2(\bar{z}_{\bar{\varphi}})$, the graph $H = H_{\bar{\varphi},\mathscr{M},\bar{c}}$ is not $H = H_{\bar{\varphi}',\mathscr{M},\bar{c}'}$ when ($\bar{\varphi}', \bar{c}'$ appropriate and) $\mathrm{Rang}(\bar{c}') \subsetneqq \mathrm{Rang}(\bar{c})$; recall $\bar{c}$ is without repetitions.

§ 1(B). **Simple Random Graph.**

Our intention is that the behaviour of $\mathscr{G}_{q,n}$ expanded by some formulas in the expanded logic will be like $\mathscr{M}_{\mathbf{s},\bar{p},n}, \bar{p} \in \mathbf{P}^2_{\mathbf{s}}$, but we need a relative as we can iterate.

Definition 1.12. For $\iota = 1, 2, 3$ let $\mathbf{U}_\iota$ be the set of objects $\mathbf{u}$ consisting of the following (we may add subscript $\mathbf{u}$):

- (a) $\bar{\mathbf{s}} = \langle \mathbf{s}_\ell : \ell \le \ell(\mathbf{u}) \rangle$
- (b) $\mathbf{s}_\ell$ is a kind sequence
- (c) $\mathbf{s}_0 = \mathbf{s}_{\rm gr}$, the graph kind sequence, see 1.3(1A)
- (d) $\mathbf{s}_\ell \subseteq \mathbf{s}_{\ell+1}$, i.e. $I_{\mathbf{s}_\ell} \subseteq I_{\mathbf{s}_{\ell+1}}$ and $t \in I_{\mathbf{s}_\ell} \Rightarrow (k_{\mathbf{s}_\ell,t}, K_{\mathbf{s}_\ell,t}) = (k_{\mathbf{s}_{\ell+1},t}, K_{\mathbf{s}_{\ell+1},t})$
- (e) notation: so we may write $(k_{\mathbf{u},t}, K_{\mathbf{u},t})$ for $t \in I_{\mathbf{s}_{\ell(\mathbf{u})}}$ and $I_{\mathbf{q}} = I_{\mathbf{s}_{\ell(\mathbf{u})}}$
- (f) for $t \in I_{\mathbf{s}_{\ell+1}} \backslash I_{\mathbf{s}_\ell}$ we have: $\bar{\varphi}_t$ is a complete reduced $\mathbf{s}_\ell$-scheme, not ι-weak such that $K_t = K_{\bar{\varphi}_t}$, see Definition 1.18(2) let $\mathbf{i}_t = \mathbf{i}(t) = \mathbf{i}(\bar{\varphi}_t)$ and similarly $\varphi_{t,2}, \varphi_{t,0,i}, \varphi_{t,1,i,j}$ but let $\varphi_t(\bar{z}_t) = \varphi_{t,2}(\bar{z}_t)$. In the case $\iota = 2, 3$ if $\bar{y}_{t,i} \neq \langle\rangle$ then $\bar{y}_{t,i}$ is a singleton so we shall write $\varphi_{t,0,i}(y, \bar{z}_{t,i})$
- (g) $q = q_{\mathbf{u}} \in (0,1)_{\mathbb{R}}$.

Definition 1.13. For $\mathbf{u} \in \mathbf{U}_\iota$ we define a random $\mathscr{M}_{\mathbf{u},n}$, i.e. a 0-1 context, as follows.

For a given $n, \mathscr{M}_{\mathbf{u},n}$ is gotten by drawing $\mathscr{M}_{\mathbf{u},n,\ell} \in \mathbf{M}_{\mathbf{s}_{\mathbf{u},\ell},n}$ by induction on $\ell \le \ell(\mathbf{u})$ and in the end $\mathscr{M}_{\mathbf{u},n} = \mathscr{M}_{\mathbf{u},n,\ell(\mathbf{u})}$.

Now

- (a) if $\ell = 0, \mathscr{M}_{\mathbf{u},n,\ell}$ is $\mathscr{G}_{q(\mathbf{u}),n}$, i.e.the random graph on n with edge probability q
- (b) if $\ell < \ell(\mathbf{u})$ and $\mathscr{M}_{\mathbf{u},n,\ell}$ has been drawn and $t \in I_{\mathbf{s}_{ell+1}} \backslash I_{\mathbf{s}_\ell}$, we draws $R_t(\mathscr{M}_{\mathbf{s}_{\ell+1}})$ as follows:
  - (α) if $\bar{c} \in \varphi_t(M)$ we draw the truth value of $\bar{c} \in R_t(\mathscr{M}_{\mathbf{s}_{\ell+1},n})$ with probability $\mathbf{h}(\sum\limits_{i<\mathbf{i}(t)} \mathrm{EXP}|\varphi_{a,i}(\mathscr{M}_{\mathbf{s}_{\ell+1},n}, \bar{c})|/|K_{t,i}|)$ recalling EXP is the expected value
  - (β) if $\bar{c}$ is a sequence of length k_t but $\notin \varphi_t(M)$ then $\bar{c} \notin R_t(\mathscr{M}_{\mathbf{s}_{\ell+1},t})$.

Claim 1.14. *For $\mathbf{u} \in \mathbf{U}_\iota, \mathscr{M}_{\mathbf{u},n}$ is like $\mathscr{M}_{\mathbf{s}_{\mathbf{q}},\bar{p}}$ for any $\bar{p} \in \mathbf{P}^2_{\mathbf{s}_{\mathbf{q}}}$ (and $M_{\mathbf{u},n,\ell}$ like $\mathscr{M}_{\mathbf{s}_{\mathbf{q}},\ell,\bar{p}}$), in particular, satisfying the zero one law:*

- $(*)$ *for any k_1 for some k_2, for any random enough $\mathscr{M}_{\mathbf{u},n}$ we have:*
  - *if $\varphi(\bar{x}), \psi(\bar{y}, \bar{z})$ are complete $\mathbb{L}(\tau_{\mathbf{s}_{\mathbf{u}}})$-formulas such that $\psi(\bar{y}, \bar{z}) \vdash \varphi(\bar{z})$ (so they respect the $K_{\mathbf{u},t}$'s!, see Definition 1.9(6)) and $\ell g(\bar{y}) + \ell g(\bar{x}) \le k_1$ and $\bar{c} \in \varphi(\mathscr{M}_{\mathbf{u},n})$ and $k_{t,i} \ge 1$ then the number of members of $\psi_{t,i}(M_{\mathbf{u},n}, \bar{c})$ is similar to $\binom{n^{\ell g(\bar{y})}}{k_t} \cdot \frac{k_t}{(K_t)}$; fully*

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- *at most*[1] $\binom{n^{\ell g(\bar{y}_{t,i})}}{k_{t,i}} \cdot \frac{k_{t,i}!}{|K_{t,i}|} \cdot (1+\frac{1}{k_2})$
- *at least* $\binom{n^{k_{t,i}}}{k_t} \cdot \frac{k_{t,i}!}{|K_{t,i}|} \cdot \mathbf{h}(n) - k_2$
- *if* $\iota = 2$*, then* $k_{t,i} = 1$*, so this is simpler.*

Remark 1.15. What is the reason for our choice in Clause $(b)(\alpha)$ of Def 1.13? There are some demands pulling in different directions.

(a) This probability should be not too small (considering it belongs to $(0, 1/2)$) such that the argument "a Σ_1 formulas $(\exists y)\varphi(y, \bar{a})$ hold when not excluded" as in $\mathscr{M}_{\mathbf{s},\bar{p},n}$

(b) but always is not so small such that Prob$((\exists \bar{y})\varphi(\bar{y}\varphi)$ converge to zero or to one

(c) The $\mathscr{M}_{\mathbf{u},n}$ are intended to imitate what we get by starting with $\mathscr{G}_{p,n}$ and expanding it by relations definable by formulas $\varphi(\bar{x})$ from our logic, so we are applying our quantifier to a definable (with parameters) graph. So such a graph even almost surely will not have exactly n nodes. In the non-degenerated case the number will be of the order of magnitude
   (*) Cn^k for some positive real C and $k \geq 1$ in the 1-low case
   (*) Cn for some positive real C in the 2/3-low case,

Proof. Should be clear. $\square_{1.14}$

§ 1(C). **Low/High Graphs.**

An $\mathbf{s}$'s scheme $\bar{\varphi}$ may be such that, e.g. the bi-partite graph with the i-th kind and the j-th kind is in the low case, see Definition 1.9(4); so we try to single out those $\bar{\varphi}$'s. Those cases are "undesirable" for us and we shall try to discard them.

Definition 1.16. 1) We say a finite graph H is $\mathbf{h}-1$-low (recall $\mathbf{h}$ is from 1.1 so can be omitted) when there are no disjoint $A, B \subseteq H$ and $\iota < 2$ such that (letting $n = |H|$)

(a) $|A|, |B| \geq |H|^{\mathbf{h}(n)}$

(b) if $a \in A$ and $b \in B$ then (a, b) is an edge of H iff $\iota = 1$.

2) We say that a finite graph H is $\mathbf{h}-2$-low when[2] letting $n = |H|, m = \lfloor \mathbf{g}(n) \rfloor = \lfloor n^{\mathbf{h}(n)} \rfloor$, there are no $\bar{a}, \bar{b}, M, \mathbf{c}$ such that:

(a) $\bar{a} = \langle a_\ell : \ell < m \rangle$

(b) $\bar{b} = \langle b_{\ell,k} : \ell < k \leq m \rangle$

(c) $\bar{a}^\frown \bar{b}$ is a sequence of nodes of H with no repetitions

(d) each $\mathbf{c}_0, \mathbf{c}_1$ is a function from $\{(\ell, j) : \ell, j \leq m\}$ to $\{0, 1, \ldots, \lfloor \mathbf{g}(n) \rfloor\}$

(e) $\mathbf{c}_2$ is a function from $\{(\ell, k) : \ell, k \leq m\}$ into $\{0, 1, \ldots, \lfloor \mathbf{g}(n) \rfloor\}$

(f) if $\ell' < k' \leq m$ and $j' < m$ and $\ell'' < k'' \leq m, j'' \leq n$ and $\mathbf{c}_0(\ell', j') = \mathbf{c}_0(\ell'', j'')$ and $\mathbf{c}_1(k', j') = \mathbf{c}_1(k'', j'')$ and $\mathbf{c}_2(k', \ell) = \mathbf{c}_2(b'', \ell'')$ then $(b_{\ell',k'}, a_{j'})$ is an edge of H iff $(b_{\ell'',k''}, a_{j''})$ is an edge of H.

---

[1] we could have allowed, e.g. when $k_t = 1$ to be near to 1 though not too closely, but if we shal use a quantifier $\mathbf{Q}$ such that $\ll \frac{1}{2}$ of the structures are in it

[2] The specific choice of m is not important, but they have to be $\leq n^{1/k}$ and $> k$ for any k, for large enough n. Similarly $|\text{Rang}(\mathbf{c}_\ell)|$ compared to m.

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3) We say the finite graph H is $\mathbf{h}-3$-low when it is $\mathbf{h}-1$-low or $\mathbf{h}-2$-low.
4) In parts (1) and (2), $\mathbf{h}-\iota$-high means the negation of $\mathbf{h}-\iota$-low.

Claim 1.17. *Assume* $\mathbf{s}$ *is an I-kind, (see Definition 1.3) and* $\bar{\varphi}$ *is a complete* $\mathbf{s}$*-scheme (see Definition1.9(2)). 1.6, 1.9(2))*

(A) *the following are equivalent:*
  (α) $\bar{\varphi}$ *is trivial*
  (β) *if* $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ *then for random enough* $\mathscr{M} = \mathscr{M}_{\mathbf{s},n,\bar{p}}$ *and* $\bar{c} \in \varphi_2(\mathscr{M})$ *the graph* $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ *has* $\leq \mathbf{i}(\bar{\varphi})(k(\bar{\varphi})!)$ *nodes*
  (γ) *if* $\varepsilon > 0$ *and* $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ *then* $0 < \limsup_n \text{Prob}$*(letting* $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$*, for some* $\bar{c} \in \varphi_2(\mathscr{M})$ *the graph* $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ *has* $\leq n^{1-\varepsilon}$ *nodes).*

(B) *the following are equivalent for non-trivial* $\bar{\varphi}$*:*
  (α) $\bar{\varphi}$ *is 1-, see Def 1.9,*
  (β) *if* $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ *then for every random enough* $\mathscr{M} = \mathscr{M}_{\mathbf{s},n,\bar{p}}$ *and for every* $\bar{c} \in \varphi_2(\mathscr{M})$ *the graph* $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ *is* $\mathbf{h}-1$*-low*
  (γ) *if* $\varepsilon > 0$ *and* $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ *then* $0 < \limsup_n \text{Prob}$*(letting* $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$*, for some* $\bar{c} \in \varphi_2(\mathscr{M})$ *the graph* $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ *is 1-low)*

(C) *Like (B), replacing 1-weak,* $\mathbf{h}-1$*-low by 2-weak,* $\mathbf{h}-2$*-low respectively*

(D) *Like (B), replacing 1-weak,* $\mathbf{h}-1$*-low by 3-weak,* $\mathbf{h}-3$*-low respectively.*

Proof. Clause (A):

Trivially $(A)(\alpha) \Rightarrow (A)(\beta)$ and $(A)(\beta) \Rightarrow (A)(\gamma)$.

So it suffices to assume $\bar{\varphi}$ is non-trivial, $\bar{p} \in \mathbf{P}^2_{\mathbf{s}}$ and let $\varepsilon > 0$ be small enough and prove that for every random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ and $\bar{c} \in \varphi_2(\mathscr{M})$ the graph $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ has $\geq \varepsilon n$ nodes.

Let $i < \mathbf{i}(\bar{\varphi})$ be such that $k_i = \ell g(\bar{x}_i) > 0$, so for n large enough and $\bar{c} \subseteq [n]$ of length $\ell g(\bar{z})$ let $S_{n,\bar{c}} = \{\bar{a} : \bar{a}$ is a sequence of length $\ell g(\bar{x}_i)$ with no repetition of members of $[n]$ not from $\bar{c}\}$. For every $\bar{a} \in S_{n,i}$, the real $\text{Prob}(\mathscr{M}_{\mathbf{s},\bar{p},n} \models$ "if $\varphi_2(\bar{c})$ then $\varphi_{1,i}(\bar{a},\bar{c})$") is the same for every $\bar{a} \in S_{n,\bar{c}}$ and is of the form $r(1)\mathbf{g}(n)^m$ for some $r(1) \in (0,1)_{\mathbb{R}}, m \in \mathbb{N}\backslash\{0\}$ not depending on n. Fixing $\bar{c}$ under the assumption $\mathscr{M}_{\mathbf{s},\bar{p},n} \models \varphi_2[\bar{c}]$, considering a maximal set of pairwise disjoint $i \in S_{n,\bar{c}}$, the events $\mathscr{M}_{\mathbf{x},\bar{p},n} \models \varphi_{1,i}[\bar{a},\bar{c}]$ are independent, such that almost surely the number $|\{\bar{a} \in S_{n,\bar{c}} : \mathscr{M}_{\mathbf{s},\bar{p},n} \models \varphi_{1,i}[\bar{a},\bar{c}]\}$ is $\geq n/(r(1)\mathbf{g}(n)^m(1-\varepsilon))$. Similarly almost surely the number of $\bar{c}$ such that $\mathscr{M} \models \varphi_2[\bar{c}]$ is large.

Clause (B):

First why $(B)(\alpha) \Rightarrow (B)(\beta)$?

Note $\bar{\varphi}$ is non-trivial; $(\mathbf{s},\bar{\varphi})$ cannot satisfy clause (a) of Definition 1.9 because in the present claim we are assuming $\bar{\varphi}$ is non-degenerated. So assume clause (b) of 1.9(3) holds as exemplified by $i_1, i_2, \mathbf{i}(\bar{\varphi}), v_1, v_2$ and truth value $\mathbf{t}$, i.e. $\ell g(\bar{x}_i), \ell g(\bar{x}_j) > 0$, etc. So assume n is large enough and $\mathscr{M} = M, \bar{c} \subseteq [n]$ has length $\ell g(\bar{z}_{\bar{\varphi}})$.

Let $A_\ell = \{\bar{a} : \bar{a} \subseteq [n]$ is of length $\ell g(\bar{x}_{i_\ell})$ for $\ell = 1,2$ with no repetition and is disjoint to $\bar{c}\}$. Choose for $\ell = 1,2$ disjoint $\bar{a}^*_\ell \in A_\ell$. So the event $\mathscr{E}_{\bar{c}} =$ "$(\bar{c}^\frown \bar{a}^*_1{}^\frown \bar{a}^*_n)$ is as in 1.9(3)" has probability $\geq r_1(\mathbf{g}(n)^{k(1)})$ for some $r_1 \in (0,1)_{\mathbb{R}}, k \in \mathbb{N}\backslash\{0\}$ not depending on n (and $\bar{c}$). Fixing $(\bar{c},a^*_1,a^*_2)$ let $C_\ell \subseteq (n^1\backslash \text{rang}(\bar{c}^\frown \bar{a}^*_1{}^\frown \bar{a}^*_2), |C_\ell| \geq (n - |\ell g|(\bar{z}^\frown \bar{x}_{i_1}{}^\frown \bar{x}_{i_1}) - \frac{1}{2}$ for $\ell = 1,2$ and $C_1 \cap C_2 = \emptyset$. Let $A'_\ell = \{\bar{a} \in A_\ell : \text{Rang}(\bar{a}) \subseteq C_\ell\}$.

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Easily for some $r(2), r(3) \in (0,1)_{\mathbb{R}}$ not depending on n, t the probability of the event $\mathscr{E}_2 = \mathscr{E}^r_{\bar{c},\bar{a}^*,\bar{a}^*_1}$ is $\geq 1 - 2^{-r(2)n}$ where

- $(*)$ $\mathscr{E}_2$ means: if $\mathscr{M} \models \varphi_2[\bar{c}] \wedge \varphi_{1,i_1}[\bar{a}^*_1] \wedge \varphi_{1,i_2}[\bar{a}^*_2]$ then $|\{\bar{a}_\ell \in A'_\ell : \mathscr{M} \models \varphi[\bar{a}_{\ell,m}]\}| \geq n^{|u(\ell)|} r(3)$ for $\ell = 1, 2$.

If $\mathscr{E}_2$ occurs, clearly $\mathbf{t}$ and $A^*_{\mathscr{M},\ell} = \{\bar{a}/E_{K_{\mathbf{s},i,\ell}} : \bar{a} \in A'_\ell$ and $\mathscr{M} \models \varphi_{0,i_\ell}[a_{n,m}, \bar{c}]\}$ for $\ell = 1, 2$ exemplifies $H_{\bar{\varphi},\mathscr{M},\bar{c}}$ is low. As the number of $\bar{c}, \bar{a}^*_1, \bar{a}^*_2$ is polynomial we can finish.

Second, why $(B)(\beta) \Rightarrow (B)(\gamma)$:

Read the clauses and Definition of 1.16.

Third, $\neg(B)(\alpha) \Rightarrow \neg(B)(\gamma)$: This suffices

Why this holds? Let $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar{p},n}$ be random enough, $\bar{c}_2 \in \varphi_2(\mathscr{M})$ and $A_1, A_2 \subseteq H = H_{\bar{\varphi},\mathscr{M},\bar{c}}$ witness H is low, so $|A_\ell| \geq n^{\mathbf{h}(n)}$. So $n^*_1 = \min\{|A^*_1|; |A^*_2|\} \geq m^{\mathbf{h}(n)}$.

Clearly for each $\ell \in \{1,2\}$ for some $i(\ell) < \mathbf{i}(\bar{\varphi})$ we have

$$|\bar{a}/K_{\mathbf{s},i,\ell} \in A_\ell : \bar{a} \in \varphi_{1,i}(\mathscr{M}, \bar{c})\}| \geq |A_\ell|/\mathbf{i}(\bar{\varphi}) = n^*_2 \geq n^{\mathbf{h}(n)}/\mathbf{i}(\bar{\varphi}).$$

So for some $r \in (0,1)_{\mathbb{R}}$ not depending on n for $\ell = 1, 2$ we can find $\langle \bar{a}_{\ell,m} : m < n^*_3 = \langle (n^*_2)^r \rangle$ and partition v_ℓ, u_ℓ of $\ell g(\bar{x}_{i(\ell)})$ such that:

- $(*)$ (a) $\bar{a}_{\ell,m} \restriction v_c = a^*_\ell$
  - (b) $\mathrm{Rang}(\bar{a}_{\ell_1,m_1} \restriction u_\ell) \cap \mathrm{Rang}(\bar{a}_{\ell_2,m_2}) = \emptyset$ when $m_1, m_2 < n^*_3(\bar{\varphi}) \wedge \ell_1, \ell_2 \in \{1,2\} \wedge (\ell_1, m_1) \neq (\ell_2, m_2)$
  - (c) $\mathrm{Rang}(\bar{a}_{\ell,m} \restriction u_\ell), \mathrm{Rang}(\bar{a}_{2,m(r)} \restriction u_2), \bar{a}^* {}^\frown \bar{a}^*_2$ are pairwise disjoint for $\ell \in \{1,2\}, m < n^*_3$.

We draw $\mathscr{M} \restriction (\bar{c}^\frown \bar{a}_{\ell,m})$ for every $\ell \in \{1,2\}$ and $m < n^*_3$ we get $\mathscr{M}'$. So ignoring events of very low probability ($\leq (\frac{1}{2})^{rn}$ for fix $r \in (0,1)_{\mathbb{R}}$)

- $(*)$ $w_\ell := \{m < n^*_3 : (M' \restriction \bar{c}^\frown \bar{a}_{\ell,m}) \models \varphi_{1,i(\ell)}[\bar{a}_{\ell,m}, \bar{c}]\}$ has $\geq n^*_4 := \sqrt{n^*_3}$ members.

So $n^*_4 \geq n^\varepsilon$ for ε small enough but let $Y_\ell = \{\bar{a}_{\ell,m}/K_{t,i(\ell)} : m \in w_\ell\}$; it is a set of $\geq n^*$ nodes of $H_{\bar{\varphi},\mathscr{M},\bar{c}}$.

Now

- (a) $m(1), m(2) < n^*_4 \Rightarrow \mathrm{Prob}(\mathscr{M} \models \varphi_{1,i(1),i(2)}(\bar{a}_{1,m(1)}, \bar{a}_{2,m(2)}, \bar{c})) = r/\mathbf{g}(n)^k$ for some $r \in \mathbb{R}_{>0}, k \in \mathbb{N}\backslash\{0\}$ not depending on n
- (b) if $i(1), i(2)$ are not as required in 1.9(3)(h) and $\mathbf{t} = 0, 1$ then with negligible probability we have for some $u_1 \subseteq w_2, u_2 \subseteq w_2$ with $\lfloor \mathbf{g}(|H|) \rfloor$ elements each we have $m(1) \in u_1 \wedge m(2) \in u_2 \Rightarrow \mathscr{M} \models \varphi_{1,i(1),i(2)}(\bar{a}_{1,m(1)}, \bar{a}_{2,m(2)}, \bar{c})^{\mathrm{if}(\mathbf{t})}$.

So this could not have occured.

Clauses (C),(D):

Also straightforward. $\square_{1.17}$

* * *

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Definition 1.18. 1) Assume $\bar\varphi^1 = \bar\varphi_1, \bar\varphi^2 = \bar\varphi_2$ are $\mathbf{s}$-schemes and $\bar\varphi_1, \bar\varphi_2$ are reduced and complete. We say $(\mathbf{s}, \bar\varphi^1), (\mathbf{s}, \bar\varphi^2)$ are explicitly isomorphic when some π and $\varkappa$ witness it which means:

- (a) $\mathbf{i}(\bar\varphi^1) = \mathbf{i}(\bar\varphi^2)$ and $\ell g(\bar z_{\bar\varphi^1}) = \ell g(\bar z_{\bar\varphi^2})$
- (b) π is a permutation of $\{0, \dots, \mathbf{i}(\bar\varphi^1) - 1\}$ such that $k_{\bar\varphi^1,i} = k_{\bar\varphi^2,\pi(i)}$ and $K_{\bar\varphi_1,i} = K_{\bar\varphi_2,i}$ for $i < \bar\varphi_1$
- (c) $\varkappa$ is a permutation of $\ell g(\bar z_{\bar\varphi_1})$
- (d) for random enough $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar p,n}$, if $\ell \in \{1,2\}, \mathscr{M} \models \varphi_2^\ell[\bar c_\ell]$ then letting $\bar c_{3-\ell}$ be such that $\bar c_2 = \varkappa(\bar c_1)$ we have $M \models \varphi_2^{3-\ell}[\bar c_{3-\ell}]$ and $\varphi_{1,i}(\mathscr{M}, \bar c_1) = \varphi_{1,\pi(i)}(\mathscr{M}, \bar c_2)$ and $\varphi_{1,i,j}(\mathscr{M}, \bar c_1) = \varphi_{1,\pi(i),\pi(j)}, (\mathscr{M}, \bar c_2)$

2) For $\mathbf{s}, \bar\varphi$ as above let $K_{\bar\varphi} = K_{\mathbf{s},\bar\varphi}$ be the group of permutations K of $\ell g(\bar z_{\bar\varphi})$ such that $\bar\varphi$ is explicitly isomorphic to itself using our $\varkappa$ in 1.18(1).

Claim 1.19. *1) For every* $\mathbf{s}$*-scheme* $\bar\varphi$ *we can find* $\langle \bar\varphi^\iota(\bar z_\iota) : \iota < \iota(*)\rangle$ *such that:*

- (a) $\bar\varphi^\iota(\bar z^\iota_i)$ *is a complete reduced* $\mathbf{s}$*-scheme such that* $\bar z^\iota$ *is a subsequence of* $\bar z$
- (b) *for every* $\mathbf{s}$*-structure* M *and* $\bar c \in \varphi_2(M)$ *for some* ι *letting* $\bar c_\iota = \langle c_j : j \in \mathrm{dom}(\bar z)$ *and* z_j *appears in* $\bar z^\iota\rangle$ *we have* $H_{\bar\varphi,M,\bar c} \cong H_{\bar\varphi^\iota,M,\bar c^\iota}$
- (c) *for every* $\mathbf{s}$*-structure* M $\iota < \iota(*)$ *and* $\bar c^\iota \in \varphi_2^\iota(M)$ *there is* $\bar c$ *such that* $(\bar c, \bar c^\iota, \bar\varphi, \bar\varphi^i)$ *are as in clause (b).*

2) For complete $\bar\varphi$ *in the definition of "trivial", "degenerated", "reduced" we can replace "some* $\bar c$*" by "* $\bar c'$*".*
3) In the definition of $\mathbb{L}(\mathbf{Q_t})(\tau)$*, see Definition 2.2, we can use* $(\mathbf{Q} \dots, \bar x_{1,i}, \bar x'_{1,i}, \dots)\bar\varphi$ *for complete reduced non-trivial, non-degenerated* $\bar\varphi$*.*

Proof. Easy. $\square_{1.19}$

The Isomorphism Claim 1.20. *Assume* $\mathbf{s}$ *is an* I*-kind and* $\bar\varphi', \bar\varphi''$ *are complete reduced* $\mathbf{s}$*-schemes as above.*
1) If $\mathscr{M} = \mathscr{M}_{\mathbf{s},\bar p,n}$ *is random enough and* $\mathscr{M} \models \varphi'_2[\bar c'] \wedge \varphi''_2[\bar c'']$ *so* $H' = H_{\bar\varphi',\mathscr{M},\bar c'}, H'' = H_{\bar\varphi'',\mathscr{M},\bar c''}$ *are well defined then* $H' \cong H''$ *iff* $\mathrm{Rang}(\bar c'), \mathrm{Rang}(\bar c'')$ *and moreover* $(\mathbf{s}, \bar\varphi'), (\mathbf{x}, \bar\varphi'')$ *are explicitly isomorphic, as witness by* $(\pi, \varkappa)$ *such that* π *maps* $\bar c'$ *to* $\bar c''$*, see Definition 1.18.*
2) Being explicitly isomorphic $\mathbf{s}$*-schemes is an equivalence relation.*

Proof. Straightforward. $\square_{1.20}$

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§ 2. The random quantifier

Hypothesis 2.1. Let $\iota \in \{1,3\}$ but $\iota = 3$ is simpler and large part is O.K. also for $\iota = 2$.

Definition 2.2. 1) We say $\mathbf{Q} = \mathbf{Q}_{\mathbf{K}}$ is a $\mathbf{h} - \iota$-high-graph quantifier when:

- (a) $\mathbf{Q}$ is a quantifier on finite graphs, i.e. it is a class of finite graphs closed under isomorphisms
- (b) if H is a finite graph and is $\mathbf{h} - \iota$-low then $H \notin \mathbf{Q}$.

2) We define a probability space on the set of high-graph quantifiers as follows: let $\bar{H}^* = \langle H^*_m : m \in \mathbb{N}\rangle$ be a sequence of pairwise non-isomorphic finite graphs such that each finite graph is isomorphic to exactly one of them.

For $\iota \in \{1,2,3\}$, we let:

- (a) $\mathbf{T} = \mathbf{T}_\iota = \{\bar{\mathbf{t}} : \bar{\mathbf{t}} = \langle \mathbf{t}_m : m \in \mathbb{N}\rangle, \mathbf{t}_m$ a truth value, $\mathbf{t}_m = 0$ if H^*_m is $\mathbf{h} - \iota$-low$\}$
- (b) we draw the $\mathbf{t}_m$'s independently, $\mathbf{t}_m = 0$ if H^*_m is $\mathbf{i} - \iota$-low and $\mathbf{t}_m = 1$ has probability $1/\mathbf{g}(|H^*_m|)$ when H^*_m is not $\mathbf{h} - \iota$-low
- (c) Let $\mu_{\mathbf{T}_\iota}$ be the derived distribution.

2A) So the probability space is $(\mathbb{B}, \mu_{\mathbf{T}}), \mathbb{B}$ is the family of Borel subsets of ${}^{\mathbb{N}}2, \mu_{\mathbf{T}}$ the measure.
3) For $\bar{\mathbf{t}} \in \mathbf{T}$ let $\mathbf{Q}^\iota_{\bar{\mathbf{t}}}$ be the quantifier $\mathbf{Q}_{\mathbf{K}_{\bar{\mathbf{t}}}}, \mathbf{K}_{\bar{\mathbf{t}}} = \{H : H$ a finite graph isomorphic to some H^*_m such that $\mathbf{t}_n = 1\}$.
4) We say H is $\mathbf{h} - \iota$-high where H is a finite graph which is not $\mathbf{h} - \iota$-low.

Claim 2.3. *For every random enough $\bar{\mathbf{t}} \in \mathbf{T}$ the following holds.*
1) $\mathbf{Q}_{\mathbf{t}}$ is a Lindström quantifier.
2) For random enough graph $\mathscr{G}_{n,p}, \mathbf{Q}_{\bar{\mathbf{t}}}$ define non-trivial quantifier, defining (with parameters) non-first order definable sets.
3) More specifically the formula $\psi(x) =$ (the graph restricted to $\{y : yRx\}$ belongs to $\mathbf{K}_{\bar{\mathbf{t}}}$)define in every random enough $\mathscr{G}_{n,p}$, a set which is not first order definable by a formula of length k.

Proof. Straightforward. $\square_{2.3}$

So

Definition 2.4. 1) The set of formulas $\varphi(\bar{x})$ of $\mathbb{L}(\mathbf{Q}_{\mathbf{t}})(\tau_{\mathbf{s}})$ for a kind sequence $\mathbf{s}$ is the closure of the set of atomic formulas of $\mathbb{L}(\tau_{\mathbf{s}})$ by negation $(\psi(\bar{x}) = \neg\varphi(\bar{x}))$, conjunction $(\psi(\bar{x})) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x}))$, existential quantification $(\psi(\bar{x}) = (\exists y)\varphi(\bar{x}, y))$ and applying $\mathbf{Q}_{\mathbf{t}}, \psi(\bar{z}) = (\mathbf{Q}_{\mathbf{t}}, \ldots, \bar{x}_{0,i}, \bar{x}'_{0,i}, \ldots)_{i<\mathbf{i}(\bar{\varphi})}\bar{\varphi}$ where $\bar{\varphi}$ is an $\mathbf{s}$-scheme of formulas which are already in $\mathbb{L}(\mathbf{Q}_{\mathbf{t}})(\tau_{\mathbf{s}})$, so as defined in 1.6(1) except that now the $\varphi_{\iota,i}$ are not necessarily quantifier free formulas from $\mathbb{L}(\tau_{\mathbf{s}})$.
2) Satisfaction, i.e. for a (finite) $\mathbf{s}$-structure M, formula $\varphi(\bar{x})$ and sequence $\bar{a}$ of elements of M of length $\ell g(\bar{x})$, we define the truth value of $M \models \varphi[\bar{a}]$ by induction on φ, the new case is when:

- $\varphi(\bar{z}_{\bar{\varphi}}) = (\mathbf{Q}_{\bar{t}}, \ldots, \bar{x}_{0,i}, \bar{x}'_{0,i}, \ldots)_{i<\mathbf{i}(\bar{\varphi})}\bar{\varphi}$.

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Now $M \models \varphi[\bar{c}]$ iff $\bar{c} \in \varphi_2(M)$ and $H_{\bar{\varphi},M,\bar{c}}$ is isomorphic to some graph from $\{H^*_m : \mathbf{t}_m = 1\}$.
3) The syntax of $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$ does not depend on $\bar{\mathbf{t}}$ so may write $\mathbb{L}(\mathbf{Q})$ that is $\mathbb{L}(\mathbf{Q})(\tau)$ is the relevant set of formulas, but the satisfaction depends so we shall write $M \models_{\bar{\mathbf{t}}} \varphi[\bar{a}]$ for $\bar{a}$ a sequence from M and formula $\varphi(\bar{x}) \in \mathbb{L}(\mathbf{Q})$; of course, such that $\ell g(\bar{a}) = \ell g(\bar{x})$.

Theorem 2.5. *1) For any $p \in (0,1)_{\mathbb{R}}$ for all but a null set of $\bar{\mathbf{t}} \in \mathbf{T}$, the random graph $\mathscr{G}_{n,p}$ satisfies the 0-1 law for the logic $\mathbb{L}(\mathbf{Q}^t_{\bar{\mathbf{t}}})$, i.e. we may allow to apply $\mathbf{Q}_{\bar{\mathbf{t}}}$ to definitions as in Definition 1.6, see Claim 1.10.*
2)The limit theory T_* is decidable modulo an oracle for the random $\mathbf{K}_{\mathbf{t}}$.

Remark 2.6. 1) Of course, we can replace the class of graphs by the class of $\mathbf{s}$-structures, $\mathbf{s}$ any kind sequence.

2) Does the limit theory depend on $\bar{\mathbf{t}}$? The problem is for when we apply the quantifier . to graphs of fixed size, so use completer $\bar{\varphi}$ with $k^*_{\mathbf{s}}(\bar{\varphi}) = 0$. So we have to decide if to include formulas in which this occurs. Does

Proof. Consider a sentence $\psi \in \mathbb{L}(\mathbf{Q})$, see 2.4.

- $\boxplus_0$ for each n we consider drawing $(\mathscr{G}_{p,n}, \bar{\mathbf{t}}) \in \text{Graph}_n \times \mathbf{T}$, that is, independently we draw
  - $\bar{\mathbf{t}} \in \mathbf{T}$ by the probability space from 2.2(2)
  - $\mathscr{G}_{n,\bar{p}} \in \text{Graph}_n$ = the set of graphs with set of nodes $[n]$ with each edge drawn with probability p_n independently of the other edges
- $\boxplus_1$ It suffices to prove that
  - (a) the probability of "$\mathscr{G}_{n,p} \models_{\bar{\mathbf{t}}} \psi$", i.e. the pair $(\mathscr{G}_{n,\bar{p}}, \bar{\mathbf{t}})$ satisfies this, either is $\geq \frac{1}{2^{nr}}$ or is $\geq 1 - \frac{1}{2^{nr}}$ for some $r = r(\psi) \in (0,1)_{\mathbb{R}}$
  - (b) which case does not depend on n
  - (c) moreover the probability is $\geq 1 - \frac{1}{2^{n\geq}}$ iff $\psi \in T_*$.

[Why? Consider the drawing of $(\langle \mathscr{G}_{n,p} : n \in \mathbb{N}\rangle, \mathbf{t}) \in \prod\limits_n \text{Graph}_n \times \mathbf{T}$. For every $\psi \in \mathbb{L}(\mathbf{Q})$, the following event $\mathscr{E}^1_\psi \wedge \mathscr{E}^2_\psi$ has probability zero, where

$$\mathscr{E}^1_\psi := (\text{for infinitely many } n, \mathscr{G}_{n,p} \models_{\mathbf{t}} \psi)$$

$$\mathscr{E}^2_\psi := (\text{for infinitely many } n, \mathscr{G}_{n,p} \models_{\mathbf{t}} \neg\psi).$$

This holds by (a)+(b) of $\boxplus_1$. Hence also the event $\mathscr{E} = \bigvee\{\mathscr{E}^1_\psi \wedge \mathscr{E}^2_\psi : \psi \in \mathbb{L}(\mathbf{Q})\}$ has probability zero. Hence, by Fubini theorem, drawing for a set of $\bar{\mathbf{t}}$'s of measure 1, the event $\mathscr{E}^1_\varphi[\mathbf{t}] \wedge \mathscr{E}^2_\psi[\mathbf{t}]$ has probability zero, where $\mathscr{E}^\ell_\psi[\mathbf{t}]$ is the event $\mathscr{E}^\ell_\psi$ fixing $\mathbf{t}$.]

To prove $\boxplus_1$, fix $\psi \in \mathbb{L}(\mathbf{Q})(\tau_{\rm gr})$. We can find a $\bar{\Delta}$ such that:

- $\boxplus_2$ (a) $\bar{\Delta} = \langle \Delta_\ell : \ell \leq \ell(*)\rangle$
  - (b) Δ_ℓ is a finite set of formulas from $\mathbb{L}(\mathbf{Q})$ increasing with ℓ
  - (c) Δ_0 is the set of quantifier free formulas
  - (d) $\psi \in \Delta_{\ell(*)}$
  - (e) every formula in $\Delta_{2\ell+1} \backslash \Delta_{2\ell}$ is gotten from formulas from $\Delta_{2\ell}$ by a first order operation $(\neg\varphi(\bar{x}), \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x}), \exists y \varphi(\bar{x}, y))$

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(f) every formula in $\Delta_{2\ell+2}\backslash\Delta_{2\ell+1}$ is of the form $\psi(\bar z) = (\mathbf{Q}\dots\bar x_i,\bar x'_i,\dots)_{i<k}\bar\varphi(\bar z)$ where $\bar\varphi = \bar\varphi(\bar z)$ recalling 1.20 is a complete reduced $\mathbf{s}'$-scheme for some $\mathbf{s}'$, i.e. is as in Definition 1.6 but the $\varphi_{0,i}(\bar x,\bar z),\varphi_{1,i,j}(\bar x_i,\bar x_j,\bar z),\varphi_2(\bar z)$ being from $\Delta_{2\ell+1}$

(g) no two distinct $\bar\varphi$'s which occur in Δ on $(\mathbf{Q},\dots)\bar\varphi$ are explicitly isomorphic (see Definition 1.18), but replacing equality of formuals by equivalence for every randm enough $\mathscr{G}_{p,n}$ (during the proof this will get a syntactical characterization).

[Why? Should be clear.]

$\boxplus_3$ let $\Delta_\ell = \{\vartheta_s(\bar x_s) : s\in I^*_\ell\}$ hence $\mathscr{T}_\ell$ is finite and $m<\ell\Rightarrow I^*_m\subseteq I^*_\ell$.

Now by induction on $\ell\le\ell(*)$ we choose $\mathbf{s}_\ell,\bar\vartheta'_\ell,\bar\vartheta''_\ell$ and the function $G\mapsto M_{G,\bar{\mathbf{t}},\ell}$ for G a graph on $[n]$ some n such that:

$\boxplus_{4,\ell}(A)$ (a) I_ℓ finite

(b) $\mathbf{s}_\ell$ is as in Definition 1.3, an I_ℓ-kind of a vocabulary

(c) (α) $\mathbf{s}_0, I_0$ are defined by $I_0=\{s_0\}$ for some $s_0\notin I^*_{\ell(*)}, n_{\mathbf{s},s_0}=2, K_{\mathbf{s}_0,s_0}=\mathrm{Sym}(2)$, the group of permutation of $\{0,1\}$

(β) $I_{2\ell+1}=I_{2\ell}$

(γ) $I_{2\ell+2}=I_{2\ell+1}\cup(I^*_{2\ell+2}\backslash I^*_{2\ell+1})$

(δ) so

- $\langle I_\ell:\ell\le\ell(*)\rangle$ is increasing
- $\mathbf{M}_{\mathbf{s}_0,n}$ is Graph_n, the set of graphs with set of nodes $[n]$

(d) $\bar\vartheta'_\ell=\langle\vartheta'_s(\bar x_s):s\in I_\ell\rangle$

(e) $\vartheta'_s(\bar x_s)$ a formula in $\mathbb{L}(\tau_{\mathbf{s}_\ell})$ for $s\in I_\ell$

(f) $\vartheta''_s(\bar x_s)$ is a quantifier free formula in $\mathbb{L}(\tau_{\mathbf{s}_\ell})$ equivalent to $\vartheta'_s(\bar x_s)$ in the limit theory $T_{\mathbf{s}_\ell}$, see Definition 1.4

(g) for any given $G\in\mathscr{G}_{n,p}$, i.e. $G\in\mathbf{M}_{\mathbf{s}_0,n}$ and $\bar{\mathbf{t}}\in\mathbf{T}$ we define $M_{G,\bar{\mathbf{t}},\ell}\in\mathbf{M}_{\mathbf{s}_\ell,n}$ by:

(α) $M_{G,\bar{\mathbf{t}},\ell}$ is a $\tau_{\mathbf{s}_\ell}$-model expanding $M_{G,\bar{\mathbf{t}},m}$ for $m<\ell$ and for $s\in I_\ell, R_s^{M_{\mathscr{G},\bar{\mathbf{t}},\ell}}$ is defined by $\vartheta_s(\bar z_s)$ and also by $\vartheta'_s(\bar z_s)$

(β) if $\ell=0, M_{G,\bar{\mathbf{t}},\ell}$ is G

(γ) if $\ell=2m+1, s\in I_\ell\backslash I_{2m}$ we apply the the first order construction of $\vartheta_s(\bar x_s)$ from the formulas $\langle\vartheta_s(\bar x_s):s\in I_{2m}\rangle$ to construct $\vartheta_s(\bar z_s)$ from $\langle\vartheta''_t(\bar x_t,\bar z_s):t\in I_{2m}\rangle$

(δ) for $\ell=2m+2$ and $s\in I_\ell\backslash I_{2m+1}$ if $\vartheta_s(\bar x_s)=(\mathbf{Q}\dots,\bar x_i,\bar x'_i,\dots)_{i<\mathbf{i}(\bar\varphi_{\mathbf{i}})}\bar\varphi_s(\bar x_s)$ we define $\vartheta'_s(\bar x_s)$ by replacing in $\bar\varphi_s$ every ϑ_t by ϑ''_t getting $\bar\varphi'_s$ and let $\vartheta'_s(\bar x_\iota)=(\mathbf{Q}\dots,\bar x_i,\bar x'_i,\dots)_{i<\mathbf{i}(\bar\varphi_\iota)}\bar\varphi'_s$

(ε) we choose $\vartheta''_i(\bar x_i)$ by 1.4 sequence clause (f) here.

Now for each $\ell\le\ell(*)$ we have two relevant ways to draw as $\mathbf{s}_\ell$-structure M with-universe = set of elements $[n]$.

First, draw $\mathbf{t}\in\mathbf{T}$ and $\mathscr{G}=\mathscr{G}_{q,n}$ (recall $q\in(0,1)_{\mathbb{R}}$ was fixed in the beginning of Theorem 2.5) and compute $M_{\mathscr{G},\bar{\mathbf{t}},n}$, a $\mathbf{s}_\ell$-structure. This induces a distribution $\mu_{q,n,\ell}$ on $\mathbf{M}_{\mathbf{s}_\ell,n}$, i.e. $\mu_{q,n,\ell}(M)=\mathrm{Prob}(M_{\mathscr{G}_{q,n},\mathbf{t},n}=M|\mu_{\mathrm{gr},q,n}\times\mu_{\mathbf{T}}=\mu_{\mathbf{s}_{\mathrm{gr}},\bar p_{\mathrm{gr},q,n}}\times\mu_{\mathbf{T}})$.

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Second, we shall choose $\bar{p}_\ell \in \mathbf{P}^2_{\mathbf{s}_\ell}$ and draw $\mathscr{M}_{\mathbf{s},\bar{p}_\ell,n}$ here the distribution is ? The interest in the first is that our aim is to prove the 0-1 law for $M_{\mathscr{G},\bar{p},n}$, in particular, for $\ell = \ell(*)$ and our sentence ψ; we use the other ℓ's in an induction.

A priori the probability of "$M_{\mathscr{G},p,n} \models \psi$" is opaque.

For the second, $\mathscr{M}_{\mathbf{s},\bar{p}_\ell,n}$ an understanding of the probability of $M_{\mathbf{s},\bar{p}_\ell,n} \models \psi$ is now well known and satisfies the 0-1 law. Hence it suffices to prove that the distribution of $M_{\mathscr{G},\mathbf{t},\ell}$ (for $\mathscr{G} \in \mathscr{G}_{p,n}$) from $\mathbf{M}_{\mathbf{s}_\ell,n}$ and $M_{\mathbf{s}_\ell,\bar{p},n} \in \mathbf{G}_{\mathbf{s}_\ell,n}$ are sufficiently similar.

Naturally we choose:

$(*)_1$ (a) $p_{\mathbf{s}_{\rm gr},s_{\rm gr},n} = p_{\mathbf{s}_0,s_{\rm gr},n} = q$

(b) $p_{\mathbf{s}_\ell,t,n} = q/\mathbf{g}(n)$ for $t \in I_{\mathbf{s}_\ell} \backslash \{s_{\rm sq}\}$.

Of course, we induct; for $\ell = 0$ there is no difference so we deal now with $\ell + 1$ if ℓ is even this is trivial so assume ℓ is odd.

There are several reasons for a difference, for a given model $M \in \mathbf{M}_{\mathbf{s}_\ell,n}$

$(*)_{M,1}$ $t \in I^*_{\ell+1} \backslash I^*_\ell$ and $\bar{c} \in \varphi_{t,2}(M)$. The graph $H_{\bar{\varphi}_t,M,\bar{c}}$ is ι-low (for a given n there are at most $n^{k(\bar{\varphi}'_t)}$ (check cases)

$(*)_{M,2}$ for some $t(1), t(2) \in I^*_{\ell+1} \backslash I^*_\ell, \bar{c}_2 \in \varphi_{t(1),2}(M)$ and $\bar{c}_2 \in \varphi_{t,2}(M)$ we have $(t(j), \bar{c}_1/E_{\bar{\varphi}'_{t(1)}} \neq (t(2), \bar{c}_2/E_{\bar{\varphi}'_{t(2)}})$ but the graphs $H_{\bar{\varphi}'_{t(1)},M,\bar{c}_1}, H_{\bar{\varphi}'_{t(2)},M,\bar{c}_2}$ are isomorphic

$(*)_{M,3}$ for some $t(1), t(2) \in I^*_{\ell+2} \backslash I^*_\ell$ and $t(2) \in \cup\{I^*_{2k+2} \backslash I^*_{2k+1} : 2k+2 \leq \ell\}$ and $\bar{c}_1 \in \varphi_{t(1),2}(M), \bar{c}_2 \in \varphi'_{t(2),2}(M)$ the graphs $H_{\bar{\varphi}'_{t(1)},M,\bar{c}_2}, H_{\bar{\varphi}'_{t(2)},M,\bar{c}_2}$ are isomorphic

$(*)_{M,4}$ the sequence $\bar{p} \in \mathbf{P}^2_{\mathbf{q}}$ try to immitate $\mathbf{t}$, but having the probability for $\mathscr{M}_{\mathbf{s}_{\ell+1},\bar{p},n} \models R_t[\bar{c}]$ is $p_{t,n} = 1/\mathbf{g}(n)$ whereas the probability $\mathbf{t}_i = 1$ is $1/\mathbf{g}(|H^*_i|)$ where i is such that $H_{\bar{\varphi}_t,\mathscr{M}_{\mathscr{G},\mathbf{t}},\bar{c}} = H^*_i$ for $\mathscr{G} = \mathscr{G}_{q,n}$.

Now there is no reason that usually $i = n$. However, if $\iota = 2$ then $|H^*_i| \leq k(\bar{\varphi}_t) \cdot n$ and if $\iota = 1, H^*_1 \leq n^{k(\bar{\varphi}_2)}$. In both cases with probability very close to 1, (for $\mu_{\mathbf{s}_{\ell+1},\bar{p},n}$), $|H^*_i| \geq n/2^{k(\bar{\varphi}_t)}$. So clearly as $\mathbf{q}$ grow slowly enough, see 1.1(2).

This is also true for $(*)_{M,1}, (*)_{M,2}, (*)_{M,3}$. Together, we have two distributions on $\mathbf{M}_{\mathbf{s}_{\ell+1},n}$ and for the second, omitting a set of M with small probability (in $\mu_{\mathbf{s}_{\ell+1},\bar{p},n}$) for any other M, the two distributions give almost the same values. The computations are easy so we are done. $\square_{2.5}$

Remark 2.7. To eliminate $(*)_4 M$, in the end of the proof we may complicate the drawing of $\mathscr{M}_{\mathbf{s}_{\ell+1},\bar{p},n}$ We draw $\mathscr{M}_{\mathbf{s}_m,\bar{p},n}$ by induction on m: if $m = 2j+2, M = M_{\mathbf{x}_{2j+1},\bar{p},n}$ given for $R_t(t \in I^*_m \backslash I^*_{2k+1})$ we consider only $\bar{c} \in \varphi'_{t,2}(M)$ let $m = m_t(\bar{c}) = m_t(\bar{c}, M)$ be the number of nodes of $H_{\bar{\varphi}'_t,M,t}$ and we draw a truth value of $R_t(\bar{c})$ with probability $1/g(m)$. Proving the 0-1 law for such drawing is easy.

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§ 3. How to get a real quantifier, i.e. definable K

Discussion 3.1. In the introduction we have considered drawing a truth value to all graphs. So replacing "converge to zero or to one" we ask only "for every $\varepsilon > 0$ for every n large enough the probability is up to ε closed to zero or to one, The point is that otherwise we can weakly express "$|\varphi_1(\mathscr{G}_{p,n}, \bar{a}_1)| = |\varphi_2(\mathscr{G}_{p,n}, \bar{a}_2)|$, e.g. for $\varphi(x,y) = xRy$. So we can find $\psi_1(x_1, x_2)$ implying $\text{valency}_{\mathscr{G}_{p,n}}(y_1) = \text{valency}_{\mathscr{G}_{p,n}}(y_n)$, this will complicate the matter.

In more details, let $\psi_\varphi(y)$ say "the empty graph on $\varphi(\mathscr{G}_{p,n}, y)$ is green".

Let $\psi_2(y_1, y_2)$ say:

- (a) $\psi_1(y_1) \equiv \psi_2(y_2)$
- (b) for $\ell \in \{1,2\}$ and y'_ℓ there is $y_{3-\ell}$ such that $|\varphi(\mathscr{G}_{p,n}, y_1) \cap \varphi(\mathscr{G}_{n,p}, y'_1)| = |\varphi(\mathscr{G}_{p,n}, y_2) \cap \varphi(\mathscr{G}_{p,n}, y'_2)|$.

This nearly expresses $|\varphi(\mathscr{G}_{p,n}, y_1)| = |\varphi(\mathscr{G}_{p,n}, y_2)|$. We can strengthen this and find approximation to $a+1$ and cases of addition.

While the above does not suffice to prove impossibility, it suffices to show the problem is not promising and is different; maybe relevant is the late [S$^+$a].

Discussion 3.2. Can we use a quantifier $\mathbf{Q}_{\mathbf{K}}$ which depends just on the number of edges via the number of nodes.

1) If it depends only on the number of nodes, it seemed that this is bad for 0-1 laws.

2) Notes that surely graphs H_1, H_2 occur up to isomorphism when H_2 is gotten by omitting one edge of H_1. So we may try that it depends only the number modulo $(\lfloor \log\log(4+1) \rfloor)$! Quite reasonable choice of the quantifier but not ideal.

3) So we may try to change the logic such that essentially just changing one edge does not matter; that is excluding some family of graphs which with probability one does not occurs for a random enough $\mathscr{G}_{p,n}$. This is a reasonable logic, even without "$H \in \mathbf{K}$ depends just on the number of edges (and nodes)"

- (A) if we forget this restriction, we need to change the flipping of coins for the logic, e.g. fixing size first, choose one randomly, do this for each neighborhood, choose with distorted probability; not clear if converge and there is a natural way
- (B) Here $n^{\mathbf{g}(h)}$ goes slowly to ∞ and is used how to make the results O.K.. Note: in $\mathscr{G}_n$ the size of a definable graph for some m, is $\approx \frac{n}{m}$ so the variance is $c\sqrt{\frac{n}{m}}$; still the edges have probability $\frac{1}{2}$ and so O.K.

However for later M^ι_n ($\iota <$ quantifier depth) the probability of each case of a relation is, i.e. $H \in \mathbf{K}$ for a structure with probability $\frac{1}{\mathbf{h}(n)}$ so manipulating $\mathbf{h}$ gives different results.

4) But we have a more profound problem: we have nicely definable H_1, H_2 getting H_2 from H_1 by, for some nodes $a \neq b$ omitting the edges (a,c) and adding the edges (b,c) whenever $(b,c))$ is an edge when (a,c) is not.

Alternatively omitting the edge (a,c) when (b,c) is an edge, The first does not change the number of edges, the second changes seriously. This may be close to the variance for the number of edges.

A medicine? ask: omitting $\log_*(H_)$ edges, what is the minimal number of edges?

The overcoming may cost: in how to make the probability computations right.

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5) Note: from random $\mathscr{G}_{n,1/2}$ we build $\mathscr{M}^9_{n_1} = \mathscr{G}_{n,H}$ an $\mathbf{s}_0$-structure $\mathscr{M}^1_n$ expands by applying the quantifier getting an $\mathbf{s}_1$-structure. But $\mathscr{M}^{\mathbf{s}_1}_n$ is different:

(a) for $\mathscr{M}^{\mathbf{s}_1}_n$ the cases are totally independent

(b) $\mathscr{M}^1_n$ is different: first we draw $R_{\rm gr}$ ($= R_0$ in the lecture) after this we draw the other relatives but their probabilities:

- depends on the drawing of $\mathscr{M}_n = \mathscr{G}_{1/2,n}$
- in particular, on the sizes of the H's which are not too far from n but are different.

This complicates our work but the estimates are not so different.

Discussion 3.3. One which seems easiest while not unreasonable is: given a finite graph G, with m points, which is reasonable - defined as in [S$^+$b] and a point b in it, compute the valency minus $m/2$, divided by square root of m (or the variance of the related normal distribution) and ask if rounding to integers is odd or even.

We may replace the valency by the number of edges of G.

What are the dangers? As we may define a variant of the graph omitting one edge, in some cases this will change the truth value. For each nod the probability goes to zero but in binomial distribution the probability of e.g. getting valency exactly half of the expected value (rounded) is about 1 divided by the square root of m.

So we should divide not by the square root of m but by a larger value (maybe instead of asking on even/odd of the rounded value just ask if it can be larger than one, or absolute value) such that:

(a) almost surely (i.e. with large probability) for some node the value is above 1

(b) the probability that it is exactly one for some node is negligible, and this is true even if we use a graph only definable (reversing edge/non-edge, omitting some, etc.).

So we should say that clearly by continuity considerations there are such choices. A danger is that the n being odd/even can be expressed.

Another avenue is to choose the more natural "the valency is at least half"; but then it seems we can express being even/odd: say change by one edge change the truth value and this is true even if we omit one node. So the number of neighborhoods is half in both cases.

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