# NONTRIVIAL AUTOMORPHISMS OF $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ FROM VARIANTS OF SMALL DOMINATING NUMBER

SAHARON SHELAH AND JURIS STEPRĀNS

ABSTRACT. It is shown that if various cardinal invariants of the continuum related to $\mathfrak{d}$ are equal to $\aleph_1$ then there is a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$. Some of these results extend to automorphisms of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ if $\kappa$ is inaccessible.

## 1. Introduction

A fundamental result in the study of the Čech–Stone compactification, due to W. Rudin [7, 8], is that, assuming the Continuum Hypothesis, there are $2^{\mathfrak{c}}$ autohomeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ and, hence, there are some that are non-trivial in the sense that they are not induced by any one-to-one function on $\mathbb{N}$. While Rudin established his result by showing that for any two P-points of weight $\aleph_1$ there is an autohomeomorphism sending one to the other, Parovičenko [6] showed that non-trivial autohomeomorphisms could be found by exploiting the countable saturation of the Boolean algebra of clopen subsets of $\beta\mathbb{N}\setminus\mathbb{N}$ — this is isomorphic to the algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$. Indeed, the duality between Stone spaces of Boolean algebras and algebras of regular open sets shows that the existence of non-trivial autohomeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ is equivalent to the existence of non-trivial isomorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself.

**Notation 1.1.** If $A$ and $B$ are subsets of $\kappa$ let $\equiv_\kappa$ denote the equivalence relation defined by $A\equiv_\kappa B$ if and only if $|A\triangle B|<\kappa$ and $A\subseteq_\kappa B$ will denote the assertion that $|A\setminus B|<\kappa$. Let $[A]_\kappa$ denote the equivalence class of $A$ modulo $\equiv_\kappa$ and let $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ denote the quotient algebra of the $\mathcal{P}(\kappa)$ modulo the congruence relation $\equiv_\kappa$. If $\kappa=\omega$ it is customary to use $\equiv^*$ instead of $\equiv_\omega$ and $\subseteq^*$ instead of $\subseteq_\omega$.

**Notation 1.2.** If $f$ is a function defined on the set $A$ and $X\subseteq A$ then the notation $f(X)$ will be used to denote $\{f(x)\mid x\in X\}$ in spite of the potential for ambiguity.

**Definition 1.1.** An isomorphism $\Phi:\mathcal{P}(\kappa)/[\kappa]^{<\kappa}\to\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ will be said to be *trivial* if there is a one-to-one function $\varphi:\kappa\to\kappa$ such that $\Phi([A]_\kappa)=[\varphi(A)]_\kappa$ for each $A\subseteq\kappa$. The isomorphism $\Phi$ will be said to be *somewhere trivial* if there is some $B\in[\kappa]^\kappa$ and a one-to-one function $\varphi:B\to\kappa$ such that $\Phi([A]_\kappa)=[\varphi(A)]_\kappa$ for each $A\subseteq B$ and $\Phi$ will be said to be *nowhere trivial* if it is not somewhere trivial.

The question of whether the Continuum Hypothesis, or some other hypothesis, is needed in order to find a non-trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself was settled in the affirmative by S. Shelah in [9]. The argument of [9] relies on an iterated oracle chain condition forcing to obtain a model where $2^{\aleph_0}=\aleph_2$ and every isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself is induced by a one-to-one function from $\mathbb{N}$ to $\mathbb{N}$. The oracle chain condition requires the addition of cofinally many Cohen reals and so $\mathfrak{d}=\aleph_2$ in this model. Subsequent work has shown that it is also possible to obtain that every isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ is trivial by other approaches [15, 11, 2] but these have always required $\mathfrak{d}>\aleph_1$ as well. However, it was shown in [12] that this cardinal inequality is not entailed by the non existence of nowhere trivial isomorphisms from $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself — in the model obtained by iterating $\omega_2$ times Sacks reals there are no nowhere trivial isomorphisms yet $\mathfrak{d}=\aleph_1$.

On the other hand, while we now know that the Continuum Hypothesis can not be completely eliminated from Rudin's result, perhaps it can be weakened to some other cardinal equality such as $\mathfrak{d}=\aleph_1$. It will be shown in this article that non-trivial isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself can indeed be constructed from hypotheses on cardinal arithmetic weaker than $2^{\aleph_0}=\aleph_1$ and reminiscent of $\mathfrak{d}=\aleph_1$. However, it is shown in [3] that it is consistent with set theory that $\mathfrak{d}=\aleph_1$ yet all isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial so some modification of the equality $\mathfrak{d}=\aleph_1$ will be required.

It will also be shown that natural generalizations of the arguments can be applied to the same question for $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ where $\kappa$ is inaccessible. The chief interest here is that, unlike $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$, the algebra $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ is not countably saturated if $\kappa>\omega$— to see this, simply consider a family $\{A_n\}_{n\in\omega}\subseteq[\kappa]^\kappa$ such that $\bigcap_{n\in\omega}A_n=\emptyset$. In other words, Parovičenko's transfinite induction argument to construct non-trivial isomorphisms from $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ to itself is not available and some other technique is needed.

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The statement and proof of Lemma 2.1 is provided for all $\kappa$ and will apply both to the case that $\kappa = \omega$ and to the case that $\kappa$ is inaccessible. However, Lemma 3.2 deals only with the case that $\kappa$ is inaccessible. It is somewhat simpler than the case for $\omega$ and so is dealt with first because the general approach is similar in the $\kappa = \omega$ case, but this case requires some technical details not needed in the inaccessible case.

## 2. A sufficient condition for a non-trivial isomorphism

The following lemma provides sufficient conditions for the existence of a nontrivial isomorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ to itself. The set theoretic requirements for the satisfaction of these conditions will be examined later. The basic idea of the lemma is that an isomorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ can be approximated by partitioning $\kappa$ into small sets $I_\nu$ and constructing isomorphisms from subalgebras of $\mathcal{P}(I_\nu)$ and taking the union of these. Unless the subalgebras of $\mathcal{P}(I_\nu)$ are all of $\mathcal{P}(I_\nu)$, this union will only be a partial isomorphism. Hence a $\kappa^+$ length sequence of ever larger families of subalgebras of $\mathcal{P}(I_\nu)$ is needed to obtain a full isomorphism. In order to guarantee that this isomorphism is not trivial the prediction principles described in Hypothesis (4) and Hypothesis (5) of Lemma 2.1 are needed.

**Lemma 2.1.** *There is a non-trivial automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ provided that there are $\{I_\nu\}_{\nu\in\kappa}$, $\{\mathfrak{B}_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$ and $\{\Phi_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$ such that:*

1. *$\{I_\nu\}_{\nu\in\kappa}$ is a partition of $\kappa$ such that $|I_\nu| < \kappa$ for each $\nu \in \kappa$.*
2. *$\mathfrak{B}_{\xi,\nu}$ is a Boolean subalgebra of $\mathcal{P}(I_\nu)$ and $\Phi_{\xi,\nu}$ is an automorphism of $\mathfrak{B}_{\xi,\nu}$ for each $\xi \in \kappa^+$ and $\nu \in \kappa$.*
3. *If $\xi \in \eta \in \kappa^+$ then there is $\beta \in \kappa$ such that $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi,\nu} \subseteq \Phi_{\eta,\nu}$ for all $\nu \in \kappa \setminus \beta$.*
4. *For any one-to-one $F : \kappa \to \kappa$ there are $\xi \in \kappa^+$ and cofinally many $\nu \in \kappa$ for which there is an $a \in \mathfrak{B}_{\xi,\nu}$ and $w \in a$ such that $F(w) \notin \Phi_{\xi,\nu}(a)$.*
5. *For any $A \subseteq \kappa$ there are $\xi \in \kappa^+$ and $\beta$ in $\kappa$ such that $A \cap I_\nu \in \mathfrak{B}_{\xi,\nu}$ for all $\nu \in \kappa \setminus \beta$.*

*Proof.* Define

$$\Phi([A]_\kappa) = \lim_{\xi\to\kappa^+}\left[\bigcup_{\nu\in\kappa}\Phi_{\xi,\nu}(A\cap I_\nu)\right]_\kappa$$

and begin by observing that this is well defined. To see this, it must first be observed that given $A$ and $B$ such that $|A\triangle B| < \kappa$ there is $\alpha \in \kappa^+$ such that for all $\nu$ in a final segment of $\kappa$ the equation

$$\Phi_{\alpha,\nu}(A\cap I_\nu) = \Phi_{\alpha,\nu}(B\cap I_\nu)$$

is defined and valid by Hypothesis (5). From Hypothesis (3) it then follows that if $\xi \geq \alpha$ then

$$\bigcup_{\nu\in\kappa}\Phi_{\xi,\nu}(A\cap I_\nu) \equiv_\kappa \bigcup_{\nu\in\kappa}\Phi_{\alpha,\nu}(B\cap I_\nu)$$

and, hence, $\Phi([A]_\kappa)$ is well defined. Since each $\Phi_{\xi,\nu}$ is an automorphism it follows that $\Phi$ is an automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$.

To see that $\Phi$ is non-trivial, suppose that there is a one-to-one function $F : \kappa \to \kappa$ such that $[F(A)]_\kappa = \Phi([A]_\kappa)$ for all $A \subseteq \kappa$. Using Hypothesis (4) choose $\xi \in \kappa^+$ for which there is $Z \in [\kappa]^\kappa$ and $a_\nu \in \mathfrak{B}_{\xi,\nu}$ and $j_\nu \in a_\nu$ such that $F(j_\nu) \notin \Phi_{\xi,\nu}(a_\nu)$ for each $\nu \in Z$. Let $W \in [Z]^\kappa$ be such that for each $\nu \in W$, if $F(j_\nu) \in I_\mu$ and $\mu \neq \nu$ then $\mu \notin W$. Let $A = \bigcup_{\nu\in W} a_\nu$. It follows from Hypothesis (3) that for any $\eta \geq \xi$

$$\{F(j_\nu) \mid \nu \in W\} \cap \bigcup_{\nu\in W}\Phi_{\eta,\nu}(a_\nu) \equiv_\kappa \{F(j_\nu) \mid \nu \in W\} \cap \bigcup_{\nu\in W}\Phi_{\xi,\nu}(a_\nu) \equiv_\kappa \emptyset$$

and, hence, $[F(A)]_\kappa = \Phi([A]_\kappa)$. □

## 3. When are the hypotheses of Lemma 2.1 satisfied?

In answering a question of A. Blass concerning the classification of cardinal invariants of the continuum based on the Borel hierarchy M. Goldstern and S. Shelah introduced a family of cardinal invariants called $\mathbf{c}(f,g)$ defined to be the least number of uniform trees with $g$-splitting needed to cover a uniform tree with $f$-splitting [4] and showed that uncountably many of these can be distinct simultaneously. The following definition is very closely related to this as well as to the notion of a *slalom* found in [1].

**Definition 3.1.** Given functions $f$ and $g$ on $\kappa$ such that $g(\xi)$ is a cardinal for each $\xi \in \mathbf{cof}(\kappa)$ define $\mathfrak{d}_{f,g}$ to be the least cardinal of a family $\mathcal{D} \subseteq \prod_{\nu\in\kappa}[f(\nu)]^{g(\nu)}$ such that for every $F \in \prod_{\nu\in\kappa} f(\nu)$ there is $G \in \mathcal{D}$ such that $F(\nu) \in G(\nu)$ for a final segment of $\nu \in \kappa$.

**Hypothesis 3.1.** For the remainder of this section fix a cardinal $\kappa$ as well as functions $f$ and $g$ from $\kappa$ to the infinite regular cardinals below $\kappa$ such that for all $\nu \in \kappa$:

(1) $2^{g(\nu)} < f(\nu)$

(2) if $\nu \in \nu^*$ then $|\nu| \leq g(\nu) \leq g(\nu^*)$
(3) $\mathfrak{d}_{2^f,g} = \kappa^+$

**Lemma 3.1.** *There are $\{Z_{\eta,\zeta}\}_{\eta\in\kappa^+,\zeta\in\kappa}$ satisfying the following:*

*(1)* $Z_{\eta,\zeta} \subseteq \eta$
*(2)* $|Z_{\eta,\zeta}| \leq g(\zeta)$
*(3) if* $\zeta < \zeta^*$ *then* $Z_{\eta,\zeta} \subseteq Z_{\eta,\zeta^*}$
*(4)* $\bigcup_{\zeta\in\kappa} Z_{\eta,\zeta} = \eta$
*(5) if* $\eta \in \eta^*$ *then there is* $\beta \in \kappa$ *such that* $Z_{\eta^*,\zeta} \cap \eta = Z_{\eta,\zeta}$ *for all* $\zeta > \beta$
*(6) if* $\eta \in Z_{\eta^*,\zeta}$ *then* $Z_{\eta^*,\zeta} \cap \eta = Z_{\eta,\zeta}$.

*Proof.* Start by letting $Z_{0,\zeta} = \varnothing$. If $Z_{\eta,\zeta}$ have been defined for $\eta \in \xi$ and all $\zeta \in \kappa$ consider three cases.

**Case One.** $\xi = \xi^* + 1$

In this case simply define $Z_{\xi,\zeta} = Z_{\xi^*,\zeta} \cup \{\xi^*\}$.

**Case Two.** $\mathbf{cof}\,\xi < \kappa$

In this case let $\{\gamma_\nu\}_{\nu\in\mathbf{cof}(\xi)}$ be an increasing sequence cofinal in $\xi$. Using Induction Hypotheses (4) and (5) let $\alpha \in \kappa\backslash\mathbf{cof}(\xi)$ be so large that

$$(\forall\nu < \nu^* < \mathbf{cof}(\xi))\ \gamma_\nu \in Z_{\gamma^*_\nu,\alpha}. \tag{3.1}$$

Note that from Condition 3.1 and Induction Hypothesis (6) it follows that

$$(\forall\zeta \geq \alpha)(\forall\nu < \nu^* < \mathbf{cof}(\xi))\ Z_{\gamma_\nu\zeta} = Z_{\gamma_{\nu^*},\zeta} \cap \gamma_\nu. \tag{3.2}$$

Then define

$$Z_{\xi,\zeta} = \begin{cases} \varnothing & \text{if } \zeta \leq \alpha \\ \bigcup_{\nu\in\mathbf{cof}(\xi)} Z_{\gamma_\nu,\zeta} & \text{if } \zeta > \alpha \end{cases}$$

and note that

$$|Z_{\xi,\zeta}| \leq \mathbf{cof}(\xi)g(\zeta) \leq |\alpha|g(\zeta) \leq |\zeta|g(\zeta) \leq g(\zeta)$$

by (2) of Hypothesis 3.1. Hence Induction Hypothesis (2) holds. To see that Induction Hypothesis (3) holds let $\zeta < \zeta^*$. Then $Z_{\gamma_\theta,\zeta} \subseteq Z_{\gamma_\theta,\zeta^*}$ for each $\theta \in \mathbf{cof}(\xi)$ by Induction Hypothesis (3) and it follows that $Z_{\xi,\zeta} \subseteq Z_{\xi,\zeta^*}$. Induction Hypothesis (4) follows from the fact that $\bigcup_{\zeta\in\kappa} Z_{\gamma_\theta,\zeta} = \gamma_\theta$ for each $\theta \in \mathbf{cof}(\xi)$. To see that Induction Hypothesis (5) holds let $\eta \in \xi$. Let $\theta$ be so large that $\eta \in \gamma_\theta$. Then let $\beta \in \kappa \setminus \alpha$ be so large that $Z_{\eta,\zeta} = Z_{\gamma_\theta,\zeta} \cap \eta$ for all $\zeta > \beta$. If $\zeta > \beta$ then

$$Z_{\xi,\zeta} \cap \eta = \left(\bigcup_{\nu\in\mathbf{cof}(\xi)} Z_{\gamma_\nu,\zeta}\right) \cap \eta = Z_{\gamma_\theta,\zeta} \cap \eta = Z_{\eta,\zeta}$$

with the second equality following from Condition 3.2 and the third equality from the choice of $\beta$. To see that Induction Hypothesis (6) holds let $\xi^* \in Z_{\xi,\zeta} \setminus \kappa$. Since $Z_{\xi,\zeta} \neq \varnothing$ it follows that $Z_{\xi,\zeta} = \bigcup_{\nu\in\mathbf{cof}(\xi)} Z_{\gamma_\nu,\zeta}$ and hence $\xi^* \in Z_{\gamma_\nu,\zeta}$ for some $\nu \in \mathbf{cof}(\xi)$. Then $Z_{\xi^*,\zeta} = Z_{\gamma_\nu,\zeta} \cap \xi^*$ by Induction Hypothesis (6). By Condition (3.2) it follows that

$$Z_{\xi,\zeta} \cap \xi^* = \bigcup_{\lambda\in\mathbf{cof}(\xi)} Z_{\gamma_\lambda,\zeta} \cap \xi^* = Z_{\gamma_\nu,\zeta} \cap \xi^* = Z_{\xi^*,\zeta}$$

as required.

**Case Three.** $\mathbf{cof}\,\xi = \kappa$

In this case let $\{\gamma_\nu\}_{\nu\in\kappa}$ be an increasing sequence cofinal in $\xi$. Then let $\{\beta(\theta)\}_{\theta\in\kappa}$ be increasing ordinals in $\kappa$ such that

$$(\forall\nu < \theta)(\forall\zeta \geq \beta(\theta))\ Z_{\gamma_\theta,\zeta} \cap \gamma_\nu = Z_{\gamma_\nu,\zeta}. \tag{3.3}$$

Let $Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta}$ for all $\zeta$ such that $\beta(\theta) \leq \zeta < \beta(\theta+1)$. It is clear that Induction Hypothesis (1) holds. To see that Induction Hypothesis (2) holds note that for each $Z_{\xi,\zeta}$ there is some $\theta$ such that $|Z_{\xi,\zeta}| = |Z_{\gamma_\theta,\zeta}| \leq g(\zeta)$. To see that Induction Hypothesis (3) holds let $\zeta < \zeta^*$. Then there are $\theta \leq \theta^*$ such that $Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta}$ and $Z_{\xi,\zeta^*} = Z_{\gamma_{\theta^*},\zeta^*}$. By Induction Hypothesis (3) it follows that $Z_{\gamma_\theta,\zeta} \subseteq Z_{\gamma_\theta,\zeta^*} \subseteq Z_{\gamma_{\theta^*},\zeta^*}$ with the last inclusion following from the choice of $\beta(\theta^*)$, Condition (refWeid1) and the fact that $\beta(\theta^*) \leq \zeta^*$. Induction Hypothesis (4) follows from the fact that $\bigcup_{\zeta\in\kappa} Z_{\gamma_\theta,\zeta} = \gamma_\theta$ for each $\theta \in \kappa$. To see that Induction Hypothesis (5) holds let $\eta \in \xi$. Let $\theta$ be so large that $\eta \in \gamma_\theta$. Then let $\beta \in \kappa$ be so large that $Z_{\eta,\zeta} = Z_{\gamma_\theta,\zeta} \cap \eta$ for all $\zeta > \beta$. Now let $\zeta > \beta \cup \beta(\theta)$. Then $Z_{\xi,\zeta} = Z_{\gamma_{\theta^*},\zeta}$ for some $\theta^* \geq \theta$ and $\beta(\theta^*) \geq \beta(\theta)$. Hence

$$Z_{\eta,\zeta} = Z_{\gamma_\theta,\zeta} \cap \eta = (Z_{\gamma_{\theta^*},\zeta} \cap \gamma_\theta) \cap \eta = Z_{\xi,\zeta} \cap \eta$$

as required. Since each $Z_{\xi,\zeta}$ is equal to $Z_{\gamma_\theta,\zeta}$ for some $\theta$ it is immediate that Induction Hypothesis (6) holds at $\xi$. □

**Lemma 3.2.** *The hypotheses of Lemma 2.1 hold at $\kappa$.*

*Proof.* To begin the argument let

- $\{Z_{\eta,\zeta}\}_{\eta\in\kappa^+,\zeta\in\kappa}$ be the family whose existence is established in Lemma 3.1
- $\{I_\xi\}_{\xi\in\kappa}$ be a partition of $\kappa$ such that $|I_\xi| = f(\xi)$
- $\{G_\xi\}_{\xi\in\kappa^+}$ witness that Equation (3) of Hypothesis (3.1) holds
- $\{\pi_{\theta,\nu}\}_{\theta\in 2^{f(\nu)}}$ enumerate all functions from $I_\nu$ to $I_\nu$
- $\{E_{\theta,\nu}\}_{\theta\in 2^{f(\nu)}}$ enumerate $\mathcal{P}(I_\nu)$.

For $\nu$ in $\kappa$ let $\mathfrak{B}_{0,\nu} = \{\varnothing, I_\nu\}$ and let $\varphi_{0,\nu}$ be the empty mapping. As the induction hypothesis assume that $\mathfrak{B}_{\xi,\nu}$ and $\varphi_{\xi,\nu}$ have been defined all $\xi \in \eta$ and $\nu \in \kappa$ such that the following conditions are satisfied:

1. for $\xi \in \eta$ and $\nu \in \kappa$:
   (a) $\varphi_{\xi,\nu} : D_{\xi,\nu} \to D_{\xi,\nu}$ is a bijection
   (b) $|D_{\xi,\nu}| < f(\nu)$
   (c) $\mathfrak{B}_{\xi,\nu}$ is an atomic Boolean subalgebra of $\mathcal{P}(I_\nu)$
   (d) $D_{\xi,\nu} \in \mathfrak{B}_{\xi,\nu}$
   (e) $|\,\mathbf{Atoms}(\mathfrak{B}_{\xi,\nu})| \le 2^{g(\nu)}$
2. if $\bar\eta \in \eta$ and $\xi \in Z_{\bar\eta,\nu}$ then:
   (a) $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\bar\eta,\nu}$
   (b) $\varphi_{\xi,\nu} \subseteq \varphi_{\bar\eta,\nu}$
   (c) $\Phi_{\xi,\nu} \subseteq \Phi_{\bar\eta,\nu}$ where $\Phi_{\xi,\nu} : \mathfrak{B}_{\xi,\nu} \to \mathfrak{B}_{\xi,\nu}$ is the isomorphism defined by $\Phi_{\xi,\nu}(A) = \varphi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup (A \setminus D_{\xi,\nu})$.

Observe that by Condition (6) of Lemma 3.1 and Induction Hypothesis (2a) it follows that if $\xi \in \xi^*$ and $\{\xi, \xi^*\} \subseteq Z_{\eta,\nu}$ then $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\xi^*,\nu}$. Since $|Z_{\eta,\nu}| \le g(\nu)$ it follows from Induction Hypothesis (1e) that if

$$\mathfrak{A}_{\eta,\nu} = \left\{ \bigcap_{\xi\in Z_{\eta,\nu}} a(\xi) \;\middle|\; a \in \prod_{\xi\in Z_{\eta,\nu}} \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu}) \right\} \setminus \{\varnothing\}$$

then

$$(\forall \nu \in \kappa)\ \ |\mathfrak{A}_{\eta,\nu}| \le (2^{g(\nu)})^{g(\nu)} = 2^{g(\nu)}. \tag{3.4}$$

Let $\mathcal{E}_{\eta,\nu}$ be the partition of $I_\nu$ generated by $\{E_{\theta,\nu}\}_{\theta\in G_\eta(\nu)}$ and observe that

$$(\forall \nu \in \kappa)\ |\mathcal{E}_{\eta,\nu}| \le 2^{|G_\eta(\nu)|} = 2^{g(\nu)}. \tag{3.5}$$

Let $\mathcal{A}_{\eta,\nu}$ be the partition of $I_\nu$ generated by $\mathcal{E}_{\eta,\nu}$ and $\mathfrak{A}_{\eta,\nu}$ and note that Inequalities (3.4) and (3.5) imply that

$$|\mathcal{A}_{\eta,\nu}| \le 2^{g(\nu)} < f(\nu). \tag{3.6}$$

Since $f(\nu)$ is regular, there is some $A_{\eta,\nu} \in \mathcal{A}_{\eta,\nu}$ such that $|A_{\eta,\nu}| = f(\nu)$.

For each $\nu \in \kappa$ let $B^0_\nu \in [A_{\eta,\nu}]^{|G_\eta(\nu)|}$ be closed under the functions $\pi_{\theta,\nu}$ for $\theta \in G_\eta(\nu)$ and let $B^1_\nu \in [A_{\eta,\nu} \setminus B^0_\nu]^{|G_\eta(\nu)|}$. Now let $\varphi : B^0_\nu \cup B^1_\nu \to B^0_\nu \cup B^1_\nu$ be any involution sending $B^0_\nu$ to $B^1_\nu$. Then define

$$D_{\eta,\nu} = \bigcup_{\xi\in Z_{\eta,\nu}} D_{\xi,\nu} \cup B^0_\nu \cup B^1_\nu$$

noting that $B^0_\nu \cup B^1_\nu$ is disjoint from $D_{\xi,\nu}$ for all $\xi \in Z_{\eta,\nu}$ because $|A_{\eta,\nu}| = f(\nu)$, $|D_{\xi,\nu}| < f(\nu)$ and Induction Hypothesis (1d) holds. Then let

$$\varphi_{\eta,\nu} = \bigcup_{\xi\in Z_{\eta,\nu}} \varphi_{\xi,\nu} \cup \varphi \tag{3.7}$$

and note that if $\xi$ and $\xi^*$ are in $Z_{\eta,\nu}$ then $\varphi_{\xi,\nu} \cup \varphi_{\xi^*,\nu}$ is a bijection by Induction Hypothesis (2b) and Condition (6) of Lemma 3.1. Hence Induction Hypotheses (1a) and (1b) are satisfied.

Let $\mathfrak{B}_{\eta,\nu}$ be the Boolean algebra generated by $\mathcal{A}_{\eta,\nu} \cup \{D_{\eta,\nu}, B^0_\nu, B^1_\nu\}$ and note that (1e) holds by Inequalities (3.6). Clearly Induction Hypotheses (1c) and (1d) also hold. To see that (2a) holds use the definition of $\mathfrak{A}_{\eta,\nu}$. To see that (2b) holds use Equation (3.7) .

To see that (2c) also holds it must be shown that if $\xi \in Z_{\eta,\nu}$ then $\Phi_{\xi,\nu} \subseteq \Phi_{\eta,\nu}$ so let $\xi \in Z_{\eta,\nu}$ and $A \in \mathfrak{B}_{\xi,\nu}$. Then $\Phi_{\xi,\nu}(A) = \varphi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup (A \setminus D_{\xi,\nu})$. On the other hand,

$$\Phi_{\eta,\nu}(A) = \varphi_{\eta,\nu}(A \cap D_{\eta,\nu}) \cup (A \setminus D_{\eta,\nu}) \tag{3.8}$$

and

$$\varphi_{\eta,\nu}(A \cap D_{\eta,\nu}) = \varphi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \left( \bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} \varphi_{\xi^*,\nu}(A \cap (D_{\xi^*,\nu} \setminus D_{\xi,\nu})) \right) \cup \varphi(A \cap (B^0_\nu \cup B^1_\nu)). \tag{3.9}$$

Using Induction Hypothesis (2c) it follows that if $\xi^* \in Z_{\eta,\nu} \setminus \xi$ then $\Phi_{\xi^*,\nu} \supseteq \Phi_{\xi,\nu}$ and so

$$\varphi_{\xi^*,\nu}(A \setminus D_{\xi,\nu}) = \Phi_{\xi^*,\nu}(A \setminus D_{\xi,\nu}) = \Phi_{\xi,\nu}(A \setminus D_{\xi,\nu}) = A \setminus D_{\xi,\nu}$$

Furthermore, $\varphi_{\xi^*,\nu}(D_{\xi^*,\nu}) = D_{\xi^*,\nu}$ and so it follows that

$$\varphi_{\xi^*,\nu}(A\cap(D_{\xi^*,\nu}\setminus D_{\xi,\nu})) = \varphi_{\xi^*,\nu}(A\setminus D_{\xi,\nu})\cap\varphi_{\xi^*,\nu}(D_{\xi^*,\nu}) = \varphi_{\xi^*,\nu}(A\setminus D_{\xi,\nu})\cap D_{\xi^*,\nu} = (A\setminus D_{\xi,\nu})\cap D_{\xi^*,\nu} = A\cap(D_{\xi^*,\nu}\setminus D_{\xi,\nu})$$

and hence

$$\bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} \varphi_{\xi^*,\nu}(A \cap (D_{\xi^*,\nu} \setminus D_{\xi,\nu})) = \bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} A \cap (D_{\xi^*,\nu} \setminus D_{\xi,\nu}) = A \cap \left( \bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} D_{\xi^*,\nu} \setminus D_{\xi,\nu} \right)$$

Hence Equation (4.10) becomes

$$\varphi_{\eta,\nu}(A \cap D_{\eta,\nu}) = \varphi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \left( A \cap \left( \bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} D_{\xi^*,\nu} \setminus D_{\xi,\nu} \right) \right) \cup \varphi(A \cap (B^0_\nu \cup B^1_\nu)). \tag{3.10}$$

Noting that either $A_{\eta,\nu} \subseteq A$ or $A_{\eta,\nu} \cap A = \varnothing$ it follows that either $B^0_\nu \cup B^1_\nu \subseteq A$ or $(B^0_\nu \cup B^1_\nu) \cap A = \varnothing$. In either case it follows that

$$\varphi(A \cap (B^0_\nu \cup B^1_\nu)) = A \cap (B^0_\nu \cup B^1_\nu)$$

and so Equation (3.10) becomes
(3.11)

$$\varphi_{\eta,\nu}(A\cap D_{\eta,\nu}) = \varphi_{\xi,\nu}(A\cap D_{\xi,\nu})\cup \left( A \cap \left( \bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} D_{\xi^*,\nu} \setminus D_{\xi,\nu} \right) \right) \cup (A\cap(B^0_\nu\cup B^1_\nu)) = \varphi_{\xi,\nu}(A\cap D_{\xi,\nu})\cup(A\cap D_{\eta,\nu}\setminus D_{\xi,\nu})$$

Combining this with Equation (4.9) it follows that

$$\Phi_{\eta,\nu}(A) = \varphi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup (A \setminus D_{\xi,\nu})$$

as required for (2c) to hold.

To see that Hypothesis (4) of Lemma 2.1 holds let $F : \kappa \to \kappa$ be one-to-one. If there are cofinally many $\nu$ such that there is $z_\nu \in I_\nu$ such that $F(z_\nu) \notin I_\nu$ then let $\xi = 0$ and note that $z_\nu \in I_\nu \in \mathfrak{B}_{0,\nu}$ for cofinally many $\nu$. If, on the other hand, $F(I_\nu) \subseteq I_\nu$ for all $\nu$ in a final segment of $\kappa$ then $F \restriction I_\nu = \pi_{J(\nu),\nu}$ for some $J(\nu) \in 2^{f(\nu)}$ for a tail of $\nu \in \kappa$. By (3) of Hypothesis 3.1 there is then some $\xi \in \kappa^+$ such that $J(\nu) \in G_\xi(\nu)$ for a final segment of $\nu \in \kappa$. By construction, for a final segment of $\nu \in \kappa$ there are disjoint $B^0_\nu$ and $B^1_\nu$ such that $\Phi_{\xi,\nu}(B^0_\nu) = B^1_\nu$ while $F(B^0_\nu) = \pi_{J(\nu),\nu}(B^0_\nu) = B^0_\nu$.

Finally, to see that Hypothesis (5) of Lemma 2.1 holds let $A \subseteq \kappa$. Then $A \cap I_\nu = E_{J(\nu),\nu}$ for some $J(\nu) \in 2^{f(\nu)}$ for all $\nu \in \kappa$. By (3) of Hypothesis 3.1 there is $\xi \in \kappa^+$ such that $J(\nu) \in G_\xi(\nu)$ for a final segment of $\nu \in \kappa$. It follows that $E_{J(\nu),\nu}$ belongs to the Boolean algebra $\mathfrak{B}_{\xi,\nu}$ for a tail of $\nu \in \kappa$. In other words, $A \cap I_\nu = E_{J(\nu),\nu} \in \mathfrak{B}_\xi(\nu)$ for all $\nu$ in a final segment of $\kappa$. □

**Corollary 3.1.** *If $\kappa$ is inaccessible and $2^\kappa = \kappa^+$ then there is a non-trivial automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$.*

*Proof.* Let $f$ and $g$ be functions on $\kappa$ satisfying Conditions (1) and (2) of Hypothesis 3.1. Let $\{H_\xi\}_{\xi\in\kappa^+}$ enumerate $\prod_{\nu\in\kappa} 2^{f(\nu)}$. Let $e_\xi : \kappa \to \xi$ be a bijection for $\xi \in \kappa^+ \setminus \kappa$. Then let $G_\xi \in \prod_{\nu\in\kappa}[2^{f(\nu)}]^{g(\nu)}$ be defined by

$$G_\xi(\nu) = \left\{ H_{e_\xi(\eta)}(\nu) \mid \eta \in g(\nu) \right\}.$$

A standard argument shows that $\{G_\xi\}_{\xi\in\kappa^+}$ witnesses that $\mathfrak{d}_{2^f,g} = \kappa^+$ so that Lemma 3.2 can be applied. □

**Corollary 3.2.** *If $\kappa$ is inaccessible then it is consistent that $2^\kappa > \kappa^+$ and there is a non-trivial automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$.*

*Proof.* Fix functions $f$ and $g$ on $\kappa$ satisfying Conditions (1) and (2) of Hypothesis 3.1. Let $\mathbb{P}$ be the partial order consisting of pairs $(d, \mathcal{H})$ such that:

- there is some $\alpha \in \kappa$ such that $d \in \prod_{\nu\in\alpha}[f(\nu)]^{g(\nu)}$
- $\mathcal{H} \subseteq \prod_{\nu\in\kappa} f(\nu)$
- $|\mathcal{H}| \leq g(\alpha)$

and define $(d, \mathcal{H}) \leq (d^*, \mathcal{H}^*)$ if

- $\mathcal{H} \supseteq \mathcal{H}^*$
- $d \supseteq d^*$
- $h(\eta) \in d(\eta)$ for all $h \in \mathcal{H} \setminus \mathcal{H}^*$ and $\eta$ in the domain of $d \setminus d^*$.

It is routine to see that if $G \subseteq \mathbb{P}$ is generic over $V$ and $d_G \in \prod_{\nu\in\alpha}[f(\nu)]^{g(\nu)}$ is defined by $d_G(\alpha) = d(\alpha)$ for some $(d, \mathcal{H}) \in G$ then for any $h \in \prod_{\nu\in\kappa} f(\nu) \cap V$ there is some $\beta \in \kappa$ such that $h(\alpha) \in d_G(\alpha)$ for all $\alpha > \beta$. It is then standard to obtain Condition (3) and $2^\kappa > \kappa^+$ by iterating this forcing $\kappa^+$ times over a model where $2^\kappa > \kappa^+$. □

## 4. When are the hypotheses of Lemma 2.1 satisfied at $\omega$?

**Hypothesis 4.1.** For the remainder of this section fix functions $f$ and $g$ from $\omega$ to $\omega$ such that

1. $f(k) \geq 2g(k)(g(k) + 3^{g(k)^2})$ for all $k$
2. $g(k) \geq 2$ for all $k$ and hence
   (a) $f(k)! \geq 2^{f(k)}$ for all $k$
   (b) $2^{g(k)}3^{g(k)(m-1)} + 2 \leq 3^{g(k)m}$ for all $k$ and $m \geq 1$
3. if $k < k^*$ then $k \leq g(k) \leq g(k^*)$
4. $\mathfrak{d}_{f!,g} = \omega_1$.

**Lemma 4.1.** *There are $\{Z_{\eta,k}\}_{\eta\in\omega_1,k\in\omega}$ satisfying the following:*

*(1) $Z_{\eta,k} \subseteq \eta$ and $\varnothing \neq Z_{\eta,k}$ if $\eta > 0$.*
*(2) $|Z_{\eta,k}| \leq g(k)$*
*(3) $\lim_{k\to\infty} |Z_{\eta,k}|/g(k) = 0$*
*(4) if $k < k^*$ then $Z_{\eta,k} \subseteq Z_{\eta,k^*}$*
*(5) $\bigcup_{k\in\omega} Z_{\eta,k} = \eta$*
*(6) if $\eta \in \eta^* \in \omega_1$ then there is $j \in \omega$ such that $Z_{\eta^*,m} \cap \eta = Z_{\eta,m}$ for all $m > j$*
*(7) if $\eta \in Z_{\eta^*,k}$ then $Z_{\eta^*,k} \cap \eta = Z_{\eta,k}$.*

*Proof.* Start by letting $Z_{0,k} = \varnothing$. If $Z_{\eta,k}$ have been defined for $\eta \in \xi$ and $k \in \omega$ consider two cases.

**Case One.** $\xi = \xi^* + 1$

In this case let $m$ be so large that $|Z_{\xi^*,k}| + 1 \leq g(k)$ for $k > m$ and define

$$Z_{\xi,k} = \begin{cases} Z_{\xi^*,k} \cup \{\xi^*\} & \text{if } k > m \\ 1 & \text{if } k \leq m. \end{cases}$$

**Case Two.** $\xi$ is a limit ordinal

In this case let $\{\gamma_n\}_{n\in\omega}$ be an increasing sequence cofinal in $\xi$ and let $\{b(t)\}_{t\in\omega}$ be an increasing sequence of intergers such that

$$(\forall t \geq 1)(\forall k \geq b(t))\ |Z_{\gamma_t,k}|/g(k) \leq 1/t. \tag{4.1}$$

$$(\forall t)(\forall n < t)(\forall k \geq b(t))\ Z_{\gamma_t,k} \cap \gamma_n = Z_{\gamma_n,k}. \tag{4.2}$$

Let

$$Z_{\xi,k} = \begin{cases} 1 & \text{if } k < b(0) \\ Z_{\gamma_t,k} & \text{if } b(t) \leq k < b(t+1). \end{cases}$$

It is clear that Induction Hypothesis (1) holds. To see that Induction Hypothesis (2) holds note that for each $Z_{\xi,k}$ there is some $t$ such that $|Z_{\xi,k}| = |Z_{\gamma_t,k}| \leq g(k)$. The fact that Induction Hypothesis (3) holds follows from Inequality (4.1). To see that Induction Hypothesis (4) holds let $k < k^*$. Then there are $t \leq t^*$ such that $Z_{\xi,k} = Z_{\gamma_t,k}$ and $Z_{\xi,k^*} = Z_{\gamma_{t^*},k^*}$. By Induction Hypothesis (4) it follows that $Z_{\gamma_t,k} \subseteq Z_{\gamma_t,k^*} \subseteq Z_{\gamma_{t^*},k^*}$ with the last inclusion following from the choice of $b(t^*)$, Condition (4.2) and the fact that $b(t^*) \leq k^*$. Induction Hypothesis (5) follows from the fact that $\bigcup_{k\in\omega} Z_{\gamma_t,k} = \gamma_t$ for each $t \in \omega$ and Condition (4.2). To see that Induction Hypothesis (6) holds let $\eta \in \xi$. Let $t$ be so large that $\eta \in \gamma_t$. Then let $b \in \omega$ be so large that $Z_{\eta,k} = Z_{\gamma_t,k} \cap \eta$ for all $k \geq b$. Now let $k > \max(b, b(t))$. Then $Z_{\xi,k} = Z_{\gamma_{t^*},k}$ for some $t^* \geq t$ and note that $b(t^*) \geq b(t)$. Hence

$$Z_{\eta,k} = Z_{\gamma_t,k} \cap \eta = (Z_{\gamma_{t^*},k} \cap \gamma_t) \cap \eta = Z_{\xi,k} \cap \eta$$

as required. Since each $Z_{\xi,k}$ is equal to $Z_{\gamma_t,k}$ for some $t \in \omega$ it is immediate that Induction Hypothesis (7) holds at $\xi$. □

**Lemma 4.2.** *The hypotheses of Lemma 2.1 hold at $\omega$.*

*Proof.* To begin the argument let

- $\{Z_{\eta,k}\}_{\eta\in\omega_1,k\in\omega}$ be the family whose existence is established in Lemma 4.1

- $\{I_k\}_{k\in\omega}$ be a partition of $\omega$ such that $|I_k| = f(k)$
- $\{G_\xi\}_{\xi\in\omega_1}$ witness that Equation (3) of Hypothesis 4.1 holds
- $\{\pi_{j,k}\}_{j\in f(k)!}$ enumerate all bijections from $I_k$ to $I_k$
- $\{E_{j,k}\}_{j\in 2^{f(k)}}$ enumerate $\mathcal{P}(I_k)$ using Inequality (2a) of Hypothesis 4.1.

For $k$ in $\omega$ let $\mathfrak{B}_{0,k} = \{\varnothing, I_k\}$ and let $\varphi_{0,k}$ be the empty mapping. As the induction hypothesis assume that $\mathfrak{B}_{\xi,k}$ and $\varphi_{\xi,k}$ have been defined all $\xi \in \eta$ and $k \in \omega$ such that the following conditions are satisfied:

(1) for $\xi \in \eta$ and $k \in \omega$:
  (a) $\varphi_{\xi,k} : D_{\xi,k} \to D_{\xi,k}$ is a bijection
  (b) $|D_{\xi,k}| \leq 2g(k)|Z_{\xi,k}|$
  (c) $\mathfrak{B}_{\xi,k}$ is a Boolean subalgebra of $\mathcal{P}(I_k)$
  (d) $D_{\xi,k} \in \mathfrak{B}_{\xi,k}$
  (e) $|\,\mathbf{Atoms}(\mathfrak{B}_{\xi,k})| \leq 3^{g(k)|Z_{\xi,k}|}$

(2) if $\bar\eta \in \eta$ and $\xi \in Z_{\bar\eta,k}$ then:
  (a) $\mathfrak{B}_{\xi,k} \subseteq \mathfrak{B}_{\bar\eta,k}$
  (b) $\varphi_{\xi,k} \subseteq \varphi_{\bar\eta,k}$
  (c) $\Phi_{\xi,k} \subseteq \Phi_{\bar\eta,k}$ where $\Phi_{\xi,k} : \mathfrak{B}_{\xi,k} \to \mathfrak{B}_{\xi,k}$ is the isomorphism defined by $\Phi_{\xi,k}(A) = \varphi_{\xi,k}(A \cap D_{\xi,k}) \cup (A \setminus D_{\xi,k})$.

Using Condition (1) of Lemma 4.1 let $\mu(k)$ denote the maximum element of $Z_{\eta,k}$. It follows that

$$|\,\mathbf{Atoms}(\mathfrak{B}_{\mu(k),k})| \leq 3^{g(k)|Z_{\mu(k),k}|} = 3^{g(k)(|Z_{\eta,k}|-1)} \tag{4.3}$$

by Condition (7) of Lemma 4.1. Also by Condition (7) of Lemma 4.1 and Induction Hypothesis (2a) it follows that

$$(\forall \xi \in Z_{\eta,k})\ \mathfrak{B}_{\xi,k} \subseteq \mathfrak{B}_{\mu(k),k}. \tag{4.4}$$

Let $\mathcal{E}_{\eta,k}$ be the partition of $I_k$ generated by $\{E_{j,k}\}_{j\in G_\eta(k)}$ and observe that

$$(\forall k \in \omega)\ |\mathcal{E}_{\eta,k}| \leq 2^{|G_\eta(k)|} = 2^{g(k)}. \tag{4.5}$$

Let $\mathcal{A}_{\eta,\nu}$ be the partition of $I_{\eta,\nu}$ generated by $\mathcal{E}_{\eta,\nu}$ and $\mathbf{Atoms}(\mathfrak{B}_{\mu(k),k})$ and note that Inequalities (4.3) and (4.5) imply that

$$(\forall k \in \omega)\ |\mathcal{A}_{\eta,k}| \leq 2^{g(k)}3^{g(k)(|Z_{\eta,k}|-1)} \leq 3^{g(k)|Z_{\eta,k}|} \leq 3^{g(k)^2}. \tag{4.6}$$

Since

$$|I_k \setminus D_{\mu(k),k}| \geq f(k) - 2g(k)|Z_{\mu(k),k}| \geq f(k) - 2g(k)^2 \geq 3^{g(k)^2}2g(k)$$

by Condition (1) of Hypothesis 4.1 there is some $A_{\eta,k} \in \mathcal{A}_{\eta,k}$ such that $|A_{\eta,k}| \geq 2g(k)$ and $A_{\eta,k} \cap D_{\mu(k),k} = \varnothing$. For each $k \in \omega$ let $w_k \in A_{\eta,k}$ and let

$$B^0_k = \{\pi_{\eta,j}(w_k)\ \mid j \in G_\eta(k)\} \cap A_{\eta,k}$$

Then let $B^1_k \in [A_{\eta,k} \setminus B^0_k]^{|B^0_k|}$ and let $\varphi : B^0_k \cup B^1_k \to B^0_k \cup B^1_k$ be any involution sending $B^0_k$ to $B^1_k$. Then define

$$D_{\eta,k} = D_{\mu(k),k} \cup B^0_k \cup B^1_k$$

noting that

$$|D_{\eta,k}| \leq 2g(k) + |D_{\mu(k),k}| \leq 2g(k) + |Z_{\mu(k),k}|2g(k) = 2g(k) + (|Z_{\eta,k}| - 1)2g(k) = |Z_{\eta,k}|2g(k)$$

Then let

$$\varphi_{\eta,k} = \varphi_{\mu(k),k} \cup \varphi \tag{4.7}$$

and note that $\varphi_{\eta,k}$ is a bijection by Induction Hypothesis (2b) and since $B^0_k \cup B^1_k$ is disjoint from $D_{\eta,k}$. Hence Induction Hypotheses (1a) and (1b) are satisfied.

Define $\mathfrak{B}_{\eta,k}$ be the Boolean algebra generated by $\mathcal{A}_{\eta,k} \cup \{B^0_k, B^1_k\}$ and note that $D_{\eta,k}, \in \mathfrak{B}_{\eta,k}$ so that Induction Hypothesis 1d holds. Using the first part of Inequality (4.6) and Inequality (2b) of Hypothesis 4.1,

$$|\,\mathbf{Atoms}(\mathfrak{B}_{\eta,k})| = |\mathcal{A}_{\eta,k}| + 2 \leq 2^{g(k)}\left(3^{g(k)}\right)^{|Z_{\eta,k}|-1} + 2 \leq 3^{g(k)|Z_{\eta,k}|} \tag{4.8}$$

so that Induction Hypothesis (1e) holds. Clearly Induction Hypotheses (1c) and (1d) also hold. To see that (2a) holds use Inclusion (4.4). To see that (2b) holds use Equation (4.7) .

To see that (2c) also holds it must be shown that if $\xi \in Z_{\eta,k}$ then $\Phi_{\xi,k} \subseteq \Phi_{\eta,k}$ so let $\xi \in Z_{\eta,k}$ and $A \in \mathfrak{B}_{\xi,k}$. Begin by noting that $\xi \leq \mu(k)$ and so from Induction Hypothesis (2c) it follows that $\Phi_{\xi,k}(A) = \Phi_{\mu(k),k}(A)$ and so it suffices to show that $\Phi_{\mu(k),k}(A) = \Phi_{\eta,k}(A)$. Start by recalling that

$$\Phi_{\eta,k}(A) = \varphi_{\eta,k}(A \cap D_{\eta,k}) \cup (A \setminus D_{\eta,k}) \tag{4.9}$$

and

$$\varphi_{\eta,k}(A \cap D_{\eta,k}) = \varphi_{\mu(k),k}(A \cap D_{\mu(k),k}) \cup \varphi(A \cap (B^0_\nu \cup B^1_\nu)). \tag{4.10}$$

Furthermore,

$$\Phi_{\mu(k),k}(A) = \varphi_{\mu(k),k}(A \cap D_{\mu(k),k}) \cup (A \setminus D_{\mu(k),k})$$

and so it suffices to show that

$$\varphi_{\mu(k),k}(A \cap D_{\mu(k),k}) \cup (A \setminus D_{\mu(k),k}) = \varphi_{\mu(k),k}(A \cap D_{\mu(k),k}) \cup \varphi(A \cap (B^0_k \cup B^1_k)) \cup (A \setminus D_{\eta,k}) \tag{4.11}$$

Noting that either $A_{\eta,k} \subseteq A$ or $A_{\eta,k} \cap A = \varnothing$ it follows that either $B^0_k \cup B^1_k \subseteq A$ or $(B^0_k \cup B^1_k) \cap A = \varnothing$. In either case it follows that

$$\varphi(A \cap (B^0_k \cup B^1_k)) = A \cap (B^0_k \cup B^1_k)$$

and so Equation (4.11) becomes

$$\varphi_{\mu(k),k}(A \cap D_{\mu(k),k}) \cup (A \setminus D_{\mu(k),k}) = \varphi_{\mu(k),k}(A \cap D_{\mu(k),k}) \cup \big(A \cap (B^0_k \cup B^1_k)\big) \cup (A \setminus D_{\eta,k}) \tag{4.12}$$

and so to see that (2c) holds it suffices to observe that $A \setminus D_{\mu(k),k} = \big(A \cap (B^0_k \cup B^1_k)\big) \cup (A \setminus D_{\eta,k})$.

The proofs of Hypothesis (4) and Hypothesis (5) of Lemma 2.1 are exactly the same as in the proof of Lemma 3.2. □

**Corollary 4.1.** *If $\mathfrak{d}_{f!,g} = \aleph_1$ then there is a nontrivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.*

**Definition 4.1.** Given functions $f$ and $g$ from $\omega$ to $\omega$ define and a filter $\mathcal{F}$ on $\omega$ define $\mathfrak{d}_{f,g}(\mathcal{F})$ to be the least cardinal of a family $\mathcal{D} \subseteq \prod_{k\in\omega}[f(k)]^{g(k)}$ such that for every $F \in \prod_{k\in\omega} f(k)$ there is $G \in \mathcal{D}$ such that $\{k \in \omega \mid F(k) \in G(k)\} \in \mathcal{F}^+$ .

*Remark* 1. It can be verified that if $\mathcal{F}$ is generated by a $\subseteq^*$ descending tower of length $\omega_1$ and then in order to obtain the conclusion of Lemma 4.2 it suffices to have the equality $\mathfrak{d}_{f!,g}(\mathcal{F}) = \aleph_1$. This yields the following corollary.

**Corollary 4.2.** *If there is an $\aleph_1$-generated filter $\mathcal{F}$ such that $\mathfrak{d}_{f!,g}(\mathcal{F}) = \aleph_1 \neq \mathfrak{d}$ then there is a nontrivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.*

*Proof.* Let $\mathcal{F}$ be generated by $\{X_\xi\}_{\xi\in\omega_1}$. Use Rothberger's argument and $\aleph_1 \neq \mathfrak{d}$ to construct a $\subseteq^*$-descending sequence $\{Y_\xi\}_{\xi\in\omega_1}$ all of whose terms are $\mathcal{F}$ positive and such that $Y_\xi \subseteq X_\xi$. Let $\mathcal{F}'$ be generated by $\{Y_\xi\}_{\xi\in\omega_1}$ and note that $\mathfrak{d}_{f!,g}(\mathcal{F}') = \aleph_1$. □

## 5. Remarks and questions

The first thing to note that there are models where $\mathfrak{d}_{f!,g} = \aleph_1 < 2^{\aleph_0}$ for $f$ and $g$ satisfying the Hypothesis 4.1 — for example, this is true in the model obtained by either iteratively adding $\omega_2$ Sacks reals[1] or adding any number of Sacks reals, greater than $\aleph_1$ of course, side-by-side. Of course $\mathfrak{d} = \aleph_1$ also in these models. It is therefore of interest to note that the Laver property implies that $\mathfrak{d}_{f!,g} = \aleph_1$ in the Laver model as well, yet $\mathfrak{d} = \aleph_2$ in this model. It should also be observed that it is possible for $\mathfrak{d}_{f,g}$ to be larger that $\mathfrak{d}$. For example, iteratively forcing $\omega_2$ times with perfect trees $T$ that are cofinally $f$ branching will yield such a model.

To be a bit more precise, given $f : \omega \to \omega$ define $\mathbb{S}(f)$ to consist of all trees $T \subseteq \bigcup_{n\in\omega}\prod_{j\in n} f(j)$ such that or each $t \in T$ there is $s \supseteq t$ such that $s^\frown j \in T$ for all $j \in f(|s|)$. So Sacks forcing is just $\mathbb{S}(2)$ where 2 is the constant 2 function. The same proof as for Sacks forcing shows that $\mathbb{S}(f)$ is proper and adds no reals unbounded by the ground model. Iterating $\mathbb{S}(f)$ with countable support $\omega_2$ times then yields model in which $\mathfrak{d} = \aleph_1$. However, if $g : \omega \to \omega$ and $\mathcal{H} \subseteq \prod_{n\in\omega}[f(n)]^{g(n)}$ has cordiality $\aleph_1$ then there is some model containing $g$ and $\mathcal{H}$ and there is $\Gamma \in \prod_{n\in\omega} f(n)$ which is generic over this model. This generically ensure that for all $h \in \mathcal{H}$ there is some $j$ such that $\Gamma(j) \notin h(j)$.

It has already been shown in Corollary 3.2 that the hypotheses of Lemma 3.2 can be satisfied for uncountable cardinals, but it is worth noting that the generalization of Sacks reals to uncountable cardinals in [5] provides an alternate argument. However, the following question does not seem to be answered.

**Question 5.1.** Is it consistent for an inaccessible cardinal $\kappa$ that $\mathfrak{d}_{f,g} = \kappa^+$ where $f(\alpha) = (2^{\aleph_\alpha})^+$ and $g(\alpha) = \aleph_\alpha$ yet $\mathfrak{d}(\kappa) > \kappa^+$ where $\mathfrak{d}(\kappa)$ is the generalization of $\mathfrak{d}$ to $\kappa$?

It has to be noted that the hypothesis of Corollary 4.2 is not vacuous in the sense that there are models of set theory in which it holds. For example, in the model obtained by iterating Miller reals $\omega_2$ times the following hold:

- $\mathfrak{d} = \aleph_2$ because the Miller reals themselves are unbounded by the ground model
- $\mathfrak{d}_{f,g} = \aleph_1$ for appropriate $f$ and $g$ because the Miller partial order satisfies the Laver property
- $\mathfrak{u} = \aleph_1$ because P-points from the ground model generate ultrafilters in the extension.

[1]See [1] for definitions of terms not defined in this section as well as for details of proofs.

However there does not seem to be any model demonstrating that the assumption that $\aleph_1 \neq \mathfrak{d}$ in Corollary 4.2 is essential. It is shown in [3] that it is consistent with set theory that $\mathfrak{d} = \aleph_1$ yet all automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial. However, $\mathfrak{u} = \aleph_2$ in that model because random reals are added cofinally often. This motivates the following question.

**Question 5.2.** Does the existence of a nontrivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ follow from the assumption that there is an $\aleph_1$-generated filter $\mathcal{F}$ such that $\mathfrak{d}_{f!,g}(\mathcal{F}) = \aleph_1$?

It is worth observing that the isomorphism of Lemma 2.1 is trivial on some infinite sets — indeed, if $\xi \in \kappa^+$ and $X \subseteq \mathbb{N}$ are such that $\{x\}$ belongs to some $\mathfrak{B}_{\xi,\nu}$ for each $x \in X$ then $\Phi$ is trivial on $\mathcal{P}(X)$. However, if $\mathcal{T}(\Phi)$ is defined to be the ideal $\{X \subseteq \mathbb{N} \mid \Phi \restriction \mathcal{P}(X) \text{ is trivial }\}$ then $\mathcal{T}(\Phi)$ is a small ideal in the sense that the quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{T}(\Phi)$ has large antichains, even modulo the ideal of finite sets — in the terminology of [2], the ideal $\mathcal{T}(\Phi)$ is not ccc by fin. To see this, simply observe that the proof of Lemma **??** actually shows that Hypothesis 4 of Lemma 2.1 can be strengthened to: For any one-to-one $F : \mathbb{N} \to \mathbb{N}$ there is $\xi \in \kappa^+$ such that for all but finitely many $\nu \in \omega$ there is an atom $a \in \mathfrak{B}_{\xi,\nu}$ and $\iota \in a$ such that $F(\iota) \notin \Phi_{\xi,\nu}(a)$. It follows that if $Z \subseteq \mathbb{N}$ is infinite then $Z^* = \bigcup_{\nu \in Z} I_\nu \notin \mathcal{T}(\Phi)$. Hence, if $\mathcal{A}$ is an almost disjoint family of subsets of $\mathbb{N}$ then $\{A^* \mid A \in \mathcal{A}\}$ is an antichain modulo the ideal of finite sets.

One should not, therefore, expect to get a nowhere trivial isomorphism by these methods. It is nevertheless, conceivable that there are some other cardinal invariants similar to $\mathfrak{d}_{f,g}$ that would, when small, imply the existence of nowhere trivial isomorhisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$. In this context it is interesting to note that it is at least consistent with small $\mathfrak{d}$ that there are nowhere trivial isomorphisms.

**Proposition 5.1.** *It is consistent that $\aleph_1 = \mathfrak{d} \neq 2^{\aleph_0}$ and there is a nowhere trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ .*

*Sketch of proof.* The partial order defined in Definition 2.1 of §2 of [13] will be used[2]. Begin with a model $V$ satisfying $2^{\aleph_0} > \aleph_1$ and construct a tower of permutations $\{(A_\xi, F_\xi, \mathfrak{B}_\xi)\}_{\xi \in \mathrm{Lim}(\omega_1)}$ such that, letting $\mathfrak{S}_\eta = \{(A_\xi, F_\xi, \mathfrak{B}_\xi)\}_{\xi \in \mathrm{Lim}(\eta)}$ and $\mathbb{P}_\eta$ be the finite support iteration of partial orders that are $\mathbb{Q}(\mathfrak{S}_\xi)$ for $\xi \in \mathrm{Lim}(\eta)$ and Hechler forcing if $\xi$ is a successor, the following holds for each $\eta$ and $G$ that is $\mathbb{P}_{\omega_1}$ generic over $V$:

- $A_\eta = A_{\mathfrak{S}_\eta}[G \cap \mathbb{Q}(\mathfrak{S}_\eta)]$
- $F_\eta = F_{\mathfrak{S}_\eta}[G \cap \mathbb{Q}(\mathfrak{S}_\eta)]$
- $\mathfrak{B}_\eta = \mathcal{P}(\mathbb{N}) \cap V[G \cap \mathbb{P}_\eta]$

The proof of Theorem 2.1 in [13] shows that there is a nowhere trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ in this model and, since $\mathbb{P}_{\omega_1}$ is ccc, it is also true that $2^{\aleph_0}$ remains larger than $\aleph_1$ in the generic extension. The Hechler reals guarantee that $\mathfrak{d} = \aleph_1$. □

It should also be noted that Lemma 2.1 actually yields $2^{(\kappa^+)}$ isomorphisms. It is shown in [14] that it is possible to have non-trivial isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ without having $2^{\mathfrak{c}}$ such isomorphisms. This motivates the following, somewhat vague, question.

**Question 5.3.** Can there be some variant of $\mathfrak{d}_{f,g}$ which, when small, yields a non-trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ without yielding the maximal possible number of such?

Given the remarks following Corollary 4.2 it is natural to ask the following.

**Question 5.4.** Is it consistent that $\mathfrak{d}_{f!,g} = \mathfrak{d}$ for $f$ and $g$ satisfying the hypothesis of Lemma **??** and to have $\mathfrak{u} = \aleph_1$ and to have that all isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial?

As a final remark it will be noted that Corollary 4.1 shows that Theorem 3.1 of [13] cannot be improved to show that in models obtained by iterating Sacks or Silver reals all isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial because the equality $\mathfrak{d}_{f!,g} = \aleph_1$ holds in these models for the necessary $f$ and $g$.

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[2]The reader is warned that the word "finite" should be removed from (3) of Definition 2.1 in [13].

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Institute of Mathematics, Hebrew University, Jerusalem, Givat Ram, Israel and Department of Mathematics, Rutgers University, New Brunswick, New Jersey, USA
*E-mail address*: shelah@math.huji.ac.il

Department of Mathematics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3
*E-mail address*: steprans@yorku.ca