Vive la Différence !

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Abstract. For a set M, $\operatorname{fin}(M)$ denotes the set of all finite subsets of M, M^2 denotes the Cartesian product $M \times M$, $[M]^2$ denotes the set of all 2-element subsets of M, and $\operatorname{seq}^{1\text{-}1}(M)$ denotes the set of all finite sequences without repetition which can be formed with elements of M. Furthermore, for a set S, let $|S|$ denote the cardinality of S. Under the assumption that the four cardinalities $|[M]^2|$, $|M^2|$, $|\operatorname{fin}(M)|$, $|\operatorname{seq}^{1\text{-}1}(M)|$ are pairwise distinct and pairwise comparable in ZF, there are six possible linear orderings between these four cardinalities. We show that at least five of the six possible linear orderings are consistent with ZF.

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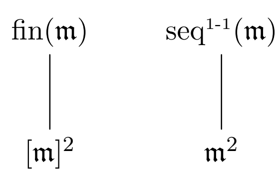
1 Introduction

Let M be a set. Then $\operatorname{fin}(M)$ denotes the set of all finite subsets of M, M^2 denotes the Cartesian product $M \times M$, $[M]^2$ denotes the set of all 2-element subsets of M, $\operatorname{seq}^{1\text{-}1}(M)$ denotes the set of all finite sequences without repetitions which can be formed with elements of M, and $\operatorname{seq}(M)$ denotes the set of all finite sequences which can be formed with elements of M (where repetitions are allowed).

Furthermore, for a set A, let $|A|$ denote the cardinality of A. We write $|A| = |B|$, if there exists a bijection between A and B, and we write $|A| \leq |B|$, if there exists a bijection between A and a subset $B' \subseteq B$ (*i.e.*, $|A| \leq |B|$ if and only if there exists an injection from A into B). Finally, we write $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$. By the Cantor-Bernstein Theorem, which is provable in ZF only (*i.e.*, without using the Axiom of Choice), we get that $|A| \leq |B|$ and $|A| \geq |B|$ implies $|A| = |B|$.

Let $\mathfrak{m} := |M|$, and let $[\mathfrak{m}]^2 := |[M]^2|$, $\mathfrak{m}^2 := |M^2|$, $\operatorname{fin}(\mathfrak{m}) := |\operatorname{fin}(M)|$, $\operatorname{seq}^{1\text{-}1}(\mathfrak{m}) := |\operatorname{seq}^{1\text{-}1}(M)|$, and $\operatorname{seq}(\mathfrak{m}) := |\operatorname{seq}(M)|$. Concerning these cardinalities, in ZF we obviously have $\operatorname{seq}^{1\text{-}1}(\mathfrak{m}) \leq \operatorname{seq}(\mathfrak{m})$, $[\mathfrak{m}]^2 \leq \operatorname{fin}(\mathfrak{m})$ and $\mathfrak{m}^2 \leq \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$, where the latter relations are visualized by the following diagram (in the diagram, $\mathfrak{n}_1$ is below $\mathfrak{n}_2$ if $\mathfrak{n}_1 \leq \mathfrak{n}_2$):



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Moreover, for *finite* cardinals $\mathfrak{m}$ with $\mathfrak{m} \geq 5$ we have

$$[\mathfrak{m}]^2 < \mathfrak{m}^2 < \mathrm{fin}(\mathfrak{m}) < \mathrm{seq}^{\text{1-1}}(\mathfrak{m}),$$

and in the presence of the Axiom of Choice (*i.e.*, in ZFC), for every infinite cardinal $\mathfrak{m}$ we have

$$[\mathfrak{m}]^2 = \mathfrak{m}^2 = \mathrm{fin}(\mathfrak{m}) = \mathrm{seq}^{\text{1-1}}(\mathfrak{m}).$$

It is natural to ask whether some of these equalities can be proved also in ZF, *i.e.*, without the aid of AC. Surprisingly, this is not the case. In [1], a permutation model was constructed in which for an infinite cardinal $\mathfrak{m}$ we have $\mathrm{seq}(\mathfrak{m}) < \mathrm{fin}(\mathfrak{m})$ (see [1, Thm. 2] or [4, Prp. 7.17]). As a consequence we obtain that the existence of an infinite cardinal $\mathfrak{m}$ such that $\mathrm{seq}^{\text{1-1}}(\mathfrak{m}) < \mathrm{fin}(\mathfrak{m})$ is consistent with ZF. This consistency result was modified to the existence of an infinite cardinal $\mathfrak{m}$ for which $\mathfrak{m}^2 < [\mathfrak{m}]^2$ (see [4, Prp. 7.18]), and later, it was strengthened to the existence of an infinite cardinal $\mathfrak{m}$ for which $\mathrm{seq}(\mathfrak{m}) < [\mathfrak{m}]^2$ (see [3] or [5, Prp. 8.28]). The consistency of $\mathrm{fin}(\mathfrak{m}) < \mathrm{seq}^{\text{1-1}}(\mathfrak{m})$ for infinite cardinals $\mathfrak{m}$ can be obtained with the *Ordered Mostowski Model* (see, for example, [5, Related Result 48, p. 217]), in which there is an infinite cardinal $\mathfrak{m}$ with

$$[\mathfrak{m}]^2 < \mathfrak{m}^2 < \mathrm{fin}(\mathfrak{m}) < \mathrm{seq}^{\text{1-1}}(\mathfrak{m}).$$

Consistency results as well as ZF-results concerning the relations between these cardinals with other cardinals can be found, for example, in [7, 8] or [2].

Concerning the four cardinalities $[\mathfrak{m}]^2$, $\mathfrak{m}^2$, $\mathrm{fin}(\mathfrak{m})$, and $\mathrm{seq}^{\text{1-1}}(\mathfrak{m})$, a question which arises naturally is whether for some infinite cardinal $\mathfrak{m}$, $\mathrm{fin}(\mathfrak{m}) < \mathfrak{m}^2$ is consistent with ZF (see [5, Related Result 20, p. 133]). Moreover, assuming that $\mathfrak{m}$ is infinite and the four cardinalities $[\mathfrak{m}]^2$, $\mathfrak{m}^2$, $\mathrm{fin}(\mathfrak{m})$, $\mathrm{seq}^{\text{1-1}}(\mathfrak{m})$ are pairwise distinct and pairwise comparable in ZF, one may ask which linear orderings on these four cardinalities are consistent with ZF.

Since for cardinals $\mathfrak{m} \geq 5$, we cannot have $[\mathfrak{m}]^2 > \mathrm{fin}(\mathfrak{m})$ or $\mathfrak{m}^2 > \mathrm{seq}^{\text{1-1}}(\mathfrak{m})$, there are only the following six linear orderings on these four cardinalities which might be consistent with ZF (where for two cardinals $\mathfrak{n}_1$ and $\mathfrak{n}_2$, $\mathfrak{n}_1 \longrightarrow \mathfrak{n}_2$ means $\mathfrak{n}_1 < \mathfrak{n}_2$).

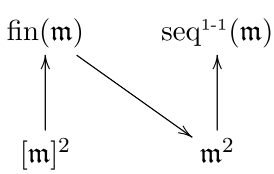


Diagram **N**

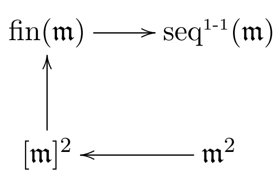


Diagram **C**

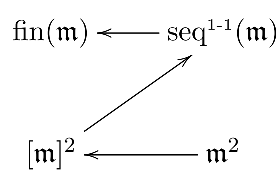


Diagram **Z**

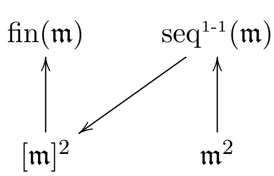


Diagram **И**

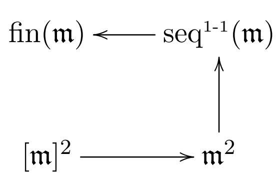


Diagram **Ↄ**

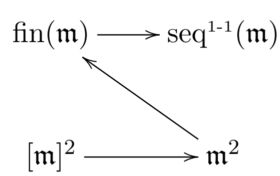


Diagram **Σ**

Below we show that each of the five diagrams **N**, **Z**, **И**, **Ↄ**, **Σ** is consistent with ZF.

2 Permutation Models

In order to show, for example, that for some infinite cardinals $\mathfrak{m}$ and $\mathfrak{n}$, $\mathfrak{m} < \mathfrak{n}$ is consistent with ZF, by the JECH-SOCHOR EMBEDDING THEOREM (see, for example, [6, Thm. 6.1] or [5, Thm. 17.2]), it is enough to construct a permutation model in which this statement holds. The underlying idea of permutation models, which will be models of set theory with atoms (ZFA), is the fact that a model $\boldsymbol{\mathcal{V}} \models \mathsf{ZFA}$ does not distinguish between the atoms, where atoms are objects which do not have any elements but which are distinct from the empty set. The theory ZFA is essentially the same as that of ZF (except for the definition of ordinals, where we have to require that an ordinal does not have atoms among its elements). Let A be a set. Then by transfinite recursion on the ordinals $\alpha \in \Omega$ we can define the α-power $\mathscr{P}^{\alpha}(A)$ of A and $\mathscr{P}^{\infty}(A) := \bigcup_{\alpha\in\Omega} \mathscr{P}^{\alpha}(A)$. Like for the cumulative hierarchy of sets in ZF, one can show that if $\boldsymbol{\mathcal{M}}$ is a model of ZFA and A is the set of atoms of $\boldsymbol{\mathcal{M}}$, then $\boldsymbol{\mathcal{M}} := \mathscr{P}^{\infty}(A)$. The class $M_0 := \mathscr{P}^{\infty}(\emptyset)$ is a model of ZF and is called the **kernel**. Notice that all ordinals belong to the kernel. By construction we obtain that every permutation of the set of atoms induces an automorphism of $\boldsymbol{\mathcal{V}}$, where the sets in the kernel are fixed.

Permutation models were first introduced by Adolf Fraenkel and, in a precise version (with supports), by Andrzej Mostowski. The version with filters, which we will follow below, is due to Ernst Specker (a detailed introduction to permutation models can be found, for example, in [5, Ch. 8] or [6]).

In order to construct a permutation model, we usually start with a set of atoms A and then define a group G of permutations or automorphisms of A.

The permutation models we construct below are of the following simple type: For each finite set $E \in \operatorname{fin}(A)$, let

$$\operatorname{Fix}_G(E) := \big\{\pi \in G : \forall a \in E\, (\pi a = a)\big\}\,,$$

and let $\mathscr{F}$ be the filter of subgroups on G generated by the subgroups $\{\operatorname{Fix}_G(E) : E \in \operatorname{fin}(A)\}$. In other words, $\mathscr{F}$ is the set of all subgroups $H \leq G$, such that the there exists a finite set $E \in \operatorname{fin}(A)$, such that $\operatorname{Fix}_G(E) \leq H$.

For a set x, let

$$\operatorname{sym}_G(x) := \{\pi \in G : \pi x \;=\; x\}$$

where

$$\pi x = \begin{cases} \emptyset & \text{if } x = \emptyset, \\ \pi a & \text{if } x = a \text{ for some } a \in A, \\ \big\{\pi y : y \in x\big\} & \text{otherwise.} \end{cases}$$

Then, a set x is *symmetric* if and only if there exists a set of atoms $E_x \in \operatorname{fin}(A)$, such that

$$\operatorname{Fix}_G(E_x) \leq \operatorname{sym}_G(x).$$

We say that E_x is a *support* of x. Finally, let $\boldsymbol{\mathcal{V}}$ be the class of all hereditarily symmetric objects; then $\boldsymbol{\mathcal{V}}$ is a transitive model of ZFA. We call $\boldsymbol{\mathcal{V}}$ a *permutation model*. So, a set x belongs to the permutation model $\boldsymbol{\mathcal{V}}$ (with respect to G and $\mathscr{F}$), if and only if x has a finite support $E_x \in \operatorname{fin}(A)$. Because every $a \in A$ is symmetric, we get that each atom $a \in A$ belongs to $\boldsymbol{\mathcal{V}}$.

2.1 A Model for Diagram N

We first show that in every model for Diagram **N**, we have that the cardinality $\mathfrak{m}$ is transfinite.

LEMMA 1. *If* $\operatorname{fin}(\mathfrak{m}) \leq \mathfrak{m}^2$ *for some* $\mathfrak{m} \geq 5$, *then* $\aleph_0 \leq \mathfrak{m}$.

Proof. Let A be a set of cardinality $\mathfrak{m} \geq 5$ and assume that $h : \operatorname{fin}(A) \to A^2$ is an injection. First we choose a 5-sequence $S_5 := \langle a_1, \ldots, a_5 \rangle$ of pairwise distinct elements of A. The ordering of S_5 induces an ordering on $P_5 := \operatorname{fin}(\{a_1, \ldots, a_5\})$, and since $|h[P_5]| = 2^5$ and $2^5 > 5^2$, there exists a first set $u \in P_5$ such that for $\langle x, y \rangle = h(u)$, the set $D_6 := \{x, y\} \setminus \{a_1, \ldots, a_5\}$ is non-empty. If $x \in D_6$, let $a_6 := x$, otherwise, let $a_6 := y$. Now, let $S_6 := \langle a_1, \ldots, a_6 \rangle$ and $P_6 := \operatorname{fin}(\{a_1, \ldots, a_6\})$. As above, we find a $u \in P_6$ such that for $\langle x, y \rangle = h(u)$, the set $D_7 := \{x, y\} \setminus \{a_1, \ldots, a_7\}$ is non-empty. If $x \in D_7$, let $a_7 := x$, otherwise, let $a_7 := y$. Proceeding this way, we finally have an injection from ω into A, which shows that $\aleph_0 \leq \mathfrak{m}$. ⊣

PROPOSITION 2. *If* $\aleph_0 \leq \mathfrak{m}$ *for some infinite cardinal* $\mathfrak{m} = |A|$, *then there exists a finite-to-one function* $g : \operatorname{seq}(A) \to \operatorname{fin}(A)$.

Proof. By LEMMA 1, there exists an injection $h : \omega \to A$, and for each $i \in \omega$, let $x_i := h(i)$, let $B = \{x_i : i \in \omega\}$, and let $C := \{x_{2i} : i \in \omega\}$. Notice that

$$\iota(a) = \begin{cases} a & \text{if } a \in A \setminus B, \\ x_{2i+1} & \text{if } a = x_i, \end{cases}$$

is a bijection between A and $A \setminus C$. Thus, it is enough to construct a finite-to-one function $g : \operatorname{seq}(A \setminus C) \to \operatorname{fin}(A)$. Let $s = \langle a_0, \ldots, a_{n-1} \rangle \in \operatorname{seq}(A \setminus C)$ and let $\operatorname{ran}(s) := \{a_0, \ldots, a_{n-1}\}$. The sequence s gives us in a natural way an enumeration of $\operatorname{ran}(s)$, and with respect to this enumeration we can encode the sequence s by a natural number $i_s \in \omega$. Now, let $g(s) := \operatorname{ran}(s) \cup \{x_{2i_s}\}$. Then, since there are just finitely many enumerations of $\operatorname{ran}(s)$, g is a finite-to-one function. ⊣

The following result is just a consequence of PROPOSITION 2 and LEMMA 1.

COROLLARY 3. *If* $\operatorname{fin}(\mathfrak{m}) \leq \mathfrak{m}^2$ *for some* $\mathfrak{m} = |A| \geq 5$, *then there exists a finite-to-one function* $g : \operatorname{seq}(A) \to \operatorname{fin}(A)$.

We now introduce the technique we intend to use in order to build a permutation model from which it will follow that for some infinite cardinal $\mathfrak{m}$, the relation $\operatorname{fin}(\mathfrak{m}) < \mathfrak{m}^2$ is consistent with ZF. Notice that this relation is the main feature of Diagram **N** and that this relation implies that $\aleph_0 \leq \mathfrak{m}$. In the next section, we shall use a similar permutation model in order to show the consistency of Diagram **Z** with ZF.

Let K be the class of all the pairs (A, h) such that A is a (possibly empty) set and h is an injection $h\colon \operatorname{fin}(A) \to A^2$. We will also refer to the elements of K as models. We define a partial ordering $\leq$ on K by stipulating

$$(A, h) \leq (B, f) \iff A \subseteq B \;\wedge\; h \subseteq f \;\wedge\; \operatorname{ran}\left(f|_{\operatorname{fin}(B)\setminus\operatorname{fin}(A)}\right) \subseteq (B \setminus A)^2 .$$

When the functions involved are clear from the context, with a slight abuse of notation we will just write $A \leq B$ instead of $(A, h) \leq (B, f)$ and $A \in K$ instead of $(A, h) \in K$. Notice that the last condition in the definition of $(A, h) \leq (B, f)$ implies that $(B \setminus A) \in K$.

PROPOSITION 4 (CH). *There is a model M_* of cardinality $\mathfrak{c}$ in K such that:*

- *M_* is $\aleph_1$-universal, i.e., if $N \in K$ is countable then N is isomorphic to some $N_* \leq M_*$.*
- *M_* is $\aleph_1$-homogeneous, i.e., if $N_1, N_2 \leq M_*$ are countable and $\pi\colon N_1 \to N_2$ is an isomorphism then there exists an automorphism π_* of M_* such that $\pi \subseteq \pi_*$.*
- *If $N \leq M_*$ and $A \subseteq M_*$ are countable, then there is an automorphism π of M_* that fixes N pointwise, such that $\pi(A) \setminus N$ is disjoint from A.*

Proof. We construct the model M_* by induction on ω_1, where we assume that $\omega_1 = \mathfrak{c}$. Let $M_0 = \emptyset$. When M_α is already defined for some $\alpha \in \omega_1$, we can define

$$C_\alpha := \big\{N \leq M_\alpha : N \in K \text{ and } N \text{ is countable}\big\}.$$

The construction of $M_{\alpha+1}$, starting from M_α, consists of a disjoint union of two differently built sets of models. First, for each element $N \in C_\alpha$, let S_N be a system of representatives for the *strong* isomorphism classes of all the models $M \in K$ such that $N \leq M$, M is countable, and for all $M_1, M_2 \in S_N$ we have $M_1 \cap M_2 = N$. Here, by *strong* we mean that, for two models M_1 and M_2 with $N \leq M_1, M_2$, it is not enough to be isomorphic in order to belong to the same class, but we require that there exists an isomorphism between M_1 and M_2 that fixes N pointwise, which we can express by saying that M_1 *is isomorphic to* M_2 *over* N. We first extend M_α by the set

$$M'_\alpha = \bigsqcup_{N \in C_\alpha} \bigsqcup_{M \in S_N} M \setminus N,$$

where "$\bigsqcup$" indicates that we have a *disjoint union*. Now, we extend M_α by a second set M''_α, where M''_α is defined as follows: for each pair $\{M_1, M_2\} \in [C_\alpha]^2$, we fix a way of constructing a common extension $M_1 * M_2$ with $M_1, M_2 \leq M_1 * M_2$. Let f^{M_1} and f^{M_2} be the functions coming with the models M_1, M_2, let $N_0 = M_1 \cup M_2$, let $f_0 = f^{M_1} \cup f^{M_2}$, and, for each finite set E in $\mathscr{E}_0 := \operatorname{fin}(N_0) \setminus \operatorname{dom}(f_0)$, choose a pair $p_E = \langle a_1, a_2\rangle$ such that $\{a_1, a_2\} \cap N_0 = \emptyset$ and for all $E, E' \in \mathscr{E}_0$, either $E = E'$ or $\operatorname{ran}(p_E) \cap \operatorname{ran}(p_{E'}) = \emptyset$, where for $p_E = \langle a_1, a_2\rangle$, $\operatorname{ran}(p_E) := \{a_1, a_2\}$. Now, by induction on ω, define

$$N_{i+1} = N_i \sqcup \bigsqcup_{E \in \mathscr{E}_i} \operatorname{ran}(p_E),$$

together with

$$f_{i+1} = f_i \cup \bigcup_{E \in \mathscr{E}_i} \langle E, p_E\rangle \qquad \text{and} \qquad \mathscr{E}_{i+1} = \operatorname{fin}(N'_{i+1}) \setminus \operatorname{dom}(f_{i+1}),$$

and set $M_1 * M_2 := \bigcup_{i \in \omega} N_i$ with $f^{M_1 * M_2} := \bigcup_{i \in \omega} f_i$. Now, let

$$M''_\alpha := \bigsqcup_{\{M_1, M_2\} \in [C_\alpha]^2} M_1 * M_2 \setminus (M_1 \cup M_2).$$

Finally, let $M_{\alpha+1} = M_\alpha \sqcup M'_\alpha \sqcup M''_\alpha$, for non-empty limit ordinals δ define $M_\delta = \cup_{\alpha\in\delta} M_\alpha$, and let

$$M_* = \bigcup_{\alpha\in\omega_1} M_\alpha.$$

It remains to show that the model M_* has the required properties: First we notice that M_* has cardinality $|M_*| = \mathfrak{c}$, as required, and since, by construction, M_1 is $\aleph_1$-universal, M_* is also $\aleph_1$-universal. In order to show that M_* is $\aleph_1$-homogeneous, let $N_1, N_2 \leq M_*$ be countable models and $\pi\colon N_1 \to N_2$ an isomorphism. Let $\{x_\alpha : \alpha \in \omega_1\}$ be an enumeration of the elements of M_* and let $I_0 := N_1$. If x_{δ_1} is the first element (w.r.t. this enumeration) in $M_* \setminus I_0$, then, by construction, there exists a model $I_1 \leq M_*$ such that $I_0 \leq I_1$ and $x_{\delta_1} \in I_1$. Again by construction, there is a model J_1 with $N_2 \leq J_1 \leq M_*$ such that there exists an isomorphism $\pi_1\colon I_1 \to J_1$ with $\pi \subseteq \pi_1$. In fact, we have that J_1 and I_1 are isomorphic over N_2. Proceed inductively with $x_{\delta_{\alpha+1}}$ being the first element in $M_* \setminus I_\alpha$, we find models $I_\alpha \leq I_{\alpha+1} \leq M_*$, $J_\alpha \leq J_{\alpha+1} \leq M_*$ and isomorphisms $\pi_{\alpha+1}\colon I_{\alpha+1} \to J_{\alpha+1}$, and finally we obtain $\pi_* = \cup_{\alpha\in\omega_1}\pi_\alpha$, which is the required automorphism.

To show the last property of the theorem, let $N \leq M_*$ and $A \subseteq M_*$ be both countable. Since the cofinality of ω_1 is greater than ω, we can find by construction both a countable model M satisfying the properties $A \subseteq M$, $N \leq M \leq M_*$ and a further model M' with $N \leq M' \leq M_*$ such that $M' \cap (A \setminus N) = \emptyset$, and such that there exists an isomorphism $i\colon M \to M'$ with i fixing N pointwise. Now, by M_* being $\aleph_1$-homogeneous we obtain an automorphism i_* extending i, as required. ⊣

As anticipated, the construction of the previous theorem does not exploit any particular property of the functions $h\colon \operatorname{fin}(A) \to A^2$. In fact, the construction is an analogue of a Fraïssé limit as it relies on similar properties, like, for example, a modified version of the *Disjoint Amalgamation Property* (DAP) of K, where we require that embeddings between structures $f\colon (A,h) \to (B,g)$ are allowed only when, according to our previous definition, $A \leq B$. Indeed, exactly the same construction can be carried out in the alternative framework of models (A,f,g,h), where A is a set and we have three injections $f\colon A^2 \to [A]^2$, $g\colon [A]^2 \to \operatorname{seq}^{1\text{-}1}(A)$ and $h\colon \operatorname{seq}^{1\text{-}1}(A) \to \operatorname{fin}(A)$, which will be used below to show the consistency of Diagram **Z** with ZF.

Given Proposition 4, we consider the permutation model $\mathcal{V}_{\mathsf{N}}$ that arises naturally by considering the elements of the $\aleph_1$-universal and $\aleph_1$-homogeneous model M_* as the set of atoms and its automorphisms $\operatorname{Aut}(M_*)$ as the group G of permutations. In particular, each permutation of M_* preserves the injection $h\colon \operatorname{fin}(M_*) \to M_*^2$ that the model (M_*, h) comes with.

We are now ready to prove the following result.

Theorem 5. *Let M_* be the set of atoms of $\mathcal{V}_{\mathsf{N}}$ and let $\mathfrak{m} = |M_*|$. Then*

$$\mathcal{V}_{\mathsf{N}} \models [\mathfrak{m}]^2 < \operatorname{fin}(\mathfrak{m}) < \mathfrak{m}^2 < \operatorname{seq}^{1\text{-}1}(\mathfrak{m}).$$

Proof. The existence of an injection $h\colon \operatorname{fin}(M_*) \to M_*^2$ in $\mathcal{V}_{\mathsf{N}}$ follows directly from the definition of the specific permutation model. So, we only need to prove that in $\mathcal{V}_{\mathsf{N}}$, there is no reverse injection from M_*^2 into $\operatorname{fin}(M_*)$, and that there are no injections from $\operatorname{fin}(M_*)$ into $[M_*]^2$ or from $\operatorname{seq}^{1\text{-}1}(M_*)$ into M_*^2.

In order to show that there is neither an injection from M_*^2 into $\mathrm{fin}(M_*)$, nor an injection from $\mathrm{seq}^{1\text{-}1}(M_*)$ into M_*^2, assume towards a contradiction that $\mathcal{V}_\mathsf{N}$ contains an injection $f_1\colon M_*^2 \to \mathrm{fin}(M_*)$ or an injection $f_2\colon \mathrm{seq}^{1\text{-}1}(M_*) \to M_*^2$. Let S be a finite support of both functions f_1 and f_2 (if they exists). In other words, $S \in \mathrm{fin}(M_*)$ and for each automorphism $\pi \in \mathrm{Fix}_G(S)$ we have $\pi(f_1) = f_1$ and $\pi(f_2) = f_2$, respectively. Let N_1 be a countable model in K with $S \subseteq N_1 \leq M_*$. Let (N_2, g) be a countable model in K such that $(N_1, h|_{N_1}) \leq (N_2, g)$, constructed as follows: The domain of N_2 is the disjoint union

$$N_2 = N_1 \sqcup \{x, y, z\} \sqcup \{a_i : i \in \omega\}\,.$$

Furthermore, we define the injection $g\colon \mathrm{fin}(N_2) \to N_2^2$ such that $g \supseteq h|_{N_1}$ and for $E \in \mathrm{fin}(N_2) \setminus \mathrm{fin}(N_1)$ we define $g(E) = \langle e_1, e_2\rangle$ such that g is injective and satisfies the following conditions (recall the since N_2 is countable, also $\mathrm{fin}(N_2)$ is countable):

- If $E \cap \{x, y, z\} = \emptyset$ then $\langle e_1, e_2\rangle = \langle a_n, a_m\rangle$ for some $n, m \in \omega$.
- If $|E \cap \{x, y, z\}| = 1$, then $\langle e_1, e_2\rangle = \langle u, a_k\rangle$ for some $k \in \omega$, where u is the unique element in $E \cap \{x, y, z\}$.
- If $|E\cap\{x, y, z\}| = 2$, then $\langle e_1, e_2\rangle = \langle v, a_k\rangle$ for some $k \in \omega$, where v is the unique element in $\{x, y, z\} \setminus (E \cap \{x, y, z\})$.
- If $|E \cap \{x, y, z\}| = 3$ then $\langle e_1, e_2\rangle = \langle a_n, a_m\rangle$ for some $n, m \in \omega$.

Notice that there are automorphisms of (N_2, g) that just permute x, y, z and fix all other elements of N_2 pointwise. By construction of M_*, we find a model $N_2' \in K$ such that $N_1 \leq N_2' \leq M_*$ and N_2' is isomorphic to N_2 over N_1. For this reason we can refer to N_2 as a legit submodel of M_* that extends N_1 in the way we described. Let us now consider $f_1(\langle x, y\rangle)$, where we assumed in $\mathcal{V}_\mathsf{N}$ the existence of an injection $f_1\colon M_*^2 \to \mathrm{fin}(M_*)$ with finite support S. If $f_1(\langle x, y\rangle) \nsubseteq N_2 \setminus N_1$, then we can apply the third property of Proposition 4 with respect to $f_1(\langle x, y\rangle)$ and N_1 and N_2, which gives us a contradiction. If $\{x, y\} \subseteq f_1(\langle x, y\rangle)$ or $\{x, y\} \cap f_1(\langle x, y\rangle) = \emptyset$, we could swap x and y while fixing every other element of N_2 pointwise and get $f(\langle x, y\rangle) = f_1(\langle y, x\rangle)$, which would imply that f_1 is not injective. So, assume that $|\{x, y\} \cap f_1(\langle x, y\rangle)| = 1$ and without loss of generality assume that $\{x, y\} \cap f_1(\langle x, y\rangle) = \{x\}$. Now, if $z \in f_1(\langle x, y\rangle)$, *i.e.*, $\{x, z\} \subseteq f_1(\langle x, y\rangle)$, we similarly obtain a contradiction by swapping z and x, while if $z \notin f_1(\langle x, y\rangle)$ we get a contradiction by swapping z and y. This shows that f_1 cannot belong to $\mathcal{V}_\mathsf{N}$.

For what concerns f_2, let us consider the set $\mathscr{S}$ consisting of sequences without repetition of $\{x, y, z\}$ of length 2 or 3. Notice that $|\mathscr{S}| = 12$. Now, for each element $s \in \mathscr{S}$, if $f_2(s) = \langle a, b\rangle$, then a and b are such that $a \neq b$ and $\langle a, b\rangle \in \{x, y, z\}^2$ — notice that otherwise, for example, if $\{a, b\} \cap \{x, y\} = \emptyset$, then we can swap x and y and hence move s without moving $\langle a, b\rangle$, which is not consistent with S being a support of f_2. We get the conclusion by noticing that, because of this restriction, there are only six possible images of elements of E, which implies that f_2 cannot be an injection.

It remains to show that in $\mathcal{V}_\mathsf{N}$ there are no injections from $\mathrm{fin}(M_*)$ into $[M_*]^2$. For this, assume towards a contradiction that there exists such a function f_3 in $\mathcal{V}_\mathsf{N}$ and assume that S is a finite support of f_3. Then, let N_1 be a countable model in K with $S \subseteq N_1 \leq M_*$. We will construct a countable model $(N_2, g) \in K$ satisfying $(N_1, h|_{N_1}) \leq (N_2, g) \leq M_*$ with a finite subset $u \in \mathrm{fin}(N_2 \setminus N_1)$ such that, for all $\langle x, y\rangle \in (N_2 \setminus N_1)^2$, one of the following holds:

- there is no finite set $E \in \mathrm{fin}(N_2)$ with $h(E) = \langle x, y\rangle$;
- there exists an automorphism π of N_2 over N_1 with $\pi(u) \neq u$ and $\pi\{x, y\} = \{x, y\}$.

Let $u = \{a_0, b_0, c_0\}$ be disjoint from N_1 and define $G_0^1 = N_1 \sqcup \{a_0, b_0, c_0\}$. Now, for each finite set $E \in \mathrm{fin}(G_0^1)$ which is not in the domain of $h_0 = h|_{N_1}$, that is, for each finite set $E \in \mathrm{fin}(G_0^1)$ with $E \cap \{a_0, b_0, c_0\} \neq \emptyset$, let $\{x_E, y_E\}$ be a pair of new elements and define

$$G_0^* := G_0^1 \sqcup \bigsqcup_{E\in\mathrm{fin}(G_0^1)\setminus\mathrm{dom}(h_0)} \{x_E, y_E\} \quad \text{and} \quad h_0^1 := h_0 \sqcup \bigsqcup_{E\in\mathrm{fin}(G_0^1)\setminus\mathrm{dom}(h_0)} \{\langle E, \langle x_E, y_E\rangle\rangle\}.$$

Let now G_0^2 be an extension of G_0^* by adding a copy of $G_0^1 \setminus N_1$, where the "copy function" is denoted τ_0. Notice that at this stage, $G_0^1 \setminus N_1 = \{a_0, b_0, c_0\}$. More formally, $G_0^2 = G_0^* \sqcup \{\tau_0(a) : a \in G_0^1 \setminus N_1\}$, together with an extension of h_0^1 defined as

$$h_0^2 := h_0^1 \sqcup \bigsqcup_{E\in\mathrm{fin}(G_0^1)\setminus\mathrm{dom}(h_0)} \{\langle \tau_0(E), \langle y_E, x_E\rangle\rangle\},$$

where, given $E \in \mathrm{fin}(G_0^1) \setminus \mathrm{dom}(h_0)$, $\tau_0(E)$ is defined as

$$\tau_0(E) := (E \cap N_1) \sqcup \{\tau_0(a) : a \in E \setminus N_1\}.$$

Notice that if $a \in G_0^1 \setminus N_1$, then $\tau_0(a) \in G_0^2 \setminus G_0^*$. The construction carried out so far is actually the first of countably many analogous extension steps we will consequently apply in order to consider the union of all the progressive extensions. That is, assume that for some $i \in \omega$ we have already defined G_i^2 and h_i^2. Define $G_{i+1}^1 = G_i^2$ and, for each finite set $E \in \mathrm{fin}(G_{i+1}^1)$ which is not in the domain of h_i^2, consider a pair of new elements $\{x_E, y_E\}$ and define

$$G_{i+1}^* := G_{i+1}^1 \sqcup \bigsqcup_{E\in\mathrm{fin}(G_{i+1}^1)\setminus\mathrm{dom}(h_i^2)} \{x_E, y_E\} \quad \text{and} \quad h_{i+1}^1 := h_i^2 \sqcup \bigsqcup_{E\in\mathrm{fin}(G_{i+1}^1)\setminus\mathrm{dom}(h_i^2)} \{\langle E, \langle x_E, y_E\rangle\rangle\}.$$

Let now G_{i+1}^2 be an extension of G_{i+1}^* by adding a copy of $G_{i+1}^1 \setminus N_1$, where the "copy function" is now τ_{i+1}. More formally, $G_{i+1}^2 := G_{i+1}^* \sqcup \{a_{i+1} : a \in G_{i+1}^1 \setminus N_1\}$, together with an extension of h_{i+1}^1 defined as

$$h_{i+1}^2 := h_{i+1}^1 \sqcup \bigsqcup_{E\in\mathrm{fin}(G_{i+1}^1)\setminus\mathrm{dom}(h_i^2)} \{\langle \tau_{i+1}(E), \langle y_E, x_E\rangle\rangle\},$$

where, again, given $E \in \mathrm{fin}(G_{i+1}^1) \setminus \mathrm{dom}(h_i^2)$, $\tau_{i+1}(E)$ is defined as $\tau_{i+1}(E) := (E \cap N_1) \sqcup \{\tau_{i+1}(a) : a \in E \setminus N_1\}$, for which we newly remark that if $a \in G_{i+1}^1 \setminus N_1$, then $\tau_{i+1}(a) \in G_{i+1}^2 \setminus G_{i+1}^*$. Notice that every automorphism of G_{i+1}^1 which moves a finite set E to E', moves the pair $\{x_E, y_E\}$ to $\{x_{E'}, y_{E'}\}$, and consequently, it moves $\langle E, \langle x_E, y_E\rangle\rangle$ to $\langle E', \langle x_{E'}, y_{E'}\rangle\rangle$. In particular, every automorphism of (G_{i+1}^1, h_i^2) can be extended to an automorphism of (G_{i+1}^*, h_{i+1}^1). Moreover, every automorphism of (G_{i+1}^*, h_{i+1}^1) can be extended to an automorphism of (G_{i+1}^2, h_{i+1}^2).

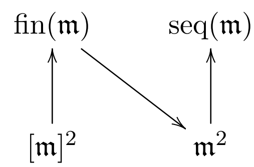
Now, let

$$N_2 := \bigcup_{i\in\omega} G_i^1 \qquad \text{and} \qquad g := \bigcup_{i\in\omega} h_i^1 \,.$$

We claim that (N_2, g) satisfies the required properties. Indeed, if $\langle x,y\rangle \in (N_2\backslash N_1)^2$ and there is some finite set $E \in \mathrm{fin}(N_2)$ with $g(E) = h(E) = \langle x, y\rangle$, then by construction there exists some index $n \in \omega$ such that $\langle x,y\rangle \in (G_n^*\backslash G_n^1)^2$, which implies that there exists an automorphism π of N_2 over N_1 acting as follows: $\pi\langle x,y\rangle = \langle y,x\rangle$ and $\pi u = \pi\{a_0, b_0, c_0\} = \big\{\tau_n(a_0), \tau_n(b_0), \tau_n(c_0)\big\}$, which in particular means $\pi\{x,y\} = \{x,y\}$ and $\pi u \neq u$, as desired. We can finally consider the image $f_3(u) = \{x,y\}$: If $\{x,y\} \nsubseteq N_2 \setminus N_1$, then we can apply the third property of Proposition 4 with respect to $\{x,y\}$ and N_1 and N_2, which gives us a contradiction. Thus $\{x,y\} \subseteq N_2 \setminus N_1$, and if there exists some finite set $E \in \mathrm{fin}(N_2)$ with $g(E) = h(E) = \langle x,y\rangle$, then by the reasoning above we find that some automorphism of N_2 over N_1 does not preserve f_3, a contradiction. In every other case, we consider $\mathrm{cl}(N_1 \cup \{x,y\}, M_*)$ and notice that, since for no $E \in \mathrm{fin}(N_2)$ we have $h(E) = \langle x,y\rangle$ or $h(E) = \langle y,x\rangle$, then u cannot be a subset of $\mathrm{cl}(N_1 \cup \{x,y\}, M_*)$, which allows us to fix $\mathrm{cl}(N_1 \cup \{x,y\}, M_*)$ pointwise, while not preserving u, a contradiction as well. ⊣

So, the model $\mathcal{V}_{\mathbf{N}}$ witnesses the following

Consistency Result 1. *The existence of an infinite cardinal* $\mathfrak{m}$ *satisfying*



is consistent with ZF.

2.2 A Model for Diagram Z

We are now going to set an analogue framework to the one for Diagram **N**, just with the definitions adapted, in order to show the consistency of Diagram **Z**. In fact, as mentioned above, we can state the same proposition, guaranteeing the existence of a suitable $\aleph_1$-universal and $\aleph_1$-homogeneous model.

Let K be the class of all the pairs (A, f, g, h) such that A is a (possibly empty) set and f, g, h are the following three injections:

$$f\colon A^2 \to [A]^2 \qquad g\colon [A]^2 \to \mathrm{seq}^{1-1}(A) \qquad h\colon \mathrm{seq}^{1-1}(A) \to \mathrm{fin}(A)$$

As before, we define a partial ordering $\leq$ on K by stipulating $(A, f_1, g_1, h_1) \leq (B, f_2, g_2, h_2)$ if and only if

- $A \subseteq B$,
- $f_1 \subseteq f_2$, $\mathrm{ran}\,\big(f_2|_{B^2\setminus A^2}\big) \subseteq [B \setminus A]^2$,

- $g_1 \subseteq g_2$, $\operatorname{ran}\left(g_2|_{[B]^2\setminus[A]^2}\right) \subseteq \operatorname{seq}^{1\text{-}1}(B \setminus A)$,
- $h_1 \subseteq h_2$, $\operatorname{ran}\left(h_2|_{\operatorname{seq}^{1\text{-}1}(B)\setminus\operatorname{seq}^{1\text{-}1}(A)}\right) \subseteq \operatorname{fin}(B \setminus A)$.

PROPOSITION 6 (CH). *There is a model M_* of cardinality $\mathfrak{c}$ in K such that:*

- *M_* is $\aleph_1$-universal, i.e., if $N \in K$ is countable then N is isomorphic to some $N_* \leq M_*$.*
- *M_* is $\aleph_1$-homogeneous, i.e., if $N_1, N_2 \leq M_*$ are countable and $\pi\colon N_1 \to N_2$ is an isomorphism then there exists an automorphism π_* of M_* such that $\pi \subseteq \pi_*$.*
- *If $N \leq M_*$ and $A \subseteq M_*$ are countable, then there is an automorphism π of M_* over N such that $\pi(A) \setminus N$ is disjoint from A.*

Proof. The proof is essentially the same as the one of PROPOSITION 4. ⊣

We define $\mathcal{V}_{\mathsf{Z}}$ as the permutation model obtained by setting the elements of the $\aleph_1$-universal and $\aleph_1$-homogeneous model M_* as the set of atoms and its automorphisms $\operatorname{Aut}(M_*)$ as the group G of permutations. In particular, each permutation of M_* preserves the injections f, g, h that the model (M_*, f, g, h) comes with.

THEOREM 7. *Let M_* be the set of atoms of $\mathcal{V}_{\mathsf{Z}}$ and let $\mathfrak{m} = |M_*|$. Then*

$$\mathcal{V}_{\mathsf{Z}} \models \mathfrak{m}^2 < [\mathfrak{m}]^2 < \operatorname{seq}^{1\text{-}1}(\mathfrak{m}) < \operatorname{fin}(\mathfrak{m})\,.$$

Proof. The existence of the required injections is clear by the definition of the model. Thus, it remains to prove that there are no reverse injections. First, we give two preliminary definitions. Given a model (M, f, g, h) and a countable subset $A \subseteq M$, we define the *closure* $\operatorname{cl}(A, M)$ as the smallest superset of A that is closed under f, g, h and pre-images with respect to the same functions. Constructively, we can characterize $\operatorname{cl}(A, M)$ as a countable union as follows: Define $\operatorname{cl}_0 = \operatorname{cl}_0(A, M) := A$ and, for all $i \in \omega$,

$$\begin{aligned}\operatorname{cl}_{i+1} = \ \operatorname{cl}_i \sqcup \bigsqcup_{p \in (\operatorname{cl}_i)^2} f(p) \sqcup \bigsqcup_{q \in [\operatorname{cl}_i]^2} \operatorname{ran}(g(q)) \sqcup \bigsqcup_{s \in \operatorname{seq}^{1\text{-}1}(\operatorname{cl}_i)} h(s) \\ \sqcup \bigsqcup_{q \in [\operatorname{cl}_i]^2} \operatorname{ran}(f^{-1}(q)) \sqcup \bigsqcup_{s \in \operatorname{seq}^{1-1}(\operatorname{cl}_i)} g^{-1}(s) \sqcup \bigsqcup_{r \in \operatorname{fin}(\operatorname{cl}_i)} \operatorname{ran}(h^{-1}(r))\,,\end{aligned}$$

in order to finally define $\operatorname{cl}(A, M) = \cup_{i \in \omega} \operatorname{cl}_i$. Furthermore, we set a standardized way to extend a *partial* model (A, f, g, h), where f, g, h are only partial functions, to an element of K: Consider (A, f', g', h'), where A is a countable set and f, g, h are injections with

$$\begin{aligned}&\operatorname{dom}(f) \subseteq A^2, \quad &&\operatorname{dom}(g) \subseteq [A]^2, \quad &&\operatorname{dom}(h) \subseteq \operatorname{seq}^{1-1}(A)\\ &\operatorname{ran}(f) \subseteq [A]^2, \quad &&\operatorname{ran}(g) \subseteq \operatorname{seq}^{1-1}(A), \quad &&\operatorname{ran}(h) \subseteq \operatorname{fin}(A).\end{aligned}$$

Let $(M_0, f'_0, g'_0, h'_0) = (A, f', g', h')$ and, for $j \in \omega$, define inductively $(M_{j+1}, f'_{j+1}, g'_{j+1}, h'_{j+1})$ as follows: M_{j+1} is the fully disjoint union

$$M_j \quad \sqcup \bigsqcup_{P \in M_j^2 \setminus \mathrm{dom}(f_j)} \{a_P, b_P\} \quad \sqcup \bigsqcup_{Q \in [M_j]^2 \setminus \mathrm{dom}(g_j)} \{a_Q, b_Q, c_Q\} \quad \sqcup \bigsqcup_{R \in \mathrm{seq}^{\text{1-1}}(M_j) \setminus \mathrm{dom}(h_j)} \{a_R, b_R, c_R\}.$$

For what concerns the injections $f'_{j+1}, g'_{j+1}, h'_{j+1}$, we naturally require the inclusions $f'_j \subseteq f'_{j+1}$, $g'_j \subseteq g'_{j+1}$, and $h'_j \subseteq h'_{j+1}$, as well as the equalities $\mathrm{dom}(f'_{j+1}) = M_j^2$, $\mathrm{dom}(g'_{j+1}) = [M_j]^2$, and $\mathrm{dom}(h'_{j+1}) = \mathrm{seq}^{1-1}(M_j)$, respectively, where for $P \in M_j^2 \setminus \mathrm{dom}(f_j)$, $Q \in [M_j]^2 \setminus \mathrm{dom}(g_j)$, and $R \in \mathrm{seq}^{1-1}(M_j) \setminus \mathrm{dom}(h_j)$, we define

$$f'_{j+1}(P) := \{a_P, b_P\}, \qquad g'_{j+1}(Q) := \langle a_Q, b_Q, c_Q \rangle, \qquad h'_{j+1}(R) := \{a_R, b_R, c_R\}.$$

We are now in the position of defining the plain extension of (A, f', g', h') as

$$(M, f, g, h) = \left(\bigcup_{j \in \omega} M_j,\ \bigcup_{j \in \omega} f'_j,\ \bigcup_{j \in \omega} g'_j,\ \bigcup_{j \in \omega} h'_j \right),$$

and we can finally prove, in three analogous steps, that neither of the three injections of the model (M_*, f, g, h) admits a reverse injection.

Assume there is an injection $i \colon [A]^2 \to A^2$ with finite support S, where A is the set of atoms. Let $N_1 \in K$ be a countable model such that $N_1 \leq M_*$ and $S \subseteq N_1$. Let $\{x, y\} \in [A]^2$ with $N_1 \cap \{x, y\} = \emptyset$, let $M_0 = N_1 \sqcup \{x, y\}$, and let N_2 be the plain extension of M_0. Without loss of generality we can assume that $N_2 \leq M_*$. Consider $\langle a, b \rangle = i(\{x, y\})$. Then $\{a, b\} \nsubseteq N_1$ and $\{a, b\} \subseteq N_2$, since otherwise, we could apply the third property of Proposition 6 with respect to $\{a, b\}$ and N_1 and N_2, respectively. Moreover, $\{a, b\} \cap \{x, y\} = \emptyset$, since otherwise, (e.g., $a = x$), we can swap x and y which would imply that $b = y$, and since $x \neq y$, we get $i(\{x, y\}) = \langle x, y \rangle \neq \langle y, x \rangle = i(\{y, x\})$ which is a contradiction to the assumption that i is a function. Furthermore, we have

$$\{x, y\} \ \subseteq\ \mathrm{cl}(N_1 \cup \{a, b\}, M_*) \tag{$*$}$$

since otherwise, we could apply the third property of Proposition 6 with respect to $\{x, y\}$ and $\mathrm{cl}\big(N_1 \cup \{a, b\}, M_*\big) \leq M_*$. Now, this last inclusion implies that $a \neq b$ and that $\{a, b\} = f(\langle x', y' \rangle)$ for some $x', y' \in N_2 \setminus N_1$. To see this, notice first that since $N_1 \cup \{a, b\} \subseteq N_2$, we build the closure $\mathrm{cl}\big(N_1 \cup \{a, b\}, M_*\big)$ within the plain extension N_2, and recall that for $\{u, v\} \in [N_2]^2 \setminus [M_0]^2$ we have $g(\{u, v\}) = \langle x_1, x_2, x_3 \rangle$ where $\langle x_1, x_2, x_3 \rangle \in \mathrm{seq}^{\text{1-1}}(N_2 \setminus M_0)$, and that for $\langle x_1, \ldots, x_n \rangle \in \mathrm{seq}^{\text{1-1}}(N_2) \setminus \mathrm{seq}^{\text{1-1}}(M_0)$ we have $h(\langle x_1, \ldots, x_n \rangle) \in [N_2 \setminus M_0]^3$. If there are no x', y' such that $f(\langle x', y' \rangle) = \{a, b\}$, then, since $\{a, b\} \subseteq N_2$, $\{a, b\}$ is a proper subset of $\mathrm{ran}\big(g(\{u, v\})\big)$ for some u, v, or of $h(\langle x_1, \ldots, x_n \rangle)$ for some $x_1, \ldots, x_n$. In both cases we have that $\{x, y\} \nsubseteq \mathrm{cl}(N_1 \cup \{a, b\}, M_*)$, which is a contradiction to $(*)$. Now, since $f(\langle x', y' \rangle) = \{a, b\}$, by the construction of the plain extension N_2 we find an automorphism π of N_2 that fixes $N_1 \cup \{x, y\}$ pointwise and for which we have $\pi(a) = b$ and $\pi(b) = a$. Hence, $i(\pi\{x, y\}) = \langle a, b \rangle \neq \langle b, a \rangle = \pi i(\{x, y\})$, which is a contradiction.

Assume there is an injection $i \colon \mathrm{seq}^{1-1}(A) \to [A]^2$ with finite support S. Let $N_1 \in K$ be a countable model such that $N_1 \leq M_*$ and $S \subseteq N_1$. Let $\langle x, y, z \rangle \in \mathrm{seq}^{\text{1-1}}(A)$ with $N_1 \cap$

$\{x, y, z\} = \emptyset$, let $M_0 = N_1 \sqcup \{x, y, z\}$, and let N_2 be the plain extension of M_0. Finally, let $\{a, b\} = i(\langle x, y, z\rangle)$. Then, a contradiction follows by noticing — with similar arguments as above — that necessarily $\{x, y, z\} \not\subseteq \operatorname{cl}\big(N_1 \cup \{a, b\}, M_*\big)$. So, similarly as above, there is an automorphism π of M_* which fixes $\operatorname{cl}(N_1 \cup \{a, b\}, M_*)$ pointwise, but $\pi(\{x, y, z\}) \neq \{x, y, z\}$.

Finally, assume there is an injection $i\colon \operatorname{fin}(A) \to \operatorname{seq}^{1-1}(A)$ with finite support S. Let $N_1 \in K$ be a countable model such that $N_1 \leq M_*$ and $S \subseteq N_1$. Let $\{x, y, z\} \in [A]^3$ be such that $N_1 \cap \{x, y, z\} = \emptyset$, let $M_0 = N_1 \sqcup \{x, y, z\}$, and let N_2 be the plain extension of M_0. Consider $\langle a_j : j \in n\rangle = i(\{x, y, z\})$ for some $n \in \omega$. It is easy to see that we must have $\{a_j : j \in n\} \cap (N_2 \setminus M_0) \neq \emptyset$, and as before it must also hold $\{x, y, z\} \subseteq \operatorname{cl}\big(N_1 \cup \{a_j : j \in n\}, M_*\big)$. The last inclusion implies that a 3-cycle π applied to $\{x, y, z\}$ cannot leave $\{a_j : j \in n\} \cap (N_2 \setminus M_0)$ unchanged, since π moves every unordered pair, ordered pair and injective sequence with values in $\{x, y, z\}$. We conclude the proof by noticing that we can easily find an automorphism of N_2 that fixes N_1 pointwise and that acts on $\{x, y, z\}$ as a 3-cycle. $\dashv$

So, the model $\mathcal{V}_{\mathsf{Z}}$ witnesses the following

Consistency Result 2. *The existence of an infinite cardinal* $\mathfrak{m}$ *satisfying*

$$\begin{array}{ccc} \operatorname{fin}(\mathfrak{m}) & \longleftarrow & \operatorname{seq}^{1\text{-}1}(\mathfrak{m}) \\ & \nearrow & \\ [\mathfrak{m}]^2 & \longleftarrow & \mathfrak{m}^2 \end{array}$$

is consistent with ZF.

2.3 A Model for Diagram **И**

We show that Diagram **И** holds in the model constructed in [3] (see also [5, p. 209 ff]), where $\mathfrak{m}$ is the cardinality of the set of atoms of that model.

The atoms of the permutation model $\mathcal{V}_{\text{И}}$ for Diagram **И** are constructed as follows:

- (α) Let A_0 be an arbitrary infinite set.
- (β) G_0 is the group of *all* permutations of A_0.
- (γ) $A_{n+1} := A_n \cup \big\{(n+1, p, \varepsilon) : p \in \bigcup_{k=0}^{n+1} A_n^k \wedge \varepsilon \in \{0, 1\}\big\}$.
- (δ) G_{n+1} is the subgroup of the permutation group of A_{n+1} containing all permutations σ for which there are $\pi_\sigma \in G_n$ and $\varepsilon_{\sigma,p} \in \{0, 1\}$ such that

$$\sigma(x) = \begin{cases} \pi_\sigma(x) & \text{if } x \in A_n, \\ (n+1, \pi_\sigma(p), \varepsilon_{\sigma,p} +_2 \varepsilon) & \text{if } x = (n+1, p, \varepsilon), \end{cases}$$

where for $p = \langle p_0, \ldots, p_{l-1}\rangle \in \bigcup_{0 \leq k \leq n+1} A_n^k$, $\pi_\sigma(p) := \langle \pi_\sigma(p_0), \ldots, \pi_\sigma(p_{l-1})\rangle$ and $+_2$ denotes addition modulo 2.

Let $A := \bigcup\{A_n : n \in \omega\}$ be the set of atoms and let $\mathrm{Aut}(A)$ be the group of all permutations of A. Then

$$G := \big\{H \in \mathrm{Aut}(A) : \forall n \in \omega\,(H|_{A_n} \in G_n)\big\}$$

is a group of permutations of A. The sets in $\mathcal{V}_{\text{И}}$ are those with finite support.

Proposition 8. *Let A be the set of atoms of $\mathcal{V}_{\text{И}}$ and let $\mathfrak{m} := |A|$. Then*

$$\mathcal{V}_{\text{И}} \models \mathfrak{m}^2 < \mathrm{seq}^{\text{1-1}}(\mathfrak{m}) < [\mathfrak{m}]^2 < \mathrm{fin}(\mathfrak{m})\,.$$

Proof. In [3] it is shown that $\mathcal{V}_{\text{И}} \models \mathrm{seq}(\mathfrak{m}) < [\mathfrak{m}]^2$, which implies that $\mathcal{V}_{\text{И}} \models \mathrm{seq}^{\text{1-1}}(\mathfrak{m}) < [\mathfrak{m}]^2$. Thus, since $\mathfrak{m}^2 \leq \mathrm{seq}^{\text{1-1}}(\mathfrak{m})$ and $[\mathfrak{m}]^2 \leq \mathrm{fin}(\mathfrak{m})$, it remains to show that in $\mathcal{V}_{\text{И}}$ we have $\mathfrak{m}^2 \neq \mathrm{seq}^{\text{1-1}}(\mathfrak{m})$ and $[\mathfrak{m}]^2 \neq \mathrm{fin}(\mathfrak{m})$.

$\mathfrak{m}^2 \neq \mathrm{seq}^{\text{1-1}}(\mathfrak{m})$: We show that there is no injection $g_1 : \mathrm{seq}^{\text{1-1}}(A) \to A^2$. Assume towards a contradiction that there is such an injection with finite support E_1.

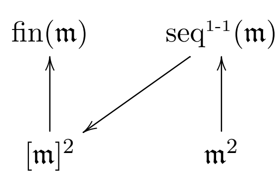
Since E_1 is finite, there is an integer $n_1 \in \omega$ such that $E_1 \subseteq A_{n_1}$. By extending E_1 if necessary, we may assume that if $(n+1, \langle a_0, \dots, a_{l-1}\rangle, \varepsilon) \in E_1$, then also $a_0, \dots, a_{l-1}$ belong to E_1 as well as the atom $(n+1, \langle a_0, \dots, a_{l-1}\rangle, 1-\varepsilon)$.

For a large enough number $k \in \omega$ choose a k-element set $X \subseteq A_0 \setminus E_1$ such that $|\,\mathrm{seq}^{\text{1-1}}(X)\,| > \big|\,(E_1 \cup X)^2\,\big|$. Notice that $|\,\mathrm{seq}^{\text{1-1}}(X)\,| \geq k!$ and that $\big|\,(E_1 \cup X)^2\,\big| = (|E_1| + k)^2$. Thus, we find a sequence $s \in \mathrm{seq}^{\text{1-1}}(X)$ such that $g_1(s) \notin (E_1 \cup X)^2$. So, there exists a $\pi \in \mathrm{Fix}_G(E_1 \cup X)$ such that $\pi g_1(s) \neq g_1(s)$ but $\pi s = s$, which contradicts the fact that E_1 is a support of g_1.

$[\mathfrak{m}]^2 \neq \mathrm{fin}(\mathfrak{m})$: We show that there is no injection $g_2 : \mathrm{fin}(A) \to [A]^2$. Assume towards a contradiction that there is such an injection with finite support E_2. After extending E_2 in the same way as above, if necessary, for a large enough number $k \in \omega$ we choose again a k-element set $X \subseteq A_0 \setminus E_2$ such that $|\,\mathrm{fin}\,(X)\,| > \big|\,[E_2 \cup X]^2\,\big|$. Then, by similar arguments as above, we can show that E_2 is not a support of g_2, which gives us the desired contradiction. ⊣

So, the permutation model $\mathcal{V}_{\text{И}}$ witnesses the following

Consistency Result 3. *The existence of an infinite cardinal $\mathfrak{m}$ satisfying*



is consistent with ZF.

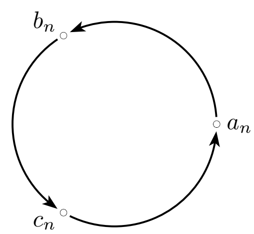
2.4 A Model for Diagram Ↄ

We show that Diagram Ↄ holds in a permutation model $\mathcal{V}_{Ↄ}$ which is similar to the *Second Fraenkel Model*, where $\mathfrak{m}$ is the cardinality of the set of atoms of $\mathcal{V}_{Ↄ}$.

The permutation model $\mathcal{V}_\gimel$ is constructed as follows (see also [5, p. 197]): The set of atoms of the model $\mathcal{V}_\gimel$ consists of countably many mutually disjoint, cyclically ordered 3-element sets. More formally,

$$A = \bigcup_{n\in\omega} P_n, \quad \text{where } P_n = \{a_n, b_n, c_n\} \text{ (for } n \in \omega),$$

and the cyclic ordering on P_n is illustrated by the following figure:



On each triple P_n, we define the cyclic distance between two elements by stipulating

$$\operatorname{cyc}(a_n, b_n) = \operatorname{cyc}(b_n, c_n) = \operatorname{cyc}(c_n, a_n) = 1$$

and

$$\operatorname{cyc}(a_n, c_n) = \operatorname{cyc}(b_n, a_n) = \operatorname{cyc}(c_n, b_n) = 2\,.$$

Let G be the group of those permutations of A which preserve the triples P_n (*i.e.*, $\pi P_n = P_n$ for $\pi \in G$ and $n \in \omega$) and their cyclic ordering. The sets in $\mathcal{V}_\gimel$ are those with finite support.

Proposition 9. *Let A be the set of atoms of $\mathcal{V}_\gimel$ and let $\mathfrak{m} := |A|$. Then*

$$\mathcal{V}_\gimel \models [\mathfrak{m}]^2 < \mathfrak{m}^2 < \operatorname{seq}^{1\text{-}1}(\mathfrak{m}) < \operatorname{fin}(\mathfrak{m})\,.$$

Proof. We first show that $[\mathfrak{m}]^2 \le \mathfrak{m}^2$, $\mathfrak{m}^2 \le \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$, and $\operatorname{seq}^{1\text{-}1}(\mathfrak{m}) \le \operatorname{fin}(\mathfrak{m})$, and then we show that $[\mathfrak{m}]^2 \ne \mathfrak{m}^2$, $\mathfrak{m}^2 \ne \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$, and $\operatorname{seq}^{1\text{-}1}(\mathfrak{m}) \ne \operatorname{fin}(\mathfrak{m})$.

$[\mathfrak{m}]^2 \le \mathfrak{m}^2$: We define an injective function $f_1 : [A]^2 \to A^2$. Let $\{x, y\} \in [A]^2$ and $m, n \in \omega$ be such that $x \in P_m$ and $y \in P_n$. Without loss of generality we may assume that $m \le n$. If $m < n$, then $f_1(\{x, y\}) := \langle x, y\rangle$, and if $m = n$, then $f_1(\{x, y\}) := \langle z, z\rangle$ where $z := P_m \setminus \{x, y\}$. It is easy to see that f_1 is an injective function, and since f_1 has empty support, f_1 belongs to $\mathcal{V}_\gimel$.

$\mathfrak{m}^2 \le \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$: The function $f_2 : A^2 \to \operatorname{seq}^{1\text{-}1}(A)$ defined by stipulating

$$f_2(\langle x, y\rangle) := \begin{cases} \langle x\rangle & \text{if } x = y, \\ \langle x, y\rangle & \text{otherwise,} \end{cases}$$

is an injective function from A^2 into $\operatorname{seq}^{1\text{-}1}(A)$ which belongs to $\mathcal{V}_\gimel$.

$\operatorname{seq}^{1\text{-}1}(\mathfrak{m}) \le \operatorname{fin}(\mathfrak{m})$: We define a injective function $f_3 : \operatorname{seq}^{1\text{-}1}(A) \to \operatorname{fin}(A)$. First, let $f_3(\langle\,\rangle) := \emptyset$. Now, let $s = \langle a_0, \dots, a_{k-1}\rangle \in \operatorname{seq}^{1\text{-}1}(A)$ be a non-empty sequence without repetition of length k.

Let $j : k \to \omega$ be such that for each $i \in k$, $a_i \in P_{j(i)}$. Let $E_0 := \emptyset$, and by induction, for $i \in k$ define

$$E_{i+1} := \begin{cases} E_i \cup \{a_i\} & \text{if } P_{j(i)} \cap E_i = \emptyset, \\ E_i & \text{otherwise,} \end{cases}$$

and

$$\varepsilon_i := \begin{cases} 2 & \text{if } |P_{j(i)} \cap E_i| = 1 \text{ and } \operatorname{cyc}\left(P_{j(i)} \cap E_i, a_i\right) = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore, let $\sigma(0) := j(0)$, and by induction, for $i \in k_1$ define $\sigma(i+1) := \sigma(i) + j(i+1) + 1$. Finally, let $\{p_i : i \in \omega\}$ be an enumeration of the prime numbers and let

$$q_s := \prod_{i \in k} p_{\sigma(i)}^{\varepsilon_i} .$$

Now, we define $f_3(s) := E_k \cup P_{q_s}$. It is easy to see that f_3 is an injective function from $\operatorname{seq}^{1\text{-}1}(A)$ into $\operatorname{fin}(A)$, and since f_3 has empty support, f_3 belongs to $\boldsymbol{\mathcal{V}}_{\text{כ}}$.

$[\mathfrak{m}]^2 \neq \mathfrak{m}^2$: It is enough to show that there is no injection from A^2 into $[A]^2$. Assume towards a contradiction that there exists an injection $g_1 : A^2 \to [A]^2$ with finite support E_1.

Let $E_1 \subseteq E$ for some non-empty set $E \in \operatorname{fin}(A)$. Then E is also a support of g_1. Now $|[E]^2| < |E^2|$, which implies that there exists a pair $\langle x, y\rangle \in E^2$ such that $g_1(\langle x, y\rangle) \notin [E]^2$. So, there exists a $\pi \in \operatorname{Fix}_G(E)$ such that $\pi g_1(\langle x, y\rangle) \neq g_1(\langle x, y\rangle)$, but $\pi\langle x, y\rangle = \langle x, y\rangle$, which contradicts the fact that E is a support of g_1.

$\mathfrak{m}^2 \neq \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$: It is enough to show that there is no injection from $\operatorname{seq}^{1\text{-}1}(A)$ into A^2. Assume towards a contradiction that there exists an injection $g_2 : \operatorname{seq}^{1\text{-}1}(A) \to A^2$ with finite support E_2. Let $E_2 \subseteq E$ for some non-empty set $E \in \operatorname{fin}(A)$. Then E is also a support of g_2. Now $|E^2| < |\operatorname{seq}^{1\text{-}1}(E)|$, and by similar arguments as above, we obtain a contradiction.

$\operatorname{seq}^{1\text{-}1}(\mathfrak{m}) \neq \operatorname{fin}(\mathfrak{m})$: It is enough to show that there is no injection from $\operatorname{fin}(A)$ into $\operatorname{seq}^{1\text{-}1}(A)$. Assume towards a contradiction that there is an injection $g_3 : \operatorname{fin}(A) \to \operatorname{seq}^{1\text{-}1}(A)$ with finite support E_3.

Since E_3 is finite, there exists an $n \in \omega$ such that $g_3(P_n) \notin \operatorname{seq}^{1\text{-}1}(E_3)$. Let $a \in A$ be the first element of the sequence $g_3(P_n)$ which does not belong to E_3. Then we find a $\pi \in \operatorname{Fix}_G(E_3)$ such that $\pi a \neq a$ (i.e., $\pi g_3(P_n) \neq g_3(P_n)$) but $\pi P_n = P_n$, which contradicts the fact that E_3 is a support of g_3. $\dashv$

So, the model $\boldsymbol{\mathcal{V}}_{\text{כ}}$ witnesses the following

Consistency Result 4. *The existence of an infinite cardinal* $\mathfrak{m}$ *satisfying*

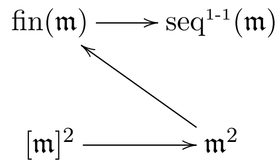
$$\begin{array}{ccc} \operatorname{fin}(\mathfrak{m}) & \longleftarrow & \operatorname{seq}^{1\text{-}1}(\mathfrak{m}) \\ & & \uparrow \\ [\mathfrak{m}]^2 & \longrightarrow & \mathfrak{m}^2 \end{array}$$

is consistent with ZF.

2.5 A Model for Diagram $\boldsymbol{\Sigma}$

As mention above, Diagram $\boldsymbol{\Sigma}$ holds in the *Ordered Mostowski Model*, where $\mathfrak{m}$ is the set of atoms (see, for example, [5, Related Result 48, p. 217]). This leads to the following

Consistency Result 5. *The existence of an infinite cardinal* $\mathfrak{m}$ *satisfying*



is consistent with ZF.

3 On Diagram C

Similar as in the proof of Lemma 1, in every model for Diagram **C**, where $\mathfrak{m} = |A|$, we have that the cardinality $\mathfrak{m}$ is transfinite.

Lemma 10. *If* $\mathfrak{m}^2 \leq [\mathfrak{m}]^2$ *and* $\operatorname{fin}(\mathfrak{m}) \leq \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$ *for some* $\mathfrak{m} \geq 1$*, then* $\aleph_0 \leq \mathfrak{m}$.

Proof. Assume that $\mathfrak{m}^2 \leq [\mathfrak{m}]^2$ and $\operatorname{fin}(\mathfrak{m}) \leq \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$ for some cardinal $\mathfrak{m} \geq 1$ and let A be a necessarily infinite set with $|A| = \mathfrak{m}$. Let $f : A^2 \to [A]^2$ and $g : \operatorname{fin}(A) \to \operatorname{seq}^{1\text{-}1}(A)$ be injections. The goal is to construct with the functions f and g an injection $h : \omega \to A$. We first construct a countably infinite set of pairwise disjoint non-empty finite subsets of A. For this, we first choose an element $a_0 \in A$, let $E_0 := \{a_0\}$, and let $\mathscr{E}_0 := \{E_0\}$.

Assume that for some $n \in \omega$ we have already constructed an $(n+1)$-element set $\mathscr{E}_n := \{E_i : i \leq n\}$ of pairwise disjoint non-empty finite subsets of A. Let

$$E_{n+1} := \bigcup_{i,j \leq n} \Big\{\{x,y\} : \exists a \in E_i\, \exists b \in E_j \big(f(\langle a,b\rangle) = \{x,y\}\big)\Big\} \setminus \bigcup_{i \leq n} E_i\,,$$

and let $\mathscr{E}_{n+1} := \mathscr{E}_n \cup \{E_{n+1}\}$. Notice that for $k := \bigcup_{i \leq n} E_i$, we have

$$\Big|\bigcup_{i \leq n} E_i{}^2\Big| \;=\; k^2 \;>\; \binom{k}{2} \;=\; \Big|\Big[\bigcup_{i \leq n} E_i\Big]^2\Big|\,,$$

which implies that $E_{n+1} \neq \emptyset$. Proceeding this way, $\{E_n : n \in \omega\}$ is a countably infinite set of pairwise disjoint non-empty finite subsets of A.

Now, we apply the function g. For every $n \in \omega$, let $S_n := g(E_n)$. Furthermore, let $\mathscr{S}_0 := S_0$, and in general, for $n \in \omega$ let $\mathscr{S}_{n+1} := \mathscr{S}_n{}^\frown S_{n+1}$. This way, we obtain an infinite sequence $\mathscr{S}_\infty$ of elements of A. Since g is injective and the sets finite sets E_n are pairwise disjoint, the sequence $\mathscr{S}_\infty$ must contain infinitely many pairwise distinct elements of A. Now, let h be the enumeration of these pairwise distinct elements in the order they appear in $\mathscr{S}_\infty$. Then $h : \omega \to A$ is an injection. ⊣

As a consequence of PROPOSITION 2 and LEMMA 10 we get

COROLLARY 11. *If* $\mathfrak{m}^2 \leq [\mathfrak{m}]^2$ *and* $\operatorname{fin}(\mathfrak{m}) \leq \operatorname{seq}^{1\text{-}1}(\mathfrak{m})$ *for some* $\mathfrak{m} = |A| \geq \mathbf{1}$*, then there exists a finite-to-one function* $g : \operatorname{seq}(A) \to \operatorname{fin}(A)$*.*

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