European Journal of Mathematics (2021) 7:1–27
https://doi.org/10.1007/s40879-020-00425-w

RESEARCH ARTICLE

On some variants of the club principle

Ashutosh Kumar[1] · **Saharon Shelah**[1,2]

Received: 25 February 2018 / Revised: 7 June 2020 / Accepted: 10 July 2020 / Published online: 25 August 2020

Abstract
We study some asymptotic variants of the club principle. Along the way, we construct some forcings and use them to separate several of these principles.

Keywords Club principle · Forcing

Mathematics Subject Classification 03E05 · 03E35

1 Introduction

For a regular uncountable cardinal κ and a stationary $S \subseteq \mathsf{Lim}(\kappa)$, the club principle $\clubsuit_S$ says the following: There exists $\overline{A} = \langle A_\delta : \delta \in S\rangle$ where each A_δ is an unbounded subset of δ of order type $\mathsf{cf}(\delta)$ such that for every $A \in [\kappa]^\kappa$, there exists some (equivalently, stationary many) $\delta \in S$ such that $A_\delta \subseteq A$. We say that $\overline{A}$ is a $\clubsuit_S$ witnessing sequence. If $\kappa = \omega_1$ and $S = \mathsf{Lim}(\omega_1)$ is the set of all countable limit ordinals, we drop the S and write $\clubsuit$.

The principle $\clubsuit$ was introduced by Andrzej Ostaszewski in [6] where he used $\clubsuit+\mathsf{CH}$ (equivalently, $\diamondsuit$) to construct an Ostaszewski space. Several variants of this principle have since been studied [1,2]. For example, in [1], it was shown that $\clubsuit^1$ does not imply

AK supported by a Postdoctoral Fellowship at the Einstein Institute of Mathematics funded by European Research Council grant 338821. SS partially supported by European Research Council grant 338821; Publication no. 1136.

✉ Ashutosh Kumar
akumar@math.huji.ac.il

Saharon Shelah
shelah@math.huji.ac.il

1 Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J Safra Campus, Givat Ram, Jerusalem 91904, Israel

2 Department of Mathematics, Rutgers, The State University of New Jersey, Hill Center-Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

$\clubsuit$ where $\clubsuit^1$ is the following statement: There exists $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ where each A_δ is an unbounded subset of δ of order type ω such that for every $A \in [\omega_1]^{\aleph_1}$, there exists δ such that $A_\delta \setminus A$ is finite.

In this work, we mostly study asymptotic versions of the club principle where the requirement $A_\delta \subseteq A$ is replaced by $A_\delta \cap A$ is a "large" subset of A_δ. Some of these principles have previously appeared in [3,4]. As a motivating example, suppose we start with a model of $\clubsuit$ and add $\aleph_2$ Cohen reals. Then it is easy to see that $\clubsuit^1$ and therefore $\clubsuit$ are destroyed. But the following continues to hold (see Remark 7.2): There exists $\langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ where each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ where $\alpha_{\delta,n}$'s are increasing cofinal in δ and for every $A \in [\omega_1]^{\aleph_1}$, here exists δ such that $\{n < \omega : \alpha_{\delta,n} \in A\}$ has upper asymptotic density 1. It follows that $\neg\clubsuit^1 \wedge \clubsuit^{\sup \geqslant 1}$ is consistent.

Definition 1.1 xxx For $a \in (0, 1]$ and a stationary set $S \subseteq \mathsf{Lim}(\omega_1)$, the principle $\clubsuit_S^{\inf \geqslant a}$ says the following: There exists $\overline{A} = \langle A_\delta : \delta \in S\rangle$ such that

(a) each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ, and
(b) for every $A \in [\omega_1]^{\aleph_1}$, there exists $\delta \in S$ such that

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} \geqslant a.$$

If $S = \mathsf{Lim}(\omega_1)$, we write $\clubsuit^{\inf \geqslant a}$. By $\clubsuit^{\lim}$, we mean $\clubsuit^{\inf \geqslant 1}$.

It is clear that $\clubsuit^1$ implies $\clubsuit^{\lim}$ and for $0 < a < b \leqslant 1$, $\clubsuit^{\inf \geqslant b}$ implies $\clubsuit^{\inf \geqslant a}$. At the end of Sect. 1, we show that under CH, all of these principles are equivalent to diamond.

Theorem 1.2 *Assume CH. Then for every $a \in (0, 1]$, $\clubsuit^{\inf \geqslant a}$ implies $\diamondsuit$.*

The bulk of the work in this paper is to show the following.

Theorem 1.3 (1) $\clubsuit^{\lim} \wedge \neg\clubsuit^1$ *is consistent.*
(2) *For every $a \in (0, 1]$, $\clubsuit^{\inf \geqslant a} \wedge (\forall b > a)\neg\clubsuit^{\inf \geqslant b}$ is consistent.*
(3) *For every $a \in (0, 1]$, $\neg\clubsuit^{\inf \geqslant a} \wedge (\forall b < a)\clubsuit^{\inf \geqslant b}$ is consistent.*

In Sects. 2–5 we introduce the necessary tools for constructing the forcings used in proving Theorem 1.3. Sections 2 and 3 introduce the class of "thin $\aleph_1$-CP's" (Definition 3.5) which constitute the main building block of these constructions. The associated forcings satisfy ccc and are somewhat intermediate between $\mathsf{Add}(\omega, \omega_1)$ and $\mathsf{Add}(\omega_1, 1)$. The main point here is that while forcing with a suitable member of this class will destroy all old $\clubsuit^1$-witnesses, it will preserve some $\clubsuit^{\lim}$-witness. Section 4 handles the next issue, namely, how to perform an iteration of thin $\aleph_1$-CP's that preserves $\clubsuit^{\lim}$. This is achieved via "guided products" (Definition 4.1). In Sect. 5, we construct our iteration and show that the resulting model witnesses Theorem 1.3 (1). Our construction scheme is quite flexible and should be useful for separating many similar principles. This is illustrated in Sect. 6 where we prove Theorem 1.3 (2),(3) by forcing with the guided product of another family of thin $\aleph_1$-CP's.

In Sect. 7, we introduce $\clubsuit^{\sup \geqslant a}$ (Definition 7.1) and prove the following in ZFC.

Theorem 1.4 *For every $a, b \in (0, 1)$, $\clubsuit^{\sup \geqslant a}$ is equivalent to $\clubsuit^{\sup \geqslant b}$.*

Finally, in Sect. 8, we prove that

Theorem 1.5 $\clubsuit^{\sup \geqslant 0.5} \wedge \neg \clubsuit^{\sup \geqslant 1}$ *is consistent.*

On notation: $\mathsf{Lim}(\kappa)$ denotes the set of all limit ordinals below κ. For a set of ordinals X, $\mathsf{otp}(X)$ denotes the order type of X. $\mathsf{cf}(\alpha)$ is the cofinality of α. $S^\kappa_\delta = \{\alpha < \kappa : \mathsf{cf}(\alpha) = \mathsf{cf}(\delta)\}$. For $k \leqslant \omega$, ω^k is the kth power of ω under ordinal exponentiation. For a, b sets of ordinals, we write $a < b$ to denote $(\forall \alpha \in a)(\forall \beta \in b)(\alpha < \beta)$. In forcing, we use the convention that a larger condition is the stronger one — so $p \geqslant q$ means p extends q.

1.1 CH and $\clubsuit^{\inf}$

Fact 1.6 *Suppose $S \subseteq \mathsf{Lim}(\omega_1)$ is stationary and $\overline{A} = \langle A_\delta : \delta \in S\rangle$ is a $\clubsuit^?_S$-witnessing sequence where $\clubsuit^?_S$ is one of club principles defined in Definitions 1.1, 6.1, 7.1, 8.1. Then for every $A \in [\omega_1]^{\omega_1}$, there are stationary many $\delta \in S$ witnessing the corresponding requirement for the pair A_δ, A.*

Proof Fix $A \in [\omega_1]^{\omega_1}$ and let $E \subseteq \omega_1$ be a club. Choose $B \in [A]^{\omega_1}$ such that between any two members of B, there is a member of E. Choose $\delta \in S$ witnessing the corresponding requirement for the pair A_δ, B. As $B \subseteq A$, this δ also works for A. Since $B \cap A_\delta$ is unbounded in δ, $E \cap \delta$ is also unbounded in δ. As E is a club, it follows that $\delta \in S \cap E$. Hence there are stationary many such $\delta \in S$. □

Recall that $\diamondsuit$ says the following: There exists $\langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ where each $A_\delta \subseteq \delta$ such that for every $A \subseteq \omega_1$, $\{\delta \in \mathsf{Lim}(\omega_1) : A_\delta = A \cap \delta\}$ is stationary. An equivalent formulation (see [5]) is the following: There exists $\langle \mathcal{A}_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ where each $\mathcal{A}_\delta$ is a countable family of subsets of δ such that for every $A \subseteq \omega_1$, $\{\delta \in \mathsf{Lim}(\omega_1) : A \cap \delta \in \mathcal{A}_\delta\}$ is stationary.

Proof of Theorem 1.2 Assume CH. Suppose $a \in (0, 1]$ and $\clubsuit^{\inf \geqslant a}$ holds as witnessed by $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$. Let $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ list A_δ in increasing order. Using CH, fix $\langle B_i : i < \omega_1\rangle$ such that each $B_i \subseteq i$ and for every $B \in [\omega_1]^{\leqslant \aleph_0}$, there are uncountably many $i < \omega_1$ for which $B = B_i$.

For $\delta \in \mathsf{Lim}(\omega_1)$, define $\mathcal{A}_\delta$ as follows. $A \in \mathcal{A}_\delta$ iff for some $u \subseteq \omega$ the following hold:

- $\liminf_n |u \cap n|/n \geqslant a$.
- For every $m < n$ in u, $B_{\alpha_{\delta,m}} = B_{\alpha_{\delta,n}} \cap \alpha_{\delta,m}$ and $A = \bigcup_{n \in u} B_{\alpha_{\delta,n}}$.

We claim that each $\mathcal{A}_\delta$ is finite. In fact, $|\mathcal{A}_\delta| \leqslant 1/a$. To see this, assume otherwise and let $\{A_k : k < K\}$ be pairwise distinct members of $\mathcal{A}_\delta$ where $Ka > 1$. Choose $\langle u_k : k < K\rangle$ witnessing $A_k \in \mathcal{A}_k$. Choose $N_1 < N_2$ such that the following hold:

(a) $\langle A_k \cap \alpha_{\delta,N_1} : k < K\rangle$ has pairwise distinct members,
(b) $|u_k \cap [N_1, N_2)| > (N_2 - N_1)/K$ for each $k < K$.

By (b), it follows that for some $j < k < K$, $[N_1, N_2) \cap u_j \cap u_k \neq \varnothing$. But if $n \in [N_1, N_2) \cap u_j \cap u_k$, then $B_{\alpha_{\delta,n}} = A_j \cap \alpha_{\delta,n} = A_k \cap \alpha_{\delta,n}$ which is impossible by (a).

To complete the proof it is enough to show the following.

Claim 1.7 *For every $X \subseteq \omega_1$, for every club $E \subseteq \omega_1$, there exists $\delta \in E$ such that $X \cap \delta \in \mathcal{A}_\delta$.*

Proof Construct $\langle \alpha_i : i < \omega_1 \rangle$ such that α_i's are increasing and for every $i < \omega_1$, $X \cap \sup_{j<i} \alpha_j = B_{\alpha_i}$. Choose $\delta \in E$ and $u \subseteq \omega$ such that $\liminf_n |u \cap n|/n \geqslant a$ and $\{\alpha_{\delta,n} : n \in u\} \subseteq \{\alpha_i : i < \omega_1\}$. It follows that $X \cap \delta = \bigcup_{n \in u} B_{\alpha_{\delta,n}} \in \mathcal{A}_\delta$. □

2 Creatures

Fix a family $\{S_k : k < \omega\}$ of pairwise disjoint stationary subsets of ω_1 consisting of limit ordinals. We describe a ccc forcing which is somewhat intermediate between adding $\aleph_1$ Cohen reals and adding a Cohen subset of ω_1.

Definition 2.1 We say that (CR, Σ) is an $\aleph_1$-CP (*creating pair*) if the following holds:

(A) We call members of CR *creatures*. For each $\mathfrak{c} \in \mathsf{CR}$,

  (i) $\mathfrak{c} = (\mathsf{dom}(\mathfrak{c}), \mathsf{pos}(\mathfrak{c}), f_\mathfrak{c})$.
  (ii) $\mathsf{dom}(\mathfrak{c})$ is a non-empty subset of ω_1 of order type $< \omega^\omega$.
  (iii) For every limit $\delta < \omega_1$, if $\mathsf{dom}(\mathfrak{c}) \cap \delta$ is unbounded in δ, then for some $k \geqslant 1$, $\delta \in S_k$ and $\mathsf{otp}(\mathsf{dom}(\mathfrak{c}) \cap \delta) = \varepsilon + \omega^j$ for some $\varepsilon < \omega^\omega$ and $1 \leqslant j \leqslant k$ — in particular, for every $\delta \in S_0$, $\mathsf{dom}(\mathfrak{c}) \cap \delta$ is bounded below δ.
  (iv) $\mathsf{pos}(\mathfrak{c})$ (*possibilities* for $\mathfrak{c}$) is a countable set of functions from $\mathsf{dom}(\mathfrak{c})$ to $\{0, 1\}$ and $f_\mathfrak{c} \in \mathsf{pos}(\mathfrak{c})$.
  (v) If $\mathsf{dom}(\mathfrak{c})$ is finite, then $\mathsf{pos}(\mathfrak{c}) = \{f_\mathfrak{c}\}$ — we call such $\mathfrak{c}$ *finite creature*.

(B) For every finite $u \subseteq \omega_1$ and $f : u \to \{0, 1\}$, there exists $\mathfrak{c} \in \mathsf{CR}$ such that $\mathsf{dom}(\mathfrak{c}) = u$ and $f_\mathfrak{c} = f$.
(C) For every $\delta < \omega_1$, $|\{\mathfrak{c} \in \mathsf{CR} : \mathsf{dom}(\mathfrak{c}) \subseteq \delta\}| \leqslant \aleph_0$.
(D) Σ is a function with domain CR that satisfies the following:

  (i) $\Sigma(\mathfrak{c})$ is a countable set of finite tuples $\bar{\mathfrak{d}} = \langle \mathfrak{d}_k : k < n \rangle$ where
    (a) $\mathfrak{d}_k \in \mathsf{CR}$,
    (b) $\mathsf{dom}(\mathfrak{c}) = \bigcup_{k<n} \mathsf{dom}(\mathfrak{d}_k)$,
    (c) $\mathsf{dom}(\mathfrak{d}_k) < \mathsf{dom}(\mathfrak{d}_{k+1})$, and
    (d) whenever $f_k \in \mathsf{pos}(\mathfrak{d}_k)$ for $k < n$, $\bigcup_{k<n} f_k \in \mathsf{pos}(\mathfrak{c})$.
  (ii) Cuts: If $\mathfrak{c} \in \mathsf{CR}$ and $\alpha \in \mathsf{dom}(\mathfrak{c})$ then for some $\overline{d} = \langle \mathfrak{d}_k : k < n \rangle \in \Sigma(\mathfrak{c})$, there exists $k < n$ such that $\min(\mathsf{dom}(\mathfrak{d}_k)) = \alpha$.
  (iii) $\langle \mathfrak{c} \rangle \in \Sigma(\mathfrak{c})$.
  (iv) Transitivity: If $\langle \mathfrak{c}_k : k < n \rangle \in \Sigma(\mathfrak{c})$ and $\langle \mathfrak{d}_{k,l} : l < n_k \rangle \in \Sigma(c_k)$ for $k < n$, then $\langle \mathfrak{d}_{k,l} : k < n, l < n_k \rangle \in \Sigma(\mathfrak{c})$.

(E) Finite joins: If $\{\mathfrak{d}_k : k < n\} \subseteq \mathsf{CR}$ and $\mathsf{dom}(\mathfrak{d}_k) < \mathsf{dom}(\mathfrak{d}_{k+1})$, then there exists $\mathfrak{c} \in \mathsf{CR}$ such that

  (i) $\mathsf{dom}(\mathfrak{c}) = \bigcup_{k<n} \mathsf{dom}(\mathfrak{d}_k)$,
  (ii) $\mathsf{pos}(\mathfrak{c}) = \left\{\bigcup_{k<n} f_k : (\forall k < n)(f_k \in \mathsf{pos}(\mathfrak{d}_k))\right\}$,

(iii) $f_{\mathfrak{c}} = \bigcup_{k<n} f_{\mathfrak{d}_k}$, and
(iv) $\Sigma(\mathfrak{c}) = \{\bigcup_{i<n} \bar{\mathfrak{f}}_i : (\forall i < n)(\bar{\mathfrak{f}}_i \in \Sigma(\mathfrak{d}_i))\}$.

Definition 2.2 Suppose (CR, Σ) is an $\aleph_1$-CP. Define $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$ to be the forcing whose conditions are $p = \{\mathfrak{c}_k : k < n\}$ where $\mathfrak{c}_k \in \mathsf{CR}$ and $\mathsf{dom}(\mathfrak{c}_k) < \mathsf{dom}(\mathfrak{c}_{k+1})$. We write $\mathsf{dom}(p)$ for $\bigcup_{\mathfrak{c}\in p} \mathsf{dom}(\mathfrak{c})$. For $p, q \in \mathbb{Q}$, define $p \leqslant q$ iff for every $\mathfrak{c} \in p$, there exists $\bar{\mathfrak{d}} = \langle \mathfrak{d}_k : k < n\rangle \in \Sigma(\mathfrak{c})$ such that $\{\mathfrak{d}_k : k < n\} \subseteq q$. Define $\mathbb{Q} \restriction \alpha = \{p \in \mathbb{Q} : \mathsf{dom}(p) \subseteq \alpha\}$. Let

$$\mathring{f}_{\mathbb{Q}} = \bigcup \{ f_{\mathfrak{d}} : (\exists\, p \in G_{\mathbb{Q}})(\mathfrak{d} \in p \text{ is a finite creature})\}.$$

It is easy to see that $\Vdash_{\mathbb{Q}} \mathring{f}_{\mathbb{Q}} : \omega_1 \to 2$ (See Remark 2.3 (ii) below).

Remark 2.3 Let (CR, Σ), $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$ and $\mathring{f}_{\mathbb{Q}}$ be as in Definition 2.2.

(i) By Clause (D) (iv) in Definition 2.1, it follows that $\leqslant_{\mathbb{Q}}$ is transitive.

(ii) Let us check that $\Vdash_{\mathbb{Q}} \mathring{f}_{\mathbb{Q}} : \omega_1 \to 2$. For suppose $p \in \mathbb{Q}$ and $\alpha < \omega_1$. It suffices to find $q \geqslant p$ such that for some finite creature $\mathfrak{d} \in q$, $\alpha \in \mathsf{dom}(\mathfrak{d})$. If $\alpha \notin \mathsf{dom}(p)$, then we can add a creature with domain $\{\alpha\}$ to p. So assume $\alpha \in \mathsf{dom}(p)$. Fix $\mathfrak{c} \in p$ with $\alpha \in \mathsf{dom}(\mathfrak{c})$. Using Clauses (D) (ii) and (D) (iv) in Definition 2.1, we can find $\langle \mathfrak{d}_k : k < n\rangle \in \Sigma(\mathfrak{c})$ such that for some k, $\mathsf{dom}(\mathfrak{d}_k) = \{\alpha\}$. Put $q = (p \setminus \{\mathfrak{c}\}) \cup \{\mathfrak{d}_k : k < n\}$. Then $q \geqslant p$ is as required.

(iii) Let CR be the set of all finite creatures $\mathfrak{c} = (F, \{f\}, f)$ — so $F \subseteq \omega_1$ is finite and $f : F \to 2$. Let $\Sigma(\mathfrak{c})$ be the set of all $\bar{\mathfrak{d}}$ such that the join of the members of $\bar{\mathfrak{d}}$ is $\mathfrak{c}$. Then forcing with $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$ is same as adding $\aleph_1$ Cohen reals. Note that this destroys all old witnesses to $\clubsuit^{\mathrm{lim}}$. We would later (Sect. 3) add more creatures to CR in such a way that while some old $\clubsuit^{\mathrm{lim}}$ witnessing sequences are preserved, all old $\clubsuit^{1}$ witnessing sequences are destroyed.

Recall that a forcing notion $\mathbb{Q}$ has $\aleph_1$ as a precaliber if whenever $\{p_i : i < \omega_1\} \subseteq \mathbb{Q}$, there exists $X \in [\omega_1]^{\aleph_1}$ such that $\{p_i : i \in X\}$ is centered — i.e., for every finite $F \subseteq X$, there exists $p \in \mathbb{Q}$ such that $(\forall i \in F)(p_i \leqslant p)$.

Claim 2.4 *Suppose* (CR, Σ) *is an* $\aleph_1$-CP. *Let* $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$. *Then* $\mathbb{Q}$ *has* $\aleph_1$ *as a precaliber.*

Proof Suppose $\{p_i : i < \omega_1\} \in [\mathbb{Q}]^{\aleph_1}$. The map $i \mapsto k(i) = \sup\left(\bigcup_{\mathfrak{c}\in p_i} \mathsf{dom}(\mathfrak{c}) \cap i\right)$ is regressive on S_0. Choose $X_1 \in [S_0]^{\aleph_1}$ and $k(\star) < \omega_1$ such that for every $i \in X_1$, $k(i) = k(\star)$ and for every $i < j$ in X_1, $\mathsf{dom}(p_i) \cap \mathsf{dom}(p_j) \subseteq k(\star)$. Using Definition 2.1 (D) (ii), by possibly extending each p_i, we can assume that for every $\mathfrak{c} \in p_i$, either $\mathsf{dom}(\mathfrak{c}) \subseteq k(\star)$ or $\inf(\mathsf{dom}(\mathfrak{c})) \geqslant k(\star)$. Since $\{\mathfrak{c} \in \mathsf{CR} : \mathsf{dom}(\mathfrak{c}) \subseteq k(\star)\}$ is countable, we can find $X \in [X_1]^{\aleph_1}$ such that for every $i \in X$, $\{\mathfrak{c} \in p_i : \mathsf{dom}(\mathfrak{c}) \subseteq k(\star)\}$ does not depend on $i \in X$. Now for any finite $F \subseteq X$, $\bigcup_{i\in F} p_i$ is a common extension of $\{p_i : i \in F\}$. ◻

Claim 2.5 *Suppose* (CR, Σ) *is an* $\aleph_1$-CP. *Let* $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$. *Let* $\langle p_i : i < \omega_1\rangle$ *be a sequence of conditions in* $\mathbb{Q}$ *such that for every* $i < j < \omega_1$, $\sup(\mathsf{dom}(p_i)) < \sup(\mathsf{dom}(p_j))$. *Then there exist* $X \in [\omega_1]^{\aleph_1}$, $\langle q_i : i \in X\rangle$, $m < n < \omega$ *such that for every* $i \in X$*:*

(a) $q_i \in \mathbb{Q}$, $q_i \geqslant p_i$ *and* $\mathsf{dom}(q_i) = \mathsf{dom}(p_i)$,
(b) $q_i = \{\mathfrak{c}_{i,k} : k < n\}$ *and for every* $k < n - 1$, $\mathsf{dom}(\mathfrak{c}_{i,k}) < \mathsf{dom}(\mathfrak{c}_{i,k+1})$,
(c) *for* $k < m$, $\mathfrak{c}_{i,k} = \mathfrak{c}_k$ *does not depend on* $i \in X$,
(d) *for every* $j < j'$ *in* X, $\mathsf{dom}(\mathfrak{c}_{j,n-1}) < \mathsf{dom}(\mathfrak{c}_{j',m})$ *and*
(e) $\mathsf{otp}(\mathsf{dom}(\mathfrak{c}_{i,k}))$ *does not depend on* $i \in X$.

Proof Just follow the argument in the proof of Claim 2.4 noting that $\mathsf{dom}(p_i)$'s are unbounded in ω_1. □

3 Countable joins

In the course of club preservation arguments, we would like to be able to form new creatures out of old ones in the following way. Suppose $\langle q_i : i \geqslant 1\rangle$ is a sequence of conditions in $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$ which forms a Δ-system of an appropriate kind — it satisfies clauses (b)–(e) in Claim 2.5. We would like to construct a new condition $q \in \mathbb{Q}$ such that $q \Vdash_{\mathbb{Q}}$ "$\lim_n |\{i < n : q_i \in G_{\mathbb{Q}}\}|/n = 1$ and $\{i < \omega : q_i \notin G_{\mathbb{Q}}\}$ is infinite". This will require us to add "countable joins" of certain sequences of creatures to CR. This section introduces the countable join construction.

Definition 3.1 For $\alpha < \omega_1$, we say that $(\mathsf{CR}_p, \Sigma_p)$ is a partial $\aleph_1$-CP at α if for some $\aleph_1$-CP (CR, Σ),

- $\mathsf{CR}_p = \mathsf{CR} \restriction \alpha = \{\mathfrak{c} \in \mathsf{CR} : \sup(\mathsf{dom}(\mathfrak{c})) < \alpha\}$, and
- $\Sigma_p = \Sigma \restriction \mathsf{CR}_p$.

Lemma 3.2 *Suppose* $\alpha < \omega_1$. *Then* $(\mathsf{CR}_\star, \Sigma_\star)$ *is a partial* $\aleph_1$-CP *at* α *iff* $(\mathsf{CR}_\star, \Sigma_\star)$ *satisfies all the clauses in Definition* 2.1 *when we replace* ω_1 *by* α *and for every* $\mathfrak{c} \in \mathsf{CR}_\star$, $\sup(\mathsf{dom}(\mathfrak{c})) < \alpha$.

Proof If $(\mathsf{CR}_\star, \Sigma_\star)$ is a partial $\aleph_1$-CP at α, then it is clear that it satisfies all the clauses in Definition 2.1 when we replace ω_1 by α. Now suppose $(\mathsf{CR}_\star, \Sigma_\star)$ satisfies all the clauses in Definition 2.1 when we replace ω_1 by α and for every $\mathfrak{c} \in \mathsf{CR}_\star$, $\sup(\mathsf{dom}(\mathfrak{c})) < \alpha$. Let CR be the set of creatures obtained by adding all finite creatures to $\mathsf{CR}_\star$ and closing it under finite joins. For $\mathfrak{c} \in \mathsf{CR}$, define $\Sigma(\mathfrak{c})$ as follows. If $\mathfrak{c} \in \mathsf{CR}_\star$, then $\Sigma(\mathfrak{c}) = \Sigma_\star(\mathfrak{c})$. If $\mathfrak{c}$ is finite, then $\Sigma(\mathfrak{c})$ is the set of all $\bar{\mathfrak{d}} = \langle \mathfrak{d}_k : k < n\rangle$ where each $\mathfrak{d}_k \in \mathsf{CR}$ and the join of $\bar{\mathfrak{d}}$ is $\mathfrak{c}$. If $\mathfrak{c}$ is neither finite nor in $\mathsf{CR}_\star$, then $\mathfrak{c}$ is the join of some $\mathfrak{c}_0 \in \mathsf{CR}_\star$ and a finite creature $\mathfrak{c}_1$ with $\mathsf{dom}(\mathfrak{c}_0) < \mathsf{dom}(\mathfrak{c}_1)$. In this case, define $\Sigma(\mathfrak{c}) = \{\bar{\mathfrak{d}} \cup \bar{\mathfrak{e}} : \bar{\mathfrak{d}} \in \Sigma_\star(\mathfrak{c}_0) \text{ and } \bar{\mathfrak{e}} \in \Sigma(\mathfrak{c}_1)\}$. Then (CR, Σ) witnesses that $(\mathsf{CR}_\star, \Sigma_\star)$ is a partial $\aleph_1$-CP at α. □

Definition 3.3 Suppose $k_\star \geqslant 1$, $\delta \in S_{k_\star}$, and $(\mathsf{CR}_p, \Sigma_p)$ is a partial $\aleph_1$-CP at δ. Suppose $m < n < \omega$ and $\bar{\mathfrak{d}}_i = \langle \mathfrak{d}_{i,k} : k < n\rangle$ satisfy the following for $1 \leqslant i < \omega$:

(a) $\mathfrak{d}_{i,k} \in \mathsf{CR}_p$.
(b) $\mathfrak{d}_{i,j} = \mathfrak{d}_j$ does not depend on i for $j < m$.
(c) $\mathsf{dom}(\mathfrak{d}_{i,k}) < \mathsf{dom}(\mathfrak{d}_{i,k+1})$.
(d) $\mathsf{dom}(\mathfrak{d}_{i,n-1}) < \mathsf{dom}(\mathfrak{d}_{i+1,m})$.
(e) $\mathsf{otp}(\mathsf{dom}(\mathfrak{d}_{i,k}))$ only depends on k.

(f) $W = \bigcup\{\mathrm{dom}(\mathfrak{d}_{i,k}) : 1 \leqslant i < \omega, k < n\}$ is unbounded in δ and has order type $\varepsilon + \omega^{j_\star}$ for some $\varepsilon < \omega_1$ and $1 \leqslant j_\star \leqslant k_\star$.

We say that $\langle \bar{\mathfrak{d}}_i : i \geqslant 1\rangle$ is a *joinable candidate* for $(\mathsf{CR}_p, \Sigma_p)$ at δ.

For each $N \geqslant 1$ where N is a power of 2, we define new creatures $\mathfrak{c}^\star_N = (\mathrm{dom}(\mathfrak{c}^\star_N), \mathrm{pos}(\mathfrak{c}^\star_N), f_{\mathfrak{c}^\star_N})$ and $\Sigma_\star(\mathfrak{c}^\star_N)$, as follows:

(1) $\mathrm{dom}(\mathfrak{c}^\star_1) = W$ and $\mathrm{dom}(\mathfrak{c}^\star_N) = \bigcup\{\mathrm{dom}(\mathfrak{d}_{i,k}) : N \leqslant i < \omega, m \leqslant k < n\}$ for $N \geqslant 2$.

(2) $f_{\mathfrak{c}^\star_1} = \bigcup\{f_{\mathfrak{d}_{i,k}} : 1 \leqslant i < \omega, k < n\}$ and $f_{\mathfrak{c}^\star_N} = \bigcup\{f_{\mathfrak{d}_{i,k}} : N \leqslant i < \omega, m \leqslant k < n\}$ for $N \geqslant 2$.

(3) $\Sigma_\star(\mathfrak{c}^?_1)$ is the smallest family satisfying the following:

  (i) $\langle \mathfrak{c}^\star_1\rangle \in \Sigma_\star(\mathfrak{c}^\star_1)$.

  (ii) Whenever $j > 1$ is a power of 2 and $\langle \mathfrak{d}'_{i,k} : i < j, m \leqslant k < n\rangle$, $\langle \bar{\mathfrak{f}}_{i,k} : i < j, m \leqslant k < n\rangle$ and $\langle \bar{\mathfrak{g}}_k : k < m\rangle$ satisfy (a)–(d) below, we have, under appropriate order

$$\bigcup\{\bar{\mathfrak{g}}_k : k < m\} \cup \bigcup\{\bar{\mathfrak{f}}_{i,k} : i < j, m \leqslant k < n\} \cup \{\mathfrak{c}^\star_j\} \in \Sigma_\star(\mathfrak{c}^\star_1).$$

   (a) $\mathfrak{d}'_{i,k} \in \mathsf{CR}_p$ and $\mathrm{dom}(\mathfrak{d}'_{i,k}) = \mathrm{dom}(\mathfrak{d}_{i,k})$.

   (b) $|\{i \in [j_1, j_2) : (\exists k \in [m, n))(\mathfrak{d}'_{i,k} \neq \mathfrak{d}_{i,k})\}| \leqslant (j_2 - j_1)/\log_2(j_1)$ for every $2 \leqslant j_1 < j_2 \leqslant j$ where j_1, j_2 are powers of 2.

   (c) $\bar{\mathfrak{f}}_{i,k} \in \Sigma(\mathfrak{d}'_{i,k})$.

   (d) $\bar{\mathfrak{g}}_k \in \Sigma(\mathfrak{d}_k)$.

(4) For $N \geqslant 2$, $\Sigma_\star(\mathfrak{c}^\star_N)$ is the smallest family satisfying the following:

  (i) $\langle \mathfrak{c}^\star_N\rangle \in \Sigma_\star(\mathfrak{c}^\star_N)$.

  (ii) Whenever $j > N$ is a power of 2 and $\langle \mathfrak{d}'_{i,k} : N \leqslant i < j, m \leqslant k < n\rangle$ and $\langle \bar{\mathfrak{f}}_{i,k} : N \leqslant i < j, m \leqslant k < n\rangle$ satisfy (a)–(c) below, we have, under appropriate order

$$\bigcup\{\bar{\mathfrak{f}}_{i,k} : i < j, m \leqslant k < n\} \cup \{\mathfrak{c}^\star_j\} \in \Sigma_\star(\mathfrak{c}^\star_N).$$

   (a) $\mathfrak{d}'_{i,k} \in \mathsf{CR}_p$ and $\mathrm{dom}(\mathfrak{d}'_{i,k}) = \mathrm{dom}(\mathfrak{d}_{i,k})$.

   (b) $|\{i \in [j_1, j_2) : (\exists k \in [m, n))(\mathfrak{d}'_{i,k} \neq \mathfrak{d}_{i,k})\}| \leqslant (j_2 - j_1)/\log_2(j_1)$ for every $N \leqslant j_1 < j_2 \leqslant j$ where j_1, j_2 are powers of 2.

   (c) $\bar{\mathfrak{f}}_{i,k} \in \Sigma(\mathfrak{d}'_{i,k})$.

(5) $\mathrm{pos}(\mathfrak{c}^\star_N) = \left\{\bigcup_{k<K} f_{\mathfrak{c}_k} : \langle \mathfrak{c}_k : k < K\rangle \in \Sigma_\star(\mathfrak{c}^\star_N)\right\}$.

Let $(\mathsf{CR}'_p, \Sigma'_p)$ be the partial $\aleph_1$-CP at $\delta+1$ such that $\mathsf{CR}'_p = \mathsf{CR}_p \cup \{\mathfrak{c}^\star_N : N \geqslant 1 \text{ is a power of } 2\}$ with $\mathrm{dom}(\mathfrak{c}^\star_N)$, $\mathrm{pos}(\mathfrak{c}^\star_N)$, and $f_{\mathfrak{c}^\star_N}$ as above, $\Sigma'_p \restriction \mathsf{CR}_p = \Sigma_p$ and $\Sigma'_p(\mathfrak{c}^\star_N) = \Sigma_\star(\mathfrak{c}^\star_N)$. We say that $(\mathsf{CR}'_p, \Sigma'_p)$ is the result of adding the countable join $\mathfrak{c}_1 = \bigoplus_{i\geqslant 1} \bar{\mathfrak{d}}_i$ of $\langle \bar{\mathfrak{d}}_i : i \geqslant 1\rangle$ to $(\Sigma_p, \mathsf{CR}_p)$.

Note that by Lemma 3.2, $(\mathsf{CR}'_p, \Sigma'_p)$ is indeed a partial $\aleph_1$-CP at $\delta+1$. The next lemma will play a key role in the proof of Theorem 5.4.

Lemma 3.4 *Let* $(\mathsf{CR}'_p, \Sigma'_p)$ *be as in Definition 3.3. Let* (CR, Σ) *be an* $\aleph_1$-CP *such that* $\mathsf{CR}'_p = \{\mathfrak{c} \in \mathsf{CR} : \mathsf{dom}(\mathfrak{c}) \subseteq \delta\}$ *and* $\Sigma'_p = \Sigma \restriction \mathsf{CR}'_p$. *Let* $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$, $p = \{\mathfrak{c}^\star_1 = \bigoplus_{i \geqslant 1} \bar{\mathfrak{d}}_i\}$ *and* $p_i = \{\mathfrak{d}_{i,k} : k < n\}$. *Then*

$$p \Vdash_{\mathbb{Q}} \lim_j \frac{|\{i < j : p_i \in G_{\mathbb{Q}}\}|}{j} = 1.$$

Proof It suffices to show that for every $q \geqslant p$ and $j_\star \geqslant 2^{10}$ there exists $r \geqslant q$ such that

$$r \Vdash_{\mathbb{Q}} \frac{|\{i < j_\star : p_i \in \mathbb{G}_{\mathbb{Q}}\}|}{j_\star} > 1 - \frac{8}{\log_2 j_\star}.$$

Since $q \geqslant p = \{\mathfrak{c}^\star_1\}$, we can find $r \geqslant q$ and $j_0 > j_\star$ such that j_0 is a power of 2 and

$$\bigcup \{\bar{\mathfrak{g}}_k : k < m\} \cup \bigcup \{\bar{\mathfrak{f}}_{i,k} : i < j_0, m \leqslant k < n\} \cup \{\mathfrak{c}^\star_{j_0}\} \subseteq r,$$

where $\langle \mathfrak{d}'_{i,k} : i < j_0, m \leqslant k < n\rangle$, $\langle \bar{\mathfrak{f}}_{i,k} : i < j_0, m \leqslant k < n\rangle$, and $\langle \bar{\mathfrak{g}}_k : k < m\rangle$ are as in Definition 3.3 (3) (ii).

Choose $N \geqslant 10$ such that $2^N \leqslant j_\star < 2^{N+1}$. Then r forces that

$$\begin{aligned}\frac{|\{i < j_\star : p_i \in G_{\mathbb{Q}}\}|}{j_\star} &\geqslant 1 - \sum_{1 \leqslant j < N} \frac{2^{j+1} - 2^j}{j j_\star} - \frac{2^{N+1} - 2^N}{N j_\star} \\ &\geqslant 1 - \sum_{1 \leqslant j < N} \frac{1}{j 2^{N-j}} - \frac{1}{N}.\end{aligned}$$

Since $\sum_{1 \leqslant j < N/2} \frac{1}{j2^{N-j}} \leqslant \frac{N}{2^{N/2}} \leqslant \frac{4}{N}$ (as $N \geqslant 10$) and $\sum_{N/2 \leqslant j < N} \frac{1}{j2^{N-j}} \leqslant \frac{2}{N}$, it follows that

$$r \Vdash_{\mathbb{Q}} \frac{|\{i < j_\star : p_i \in G_{\mathbb{Q}}\}|}{j_\star} \geqslant 1 - \left(\frac{4}{N} + \frac{2}{N} + \frac{1}{N}\right) > 1 - \frac{8}{N}. \qquad \square$$

Definition 3.5 (CR, Σ) is a *thin* $\aleph_1$-CP if (CR, Σ) is an $\aleph_1$-CP and there exist S and $\langle \mathfrak{c}_\delta : \delta \in S\rangle$ such that the following hold:

(a) $S \subseteq \bigcup_{k \geqslant 1} S_k$.
(b) $\mathfrak{c}_\delta \in \mathsf{CR}$.
(c) For every $k_\star \geqslant 1$ and $\delta \in S \cap S_{k_\star}$, letting $(\mathsf{CR}_P, \Sigma_p)$ be the partial $\aleph_1$-CP at δ satisfying $\mathsf{CR}_p = \mathsf{CR} \restriction \delta = \{\mathfrak{c} \in \mathsf{CR}: \sup(\mathsf{dom}(\mathfrak{c})) < \delta\}$ and $\Sigma_p = \Sigma \restriction \mathsf{CR}_p$, there exists a joinable candidate $\langle \bar{\mathfrak{d}}_i : i \geqslant 1\rangle$ for $(\mathsf{CR}_p, \Sigma_p)$ at δ such that
   (i) $\mathfrak{c}_\delta = \bigoplus_{i \geqslant 1} \bar{\mathfrak{d}}_i$ and
   (ii) $\mathsf{CR}'_p = \{\mathfrak{c} \in \mathsf{CR}: \mathsf{dom}(\mathfrak{c}) \subseteq \delta\}$ and $\Sigma'_p = \Sigma \restriction \mathsf{CR}'_p$ where $(\mathsf{CR}'_p, \Sigma'_p)$ is the result of adding $\bigoplus_{i \geqslant 1} \bar{\mathfrak{d}}_i$ to $(\mathsf{CR}_p, \Sigma_p)$.
(d) $\mathfrak{c} \in \mathsf{CR}$ iff $\mathfrak{c}$ is a finite join of $\{\mathfrak{d} \in \mathsf{CR}: \mathfrak{d}$ is finite$\} \cup \bigcup \{\Sigma(\mathfrak{c}_\delta) : \delta \in S\}$.

Claim 3.6 *Suppose* (CR, Σ) *is an* $\aleph_1$-CP *as witnessed by* S, $\langle \mathfrak{c}_\delta : \delta \in S\rangle$. *Suppose* $\mathfrak{c} \in \mathsf{CR}$, $k_\star \geqslant 1$, $\delta \in S_{k_\star}$, $\mathsf{dom}(\mathfrak{c})$ *is an unbounded subset of* δ. *Then there exist* $\overline{c} = \langle \mathfrak{c}_k : k \leqslant k_1\rangle \in \Sigma(\mathfrak{c})$ *and* $\overline{d} = \langle \mathfrak{d}_k : k \leqslant k_2\rangle \in \Sigma(\mathfrak{c}_\delta)$ *such that* $\mathfrak{c}_{k_1} = \mathfrak{d}_{k_2}$.

Proof Easily follows from Definition 3.5. □

We will later see (Lemma 4.4) that the forcing $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$ associated with a thin $\aleph_1$-CP (CR, Σ) destroys all old witnesses for $\clubsuit^1$. It would therefore follow, for example, that forcing with a finite support product/iteration of ω_2 such $\mathbb{Q}$'s will yield a model of $\neg\clubsuit^1$. Unfortunately, a finite support product/iteration of length ω_2 will always destroy $\clubsuit^{\mathrm{lim}}$ since we will be adding Cohen reals at each stage of cofinality $\aleph_0$. To overcome this issue, in the following section, we introduce the notion of a "guided product" (Definition 4.1).

4 Guided products

Definition 4.1 Suppose $\langle \mathbb{Q}_\alpha : \alpha < \omega_2\rangle$ and $\langle p^\star_\delta : \delta \in S^{\omega_2}_{\aleph_0}\rangle$ satisfy the following:

(i) $\mathbb{Q}_\alpha = \mathbb{Q}_{\mathsf{CR}_\alpha,\Sigma_\alpha}$ where $(\mathsf{CR}_\alpha, \Sigma_\alpha)$ is a thin $\aleph_1$-CP.
(ii) $p^\star_\delta$ is a function whose domain is a countable unbounded subset of δ and for every $\alpha \in \mathsf{dom}(p^\star_\delta)$, $p^\star_\delta(\alpha) \in \mathbb{Q}_\alpha$.

For $\gamma \leqslant \omega_2$, define a forcing $\mathbb{P}_\gamma$ as follows:

(1) $p \in \mathbb{P}_\gamma$ iff

  (a) p is a function, $\mathsf{dom}(p) \subseteq \gamma$ and $\mathsf{otp}(\mathsf{dom}(p)) < \omega^\omega$,
  (b) for every $\alpha \in \mathsf{dom}(p)$, $p(\alpha) \in \mathbb{Q}_\alpha$, and
  (c) for every $\delta \leqslant \gamma$ with $\mathsf{cf}(\delta) = \aleph_0$, if $\mathsf{dom}(p) \cap \delta$ is unbounded in δ, then for some $\eta < \delta$, $p \restriction (\eta, \delta) = p^\star_\delta \restriction (\eta, \delta)$.

(2) For $p, q \in \mathbb{P}_\gamma$, define $p \leqslant q$ iff $\mathsf{dom}(p) \subseteq \mathsf{dom}(q)$ and for every $\alpha \in \mathsf{dom}(p)$, $p(\alpha) \leqslant_{\mathbb{Q}_\alpha} q(\alpha)$.

We say that $\mathbb{P}_{\omega_2}$ is the product of $\langle \mathbb{Q}_\alpha : \alpha < \omega_2\rangle$ guided by $\langle p^\star_\delta : \delta \in S^{\omega_2}_{\aleph_0}\rangle$. Note that for $\mathsf{cf}(\gamma) = \aleph_1$, $\mathbb{P}_\gamma$ is completely determined by $\langle \mathbb{Q}_\alpha : \alpha < \gamma\rangle$ and $\langle p^\star_\delta : \delta < \gamma, \mathsf{cf}(\delta) = \aleph_0\rangle$.

Claim 4.2 *Let* $\langle \mathbb{Q}_\alpha : \alpha < \omega_2\rangle$, $\langle p^\star_\delta : \delta \in S^{\omega_2}_{\aleph_0}\rangle$, *and* $\mathbb{P}_\gamma$ *for* $\gamma \leqslant \omega_2$ *be as in Definition* 4.1. *Then the following hold:*

(a) $\mathbb{P}_{\gamma+1} = \mathbb{P}_\gamma \times \mathbb{Q}_\gamma$.
(b) *If* $\beta < \gamma \leqslant \omega_2$, *then* $\mathbb{P}_\beta \lessdot \mathbb{P}_\gamma$.
(c) $\mathbb{P}_\gamma$ *satisfies ccc.*

Proof (a) and (b) are obvious from the definition of $\mathbb{P}_\gamma$. To show (c), we will use the following.

Lemma 4.3 *Suppose* $\gamma \leqslant \omega_2$ *and* $\langle p_i : i < \omega_1\rangle$ *is a sequence of conditions in* $\mathbb{P}_\gamma$. *Then there exists* $X \in [\omega_1]^{\aleph_1}$ *and a finite* $F \subseteq \gamma$ *such that for every* $\alpha \in \gamma \setminus F$, *if there are* $i < j$ *in* X *such that* $\alpha \in \mathsf{dom}(p_i) \cap \mathsf{dom}(p_j)$, *then* $(\forall i \in X)(\alpha \in \mathsf{dom}(p_i)$ *and* $p_i(\alpha)$ *does not depend on* $i \in X)$.

Proof By induction on $\gamma \leqslant \omega_2$. If γ is a successor or $\gamma = \omega_2$, this is trivial.

Next suppose $\mathsf{cf}(\gamma) = \aleph_0$ and $\langle p_i : i < \omega_1 \rangle$ is a sequence of conditions in $\mathbb{P}_\gamma$. Let $\langle \gamma_n : n < \omega \rangle$ be increasing cofinal in γ. For each $i < \omega_1$, choose $n = n_i < \omega$ such that either $p_i \in \mathbb{P}_{\gamma_n}$ or $p_i \upharpoonright (\gamma_n, \gamma) = p^\star_\gamma \upharpoonright (\gamma_n, \gamma)$. Choose $Z \in [\omega_1]^{\aleph_1}$ and $n_\star < \omega$ such that $(\forall i \in Z)(n_i = n_\star)$. Apply the inductive hypothesis to $\langle p_i \upharpoonright \gamma_{n_\star} : i \in Z \rangle$ to get $Y \in [Z]^{\aleph_1}$ and a finite $F \subseteq \gamma_{n_\star}$ such that for every $\alpha \in \gamma_{n_\star} \setminus F$, if there are $i < j$ in Y such that $\alpha \in \mathsf{dom}(p_i) \cap \mathsf{dom}(p_j)$, then $(\forall i \in Y)(\alpha \in \mathsf{dom}(p_i)$ and $p_i(\alpha)$ does not depend on $i \in Z)$. Choose $X \in [Y]^{\aleph_1}$ such that either $(\forall i \in X)(\mathsf{dom}(p_i) \subseteq \gamma_{n_\star})$ or $(\forall i \in X)(p_i \upharpoonright (\gamma_{n_\star}, \gamma) = p^\star_\gamma \upharpoonright (\gamma_{n_\star}, \gamma)$. Then X, $F \cup \{\gamma_{n_\star}\}$ are as required.

Finally, suppose $\mathsf{cf}(\gamma) = \omega_1$ and $\langle p_i : i < \omega_1 \rangle$ is a sequence of conditions in $\mathbb{P}_\gamma$. We claim that there are $\gamma_\star < \gamma$ and $W \in [\omega_1]^{\aleph_1}$ such that $\langle \mathsf{dom}(p_i) \setminus \gamma_\star : i \in W \rangle$ is a sequence of pairwise disjoint sets. This suffices since we can then apply the inductive hypothesis to $\langle p_i \upharpoonright \gamma_\star : i \in W \rangle$ to get $X \in [W]^{\aleph_1}$ and a finite $F \subseteq \gamma_\star$ such that for every $\alpha \in \gamma_\star \setminus F$, if there are $i < j$ in X such that $\alpha \in \mathsf{dom}(p_i) \cap \mathsf{dom}(p_j)$, then $(\forall i \in X)(\alpha \in \mathsf{dom}(p_i)$ and $p_i(\alpha)$ does not depend on $i \in X)$. It follows that X, $F \cup \{\gamma_\star\}$ will be as required.

Fix a continuously increasing sequence $\langle \gamma(i) : i < \omega_1 \rangle$ cofinal in γ. Let $E = \{i \in \mathsf{Lim}(\omega_1) : (\exists j \in \mathsf{Lim}(\omega_1))(j > i$ and $\sup(\mathsf{dom}(p_j) \cap \gamma(i)) < \gamma(i))\}$. We claim that $W = E \setminus \mathsf{Lim}(\omega_1)$ is countable and therefore E contains a club. Suppose not and fix an increasing sequence $\langle i_\xi : \xi < \omega_1 \rangle$ in W. Choose $j \in \mathsf{Lim}(\omega_1)$ such that $j > i_\xi$ for every $\xi < \omega^\omega$. Then $\sup(\mathsf{dom}(p_j) \cap \gamma(i_\xi)) = \gamma(i_\xi)$ for every $\xi < \omega^\omega$. But this implies that $\mathsf{otp}(\mathsf{dom}(p_j)) \geqslant \omega^\omega$ which is impossible. Fix $h\colon E \to \omega_1$ such that for every $i \in E$, $h(i) \in \mathsf{Lim}(\omega_1)$, $h(i) > i$ and $\mathsf{dom}(p_{h(i)}) \cap \gamma(i)$ is bounded below $\gamma(i)$. Let $E_1 \subseteq E$ be a club such that for every $i < j$ in E, $h(i) < j$ and $\sup(\mathsf{dom}(p_{h(i)})) < \gamma(j)$. For each $i \in E_1$, fix $k(i) < i$ such that $\sup(\mathsf{dom}(p_{h(i)}) \cap \gamma(i)) < \gamma(k(i))$. As the map $i \mapsto k(i)$ is regressive on E_1, by Fodor's lemma, we can find a stationary $S \subseteq E_1$ and $i_\star < \min(S)$ such that for every $i \in S$, $\sup(\mathsf{dom}(p_{h(i)}) \cap \gamma(i)) < \gamma(i_\star)$. It follows that if $i < j$ are in S, then $\mathsf{dom}(p_{h(i)}) \cap \mathsf{dom}(p_{h(j)}) \subseteq \gamma(i_\star)$. So take $\gamma_\star = \gamma(i_\star)$ and $W = h[S]$. ■

Fix $\{p_i : i < \omega_1\} \subseteq \mathbb{P}_\gamma$. Fix X, F as in Lemma 4.3. Since F is finite and each $\mathbb{Q}_\alpha$ has $\aleph_1$ as a precaliber, the product of $\{\mathbb{Q}_\alpha : \alpha \in F\}$ is ccc. Choose $Y \in [X]^{\aleph_1}$ such that $\{p_i \upharpoonright F : i \in Y\}$ is centered. It follows that $\{p_i : i \in Y\}$ is also centered. Hence $\mathbb{P}_\gamma$ satisfies ccc. □

Lemma 4.4 *Let* $\langle \mathbb{Q}_\alpha : \alpha < \omega_2 \rangle$, $\langle p^\star_\delta : \delta \in S^{\omega_2}_{\aleph_0} \rangle$, *and* $\mathbb{P}_{\omega_2}$ *be as in Definition 4.1. Then* $V^{\mathbb{P}_{\omega_2}} \models \neg \clubsuit^1$.

Proof Towards a contradiction, suppose $p_0 \in \mathbb{P}_{\omega_2}$, $\langle \mathring{A}_\delta = \{\mathring{\alpha}_{\delta,n} : n < \omega\} : \delta \in \mathsf{Lim}(\omega_1) \rangle \in V^{\mathbb{P}_{\omega_2}}$ are such that $p_0 \Vdash$ "$(\forall \delta \in \mathsf{Lim}(\omega_1))(\{\mathring{\alpha}_{\delta,n} : n < \omega\}$ is increasing cofinal in $\delta)$ and $\langle \mathring{A}_\delta : \delta < \omega_1 \rangle$ is a $\clubsuit^1$ witnessing sequence". Since $\mathbb{P}_{\omega_2}$ satisfies ccc, we can find $\gamma < \omega_2$ such that $p_0 \in \mathbb{P}_\gamma$ and each $\mathring{\alpha}_{\delta,n}$ is a $\mathbb{P}_\gamma$-name.

Let $\mathring{X} = \{\alpha < \omega_1 : \mathring{f}_{\mathbb{Q}_\gamma}(\alpha) = 1\}$. Then $\mathring{X} \in V^{\mathbb{P}_{\gamma+1}}$ and $V^{\mathbb{P}_{\gamma+1}} \Vdash \mathring{X} \in [\omega_1]^{\aleph_1}$. So there exist $p_1 \in \mathbb{P}_\gamma$, $q \in \mathbb{Q}_\gamma$, $\delta \in \mathsf{Lim}(\omega_1)$, and $n_\star < \omega$ such that $p_1 \geqslant p_0$ and $(p_1, q) \Vdash_{\mathbb{P}_{\gamma+1}} (\forall n \geqslant n_\star)(\mathring{\alpha}_{\delta,n} \in \mathring{X})$. Note that we must have that $\mathsf{dom}(q) \cap \delta$ is unbounded in δ otherwise we can easily extend (p_1, q) to get a contradiction. By possibly extending q, by Definition 2.1 (D) (ii), we can assume that $q = \{\mathfrak{c}_k : k < K_\star\}$ where $\mathrm{dom}(\mathfrak{c}_k) < \mathrm{dom}(\mathfrak{c}_{k+1})$ for every $k < K_\star - 1$ and for some $K < K_\star$, $\mathrm{dom}(\mathfrak{c}_K)$ is an unbounded subset of δ. Let S_γ and $\langle \mathfrak{c}_{\gamma,\delta} : \delta \in S_\gamma \rangle$ witness that $(\mathrm{CR}_\gamma, \Sigma_\gamma)$ is a thin $\aleph_1$-CP. By Claim 3.6, we can further assume that $\mathfrak{c}_K = \mathfrak{c}'_{K'}$ for some $\langle \mathfrak{c}'_n : n \leqslant K' \rangle \in \Sigma(\mathfrak{c}_{\gamma,\delta})$.

Let $m < n < \omega$ and $\bar{\mathfrak{d}}_i = \langle \mathfrak{d}_{i,k} : k < n \rangle$ for $i \geqslant 1$ be as in Definition 3.3 and $\mathfrak{c}_{\gamma,\delta} = \bigoplus_{i \geqslant 1} \bar{\mathfrak{d}}_i$. Then as $\langle \mathfrak{c}'_n : n \leqslant K' \rangle \in \Sigma(\mathfrak{c}_{\gamma,\delta})$, we can find $N \geqslant 1$ a power of 2 such that $\mathfrak{c}_K = \mathfrak{c}'_{K'} = \mathfrak{c}^\star_N$ in the notation of Definition 3.3.

Choose $p_2 \in \mathbb{P}_\gamma$, $p_2 \geqslant p_1$, $n(1) > n_\star$, and $\alpha > \min(\mathrm{dom}(\mathfrak{c}^\star_N))$ such that $p_2 \Vdash_{\mathbb{P}_\gamma} \mathring{\alpha}_{\delta,n(1)} = \alpha$. We can assume that $\alpha \in \mathrm{dom}(\mathfrak{c}^\star_N)$ — otherwise letting $\mathfrak{c}_\star$ be a creature with domain $\{\alpha\}$ and $f_{\mathfrak{c}_\star}(\alpha) = 1$, we have $q' = q \cup \{\mathfrak{c}_\star\} \Vdash_{\mathbb{Q}_\gamma} \alpha \notin \mathring{X}$ so that (p_2, q') forces a contradiction. Choose $i(\star) \geqslant N$ and $m \leqslant k(\star) < n$ such that $\alpha \in \mathrm{dom}(\mathfrak{d}_{i(\star),k})$. Let $N_1 \geqslant N$ be the largest power of 2 such that $i(\star) \geqslant N_1$ and let $j > i(\star)$ be a power of 2. Choose a creature $\mathfrak{d}'_{i(\star),k(\star)}$ such that $\mathrm{dom}(\mathfrak{d}'_{i(\star),k(\star)}) = \mathrm{dom}(\mathfrak{d}_{i(\star),k(\star)})$ and $\bar{\mathfrak{f}} \in \Sigma(\mathfrak{d}'_{i(\star),k(\star)})$ such that for some finite $\mathfrak{c}_\star \in \bar{\mathfrak{f}}$, $\mathrm{dom}(\mathfrak{c}_\star) = \{\alpha\}$ and $f_{\mathfrak{c}_\star}(\alpha) = 0$. It follows that, under appropriate order

$$\left\{\mathfrak{d}_{i,k} : N \leqslant i < j, m \leqslant k < n, (i,k) \neq (i(\star), k(\star))\right\} \cup \bar{\mathfrak{f}} \cup \{\mathfrak{c}^\star_j\} \in \Sigma(\mathfrak{c}^\star_N).$$

Let $q' = (q \setminus \{\mathfrak{c}_K\}) \cup \{\mathfrak{d}_{i,k} : N \leqslant i < j, m \leqslant k < n, (i,k) \neq (i(\star), k(\star))\} \cup \bar{\mathfrak{f}} \cup \{\mathfrak{c}^\star_j\}$. Then $(p_2, q') \geqslant (p, q)$ and $q' \Vdash_{\mathbb{Q}_\gamma} \alpha \notin \mathring{X}$ — contradiction. □

We would now like to construct $\langle \mathbb{Q}_\alpha : \alpha < \omega_2 \rangle$, $\langle p^\star_\delta : \delta \in S^{\omega_1}_{\aleph_0} \rangle$, and $\mathbb{P}_{\omega_2}$ as in Definition 4.1 and a sequence $\overline{A} = \langle A_\delta : \delta \in \mathrm{Lim}(\omega_1) \rangle$ such that in $V^{\mathbb{P}_{\omega_2}} \Vdash \overline{A}$ is a $\clubsuit^{\lim}$-witnessing sequence. To motivate this construction, let us first consider a simpler situation where we want to find $\overline{A}$ and a thin $\aleph_1$-CP (CR, Σ) such that after forcing with $\mathbb{Q} = \mathbb{Q}_{\mathrm{CR},\Sigma}$, $\overline{A}$ remains a $\clubsuit^{\lim}$-sequence. It turns out that choosing $\overline{A}$ and $\mathbb{Q}$ generically is sufficient to guarantee this. More precisely, if we force using a product $\mathbb{R} = \mathbb{A} \times \mathbb{S}$ where $\mathbb{A}$ adds $\overline{A}$ and $\mathbb{S}$ adds (CR, Σ) both via countable approximations, then $V^{\mathbb{R} \star \mathbb{Q}} \Vdash \overline{A}$ is a $\clubsuit^{\lim}$-witnessing sequence. The preparatory forcing $\mathbb{R}$ in the next section does exactly this for adding $\overline{A}$ and the sequences $\langle \mathbb{Q}_\alpha : \alpha < \omega_2 \rangle$, $\langle p^\star_\delta : \delta \in S^{\omega_1}_{\aleph_0} \rangle$.

5 $\clubsuit^{\lim}$ and $\neg\clubsuit^1$

We define a preparatory forcing $\mathbb{R}$ which generically adds $\langle \mathbb{Q}_\alpha : \alpha < \omega_2 \rangle$ and $\langle p^\star_\delta : \delta \in S^{\omega_2}_{\aleph_0} \rangle$ satisfying Definition 4.1 (i)–(ii) using countable approximations. This ensures that the resulting guided product $\mathbb{P}_{\omega_2}$ preserves a $\clubsuit^{\lim}$ witnessing sequence $\overline{A}$ which is also added by $\mathbb{R}$ via countable approximations.

Definition 5.1 Let $\mathbb{R}$ be a forcing whose conditions are $r = (u_r, \delta_r, \langle \mathbb{Q}_{r,\alpha} : \alpha \in u_r \rangle, v_r, \langle p^\star_{r,\alpha} : \alpha \in v_r \rangle, \overline{A}_r)$ where

(a) $u_r \in [\omega_2]^{\leqslant \aleph_0}$, $\delta_r < \omega_1$,

(b) $\mathbb{Q}_{r,\alpha} = \bigcup_{\xi<\delta_r}(\mathbb{Q}_{\mathsf{CR}_{r,\alpha},\Sigma_{r,\alpha}} \restriction \xi)$ for some thin $\aleph_1$-CP $(\mathsf{CR}_{r,\alpha}, \Sigma_{r,\alpha})$ as witnessed by $(S_{r,\alpha}, \langle \mathfrak{c}_{r,\alpha,\delta} : \delta \in S_{r,\alpha}\rangle)$ — so only $S_{r,\alpha} \cap \delta_r$ and $\langle \mathfrak{c}_{r,\alpha,\delta} : \delta \in S_{r,\alpha} \cap \delta_r\rangle$ are relevant,

(c) $v_r \subseteq u_r \cap \mathsf{Lim}(\omega_1)$ and for every $\alpha \in v_r$, $u_r \cap \alpha$ is unbounded in α,

(d) $p^\star_{r,\alpha}$ is a function with domain an unbounded subset of $u_r \cap \alpha$ and for each $\xi \in \mathsf{dom}(p^\star_{r,\alpha})$, $p^\star_{r,\alpha}(\xi) \in \mathbb{Q}_{r,\alpha}$, and

(e) $\overline{A}_r = \langle A_{r,\gamma} : \gamma \in \mathsf{Lim}(\omega_1) \cap \delta_r\rangle$ where each $A_{r,\gamma}$ is an unbounded subset of γ of order type ω.

For $r, s \in \mathbb{R}$, define $r \leqslant s$ iff the following hold:

(i) $u_r \subseteq u_s$, $\delta_r \leqslant \delta_s$.

(ii) For every $\alpha \in u_r$, $S_{r,\alpha} \cap \delta_r = S_{s,\alpha} \cap \delta_r$, and $\mathfrak{c}_{r,\alpha,\delta} = \mathfrak{c}_{s,\alpha,\delta}$ for every $\delta \in S_r \cap \delta_r$. It follows that $\mathbb{Q}_{r,\alpha} \subseteq \mathbb{Q}_{s,\alpha}$ and for every $p \in \mathbb{Q}_{s,\alpha}$, if $\mathsf{dom}(p)$ is bounded below δ_r, then $p \in \mathbb{Q}_{r,\alpha}$.

(iii) $v_r \subseteq v_s$ and for every $\alpha \in v_r$, $p^\star_{s,\alpha} = p^\star_{r,\alpha}$.

(iv) $\overline{A}_r = \overline{A}_s \restriction (\mathsf{Lim}(\omega_1) \cap \delta_r)$.

Claim 5.2 *$\mathbb{R}$ is countably closed and hence it preserves stationary subsets of ω_1. Under CH, it satisfies $\aleph_2$-c.c. and therefore preserves all cofinalities.*

Proof It is clear that $\mathbb{R}$ is countably closed. Next let $\{r_i : i < \omega_2\} \subseteq \mathbb{R}$. Using CH, we can find $X_0 \in [\omega_2]^{\aleph_2}$ such that $\langle u_{r_i} : i \in X_0\rangle$ forms a Δ-system with root $u_\star$. By possibly extending each r_i, we can assume that $u_{r_i} \setminus u_\star \neq \varnothing$ for every $i \in X_0$. Choose $X \in [X_0]^{\aleph_2}$ such that the following hold:

(i) For every $i, j \in X$ with $i < j$, $\sup(u_\star) < \min(u_{r_i} \setminus u_\star) \leqslant \sup(u_{r_i} \setminus u_\star) < \inf(u_{r_j} \setminus u_\star)$.

(ii) $\langle v_{r_i} : i \in X\rangle$ forms a Δ-system with root $v_\star \subseteq u_\star$.

(iii) $\delta_{r_i} = \delta_\star$ does not depend on $i \in X$.

(iv) For every $\alpha \in u_\star$, $\mathbb{Q}_{r_i,\alpha} = \mathbb{Q}_\alpha$ does not depend on $i \in X$.

(v) For every $\alpha \in v_\star$, $p^\star_{r_i,\alpha} = p^\star_\alpha$ does not depend on $i \in X$.

(vi) $\overline{A}_{r_i} = \overline{A}_\star$ does not depend on $i \in X$.

For clauses (iv), (v) and (vi), we use CH. It is clear that any two conditions in $\{r_i : i \in X\}$ have a common extension. □

From now on we assume CH. The next claim is easily verified.

Claim 5.3 *Each of the following sets is dense in $\mathbb{R}$:*

(a) $\{r \in \mathbb{R} : \alpha \in u_r\}$ *for* $\alpha < \omega_2$.

(b) $\{r \in \mathbb{R} : \delta_r > \delta\}$ *for* $\delta < \omega_1$.

(c) $\{r \in \mathbb{R} : \delta \in v_r\}$ *for* $\delta \in S^{\omega_2}_{\aleph_0}$.

Let $G_{\mathbb{R}}$ be $\mathbb{R}$-generic over V. Work in $V_1 = V[G_{\mathbb{R}}]$. For $\alpha < \omega_2$, define $\mathbb{Q}_\alpha = \bigcup\{\mathbb{Q}_{r,\alpha} : r \in G_{\mathbb{R}}, \alpha \in u_r\}$. Note that for every $\alpha < \omega_2$, $S_\alpha = \bigcup\{S_{r,\alpha} \cap \delta_r : r \in G_{\mathbb{R}}, \alpha \in u_r\}$ is a stationary subset of $\bigcup_{k\geqslant 1} S_k$ and $V_1 \Vdash$ "$(\forall\alpha < \omega_2)(\mathbb{Q}_\alpha = \mathbb{Q}_{\mathsf{CR}_\alpha,\Sigma_\alpha}$ for some thin $\aleph_1$-CP $(\mathsf{CR}_\alpha, \Sigma_\alpha))$". For $\delta \in S^{\omega_2}_{\aleph_0}$, let $p^\star_\delta = p^\star_{r,\delta}$ for some $r \in G_{\mathbb{R}}$ with

$\delta \in v_r$. Let $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle = \bigcup\{\overline{A}_r : r \in G_{\mathbb{R}}\}$. Let $\{\alpha_{\delta,n} : n < \omega\}$ list A_δ in increasing order.

Let $\mathbb{P}_{\omega_2} \in V_1$ be the product of $\langle \mathbb{Q}_\alpha : \alpha < \omega_2\rangle$ guided by $\langle p^\star_\delta : \delta \in S^{\omega_2}_{\aleph_0}\rangle$. Note that, since $\mathbb{R}$ is countably closed, the set of conditions $(r, p) \in \mathbb{R}{\star}\mathbb{P}_{\omega_2}$ satisfying the following is dense in $\mathbb{R}{\star}\mathbb{P}_{\omega_2}$:

(a) p is an actual object.
(b) $\mathsf{dom}(p) \subseteq u_r$.
(c) $(\forall \alpha \in \mathsf{dom}(p))(p(\alpha) \in \mathbb{Q}_{r,\alpha})$.
(d) For every $\alpha < \omega_2$ of cofinality $\aleph_0$, if $\mathsf{dom}(p) \cap \alpha$ is unbounded in α, then $\alpha \in v_r$.

So we can assume that our conditions in $\mathbb{R}{\star}\mathbb{P}_{\omega_2}$ have this form.

Theorem 5.4 $V_1^{\mathbb{P}_{\omega_2}} \Vdash \clubsuit^{\mathrm{lim}} \wedge \neg \clubsuit^1$.

Proof That $V_1^{\mathbb{P}_{\omega_2}} \Vdash \neg\clubsuit^1$ follows from Lemma 4.4. We will show that $\overline{A}$ witnesses $\clubsuit^{\mathrm{lim}}$ in $V_1^{\mathbb{P}_{\omega_2}}$. Suppose $(r_\star, p_\star) \Vdash_{\mathbb{R}\star\mathbb{P}_{\omega_2}} \mathring{A} \in [\omega_1]^{\aleph_1}$. We will construct $(r, p) \geqslant (r_\star, p_\star)$ and $\delta < \omega_1$ such that

$$(r, p) \Vdash_{\mathbb{R}\star\mathbb{P}_{\omega_2}} \lim_n \frac{|\{k < n : \mathring{\alpha}_{\delta,k} \in \mathring{A}\}|}{n} = 1.$$

Choose $\langle (r_i, p_i, \gamma_i) : i < \omega_1\rangle$ such that the following hold:

(i) $(r_i, p_i) \geqslant (r_\star, p_\star)$.
(ii) For all $i < j < \omega_1$, $r_i \leqslant_{\mathbb{R}} r_j$, $\sup(u_{r_i}) < \sup(u_{r_j})$ and $i \leqslant \delta_{r_i} < \delta_{r_j}$.
(iii) For some $N < \omega$, for every $i < \omega_1$, $\sup\big(\bigcup_{j<i} \mathsf{dom}(p_j)\big) < \sup(\mathsf{dom}(p_i))$ and $\mathsf{otp}(\mathsf{dom}(p_i)) \leqslant \omega^N$.
(iv) For every $i < \omega_1$ $i \in u_i$ and for every $\alpha < \sup(u_{r_i})$, there exists $j \in (i, \omega_1)$ such that $\alpha \in u_{r_j}$. So $\bigcup_{i<\omega_1} u_{r_i} = \alpha_\star \in [\omega_1, \omega_2)$ and $\mathsf{cf}(\alpha_\star) = \aleph_1$.
(v) For every $\delta < \alpha_\star$ with $\mathsf{cf}(\delta) = \aleph_0$, there exists $i < \omega_1$ such that $\delta \in v_{r_i}$. Hence $\bigcup_{i<\omega_1} v_{r_i} = \{\delta < \alpha_\star : \mathsf{cf}(\delta) = \aleph_0\}$.
(vi) $\langle \gamma_i : i < \omega_1\rangle$ is a strictly increasing sequence in ω_1.
(vii) $(r_i, p_i) \Vdash \gamma_i \in \mathring{A}$.

Claim 5.5 *There exist $F \subseteq \omega_2$ finite and $X \in [\omega_1]^{\aleph_1}$ such that for every $\alpha \in \omega_2 \setminus F$, if $\alpha \in \mathsf{dom}(p_i) \cap \mathsf{dom}(p_j)$ for some $i < j$ in X, then $(\forall i \in X)(\alpha \in \mathsf{dom}(p_i)$ and $p_i(\alpha)$ does not depend on $i \in X)$.*

Proof For $\alpha < \alpha_\star$, let $\mathbb{Q}'_\alpha = \bigcup\{\mathbb{Q}_{r_i,\alpha} : i < \omega_1, \alpha \in u_{r_i}\}$. Then $\mathbb{Q}'_\alpha$ is a thin $\aleph_1$-CP. For $\delta < \alpha_\star$ with $\mathsf{cf}(\delta) = \aleph_0$, let $p^\star_\delta = p^\star_{r_i,\delta}$ where $i < \omega_1$ and $\delta \in v_{r_i}$. Let $\mathbb{P}_{\alpha_\star}$ be the countable support product of $\langle \mathbb{Q}'_\alpha : \alpha < \alpha_\star\rangle$ guided by $\langle p^\star_\delta : \delta < \alpha_\star, \mathsf{cf}(\delta) = \aleph_0\rangle$ so that each $p_i \in \mathbb{P}_{\alpha_\star}$. Now apply Lemma 4.3. ∎

By shrinking X and F, we can assume that for every $i \in X$, $F \subseteq \mathsf{dom}(p_i)$. Let $W = \bigcap_{i\in X}(\mathsf{dom}(p_i) \setminus F)$ and $Y_i = \mathsf{dom}(p_i) \setminus (F \cup W)$. Then $\langle Y_i : i \in X\rangle$ is a sequence of pairwise disjoint non-empty countable sets. By shrinking X, we can also assume that for every $i < j$ in X, $\sup(Y_i) < \min(Y_j)$ and $\mathsf{otp}(\mathsf{dom}(p_i))$ does not depend on $i \in X$.

By Claim 2.5, we can find $X_1 \in [X]^{\aleph_1}$ such that for every $\alpha \in F$ exactly one of the following holds:

(A) For every $i \in X_1$, $p_i(\alpha) = q_\alpha$ does not depend on i.
(B) There are $m = m_\alpha$, $n = n_\alpha$, $m < n < \omega$ and $\langle q_{i,\alpha} : i \in X_1\rangle$ such that for every $i \in X_1$:

  (i) $q_{i,\alpha} \in \mathbb{Q}_{r_i,\alpha}$, $\mathsf{dom}(q_{i,\alpha}) = \mathsf{dom}(p_i(\alpha))$ and $r_i \Vdash_{\mathbb{R}} p_i(\alpha) \leqslant_{\mathbb{Q}_\alpha} q_{i,\alpha}$,
  (ii) $q_{i,\alpha} = \{\mathfrak{d}_{i,\alpha,k} : k < n\}$ and for every $k < n-1$, $\mathsf{dom}(\mathfrak{d}_{i,\alpha,k}) < \mathsf{dom}(\mathfrak{d}_{i,\alpha,k+1})$,
  (iii) for every $k < m$, $\mathfrak{d}_{i,\alpha,k} = \mathfrak{d}_{\alpha,k}$ does not depend on $i \in X_1$,
  (iv) for every $j < j'$ in X, $\mathsf{dom}(\mathfrak{d}_{j,\alpha,n-1}) < \mathsf{dom}(\mathfrak{d}_{j',\alpha,m})$, and
  (v) $\mathsf{otp}(\mathfrak{d}_{i,\alpha,k}) = \theta_{\alpha,k}$ does not depend on $i \in X_1$ and $1 \leqslant k_\alpha < \omega$ is such that $\theta_{\alpha,k} < \omega^{k_\alpha}$.

Let F_0 be the set of $\alpha \in F$ for which case (A) holds and $F_1 = F \setminus F_0$.

By reindexing, we can assume that $X_1 = \omega_1$. Let $k_\star = \max(\{k_\alpha + 2 : \alpha \in F\})$. Put $Y = \bigcup_{i<\omega_1} Y_i$. Choose a club $E \subseteq \omega_1$ such that for every $\delta \in E$, the following hold:

(a) For every $i < \delta$, there exists $j < \delta$ such that $\sup(u_{r_i} \cap Y) < \sup(Y_j)$.
(b) $\sup(\{\delta_{r_i} : i < \delta\}) = \delta$.
(c) For every $\alpha \in F_1$, $\sup(\{\mathsf{dom}(q_i(\alpha)) : i < \delta\}) = \delta$.
(d) $\sup(\{\gamma_i : i < \delta\}) = \delta$.

Fix $\delta \in S_{k_\star} \cap E$ and let $\langle i(n) : n < \omega\rangle$ be increasing cofinal in δ. Let $\alpha_\star = \sup(\{Y_{i(n)} : n < \omega\})$. We can assume that $\alpha_\star \notin F \cup W$ — just pick a sufficiently large $\delta \in S_{k_\star} \cap E$. Define $r \in \mathbb{R}$ as follows:

(a) $u_r = \bigcup_{n<\omega} u_{r_{i(n)}} \cup \{\alpha_\star\}$, $\delta_r = \delta + 1$.
(b) For $\alpha \in u_r$, choose $\mathbb{Q}_{r,\alpha}$, $(\mathsf{CR}_{r,\alpha}, \Sigma_{r,\alpha})$ and $(S_{r,\alpha}, \langle \mathfrak{c}_{r,\alpha,\delta} : \delta \in S_{r,\alpha}\rangle)$ as follows:

  (i) If $\alpha \in u_r \setminus (F_1 \cup \{\alpha_\star\})$, choose a thin $\aleph_1$-CP $(\mathsf{CR}_{r,\alpha}, \Sigma_{r,\alpha})$ with witnessing pair $(S_{r,\alpha}, \langle \mathfrak{c}_{r,\alpha,\delta} : \delta \in S_{r,\alpha}\rangle)$ such that for every $n < \omega$, $S_{r,\alpha} \cap \delta_{i(n)} = S_{r_{i(n)},\alpha} \cap \delta_{i(n)}$ and $\mathfrak{c}_{r,\alpha,\delta} = \mathfrak{c}_{r_{i(n)},\alpha,\delta}$ for every $\delta \in S_{r,\alpha} \cap \delta_{i(n)}$. So $\bigcup_{n<\omega} \mathbb{Q}_{r_{i(n)},\alpha} \subseteq \mathbb{Q}_{r,\alpha} = \mathbb{Q}_{\mathsf{CR}_{r,\alpha},\Sigma_{r,\alpha}} \restriction \delta$.
  (ii) If $\alpha = \alpha_\star$, choose $\mathbb{Q}_{r,\alpha}$, $(\mathsf{CR}_{r,\alpha}, \Sigma_{r,\alpha})$ and $(S_{r,\alpha}, \langle \mathfrak{c}_{r,\alpha,\delta} : \delta \in S_{r,\alpha}\rangle)$ arbitrarily.
  (iii) If $\alpha \in F_1$, choose a thin $\aleph_1$-CP $(\mathsf{CR}_{r,\alpha}, \Sigma_{r,\alpha})$ with witnessing pair $(S_{r,\alpha}, \langle \mathfrak{c}_{r,\alpha,\delta} : \delta \in S_{r,\alpha}\rangle)$ such that for every $n < \omega$, $S_{r,\alpha} \cap \delta_{i(n)} = S_{r_{i(n)},\alpha} \cap \delta_{i(n)}$, $\mathfrak{c}_{r,\alpha,\delta} = \mathfrak{c}_{r_{i(n)},\alpha,\delta}$ for every $\delta \in S_{r,\alpha} \cap \delta_{i(n)}$, $\delta \in S_{r,\alpha}$ and $\mathfrak{c}_{r,\alpha,\delta} = \bigoplus_{n\geqslant 1} \langle \mathfrak{d}_{i(n),\alpha,k} : k < n_\alpha\rangle$ where $\langle \mathfrak{d}_{i(n),\alpha,k} : k < n_\alpha\rangle$ is from clause (B)(ii) above. Put $\mathbb{Q}_{r,\alpha} = \mathbb{Q}_{\mathsf{CR}_{r,\alpha},\Sigma_{r,\alpha}} \restriction \delta$.

(c) $v_r = \bigcup_{n<\omega} v_{r_{i(n)}} \cup \{\alpha_\star\}$.
(d) For $\alpha \in v_{r_{i(n)}}$, $p^\star_{r,\alpha} = p^\star_{r_{i(n)},\alpha}$, and $p^\star_{r,\alpha_\star} = \bigcup_{n<\omega} p_{i(n)} \restriction Y_{i(n)}$. So $\mathsf{dom}(p^\star_{r,\alpha_\star})$ is an unbounded subset of $u_r \cap \alpha_\star$.
(e) $\overline{A}_r = \bigcup_{n<\omega} \overline{A}_{r_{i(n)}} \cup \{(\delta, \{\gamma_{i(n)} : n < \omega\})\}$.

Next define p as follows:

  (i) $\mathsf{dom}(p) = F \cup W \cup \bigcup_{n<\omega} Y_n$.

(ii) If $\alpha \in F_0$, then $p(\alpha) = q_\alpha$ where q_α is from clause (B) above.
(iii) If $\alpha \in F_1$, then $p(\alpha) = \{\mathfrak{c}_{r,\alpha,\delta}\}$.
(iv) If $\alpha \in W$, then $p(\alpha) = p_{i(n)}(\alpha)$ which does not depend on $n < \omega$.
(v) For every $n < \omega$, $p \restriction Y_{i(n)} = p_{i(n)} \restriction Y_i$.

It is clear that $(r_\star, p_\star) \leqslant_{\mathbb{R}\star\mathbb{P}_{\omega_2}} (r, p)$. By Lemma 3.4,

$$(r, p) \Vdash_{\mathbb{R}\star\mathbb{P}_{\omega_2}} \lim_n \frac{|\{k < n : (r_{i(k)}, p_{i(k)}) \in G_{\mathbb{R}\star\mathbb{P}_{\omega_2}}\}|}{n} = 1.$$

Hence

$$(r, p) \Vdash_{\mathbb{R}\star\mathbb{P}_{\omega_2}} \lim_n \frac{|\{k < n : \gamma_{i(k)} \in \mathring{A}\}|}{n} = 1.$$

Since $A_{r,\delta} = \{\gamma_{i(n)} : n < \omega\}$, the result follows. □

6 On $\clubsuit^{\mathrm{inf}} \geqslant a$

Definition 6.1 For $a \in (0, 1]$, the principle $\clubsuit^{\mathrm{inf}>a-}$ says the following. There exists $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ such that each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ where $\alpha_{\delta,n}$'s are increasing cofinal in δ and for every $A \in [\omega_1]^{\aleph_1}$ and $b < a$, there exists some δ such that

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} \geqslant b.$$

Theorem 6.2 *Let $0 < a \leqslant 1$ and suppose for every $b < a$, $\clubsuit^{\mathrm{inf}\geqslant b}$ holds. Then $\clubsuit^{\mathrm{inf}>a-}$ holds.*

We need some lemmas.

Lemma 6.3 *Suppose $0 < a \leqslant 1$, $S \subseteq \mathsf{Lim}(\omega_1)$ is stationary, $B \in [\omega_1]^{\aleph_1}$, and $\overline{A} = \langle A_\delta : \delta \in S\rangle$ satisfy the following:*

(a) *Each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ where $\alpha_{\delta,n}$'s are increasing cofinal in δ.*
(b) *For every $A \in [B]^{\aleph_1}$, there exists $\delta \in S$ such that*

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} \geqslant a.$$

Then $\clubsuit_S^{\mathrm{inf}\geqslant a}$ holds.

Proof For each $A \in [B]^{\aleph_1}$, define $T(A)$ to be the set of $\delta \in S$ satisfying

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} \geqslant a.$$

Note that each $T(A)$ is stationary (see the proof of Fact 1.6) and $A_1 \subseteq A_2$ implies $T(A_1) \subseteq T(A_2)$. Put $T = T(B)$. Let $\{\alpha(\xi) : \xi < \omega_1\}$ list the members of B in increasing order. Let $E \subseteq \mathsf{Lim}(\omega_1)$ be a club such that for every $\delta \in E$ and $\xi < \delta$, $\xi \leqslant \alpha(\xi) < \delta$. Define $\overline{C} = \langle C_\delta : \delta \in T \rangle$ by $C_\delta = \{\xi : \alpha(\xi) \in A_\delta\}$. Note that $\sup(C_\delta) = \delta$ for every $\delta \in T$. We claim that $\overline{C}$ witnesses $\clubsuit_T^{\inf \geqslant a}$ and since $T \subseteq S$, $\clubsuit_S^{\inf \geqslant a}$ also holds. To see this, fix $W \in [\omega]^{\aleph_1}$ and put $A = \{\alpha(\xi) : \xi \in W\}$. Fix $\delta \in E \cap T(A)$. It suffices to show that

$$\liminf_{\eta \to \delta} \frac{|C_\delta \cap W \cap \eta|}{|C_\delta \cap \eta|} \geqslant a.$$

For each $\eta \in (\min(C_\delta), \delta)$, let $\beta_\eta = \max(\{\alpha(\xi) : \xi \in C_\delta \cap \eta\}) + 1$. Observe that $|C_\delta \cap W \cap \eta| \geqslant |A_\delta \cap A \cap \beta_\eta|$ and $|C_\delta \cap \eta| = |A_\delta \cap B \cap \beta_\eta| \leqslant |A_\delta \cap \beta_\eta|$. It follows that

$$\frac{|C_\delta \cap W \cap \eta|}{|C_\delta \cap \eta|} \geqslant \frac{|A_\delta \cap A \cap \beta_\eta|}{|A_\delta \cap \beta_\eta|}.$$

Taking lim inf as $\eta \to \delta$, the result follows. □

Lemma 6.4 *Suppose $\clubsuit_S^{\inf \geqslant a}$ holds. Then there exists a partition $\langle S_i : i < \omega_1 \rangle$ of S into stationary sets such that for every $i < \omega_1$, $\clubsuit_{S_i}^{\inf \geqslant a}$ holds.*

Proof Fix a witness $\overline{A} = \langle A_\delta : \delta \in S \rangle$ for $\clubsuit_S^{\inf \geqslant a}$ where each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ. Note that if $a \in (0.5, 1]$, this is easy — choose $\langle X_i : i < \omega_1 \rangle$ where X_i's are pairwise disjoint unbounded subsets of ω_1 and let

$$S_i = \left\{ \delta \in S : \liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in X_i\}|}{n} \geqslant a \right\}.$$

Since $a > 0.5$, S_i's are pairwise disjoint and by Fact 1.6, for every $Y \in [X_i]^{\aleph_1}$, there are stationary many $\delta \in S_i$ such that

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in Y\}|}{n} \geqslant a.$$

By Lemma 6.3, it follows that for every $i < \omega_1$, $\clubsuit_{S_i}^{\inf \geqslant a}$ holds.

In the general case, S_i's may not be pairwise disjoint but for any $F \in [\omega_1]^K$, where $Ka > 1$, we have $\bigcap_{i \in F} S_i = \varnothing$. For $Y \subseteq \omega_1$, let $S(Y)$ be the set of $\delta \in S$ such that

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in Y\}|}{n} \geqslant a.$$

Claim 6.5 *There exists $\langle Y_i : i \in W \rangle$ such that $W \in [\omega_1]^{\aleph_1}$, each $Y_i \in [X_i]^{\aleph_1}$ and for every $i \in W$ and $Z \in [Y_i]^{\aleph_1}$, $S(Z) \setminus \bigcup_{j \in W \cap i} S(Y_j)$ is stationary.*

Proof Let $\mathcal{F}$ be the set of $\overline{Y} = \langle Y_i : i \in W \rangle$ where $W \in [\omega_1]^{\aleph_1}$ and each $Y_i \in [X_i]^{\aleph_1}$. For $\overline{Y} = \langle Y_i : i \in W \rangle \in \mathcal{F}$, let $n(\overline{Y})$ be the least n such that for every $F \in [W]^n$, $\bigcap_{i \in F} S(Y_i)$ is non-stationary — so $2 \leqslant n(\overline{Y}) \leqslant K$. Let $N = \min\{n(\overline{Y}) : \overline{Y} \in \mathcal{F}\}$ and fix $\overline{Y} = \langle Y_i : i \in W \rangle$ with $n(\overline{Y}) = N$. It suffices to show that for every $i_\star \in W$, there exists $j \in W$ such that $j > i_\star$ and for every $Z \in [Y_j]^{\aleph_1}$, $S(Z) \setminus \bigcup \{S(Y_i) : i \leqslant i_\star, i \in W\}$ is stationary. Towards a contradiction, suppose this fails for some $i_\star \in W$. Let $W' = W \setminus (i_\star + 1)$. For each $j \in W'$, choose $Z_j \in [Y_j]^{\aleph_1}$ such that $S(Z_j) \setminus \bigcup \{S(Y_i) : i \leqslant i_\star, i \in W\}$ is non-stationary. Let $\overline{Z} = \langle Z_j : j \in W' \rangle$. Then $n(\overline{Z}) \geqslant N$, so we can find $F \in [W']^{N-1}$ and such that $\bigcap_{j \in F} S(Z_j)$ is stationary. It follows that there exists $i \in W$ such that $i \leqslant i_\star$ and $\bigcap_{j \in F} S(Z_j) \cap S(Y_i)$ is stationary. Hence $\bigcap_{j \in F \cup \{i_\star\}} S(Y_j)$ is also stationary: Contradiction. ■

Let $\langle Y_i : i \in W \rangle$ be as in Claim 6.5. For $i \in W$, let $T_i = S(Y_i) \setminus \bigcup_{j \in W \cap i} S(Y_j)$. Then $\{T_i : i \in W\}$ is a family of pairwise disjoint stationary sets and for every $Z \in [Y_i]^{\aleph_1}$, there are stationary many $\delta \in T_i$ such that

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in Z\}|}{n} \geqslant a.$$

By Lemma 6.3, it follows that $\clubsuit_{T_i}^{\inf \geqslant a}$ holds for every $i \in W$. This completes the proof of Lemma 6.4. □

Lemma 6.6 *Suppose* $\clubsuit_S^{\inf \geqslant a}$ *holds and* $S = S_1 \cup S_2$. *Then one of* $\clubsuit_{S_1}^{\inf \geqslant a}$, $\clubsuit_{S_2}^{\inf \geqslant a}$ *holds.*

Proof Fix a witness $\overline{A} = \langle A_\delta : \delta \in S \rangle$ for $\clubsuit_S^{\inf \geqslant a}$ where each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ. Suppose $\clubsuit_{S_1}^{\inf \geqslant a}$ fails and choose $A \in [\omega_1]^{\aleph_1}$ such that for every $\delta \in S_1$,

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} < a.$$

Since $\overline{A}$ is $\clubsuit_S^{\inf \geqslant a}$ witnessing sequence, it follows that for every $B \in [A]^{\aleph_1}$, there exists $\delta \in S_2$ such that

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in B\}|}{n} \geqslant a.$$

Now apply Lemma 6.3 to get $\clubsuit_{S_2}^{\inf \geqslant a}$. □

Proof of Theorem 6.2 Let $\langle a_n : n < \omega \rangle$ be an increasing sequence with $\lim_n a_n = a$. For each n, using Lemma 6.4, choose a sequence $\langle S_{n,i} : i < \omega_1 \rangle$ of pairwise disjoint stationary sets such that $\clubsuit_{S_{n,i}}^{\inf \geqslant a_n}$ holds. For $m < n < \omega$, define $W_{m,n} = \{i < \omega_1 : \clubsuit_{S_{m,i}}^{\inf \geqslant a_n} \text{ holds}\}$.

First suppose that for some $m < \omega$, there are infinitely many $n > m$ such that $W_{m,n}$ is infinite. Let $\langle n(k) : k < \omega \rangle$ list such n's in increasing order. Inductively choose

$i(k) \in W_{m,n(k)}$ such that $i(k)$'s are pairwise distinct and $\clubsuit^{\inf \geqslant a_{n(k)}}_{S_{m,i(k)}}$ holds. For each $k < \omega$, choose $\langle A_\delta : \delta \in S_{m,i(k)} \rangle$ witnessing $\clubsuit^{\inf \geqslant a_{n(k)}}_{S_{m,i(k)}}$. Put $T = \bigcup\{S_{m,i(k)} : k < \omega\}$. Then $\langle A_\delta : \delta \in T\rangle$ witnesses $\clubsuit^{\inf > a-}_T$. Since $T \subseteq S$, $\clubsuit^{\inf > a-}_S$ also holds.

So we can assume that there is no such m. Inductively choose a strictly increasing sequence $\langle m(k) : k < \omega\rangle$ such that for every $n \geqslant m(k+1)$, $W_{m(k),n}$ is finite. Let $W = \bigcup\{W_{m(j),m(k)} : j < k < \omega\}$ and choose $i > \sup(W)$. Put $T_k = S_{m(k),i} \setminus \bigcup_{l<k} S_{m(l),i}$ and $T'_k = S_{m(k),i} \setminus T_k$. Then T_k's are pairwise disjoint, $S_{m(k),i} = T_k \cup T'_k$ and by our choice of i, $\clubsuit^{\inf \geqslant a_{m(k)}}_{T'_k}$ does not hold. Hence, by Lemma 6.6, $\clubsuit^{\inf \geqslant a_{m(k)}}_{T_k}$ holds. Put $T = \bigcup\{T_k : k < \omega\}$. Since T_k's are pairwise disjoint, we can take the union of the witnesses for $\clubsuit^{\inf \geqslant a_{m(k)}}_{T_k}$'s to get a witness for $\clubsuit^{\inf > a-}_T$. As $T \subseteq S$, $\clubsuit^{\inf > a-}_S$ also holds. $\square$

Proof of Theorem 1.3 (2) Fix $0 < a < 1$. We indicate the essential changes in the proof of Theorem 1.3(1) to get a model of $\clubsuit^{\inf \geqslant a} \wedge (\forall b \in (a, 1]) \neg \clubsuit^{\inf \geqslant b}$. Define a modified countable join as follows. In Definition 3.3, replace Clause (3)(ii)(b) by ($\mathrm{b}_\star$) and Clause (4)(ii)(b) by ($\mathrm{b}_{\star\star}$) below.

($\mathrm{b}_\star$) $|\{i \in [2, j_1) : (\exists k \in [m, n))(\mathfrak{d}'_{i,k} \neq \mathfrak{d}_{i,k})\}| \leqslant j_1(1-a)$ for every $2 < j_1 \leqslant j$.

($\mathrm{b}_{\star\star}$) $|\{i \in [N, j_1) : (\exists k \in [m, n))(\mathfrak{d}'_{i,k} \neq \mathfrak{d}_{i,k})\}| \leqslant (j_1 - N)(1-a)$ for every $N < j_1 \leqslant j$.

Note that this gives rise to a transitive Σ'_p there. Lemma 3.4 gets modified to the following.

Lemma 6.7 *Let* $(\mathsf{CR}'_p, \Sigma'_p)$ *be as in Definition 3.3 with* ($\mathrm{b}_\star$) *in place of Clause* (3)(ii)(b) *and* ($\mathrm{b}_{\star\star}$) *in place of Clause* (4)(ii)(b). *Let* (CR, Σ) *be an* $\aleph_1$-CP *such that* $\mathsf{CR}'_p = \{\mathfrak{c} \in \mathsf{CR} : \mathrm{dom}(\mathfrak{c}) \subseteq \delta\}$ *and* $\Sigma'_p = \Sigma \restriction \mathsf{CR}'_p$. *Let* $\mathbb{Q} = \mathbb{Q}_{\mathsf{CR},\Sigma}$, $p = \{\mathfrak{c}^\star_1 = \bigoplus_{i \geqslant 1} \mathfrak{d}_i\}$ *and* $p_i = \{\mathfrak{d}_{i,k} : k < n\}$. *Then*

$$p \Vdash_{\mathbb{Q}} \liminf_j \frac{|\{i < j : p_i \in G_{\mathbb{Q}}\}|}{j} \geqslant a.$$

Next, Lemma 4.4 gets replaced by the following.

Lemma 6.8 *For every* $b \in (a, 1]$, $V^{\mathbb{P}_{\omega_2}} \Vdash \neg \clubsuit^{\inf \geqslant b}$.

Proof Fix $b' \in (a, 1]$. Towards a contradiction, suppose $p_0 \in \mathbb{P}_{\omega_2}$, $\langle \mathring{A}_\delta = \{\mathring{\alpha}_{\delta,n} : n < \omega\} : \delta \in \mathrm{Lim}(\omega_1)\rangle \in V^{\mathbb{P}_{\omega_2}}$ are such that $p_0 \Vdash$ "$(\forall \delta \in \mathrm{Lim}(\omega_1))(\{\mathring{\alpha}_{\delta,n} : n < \omega\}$ is increasing cofinal in δ) and $\langle \mathring{A}_\delta : \delta < \omega_1\rangle$ is a $\clubsuit^{\inf \geqslant b'}$ witnessing sequence". Since $\mathbb{P}_{\omega_2}$ satisfies ccc, we can find $\gamma < \omega_2$ such that $p_0 \in \mathbb{P}_\gamma$ and each $\mathring{\alpha}_{\delta,n}$ is a $\mathbb{P}_\gamma$-name. Fix $b \in (a, b')$.

Let $\mathring{X} = \{\alpha < \omega_1 : \mathring{f}_{\mathbb{Q}_\gamma} = 1\}$. Then $\mathring{X} \in V^{\mathbb{P}_{\gamma+1}}$ and $V^{\mathbb{P}_{\gamma+1}} \Vdash \mathring{X} \in [\omega_1]^{\aleph_1}$. So there exist $p_1 \in \mathbb{P}_\gamma$, $q \in \mathbb{Q}_\gamma$, $\delta \in \mathrm{Lim}(\omega_1)$, and $n_0 < \omega$ such that $p_1 \geqslant p_0$ and $(p_1, q) \Vdash_{\mathbb{P}_{\gamma+1}} (\forall j \geqslant n_0)(|\{i < j : \mathring{\alpha}_{\delta,i} \in \mathring{X}\}| \geqslant jb)$. We must have that $\mathrm{dom}(q) \cap \delta$ is unbounded in δ otherwise we can easily extend (p_1, q) to get a contradiction. By possibly extending q, by Definition 2.1 (D)(ii), we can assume that $q = \{\mathfrak{c}_k : k < K_\star\}$

where $\sup(\mathrm{dom}(\mathfrak{c}_k)) < \inf(\mathrm{dom}(\mathfrak{c}_{k+1}))$ for every $k < K_\star - 1$ and for some $K < K_\star$, $\mathrm{dom}(\mathfrak{c}_K)$ is an unbounded subset of δ. Let S_γ and $\langle \mathfrak{c}_{\gamma,\delta} : \delta \in S_\gamma \rangle$ witness that $(\mathsf{CR}_\gamma, \Sigma_\gamma)$ is a thin $\aleph_1$-CP. By Claim 3.6, we can further assume that $\mathfrak{c}_K = \mathfrak{c}'_{K'}$ for some $\langle \mathfrak{c}'_n : n \leqslant K' \rangle \in \Sigma(\mathfrak{c}_{\gamma,\delta})$.

Let $m < n < \omega$ and $\bar{\mathfrak{d}}_i = \langle \mathfrak{d}_{i,k} : k < n \rangle$ for $i \geqslant 1$ be as in Definition 3.3 and $\mathfrak{c}_{\gamma,\delta} = \bigoplus_{i \geqslant 1} \bar{\mathfrak{d}}_i$. Then as $\langle \mathfrak{c}'_n : n \leqslant K' \rangle \in \Sigma(\mathfrak{c}_{\gamma,\delta})$, we can find $N \geqslant 1$ a power of 2 such that $\mathfrak{c}_K = \mathfrak{c}'_{K'} = \mathfrak{c}^\star_N$ in the notation of Definition 3.3.

Choose $p_2 \in \mathbb{P}_\gamma$, $p_2 \geqslant p_1$, $n_\star > n_0$ a power of 2, and $\alpha_{n_\star} > \min(\mathrm{dom}(\mathfrak{c}^\star_N))$ such that $p_2 \Vdash_{\mathbb{P}_\gamma} \mathring{\alpha}_{\delta,n_\star} = \alpha_{n_\star}$. Put $c = (a+b)/2$. Let $n_{\star\star} > n_\star$ be a power of 2 such that $n_\star/n_{\star\star} < (b-c)/(1-c)$. Choose $p_3 \geqslant p_2$ and $\langle \alpha_n : n \in [n_\star, n_{\star\star}) \rangle$ such that for every $n \in [n_\star, n_{\star\star})$, $p_3 \Vdash_{\mathbb{P}_\gamma} \mathring{\alpha}_{\delta,n} = \alpha_n$. Let $F = \{\alpha_n \notin \mathrm{dom}(q) : n \in [n_\star, n_{\star\star})\}$. Let $q' = q \cup \bigcup_{\alpha \in F} \{\mathfrak{d}_\alpha\}$ where $\mathrm{dom}(\mathfrak{d}_\alpha) = \{\alpha\}$ and $f_{\mathfrak{d}_\alpha}(\alpha) = 0$. If F is empty, put $q' = q$.

Now it is possible to choose $\bar{\mathfrak{g}} \in \Sigma(\mathfrak{c}^\star_N)$ such that letting $q'' = (q' \setminus \{\mathfrak{c}^\star_N\}) \cup \bar{\mathfrak{g}}$ forces $\{n \in [n_\star, n_{\star\star}) : \alpha_n \notin \mathring{X}\} \geqslant (1-c)(n_{\star\star} - n_\star)$ — we leave the details of this to the reader. This means that (p_3, q'') forces that $|\{i < n_{\star\star} : \mathring{\alpha}_{\delta,i} \in \mathring{X}\}| \leqslant n_\star + c(n_{\star\star} - n_\star) < b n_{\star\star}$ which is a contradiction. ■

Now the remainder of the proof is exactly the same except for the fact that at the end of the proof of $\clubsuit^{\inf \geqslant a}$, we use Lemma 6.7 in place of Lemma 3.4. □

Proof of Theorem 1.3 (3) Let $\langle a_k : k \geqslant 1 \rangle$ be an increasing sequence with limit a. Proceed as in the proof of Theorem 1.3 (2) with the following modification for countable joins. In Definition 3.3, replace Clause (3) (ii) (b) by (b*) and Clause (4) (ii) (b) by (b**) below.

(b*) $|\{i \in [2, j_1) : (\exists k \in [m,n))(\mathfrak{d}'_{i,k} \neq \mathfrak{d}_{i,k})\}| \leqslant j_1(1 - a_{k_\star})$ for every $2 < j_1 \leqslant j$.

(b**) $|\{i \in [N, j_1) : (\exists k \in [m,n))(\mathfrak{d}'_{i,k} \neq \mathfrak{d}_{i,k})\}| \leqslant (j_1 - N)(1 - a_{k_\star})$ for every $N < j_1 \leqslant j$.

The rest of the proof is similar to that of Theorem 1.3 (2). We leave the details to the reader. □

7 On $\clubsuit^{\sup \geqslant a}$

Definition 7.1 For $a \in (0, 1]$ and $S \subseteq \mathrm{Lim}(\omega_1)$ stationary, the principle $\clubsuit_S^{\sup \geqslant a}$ says the following: There exists $\bar{A} = \langle A_\delta : \delta \in S \rangle$ such that

(a) each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ, and

(b) for every $A \in [\omega_1]^{\aleph_1}$, there exists $\delta \in S$ such that

$$\limsup_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} \geqslant a.$$

As usual, if $S = \mathrm{Lim}(\omega_1)$, we just write $\clubsuit^{\sup \geqslant a}$.

The following remark describes the situation in the Cohen and the random real models.

Remark 7.2 (1) Suppose $V \Vdash \clubsuit$ and let $\mathbb{P}$ be the forcing for adding $\aleph_2$ Cohen reals. Then $V^{\mathbb{P}} \Vdash \clubsuit^{\sup \geqslant 1} \wedge (\forall a > 0) \neg \clubsuit^{\inf \geqslant a}$. Moreover, the following holds in $V^{\mathbb{P}}$: For every $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ where each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ, there exists $A \in [\omega_1]^{\aleph_1}$ such that for every $\delta \in \mathsf{Lim}(\omega_1)$,

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} = 0.$$

(2) Suppose $V \Vdash \clubsuit$ and let $\mathbb{P}$ be the forcing for adding $\aleph_2$ random reals. Then $V^{\mathbb{P}} \Vdash (\forall a > 0)\neg\clubsuit^{\sup \geqslant a}$. Furthermore, the following holds in $V^{\mathbb{P}}$: There exists $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ where each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ such that for every $A \in [\omega_1]^{\aleph_1}$, there exists $\delta \in \mathsf{Lim}(\omega_1)$ such that

$$\limsup_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} > 0.$$

Proof (1) Fix a $\clubsuit$-witnessing sequence $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ in V where each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ. $\mathbb{P}$ is the set of all finite partial maps from ω_2 to 2 ordered by inclusion. We first check that $V^{\mathbb{P}} \Vdash \overline{A}$ is a $\clubsuit^{\sup \geqslant 1}$-witnessing sequence. Suppose $p \Vdash_{\mathbb{P}} \mathring{A} \in \bigcap [\omega_1]^{\aleph_1}$. It suffices to find $\delta \in \mathsf{Lim}(\omega_1)$ and $q \geqslant p$ such that

$$q \Vdash \limsup_n \frac{|\{k < n : \alpha_{\delta,k} \in \mathring{A}\}|}{n} = 1.$$

Choose $\langle (p_i, \gamma_i) : i < \omega_1\rangle$ such that γ_i's are strictly increasing and for every $i < \omega_1$, $p \leqslant p_i$ and $p_i \Vdash \gamma_i \in \mathring{A}$. Using the Δ-system lemma, choose $X \in [\omega_1]^{\aleph_1}$ and $R \in [\omega_2]^{<\aleph_0}$ such that $\langle \mathsf{dom}(p_i) : i \in X\rangle$ is a Δ-system with root R and $p_i \restriction R = q$ does not depend on $i \in X$. Clearly, $q \geqslant p$. Put $B = \{\gamma_i : i \in X\}$. Since $\overline{A}$ is a $\clubsuit$-witnessing sequence in V, there exists $\delta \in \mathsf{Lim}(\omega_1)$ such that $A_\delta \subseteq B$. We claim that q, δ are as required. Suppose not and fix $r \geqslant q$, $\varepsilon > 0$ rational and $N < \omega$ such that

$$r \Vdash (\forall n > N)\left(\frac{|\{k < n : \alpha_{\delta,k} \in \mathring{A}\}|}{n} < 1 - \varepsilon\right).$$

Note that r is compatible with all but finitely many conditions in $\{p_i : \gamma_i \in A_\delta\}$. Taking the union of r with a sufficiently large number of these conditions, we get an extension $s \geqslant r$ and $n_\star > N$ such that

$$s \Vdash \frac{|\{k < n_\star : \alpha_{\delta,k} \in \mathring{A}\}|}{n_\star} > 1 - \varepsilon$$

which is a contradiction. So $V^{\mathbb{P}} \Vdash \clubsuit^{\sup \geqslant 1}$. Next, we check $V^{\mathbb{P}} \Vdash (\forall a > 0)\neg\clubsuit^{\inf \geqslant a}$. Let $\mathbb{P}_\gamma$ be the poset whose conditions are finite partial maps from γ to 2. So $\mathbb{P} = \mathbb{P}_{\omega_2}$. Since each subset of ω_1 in $V^{\mathbb{P}}$ appears in $V^{\mathbb{P}_\gamma}$ for some $\gamma < \omega_2$, it suffices to show the following. Whenever $a > 0$ and $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ are in V, $V^{\mathbb{P}_{\omega_1}} \Vdash \overline{A}$ is not a $\clubsuit^{\inf \geqslant a}$-witnessing sequence. Let G be $\mathbb{P}_{\omega_1}$-generic over V. Put $g = \bigcup G$. Then

$g: \omega_1 \to 2$. Define $A = \{\alpha < \omega_1 : g(\alpha) = 1\}$. Then $A \in [\omega_1]^{\aleph_1}$. We claim that for every $\delta \in \mathsf{Lim}(\omega_1)$,

$$\liminf_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} = 0.$$

Note that this also gives us the "moreover part". Towards a contradiction, suppose this fails. Then there are $p \in G$, $\delta \in \mathsf{Lim}(\omega_1)$, $N < \omega$ and $b > 0$ such that

$$p \Vdash_{\mathbb{P}_{\omega_1}} (\forall n > N)\left(\frac{|\{k < n : \alpha_{\delta,k} \in \mathring{A}\}|}{n} \geqslant b\right).$$

Now since $\mathsf{dom}(p)$ is finite, we can easily find $q \geqslant p$ and $n_\star > N$ such that

$$q \Vdash_{\mathbb{P}_{\omega_1}} \frac{|\{k < n_\star : \alpha_{\delta,k} \in \mathring{A}\}|}{n} < \frac{b}{2}$$

which is a contradiction.

(2) Next let $\mathbb{P}$ be the forcing for adding $\aleph_2$ random reals. So $\mathbb{P} = \mathbb{P}_{\omega_2}$ where $\mathbb{P}_\gamma$ is the measure algebra on $(2^\gamma, \mu_\gamma)$ w.r.t. the standard product measure μ_γ. Note that every subset of ω_1 in $V^{\mathbb{P}}$ appears in $V^{\mathbb{P}_\gamma}$ for some $\gamma < \omega_2$. So to show that $V^{\mathbb{P}} \Vdash (\forall a > 0)(\neg \clubsuit^{\sup \geqslant a})$, it suffices to show the following. Whenever $a > 0$ and $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ are in V, $V^{\mathbb{P}_{\omega_1}} \Vdash \overline{A}$ is not a $\clubsuit^{\sup \geqslant a}$-witnessing sequence. Let G be $\mathbb{P}_{\omega_1}$-generic over V and $g \in 2^{\omega_1}$ be the generic random sequence added by G. Fix $N > 1/a$. In $V[G]$, define $A = \{\alpha < \omega_1 : (\forall n < N)(g(\omega\alpha_{\delta,k} + n) = 1)\}$. Then $A \in [\omega_1]^{\aleph_1}$. It suffices to show that for every $\delta \in \mathsf{Lim}(\omega_1)$, the asymptotic density of $\{k < \omega : \alpha_{\delta,k} \in A\}$ in ω is strictly less than a. Fix $\delta \in \mathsf{Lim}(\omega_1)$. For $x \in 2^{\omega_1}$, define

$$T_x = \{k < \omega : (\forall n < N)(x(\omega\alpha_{\delta,k} + n) = 1)\}.$$

As $A_\delta \in V$, $T_x \in V$. By the law of large numbers, for almost all $x \in 2^{\omega_1}$, the asymptotic density of T_x in ω is 2^{-N}. Since g is random over V, it follows that $\{k < \omega : \alpha_{\delta,k} \in A\} = T_g$ has asymptotic density $2^{-N} < 1/N < a$.

Finally, fix a $\clubsuit$-witnessing sequence $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$. Suppose $V^{\mathbb{P}} \Vdash \mathring{A} \in [\omega_1]^{\aleph_1}$. We will find $p \in \mathbb{P}$ and $\delta \in \mathsf{Lim}(\omega_1)$ such that

$$p \Vdash \limsup_n \frac{|\{k < n : \alpha_{\delta,k} \in \mathring{A}\}|}{n} > 0.$$

Choose $\langle (p_i, \gamma_i) : i < \omega_1\rangle$ such that γ_i's are strictly increasing and for every $i < \omega_1$, $p \leqslant p_i$ and $p_i \Vdash \gamma_i \in \mathring{A}$. Choose $X \in [\omega_1]^{\aleph_1}$ and $b > 0$ such that for every $i \in X$, $\mu(p_i) \geqslant b$. Choose $\delta \in \mathsf{Lim}(\omega_1)$ such that $A_\delta \subseteq \{\gamma_i : i \in X\}$. For each $n < \omega$, fix $i(n) \in X$ such that $\alpha_{\delta,n} = \gamma_{i(n)}$. Put $q_n = p_{\gamma_{i(n)}}$. Define $f_n = 1/n \sum_{k<n} 1_{q_k}$ where $1_{q_k} : 2^{\omega_2} \to 2$ is the characteristic function of q_k. Let $f = \limsup_n f_n$. Note that

$0 \leqslant f \leqslant 1$. By Fatou's lemma,

$$\int f = \int \limsup_n f_n \geqslant \limsup_n \int f_n \geqslant b.$$

Let $p = \{x \in 2^{\omega_2} : f(x) \geqslant b/2\}$. Then $\mu(p) \geqslant b/2$ otherwise

$$\int f = \int_p f + \int_{2^{\omega_2} \setminus p} f \leqslant \mu(p) + (1 - \mu(p)) \frac{b}{2} < \frac{b}{2} + \frac{b}{2} = b.$$

It is easy to see that p, δ are as required. This completes the proof of Remark 7.2. $\square$

We now prove Theorem 1.4 — for all $a, b \in (0, 1)$, $\clubsuit_S^{\sup \geqslant a}$ is equivalent to $\clubsuit_S^{\sup \geqslant b}$. For this, it is clearly enough to show the following.

Lemma 7.3 *Let $a \in (0, 1)$ and $a \leqslant b < \sqrt{a}$. Then $\clubsuit_S^{\sup \geqslant a}$ implies $\clubsuit_S^{\sup \geqslant b}$.*

Proof Let $\overline{A} = \langle A_\delta : \delta \in S \rangle$ witness $\clubsuit_S^{\sup \geqslant a}$. We can assume that $\overline{A}$ is not a $\clubsuit_S^{\sup \geqslant b}$-witnessing sequence. Choose $A \in [\omega_1]^{\aleph_1}$ such that for every $\delta \in S$, for every large enough $\alpha < \delta$,

$$\frac{|A \cap A_\delta \cap \alpha|}{|A_\delta \cap \alpha|} < b.$$

Let S' be the set of $\delta \in S$ such that

$$\limsup_{\alpha \to \delta} \frac{|A \cap A_\delta \cap \alpha|}{|A_\delta \cap \alpha|} \geqslant a.$$

Then S' is stationary. For $\delta \in S'$, define $B_\delta = A_\delta \cap A$.

Claim 7.4 *For every $B \in [A]^{\aleph_1}$ there are stationary many $\delta \in S'$ such that*

$$\limsup_{\alpha \to \delta} \frac{|B \cap B_\delta \cap \alpha|}{|B_\delta \cap \alpha|} \geqslant b.$$

Proof Suppose not. Choose $B \in [A]^{\aleph_1}$ and $W \subseteq S'$ non-stationary such that for every $\delta \in S' \setminus W$, for every large enough $\alpha < \delta$, we have

$$\frac{|B \cap B_\delta \cap \alpha|}{|B_\delta \cap \alpha|} < b.$$

Since $B \subseteq A$, we can choose $\delta \in S' \setminus W$ such that

$$\limsup_{\alpha \to \delta} \frac{|B \cap A_\delta \cap \alpha|}{|A_\delta \cap \alpha|} \geqslant a.$$

Now for every large enough $\alpha < \delta$, we have

$$\left(\frac{|B \cap B_\delta \cap \alpha|}{|B_\delta \cap \alpha|}\right)\left(\frac{|A \cap A_\delta \cap \alpha|}{|A_\delta \cap \alpha|}\right) < b^2.$$

Since $B \cap B_\delta = B \cap A_\delta$ and $B_\delta \cap \alpha = A \cap A_\delta \cap \alpha$, we get

$$\frac{B \cap A_\delta \cap \alpha}{A_\delta \cap \alpha} < b^2 < a$$

which is impossible. ■

Let $\{\alpha_i : i < \omega_1\}$ list A in increasing order. Let $E \subseteq \omega_1$ be a club such that for every $i \in E$, $\sup_{j<i} \alpha_j = i$. Define $\overline{C} = \langle C_\delta : \delta \in S \rangle$ as follows. If $\delta \in E \cap S'$, then $C_\delta = \{j < \delta : \alpha_j \in B_\delta\}$. Otherwise, choose C_δ to be an arbitrary unbounded subset of δ of order type ω. It is easy to check that $\overline{C}$ witnesses $\clubsuit_S^{\sup \geqslant b}$. □

8 $\neg \clubsuit^{\sup \geqslant 1}$ and $\clubsuit^{\sup > 1-}$

Definition 8.1 The principle $\clubsuit^{\sup>1-}$ says the following: There exists $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1) \rangle$ such that

(a) each $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ and $\alpha_{\delta,n}$'s are increasing cofinal in δ, and
(b) for every $A \in [\omega_1]^{\aleph_1}$ and $\varepsilon > 0$, there exists some δ such that

$$\limsup_n \frac{|\{k < n : \alpha_{\delta,k} \in A\}|}{n} \geqslant 1 - \varepsilon.$$

To prove Theorem 1.5, it is enough to show that

Theorem 8.2 $\neg \clubsuit^{\sup \geqslant 1} \wedge \clubsuit^{\sup>1-}$ *is consistent.*

Definition 8.3 Suppose $\overline{A} = \langle A_\delta : \delta \in \mathsf{Lim}(\omega_1) \rangle$ satisfies: For every δ, $A_\delta = \{\alpha_{\delta,n} : n < \omega\}$ where $\alpha_{\delta,n}$'s are increasing and cofinal in δ. Define $\mathbb{Q} = \mathbb{Q}_{\overline{A}}$ as follows: $p \in \mathbb{Q}$ iff $p = (f_p, u_p, \overline{\varepsilon}_p)$ where

(i) f_p is a finite partial function from ω_1 to $\{0, 1\}$,
(ii) u_p is a finite subset of $\mathsf{Lim}(\omega_1)$, and
(iii) $\overline{\varepsilon}_p = \langle \varepsilon_{p,\delta} : \delta \in u_p \rangle$ where each $\varepsilon_{p,\delta}$ is a positive rational < 1.

For $p, q \in \mathbb{Q}$ define $p \leqslant q$ iff

(a) $f_p \subseteq f_q$,
(b) $u_p \subseteq u_q$,
(c) $\overline{\varepsilon}_p = \overline{\varepsilon}_q \restriction u_p$, and
(d) for every $\delta \in u_p$, letting $W = \{n < \omega : \alpha_{\delta,n} \in \mathsf{dom}(f_q) \setminus \mathsf{dom}(f_p)\}$, for every $N < \omega$ either $W \cap [0, N) = \varnothing$ or

$$\frac{|\{n \in W \cap [0, N) : f_q(\alpha_{\delta,n}) = 1\}|}{|W \cap [0, N)|} \leqslant 1 - \varepsilon_{p,\delta}.$$

Claim 8.4 *Suppose p, q, r are in $\mathbb{Q}$ where $\mathbb{Q} = \mathbb{Q}_{\overline{A}}$ is as in Definition 8.3. If $p \leqslant q$ and $q \leqslant r$, then $p \leqslant r$.*

Proof Clauses (a)–(c) are clear. So we only need to check Clause (d) for p, r. Fix $\delta \in u_p$ and $N < \omega$. Put $\varepsilon = \varepsilon_{p,\delta} = \varepsilon_{q,\delta} = \varepsilon_{r,\delta}$. Let $W_{p,r} = \{n < N : \alpha_{\delta,n} \in \mathsf{dom}(f_r) \setminus \mathsf{dom}(f_p)\}$ and $W^1_{p,r} = \{n \in W_{p,r} : f_r(\alpha_{\delta,n}) = 1\}$. Define $W_{p,q}, W_{q,r}, W^1_{p,q}$ and $W^1_{q,r}$ analogously and note that $W_{p,r} = W_{p,q} \sqcup W_{q,r}$ and $W^1_{p,r} = W^1_{p,q} \sqcup W^1_{q,r}$. Now using Clause (d) for the pairs p, q and q, r we get

$$|W^1_{p,r}| = |W^1_{p,q}| + |W^1_{q,r}| \leqslant |W_{p,q}|(1 - \varepsilon) + |W_{q,r}|(1 - \varepsilon) = |W_{p,r}|(1 - \varepsilon_{p,\delta}).$$

Hence Clause (d) also holds for p, r. □

Claim 8.5 *Let $\overline{A}$ and $\mathbb{Q} = \mathbb{Q}_{\overline{A},a}$ be as in Definition 8.3. Then $\mathbb{Q}$ has $\aleph_1$ as a precaliber.*

Proof Suppose $\{p_i = (f_i, u_i, \overline{\varepsilon}_i) : i < \omega_1\} \subseteq \mathbb{Q}$. By thinning down we can assume the following:

(a) $\langle \mathsf{dom}(f_i) : i < \omega_1\rangle$ is a Δ-system with root R and $f_i \restriction R$ does not depend on i.
(b) $\langle u_i : i < \omega_1\rangle$ is a Δ-system with root $u_\star$ and $\overline{\varepsilon}_i \restriction u_\star$ does not depend on i.
(c) For every $i < j < \omega_1$ and $\delta \in u_i$, $\mathsf{dom}(f_j) \cap A_\delta \subseteq R$.

Let $E \subseteq \omega_1$ be a club such that for every $i \in E$, for every $j < i$, $\mathsf{dom}(f_j) \cup u_j \subseteq i$. Choose $S \subseteq E$ stationary such that for every $i \in S$, $\mathsf{dom}(f_i) \cap i = R$, $u_i \cap i = u_\star$ and $\bigcup\{A_\delta \cap i : \delta \in u_i, \delta > i\} = F$ where F does not depend on $i \in S$. Note that for every infinite $X \subseteq S$ and $i \in S$, if $i > \sup(X)$, then for all but finitely many $j \in X$, $\mathsf{dom}(f_j) \cap A_i \subseteq R$. Let $X \in [S]^{\aleph_1}$ be such that for every increasing sequence $\langle \alpha_n : n < \omega\rangle$ in X, $\sup_n \alpha_n \notin X$. Define $c \colon [X]^2 \to \{0, 1\}$ by $c(\{i, j\}) = 1$ iff $i < j$ and $A_j \cap \mathsf{dom}(f_i) \subseteq R$. By the Erdős–Dushnik–Miller theorem, either there exists $Y \in [X]^{\aleph_1}$ such that $c[[Y]^2] = \{1\}$ or there exists $Y' \subseteq X$ such that $\mathsf{otp}(Y') = \omega + 1$ and $c[[Y']^2] = \{0\}$. Since the latter is impossible, we can find $Y \in [X]^{\aleph_1}$ such that $c[[Y]^2] = \{1\}$. Hence

(d) For every $i \neq j$ in Y and $\delta \in u_j$, $\mathsf{dom}(f_i) \cap A_\delta \subseteq R$.

It follows that $\{p_i : i \in Y\}$ is centered. □

Let $\mathring{f}_{\mathbb{Q}} = \bigcup\{f_p : p \in G_{\mathbb{Q}}\}$. Then $\Vdash_{\mathbb{Q}} \mathring{f}_{\mathbb{Q}} : \omega_1 \to \{0, 1\}$. Let $\mathring{X}_{\mathbb{Q}} = \{\alpha < \omega_1 : \mathring{f}_{\mathbb{Q}}(\alpha) = 1\}$. Note that $\Vdash_{\mathbb{Q}} \mathring{X}_{\mathbb{Q}} \in [\omega_1]^{\aleph_1}$. To see this, suppose $p \in \mathbb{P}$ and $\alpha < \omega_1$. Choose $\beta > \max(u_p \cup \{\alpha\})$ and define q by $f_q = f_p \cup \{(\beta, 1)\}$, $u_q = u_p$ and $\overline{\varepsilon}_q = \overline{\varepsilon}_p$. Then $q \geqslant p$ and $q \Vdash \beta \in \mathring{X}_{\mathbb{Q}}$.

Claim 8.6 *$\mathring{X}_{\mathbb{Q}}$ witnesses that $\overline{A}$ is not a $\clubsuit^{\sup \geqslant 1}$ witnessing sequence in $V^{\mathbb{Q}}$.*

Proof Easy. □

Claim 8.7 *Suppose $V \Vdash \clubsuit^{\sup > 1-}$ holds and let $\overline{C} = \langle C_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ be a witness where $C_\delta = \{\beta_{\delta,n} : n < \omega\}$ and $\beta_{\delta,n}$'s are increasing cofinal in δ. Then $V^{\mathbb{Q}} \Vdash \clubsuit^{\sup > 1-}$ holds with $\overline{C}$ as witness.*

Proof Suppose $p \Vdash_{\mathbb{Q}} \mathring{A} \in [\omega_1]^{\aleph_1}$ and $\varepsilon > 0$ is rational. Choose $\langle (p_i, \gamma_i) : i < \omega_1 \rangle$ such that γ_i's are increasing and for each $i < \omega_1$, $p \leqslant p_i \Vdash_{\mathbb{Q}} \gamma_i \in \mathring{A}$. Arguing as in the proof of Claim 8.5, we can assume the following:

(a) $\langle \mathsf{dom}(f_i) : i < \omega_1 \rangle$ is a Δ-system with root R, $f_i \restriction R = f_\star$ and $|\mathsf{dom}(f_i) \setminus R| = n_\star$ do not depend on i.
(b) If $i < j$, then $R < \mathsf{dom}(f_i) \setminus R < \mathsf{dom}(f_j) \setminus R$.
(c) $\langle u_i : i < \omega_1 \rangle$ is a Δ-system with root $u_\star$, $\overline{\varepsilon}_i \restriction u_\star = \overline{e}_\star$ does not depend on i and $i < j$ implies $u_i \setminus u_\star < u_j \setminus u_\star$.
(d) For every $i \neq j$ and $\delta \in u_i$, $\mathsf{dom}(f_j) \cap A_\delta \subseteq R$.

Put $X = \{\gamma_i : i < \omega_1\}$. Let $E \subseteq \omega_1$ be a club such that for every $i \in E$ and $j < i$, $\gamma_j < i$ and $u_\star \cup \mathsf{dom}(f_j) \subseteq i$. Choose $\delta \in E$ such that

$$\limsup_n \frac{|\{k < n : \beta_{\delta,k} \in X\}|}{n} \geqslant 1 - \frac{\varepsilon}{10}.$$

Let $q = (f_\star, u_\star \cup \{\delta\}, \overline{\varepsilon}_\star \cup \{(\delta, \varepsilon/5)\})$. It suffices to show that for any $q_1 \geqslant q$ and $N_0 < \omega$, there exist $r \geqslant q_1$ and $N_2 > N_0$ such that

$$r \Vdash_{\mathbb{Q}} \frac{|\{n < N_2 : \beta_{\delta,n} \in \mathring{A}\}|}{N_2} \geqslant 1 - \varepsilon.$$

So fix $q_1 \geqslant q$ and $N_0 < \omega$. For each $n < \omega$, define

$$r_n = \begin{cases} p_i & \text{if } \beta_{\delta,n} = \gamma_i, \\ q & \text{if } \beta_{\delta,n} \notin X. \end{cases}$$

Let $W'_n = \mathsf{dom}(f_{r_n}) \setminus R$ and $W_n = W'_n \cap A_\delta$. Choose $N_1 > N_0$ such that for every $n \geqslant N_1$, if $\delta' \in u_{q_1} \setminus \{\delta\}$, then $W'_n \cap A_{\delta'} = \phi$. We need a lemma.

Lemma 8.8 *Suppose* $0 < a_1 < a_2 < 1$ *and* $1 \leqslant K < \omega$. *Then for all sufficiently large* $N < \omega$, *the following holds. For every* $\langle W_k : k < N \rangle$ *where each* W_k *is an interval in* ω *such that* $|W_k| \leqslant K$, $W_k < W_{k+1}$ *and* $\bigcup_{k<n} W_k = [0, M)$, *there exists* $F \subseteq N$ *such that*

(i) $|F| \geqslant N a_1$, *and*
(ii) *for every* $m \leqslant M$, $\left|[0, m) \cap \bigcup_{k \in F} W_k\right| \leqslant m a_2$.

Proof First assume that $|W_k| = K$ for every $k < N$ — so $M = NK$. Let $m_1 < N$ be least such that $K m_1 \geqslant M(1 - a_2)$. Then $F = [m_1, N)$ is as required. For the general case, for each $K' \leqslant K$, put $S_{K'} = \{k < N : |W_k| = K'\}$ and find a suitable $F_{K'} \subseteq S_{K'}$ for $\langle W_k : k \in S_{K'} \rangle$. Then $F = \bigcup \{F_{K'} : 1 \leqslant K' \leqslant K\}$ is as required. □

Choose $N_2 > N_1$ such that $(1 - N_1/N_2)(1 - \varepsilon/2) \geqslant 1 - \varepsilon$ and $|\{k \in [N_1, N_2) : \beta_{\delta,k} \in X\}| \geqslant (1 - \varepsilon/4)(N_2 - N_1)$. Using Lemma 8.8, choose $F \subseteq [N_1, N_2)$ such that the following hold:

(a) $|F| \geqslant (N_2 - N_1)(1 - \varepsilon/4)$.

(b) $r = (f_r, u_r, \overline{\varepsilon}_r)$ extends each condition in $\{q_1, r_n : n \in F\}$ where

(i) $u_r = u_{q_1} \cup \bigcup_{n \in F} u_{r_n}$,
(ii) $\mathsf{dom}(f_r) = \mathsf{dom}(f_{q_1}) \cup \bigcup_{n \in F} W'_n \cup \bigcup\{W_n : n \in [N_1, N_2) \setminus F\}$,
(iii) $f_{q_1} \subseteq f_r$,
(iv) $f_r \restriction \bigcup\{W_n : n \in [N_1, N_2) \setminus F\} \equiv 0$,
(v) for every $n \in F$, $f_r \restriction W'_n = f_{r_n}$, and
(vi) $\overline{\varepsilon}_r = \overline{\varepsilon}_{q_1} \cup \bigcup_{n \in F} \overline{\varepsilon}_{r_n}$.

Note that $r \Vdash |\{k < N_2 : \beta_{\delta,k} \in \mathring{A}\}| \geqslant (N_2 - N_1)(1 - \varepsilon/2)$. By our choice of N_2, it follows that

$$r \Vdash_{\mathbb{Q}} \frac{|\{n < N_2 : \beta_{\delta,n} \in \mathring{A}\}|}{N_2} \geqslant 1 - \varepsilon. \qquad \square$$

Let $\eta \geqslant 1$ and suppose $\langle(\mathbb{P}_\xi, \mathbb{Q}_\xi, \overline{A}_\xi) : \xi < \eta\rangle$ satisfies the following:

(1) $\langle(\mathbb{P}_\xi, \mathbb{Q}_\xi) : \xi < \eta\rangle$ is a finite support iteration with limit $\mathbb{P}_\eta$.
(2) $\overline{A}_\xi \in V^{\mathbb{P}_\xi}$ and $\Vdash_{\mathbb{P}_\xi}$ "$\overline{A}_\xi = \langle A_{\xi,\delta} : \delta \in \mathsf{Lim}(\omega_1)\rangle$, $A_{\xi,\delta} = \{\alpha_{\xi,\delta,n} : n < \omega\}$ where $\alpha_{\xi,\delta,n}$'s are increasing cofinal in δ".
(3) $V^{\mathbb{P}_\xi} \Vdash \mathbb{Q}_\xi = \mathbb{Q}_{\overline{A}_\xi}$.

Note that $\mathbb{P}_\eta$ is ccc.

Claim 8.9 *Suppose $V \Vdash \clubsuit^{\sup > 1-}$ holds and let $\overline{C} = \langle C_\delta : \delta \in \mathsf{Lim}(\omega_1)\rangle$ be a witness where $C_\delta = \{\beta_{\delta,n} : n < \omega\}$ and $\beta_{\delta,n}$'s are increasing cofinal in δ. Then $V^{\mathbb{P}_\eta} \Vdash \clubsuit^{\sup \geqslant 1-}$ via the same witness.*

Proof By induction on η. If η is a successor or $\mathsf{cf}(\eta) > \aleph_1$, this follows from Claim 8.7.

Suppose $\mathsf{cf}(\eta) = \aleph_0$. Let $\langle\eta(n) : n < \omega\rangle$ be increasing cofinal in η. Suppose $p \Vdash_{\mathbb{P}_\eta} \mathring{X} \in [\omega_1]^{\aleph_1}$. Choose $n_\star < \omega$ such that $p \in \mathbb{P}_{\eta(n_\star)}$. For each $n < \omega$, let $\mathring{X}_n = \{\alpha < \omega_1 : (\exists p \in G_{\mathbb{P}_{\eta(n)}})(p \Vdash_{\mathbb{P}_\eta} \alpha \in \mathring{X})\}$ — so $\mathring{X}_n \in V^{\mathbb{P}_{\eta(n)}}$ and $\Vdash_{\mathbb{P}_\eta} \mathring{X}_n \subseteq \mathring{X}$. Then for some $n \in [n_\star, \omega)$, $p \Vdash_{\mathbb{P}_{\eta(n)}} \mathring{X}_n \in [\omega_1]^{\aleph_1}$. Now apply the inductive hypothesis.

Next suppose $\mathsf{cf}(\eta) = \aleph_1$, $\varepsilon > 0$, and $p \Vdash_{\mathbb{P}_\eta} \mathring{X} \in [\omega_1]^{\aleph_1}$. Choose $\langle(p_i, \gamma_i) : i < \omega_1\rangle$ such that the following hold:

(a) γ_i's are increasing.
(b) $p_i \in \mathbb{P}_\eta$, $p_i \geqslant p$ and $p_i \Vdash_{\mathbb{P}_\eta} \gamma_i \in \mathring{X}$.
(c) $\langle\mathsf{dom}(p_i) : i < \omega_1\rangle$ is a Δ-system with root W.

Choose $\theta < \eta$ such that $W \subseteq \theta$. Since $\mathbb{P}_\theta$ is ccc, we can find $q \in \mathbb{P}_\theta$ such that $q \geqslant p$ and $q \Vdash_{\mathbb{P}_\theta}$ "$\{i < \omega_1 : p_i \restriction \theta \in G_{\mathbb{P}_\theta}\}$ is uncountable". Let $\mathring{Y} = \{\gamma_i : i < \omega_1 \wedge p_i \restriction \theta \in G_{\mathbb{P}_\theta}\}$. Then $\mathring{Y} \in V^{\mathbb{P}_\theta}$ and $q \Vdash_{\mathbb{P}_\theta} \mathring{Y} \in [\omega_1]^{\aleph_1}$. By the inductive hypothesis, we can find $r \in \mathbb{P}_\theta$ and $\delta \in \mathsf{Lim}(\omega_1)$ such that $r \geqslant q$ and

$$r \Vdash_{\mathbb{P}_\theta} \limsup_n \frac{|\{k < n : \beta_{\delta,k} \in \mathring{Y}\}|}{n} \geqslant 1 - \frac{\varepsilon}{2}.$$

Sh:1136

Since $\langle \mathrm{dom}(p_i) \setminus \theta : i < \omega_1 \rangle$ is a sequence of pairwise disjoint sets, it also follows that

$$r \Vdash_{\mathbb{P}_\eta} \limsup_n \frac{|\{k < n : \beta_{\delta,k} \in \mathring{X}\}|}{n} \geqslant 1 - \varepsilon. \qquad \square$$

Proof of Theorem 8.2 Starting with a model of $2^{\aleph_1} = \aleph_2$ and $\clubsuit^{\sup > 1-}$ construct $\langle (\mathbb{P}_\xi, \mathbb{Q}_\xi, \overline{A}_\xi) : \xi < \omega_2 \rangle$ such that the following hold:

(1) $\langle (\mathbb{P}_\xi, \mathbb{Q}_\xi) : \xi < \omega_2 \rangle$ is a finite support iteration with limit $\mathbb{P}_{\omega_2}$.
(2) $\overline{A}_\xi \in V^{\mathbb{P}_\xi}$ and $\Vdash_{\mathbb{P}_\xi}$ "$\overline{A}_\xi = \langle A_{\xi,\delta} : \delta \in \mathrm{Lim}(\omega_1) \rangle$, $A_{\xi,\delta} = \{\alpha_{\xi,\delta,n} : n < \omega\}$ where $\alpha_{\xi,\delta,n}$'s are increasing cofinal in δ".
(3) $V^{\mathbb{P}_\xi} \Vdash \mathbb{Q}_\xi = \mathbb{Q}_{\overline{A}_\xi}$.
(4) For every $\eta < \omega_2$ and $\overline{A} \in V^{\mathbb{P}_\eta}$ satisfying $\Vdash_{\mathbb{P}_\eta}$ "$\overline{A} = \langle A_\delta : \delta \in \mathrm{Lim}(\omega_1) \rangle$ where each A_δ is an unbounded subset of δ of order type ω", there exists $\xi \in [\eta, \omega_2)$ such that $\Vdash_{\mathbb{P}_\xi} \overline{A} = \overline{A}_\xi$.

To see why clause (4) can be satisfied, use $2^{\aleph_1} = \aleph_2$ and the fact that for each $\eta < \omega_2$, $\mathbb{P}_\eta$ is a ccc forcing with a dense subset of size $\aleph_1$. $\square$

We conclude with some questions.

Question 8.10 (1) Is $\clubsuit^{\sup \geqslant 0.5} \wedge \neg \clubsuit^{\sup > 1-}$ consistent? What if CH holds?
(2) Assume CH. Does $\clubsuit^{\sup \geqslant 0.5}$ imply $\clubsuit^{\sup \geqslant 1}$? Does $\clubsuit^{\sup > 1-}$ imply $\clubsuit^{\sup \geqslant 1}$?
(3) For $a \in (0, 1)$, is $\clubsuit^{\inf \geqslant a} \wedge \neg \clubsuit^{\sup \geqslant 1}$ consistent?

References

1. Džamonja, M., Shelah, S.: Similar but not the same: various versions of ♣ do not coincide. J. Symbolic Logic **64**(1), 180–198 (1999)
2. Juhász, I.: A weakening of ♣, with applications to topology. Comment. Math. Univ. Carolin. **29**(4), 767–773 (1988)
3. Kumar, A., Shelah, S.: Clubs on quasi measurable cardinals. MLQ Math. Log. Q. **64**(1–2), 44–48 (2018)
4. Kumar, A., Shelah, S.: RVM, RVC revisited: Clubs and Lusin sets (submitted)
5. Kunen, K.: Set Theory. Studies in Logic and the Foundations of Mathematics, vol. 102. North-Holland, Amsterdam (1980)
6. Ostaszewski, A.J.: On countably compact perfectly normal spaces. J. London Math. Soc. **14**(3), 505–516 (1975)