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# MORE ON WEAK DIAMOND

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**Abstract.** We deal with the combinatorial principle Weak Diamond. We prove that, if it holds for a given cardinal, we can get this principle with more than two colours or some relevant ideal is not too saturated. Then we point out a model theoretic consequence of Weak Diamond.

## 0. Basic definitions

In this section we present basic notations, definitions and results.

The paper was circulated (including the math arXiv) and accepted to the East-West Journal of Math around 2000, but due to some problems between the editors has not appeared. Meanwhile Aspero, Larson and Moore [1] with a related result was done. Weak diamond was introduced in [2], lately see [3].

Notation 0.1. (1) $\kappa$, $\lambda$, $\theta$, $\mu$ will denote cardinal numbers and $\alpha$, $\beta$, $\delta$, $\varepsilon$, $\xi$, $\zeta$, $\gamma$ will be used to denote ordinals.

(2) Sequences of ordinals are denoted by $\nu$, $\eta$, $\rho$ (with possible indexes).

(3) The length of a sequence $\eta$ is $\ell g(\eta)$.

(4) For a sequence $\eta$ and $\ell \leq \ell g(\eta)$, $\eta{\restriction}\ell$ is the restriction of the sequence $\eta$ to $\ell$ (so $\ell g(\eta{\restriction}\ell) = \ell$). If a sequence $\nu$ is a proper initial segment of a sequence $\eta$ then we write $\nu \vartriangleleft \eta$ (and $\nu \trianglelefteq \eta$ has the obvious meaning).

(5) For a set $A$ and an ordinal $\alpha$, $\alpha_A$ stands for the function on $A$ which is constantly equal to $\alpha$.

(6) For a model $M$, $|M|$ stands for the universe of the model.

(7) The cardinality of a set $X$ is denoted by $\|X\|$. The cardinality of the universe of a model $M$ is denoted by $\|M\|$.

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Definition 0.2. Let $\lambda$ be a regular uncountable cardinal and $\theta$ be a cardinal number, possibly finite.

(1) *A $(\lambda,\theta)$-colouring* is a function $F\colon \mathrm{DOM} \longrightarrow \theta$, where DOM is either ${}^{<\lambda}2 = \bigcup_{\alpha<\lambda} {}^{\alpha}2$ or $\bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$. In the first case we will write $\mathrm{DOM}_\alpha = \mathrm{DOM}_\alpha(F) = {}^{1+\alpha}2$, in the second case we let

$$\mathrm{DOM}_\alpha = \mathrm{DOM}_\alpha(F) = {}^{1+\alpha}(\mathcal{H}(\lambda)) \quad (\text{for } \alpha \le \lambda).$$

If the choice does not matter we shall not mention it; for the main definitions the choice does not matter, see 1.10.

If $\lambda$ is understood from the context, we may omit it; if $\theta = 2$ then we may omit it (thus a $\lambda$-colouring means a $(\lambda,2)$-colouring and *a colouring* is a $(\lambda,2)$-colouring).

(2) For a $(\lambda,\theta)$-colouring $F$ and a set $S \subseteq \lambda$, we say that a function $\eta \in {}^{S}\theta$ is *an $F$-weak diamond sequence for $S$ when* for every $f \in \mathrm{DOM}_\lambda$ the set

$$\{\,\delta \in S : \eta(\delta) = F(f{\restriction}\delta)\,\}$$

is stationary.

(3) $\mathrm{WDmId}_\lambda$ is the collection of all sets $S \subseteq \lambda$ such that for some colouring $F$ there is no $F$-weak diamond sequence for $S$.

Remark 0.3. In the definition of $\mathrm{WDmId}_\lambda$ (0.2(3)), the choice of DOM (see 0.2(1)) does not matter; see [10, AP, §1], remember that $\|\mathcal{H}(\lambda)\| = 2^{<\lambda}$.

Theorem 0.4 (Devlin and Shelah [2]; see [10, AP, §1] too). *Assume that $2^\theta = 2^{<\lambda} < 2^\lambda$ (e.g. $\lambda = \mu^+$, $2^\mu < 2^\lambda$). Then for every $\lambda$-colouring $F$ there exists an $F$-weak diamond sequence for $\lambda$. Moreover,* $\mathrm{WDmId}_\lambda$ *is a normal ideal on $\lambda$ (and $\lambda \notin \mathrm{WDmId}_\lambda$).*

Remark 0.5. One could wonder why the weak diamond (and $\mathrm{WDmId}_\lambda$) is interesting. Below we list some of the applications, limitations and related problems.

(1) Weak diamond is really weaker than diamond, but provably (in ZFC) it holds true for some cardinals $\lambda$. Note that under GCH, $\diamondsuit_{\mu^+}$ holds true for each $\mu > \aleph_0$, so the only interesting case then is $\lambda = \aleph_1$.

(2) Original interest in this combinatorial principle comes from interest in Whitehead groups: *if $G$ is a strongly $\lambda$-free Abelian group and $\Gamma(G) \notin$* $\mathrm{WDmId}_\lambda$ *then $G$ is not Whitehead.*

(3) A related question was: can we have stationary subsets $S_1, S_2 \subseteq \omega_1$ such that $\diamondsuit_{S_1}$ but $\neg\diamondsuit_{S_2}$? (See [5].)

(4) Weak diamond has been helpful particularly in problems where we have some uniformity, e.g.:

$(*)_1$ *Assume $2^\lambda < 2^{\lambda^+}$. Let $\psi \in \mathbb{L}_{\lambda^+,\omega}$ be categorical in $\lambda, \lambda^+$. Then $(\mathrm{MOD}_\psi, \prec_{\mathrm{Frag}(\psi)})$ has the amalgamation property in $\lambda$.*

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$(*)_2$ *If $G$ is a group of cardinality $\lambda > \aleph_0$ then we can find subgroups $G_i$ of $G$ (for $i < \lambda$) non-conjugate in pairs (see* [9]*).*

(5) One may wonder if assuming $\lambda = \mu^+$, $2^\lambda > 2^\mu$ (and e.g. $\mu$ regular) we may find a regular $\sigma < \mu$ such that

$$\{\delta < \lambda : \mathrm{cf}(\delta) = \sigma\} \notin \mathrm{WDmId}_\lambda.$$

Of course, by the ''normal ideal" result (see 0.4), it follows that there is such $\sigma$, but does $\sigma$ depend on the present set theory? e.g. does it hold for every regular $\sigma \neq \mathrm{cf}(\mu)$ below $\mu$?

Unfortunately, this is not the case (see [7] even for $\mu = \aleph_1$).

(6) We would like to prove

(a) $\mathrm{WDmId}_\lambda$ is not $\lambda^+$-saturated or

(b) a strengthening, e.g. weak diamond for more (than two) colours.

We will get (a variant of) a local version of the disjunction, where we essentially fix $F$. There are two reasons for interest in **(a)**: understanding $\lambda^+$-saturated normal ideals (e.g. we get more information on the case CH + "$\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated"; see also Zapletal and Shelah [16]), and non $\lambda^+$-saturation helps in "non-structure theorems in model thery" (see [6], [11], [13], [14]). That is, having $2^\mu < 2^{\mu^+} < 2^{\mu^{++}}$ and some "bad" (i.e. "non-structure") properties for models in $\mu$ we get $2^{\mu^{++}}$ models in $\mu^{++}$ when $\mathrm{WDmId}_{\lambda^+}$ is not $\lambda^{++}$-saturated (and using the local version does not hurt).

(7) Note that for $S \notin \mathrm{WDmId}_\lambda$ we have a weak diamond sequence $f \in {}^S2$ such that the set of "successes" (=equalities) is stationary, but it does not have to be in $(\mathrm{WDmId}_\lambda)^+$. We would like to start and end in the same place: being positive for the same ideal. Also, in **(b)** above the set of places we guess was stationary, when we start with $S \in (\mathrm{WDmId}_\lambda)^+$.

Note that it may well be that $\lambda \in \mathrm{WDmId}_\lambda$ (if $(\exists\theta < \lambda)(2^\theta = 2^\lambda)$ this holds), but some "local" versions may still hold. E.g. in the Easton model, we have $F$-weak diamond sequences for all $F$ which are reasonably definable (see [10, AP, §1]; define e.g.

$$F(f) = 1 \;\Leftrightarrow\; \mathrm{L}[X, f] \models \varphi(X, f)$$

for a fixed first order formula $\varphi$, where $X \subseteq \lambda$ depends on $F$ only). So the case $\mathrm{WDmId}_\lambda = \mathcal{P}(\lambda)$ has some interest.

(8) Related later works are [12], [15].

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## 1. When colourings are almost constant

In 1.1, 1.2 we now "slice" $\mathrm{WDmId}_\lambda$ by finer approximations meaningful even when $\mathrm{WDmId}_\lambda$ fails.

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DEFINITION 1.1. Let $\lambda$ be a regular uncountable cardinal.

(1) Let $F$ be a $(\lambda, \theta)$-colouring.

(a) Let $S \subseteq \lambda$ . We say that *a sequence* $\eta \in {}^S\theta$ *is coded by* $F$ if there exists $f \in \mathrm{DOM}_\lambda$ such that

$$\alpha \in S \;\Leftrightarrow\; \eta(\alpha) = F(f{\restriction}(1+\alpha)).$$

(b) We let

$$\mathfrak{B}(F) \stackrel{\mathrm{def}}{=} \big\{\eta \in {}^\lambda\theta : \eta \text{ is coded by } F\big\}.$$

(2) For a family $\mathcal{A}$ of subsets of $\lambda$ let $\mathrm{ideal}_\lambda(\mathcal{A})$ be the $\lambda$-complete normal ideal on $\lambda$ generated by $\mathcal{A}$ (i.e. it is the closure of $\mathcal{A}$ under unions of $< \lambda$ elements, diagonal unions, containing singletons, and subsets).

[Note that $\mathrm{ideal}_\lambda(\mathcal{A})$ does not have to be a proper ideal.]

(3) For a $\lambda$-colouring $F$ (so $\theta = 2$) we define by induction on $\alpha$: $\mathrm{ID}_0^-(F) = \emptyset$, $\mathrm{ID}_0(F) = \{S \subseteq \lambda : S \text{ is not stationary}\}$, for a limit $\alpha$

$$\mathrm{ID}_\alpha^-(F) = \bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F), \quad \mathrm{ID}_\alpha(F) = \mathrm{ideal}_\lambda\Big(\bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F)\Big),$$

and[1] for $\alpha = \beta + 1$

$$\mathrm{ID}_\alpha^-(F) = \left\{\begin{matrix} S \subseteq \lambda : \text{ for each } S^* \subseteq S \text{ there is } f \in \mathrm{DOM}_\lambda \text{ such} \\ \text{that } \{\delta < \lambda : \delta \in S^* \Leftrightarrow F(f{\restriction}\delta) = 0\} \in \mathrm{ID}_\beta(F)\big\}; \end{matrix}\right\}$$

$$\mathrm{ID}_\alpha(F) = \mathrm{ideal}_\lambda\big(\mathrm{ID}_\alpha^-(F)\big).$$

Finally we let $\mathrm{ID}^{\mathrm{a}}(F) = \bigcup_\alpha \mathrm{ID}_\alpha(F)$.

(4) We say that $F$ is *rich when* $\mathrm{DOM}(F) = \bigcup_{\alpha<\lambda} {}^\alpha\mathcal{H}(\lambda)$, and for every function $f \in \mathrm{DOM}_\lambda$ and $\alpha < \lambda$ and a set $A \subseteq \alpha$ there is $f' \in \mathrm{DOM}_\lambda$ such that:

$$(\forall i \in (\lambda \setminus \alpha))(f(1+i) = f'(1+i) \;\&\; F(f{\restriction}(1+i)) = F(f'{\restriction}(1+i)))$$

and $(\forall j \in \alpha \setminus \{0\})(F(f'{\restriction}j) = 1 \;\Leftrightarrow\; j \in A)$.

DEFINITION 1.2. Let $\lambda$ be a regular uncountable cardinal and let $F$ be a $\lambda$-colouring.

(1) $\mathrm{WDmId}_\lambda(F)$ is the family of all sets $S \subseteq \lambda$ with the property that for every $S^* \subseteq S$ there is $f \in \mathrm{DOM}_\lambda$ such that the set

$$\big\{\delta \in S : \delta \in S^* \;\Leftrightarrow\; F(f{\restriction}\delta) = 1\big\}$$

is not stationary, (note, the difference with 0.2(3)).

[1] Note that $\mathrm{ID}_\alpha^-(F) \neq \emptyset$ iff $\alpha > 0$.

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(2) $\mathfrak{B}^+(F)$ is the closure of (see 1.1(1)):

$$\mathfrak{B}(F) \cup \{S \subseteq \lambda : S \text{ is not stationary}\}$$

under unions of $< \lambda$ sets, complement and diagonal unions (here, in $\mathfrak{B}(F)$, we identify a subset of $\lambda$ with its characteristic function).

(3) $\mathrm{ID}^{\mathrm{b}} \stackrel{\text{def}}{=} \{S \subseteq \lambda :$ for every $Y \subseteq S$ for some $B \in \mathfrak{B}^+(F)$ we have $B \cap S = Y\}$

(4) $\mathrm{ID}^{\mathrm{c}}(F)$ is the collection of all $S \subseteq \lambda$ such that for some $X \in \mathfrak{B}^+(F)$ we have: $S \subseteq X$ and there is a partition $X_0, X_1$ of $X$ such that

($\alpha$) $\mathcal{P}(X_\ell) = \{Y \cap X_\ell : Y \in \mathfrak{B}^+(F)\}$ for $\ell = 0, 1$, and

($\beta$) for $\ell < 2$ there is no $Y \in \mathfrak{B}^+(F)$ satisfying

$$Y \setminus X_\ell \in \mathrm{ID}^{\mathrm{b}}(F) \ \ \& \ \ Y \notin \mathrm{ID}^{\mathrm{b}}(F).$$

Proposition 1.3. *Assume $\lambda$ is a regular uncountable cardinal and $F$ is a $\lambda$-colouring.*

(1) $\mathrm{ideal}_\lambda(\mathcal{A})$ *is the collection of all diagonal unions $\nabla_{\xi<\lambda} A_\xi$ such that $A_\xi \in \mathcal{A}$ for $\xi < \lambda$, when $\mathcal{A}$ is a family of subsets of $\lambda$ such that*

($\circledast_{\mathcal{A}}$) *if $S_0 \subseteq S_1$ and $S_1 \in \mathcal{A}$ and $A \in [\lambda]^{<\lambda}$ then $S_0 \cup A \in \mathcal{A}$,*

(2) *The condition* ($\circledast_{\mathrm{ID}^-_\alpha(F)}$) *(see above) is true for each $\alpha$. Consequently, if $\alpha = \beta + 1$ then $\mathrm{ID}_\alpha(F) = \{\nabla_{i<\lambda} A_i : \langle A_i : i < \lambda\rangle \subseteq \mathrm{ID}^-_\alpha(F)\}$, and if $\alpha$ is limit then $\mathrm{ID}_\alpha(F) = \{\nabla_{i<\lambda} A_i : \langle A_i : i < \lambda\rangle \subseteq \bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F)\}$.*

(3) $\mathrm{ID}^{\mathrm{a}}(F)$ *and* $\mathrm{ID}_\alpha(F)$ *are $\lambda$-complete normal ideals on $\lambda$ extending the ideal of non-stationary subsets of $\lambda$ (but they do not have to be proper). For $\alpha < \gamma$ we have $\mathrm{ID}_\alpha(F) \subseteq \mathrm{ID}_\gamma(F)$ and hence $\mathrm{ID}^{\mathrm{a}}(F) = \mathrm{ID}_\alpha(F)$ for every large enough $\alpha < (2^\lambda)^+$.*

(4) *Suppose $\bar{B} = \langle B_\ell : \ell \le m\rangle$, where $B_\ell \subseteq B_{\ell+1}$ (for $\ell < m$) and $B_m \in \mathrm{ID}^{\mathrm{a}}(F)$. Then $\bar{B}$ has an $F$-representation, which means that there are a well founded tree $T \subseteq {}^{\omega>}\lambda$, sequences $\langle B^\ell_\eta : \eta \in T,\ \ell \le \ell_\eta\rangle$, and $\langle f^k_\eta : \eta \in T, k \le k_\eta\rangle$ such that $k_\eta \le \ell_\eta + 1 \le m + 1$ and*

(a) $B^\ell_{\langle\rangle} = B_\ell$, $\ell_{\langle\rangle} = m$, $B^\ell_\eta \subseteq B^{\ell+1}_\eta \subseteq \lambda$, $f^\ell_\eta \in {}^\lambda 2$,

(b) $(\forall \eta \in T \setminus \max(T))\ (\forall i < \lambda)(\eta^\frown\langle i\rangle \in T)$,

(c) *for each $\eta \in T \setminus \max(T)$ there is $\alpha_\eta < \lambda$ such that for all $\ell \le \ell_\eta$, $\delta \in \lambda \setminus \alpha_\eta$ we have:*

($\oplus$) $\delta \in B^\ell_\eta$ *iff*

$(\exists i < \delta)(\delta \in B^\ell_{\eta^\frown\langle i\rangle})$ *or*

$F(f^\ell_\eta \restriction \delta) = 1 \ \&\ \neg(\exists i < \delta)(\exists k)(\delta \in B^k_{\eta^\frown\langle i\rangle})$,

(d) *for each $\eta \in \max(T)$, $B_\eta$ is a bounded subset of $\lambda$ with $\min(B_\eta) > \sup(\{\eta(n) : n < \ell g(\eta)\})$.*

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(5) *If for some* $f^* \in {}^{\lambda}2$ *we have* $(\forall\alpha < \lambda)(F(f^*{\restriction}\alpha) = 0)$ *then in part* (4) *above we can demand that* $k_\eta = \ell_\eta + 1$.

(6) *If* $F$ *is rich then in part* (4) *above we can add*

(e) $\alpha_\eta$=0 *for* $\eta \in T \setminus \max(T)$ *and* $B_\eta = \emptyset$ *for* $\eta \in \max(T)$.

(7) $\mathrm{ID}^{\mathrm{a}}(F)$ *is the minimal normal ideal* $D$ *on* $\lambda$ *such that there is no* $S \in D^+$ *satisfying*

$$(\forall S^* \subseteq S)(\exists A \in \mathfrak{B}(F))(S^* \triangle A \in D).$$

(8) *If* $X \in \mathscr{P}(\lambda) \setminus \mathrm{ID}^{\mathrm{b}}(F)$ *then there is* $\eta \in {}^{\lambda}2$ *which is a weak diamond even modulo* $\mathrm{ID}^{\mathrm{b}}(F)$ *which means that*:
*for every* $f \in \mathrm{DOM}(F)$ *we have* $\{\delta \in X : F(f{\restriction}\delta) = \eta(\delta)\} \neq \emptyset \mod \mathrm{ID}^{\mathrm{b}}(F)$.

(9) $\mathrm{ID}^1(F) = \{S \subseteq \lambda : (\exists X \in \mathfrak{B}^+(F))(S \subseteq X \ \&\ \mathcal{P}(X) \subseteq \mathfrak{B}^+(F))\}$.

Proof. (1) Should be clear.

(2) If $\alpha = 0$ then $(\circledast_{\mathcal{A}})$ holds trivially because there is no such $S_1$.

If $\alpha$ is a limit ordinal then the condition holds because for every $S_1 \in \mathrm{ID}^-_\alpha(F)$ there is $\beta < \alpha$ such that $S_1 \in \mathrm{ID}^-_\beta(F)$ and we can use the induction hypothesis.

Lastly, if $\alpha = \beta + 1$ this is easy too.

(3) For the first sentence $\mathrm{ID}^-_\alpha(F)$ is a normal ideal by its definition; this implies $\mathrm{ID}^a(F)$ is a normal ideal by the second sentence. We still have to prove the second sentence.

By induction on $\gamma < \lambda$ and then by induction on $\alpha < \gamma$ we show that $(\forall\gamma < \lambda)(\forall\alpha < \gamma)(\mathrm{ID}_\alpha(F) \subseteq \mathrm{ID}_\gamma(F))$. If $\gamma = 1$ then this follows immediately from definitions; similarly if $\gamma$ is limit. So suppose now that $\gamma = \gamma_0 + 1$ and we proceed by induction on $\alpha \leq \gamma_0$. There are no problems neither when $\alpha = 0$ nor when $\alpha$ is limit. So suppose that $\alpha = \beta + 1 < \gamma$ (so $\beta < \gamma_0$). By the inductive hypothesis we know that $\mathrm{ID}_\beta(F) \subseteq \mathrm{ID}_{\gamma_0}(F)$. Let $A \in \mathrm{ID}_{\beta+1}(F)$. By (2) there are $A_\xi \in \mathrm{ID}^-_{\beta+1}$ (for $\xi < \lambda$) such that $A = \nabla_{\xi<\lambda} A_\xi$. Now look at the definition of $\mathrm{ID}^-_{\beta+1}(F)$: since $\mathrm{ID}_\beta(F) \subseteq \mathrm{ID}_{\gamma_0}(F)$ we see that $A_\xi \in \mathrm{ID}^-_{\gamma_0+1}(F)$. Hence $A \in \mathrm{ID}_\gamma$.

(4) By induction on $\alpha$ we show that: if $\bar{B} = \langle B_\ell : \ell \leq m\rangle$, where $B_\ell \subseteq B_{\ell+1}$ (for $\ell < m$) and $B_m \in \mathrm{ID}_\alpha(F)$ then $\bar{B}$ has an $F$-representation.

*Case* 1: $\alpha = 0$. Thus the set $B_m$ is not stationary and we may pick up a club $E$ of $\lambda$ disjoint from $B_m$. Let $E = \{\alpha_\zeta : \zeta < \lambda\}$ be the increasing enumeration. Put $T = \{\langle\rangle\} \cup \{\langle i\rangle : i < \lambda\}$, $\alpha_{\langle\rangle} = 1$, $\ell_{\langle\rangle} = \ell_{\langle i\rangle} = m$, $B^\ell_{\langle\rangle} = B_\ell$ and $B^\ell_{\langle i\rangle} = B_\ell \cap \alpha_{i+1}$. Now check.

*Case* 2: $\alpha$ is limit. It follows from (2) that $B_\ell = \nabla_{i<\lambda} B_{\ell,i}$ for some $B_{\ell,i} \in \bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F)$. Let $B'_{\ell,i}$ be defined as follows:

if $i = (m+1)j + t$, $\ell < t \leq m$ then $B'_{\ell,i} = \emptyset$,

if $i = (m+1)j + t$, $t \leq m$, $t \leq \ell$ then $B'_{\ell,i} = B_{\ell,i}$.

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Then for each $i,\ell$ we may find $\langle B^{i,\ell}_\eta, f^{i,\ell'}_\eta, \alpha^i_\eta : \eta \in T_i,\ \ell < \ell^{i,1}_\eta,\ \ell' < \ell^{i,2}_\eta\rangle$ satisfying clauses (a)–(d) and such that $\langle B^{\ell,i,k}_{\langle\rangle} : k \le k^1_\eta\rangle = \langle B'_{\ell,i} : \ell \le m\rangle$ (by the induction hypothesis). Put

$$T = \{\langle\rangle\} \cup \{\langle i\rangle^\frown\eta : \eta \in T_i\},$$

$$\ell_{\langle\rangle} = m, \quad \ell'_{\langle\rangle} = 0, \quad \ell_{\langle i\rangle^\frown\eta} = \ell^{i,1}_\eta, \quad \ell_{\langle i\rangle^\frown\eta} = \ell^{i,2}_\eta,$$

$$B^\ell_{\langle\rangle} = B_\ell, \quad B^\ell_{\langle i\rangle^\frown\eta} = B^{i,\ell}_\eta, \quad f^{\ell'}_{\langle i\rangle^\frown\eta} = f^{i,\ell'}_\eta,$$

$$\alpha_{\langle\rangle} = \omega, \quad \alpha_{\langle i\rangle^\frown\eta} = \alpha^i_\eta.$$

Checking that $\langle B^\ell_\eta, f^{\ell'}_\eta, \alpha_\eta : \eta \in T,\ \ell \le \ell_\eta,\ \ell' \le \ell'_\eta\rangle$ is as required is straightforward.

*Case* 3: $\alpha = \beta + 1$. By (2) above and the proof of Case 2 we may assume that $B_m \in \mathrm{ID}^-_\alpha(F)$. It follows from the definition of $\mathrm{ID}^-_\alpha(F)$ that there are $f_\ell \in {}^\lambda 2$ (for $\ell \le m$) such that

$$B^\oplus_\ell \stackrel{\mathrm{def}}{=} \big\{\delta < \lambda : \delta \text{ is limit and } F(f_\ell{\restriction}\delta) = 0 \Leftrightarrow \delta \in B_\ell\big\} \in \mathrm{ID}_\beta(F),$$

and hence $B^\oplus \stackrel{\mathrm{def}}{=} \bigcup_{\ell \le m} B^\oplus_\ell \in \mathrm{ID}_\beta(F)$. Therefore $B^*_\ell \stackrel{\mathrm{def}}{=} B_\ell \cap B^\oplus \in \mathrm{ID}_\beta(F)$. Now apply the inductive hypothesis for $\beta$ and $\bar B^* = \langle B^*_\ell : \ell \le m\rangle$ to get the sequences $\langle B^{\ell,*}_\eta,\ f^{k,*}_\eta : \eta \in T^*,\ \ell \le \ell^*_\eta,\ k \le k^*_\eta\rangle$ satisfying clauses (a)–(d) and such that $\langle B^{\ell,*}_{\langle\rangle} : \ell \le \ell^*_\eta\rangle = \langle B^*_\ell : \ell \le m\rangle$. Put

$$T = \{\langle\rangle\} \cup \{\langle i\rangle : i < \lambda\} \cup \{\langle 0\rangle^\frown\eta : \eta \in T^*\},$$

$$\ell_{\langle 0\rangle^\frown\eta} = \ell^*_\eta, \quad k_{\langle\rangle} = m+1, \quad k_{\langle 0\rangle^\frown\eta} = k_\eta,$$

$$B^\ell_{\langle 0\rangle^\frown\eta} = B^{\ell,*}_\eta, \quad B^\ell_{\langle 0\rangle^\frown\langle i\rangle} = B_\ell \cap (i+\omega),$$

$$f^k_{\langle\rangle} = f_k, \quad f^k_{\langle 0\rangle^\frown\eta} = f^{k,*}_\eta,$$

$$\alpha_{\langle\rangle} = \omega, \quad \alpha_{\langle 0\rangle^\frown\eta} = \alpha^*_\eta.$$

(5) If $f^\ell_\eta$ is not defined then choose $f^*$ as it.

(6), (7), (8), (9) Easy too. $\square$

REMARK 1.4. Note that it may happen that $\lambda \in \mathrm{ID}^{\mathrm{a}}(F)$. However, if $\eta \in {}^\lambda 2$ is a weak diamond sequence for $F$ then the set $\{\gamma < \lambda : \eta(\gamma) = 0\}$ witnesses $\lambda \notin \mathrm{ID}^-_1(F)$. And conversely, if $\lambda \notin \mathrm{ID}^-_1(F)$ and $S^* \subseteq \lambda$ witnesses it, then the function $0_{S^*} \cup 1_{\lambda\setminus S^*}$ is a weak diamond sequence for $F$.

DEFINITION 1.5. For a $\lambda$-colouring $F$ we define $\lambda$-colourings $F^\oplus$ and $F^\otimes$ as follows.

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(1) A function $g \in {}^{\gamma}(\mathcal{H}(\lambda))$ is called $F^{\oplus}$-standard if there is a tuple $(T, \bar{f}, \bar{\alpha}, \bar{A})$ (called a witness) such that

(i) $T \subseteq {}^{\omega >}\gamma$ is a well founded tree (so $\langle\rangle \in T$, $\nu \triangleleft \eta \in T \ \Rightarrow \nu \in T$ and $T$ has no $\omega$-branch);

(ii) $\bar{f} = \langle f^{\ell}_{\eta} : \eta \in T,\ \ell \le k_{\eta}\rangle$, where $f^{\ell}_{\eta} \in \mathrm{DOM}(F) \cap {}^{\gamma}(\mathcal{H}(\lambda))$;

(iii) $\bar{\alpha} = \langle \alpha_{\eta} : \eta \in T\rangle$, where $\alpha_{\eta} < \lambda$;

(iv) $\bar{A} = \langle A^{\ell}_{\eta} : \eta \in T,\ \ell \le \ell_{\eta}\rangle$, where $A^{\ell}_{\eta} \subseteq \alpha_{\eta}$;

(v) $g(\beta) = \big(T \cap {}^{\omega >}\beta, \langle f^{\ell}_{\eta}{\restriction}\beta : \eta \in T \cap {}^{\omega >}\beta, \ell \le k_{\eta}\rangle, \langle \alpha_{\eta} : \eta \in T \cap {}^{\omega >}\beta\rangle,$ $\langle A^{\ell}_{\eta} : \eta \in T \cap {}^{\omega >}\beta,\ \ell \le \ell_{\eta}\rangle\big)$ for each $\beta < \gamma$.

(2) $\mathrm{DOM}(F^{\oplus}) = \bigcup_{\alpha < \lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ and for $g \in {}^{\gamma}(\mathcal{H}(\lambda))$:

$(\oplus)_{\alpha}$ if $\gamma = 0$ then $F^{\oplus}(g) = 0$,

$(\oplus)_{\beta}$ if $\gamma > 0$ and $g$ is not standard then $F^{\oplus}(g) = 0$,

$(\oplus)_{\gamma}$ if $\gamma > 0$ and $g$ is standard as witnessed by $\langle \bar{T}, \bar{f}, \bar{\alpha}, \bar{A}\rangle$ then $F^{\oplus}(g) = \mathbf{t}^{0}_{F,g}(\langle\rangle)$, where $\mathbf{t}^{\ell}_{F,g}(\eta) \in \{0,1\}$ (for $\eta \in T$, $\ell = 0,1$) are defined by downward induction as follows.

If $\eta \in \max(T)$ then $\mathbf{t}^{\ell}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,
if $\eta \in T \setminus \max(T)$, $\gamma < \alpha_{\eta}$ then $\mathbf{t}^{\ell}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,
if $\eta \in T \setminus \max(T)$, $\gamma \ge \alpha_{\eta}$ then

$$\mathbf{t}^{1}_{F,g}(\eta) = 1 \ \text{ iff } \ F(f_{\eta}) = 1 \ \text{ or } \ (\exists i < \gamma)(\mathbf{t}^{1}_{F,g}(\eta^\frown\langle i\rangle) = 1),$$

$$\begin{aligned}\mathbf{t}^{0}_{F,g}(\eta) = 1 \ \text{ iff } \ & (\exists i < \gamma)(\mathbf{t}^{0}_{F,g}(\eta^\frown\langle i\rangle) = 1) \text{ or} \\ & F(f_{q}) = 1 \ \&\ (\forall i < \gamma)(\mathbf{t}^{1}_{F,g}(\eta^\frown\langle i\rangle) = 0).\end{aligned}$$

(3) A function $g \in {}^{\gamma}(\mathcal{H}(\lambda))$ is called $F^{\otimes}$-standard if there is a tuple $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ (called a witness) such that

(i) $T \subseteq {}^{\omega >}\gamma$ is a well founded tree;

(ii) $\bar{f} = \langle f_{\eta} : \eta \in T\rangle$, where $f_{\eta} \in \mathrm{DOM}(F) \cap {}^{\gamma}(\mathcal{H}(\lambda))$;

(iii) $\bar{\ell} = \langle \ell_{\eta} : \eta \in T\rangle$, where $\ell_{\eta} : {}^{3}\{0,1\} \longrightarrow \{0,1\}$;

(iv) $\bar{\alpha} = \langle \alpha_{\eta} : \eta \in T\rangle$, where $\alpha_{\eta} < \lambda$;

(v) $\bar{A} = \langle A_{\eta} : \eta \in T\rangle$, where $A_{\eta} \subseteq \alpha_{\eta}$;

(vi) $g(\beta) = (T \cap {}^{\omega >}\beta, \langle f_{\eta}{\restriction}\beta : \eta \in T \cap {}^{\omega >}\beta\rangle, \langle \ell_{\eta} : \eta \in T \cap {}^{\omega >}\beta\rangle, \langle \alpha_{\eta} : \eta \in T \cap {}^{\omega >}\beta\rangle, \langle A_{\eta} : \eta \in T \cap {}^{\omega >}\beta\rangle)$ for each $\beta < \gamma$.

(4) $\mathrm{DOM}(F^{\otimes}) = \bigcup_{\alpha < \lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ and for $g \in {}^{\gamma}(\mathcal{H}(\lambda))$:

$(\otimes)_{\alpha}$ if $\gamma = 0$ then $F^{\otimes}(g) = 0$,

$(\otimes)_{\beta}$ if $\gamma > 0$ and $g$ is not $F^{\otimes}$-standard then $F^{\otimes}(g) = 0$,

$(\otimes)_{\gamma}$ if $\gamma > 0$ and $g$ is $F^{\otimes}$-standard as witnessed by $\langle \bar{T}, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A}\rangle$ then $F^{\otimes}(g) = \mathbf{t}_{F,g}(\langle\rangle)$, where $\mathbf{t}_{F,g}(\eta) \in \{0,1\}$ (for $\eta \in T$) are defined by downward induction as follows.

If $\eta \in \max(T)$ then $\mathbf{t}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,
if $\eta \in T \setminus \max(T)$, $1 + \gamma < \alpha_{\eta}$ then $\mathbf{t}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,

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if $\eta \in T \setminus \max(T)$, $1 + \gamma \geq \alpha_\eta$ then

$$\mathbf{t}_{F,g}(\eta) = \ell_\eta\big(F(f_\eta), \max\{\mathbf{t}_{F,g}(\eta^\frown\langle\beta\rangle) : \beta < \gamma\}, \min\{\mathbf{t}_{F,g}(\eta^\frown\langle\beta\rangle) : \beta < \gamma\}\big).$$

Remark 1.6. On $F^\oplus$, $F^\otimes$ see 1.7, 1.12 below.

Proposition 1.7. *Let $F$ be a $\lambda$-colouring. Then $F^\oplus$ is a $\lambda$-colouring and*
(a) *if* $S \in \mathrm{ID}^{\mathrm{a}}(F)$ *then* $0_S \cup 1_{\lambda\setminus S} \in \mathfrak{B}(F^\oplus)$ *and* $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^\oplus)$,
(b) $\mathrm{ID}^{\mathrm{a}}(F) \subseteq \mathrm{ID}_1^-(F^\oplus) = \mathrm{ID}_1(F^\oplus) = \mathrm{ID}^{\mathrm{a}}(F^\oplus)$,

Proof. (a) Check.
(b) The main point is proving $\mathrm{ID}^{\mathrm{a}}(F) \subseteq \mathrm{ID}_1(F^\oplus)$.
Suppose that $B \in \mathrm{ID}^{\mathrm{a}}(F)$. We are going to show that then $B \in \mathrm{ID}_1^-(F^\oplus)$. So suppose that $B' \subseteq B$. We want to find $g \in \mathrm{DOM}_\lambda(F^\oplus)$ such that the set

$$\big\{\delta < \lambda : \delta \text{ is limit and } F(g{\restriction}\delta) = 0 \Leftrightarrow \delta \in B'\big\}$$

is in $\mathrm{ID}_0(F^\oplus)$ (what just means that it is non-stationary). Since $B \in \mathrm{ID}^{\mathrm{a}}(F)$ we have $B' \in \mathrm{ID}^{\mathrm{a}}(F)$, so by 1.3(4) we may find $\langle B^\ell_\eta, f^k_\eta, \alpha_\eta : \eta \in T,\ \ell \leq \ell_\eta,\ k < k_\eta\rangle$ such that the clauses (a)–(d) of 1.3(4) are satisfied with $\ell_{\langle\rangle} = 0$, $B' = B^0_{\langle\rangle}$. Define $g$ as follows. For $\beta < \lambda$ let $T_\beta = T \cap {}^{\omega>}\beta$ and

$$g(\beta) = \big(T_\beta,\ \langle f^k_\eta : \eta \in T_\beta, k \leq k_\eta\rangle,\ \langle\alpha_\eta : \eta \in T_\beta\rangle,\ \langle B^\ell_\eta \cap \alpha_\eta : \ell \leq \ell_\eta, \eta \in T_\beta\rangle\big).$$

Now look at the demands in 1.5(2) – they are exactly what 1.3(4) guarantees us. $\square$

Definition 1.8. Let $F_1$, $F_2$ be $\lambda$-colourings (with $\mathrm{DOM}(F_\ell)$ being either ${}^{\lambda>}2$ or $\bigcup_{\alpha<\lambda} {}^\alpha(\mathcal{H}(\lambda))$, see 0.2(1)).
(1) We say that $F_1 \leq F_2$ *when* there is $h : \mathrm{DOM}(F_1) \longrightarrow \mathrm{DOM}(F_2)$ such that:
(a) $\eta \trianglelefteq \nu \ \Rightarrow\ h(\eta) \trianglelefteq h(\nu)$,
(b) $h(\eta) = \lim_{\alpha<\delta} h(\eta \restriction \alpha)$, for every $\eta \in {}^\delta 2$, $\delta$ a limit ordinal,
(c) $(\forall \eta \in \mathrm{DOM}(F_1))(0 < \ell g(\eta) = \ell g(h(\eta)) \ \Rightarrow\ F_1(\eta) = F_2(h(\eta)))$.
(2) We say that $F_1 \leq^* F_2$ *when* there is $h\colon \mathrm{DOM}(F_1) \longrightarrow \mathrm{DOM}(F_2)$ such that the clauses (a)–(c) above hold but
(d) if $\eta \in \mathrm{DOM}_\lambda(F_1)$ and $\lim_{\alpha<\lambda} h(\eta{\restriction}\alpha)$ has length $< \lambda$ then $F_1(\eta{\restriction}\alpha) = 0$ for every large enough $\alpha$.

Proposition 1.9. (1) *$\leq^*$ and $\leq$ are transitive relations on $\lambda$-colourings, satisfying $\leq^* \subseteq \leq$.*
(2) *$\leq$ is $\lambda^+$-directed.*

Proposition 1.10. (1) *For every colouring $F_1\colon \bigcup_{\alpha<\lambda} {}^\alpha(\mathcal{H}(\lambda)) \longrightarrow 2$ there is a colouring $F_2\colon {}^{\lambda>}2 \longrightarrow 2$ such that $F_1 \leq F_2 \leq^* F_1$.*

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(2) *For every $\lambda$-colouring $F_2\colon {}^{\lambda>}2 \longrightarrow 2$ there is a $\lambda$-colouring*

$$F_1\colon \bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \to 2$$

*such that $F_2 \le F_1 \le^* F_2$.*

Proof. (1) Let $F_1\colon \bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$. Let $h_0$ be a one-to-one function from $\mathcal{H}(\lambda)$ to ${}^{\lambda>}2$, say $h_0(\eta) = \langle \ell_{\eta,i} : i < \ell g(h_0(\eta))\rangle$. Define a function $h_1 : \mathcal{H}(\lambda) \longrightarrow {}^{\lambda>}2$ by:

$$\begin{gathered}\ell g(h_1(\eta)) = 2\ell g(h_0(\eta)) + 2,\\ h_1(\eta)(2i) = h_0(\eta)(i),\ h_1(\eta)(2i+1) = 0 \quad \text{for } i < \ell g(h_0(\eta)), \quad \text{and}\\ h_1(\eta)(2\ell g(h_0(\eta))) = h_1(\eta)(2\ell g(h_0(\eta)+1)) = 1.\end{gathered}$$

Next, by induction on $\ell g(\eta)$, we define a function $h^+ : \bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow {}^{\lambda>}2$ as follows:

$$h^+(\langle\rangle) = \langle\rangle, \quad h^+(\eta^\frown\langle x\rangle) = h^+(\eta)^\frown h_1(x).$$

and if $\eta \in \bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ has length the limit ordinal $\delta$ then $h^+(\eta) = \bigcup\{h^+(\eta{\restriction}\beta) : \beta < \delta\}$

Clearly $h^+$ is one to one with the right domain and range.

Finally we define a colouring $F_2\colon {}^{\lambda>}2 \longrightarrow 2$ by

$$F_2(\nu) = \begin{cases} F_1(\eta) & \text{if } \nu = h^+(\eta),\\ 0 & \text{if } \nu \notin \operatorname{rng}(h^+).\end{cases}$$

It is easy to check that $F_2$ is as required.

(2) Similar to part (1). $\square$

Proposition 1.11. *Assume that $F_1$, $F_2$ are $\lambda$-colourings such that $F_1 \le F_2$, or $F_1 \le^* F_2$. Then:*

(1) *For every $\eta \in \mathrm{DOM}_\lambda(F)$ there are $\nu \in \mathrm{DOM}_\lambda(F)$ and a club $E$ of $\lambda$ such that*

$$(\forall\delta \in E)(F_1(\eta{\restriction}\delta) = F_2(\nu{\restriction}\delta)).$$

(2) $\mathrm{ID}_\alpha(F_1) \subseteq \mathrm{ID}_\alpha(F_2)$, $\mathrm{ID}^-_\alpha(F_1) \subseteq \mathrm{ID}^-_\alpha(F_2)$; *hence* $\mathrm{ID}^{\mathrm{a}}(F_1) \subseteq \mathrm{ID}^{\mathrm{a}}(F_2)$ *and* $\mathfrak{B}^+(F_1) \subseteq \mathfrak{B}^+(F_2)$.

(3) *For every colouring $F$ we have* $\mathrm{ID}^{\mathrm{c}}(F) \subseteq \mathrm{WDmId}_\lambda$

Proof. Straightforward. $\square$

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Conclusion 1.12. Assume that $\lambda$ is a regular uncountable cardinal and $F\colon {}^{\lambda>}2 \longrightarrow 2$ is a $\lambda$-colouring. Let

$$F^{\otimes}\colon \bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$$

be the colouring defined for $F$ in Definition 1.5(4). Let $\iota \in \{a, b\}$. *Then*:

(a) $F \le F^{\otimes}$.

(b) $\mathrm{ID}^{\iota}(F^{\otimes})$ is a normal ideal on $\lambda$.

(c) $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^{\otimes})$ and $\mathrm{ID}^{\iota}(F) \subseteq \mathrm{ID}^{\iota}(F^{\otimes}) = \mathrm{WDmId}_{\lambda}(F^{\otimes})$.

(d) $F^{\otimes}$ relates to itself as it relates to $F$, i.e. *if* $\alpha^* < \lambda^+$, $\langle S_\alpha : \alpha < \alpha^*\rangle$ is increasing continuous modulo $\mathrm{ID}^{\iota}(F^{\otimes})$, $S_{\alpha+1} = S_\alpha \cup A_\alpha \mod \mathrm{ID}^{\iota}(F^{\otimes})$, $A_\alpha \in \mathfrak{B}(F^{\otimes})$, $\ell_\alpha \in 2$, *then* for some $f \in {}^{\lambda}(\mathcal{H}(\lambda))$

$$\{\alpha < \lambda : F(f{\restriction}\alpha) = 1\}/\mathcal{D}_\lambda$$

is, in $\mathcal{P}(\lambda)/\mathcal{D}_\lambda$, the least upper bound of the family $\{(A_\alpha \setminus S_\alpha)/\mathcal{D}_\lambda : \ell_\alpha = 1\}$ (where $\mathcal{D}_\lambda$ stands for the club filter).

(e) The family $\mathfrak{B}(F^{\otimes})$ is closed under complements, unions and intersections of less than $\lambda$ sets, diagonal unions and diagonal intersections and it includes bounded subsets of $\lambda$. Moreover $\mathfrak{B}^{+}(F^{\otimes}) = \mathfrak{B}(F^{\otimes})$.

(f) If $\mathcal{P}(\lambda)/\mathrm{ID}^{\iota}(F^{\otimes})$ is $\lambda^+$-saturated *then* for every set $X \subseteq \lambda$ there are sets $A, B \in \mathfrak{B}(F^{\otimes})$ such that:

($\alpha$) $A \subseteq X \subseteq B$,

($\beta$) for every $Y \in \mathfrak{B}(F^{\otimes})$ one of the following occurs:

(i) the sets $(X \setminus A) \cap Y$, $(X \setminus A) \setminus Y$, $(B \setminus X) \cap Y$, $(B \setminus X) \setminus Y$ are not[2] in $\mathrm{ID}^{\iota}(F^{\otimes})$,

(ii) $Y \cap (B \setminus A) \in \mathrm{ID}^{\iota}(F^{\otimes})$,

(iii) $(B \setminus A) \setminus Y \in \mathrm{ID}^{\iota}(F^{\otimes})$.

In the situation as above we denote $A = \max_{F^{\otimes}}(X)$, $B = \min_{F^{\otimes}}(X)$ (note that these sets are unique only modulo $\mathrm{ID}^{\iota}(F^{\otimes})$). Moreover

(g) In clause (f), if $A \subseteq \min_{F^{\otimes}}(B)$ then

$$\min_{F^{\otimes}}(A) \subseteq \min_{F^{\otimes}}(B) \mod \mathrm{ID}^{\iota}(F^{\otimes}).$$

(h) In clause (f), when $\iota = b$, if $X \subseteq \lambda$, $X \notin \mathrm{ID}^{\iota}(F^{\otimes})$ then for some $Y_1, Y_2 \subseteq X$ which are not in $\mathrm{ID}^{\iota}(F^{\otimes})$ we have

$$\max_{F^{\otimes}}(Y_1) = \max_{F^{\otimes}}(Y_2) = \emptyset \quad\text{and}\quad \min_{F^{\otimes}}(Y_1) = \min_{F^{\otimes}}(Y_2) \notin \mathrm{ID}^{\iota}(F^{\otimes}).$$

(i) In clause (f), $\min_{F^{\otimes}}$ and $\max_{F^{\otimes}}$ commute with the union of $< \lambda$ and the intersection of $< \lambda$ sets.

[2] hence none of $X \setminus A$, $B \setminus A$ includes (modulo $\mathrm{ID}^{\iota}(F^{\otimes})$) a member of $\mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}^{\iota}(F^{\otimes})$

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PROOF. Clauses (a) and (b): Should be clear.

Clause (e): Note that as $\theta = 2$ we identify a sequence $\eta \in {}^{\lambda}2$ with $\{i < \lambda : \eta(i) = 1\}$.

$\mathfrak{B}(F^{\otimes})$ *is closed under complementation.* Suppose that $A \in \mathfrak{B}(F^{\otimes})$. First, assume $A$ is bounded then let $g$, $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ be as in 1.5(3) with $T = \{\langle\rangle\} \cup \{\langle i\rangle : i < \lambda\}$, $A_{\langle\rangle} = \alpha_{\langle\rangle} \setminus A$, $\alpha_{\langle\rangle} > \sup(A)$, $\ell_{\langle\rangle}$ constantly 1. Then $(\forall \alpha < \lambda)(F^{\otimes}(g{\restriction}(1+\alpha)) = 1 \Leftrightarrow \alpha \in A)$, so $F^{\otimes}$ codes $\lambda \setminus A$.

Second, suppose that $\sup(A) = \lambda$. Pick $g$ such that

$$(\forall \alpha < \lambda)(F^{\otimes}(g{\restriction}(1+\alpha)) = 1 \ \Leftrightarrow \ \alpha \in A).$$

By our assumption, for arbitrarily large $\beta < \lambda$ we have $F^{\otimes}(g{\restriction}\beta) = 1$, so $g(\beta)$ is

$$\big(T_\beta, \langle f^\beta_\eta : \eta \in T_\beta\rangle, \langle \alpha^\beta_\eta : \eta \in T_\beta\rangle, \langle \ell^\beta_\eta : \eta \in T_\beta\rangle, \langle A^\beta_\eta : \eta \in T_\beta\rangle\big)$$

and it is as in 1.5(3). If $\beta_1 < \beta_2$ then the two values necessarily cohere, in particular $T_{\beta_1} = T_{\beta_2} \cap {}^{\omega >}(\beta_1)$. Consequently there is $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ such that $T = \bigcup_{\beta<\lambda} T_\beta \subseteq {}^{\omega>}\lambda$ is closed under initial segments and is well founded (as $T_\beta$ increase with $\beta$ and $\mathrm{cf}(\lambda) > \aleph_0$). Thus we have proved

($\boxtimes$) if $A \subseteq \lambda$ is unbounded and $A$ is coded by $F^{\otimes}$, $g$, then there is $\mathbf{p} = (T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ such that the clauses (i)–(vi) of 1.5(3) hold for $\gamma = \lambda$ and $g(\beta) = \mathbf{p}{\restriction}\beta$.

Now define $\mathbf{p}'$ like $\mathbf{p}$ (with the same $T$ etc) except that $\ell^{\mathbf{p}'}_{\langle\rangle} = 1 - \ell^{\mathbf{p}}_{\langle\rangle}$ and $A^{\mathbf{p}'}_{\langle\rangle} = A^{\mathbf{p}}_{\langle\rangle}$.

$\mathfrak{B}(F^{\otimes})$ *contains all bounded subsets of* $\lambda$. By the first part of the arguments above all co-bounded subsets of $\lambda$ are in $\mathfrak{B}(F^{\otimes})$, so (by the above) their complements are there too.

$\mathfrak{B}(F^{\otimes})$ *is closed under unions of length* $< \lambda$. Let $B = \bigcup_{i<\alpha} B_i$ where $\alpha < \lambda$ and $B_i \in \mathfrak{B}(F^{\otimes})$. Let $w = \{i < \alpha : \sup(B_i) = \lambda\}$ and for $i \in w$ let $B_i$ be represented by $g_i \in {}^{\lambda}(\mathcal{H}(\lambda))$ which, by ($\boxtimes$), comes from $\mathbf{p}^i = (T^i, \bar{f}^i, \bar{\ell}^i, \bar{\alpha}^i, \bar{A}^i)$. We may assume that $w = \beta \leq \alpha$. Let

$$T = \{\langle\rangle\} \cup \{\langle i\rangle : i < \lambda\} \cup \{\langle i\rangle^\frown \eta : \eta \in T^i,\ i < \beta\},$$

$$f_{\langle i\rangle^\frown\eta} = f^i_\eta, \ \text{ etc}$$

$$\alpha_{\langle\rangle} \text{ is the first } \gamma \geq \omega \text{ such that } \gamma \geq \alpha \ \& \ (\forall i \in [\beta, \alpha))(B_i \subseteq \gamma),$$

$$B_{\langle i\rangle} = \emptyset \quad \text{ if } i \geq \beta,$$

$$A_{\langle\rangle} = \bigcup_{i<\alpha} B_i \cap \alpha_{\langle\rangle},$$

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$$\ell_{\langle\rangle}(i_0, i_1, i_2) = i_1.$$

Checking is straightforward.

$\mathfrak{B}(F^{\otimes})$ *is closed under diagonal unions.* Let $B = \nabla_{i<\lambda} B_i$, where each $B_i \in \mathfrak{B}(F^{\otimes})$ is represented by $g_i \in {}^{\lambda}(\mathcal{H}(\lambda))$ which, by $(\boxtimes)$, comes from $\mathbf{p}^i = (T^i, \bar{f}^i, \bar{\ell}^i, \bar{\alpha}^i, \bar{A}^i)$. Let $T = \{\langle\rangle\} \cup \{\langle i\rangle^\frown\eta : \eta \in T_i,\ i < \lambda\}$, $f_{\langle i\rangle^\frown\eta} = f^i_\eta$, etc, $\alpha_{\langle\rangle} = \omega$, $B_{\langle\rangle} = B \cap \omega$ and $\ell_{\langle\rangle}(i_0, i_1, i_2) = i_1$.

So we have proved the first sentence in clause (e). The second sentence there follows by it and Def. 1.2(2). Note that including the family of non-stationary sets follows by including the family of bounded subsets and being closed under diagonal unions.

Clause (c): We concentrate on the case $\mathrm{ID}^\iota = \mathrm{ID}^{\mathrm{a}}$. First note that $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^{\otimes})$ as $\mathfrak{B}(F) \subseteq \mathfrak{B}^+(F) \subseteq \mathfrak{B}^+(F^{\otimes}) = \mathfrak{B}(F^{\otimes})$ (the second inclusion by (a) and 1.11, the last equality by (e)). Next note that

$$\mathrm{WDmId}_\lambda(F^{\otimes}) \subseteq \mathrm{ID}^-_1(F^{\otimes}) \subseteq \mathrm{ID}_1(F^{\otimes}) \subseteq \mathrm{ID}^\iota(F^{\otimes}).$$

Now by induction on $\alpha$ we are proving that $\mathrm{ID}_\alpha(F^{\otimes}) \subseteq \mathrm{WDmId}_\lambda(F^{\otimes})$. So suppose that we have arrived at a stage $\alpha$.

If $\alpha = 0$ then we use the fact that every non-stationary subset of $\lambda$ is in $\mathfrak{B}(F^{\otimes})$ (by (e)).

If $\alpha$ is limit then, by the induction hypothesis, $\mathrm{ID}^-_\alpha(F^{\otimes}) \subseteq \mathfrak{B}(F^{\otimes})$ and hence $\mathrm{ID}_\alpha(F^{\otimes}) \subseteq \mathfrak{B}(F^{\otimes})$ (as $\mathfrak{B}(F^{\otimes})$ is closed under diagonal unions by (e); remember 1.3(3)).

So suppose that $\alpha = \beta + 1$ and $B \in \mathrm{ID}_\alpha(F^{\otimes})$. Suppose $B' \subseteq B$ (so $B' \in \mathrm{ID}^-_\alpha(F^{\otimes})$). There is $B'' \in \mathfrak{B}(F)$ such that $B'' \triangle B' \in \mathrm{ID}_\beta(F)$. By the first part we know that $B'' \in \mathfrak{B}(F^{\otimes})$ and by the induction hypothesis $B' \triangle B'' \in \mathfrak{B}(F^{\otimes})$. Consequently $B' \in \mathfrak{B}(F^{\otimes})$.

Together we have proved that $\mathrm{ID}^\iota(F^{\otimes}) = \mathrm{WDmId}_\lambda(F^{\otimes})$. The inclusion $\mathrm{ID}^\iota(F) \subseteq \mathrm{ID}^\iota(F^{\otimes})$ is easy.

Clause (d): Easy.

Clause (f): Let $\mathscr{D}_1$ be $\{A \subseteq X : A \in \mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}^\iota(F^{\otimes})\}$ and let $\{A_i : i < i_*\}$ be a maximal sub-family of $\mathscr{D}_1$ such that $i < j < i_* \Rightarrow A_i \cap A_j \in \mathrm{ID}^\iota(F^{\otimes})$. By the assumption of clause (f) necessarily $i_* < \lambda^+$ so without lose of generality $i_* \le \lambda$. Let $A$ be $\cup\{A_i : i < i_*\}$ if $i_* < \lambda$ and the diagonal union if $i_* = \lambda$. Clearly $A \in \mathfrak{B}^+(F) = \mathfrak{B}(F^{\otimes})$.

Let $A'$ be chosen similarly replacing $X$ by $\lambda \setminus X$ and let $B = \lambda \setminus A'$. Clearly $A$, $B$ are as required.

Clauses (g), (h), (i): Easy. $\square$

PROPOSITION 1.13. *Let $\lambda$ be a regular uncountable cardinal and $F$ be a $\lambda$-colouring.*

(1) *If $\mathrm{ID}_\alpha(F)$ is $\lambda^+$-saturated then for some $\beta < \lambda^+$ we have* $\mathrm{ID}_{\alpha+\beta}(F) = \mathrm{ID}^{\mathrm{a}}(F)$.

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(2) $\mathrm{ID}_\alpha(F) \subseteq \mathrm{WDmId}_\lambda$, *see* 1.2(3);

(3) *If* $\mathrm{ID}_\alpha(F)$ *is* $\lambda^+$*-saturated and* $\lambda \notin \mathrm{WDmId}_\lambda$ *then* $\mathrm{WDmId}_\lambda = \mathrm{ID}_1(F')$ *for some* $\lambda$*-colouring* $F'$.

(4) $\mathrm{ID}^{\mathrm{b}}(F)$, $\mathrm{ID}^{\mathrm{c}}(F)$ *are normal ideals, and* $\mathrm{ID}^1(F) \subseteq \mathrm{ID}^{\mathrm{b}}(F) \subseteq \mathrm{ID}^{\mathrm{c}}(F) \subseteq \mathrm{WDmId}_\lambda$.

(5) $\mathrm{ID}^1(F^{\otimes}) = \mathrm{WDmId}_\lambda(F^{\otimes})$.

(6) $\mathrm{WDmId}_\lambda = \bigcup\{\mathrm{ID}^\iota(F) : F$ *a function from* ${}^{\lambda>}2$ *to* $2\} = \bigcup\{\mathrm{ID}_1(F) : F$ *a function from* ${}^{\lambda>}2$ *to* $2\} \cup \{\mathrm{WDmId}_\lambda(F) : F$ *a function from* ${}^{\lambda>}2$ *to* $2\}$ *for* $\iota = a, b, c$.

Proof. (1) It follows from 1.3(3) that $\mathrm{ID}_\gamma(F)$ increases with $\gamma$ and $\beta < \gamma$, $\mathrm{ID}_\beta(F) = \mathrm{ID}_{\beta+1}$ implyies $\mathrm{ID}_\beta(F) = \mathrm{ID}_\gamma$; so the assertion should be clear.

(2) By the definition (and 1.12(c)).

(3) Assume that $\mathrm{ID}_\alpha(F)$ is $\lambda^+$-saturated and $\lambda \notin \mathrm{WDmId}_\lambda$. By induction on $\beta < \lambda^+$ we try to choose colourings $F_\beta$ such that

(a) $\mathrm{ID}(F) \subseteq \mathrm{ID}(F_\beta)$

(b) if $\beta < \gamma$ then $\mathrm{ID}^{\mathrm{a}}(F_\beta) \subseteq \mathrm{ID}^{\mathrm{a}}(F_\gamma)$,

(c) $\mathrm{ID}^{\mathrm{a}}(F_\beta) \neq \mathrm{ID}^{\mathrm{a}}(F_{\beta+1})$.

So we let $F_0 = F$. If $\beta$ is limit then we use 1.9(2) to choose $F_\beta$ so that $(\forall \gamma < \beta)(F_\gamma \leq F_\beta)$. Finally, if $\beta = \gamma + 1$ then we let $F'_\beta = (F_\gamma)^{\otimes}$ (so $\mathrm{ID}^{\mathrm{a}}(F_\gamma) \subseteq \mathrm{ID}_1(F'_\beta) = \mathrm{ID}^{\mathrm{a}}(F'_\beta) \subseteq \mathrm{WDmId}_\lambda$). If $\mathrm{ID}^{\mathrm{a}}(F'_\beta) \neq \mathrm{WDmId}_\lambda$ then we choose a set $A \in \mathrm{WDmId}_\lambda \setminus \mathrm{ID}^{\mathrm{a}}(F'_\beta)$ and $F^*_\beta$ witnessing $A \in \mathrm{WDmId}_\lambda$. We may assume that $(\forall \alpha \in \lambda \setminus A)(\forall \eta \in {}^{\alpha}2)(F^*_\beta(\eta) = 0)$. Now take a colouring $F_\beta$ such that $F'_\beta, F^*_\beta \leq F_\beta$.

After carrying out the construction choose $S^0_\beta \in \mathrm{ID}^{\mathrm{a}}(F_{\beta+1}) \setminus \mathrm{ID}^{\mathrm{a}}(F_\beta)$ (for $\beta < \lambda^+$) and let $S_\beta = S^0_\beta \setminus \nabla_{\gamma<\beta} S^0_\gamma$. Then $\langle S_\beta : \beta < \lambda^+ \rangle$ is a sequence of pairwise disjoint members of $\mathcal{P}(\lambda) \setminus \mathrm{ID}^{\mathrm{a}}(F_0) \subseteq \mathcal{P}(\lambda) \setminus \mathrm{ID}_\alpha(F)$, contradicting our assumptions.

(4), (5), (6) Easy too. $\square$

For the rest of this section we will assume the following

Hypothesis 1.14. (1) We assume that

(a) $\lambda$ is a regular uncountable cardinal,

(b) $F$ is a $\lambda$-colouring,

(c) $\lambda \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$, and

(d) $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ is $\lambda^+$-saturated, that is there is no sequence $\langle A_\alpha : \alpha < \lambda^+ \rangle$ such that for each $\alpha < \beta < \lambda^+$

$$A_\alpha \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \quad \text{and} \quad \|A_\alpha \cap A_\beta\| < \lambda.$$

(2) For each limit ordinal $\alpha \in [\lambda, \lambda^+)$ fix an enumeration $\langle \varepsilon^\alpha_i : i < \lambda \rangle$ of $\alpha$.

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Construction 1.15. Fix a sequence $\eta \in {}^{\lambda}2$ for a moment. We choose a sequence

$$\left\langle S_\alpha[\eta], A_\alpha[\eta], B_\alpha[\eta], \ell_\alpha[\eta], m_\alpha[\eta], f_\alpha[\eta] : \alpha < \alpha^*[\eta]\right\rangle$$

as follows. By induction on $\alpha < \lambda^+$ we try to choose $S_\alpha[\eta] = S_\alpha$, $A_\alpha[\eta] = A_\alpha$, $B_\alpha[\eta] = B_\alpha$, $\ell_\alpha[\eta] = \ell_\alpha$, $m_\alpha[\eta] = m_\alpha$, $f_\alpha[\eta] = f_\alpha$ such that:

(a) $S_\alpha, A_\alpha, B_\alpha \subseteq \lambda$, $\ell_\alpha, m_\alpha \in \{0,1\}$, $f_\alpha \in {}^{\lambda}2$,

(b) $A_\alpha \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$, $A_\alpha \cap S_\alpha = \emptyset$,

(c) if $\alpha = 0$ then $S_\alpha = \emptyset$

(d) $S_{\alpha+1} = S_\alpha \cup A_\alpha$;

(e) if $\alpha < \lambda$ is limit then $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$; if $\alpha \in [\lambda, \lambda^+)$ is limit then $S_\alpha = \{\gamma < \lambda : (\exists i < \gamma)(\gamma \in S_{\varepsilon_i^\alpha})\}$,

(f) $B_\alpha \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$,

(g) for every $\delta \in \lambda \setminus (S_\alpha \cup B_\alpha)$

$$\eta(\delta) = m_\alpha \ \Rightarrow\ F(f_\alpha{\restriction}\delta) = \ell_\alpha,$$

(h) $A_\alpha = \{\delta \in \lambda \setminus S_\alpha : F(f_\alpha{\restriction}\delta) = 1 - \ell_\alpha\}$.

It follows from 1.14 that at some stage $\alpha^* = \alpha^*[\eta] < \lambda^+$ we get stuck (remember clause (b) above). Still, we may define $S_{\alpha^*}$ as in clause (c).

Proposition 1.16. *Assume* 1.14. *Then*:

(1) *There exists* $\eta \in {}^{\lambda}2$ *such that*

$$\lambda \setminus S_{\alpha^*[\eta]}[\eta] \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}).$$

(2) *If* $S \in \mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *then we can demand* $S \subseteq S_{\alpha^*[\eta]}[\eta]$.

Proof. Assume not. Then for each $\eta \in {}^{\lambda}2$ the set $B_{\alpha^*}[\eta] \stackrel{\text{def}}{=} \lambda \setminus S_{\alpha^*[\eta]}$ is in $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$. Now,

$$\left\{\alpha \in B_{\alpha^*}[\eta] : \eta(\alpha) = 1\right\} \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \subseteq \mathfrak{B}(F^{\otimes})$$

(see 1.7).

Claim 1.16.1. *For each* $\alpha$, $S_\alpha \in \mathfrak{B}(F^{\otimes})$.

Proof. We show it by induction on $\alpha$. If $\alpha = 0$ then $S_\alpha = \emptyset \in \mathfrak{B}(F^{\otimes})$ (see 1.15(c)). If $\alpha < \lambda$ is a limit ordinal then $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ and by the inductive hypothesis $S_\beta \in \mathfrak{B}(F^{\otimes})$, so by 1.15(e) we are done (as $\mathfrak{B}(F^{\otimes})$ is closed under unions of $< \lambda$ elements). If $\alpha \in [\lambda, \lambda^+)$ is limit then we use the fact that $\mathfrak{B}(F^{\otimes})$ is closed under diagonal unions. If $\alpha = \beta + 1$ then $A_\beta \in \mathfrak{B}(F^{\otimes})$ or $\lambda \setminus A_\beta \in \mathfrak{B}(F^{\otimes})$ and hence we may conclude that $A_\beta \in \mathfrak{B}(F^{\otimes})$ (remember 1.12(e)). Since $\mathfrak{B}(F^{\otimes})$ is closed under unions of length $< \lambda$ we are done. $\square$

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CLAIM 1.16.2. *For each* $\alpha$, $Y_\alpha \stackrel{\text{def}}{=} \{\beta < \lambda : \eta(\beta) = 1\} \cap S_\alpha \in \mathfrak{B}(F^{\otimes})$.

PROOF. We prove it by induction on $\alpha$. If $\alpha = 0$ then $Y_\alpha = \emptyset$ and there is nothing to do. The case of limit $\alpha$ is handled like that in the proof of 1.16.1. So suppose that $\alpha = \beta + 1$. It suffices to show that the set $Y_\alpha \cap (S_\alpha \setminus S_\beta)$ is in $\mathfrak{B}(F^{\otimes})$, which means that $Y_\alpha \cap A_\alpha$ is there (remember clauses (g) and (h) of 1.15). Note that if $\delta \in A_\alpha \setminus B_\alpha$ then $F(f_\alpha{\restriction}\delta) = 1 - \ell_\alpha \neq \ell_\alpha$ and hence $\eta(\delta) \neq m_\alpha$ so $\eta(\delta) = 1 - m_\alpha$. Consequently $Y_\alpha \cap (A_\alpha \setminus B_\alpha) \in \{A_\alpha \setminus B_\alpha, \emptyset\}$. But $\mathcal{P}(B_\alpha) \subseteq \mathfrak{B}(F^{\otimes})$ so together we are done. $\square$

It follows from 1.16.1, 1.16.2 that

$$\{\beta : \eta(\beta) = 1\} \cap S_{\alpha^*[\eta]}[\eta] \in \mathfrak{B}(F^{\otimes}).$$

But $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$, so $\mathcal{P}(\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \subseteq \mathfrak{B}(F^{\otimes})$ so we get a contradiction. $\square$

CONCLUSION 1.17. Assume 1.14. Let $\eta \in {}^{\lambda}2$, $X_\ell[\eta] = (\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \cap \eta^{-1}(\{\ell\})$ (for $\ell = 0, 1$). Then one of the following occurs:

(A) $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$,

(B) $X_0[\eta], X_1[\eta] \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$, and $X_0[\eta] \cup X_1[\eta] \in \mathfrak{B}(F^{\otimes})$, $X_0[\eta] \cap X_1[\eta] = \emptyset$, and for every $f \in {}^{\lambda}2$,

*either* the sequence $\langle F(f{\restriction}\delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}[\eta])\rangle$ is $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$-almost constant

*or* both sequences $\langle F(f{\restriction}\delta) : \delta \in X_0[\eta]\rangle$ and $\langle F(f{\restriction}\delta) : \delta \in X_1[\eta]\rangle$ are not $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$-almost constant.

PROOF. Assume that the first possibility fails, so $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$.

Assume $X_0[\eta] \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$. Take any $f_{\alpha^*[\eta]} \in {}^{\lambda}2$ and choose $\ell_{\alpha^*[\eta]} \in \{0, 1\}$ so that

$$\big\{\delta \in \lambda \setminus S_{\alpha^*[\eta]}[\eta] : F(f_{\alpha^*[\eta]}{\restriction}\delta) = 1 - \ell_{\alpha^*[\eta]}\big\} \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}).$$

Putting $m_{\alpha^*[\eta]} = 0$ and $B_{\alpha^*[\eta]} = X_0[\eta]$ we get a contradiction with the definition of $\alpha^*[\eta]$. Similarly one shows that $X_1[\eta] \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$.

Suppose now that $f \in {}^{\lambda}2$ and the sequence $\langle F(f{\restriction}\delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}[\eta])\rangle$ is not $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$-almost constant but, say, the sequence $\langle F(f{\restriction}\delta) : \delta \in X_0[\eta]\rangle$ is $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$-almost constant (and let the constant value be $\ell_{\alpha^*[\eta]}$). Let $m_{\alpha^*[\eta]} = 0$, $B_{\alpha^*[\eta]} = \{\delta \in X_0[\eta] : F(f{\restriction}\delta) = 1 - \ell_{\alpha^*[\eta]}\}$. Then $B_{\alpha^*[\eta]} \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ and since necessarily

$$\big\{\delta \in X_0[\eta] \cup X_1[\eta] : F(f{\restriction}\delta) = 1 - \ell_{\alpha^*[\eta]}\big\} \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}),$$

we immediately get a contradiction. Similarly in the symmetric case. $\square$

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REMARK 1.18. Note that, in 1.17, if $S \in \mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ then there is $\eta \in {}^{\lambda}2$ such that $\eta^{-1}[\{0\}] \supseteq \lambda \setminus S$ and above $X_0, X_1 \subseteq S$ and possibility (A) fails.

PROPOSITION 1.19. *Assume* 1.14.

(1) *We can find* $S^*_F$, $S^*_0$ *and* $S^*_1$ *such that:*

(a) $S^*_F \in \mathfrak{B}(F^{\otimes})$,

(b) $S^*_F = S^*_0 \cup S^*_1$, $S^*_0 \cap S^*_1 = \emptyset$, *and* $S^*_0, S^*_1$ *witness* $S^*_F \in \mathrm{ID}^{\mathrm{c}}(F^{\otimes})$

(c) *if* $S^*_F \neq \lambda$ *then* $\mathrm{ID}^{\mathrm{c}}(F^{\otimes}){\restriction}\mathcal{P}(\lambda \setminus S^*_F) = \mathrm{WDmId}_\lambda(F^{\otimes}){\restriction}\mathcal{P}(\lambda \setminus S^*_F)$, $\lambda \setminus S^*_F \notin \mathrm{ID}^{\mathrm{c}}(F^{\otimes})$.

(d) *if* $S^*_F \neq \emptyset$ *then* $S^*_F \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *and*

$$\big\{\big(S^*_0 \cap F^{\otimes}(f)/\mathrm{ID}^{\mathrm{b}}(F^{\otimes}), S^*_1 \cap F^{\otimes}(f)/\mathrm{ID}^{\mathrm{b}}(F^{\otimes})\big) : f \in \mathrm{DOM}_\lambda\big\}$$

*is an isomorphism from* $\mathcal{P}(S^*_0)/\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *onto* $\mathcal{P}(S^*_1)/\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$.

(2) *If in* 1.17, $S^*_F \subseteq S_{\alpha^*[\eta]}[\eta]$ mod $\mathrm{ID}^{\mathrm{b}}(F)$ *then we can add*

($\circledast$) *for some* $\rho \in {}^{X_1}2$ *for every* $f \in {}^{\lambda}2$ *we have*

$$\big\{\delta \in X_1 : F(f{\restriction}\delta) = \rho(\delta)\big\} \neq \emptyset \mod \mathrm{ID}^{\mathrm{b}}(F^{\otimes}).$$

PROOF. (1) We try to choose by induction on $\alpha < \lambda^+$ sets $S_\alpha$, $S_{\alpha,0}$, $S_{\alpha,1}$ such that

(a) $S_\alpha \subseteq \lambda$,

(b) $S_\alpha = S_{\alpha,0} \cup S_{\alpha,1}$, $S_{\alpha,0} \cap S_{\alpha,1} = \emptyset$,

(c) if $\beta < \alpha$ and $\ell < 2$ then

$$S_\beta \subseteq S_\alpha \mod \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \ \text{ and } \ S_{\beta,\ell} \subseteq S_{\alpha,\ell} \mod \mathrm{ID}^{\mathrm{b}}(F^{\otimes}),$$

(d) the sets $S_{\alpha,0}, S_{\alpha,1}$ witness that $S_\alpha \in \mathrm{ID}^{\mathrm{c}}(F^{\otimes})$ (see 1.2(4)).

At some stage $\alpha < \lambda^+$ we have to be stuck (as $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ is $\lambda^+$-saturated) and then $(S_\alpha, S_{\alpha,0}, S_{\alpha,1})$ can serve as $(S^*_F, S^*_0, S^*_1)$.

(2) By the choice of $S^*_F$, for some $\ell < 2$ we have

$$\mathcal{P}(X_\ell) \neq \{F^{\otimes}(f) \cap X_\ell : f \in {}^{\lambda}2\},$$

so let $Y \subseteq X_\ell$ be such that $Y \notin \{F^{\otimes}(f) \cap X_\ell : f \in {}^{\lambda}2\}$. Let $\rho = 0_Y \cup 1_{X_\ell \setminus Y}$. Since without loss of generality $\ell = 1$, we are done. $\square$

REMARK 1.20. (1) Recall that if $\lambda \notin \mathrm{WDmId}_\lambda$ then $S^*_F \neq \lambda$.

(2) Recall: $\mathrm{ID}^{\mathrm{a}}(F^{\otimes})) = \mathrm{WDmId}_\lambda(F^{\otimes})$ is a normal ideal and $\mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \subseteq \mathrm{ID}^{\mathrm{c}}(F^{\otimes})$ are normal ideals extending it.

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## 2. Weak diamond for more colours

In this section we deduce a weak diamond for, say, three colours, assuming the weak diamond for two colours and assuming that a certain ideal is saturated.

PROPOSITION 2.1. *Assume that $\lambda$ is a regular uncountable cardinal and $\mu \leq 2^{<\lambda}$. Let $F_i\colon {}^{\lambda>}2 \longrightarrow \{0,1\}$ be $\lambda$-colourings for $i < \mu$. Then there is a colouring $F\colon {}^{\lambda>}2 \longrightarrow \{0,1\}$ such that $F_i \leq F$ for every $i < \mu$.*

PROOF. *Case* 1: $\mu \leq 2^{\|\alpha\|}$ for some $\alpha < \lambda$. Let $\rho_i \in {}^{\alpha}2$ for $i < \mu$ be pairwise distinct. For $\eta \in {}^{\lambda>}2$ let $h_i(\eta) = \rho_i{}^\frown\eta$. Define $F$ by:

$$F(\nu) = \begin{cases} 0 & \text{if } \ell g(\nu) < \alpha, \text{ or } \ell g(\nu) \geq \alpha \\ & \text{but } \nu \restriction \alpha \notin \{\rho_i : i < \mu\}, \\ F_i(\langle \nu(\alpha+\varepsilon) : \varepsilon < \ell g(\nu) - \alpha\rangle) & \text{if } \ell g(\nu) \geq \alpha \\ & \text{and for some } i < \mu,\ \nu \restriction \alpha = \rho_i. \end{cases}$$

It is easy to see that $F\colon {}^{\lambda>}2 \longrightarrow \{0,1\}$ and $h_i$ exemplifies that $F_i \leq F$.

*Case* 2: $\mu = \lambda$. For $\eta \in {}^{\lambda>}2$, $i < \mu$, we define $h_i(\eta) \in {}^{i+1+\ell g(\eta)}2$ as follows: for $\gamma < i + 1 + \ell g(\eta)$ we let

$$h_i(\eta)(\gamma) = \begin{cases} 0 & \text{if } \gamma < i, \\ 1 & \text{if } \gamma = i, \\ \eta(\gamma - (i+1)) & \text{otherwise.} \end{cases}$$

Next, for $\nu \in {}^{\lambda>}2$ define:

$$F(\nu) = \begin{cases} F_i(\langle \nu(i+1+\gamma) : \gamma < \ell g(\nu) - (i+1)\rangle) & \text{if } i = \min\{j : \nu(j) = 1\} \\ 0 & \text{if there is no such } i. \end{cases}$$

Now check.

*Case* 3: Otherwise, for each $\alpha < \lambda$ choose $F^\alpha\colon {}^{\lambda>}2 \longrightarrow \{0,1\}$ such that $(\forall i < 2^{\|\alpha\|})(F_i \leq F^\alpha)$ (exists by Case 1). Let $F\colon {}^{\lambda>}2 \longrightarrow \{0,1\}$ be such that $(\forall \alpha < \lambda)(F^\alpha \leq F)$ (exists by Case 2).

The proposition follows. $\square$

THEOREM 2.2. *If* (A) *then* (B) *where*

(A) (a) *$\lambda$ is a regular uncountable cardinal.*

(b) $F^{\mathrm{tr}} : {}^{\lambda>}2 \longrightarrow 3$.

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(c) *For* $i < 3$ *let* $F_i\colon {}^{\lambda >}2 \longrightarrow \{0,1\}$ *be such that*

$$F_i(\eta) = \begin{cases} 1 & \textit{if } F^{\mathrm{tr}}(\eta) = i, \\ 0 & \textit{otherwise}, \end{cases}$$

(d) *Let* $F\colon {}^{\lambda >}2 \longrightarrow \{0,1\}$ *be such that* $(\forall i < 3)(F_i \le F)$. *Moreover* $F_2' \le F$ *where* $F_2'$ *is the function with domain* ${}^{\lambda <}2$ *defined by* $F_2'(\eta) = \min\{1, F(\eta)\}$ *(remember* 1.11(3)*).*

(e) $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *is* $\lambda^+$*-saturated, i.e. there is no sequence* $\langle A_\alpha : \alpha < \lambda^+\rangle$ *such that*[3]

$$(\forall \alpha < \beta < \lambda^+)(A_\alpha \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \;\&\; \|A_\alpha \cap A_\beta\| < \lambda).$$

(B) *Then there is a weak diamond sequence for* $F^{\mathrm{tr}}$*, even for every* $S \in \mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}^{\mathrm{c}}(F^{\otimes})$.

Proof. Let $S_F^*$ be as in 1.19. Since we are assuming $\lambda \notin \mathrm{ID}^{\mathrm{c}}(F^{\otimes})$ necessarily $\lambda \setminus S_F^* \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$.

[Why? because by 1.20(1)(b) we have $\mathrm{ID}^{\mathrm{c}}(F^{\otimes}) = \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) + S_F^*$].

It follows from 1.16 and 1.17 that there are disjoint sets $X_0, X_1 \subseteq \lambda$ (even disjoint from $S_F^*$ from 1.19) such that $X_0, X_1 \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$, $X_0 \cup X_1 \in \mathfrak{B}(F^{\otimes})$ and for every $f \in {}^{\lambda}2$ we have one of the following:

(a) the sequence $\langle F(f{\restriction}\delta) : \delta \in X_0 \cup X_1\rangle$ is $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$-almost constant, or

(b) both sequences $\langle F(f{\restriction}\delta) : \delta \in X_0\rangle$ and $\langle F(f{\restriction}\delta) : \delta \in X_1\rangle$ are not $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$-almost constant.

It follows from 1.19(2) that we may assume that there is $\eta \in {}^{X_1}2$ such that for every $f \in {}^{\lambda}2$ the set

$$\{\delta \in X_1 : F(f{\restriction}\delta) = \eta(\delta)\}$$

is stationary. Define a function $\rho \in {}^{\lambda}2$ as follows:

$$\rho(\alpha) = \begin{cases} 1 + \eta(\alpha) & \text{if } \alpha \in X_1, \\ 0 & \text{otherwise.} \end{cases}$$

Claim 2.2.1. *$\rho$ is a weak diamond sequence for $F^{\mathrm{tr}}$ even on $X_0 \cup X_1$.*

Proof of the claim. Let $f \in {}^{\lambda}2$ and we shall prove that $Y_f = \{\delta \in X_0 \cup X_1 : F^{\mathrm{tr}}(\eta{\restriction}\delta) = \eta(\delta)\} \neq \emptyset \mod \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$.

[3] As is well known, writing below $A_\alpha \cap A_\beta \in \mathrm{ID}^b(F^{\otimes})$ instead $\|A_\alpha \cap A_\beta\| < \lambda$ does not change anything.

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If $\{\alpha \in X_0 : F^{\rm tr}(f{\restriction}\alpha) = 2\} \notin {\rm ID}^{\rm b}(F)$ then we are done (remember 1.3(3)). Otherwise (by the definition of $F_0$), we have

$$\{\alpha \in X_0 : F_2(f \restriction \alpha) = 1\} \in {\rm ID}^{\rm b}(F).$$

For $\ell < 3$, as $F_\ell \le F$ let $f_\ell \in {}^\lambda 2$ be such that the set $\{\alpha < \lambda : F_\ell(f{\restriction}\alpha) = F(f_\ell{\restriction}\alpha)\}$ contains a club of $\lambda$ and $g \in {}^\lambda 2$ such that the set $\{\alpha < \lambda : F_2'(g{\restriction}\alpha) = F(f{\restriction}\alpha)\}$ contains a club of $\lambda$, exist by clause (A)(d) of the assumption of the theorem.

We now use $f_2$. Then

$$\{\alpha \in X_0 : F(f_2{\restriction}\alpha) = 1\} \in {\rm ID}^{\rm b}(F^{\otimes}),$$

and hence, by the choice of the sets $X_0, X_1$, clause (b) there fails hence clause (a) holds, so

$$\{\alpha \in X_1 \cup X_1 : F(f_2{\restriction}\alpha) = 1\} \in {\rm ID}^{\rm b}(F^{\otimes}).$$

Consequently,

$$Z = \{\alpha \in X_1 : F^{\rm tr}(f{\restriction}\alpha) = 2\} = \{\alpha \in X_1 : F_2(f{\restriction}\alpha) = 1\} \in {\rm ID}^{\rm b}(F^{\otimes}).$$

Now we use the choice of $\eta$, by it we know that the set

$$Y = \{\delta \in X_1 : F(g{\restriction}\delta) = \eta(\delta)\}$$

is stationary and even $\neq \emptyset \mod {\rm ID}^{\rm b}(F^{\otimes})$. Hence for some $k \in \{0,1\}$ the set

$$Y_k = \{\delta \in X_1 : F(g{\restriction}\delta) = k = \eta(\delta)\}$$

is stationary and even $\neq \emptyset \mod {\rm ID}^{\rm b}(F^{\otimes})$, but $\{\delta \in X_1 : F(g{\restriction}\delta) = F_2'(f{\restriction}\delta)\}$ contains a club. Hence

$$Y_k^* = \{\delta \in X_1 : F(g{\restriction}\delta) = k = \eta(\delta) \ \text{ and } \ F(g{\restriction}\delta) = F_2'(f{\restriction}\delta) \text{ and } \delta \notin Z\}$$

is stationary and even $\neq \emptyset \mod {\rm ID}^{\rm b}(F^{\otimes})$. Finally note that

$$\delta \in Y_k^* \quad \Rightarrow \quad F(f_1{\restriction}\delta) = \eta(\delta) = F_1(f{\restriction}\delta) = k \quad \Rightarrow \quad F^{\rm tr}(f{\restriction}\delta) = k.$$

Thus the claim and the theorem are proved. $\square$ $\square$

THEOREM 2.3. *Suppose $F^{\rm tr}$ is a $(\lambda,\theta)$-colouring, $\theta \le \lambda$ and $F_i$ (for $i < \theta$) are given by*

$$F_i(f) = \begin{cases} 1 & \text{if } F(f) = i, \\ 0 & \text{otherwise.} \end{cases}$$

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*Let* $F\colon {}^{\lambda >}2 \longrightarrow 2$ *be such that* $(\forall i < \theta)(F_i \le F)$ *and let* $F^{\otimes}$ *be as in* 1.5 *for* $F$. *Suppose that* $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *is* $\lambda^+$*-saturated, and* $S^*_{F^{\otimes}} \ne \lambda$ (*equivalently* $\lambda \notin \mathrm{ID}^{\mathrm{c}}(F^{\otimes})$).

*If* $(\otimes)$ *then* $(\bigstar)$ *where:*

$(\otimes)$ *there are sets* $Y_i \subseteq \lambda \setminus S^*_{F^{\otimes}}$ *for* $i < \theta$ *such that*

(a) $(\forall i < \theta)(Y_i \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}))$,

(b) *the sets* $Y_i$ *are pairwise disjoint or at least*

$$(\forall i < j < \theta)(Y_i \cap Y_j \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})),$$

(c) $\bigcap_{i<\theta} \min_{F^{\otimes}}(Y_i) \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$, *see* 1.12(h).

$(\bigstar)$ *there is a weak diamond sequence* $\eta \in {}^{\lambda}\theta$ *for* $F^{\mathrm{tr}}$, *i.e.*

$$(\forall f \in {}^{\lambda}2) \begin{pmatrix} \{\delta < \lambda : F^{\mathrm{tr}}(f{\restriction}\delta) = \eta(\delta)\} \text{ is stationary} \\ \text{and even} \ne \emptyset \mod \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \end{pmatrix};$$

*moreover*

$$(\forall f \in {}^{\lambda}2)(\{\delta < \lambda : F^{\mathrm{tr}}(f{\restriction}\delta) = \eta(\delta)\} \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})).$$

PROOF. We may assume that the sets $\langle Y_i : i < \theta\rangle$ are pairwise disjoint (otherwise we use $Y_i' = Y_i \setminus \bigcup_{j<i} Y_j$). Let $\eta \in {}^{\lambda}\theta$ be such that $(\forall i < \theta)(\eta{\restriction}Y_i = i)$. Note that if

$$\{\delta \in Y_i : F^{\mathrm{tr}}(f{\restriction}\delta) = i\} \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$$

then we also have

$$\{\delta < \lambda : F^{\mathrm{tr}}(f{\restriction}\delta) = i\} \in \mathfrak{B}(F^{\otimes})$$

(use $F_i \le F \le F^{\otimes}$). Consequently, in this case, we have

$$\{\delta \in \min_{F^{\otimes}}(Y_i) : F^{\mathrm{tr}}(f{\restriction}\delta) = i\} \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes}).$$

If this occurs for every $i < \theta$ then

$$\Big\{\delta \in \bigcap_{i<\theta} \min_{F^{\otimes}}(Y_i) : (\exists i < \theta)(F(f{\restriction}\delta) = i)\Big\} \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes}),$$

but for each $\delta$, for some $i < \theta$ we have $F(f{\restriction}\delta) = i$, a contradiction. $\square$

PROPOSITION 2.4. *Under the assumptions of* 2.3 (*so the ideal* $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *is* $\lambda^+$*-saturated*), *if* $X \subseteq \lambda \setminus S^*_{F^{\otimes}}$, $X \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *then there is a partition* $(X_0, X_1)$ *of* $X$ (*so* $X_0 \cup X_1 = X$, $X_0 \cap X_1 = \emptyset$) *such that*

$$X_0, X_1 \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}), \quad \textit{and} \quad \min_{F^{\otimes}}(X_0) = \min_{F^{\otimes}}(X_1) = \min_{F^{\otimes}}(X).$$

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PROOF. Let

$$\mathcal{A}_{F^\otimes} \stackrel{\text{def}}{=} \left\{ \begin{matrix} Z \subseteq \lambda : Z \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \text{ and there is a partition } (Z_0, Z_1) \\ \text{of } Z \text{ such that } \min_{F^\otimes}(Z_0) = \min_{F^\otimes}(Z_1) \mod \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \end{matrix} \right\}.$$

Note that, by 1.12(h),

$(*)$ $(\forall Y \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})^+)(\exists Z \in \mathcal{A}_{F^\otimes})(Z \subseteq Y)$.

Let $X \subseteq \lambda$, $X \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ and let $\langle Z_\alpha : \alpha < \alpha^* \rangle$ be a maximal sequence such that for each $\alpha < \alpha^*$:

$$Z_\alpha \in \mathcal{A}_{F^\otimes}, \quad Z_\alpha \subseteq X, \quad \text{and} \quad (\forall \beta < \alpha)(Z_\alpha \cap Z_\beta \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})).$$

Necessarily $\alpha^* < \lambda^+$, so without loss of generality $\alpha^* \leq \lambda$, $\min(Z_\alpha) > \alpha$ and $Z_\alpha \cap Z_\beta = \emptyset$ for $\alpha < \beta < \alpha^*$. Let $\langle Z^0_\alpha, Z^1_\alpha \rangle$ be a partition of $Z_\alpha$ witnessing $Z_\alpha \in \mathcal{A}_{F^\otimes}$. Put

$$Z_0 \stackrel{\text{def}}{=} \bigcup_{\alpha < \alpha^*} Z^0_\alpha \quad \text{and} \quad Z_1 \stackrel{\text{def}}{=} \bigcup_{\alpha < \alpha^*} Z^1_\alpha.$$

Then $Z_0 \cap Z_1 = \emptyset$, $Z_0 \cup Z_1 \subseteq X$. Note that $\bigcup_{\alpha<\alpha^*} Z_\alpha$ is equal to the diagonal union and, by $(*)$ above, $X \setminus \bigcup_{\alpha<\alpha^*} Z_\alpha \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$. Consequently we may assume $Z_0 \cup Z_1 = \bigcup_{\alpha<\alpha^*} Z_\alpha = X$. Next, since

$$\min_{F^\otimes}(Z_0) \supseteq \min_{F^\otimes}(Z^0_\alpha) \supseteq Z^0_\alpha \cup Z^1_\alpha = Z_\alpha,$$

we get

$$\min_{F^\otimes}(Z_0) \supseteq \bigcup_{\alpha<\alpha^*} Z_\alpha = X = Z_0 \cup Z_1,$$

and similarly one shows that $\min_{F^\otimes}(Z_1) \supseteq X$. Now we use 1.12(h) to finish the proof. $\square$

PROPOSITION 2.5. *Under the assumptions of* 2.3:

(1) *If* $2^\theta < \lambda$ *then there is a sequence* $\langle Y_i : i < \theta \rangle$ *as required in* 2.3($\otimes$)

(2) *Similarly if* $\theta \leq \aleph_0$.

(3) *In both cases, if* $S \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ *then we can demand* $(\forall i < \theta)(Y_i \subseteq S)$.

PROOF. (1) By induction on $\alpha \leq \theta$ we choose sets $X_\eta \subseteq \lambda$ for $\eta \in {}^\alpha 2$ such that:

(i) $X_{\langle\rangle} \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$,

(ii) if $\alpha$ is limit then $X_\eta = \bigcap_{i<\alpha} X_{\eta \restriction i}$,

(iii) if $\alpha = \beta + 1$, $\eta \in {}^\beta 2$ and $X_\eta \in \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ then $X_{\eta^\frown\langle 0\rangle} = X_\eta$, $X_{\eta^\frown\langle 1\rangle} = \emptyset$;

if $\alpha = \beta + 1$, $\eta \in {}^\beta 2$ and $X_\eta \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ then $(X_{\eta^\frown\langle 0\rangle}, X_{\eta^\frown\langle 1\rangle})$ is a partition of $X_\eta$ such that $\min_{F^\otimes}(X_{\eta^\frown\langle 0\rangle}) = \min_{F^\otimes}(X_{\eta^\frown\langle 1\rangle}) = \min_{F^\otimes}(X_\eta)$. It follows from 2.4 that we can carry out the construction.

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Clearly $\langle X_\eta : \eta \in {}^\theta 2\rangle$ is a partition of $X_{\langle\rangle}$, so (as $2^\theta < \lambda$ and $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ is $\lambda$-complete) we can find a sequence $\eta \in {}^\theta 2$ such that $X_\eta \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$. Then

$$(\forall \alpha < \theta)(X_{\eta\restriction\alpha} \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}))$$

(as each of these sets includes $X_\eta$). Moreover, for each $\alpha < \theta$ and for $\ell = 0, 1$ we have

$$\min_{F^{\otimes}}(X_{\eta\restriction\alpha^\frown\langle\ell\rangle}) \supseteq X_{\eta\restriction\alpha} \supseteq X_\eta.$$

Put $Y_\alpha = X_{\eta\restriction\alpha^\frown\langle 1-\eta(\alpha)\rangle}$. Then $\langle Y_\alpha : \alpha < \theta\rangle$ is a sequence of pairwise disjoint sets (as $X_{\eta\restriction\alpha^\frown\langle 0\rangle} \cap X_{\eta\restriction\alpha^\frown\langle 1\rangle} = \emptyset$) and for every $\alpha < \theta$

$$Y_\alpha \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes}) \quad \text{and} \quad \min_{F^{\otimes}}(Y_\alpha) \supseteq X_{\eta\restriction\alpha} \supseteq X_\eta.$$

Hence $\bigcap_{\alpha<\theta} \min_{F^{\otimes}}(Y_\alpha) \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$. Let $Z_\alpha = Y_\alpha \cap \min_{F^{\otimes}}(X_\eta)$. Note that $\min_{F^{\otimes}}(Z_\alpha) = \min_{F^{\otimes}}(X_\eta)$ (the "$\leq$" is clear; if $\min_{F^{\otimes}}(Z_\alpha) < \min_{F^{\otimes}}(X_\eta)$ then $\min_{F^{\otimes}}(X_\eta) \setminus \min_{F^{\otimes}}(Z_\alpha)$ contradicts the definition of $\min_{F^{\otimes}}(Y_\alpha)$). Thus the sequence $\langle Z_\alpha : \alpha < \theta\rangle$ is as required. Moreover

$$\min_{F^{\otimes}}(Z_\alpha) = \bigcup_{\beta<\alpha} \min_{F^{\otimes}}(Z_\beta).$$

(2) Let $X \subseteq \lambda$, $X \notin \mathrm{ID}^{\mathrm{b}}(F^{\otimes})$. By induction on $n$ we choose sets $X'_n$, $X''_n$ such that $X'_n \cap X''_n = \emptyset$, $X'_n \cup X''_n \supseteq X$, and

$$\min_{F^{\otimes}}(X'_n) = \min_{F^{\otimes}}(X''_n) = \min_{F^{\otimes}}(X).$$

For $n = 0$ we use 2.4 for $X$ to get $X'_0, X''_0$. For $n + 1$ we use 2.4 for $X''_n$ to get $X'_{n+1}, X''_{n+1}$.

Finally we let $Y_n = X''_n$ (note that $\min_{F^{\otimes}}(Y_n) = \min_{F^{\otimes}}(X)$). $\square$

Conclusion 2.6. Assume that

(A) $\lambda$ is a regular uncountable cardinal,

(B) $F$ is a $(\lambda, \theta)$-colouring such that $\lambda \notin \mathrm{ID}^{\mathrm{b}}(F)$ and $\mathrm{ID}^{\mathrm{b}}(F)$ is $\lambda^+$-saturated,

(C) $2^\theta < \lambda$ or $\theta = \aleph_0$,

(D) $(\exists \mu < \lambda)(2^\mu = 2^{<\lambda} < 2^\lambda)$ or at least $\lambda \notin \mathrm{WDmId}_\lambda$ or at least $\lambda \notin \mathrm{ID}^{\mathrm{c}}(F)$.

Then there is a weak diamond sequence for $F$. Moreover, there is $\eta \in {}^\lambda\theta$ such that for each $f \in \mathrm{DOM}_\lambda(F)$ we have

$$\{\delta < \lambda : F(f\restriction\delta) = \eta(\delta)\} \notin \mathrm{ID}^{\mathrm{b}}(F).$$

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## 3. An application of Weak Diamond

In this section we present an application of Weak Diamond in model theory. For more on model–theoretic investigations of this kind we refer the reader to [11] and earlier work [8] (and see [13]), and to an excellent survey by Makowsky, [4].

Definition 3.1. Let $\mathfrak{K}$ be a collection of models.

(1) For a cardinal $\lambda$, $\mathfrak{K}_\lambda$ stands for the collection of all members of $\mathfrak{K}$ of size $\lambda$.

(2) We say that a partial order $\leq_{\mathfrak{K}}$ on $\mathfrak{K}_\lambda$ is $\lambda$-nice if

($\alpha$) $\leq_{\mathfrak{K}}$ is a suborder of $\subseteq$ and it is closed under isomorphisms of models (i.e. if $M, N \in \mathfrak{K}_\lambda$, $M \leq_{\mathfrak{K}} N$ and $f \colon N \longrightarrow N' \in \mathfrak{K}_\lambda$ is an isomorphism then $f[M] \leq_{\mathfrak{K}} N'$),

($\beta$) $(\mathfrak{K}_\lambda, \leq_{\mathfrak{K}})$ is $\lambda$-closed (i.e. any $\leq_{\mathfrak{K}}$-increasing sequence of length $\leq \lambda$ of elements of $\mathfrak{K}_\lambda$ has a $\leq_{\mathfrak{K}}$-upper bound in $\mathfrak{K}_\lambda$) and

($\gamma$) if $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$ is an $\leq_{\mathfrak{K}}$-increasing sequence of elements of $\mathfrak{K}_\lambda$ then $\bigcup_{\alpha<\lambda} M_\alpha$ is the $\leq_{\mathfrak{K}}$-upper bound to $\bar{M}$ (so $\bigcup_{\alpha<\lambda} M_\alpha \in \mathfrak{K}_\lambda$).

(3) Let $N \in \mathfrak{K}_\lambda$, $A \subseteq |N|$. We say that *the pair* $(A, N)$ *has the w. amalgmation property in* $\mathfrak{K}_\lambda$ if for every $N_1, N_2 \in \mathfrak{K}_\lambda$ such that $N \leq_{\mathfrak{K}} N_1$, $N \leq_{\mathfrak{K}} N_2$ there are $N^* \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $F_1$, $F_2$ of $N_1, N_2$ into $N^*$, respectively, such that $f_1(A) = f_2(A)$. (In words: $N_1, N_2$ can be amalgamated over $(A, N)$ setwise.)

(4) We say that $(\mathfrak{K}, \leq_{\mathfrak{K}})$ has *the amalgamation property for* $\lambda$ if for every $M_0, M_1, M_2 \in \mathfrak{K}_\lambda$ such that $M_0 \leq_{\mathfrak{K}} M_1$, $M_0 \leq_{\mathfrak{K}} M_2$ there are $M \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $F_1$, $F_2$ of $M_1, M_2$ into $M$, respectively, such that

$$M_0 \leq_{\mathfrak{K}} M \quad \text{and} \quad f_1{\restriction}M_0 = f_2{\restriction}M_0 = \mathrm{id}_{M_0}.$$

Theorem 3.2. *Assume that $\lambda$ is a regular uncountable cardinal for which the weak diamond holds (i.e. $\lambda \notin \mathrm{WDmId}_\lambda$). Suppose that $\mathfrak{K}$ is a class of models, $\mathfrak{K}$ is categorical in $\lambda$ (i.e. all models from $\mathfrak{K}_\lambda$ are isomorphic), it is closed under isomorphisms of models, and $\leq_{\mathfrak{K}}$ is a $\lambda$-nice partial order on $\mathfrak{K}_\lambda$ and $M \in \mathfrak{K}_\lambda$. Let $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence of subsets of $|M|$ such that*

$$(\forall \alpha < \lambda)(\|A_\alpha\| < \lambda) \quad \textit{and} \quad \bigcup_{\alpha<\lambda} A_\alpha = M.$$

*Then the set*

$$S^{\bar{A}}_M \stackrel{\text{def}}{=} \big\{ \alpha < \lambda \colon (A_\alpha, M) \textit{ does not have the w. amalgmation property} \big\}$$

*is in* $\mathrm{WDmId}_\lambda$.

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PROOF. Assume that $S_M^{\bar{A}} \notin \mathrm{WDmId}_\lambda$.

We can fix a partition $\langle D_i : i < \lambda \rangle$ of $\lambda$ to sets each of cardinality $\lambda$.

We may assume that $|M| = \lambda$. By induction on $i < \lambda$ we choose pairs $(B_\eta, N_\eta)$ and sequences $\langle C_j^\eta : j < \lambda \rangle$ for $\eta \in {}^i 2$ such that

(a) $\|B_\eta\| < \lambda$, $N_\eta \in \mathfrak{K}_\lambda$, $B_\eta \subseteq |N_\eta| \subseteq \cup\{D_j : j < 1 + \ell g(\eta)\}$

(b) $\langle C_j^\eta : j < \lambda \rangle$ is increasing continuous, $\bigcup_{j<\lambda} C_j^\eta = |N_\eta|$, $\|C_j^\eta\| < \lambda$,

(c) if $\nu \triangleleft \eta$ then $N_\nu \leq_{\mathfrak{K}} N_\eta$ and $B_\nu \subseteq B_\eta$,

(d) if $j_1, j_2 < i$ then $C_{j_2}^{\eta \restriction j_1} \subseteq B_\eta$,

(e) if the pair $(B_\eta, N_\eta)$ does not have the w. amalgmation property in $\mathfrak{K}_\lambda$ then $N_{\eta^\frown\langle 0 \rangle}$, $N_{\eta^\frown\langle 1 \rangle}$ witness it (i.e. they cannot be w. amalgmated over $B_\eta$),

(f) if $i$ is limit and $\eta \in {}^i 2$ then $B_\eta = \bigcup_{j<i} B_{\eta \restriction j}$, $\bigcup_{j<i} N_{\eta \restriction j} \subseteq N_\eta$.

There are no problems with carrying out the construction (remember that $\leq_{\mathfrak{K}}$ is a nice partial order. Finally, for $\eta \in {}^\lambda 2$ we let $B_\eta = \bigcup_{i<\lambda} B_{\eta \restriction i}$ and $N_\eta = \bigcup_{i<\lambda} N_{\eta \restriction i}$. Clearly, by 3.1(2)($\gamma$), we have $N_\eta \in \mathfrak{K}$ and $B_\eta \subseteq |N_\eta|$ for each $\eta \in {}^\lambda 2$. Moreover,

$$|N_\eta| = \bigcup_{j<\lambda} |N_{\eta \restriction j}| = \bigcup_{j<\lambda} \bigcup_{i<\lambda} C_i^{\eta \restriction j} = \bigcup_{j^*<\lambda} \bigcup_{j_1, j_2 < j^*} C_{j_2}^{\eta \restriction j_1} \subseteq \bigcup_{j^*<\lambda} B_{\eta \restriction j^*} = B_\eta,$$

and thus $B_\eta = |N_\eta|$. Since $\mathfrak{K}$ is categorical in $\lambda$, for each $\eta \in {}^\lambda 2$ there is an isomorphism $f_\eta \colon N_\eta \xrightarrow{\text{onto}} M$.

Fix $\eta \in {}^\lambda 2$ for a moment.

Let $E_\eta = \{\delta < \lambda : f_\eta[B_{\eta \restriction \delta}] = A_\delta = \delta\}$. Clearly, $E_\eta$ is a club of $\lambda$. Note that if $\delta \in E_\eta$ then:

$$(\boxtimes) \quad \begin{aligned} \delta \in S_M^{\bar{A}} \Rightarrow\ & (A_\delta, M) \text{ does not have the w. amalgmation property} \\ \Rightarrow\ & (B_{\eta \restriction \delta}, N_\eta) \text{ fails the w. amalgmation property} \\ \Rightarrow\ & (B_{\eta \restriction \delta}, N_{\eta \restriction \delta}) \text{ fails the w. amalgmation property} \\ \Rightarrow\ & N_{\eta \restriction \delta^\frown \langle 0 \rangle},\ N_{\eta \restriction \delta^\frown \langle 1 \rangle} \text{ cannot be w. amalgmated} \\ & \text{over } (B_{\eta \restriction \delta}, N_{\eta \restriction \delta}) \\ \Rightarrow\ & \text{for each } \nu \in {}^\lambda 2 \text{ such that } \eta \restriction \delta^\frown \langle 1 - \eta(\delta) \rangle \triangleleft \nu \\ & \text{we have } f_\nu \restriction B_{\eta \restriction \delta} \neq f_\eta \restriction B_{\eta \restriction \delta}. \end{aligned}$$

We define a colouring

$$F \colon \bigcup_{\alpha<\lambda} {}^\alpha(\mathcal{H}(\lambda)) \longrightarrow \{0, 1\}$$

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by letting, for $f \in \mathrm{DOM}_\alpha$, $\alpha < \lambda$,

$$F(f) = 1 \quad \text{iff} \quad (\exists \eta \in {}^{\lambda}2)\big(\eta(\alpha) = 0 \ \& \ (\forall i < \alpha)(f(i) = (\eta(i), f_\eta^{-1}(i)))\big).$$

We have assumed $S_M^{\bar{A}} \notin \mathrm{WDmId}_\lambda$, so there is $\rho \in {}^{\lambda}2$ such that for each $f \in \mathrm{DOM}_\lambda$ the set

$$S_f = \{\delta \in S_M^{\bar{A}} : \rho(\delta) = F(f{\restriction}\delta)\}$$

is stationary and even $\neq \emptyset$ mod $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$. Let $f \in \mathrm{DOM}_\lambda$ be defined by $f(i) = (\rho(i), f_\rho^{-1}(i))$ (for $i < \lambda$). Note that if $\alpha \in E_\rho$, $\rho(\alpha) = 0$ then $\rho$ is a witness to $F(f{\restriction}\alpha) = 1$ and hence $\alpha \notin S_f$.

Since $S_f$ is stationary and even $\neq \emptyset$ mod $\mathrm{ID}^{\mathrm{b}}(F^{\otimes})$ and $E_\rho$ is a club of $\lambda$ we may pick $\delta \in S_f \cap E_\rho$. Then $\rho(\delta) = 1$ and hence $F(f{\restriction}\delta) = 1$, so let $\eta_\delta \in {}^{\lambda}2$ be a witness for it. It follows from the definition of $F$ that then $\eta_\delta(\delta) = 0$, and $\eta_\delta{\restriction}\delta = \rho{\restriction}\delta$, and $f_{\eta_\delta}^{-1}{\restriction}\delta = f_\rho^{-1}{\restriction}\delta$. Hence $f_{\eta_\eta}{\restriction}B_{\eta_\delta{\restriction}\delta} = f_\rho{\restriction}B_{\rho{\restriction}\delta}$, so both have range $A_\delta = \delta$ (and $\delta \in E_{\eta_\delta} \cap E_\rho \cap S_M^{\bar{A}}$). But now we get a contradiction with ($\boxtimes$). $\square$

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