Madness and regularity properties

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Abstract

Starting from an inaccessible cardinal, we construct a model of $ZF + DC$ where there exists a mad family and all sets of reals are $\mathbb{Q}$-measurable for ω^ω-bounding sufficiently absolute forcing notions $\mathbb{Q}$. As a corollary, we obtain answers to questions of Enayat and Henle-Mathias-Woodin.[1]

Introduction

Our study concerns the interactions between mad families and other types of pathological sets of reals. Given a forcing notion $\mathbb{Q}$ whose conditions are subtrees of $\omega^{<\omega}$ ordered by reverse inclusion, the notion of $\mathbb{Q}-$measurability is naturally defined. As the existence of mad families and non-$\mathbb{Q}-$measurable sets follows from the axiom of choice, one may consider the possible implications between the existence of mad families and the existence of non-$\mathbb{Q}-$measurable sets. The study of models of $ZF + DC$ where no mad families exist was initiated by Mathias in [Ma], more results were obtained recently in [HwSh1090], [NN] and [To]. Models of $ZF + DC$ where all sets of reals are $\mathbb{Q}$-measurable for various forcing notions $\mathbb{Q}$ were first studied by Solovay in [So].

Our main goal is to show that $\mathbb{Q}-$measurability for ω^ω-boundning sufficiently absolute forcing notions does not imply the non-exsitence of mad families. In particular, as Random real forcing is ω^ω-bounding, it will follow that Lebesgue measurability for all sets of reals does not imply the non-existence of mad families.

We follow the strategy of [Sh218], where a model of $ZF + DC +$ "all sets of reals are Lebesgue measurable but there is a set without the Baire Property" was constructed. Fixing an inaccessible cardinal κ, we define a partial order AP consisting of pairs $(\mathbb{P}, \Gamma)$, where $\mathbb{P}$ is a forcing notion from $H(\kappa)$ and Γ is an approximation of the desired mad family such that finite unions of members of Γ are not dominated by reals from V. We shall obtain our model by forcing with this partial order and then with the partial order introduced generically by AP. The main point will be an amalgamation argument for AP (over $\mathbb{Q}$-generic reals for an appropriate $\mathbb{Q}$), which will allow us to repeat Solovay's argument from [So].

Remark: It was brought to our attention by Paul Larson and Jindra Zapletal that a model of "every set of reals is Lebesgue measurable and there is a mad family" can also be constructed using the arguments from Section 5 of their paper [LZ]. However, they assume the existence of a proper class of Woodin cardinals, while in this paper we only assume the existence of an inaccessible cardinal.

The main result

Hypothesis 1: Throughout the paper, $\mathbf{f}$ will be a fixed forcing frame (defined below) with $\kappa_{\mathbf{f}} = \kappa$ a fixed inaccessible cardinal.

Definition 2: Let $\mathbf{f} = (\kappa_{\mathbf{f}}, \mathbf{P}_{\mathbf{f}}, \mathbf{Q}_{\mathbf{f}}) = (\kappa, \mathbf{P}, \mathbf{Q})$ be a forcing frame when:

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a. κ is the inaccessible cardinal from Hypothesis 1.

b. $\mathbf{P}$ is the set of forcing notions from $H(\kappa)$.

c. $\mathbf{Q}$ is a family of $\omega^\omega-$bounding forcing notions with sufficiently absolute definitions.

d. If $\mathbb{P} \in \mathbf{P}$ and $V^{\mathbb{P}} \models$ "$\mathbb{Q} \in \mathbf{Q}$", then $\mathbb{Q} \in H(\kappa)^{(V^{\mathbb{P}})}$.

Definition 3: Let $AP = AP_{\mathbf{f}}$ be the partial order defined as follows:

a. $a \in AP$ iff a has the form $(\mathbb{P}, \Gamma) = (\mathbb{P}_a, \Gamma_a)$ where:

1. $\mathbb{P} \in \mathbf{P}$ and Γ is an infinite set of canonical $\mathbb{P}-$names of reals such that $\Vdash_{\mathbb{P}}$ "Γ is almost disjoint".

2. If $\underset{\sim}{\tau} \in \Gamma$, then $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\tau}$ is an infinite subset of ω".

3. For $a \in AP$, let Ω_a be the set of $\underset{\sim}{\tau} \in \Gamma_a$ which are objects and not just names.

4. If $1 \leq n$, $\underset{\sim}{a_0}, ..., \underset{\sim}{a_{n-1}} \in \Gamma_a \setminus \Omega_a$, $\underset{\sim}{a} = \underset{l<n}{\cup} \underset{\sim}{a_l}$ and $\underset{\sim}{f_a} : \omega \to \omega$ is the function enumerating $\underset{\sim}{a}$ in an increasing order, then $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f_a}$ is not dominated by any $f \in (\omega^\omega)^V$".

b. $a \leq_{AP} b$ iff

1. $\mathbb{P}_a \lessdot \mathbb{P}_b$.

2. $\Gamma_a \subseteq \Gamma_b$.

3. If $\underset{\sim}{a_0}, ..., \underset{\sim}{a_{n-1}} \in \Gamma_b \setminus \Gamma_a$, $\underset{\sim}{a} = \underset{l<n}{\cup} \underset{\sim}{a_l}$ and $\underset{\sim}{f_a}$ enumerates $\underset{\sim}{a}$ in an increasing order, then $\Vdash_{\mathbb{P}_b}$ "$\underset{\sim}{f_a}$ is not dominated by any member of $(\omega^\omega)^{V[G \cap \mathbb{P}_a]}$.

Observation 4: $(AP, \leq)$ is indeed a partial order.

Proof: Suppose that $a \leq b$ and $b \leq c$. Let $\underset{\sim}{a_0}, ..., \underset{\sim}{a_{n-1}} \in \Gamma_c \setminus \Gamma_a$, and let $\underset{\sim}{a}$ and $\underset{\sim}{f_a}$ be as in Definition 3(b)(3). We may assume wlog that for some $0 < m < n$, $\underset{\sim}{a_l} \in \Gamma_b$ iff $l < m$ (the cases $m = 0$ and $m = n$ are trivial). Let $G_c \subseteq \mathbb{P}_c$ be V-generic and let $G_a = G_c \cap \mathbb{P}_a$ and $G_b = G_c \cap \mathbb{P}_b$. Let $g = (n_i : i < \omega) \in V[G_a]$, wlog g is increasing. We shall prove that $\underset{\sim}{f_a}$ is not dominated by g.

Let $a_i = \underset{\sim}{a_i}[G_c]$, $a = \underset{\sim}{a}[G_c]$ and $b = \underset{l<m}{\cup} a_l$.

Subclaim 1: For infinitely many i, $[n_i, n_{i+1}) \cap (\underset{l<n}{\cup} a_l) = \emptyset$.

Subclaim 2: Subclaim 1 is equivalent to "f_a is not dominated by g".

Proof of Subclaim 1: Let $u = \{i : [n_i, n_{i+1}) \cap b = \emptyset\} \in V[G_b]$. By the fact that $a \leq b$ and by subclaim 2, u is infinite. Let $(i(l) : l < \omega) \in V[G_b]$ be an increasing enumeration of u, so $(n_{i(l)} : l < \omega) \in V[G_b]$ is increasing. Let $c = \underset{m \leq l < n-1}{\cup} a_l$ and $v = \{l : [n_{i(l)}, n_{i(l+1)}) \cap c = \emptyset\}$. As before, v is infinite. If $l \in v$ then $c \cap [n_{i(l)}, n_{i(l+1)}) = \emptyset$ and therefore, $c \cap [n_{i(l)}, n_{i(l)+1}) = \emptyset$. Similarly, if $l \in v$ then $i(l) \in u$ and therefore $b \cap [n_{i(l)}, n_{i(l)+1})$. It follows that $l \in v \to (b \cup c) \cap [n_{i(l)}, n_{i(l)+1}) = \emptyset$, and as v is infinite, we're done.

Proof of Subclaim 2: Suppose that $\underset{\sim}{f_a}$ is not dominated by any $g \in (\omega^\omega)^{V^{\mathbb{P}_a}}$ and let $g = (n_i : i < \omega) \in V^{\mathbb{P}_a}$ be increasing. Choose $f \in V^{\mathbb{P}_c}$ such that f is increasing, $l < f(l)$ for every l and $|\{i : n_i \in [l, f(l))\}|$ is sufficiently large (e.g. $> 2^l$). By our assumption, for infinitely many l, $f(l) \leq$ the lth member of $\underset{\sim}{a}$, and therefore

$|\underset{\sim}{a} \cap f(l)| \leq l$. Let $u = \{l : |\underset{\sim}{a} \cap f(l)| \leq l\}$, so u is infinite. For $l \in u$, $l + 1 < |\{i : l \leq i, [n_i, n_{i+1}) \subseteq [l, f(l))\}|$, and as u is infinite, for some i such that $l \leq i$, $[n_i, n_{i+1}) \subseteq [l, f(l))$ and $[n_i, n_{i+1}) \cap \underset{\sim}{a} = \emptyset$. Therefore, for infinitely many i, $[n_i, n_{i+1}) \cap \underset{\sim}{a} = \emptyset$.

In the other direction, suppose that $\underset{\sim}{f_a}$ satisfies the condition of Subclaim 1. Let $g \in (\omega^\omega)^{V^{\mathbb{P}_a}}$, we shall prove that $\underset{\sim}{f_a}$ is not dominated by g. We may assume wlog that g is increasing. Choose the sequence $(n_i : i < \omega)$ by induction such that $n_0 = 0$ and $n_{i+1} > n_i + g(n_i)$, so $(n_i : i < \omega) \in V^{\mathbb{P}_a}$. By the assumption, the set $u = \{i : [n_i, n_{i+1}) \cap \underset{l<n}{\cup} a_l = \emptyset\}$ is infinite. For every $i \in u$, $|a \cap n_i| \leq n_i$, therefore $n_i < f_a(n_i)$. As $[n_i, n_{i+1}) \cap a = \emptyset$, it follows that $n_{i+1} \leq f_a(n_i)$, therefore $g(n_i) < n_{i+1} \leq f_a(n_{n_i})$, so f_a is not dominated by g. $\square$

Observation 4: a. Every $\mathbb{P} \in \mathbf{P}$ is $\kappa - cc$, and $\mathbf{P}$ is closed under $\lessdot-$increasing unions of length $< \kappa$.

b. If $\mathbb{P} \in \mathbf{P}$ and $\underset{\sim}{\mathbb{Q}}$ is a canonical $\mathbb{P}-$name of a case of $\mathbf{Q}$ which is in $H(\kappa)$, then $\mathbb{P} \star \underset{\sim}{\mathbb{Q}} \in \mathbf{P}$. $\square$

Observation 5: a. If $a \in AP$ then $(\{0\}, \Omega_a) \in AP$ and $(\{0\}, \Omega_a) \leq a$.

b. AP is $(< \kappa)-$complete. $\square$

Claim 6: $(AP, \leq)$ has the division property, namely, if $a \leq b$ and $\underset{\sim}{x}$ is a $\mathbb{P}_b$-name of a real such that $\Vdash_{\mathbb{P}_b}$ "$(\omega^\omega)^{V[\mathbb{P}_a]}$ is cofinal in $(\omega^\omega)^{V[\mathbb{P}_a, \underset{\sim}{x}]}$", then there is $a_1 \in AP$ such that:

a. $a \leq a_1 \leq b$.

b. $\Gamma_{a_1} = \Gamma_a$.

c. $\mathbb{P}_{a_1} = \mathbb{P}_a \star \underset{\sim}{x}$ in the natural sense. $\square$

Claim 7 ($(AP, \leq)$ has the amalgamation property): Assume that $a_0 \leq a_l$ $(l = 1, 2)$, then there are b_l $(l \leq 3)$ and g_l $(l \leq 2)$ such that:

a. $b_0 \leq b_l \leq b_3$ $(l = 1, 2)$.

b. g_l is an isomorphism from b_l to a_l.

c. $g_0 \subseteq g_l$ $(l = 1, 2)$.

Proof: We may assume wlog that $\mathbb{P}_{a_0}$ is trivial and that $\Omega_{a_1} = \Omega_{a_2} = \Gamma_{a_0}$ (as we can simply take the quotients).

We define $\mathbb{P}_{b_3}$ as follows:

a. $p \in \mathbb{P}_{b_3}$ iff $p = (p_1, p_2) \in \mathbb{P}_{a_1} \times \mathbb{P}_{a_2}$ and for some $l(p), n_p, A_{p,1}, A_{p,2}, \underset{\sim}{a_{p,1}}, \underset{\sim}{a_{p,2}}$ the following hold:

1. $l(p) \in \{1, 2\}$ and $n_p < \omega$.

2. $A_{p,l}$ is a finite subset of Γ_{a_l} with union $\underset{\sim}{a_{p,l}}$ $(l = 1, 2)$.

3. For every $n > n_p$, there is $r_n \in \mathbb{P}_{a_{l(p)}}$ such that $\mathbb{P}_{a_{l(p)}} \models p_{l(p)} \leq r_n$ and $r_n \Vdash$ "$\underset{\sim}{a_{p,l(p)}} \cap n \subseteq n_p$".

b. $\mathbb{P}_{b_3} \models p \leq q$ iff

1. $p = (p_1, p_2), q = (q_1, q_2) \in \mathbb{P}_{a_1} \times \mathbb{P}_{a_2}$.

2. $p_l \leq q_l$ $(l = 1, 2)$.

3. $n_p \leq n_q$.

4. $A_{p,l} \subseteq A_{q,l}$ $(l = 1, 2)$.

5. There is no $n \in [n_p, n_q)$ such that $q_1 \nVdash$ "$n \notin \underset{\sim}{a}_{q,1}$" and $q_2 \nVdash$ "$n \notin \underset{\sim}{a}_{q,2}$".

We shall now define embeddings $f_l : \mathbb{P}_{a_l} \to \mathbb{P}_{b_3}$ $(l = 1, 2)$ as follows: For $p \in \mathbb{P}_{a_l}$, $f_l(p) = q \in \mathbb{P}_{b_3}$ will be the condition defined as follows:

a. $q_l = p$ and $q_{3-l} = 0_{\mathbb{P}_{a_{3-l}}} \in \mathbb{P}_{a_{3-l}}$.

b. $l(q) = l$, $n_p = 0$.

c. $A_{q,1} = \emptyset = A_{q,2}$.

Subclaim 0: $\mathbb{P}_{b_3}$ is a partial order.

Subclaim 1: For every $p = (p_1, p_2) \in \mathbb{P}_{b_3}$ and open dense $I \subseteq \mathbb{P}_{3-l(p)}$, there is $q \in \mathbb{P}_{b_3}$ above p such that $l(q) = 3 - l(p)$ and $q_{3-l(p)} \in I$.

Proof: Let $i = 3 - l(p)$ and let $p'_i \in I$ be above p_i. By the definition of AP, $f_{\underset{\sim}{a}_{p,i}}$ is not dominated by any function from V. We shall prove that there are $q_i \in \mathbb{P}_{a_i}$ above p'_i and $n_* > n_p$ such that for every $n > n_*$, there is q' above q_i such that $q' \Vdash$ "$\underset{\sim}{a}_{p,i} \cap [n_*, n) = \emptyset$". Actually, $q_i = p'_i$ should work. Suppose not, then for every $n_* > n_p$ there is $n > n_*$ such that there is no q' above q_i forcing that $\underset{\sim}{a}_{p,i} \cap [n_*, n) = \emptyset$. Now choose $(n_j : j < \omega)$ by induction on j as follows: $n_0 = n_p + 1$, and n_{j+1} is the minimal $n > n_j$ such that there is no q' above q_i forcing that $\underset{\sim}{a}_{p,i} \cap [n_j, n) = \emptyset$. By the same argument as in the proof of observation 4, as $(n_j : j < \omega) \in V$, $p'_i \Vdash$ "$\underset{\sim}{a}_{p,i} \cap [n_j, n_{j+1}) = \emptyset$ for infinitely many j". Therefore, there is q' above p'_i and i_* such that $q' \Vdash$ "$\underset{\sim}{a} \cap [n_{i_*}, n_{i_*+1}) = \emptyset$", contradicting the choice of n_{i_*+1}.

Now define $q \in \mathbb{P}_{b_3}$ as follows:

1. q_i is as above.

2. $q_{l(p)}$ is any member of $\mathbb{P}_{l(p)}$ which is above $p_{l(p)}$ and forces that $[n, n_*) \cap \underset{\sim}{a}_{p,l(p)} = \emptyset$ (such condition exists by clause (a)(3) in the definition of $\mathbb{P}_{b_3}$).

3. $l(q) = i$.

4. $n_q = n_*$.

5. $A_{q,l} = A_{p,l}$ and $\underset{\sim}{a}_{q,l} = \underset{\sim}{a}_{p,l}$ for $l = 1, 2$.

It's now easy to check that q is as required.

Subclaim 2: a. $\{p \in \mathbb{P}_{b_3} : l(p) = i\}$ is dense in $\mathbb{P}_{b_3}$ for $i = 1, 2$.

b. $I_n := \{p \in \mathbb{P}_{b_3} : n_p > n\}$ is dense in $\mathbb{P}_{b_3}$.

Proof: (a) follows from Subclaim 1. (b) follows from the proof of Subclaim 1, as we note that $n_q = n_* > n_p$ in that proof.

Subclaim 3: $f_l : \mathbb{P}_{a_l} \to \mathbb{P}_{b_3}$ is a complete embedding for $l = 1, 2$.

Proof: It suffices to show that f_l is a complete embedding into $\{p \in \mathbb{P}_{b_3} : l(p) = l\}$, which follows from the existence of a projection $\pi : \{p \in \mathbb{P}_{b_3} : l(p) = l\} \to \mathbb{P}_{a_l}$ defined in the natural way.

Subclaim 4: For every finite $A_1 \subseteq \Gamma_{a_1}$ and $A_2 \subseteq \Gamma_{a_2}$, the set $\{p \in \mathbb{P}_{b_3} : \bigwedge_{i=1,2} A_i \subseteq A_{p,i}\}$ is open dense.

Proof: In order to prove the claim by induction on $|A_1| + |A_2|$, it suffices to prove it when $A_i = \{\underset{\sim}{b}\}$ and $A_{3-i} = \emptyset$ for $i \in \{1,2\}$. Let $p \in \mathbb{P}_{b_3}$ and suppose that $l(p) = 3 - i$, it's now easy to extend p simply by adding $\underset{\sim}{b}$ to $A_{p,i}$. If $l(p) = i$, then by previous claims, there is q above p such that $l(q) = 3 - l(p)$, and now extend q as in the previous case.

Subclaim 5: Let $\Gamma := f_1(\Gamma_{a_1}) \cup f_2(\Gamma_{a_2})$, then Γ is a set of canonical $\mathbb{P}_{b_3}$-names of infinite subsets of ω and $\Vdash_{\mathbb{P}}$ "Γ is almost disjoint".

Proof: The first part follows by the fact that f_1 and f_2 are complete embeddings. In order to prove the second part, it suffices to show that if $\underset{\sim}{r} \in \Gamma_{a_1}$ and $\underset{\sim}{s} \in \Gamma_{a_2}$, then $\Vdash_{\mathbb{P}}$ "$|\underset{\sim}{r} \cap \underset{\sim}{s}| < \aleph_0$". Given $p \in \mathbb{P}_{b_3}$, by Subclaim 4, there is a stronger condition q such that $\underset{\sim}{r} \in A_{q,1}$ and $\underset{\sim}{s} \in A_{q,2}$. We shall prove that $q \Vdash$ "$|\underset{\sim}{r} \cap \underset{\sim}{s}| < \aleph_0$".

Recall that for every n, the set $I_n = \{r \in \mathbb{P}_{b_3} : n \leq n_r\}$ is dense. Now let $G \subseteq \mathbb{P}_{b_3}$ be generic over V such that $q \in G$, then for every $n_q < n$, there is $q_n \in G$ such that $n \leq n_{q_n}$. By the definition of the partial order $\leq_{\mathbb{P}_{b_3}}$ (clause (b)(5)), it follows that $q \Vdash_{\mathbb{P}_{b_3}}$ "$|\underset{\sim}{r} \cap \underset{\sim}{s}| < \aleph_0$".

Subclaim 6: Let $b_3 = (\mathbb{P}_{b_3}, \Gamma_{b_3})$ where Γ_{b_3} is Γ from the previous subclaim, then b_3 satisfies clauses $(1) + (2)$ from Definition (3)(a). As $\Omega_{a_1} = \Omega_{a_2}$, it follows that $\Omega_{b_3} = \Omega_{a_1} = \Omega_{a_2}$.

For $l = 1,2$, let $b_l = f_l(a_l) \in AP$, then clauses $(1) + (2)$ from Definition (3)(b) hold for b_l and b_3.

Subclaim 7: $b_3 \in AP$.

Proof: Let $A \subseteq \Gamma_{b_3} \setminus \Omega_{b_3}$ be finite, so there are finite sets $A_l \subseteq \Gamma_{a_l} \setminus \Omega_{a_l}$ $(l = 1,2)$ such that $A = f_1(A_1) \cup f_2(A_2)$. Let $(n_i : i < \omega) \in (\omega^\omega)^V$ be increasing and let $\underset{\sim}{u} = \{i : [n_i, n_{i+1}) \cap (\cup\{\underset{\sim}{a} : \underset{\sim}{a} \in A\}) = \emptyset\}$. Let $(p_1, p_2) \in \mathbb{P}_{b_3}$ and $n < \omega$, we shall find (q_1, q_2) and $i > n$ such that $(p_1, p_2) \leq (q_1, q_2) \in \mathbb{P}_{b_3}$ and $(q_1, q_2) \Vdash_{\mathbb{P}_{b_3}}$ "$i \in \underset{\sim}{u}$". Without loss of generality, $l((p_1, p_2)) = 2$, and by Subclaim 4, wlog $A_i \subseteq A_{(p_1,p_2),i}$ $(i = 1,2)$. For $l = 1,2$, let $\underset{\sim}{a_l} = \cup\{\underset{\sim}{a} : \underset{\sim}{a} \in A_l\}$, so $\underset{\sim}{a_l}$ is a $\mathbb{P}_{a_l}-$name and $\Vdash_{\mathbb{P}_{a_l}}$ "$(\exists^\infty i)(\underset{\sim}{a_l} \cap [n_i, n_{i+1}) = \emptyset)$". Choose $(p_{1,l}, j_{1,l} : l < \omega)$ by induction on $l < \omega$ such that:

1. $p_{1,0} = p_1$.
2. $\mathbb{P}_{a_1} \models p_{1,l} \leq p_{1,l+1}$.
3. $j_{1,l} > l + \sum_{k<l} j_{1,k}$.
4. $p_{1,l+1} \Vdash_{\mathbb{P}_{a_1}}$ "$\underset{\sim}{a_1} \cap [n_{j_{1,l}}, n_{j_{1,l}+1}) = \emptyset$".

For $l < \omega$, let $m_l = n_{j_{1,l}}$, so $(m_l : l < \omega) \in (\omega^\omega)^V$ is increasing. Let j be the minimal $j > n$ such that $n_{(p_1,p_2)} \leq m_j$. By the proof of Subclaim 1, there are p'_1 above $p_{1,j+1}$ and $k_* > n_{(p_1,p_2)}$ such that for every $k > k^*$ there is p'' above p'_1 forcing that $\underset{\sim}{a}_{(p_1,p_2),1} \cap [k^*, k) = \emptyset$. As $l((p_1, p_2)) = 2$, there is p'_2 above p_2 forcing that $\underset{\sim}{a}_{(p_1,p_2),2} \cap [n_{(p_1,p_2)}, k^* + m_{j+1}) = \emptyset$. Now let $(q_1, q_2) = (p'_1, p'_2)$, $n_{(q_1,q_2)} = k^*$, $l((q_1, q_2)) = 1$, $A_{(q_1,q_1),i} = A_{(p_1,p_2),i}$ $(i = 1,2)$, it's easy to see that (q_1, q_2) and j are as required.

Subclaim 8: $b_l \leq b_3$ where $b_l = f_l(a_l)$ $(l = 1, 2)$.

Proof: By symmetry, it suffices to prove the claim for $l = 1$. Let $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in \Gamma_{b_3} \setminus \Gamma_{b_1}$, $\underset{\sim}{a} = \underset{l<n}{\cup} \underset{\sim}{a}_l$ and let $\underset{\sim}{g}$ be a $\mathbb{P}_{b_1}$-name of an increasing sequence from ω^ω, we shall prove that $\Vdash_{\mathbb{P}_{b_3}}$ "$\underset{\sim}{u} := \{i : \underset{\sim}{a} \cap [g(i), g(i+1)) = \emptyset\}$ is infintie". There are $\underset{\sim}{a}'_l \in \Gamma_{a_2} \setminus \Omega_{a_2}$ $(l < n)$ such that $\underset{l<n}{\wedge} f_2(\underset{\sim}{a}'_l) = \underset{\sim}{a}_l$, let $\underset{\sim}{a}' = \underset{l<n}{\cup} \underset{\sim}{a}'_l$. Let $(\underset{\sim}{m}_i : i < \omega)$ be the $\mathbb{P}_{a_1}$-name for $f_1^{-1}((\underset{\sim}{g}(i) : i < \omega))$. Let $(p_1, p_2) \in \mathbb{P}_{b_3}$ and $n_* < \omega$, we shall find $(q_1, q_2) \in \mathbb{P}_{b_3}$ above (p_1, p_2) and $n > n_*$ such that $(q_1, q_2) \Vdash_{\mathbb{P}_{b_3}}$ "$n \in \underset{\sim}{u}$". We can choose $(p_{1,i}, m_{1,i} : i < \omega)$ by induction on $i < \omega$ such that $p_1 \leq p_{1,i} \in \mathbb{P}_{a_1}$, $p_{1,i} \leq p_{1,i+1}$ and $p_{1,i+1} \Vdash_{\mathbb{P}_{a_1}}$ "$\underset{\sim}{m}_i = m_{1,i}$". The rest of the proof is as in the previous subclaim. $\square$

Claim 8: For a dense set of $a \in AP$, $\Vdash_{\mathbb{P}_a}$ "Γ_a is mad".

Proof: Let $\lambda_0 = |\mathbb{P}_a|$ and $\lambda_1 = 2^{\lambda_0}$. Let $\mathbb{R}_1 = Col(\aleph_0, \lambda_1)$ and $\mathbb{P} = \mathbb{P}_a \times \mathbb{R}_1 \in H(\kappa)$. In $V^{\mathbb{P}}$, $\aleph_1^{V^{\mathbb{P}}} = \lambda_1^+$ and $\mathbb{P}_a \cup \mathcal{P}(\mathbb{P}_a)$ is countable, so $(\omega^\omega)^{V^{\mathbb{P}_a}}$ is countable and $\Gamma := \{\underset{\sim}{\tau} : \underset{\sim}{\tau}$ is a canonical $\mathbb{P}-$name of a real such that the function listing $\underset{\sim}{\tau}$ dominates $(\omega^\omega)^{V^{\mathbb{P}_a}}\}$ is dense in $[\omega]^\omega$. By the density of Γ, we can find $\Gamma' \subseteq \Gamma$ such that $\Vdash_{\mathbb{P}}$ "$\Gamma' \cup \Gamma_a$ is mad". Now let $b = (\mathbb{P}, \Gamma' \cup \Gamma_a)$, then (ignoring the obvious clauses) we need to prove that b satisfies definition 3(a)(4) and that $a \leq b$ (for which we need to prove that the requirement from 3(b)(3) is satisfied). We shall prove that a and b satisfy requirement 3(b)(3), the proof that b satisfies 3(a)(4) is similar. We shall work in $V^{\mathbb{P}_b}$. Let $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_n \in \Gamma_b \setminus \Gamma_a$ and let $\underset{\sim}{a} = \underset{l\leq n}{\cup} \underset{\sim}{a}_l$. Suppose that $(m_i : i < \omega) \in V^{\mathbb{P}_a}$ is increasing, choose a sequence $(i(k) : k < \omega) \in V^{\mathbb{P}_a}$ such that $i(k+1) > m_{i(k)+1} + i(k) + (n+1)k$ and let $m'_k = m_{i(k)+1}$ $(k < \omega)$. For each $l \leq n$, the set $\underset{\sim}{u}_l = \{k < \omega : \underset{\sim}{f}_{a_l}(k) > m_{i(k+1)}\}$ is cofinite (by the definition of Γ). Therefore, for every k large enough, $|\underset{\sim}{a}_l \cap m_{i(k+1)}| < k$ (for every $l \leq n$), hence $|\underset{\sim}{a} \cap m_{i(k+1)}| < (n+1)k$. For each such k, $|\{i : i \in [i(k), i(k+1)) \wedge \underset{\sim}{a} \cap [m_i, m_{i+1}) \neq \emptyset\}| < (n+1)k$. As $i(k+1) - i(k) > (n+1)k$, there is $i \in [i(k), i(k+1))$ such that $\underset{\sim}{a} \cap [m_i, m_{i+1}) = \emptyset$. Therefore, $\underset{\sim}{f}_a$ is not dominated by a real from $V^{\mathbb{P}_a}$. $\square$

Claim 9: For every $a \in AP$ and a $\mathbb{P}_a$-name $\underset{\sim}{r}$ of a member of $[\omega]^\omega$, there is $b \in AP$ above a such that $\Vdash_{\mathbb{P}_b}$ "there is $\underset{\sim}{s} \in \Gamma_b$ such that $|\underset{\sim}{r} \cap \underset{\sim}{s}| = \aleph_0$".

Proof: Follows directly from Claim 8. $\square$

Observation 10: Let $\mathbb{Q}$ be a forcing notion from **Q**. Assume that $a_0 \leq a_l$, $\underset{\sim}{\eta}_l$ is a $\mathbb{P}_{a_l}$-name of a $\mathbb{Q}-$generic real over $V^{\mathbb{P}_{a_0}}$ $(l = 1, 2)$, and $\mathbb{P}_{a_0} \star \underset{\sim}{\eta}_1$ is isomorphic to $\mathbb{P}_{a_0} \star \underset{\sim}{\eta}_2$ over $\mathbb{P}_{a_0}$ (so wlog they're equal to each other and we may denote the generic real by $\underset{\sim}{\eta}$). By Claim 6, there is $a'_0 \in AP$ such that $a_0 \leq a'_0 \leq a_l$ $(l = 1, 2)$, $\mathbb{P}_{a'_0} = \mathbb{P}_{a_0} \star \underset{\sim}{\eta}$ and $\Gamma_{a'_0} = \Gamma_{a_0}$. By Claim 7, there are b_l $(l \leq 3)$ and g_l $(l \leq 2)$ as there for (a'_0, a_1, a_2) here. $\square$

Definition 11: Let $H \subseteq AP$ be generic over V and let $V_1 = V[H]$. In V_1, let $\underset{\sim}{\mathbb{P}}[H]$ be $\underset{a\in H}{\cup} \mathbb{P}_a$.

Claim 12: $\Vdash_{AP}$ "$\underset{\sim}{\mathbb{P}} \models \kappa - cc$".

Proof: Suppose towards contradiction that $\Vdash_{AP}$ "$\underset{\sim}{I} \subseteq \underset{\sim}{\mathbb{P}}$ is a maximal antichain of cardinality κ". Choose by induction on $\alpha < \kappa$ a sequence $(a_\alpha, p_\alpha : \alpha < \kappa)$ such that:

a. $a_\alpha \in AP$.

b. $(a_\beta : \beta < \alpha)$ is $\leq_{AP}$-increasing cotinuous.

c. $a_{\beta+1} \Vdash_{AP}$ "$p_\beta \in \underset{\sim}{I} \setminus \{p_\gamma : \gamma < \beta\}$".

d. $p_\beta \in \mathbb{P}_{\beta+1}$.

For every $\alpha < \kappa$, there is $q_\alpha \in \mathbb{P}_{a_{<\alpha}} := \underset{\gamma<\alpha}{\cup} \mathbb{P}_{a_\gamma}$ such that p_α is compatible with every $r \in \mathbb{P}_{a_{<\alpha}}$ above q_α. Let $\gamma(\alpha) < \alpha$ be the least γ such that $q_\alpha \in \mathbb{P}_{a_\gamma}$. For some $\gamma(*) < \kappa$, $S := \{\alpha : \gamma(\alpha) = \gamma(*)\}$ is stationary. As $|\mathbb{P}_{a_{\gamma(*)}}| < \kappa$, there is $S' \subseteq S$ of cardinality κ such that $\alpha_1 < \alpha_2 \in S' \to q_{\alpha_1} = q_{\alpha_2}$, which leads to a contradiction. $\square$

Definition 13: Let V_1 be as in Definition 11 and let $G \subseteq \mathbb{P}[\underset{\sim}{H}]$ be generic over V_1, we shall denote $V[H, G]$ by V_2.

Caim 14: Every real in V_2 is from $V_1[G \cap \mathbb{P}_a]$ for some $a \in H$.

Proof: Let $\underset{\sim}{r}$ be a $AP \star \underset{\sim}{\mathbb{P}}$-name of a real. By Claim 12, $\underset{\sim}{\mathbb{P}}[H] \models \kappa - cc$ in V_1. Therefore, for every $n < \omega$ there are $AP-$names $\bar{p_n} = (\underset{\sim}{p_{n,\alpha}} : \alpha < \underset{\sim}{\alpha_n})$ and $\bar{t_n} = (\underset{\sim}{t_{n,\alpha}} : \alpha < \underset{\sim}{\alpha_n})$ such that:

a. $\underset{\sim}{\alpha_n} < \kappa$.

b. $\bar{p_n}$ is a maximal antichain in $\underset{\sim}{\mathbb{P}}[H]$.

c. $\underset{\sim}{t_{n,\alpha}}$ is a $\underset{\sim}{\mathbb{P}}[H]-$name of an element of $\{0, 1\}$.

d. $\underset{\sim}{p_{n,\alpha}} \Vdash$ "$n \in \underset{\sim}{r}$ iff $\underset{\sim}{t_{n,\alpha}} = 1$".

For every $n < \omega$ and $\alpha < \underset{\sim}{\alpha_n}$, there is $\underset{\sim}{a_{n,\alpha}} \in \underset{\sim}{H}$ such that $\underset{\sim}{p_{n,\alpha}} \in \mathbb{P}_{\underset{\sim}{a_{n,\alpha}}}$. Now let $a_0 \in AP$, we can find $\leq_{AP}$-increasing sequence $(a_n : n < \omega)$ such that $a_{n+1} \Vdash$ "$\underset{\sim}{\alpha_n} = \alpha_n^*$" for some $\alpha_n^* < \kappa$. Let $a_\omega \in AP$ be an upper bound, and now choose an increasing sequence $(a_{\omega+\alpha} : \alpha \leq \underset{n<\omega}{\Sigma} \alpha_n^*)$ by induction on $\alpha \leq \underset{n<\omega}{\Sigma} \alpha_n^*$ such that for every $n < \omega$ and $\beta < \alpha_n^*$, $a_{\omega + \underset{l<n}{\Sigma} \alpha_l^* + \beta + 1} \Vdash$ "$\underset{\sim}{a_{n,\beta}} = a_{n,\beta}^*$ and $\underset{\sim}{p_{n,\beta}} = p_{n,\beta}^*$". We may assume wlog that $a_{n,\beta}^* \leq_{AP} a_{\omega + \underset{l<n}{\Sigma} \alpha_l^* + \beta + 1}$, so $p_{n,\beta}^* \in \mathbb{P}_{a_{\omega + + \underset{l<n}{\Sigma} \alpha_l^* + \beta + 1}}$. It's now easy to see that $\underset{\sim}{r}$ is a $\mathbb{P}_{a_{\omega + \underset{n<\omega}{\Sigma} \alpha_n^*}}$-name. $\square$

Theorem 15: a. In V_2, let $\mathcal{A} = \{\underset{\sim}{a}[G] : \underset{\sim}{a} \in \Gamma_b$ for some $b \in H\}$ and let $V_3 = HOD(\mathbb{R}, \mathcal{A})$, then $V_3 \models ZF + DC +$ "there exists a mad family" $+$ "all sets of reals are $\mathbb{Q}-$measurable for every $\mathbb{Q} \in \mathbf{Q}$".

b. $ZF + DC +$ "every set of reals is Lebesgue measurable" $+$ "there exists a mad family" is consistent relative to an inaccessible cardinal.

Proof: a. The existence of a mad family follows by Claim 8. $\mathbb{Q}-$measurability for $\mathbb{Q} \in \mathbf{Q}$ follows from Claim 14 and Observation 10 as in Solovay's proof.

b. Apply the previous clause to $\mathbb{Q}$ =Random real forcing. $\square$

As a corollary to the above theorem, we obtain an answer to a question of Henle, Mathias and Woodin from [HMW]:

Corollary 16 ($ZF + DC$)**:** The existence of a mad family does not imply that $\aleph_1 \leq \mathbb{R}$.

Proof: By Theorem 15 (applied to Random real forcing) and the fact that the existence of an ω_1-sequence of distinct reals implies the existence of a non-Lebesgue measurable set of reals (see [Sh176]). $\square$

Remark: The above result was also obtained by Larson and Zapletal in [LZ] assuming the existence of a proper class of Woodin cardinals.

We conclude with a somewhat surprising observation, showing that the analog of Theorem 15 fails at the lower levels of the projective hierarchy:

Observation 17: If every Σ^1_3 set of reals is Lebesgue measurable, then there are no Σ^1_2-mad families.

Proof: By $[Sh176]$, Σ^1_3-Lebesgue measurability implies that $\omega_1^{L[x]} < \omega_1$ for every $x \in \omega^\omega$. By Theorem 1.3(2) in [To], it follows that there are no Σ^1_2-mad families. $\square$

On a question of Enayat

We now address a question asked by Ali Enayat in [En]. The question is motivated by the problem of understanding the relationship between Freiling's axiom of symmetry, the continuum hypothesis and the Lebesgue measurability of all sets of reals (see discussion in [Ch]).

As with the previous results, we were informed by Paul Larson that the following results can also be obtained under the assumption of a proper class of Woodin cardinals using the arguments from [LZ].

Definition 18: a. Let WCH (weak continuum hypothesis) be the statement that every uncountable set of reals can be put into 1-1 correspondence with $\mathbb{R}$.

b. Let AX (Freiling's axiom of symmetry) be the following statement: Let $\mathcal{F}$ be the set of functions $f : [0,1] \to \mathcal{P}_{\omega_1}([0,1])$, then for every $f \in \mathcal{F}$ there exist $x, y \in [0,1]$ such that $x \notin f(y)$ and $y \notin f(x)$.

Remark: The term WCH has a different meaning in several papers by other authors.

Theorem 19: $ZF + DC + \neg WCH +$ "every set of reals is Lebesgue measurable" is consistent relative to an inaccessible cardinal.

Proof: Let V_3 be the model from Theorem 15(b), we shall prove that $V_3 \models \neg WCH$ by showing that there is no injection from $\mathbb{R}$ to the mad family $\mathcal{A}$. Suppose toward contradiction that for some $(a, \underset{\sim}{p}) \in AP\star\underset{\sim}{\mathbb{P}}$ (where $\underset{\sim}{\mathbb{P}}$ is as in Definition 11), a canonical name for a real $\underset{\sim}{r}$ and a first order formula $\phi(x, y, z, \mathcal{A})$, $(a, \underset{\sim}{p}) \Vdash$ "$\phi(x, y, \underset{\sim}{r}, \mathcal{A})$ defines an injection $F_{\underset{\sim}{r}}$ from $\mathbb{R}$ to $\mathcal{A}$". We may assume wlog that $\underset{\sim}{r}$ is a canonical $\mathbb{P}_a$-name. We may also assume wlog that, for every $\underset{\sim}{s} \in \Gamma_a$, $(a, \underset{\sim}{p}) \Vdash$ "if $\underset{\sim}{s} \in Ran(F_{\underset{\sim}{r}})$, then $\underset{\sim}{s} = F_{\underset{\sim}{r}}(t)$ for some $t \in \mathbb{R}^{V^{\mathbb{P}_a}}$". This is possible as $|\Gamma_a| < \kappa$, so we may construct an increasing sequence $(a_\gamma : \gamma < \beta)$ of length $< \kappa$, such that $a_0 = a$ and such that the upper bound $(a_\beta, \Gamma_{a_\beta})$ satisfies the above requirement. $((a_\beta, \Gamma_a), \underset{\sim}{p})$ is then as required. By increasing a, we may assume wlog that $\underset{\sim}{p}$ is an object p (and not just an AP-name) from $\mathbb{P}_a$. Now let $a_2 \in AP$ be defined as $a_2 = (\mathbb{P}_a \star Cohen, \Gamma_a)$

and let $\underset{\sim}{\eta}$ be the $\mathbb{P}_{a_2}$-name for the Cohen real. There are $a_3 \in AP$ and a name $\underset{\sim}{\nu}$ such that $a_2 \leq a_3$ and $a_3 \Vdash$ "$p \Vdash$ "$\phi(\underset{\sim}{\eta}, \underset{\sim}{\nu}, \underset{\sim}{r}, \mathcal{A})$"", so $\underset{\sim}{\nu} \in \mathcal{A}$, and by the injectivity of $F_{\underset{\sim}{r}}$, $\underset{\sim}{\nu} \notin \Gamma_a$. We may assume wlog that $\underset{\sim}{\nu} \in \Gamma_{a_3}$.

Let a_4 be the amalgamation of two copies of a_3 over a_2 (i.e. as in the proof of Claim 7) and let $f_0 : \mathbb{P}_{a_3} \to \mathbb{P}_{a_4}$ and $f_1 : \mathbb{P}_{a_3} \to \mathbb{P}_{a_4}$ be the corresponding complete embeddings. As the amalgamation is over a_2, it follows that $f_0(\underset{\sim}{\eta}) = f_1(\underset{\sim}{\eta})$ and $f_0(\underset{\sim}{r}) = f_1(\underset{\sim}{r})$, and by the argument from the proof of Claim 7 (Subclaim 5), $f_0(\underset{\sim}{\nu}) \neq f_1(\underset{\sim}{\nu})$. As f_l $(l = 0, 1)$ are isormorphisms between a_3 and $f_l(a_3) \leq a_4$ such that $f_l \restriction \mathbb{P}_{a_2} = Id$, they induce an automorphism of $(AP, \leq_{AP})$ mapping a_3 to $f_l(a_3)$ and a_2 to itself. Therefore, $a_4 \Vdash$ ""$f_0(p) \Vdash$ "$\phi(f_0(\underset{\sim}{\eta}), f_0(\underset{\sim}{\nu}), f_0(\underset{\sim}{r}), \mathcal{A})$"",

$a_4 \Vdash$ ""$f_1(p) \Vdash$ "$\phi(f_1(\underset{\sim}{\eta}), f_1(\underset{\sim}{\nu}), f_1(\underset{\sim}{r}), \mathcal{A})$"" and $f_0(p) = f_1(p)$, a contradiction. $\square$

Theorem 20: WCH is independent of $ZF + DC + AX +$ "all sets of reals are Lebesgue measurable".

Proof: By [We], AX is implied by $ZF + DC +$ "all sets of reals are Lebesgue measurable". Therefore, AX holds in the model V_3 from Theorem 15(b) and in Solovay's model. By Corollary 19, $V_3 \models \neg WCH$. By the fact that all sets of reals in Solovay's model have the perfect set property, it follows that WCH holds in that model. $\square$

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