κ-Madness and Definability

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Abstract

Assuming the existence of a supercompact cardinal, we construct a model where, for some uncountable regular cardinal κ, there are no $\Sigma^1_1(\kappa)-\kappa-$mad families.[1]

Introduction

The study of higher analogs of descriptive set theoretic results has gained considerable attention during the past few years. Recent work includes new results on regularity properties, definable equivalence relations and the connections with classification theory (see [KLLS] for a survey and a list of relevant open problems).

In this paper we consider the definability of mad families from the point of view of generalised descriptive set theory. Our basic objects of study are the following:

Definition 1: a. A family $\mathcal{F} \subseteq [\kappa]^\kappa$ is called $\kappa-$mad if $|A \cap B| < \kappa$ for every distinct $A, B \in \mathcal{F}$, and $\mathcal{F}$ is $\subseteq-$maximal with respect to this property.

b. We say that $X \subseteq 2^\kappa$ is $\Sigma^1_1(\kappa)$ if there is a tree $T \subseteq \underset{\alpha<\kappa}{\cup} \kappa^\alpha \times 2^\alpha$ such that $X = \{\eta \in 2^\kappa :$ there is $\nu \in \kappa^\kappa$ such that $(\nu \restriction \alpha, \eta \restriction \alpha) \in T$ for every $\alpha < \kappa\}$.

Following Mathias' classical result that there are no analytic mad families ([Ma]), it's natural to investigate the higher analogs of Mathias' result for a regular uncountable cardinal κ. It turns out that under suitable large cardinal assumptions, it's possible to construct a model where no $\Sigma^1_1(\kappa)-\kappa-$mad families exist, thus consistently obtaining a higher version of the result of Mathias.

The main result of the paper is Theorem 10, which will also be stated here:

Main result: The existence of a regular uncountable cardinal κ such that there are no $\Sigma^1_1(\kappa)-\kappa-$mad families is consistent relative to a supercompact cardinal.

An important ingredient of the proof is the forcing $\mathbb{Q}_D$ in Definition 3. $\mathbb{Q}_D$ is a $(<\kappa)$-complete forcing adding a generic subset of κ that is almost contained in every set from the normal ultrafilter D on κ. We shall prove that such forcing notions destroy $\Sigma^1_1(\kappa)-\kappa-$mad families. Using a Laver-indestructible supercompact cardinal, we shall iterate those forcings to obtain the desired model

The rest of the paper will be devoted to the proof of the above result.

Proof of the main result

Hypothesis 2: We fix a measurable cardinal κ and a normal ultrafilter D on κ.

We shall now define a variant of Mathias forcing:

Definition 3: A. Let $\mathbb{Q} = \mathbb{Q}^\kappa_D$ be the forcing notion defined as follows:

a. $p \in \mathbb{Q}$ iff $p = (u, A) = (u_p, A_p)$ where $u \in [\kappa]^{<\kappa}$ and $A \in D$.

b. $\leq = \leq_{\mathbb{Q}}$ is defined as follows: $p \leq q$ iff

1. $u_p \subseteq u_q$.

2. $A_q \subseteq A_p$.

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3. $u_q \setminus u_p \subseteq A_p$.

4. $\alpha < \beta$ for every $\alpha \in u_p$ and $\beta \in u_q \setminus u_p$.

B. Let $\underset{\sim}{u}$ be the $\mathbb{Q}$-name for $\cup\{u_p : p \in \underset{\sim}{G}\}$.

C. $p \leq^{pr} q$ iff $p \leq q$ and $u_p = u_q$.

Observation 4: a. $\mathbb{Q}$ is $(< \kappa)$-complete.

b. The sequence $(p_i : i < \kappa)$ has an upper bound if the following conditions holds:

1. $(p_i : i < \kappa)$ is $\leq^{pr}$-increasing.

2. If $i \in \underset{j<i}{\cap} A_j$ and $i > sup(u_{p_0})$ then $j \in [i, \kappa) \rightarrow i \in A_{p_j}$.

Proof: a. By the κ-completeness of D.

b. By the normality of D, $(u_{p_0}, \underset{i<\kappa}{\Delta} A_{p_i} \setminus u_{p_0})$ is a condition in $\mathbb{Q}$, it's easy to see that it's the desired upper bound. $\square$

Claim 5: Suppose that $p \in \mathbb{Q}$, $sup(u_p) \leq \alpha < \kappa$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}$-name of a member of V, then there is $q \in \mathbb{Q}$ such that:

a. $p \leq^{pr} q$.

b. $A_q \cap (\alpha + 1) = A_p \cap (\alpha + 1)$.

c. If $v \subseteq \alpha + 1$ and there is $r \in \mathbb{Q}$ forcing a value to $\underset{\sim}{\tau}$ such that $u_r = v$, then $q^{[v,\alpha]} := (v, A_q \setminus (\alpha + 1)) \in \mathbb{Q}$ forces the same value to $\underset{\sim}{\tau}$.

Proof: Fix an enumeration $(v_\beta : \beta < 2^{|\alpha|})$ of $\mathcal{P}(\alpha + 1)$. We shall construct by induction a decreasing sequence $(A_\beta : \beta < 2^{|\alpha|})$ of elements of D as follows:

a. $\beta = 0$: Without loss of generality, there is $r \in \mathbb{Q}$ as in clause (c) for v_0. Let $A_0 = A_r \cap A_p$.

b. β is a limit ordinal: Let $A_\beta = \underset{\gamma<\beta}{\cap} A_\gamma \in D$ (recall that $2^{|\alpha|} < \kappa$).

c. $\beta = \gamma + 1$: Without loss of generality, there is $r \in \mathbb{Q}$ as in clause (c) for v_β. Let $A_\beta = A_\gamma \cap A_r$.

Now let $A_q := ((\underset{\beta<2^{|\alpha|}}{\cap} A_\beta) \setminus (\alpha + 1)) \cup (A_p \cap (\alpha + 1)) \in D$ and $u_q := u_p$. It's now easy to verify that q is as required. $\square$

Claim 6: If $p \in \mathbb{Q}$, $p \Vdash$ "$\underset{\sim}{\tau} \in V^\kappa$" and $sup(u_p) \leq \alpha < \kappa$, then there is $q \in \mathbb{Q}$ satisfying clause (a) from Claim 5, and in addition: If $i < \kappa$, $v \subseteq (\alpha + 1)$ and there is $r \in \mathbb{Q}$ forcing a value to $\underset{\sim}{\tau}(i)$ such that $u_r = v$, then $q^{[v,\alpha]}$ forces the same value to $\underset{\sim}{\tau}(i)$.

Proof: We construct a $\leq^{pr}$ increasing sequence $(p_i : i < \kappa)$ by induction on $i < \kappa$ as follows:

a. i=0: Let p_0 be q from the previous claim, where $\underset{\sim}{\tau}(0)$ here stands for $\underset{\sim}{\tau}$ there.

b. $i = j + 1$: Similarly, letting $(p_j, \underset{\sim}{\tau}(j))$ here stand for $(p, \underset{\sim}{\tau})$ in Claim 5, let p_i be the corresponding q from Claim 5.

c. i is a limit ordinal: Let p'_i be an upper bound for $(p_j : j < i)$ (see Observation 4). It's easy to see that if the sequence is $\leq^{pr}$-increasing, then we can get a $\leq^{pr}$-upper bound. Now construct p_i as in the previous case.

Finally, let q be a $\leq^{pr}$-upper bound for $(p_i : i < \kappa)$ (such q exists by Observation 4(b)). q is obviously as required. $\square$

Claim 7: If $p \in \mathbb{Q}$ and $p \Vdash$ "$\underset{\sim}{\tau} \in V^{\kappa}$", then there is $q \in \mathbb{Q}$ that satisfies the conclusion of Claim 6 for every $\alpha \in [sup(u_p), \kappa)$.

Proof: By Claim 6 and Observation 4(b). $\square$

Claim 8: (α) (A) implies (B) where:

A. a. $\mathbf{B}$ is a $\Sigma^1_1(\kappa)$ subset of $[\kappa]^{\kappa}$ and $\Vdash$ "$\underset{\sim}{X} \in \mathbf{B}$".

b. $\chi > 2^{\kappa}$, $N \prec (H(\chi), \in)$, $\{\mathbf{B}, D, \underset{\sim}{X}\} \subseteq N$, $|N| = \kappa$ and $[N]^{<\kappa} \subseteq N$.

c. $\mathbb{Q}$ is a $(< \kappa)$-complete forcing notion.

d. $\mathbb{Q} \in N$.

e. $G \subseteq \mathbb{Q} \restriction N$ is generic over N.

B. $\underset{\sim}{X}[G]$ is well defined and belongs to $\mathbf{B}$.

(β) (A) implies (B) where:

A. a. $\mathbf{B}$ is a $\Sigma^1_1(\kappa)$ subset of $[\kappa]^{\kappa}$ defined by the tree $T \in V$.

b. $\mathbb{Q}$ is a $(< \kappa)$-complete forcing notion.

c. $\mathbf{B}^{V^{\mathbb{Q}}}$ is κ-mad in $V^{\mathbb{Q}}$.

B. $\mathbf{B}^V$ is κ-mad in V.

Proof: (α) For $\alpha < \kappa$, let $T_\alpha = 2^{\alpha} \times \kappa^{\alpha}$, and for $\alpha < \beta \leq \kappa$ and $(\eta, \nu) \in T_\beta$, let $(\eta, \nu) \restriction \alpha = (\eta \restriction \alpha, \nu \restriction \alpha) \in T_\alpha$. Let $T_* = \underset{\alpha<\kappa}{\cup} T_\alpha$, then T_κ is the set of $\kappa-$branches through T_*. There is a subtree $T \subseteq T_*$ such that $\{\eta : (\eta, \nu) \in lim(T)\} = \mathbf{B}$ (where η is interpreted as $\{\alpha : \eta(\alpha) = 1\}$), hence there are $(\underset{\sim}{\eta}, \underset{\sim}{\nu})$ such that $\Vdash$ "$(\underset{\sim}{\eta}, \underset{\sim}{\nu}) \in lim(T)$ and $\underset{\sim}{X} = \{\alpha : \underset{\sim}{\eta}(\alpha) = 1\}$". Without loss of generality, $\underset{\sim}{\eta}, \underset{\sim}{\nu} \in N$. For each $\alpha < \kappa$, let $I_\alpha \in N$ be a dense open subset of $\mathbb{Q}$ where $I_\alpha = \{p \in \mathbb{Q} : p$ forces a value to $(\underset{\sim}{\eta}, \underset{\sim}{\nu}) \restriction \alpha\}$. For each $\alpha < \kappa$, choose $p_\alpha \in G \cap I_\alpha$ and let $(\eta_\alpha, \nu_\alpha) \in T_\alpha$ be the valued forced by p_α for $(\underset{\sim}{\eta}, \underset{\sim}{\nu}) \restriction \alpha$. For every $\alpha < \beta < \kappa$, p_α and p_β are compatible and hence $\eta_\alpha \leq \eta_\beta$ and $\nu_\alpha \leq \nu_\beta$. Let $(\eta, \nu) := (\underset{\alpha<\kappa}{\cup} \eta_\alpha, \underset{\alpha<\kappa}{\cup} \nu_\alpha) \in lim(T)$, then $N[G] \models$ "$\underset{\sim}{X}[G] = \{\alpha : \eta(\alpha) = 1\}$", hence $\underset{\sim}{X}[G] \in \mathbf{B}$. This completes the proof of (α).

(β) Obviously, each element of $\mathbf{B}^V$ has cardinality κ and $\mathbf{B}^V$ is a κ-almost disjoint family. Let $C \in [\kappa]^{\kappa}$, by assumption (A)(c), $\Vdash_{\mathbb{Q}}$ "there is $D \in \mathbf{B}$ such that $|C \cap D| = \kappa$". Therefore, for some $\mathbb{Q}$-name $\underset{\sim}{\tau}$, $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in \mathbf{B}$ and $|C \cap \underset{\sim}{\tau}| = \kappa$". Fix a large enough χ and $N \prec (H(\chi), \in)$ such that $|N| = \kappa$, $[N]^{<\kappa}$ and $\{\underset{\sim}{\tau}, \mathbf{B}, C\} \subseteq N$. By the $(< \kappa)$-completeness of $\mathbb{Q}$, there is $G \subseteq \mathbb{Q} \restriction N$ which is generic over N. By part (α) of the claim, $\underset{\sim}{\tau}[G] \in \mathbf{B}^V$ and $|C \cap \underset{\sim}{\tau}[G]| = \kappa$, hence $\mathbf{B}^V$ is κ-mad in V. $\square$

Claim 9: There are no $(\mathbb{Q}, \underset{\sim}{u}, D, \mathbf{B})$ such that:

a. $\mathbb{Q}$ is a $(< \kappa)$-complete forcing notion.

b. D is a normal ultrafilter on κ.

c. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{u} \in [\kappa]^{\kappa}$ and $\underset{\sim}{u} \subseteq^* A$ for every $A \in D$".

d. $\mathbf{B} \in V$ is a $\Sigma^1_1(\kappa)$ subset of $[\kappa]^{\kappa}$.

e. $\mathbf{B}^V$ is κ-mad in V.

f. $\mathbf{B}^{V^{\mathbb{Q}}}$ is κ-mad in $V^{\mathbb{Q}}$.

Proof: Suppose towards contradiction that there are $(\mathbb{Q}, \underset{\sim}{u}, D, \mathbf{B})$ as above. Hence $\mathbf{B}$ is a $\Sigma^1_1(\kappa)$-κ-mad family in V. Fix a sequence $(A^*_i : i < \kappa) \in V$ of pairwise distinct members of $\mathbf{B}$. Let $F : \kappa \times \kappa \to \kappa$ be the function defined as $F(i, \alpha) :=$the αth member of $A^*_i \setminus \bigcup\limits_{j<i} A^*_j \in [\kappa]^\kappa$ (recalling that κ is regular and $\mathbf{B}$ is κ-almost disjoint).

Now define the following $\mathbb{Q}$-names:

1. $\underset{\sim}{\alpha}_i$ is $min\{\underset{\sim}{u} \setminus (i+1)\}$.
2. $\underset{\sim}{\beta}_i$ is $F(i, \underset{\sim}{\alpha}_i)$.
3. $\underset{\sim}{v} = \{\underset{\sim}{\beta}_i : i \in \underset{\sim}{u}$ satisfies that $otp(i \cap \underset{\sim}{u})$ is even $\}$.

Let E be the ultrafilter on κ generated by the sets $\{\{F(i, \alpha) : i < \alpha$ are from $A\} : A \in D\}$. By Rowbottom's theorem, for every $A \in D$ and $X \subseteq \kappa$, if $f_X : [A]^2 \to \{0, 1\}$ is defined by $f_X(i, \alpha) = 0$ iff $F(i, \alpha) \in X$, then there exists a monochromatic $B \subseteq A$ such that $B \in D$. It follows that E is indeed an ultrafilter. As F is injective, each set in E has cardinality κ. By the κ-completeness of D, E is also κ-complete.

Subclaim 1: $E \cap \mathbf{B} = \emptyset$.

Proof: Let $C \in \mathbf{B}$.

Case I: $C = A^*_j$ for some $j < \kappa$. Let $A \in D$ such that $min(A) > j$, then by the definition off F, $\{F(i, \alpha) : i < \alpha$ are from $A\} \cap A^*_j = \emptyset$. It follows that $C \notin E$.

Case II: $C \in \mathbf{B} \setminus \{A^*_i : i < \kappa\}$. In this case, define $f : \kappa \to \kappa$ by $f(i) = sup(A^*_i \cap C) + i + 1$ and let $H = \{\delta < \kappa : \delta$ is a limit ordinal such that $f(i) < \delta$ for all $i < \delta\}$. $H \subseteq \kappa$ is a club, hence $H \in D$ and $H^* := \{F(i, \alpha) : i < \alpha$ are from $H\} \in E$. Suppose that $F(i, \alpha) \in H^*$, if $F(i, \alpha) \in C$ then $\alpha \leq F(i, \alpha) < f(i) < \alpha$, a contradiction. It follows that $C \notin E$.

This proves the subclaim. We shall now return to the proof of the main claim. Suppose towards contradiction that $\mathbf{B}^{V^{\mathbb{Q}}}$ is $\kappa-$mad in $V^{\mathbb{Q}}$. As $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{v} \in [\kappa]^\kappa$, there is a $\mathbb{Q}$-name $\underset{\sim}{\tau}$ of a member of $\mathbf{B}^{V^{\mathbb{Q}}}$ such that $\Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{v} \cap \underset{\sim}{\tau}| = \kappa$". For every $p \in \mathbb{Q}$, let $B^+_p = \{\alpha < \kappa : p \nVdash$ "$\alpha \notin \underset{\sim}{\tau}$"$\}$.

Subclaim 2: $B^+_p \in E$.

Proof: Suppose towards contradiction that $B^+_p \notin E$, then there is some $C_p \in D$ such that $B^+_p \cap \{F(i, \alpha) : i < \alpha$ are from $C_p\} = \emptyset$. Therefore, if $i < \alpha$ are from C_p then $p \Vdash$ "$F(i, \alpha) \notin \underset{\sim}{\tau}$". Recalling that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{u} \subseteq^* C_p$", it follows that $p \Vdash$ "$\underset{\sim}{\alpha}_i \in C_p$ for i large enough", and also $p \Vdash$ "for i large enough, $i \in \underset{\sim}{u} \to i \in C_p$". Therefore, $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\beta}_i = F(i, \underset{\sim}{\alpha}_i) \notin \underset{\sim}{\tau}$ for every large enough $i \in \underset{\sim}{u}$". Recalling the definition of $\underset{\sim}{v}$, it follows that $p \Vdash$ "$|\underset{\sim}{v} \cap \underset{\sim}{\tau}| < \kappa$", contradicting the choice of $\underset{\sim}{\tau}$. It follows that $B^+_p \in E$, which completes the proof of Subclaim 2.

For every $p \in \mathbb{Q}$, let $B^-_p = \{\alpha < \kappa : p \nVdash$ "$\alpha \in \underset{\sim}{\tau}$"$\}$.

Subclaim 3: $B^-_p \in E$.

Proof: Suppose not, then $B_* := \kappa \setminus B^-_p \in E$ (hence $B_* \in [\kappa]^\kappa$) and $p \Vdash$ "$B_* \subseteq \underset{\sim}{\tau}$". By the κ-madness of $\mathbf{B}$, there is $C \in \mathbf{B}$ (in V) such that $|C \cap B_*| = \kappa$. As

$p \Vdash$ "$B_* \cap C \subseteq \underset{\sim}{\tau}$, $\underset{\sim}{\tau} \in \mathbf{B}$ and $\mathbf{B}$ is κ-mad", it follows that $p \Vdash$ "$\underset{\sim}{\tau} = C$". We shall derive a contradiction by showing that $\Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{\nu} \cap C| < \kappa$": Choose i_* such that $C \neq A^*_i$ for every $i \in [i_*, \kappa)$. It follows that $|C \cap A^*_i| < \kappa$ for every $i \in [i_*, \kappa)$. Now repeat the argument of Case II in the proof of Subclaim 1 and choose f, H and H^* as there. As $H \in D$, $\Vdash_{\mathbb{Q}}$ "for large enough i, $i \in \underset{\sim}{u} \to i, \underset{\sim}{\alpha}_i \in H$". Repeating the same argument as in Subclaim 1, $\Vdash_{\mathbb{Q}}$ "for large enough $i \in \underset{\sim}{u}$, $\underset{\sim}{\beta}_i = F(i, \underset{\sim}{\alpha}_i) \in H^*$, hence $\underset{\sim}{\beta}_i \notin C$". It follows that $\Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{v} \cap C| < \kappa$", leading to a contradiction. This completes the proof of Subclaim 3.

Observation 4: A. Given $p_1, p_2 \in \mathbb{Q}$ and $\alpha < \kappa$, there exist (q_1, q_2, β) such that:

a. $p_l \leq_{\mathbb{Q}} q_l$ $(l = 1, 2)$.

b. $\beta \in [\alpha, \kappa)$.

c. $p_1 \Vdash$ "$\beta \in \underset{\sim}{\tau}$".

d. $p_2 \Vdash$ "$\beta \notin \underset{\sim}{\tau}$".

B. As in (A), with (d) replaced by the following:

d'. $p_2 \Vdash$ "$\beta \in \underset{\sim}{\tau}$".

Proof: By the previous subclaims, $B^+_{p_1} \cap B^-_{p_2}, B^+_{p_1} \cap B^+_{p_2} \in E$, hence there exist $\beta \in (B^+_{p_1} \cap B^-_{p_2}) \setminus \alpha$ and $\gamma \in (B^+_{p_1} \cap B^+_{p_2}) \setminus \alpha$. By the definitions of $B^{+/-}_p$, there exist $q_1 \geq p_1$ and $q_2 \geq p_1$ such that (q_1, q_2, β) are as required, and similarly for γ and (B). This proves the observation.

Let $\chi = (2^\kappa)^+$ and $N \prec (H(\chi), \in)$ such that $|N| = \kappa$, $N^{<\kappa} \subseteq N$, $\kappa \subseteq N$ and $\underset{\sim}{\tau}, D, \mathbf{B} \in N$. Let $(I_i : i < \kappa)$ list the dense open subsets of $\mathbb{Q}$ from N. We shall now choose (p^1_i, p^2_i, β_i) by induction on $i < \kappa$ such that:

a. $p^1_i, p^2_i \in \mathbb{Q} \cap N$ and $\beta_i \in N$.

b. $i < j \to p^l_i \leq_{\mathbb{Q}} p^l_j$ $(l = 1, 2)$.

c. If $i = 4j + 1$ then $p^1_i, p^2_i \in I_j$.

d. $\beta_i \in \kappa \setminus \underset{j<i}{\cup} (\beta_j + 1)$.

e. If $i = 4j + 2$ then $p^1_i \Vdash$ "$\beta_{4j+2} \in \underset{\sim}{\tau}$" and $p^2_i \Vdash$ "$\beta_{4j+2} \in \underset{\sim}{\tau}$".

f. If $i = 4j + 3$ then $p^1_i \Vdash$ "$\beta_{4j+3} \in \underset{\sim}{\tau}$" and $p^2_i \Vdash$ "$\beta_{4j+3} \notin \underset{\sim}{\tau}$".

g. If $i = 4j + 4$ then $p^1_i \Vdash$ "$\beta_{4j+4} \notin \underset{\sim}{\tau}$" and $p^2_i \Vdash$ "$\beta_{4j+4} \in \underset{\sim}{\tau}$".

Observation 5: It is possible to choose (p^1_i, p^2_i, β_i) as above for each $i < \kappa$.

Proof:

Case I: $i = 0$. This is trivial.

Case II: i is a limit ordinal: As $N^{<\kappa} \subseteq N$ and $(p^l_j : j < i), (\beta_j : j < i) \in N$, we can find p^1_i and p^2_i using the $(< \kappa)$-completeness of $\mathbb{Q}$ and elementarity. As κ is regular, there is no problem to choose β_i.

Case III: $i = 4j + 1$: As $p^1_j, p^2_j, I_j \in N$, by elementarity there exist p^1_i and p^2_i as required.

Case IV: $i = 4j + 2$: Use Observation 4(B).

Case V: $i = 4j + 3$: Use Observation 4(A).

Case VI: $i = 4j + 4$: Use Observation 4(A), with (p_i^2, p_i^1) here standing for (p_1, p_2) there.

Finally, let $G_l = \{q \in \mathbb{Q} \cap N : q \leq_{\mathbb{Q}} p_i^l$ for some $i < \kappa\}$ $(l = 1, 2)$, then $G_l \subseteq \mathbb{Q} \cap N$ is generic over N. By Claim 8(α), $C_l := \underset{\sim}{\tau}[G_l] \in \mathbf{B}$. By the choice of (p_i^1, p_i^2, β_i), $\{\beta_{4i+2} : i < \kappa\} \subseteq C_1 \cap C_2$, hence $C_1 \cap C_2 \in [\kappa]^\kappa$. Similarly, $|\{\beta_{4i+3} : i < \kappa\}| = \kappa$ and $\{\beta_{4i+3} : i < \kappa\} \subseteq C_1 \setminus C_2$, hence $C_1 \neq C_2$. This contradicts the κ-madness of $\mathbf{B}$ in V, which completes the proof of Claim 9. $\square$

Theorem 10: If κ is a Laver-indestructible supercompact cardinal then there is a generic extension where κ is supercompact and there are no $\Sigma_1^1(\kappa)$-κ-mad families.

Proof: We recall the following strong version of $\kappa^+ - c.c.$ (see e.g. [Sh:80] and [Sh:1036]): A forcing $\mathbb{Q}$ satisfies $*^1_{\kappa,\mathbb{Q}}$ if:

a. $\mathbb{Q}$ is $(< \kappa)$-complete.

b. If $\{p_\alpha : \alpha < \kappa^+\} \subseteq \mathbb{Q}$, then for some club $E \subseteq \kappa^+$ and pressing down function f on E we have $(\delta_1, \delta_2 \in E \wedge f(\delta_1) = f(\delta_2)) \to p_{\delta_1}, p_{\delta_2}$ are compatible.

c. Every two compatible conditions in $\mathbb{Q}$ have a least upper bound.

Obviously, $*^1_{\kappa,\mathbb{Q}}$ implies $\kappa^+ - c.c.$. By [Sh:80], $*^1_{\kappa,\mathbb{Q}}$ is preserved under $(< \kappa)$-support iterations.

It's easy to verify that $\mathbb{Q} = \mathbb{Q}_D$ satisfies $*^1_{\kappa,\mathbb{Q}}$ when D is a normal ultrafilter on κ (e.g. fix a bijection $g : [\kappa]^{<\kappa} \to \kappa$, and for every $\{p_\alpha : \alpha < \kappa^+\}$, let $E = (\kappa, \kappa^+)$ and let $f : E \to \kappa^+$ be defined by $f(\alpha) = g(u_\alpha)$ where $p_\alpha = (u_\alpha, A_\alpha)$)

Let $(\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta)$ be a $(< \kappa)$-support iteration such that:

a. $cf(\delta) > \kappa$.

b. Each $\underset{\sim}{\mathbb{Q}}_\beta$ is $*^1_{\kappa,\underset{\sim}{\mathbb{Q}}_\beta}$.

c. $\delta = sup\{\alpha < \delta$: in $V^{\mathbb{P}_\alpha}$, $\underset{\sim}{\mathbb{Q}}_\alpha = \mathbb{Q}_{\underset{\sim}{D}_\alpha}$ where $\underset{\sim}{D}_\alpha$ is a $\mathbb{P}_\alpha$-name of a normal ultrafilter on $\kappa\}$.

As κ is a Laver indestructible supercompact cardinal, there is an iteration as above. Suppose towards contradiction that there is a $\Sigma_1^1(\kappa) - \kappa$−mad family $\mathbf{B}$ in $V^{\mathbb{P}_\delta}$. $\mathbf{B} = \{\eta : (\eta, \nu) \in lim(T)\}$ for a suitable tree T. By the fact that $cf(\delta) > \kappa$ and $\mathbb{P}_\delta$ is $\kappa^+ - c.c.$, it follows that $T \in V^{\mathbb{P}_\beta}$ for some $\beta < \delta$. Let $\gamma \in [\beta, \delta)$ such that $\underset{\sim}{\mathbb{Q}}_\gamma = \mathbb{Q}_{\underset{\sim}{D}_\gamma}$ where $\underset{\sim}{D}_\gamma$ is a $\mathbb{P}_\gamma$-name of a normal ultrafilter on κ. By Claim 8(β), $\mathbf{B}^{V^{\mathbb{P}_\gamma}}$ is κ−mad in $V^{\mathbb{P}_\gamma}$.

Applying Claim 9 to $V_1 = V^{\mathbb{P}_\gamma}$, $\mathbb{Q} = \mathbb{P}_\delta / \mathbb{P}_\gamma$ and $D = \underset{\sim}{D}_\gamma$, it follows that $\mathbf{B}$ is not κ−mad in $V^{\mathbb{P}_\delta}$, a contradiction. It follows that there are no $\Sigma_1^1(\kappa) - \kappa$−mad families in $V^{\mathbb{P}_\delta}$. $\square$

Open problems

We conclude by listing some of the open problems following from our work:

Following the main result of the paper, one may ask whether it's possible to get an implication instead of just consistency:

Question 1: Suppose that κ is supercompact, is there a $\Sigma_1^1(\kappa) - \kappa$-mad family?

Question 2: What is the consistency strength of $ZFC +$ "for some uncountable regular cardinal κ, there are no $\Sigma_1^1(\kappa) - \kappa$−mad families"?

It's known by [Ma], [To] and [HwSh:1090] that $ZF + DC +$ "there are no mad families" is consistent ([To] shows that it holds in Solovay's model while in [HwSh:1090] we obtain a consistency result relative to ZFC).

Question 3: a. What's the consistency strength of $ZF + DC+$there exists a regular uncountable cardinal κ such that there are no $\kappa-$mad families"?

b. Suppose that $\kappa > \aleph_0$ is regular, does DC_κ imply the existence of a $\kappa-$mad family?

It's known by [HwSh:1089] and [HwSh:1095] that Borel maximal eventually different families and maxima cofinitary groups exist, therefore it's natural to investigate the κ-version of those results:

Question 4: a. Does ZFC imply that there are $\kappa-$Borel $\kappa-$maximal eventually different families for every (or at least for some) regular uncountable cardinal κ?

b. Similarly, replacing regular uncountable cardinals by successor cardinals, inaccessible non-Mahlo cardinals, etc.

Question 5: Does ZFC imply that there are $\kappa-$Borel $\kappa-$maximal cofinitary groups for every (or at least for some) regular uncountable cardinal κ?

b. Similarly, replacing regular uncountable cardinals by successor cardinals, inaccessible non-Mahlo cardinals, etc.

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