Sh:1192

Annals of Pure and Applied Logic 172 (2021) 102992

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Criteria for exact saturation and singular compactness ☆

Itay Kaplan, Nicholas Ramsey *, Saharon Shelah

A R T I C L E I N F O

Article history:
Received 24 March 2020
Received in revised form 20 April 2021
Accepted 10 May 2021
Available online 13 May 2021

MSC:
03C45
03C50
03C55
03C30

Keywords:
Model theory
Classification theory
Exact saturation
Singular compactness

A B S T R A C T

We introduce the class of *unshreddable theories*, which contains the simple and NIP theories, and prove that such theories have exactly saturated models in singular cardinals, satisfying certain set-theoretic hypotheses. We also give criteria for a theory to have singular compactness.

© 2021 Elsevier B.V. All rights reserved.

Contents

☆ This work was supported by the European Research Council grant 338821. Paper No. 1192 in Shelah's publication list. The first author would like to thank the Israel Science Foundation for its support of this research (grants No. 1533/14 and 1254/18).
* Corresponding author.

https://doi.org/10.1016/j.apal.2021.102992
0168-0072/© 2021 Elsevier B.V. All rights reserved.

Sh:1192

1. Introduction

The construction of saturated models of a theory T is sensitive to the combinatorial properties of sets definable in T. Consequently, properties of saturated models and their constructions are often reflected in model-theoretic dividing lines, defined in terms of synactic properties of a formula. For example, it is well known that a stable theory has a saturated model in every cardinal in which it is stable [11, Theorem III.3.12]. In a similar vein, the third-named author characterized the simple theories in terms of the *saturation spectrum* of a theory, namely, the set of cardinal pairs (λ, κ) with $\lambda \geq \kappa$ and every model of size λ extends to a κ-saturated model of the same size [10, Theorem 4.10]. Subsequent work on transferring saturation, Keisler's order, and the interpretability order all suggest that comparisons between saturated models and their constructions yield meaningful measures of model-theoretic complexity [1,4,8].

A theory T is said to have *exact saturation* at the cardinal κ if there is a κ-saturated model of T which is not κ^+-saturated. If κ is regular and $> |T|$, *every* theory has models with exact saturation at κ [7, Theorem 2.4, Fact 2.5], but for singular κ, this property connects with notions from classification theory. The simplest example of a theory without exact saturation at singular κ is the theory of dense linear orders. Given a singular cardinal κ and a κ-saturated dense linear order I and given any subsets $A < B$ from I with $|A| = |B| = \kappa$, there are cofinal and coinitial subsets A_0 and B_0 of A and B respectively with $|A_0| = |B_0| < \kappa$. It follows from the κ-saturation of I that there is some $c \in I$ with $a < c < b$ for all $a \in A_0$ and $b \in B_0$, hence for all $a \in A$ and $b \in B$. By quantifier elimination for the theory of dense linear orders, it follows that I is κ^+-saturated. This example suggests that failures of exact saturation are related to the presence of orders. Indeed, it was shown in [7, Theorem 4.10] that an NIP theory T has exact saturation at a singular cardinal κ if and only if T is not distal (assuming $2^\kappa = \kappa^+$ and $\kappa > |T|$).

Additionally, [7, Theorem 3.3] showed that if T is simple then T has exactly μ-saturated models for singular μ of cofinality greater than $|T|$ (again assuming $2^\mu = \mu^+$ and, additionally, $\square_\mu$). In the unstable case, this argument started from a witness $\varphi(x; y)$ to the independence property along an indiscernible sequence I of length κ and inductively constructed a model M containing I so that every type over fewer than μ parameters is realized and also so that, for every tuple c from M, there is an interval from the indiscernible sequence that is indiscernible over c. This ensures that the model is both μ-saturated yet omits the type $\{\varphi(x; a_i)^{i \text{ even}} : i \in I\}$. Simplicity theory, via the independence theorem and the forking calculus, played an important role in that argument.

Here, we are interested in both finding criteria for exact saturation in broader model-theoretic contexts but also understanding the reach of the argument of [7], which was tailored to simple theories. We introduce *shredding*, a notion that refines forking and exactly captures the obstacle to ensuring that one can realize a formula such that a large interval of a given indiscernible sequence is additionally indiscernible over the realization. This notion is defined with exact saturation in mind, but it appears to be a fairly fundamental notion and may have uses beyond the context explored here. We use shredding to define the class of *unshreddable theories*, which are roughly the theories with a bound on the number of times a type can shred, and observe that both NIP and simple theories are unshreddable. Our main theorem is that one may construct exactly saturated models of unshreddable theories with the independence property for singular cardinals satisfying certain set-theoretic hypotheses. We follow the rough outline of the argument of [7] but, in contrast to the approach taken there, which faced considerable technical issues in adapting the tools of simplicity theory for the construction of an exactly saturated model, our proof, in addition to being more general, is considerably simpler and more direct.

In section 4, we focus on the way that the class of unshreddable theories compares to other classes from classification theory. We show that there is an unshreddable theory with SOP_3, which suggests that the class of unshreddable theories is substantially broader than the simple theories. However, we show subsequently that neither NSOP_1 nor NTP_2 imply that a theory is unshreddable.

In section 5, we consider the dual problem of which conditions on a theory imply the inability to construct exactly saturated models, which we call *singular compactness*. We formulate one such criterion and show that this condition entails a considerable amount of complexity: theories that meet our condition for every formula have TP_2 and SOP_n for all n. Nonetheless, we show that our condition restricted to a fixed finite set of formulas implies a local version of singular compactness. For this local variant, we show that there is an example which satisfies the condition for a fixed finite set of formulas which is NSOP_4.

2. Shredding

2.1. Basic definitions

From now on, T will denote a complete first-order theory with monster model $\mathbb{M}$. Our model-theoretic notation and terminology is standard. Following standard model-theoretic usage, we say the A-indiscernible sequence I is *extracted* from J if I realizes the EM-type of I over A. The existence of such a sequence follows by Ramsey and compactness. In this subsection, we will describe shredding and show that it can be given a finitary characterization.

Definition 2.1. Let A be a set of parameters and λ an infinite cardinal.

(1) We say that $\varphi(x;a)$ λ-*shreds* over A when there is $\overline{b}$ such that:

  (a) $\overline{b} = \langle b_\alpha : \alpha < \lambda\rangle$ is an indiscernible sequence over A.
  (b) For no $\alpha < \lambda$ and $c \in \varphi(\mathbb{M}, a)$ is $\overline{b}_{\geq\alpha}$ an indiscernible sequence over Ac.

(2) We say a type λ-shreds over A if it implies a formula that λ-shreds over A, respectively.
(3) We say $p \in S(B)$ λ-*shreds* over A with *a built-in witness* if $A \subseteq B$ and an indiscernible sequence witnessing λ-shredding is contained in B.
(4) For the above notions, we may omit λ when $\lambda = (|T| + |A|)^+$.
(5) We define $\kappa^m_{\text{shred}}(T)$ to be the minimal regular cardinal κ such that there is no increasing continuous sequence of models $\langle M_i : i \leq \kappa\rangle$ and $p \in S^m(M_\kappa)$ so that $p \restriction M_{i+1}$ shreds over M_i with a built-in witness, if such a cardinal exists (where *continuous* means $M_\delta = \bigcup_{i<\delta} M_i$ for limit δ). Otherwise, we set $\kappa^m_{\text{shred}}(T) = \infty$. The cardinal $\kappa_{\text{shred}}(T) = \sup_m \kappa^m_{\text{shred}}(T)$.
(6) We say T is *unshreddable* if $\kappa_{\text{shred}}(T) < \infty$.

Remark 2.2. Though we do not use it, it is natural to additionally introduce an associated notion of forking: say $\varphi(x;a)$ λ-*shred-forks* over A if $\varphi(x;a) \vdash \bigvee_{i<k} \psi_i(x;a_i)$ where each $\psi_i(x;a_i)$ λ-shreds over A. This satisfies extension, by the same argument as for forking. Note that, if $\varphi(x;a)$ λ-shreds over A, then, unless $\varphi(x;a)$ is inconsistent, we know a is not contained in A.

The following lemma gives a finitary equivalent to λ-shredding.

Lemma 2.3. *Assume $\lambda = cf(\lambda) > |T| + |A|$. The following are equivalent:*

(1) The formula $\varphi(x;a)$ λ-shreds over A.
(2) There are n, $\overline{b}$, $\overline{\eta}$, and $\overline{\psi}$ satisfying:

  (a) $\overline{b} = \langle b_\alpha : \alpha < \lambda\rangle$ is an A-indiscernible sequence.
  (b) $\overline{\eta} = \langle \eta_i : i < k\rangle$ is a finite sequence of increasing functions in ${}^n(2n)$.

Sh:1192

(c) $\overline{\psi} = \langle \psi_l(x; y_0, \ldots, y_{n-1}; a'_l) : l < k\rangle$ *is a sequence of formulas with* $a'_l \in A$.
(d) For every $\delta < \lambda$ *divisible by* $2n$ *(or just for every limit* $\delta < \lambda$*), we have*

$$\varphi(x; a) \vdash \bigvee_{l<k} \left[\psi_l(x; b_\delta, \ldots, b_{\delta+n-1}, a'_l) \leftrightarrow \neg\psi_l(x; b_{\delta+\eta_l(0)}, \ldots, b_{\delta+\eta_l(n-1)}, a'_l)\right].$$

Proof. (2) $\Longrightarrow$ (1) is clear by definition of λ-shredding.

(1) $\Longrightarrow$ (2). Suppose $\varphi(x; a)$ λ-shreds over A witnessed by the indiscernible sequence $\overline{b} = \langle b_\alpha : \alpha < \lambda\rangle$. Then for each $\delta < \lambda$ consider the set of formulas $\Gamma_\delta(x)$ containing $\varphi(x; a)$ and every formula of the form

$$\chi(x; b_\delta, \ldots, b_{\delta+m-1}) \leftrightarrow \chi(x; b_{\delta+\nu(0)}, \ldots, b_{\delta+\nu(m-1)})$$

for every $m < \omega, \chi \in L(A)$, and increasing function $\nu \in {}^m\lambda$. Note that if $c \models \Gamma_\delta(x)$, then $b_{\geq\delta}$ is Ac-indiscernible so $\Gamma_\delta(x)$ is inconsistent for all $\delta < \lambda$ by the definition of λ-shredding. It follows by compactness that, for each $\delta < \lambda$, there is a finite sequence $\overline{\chi}^\delta = \langle \chi^\delta_l(x; y_\delta) : l < k_\delta\rangle$ with each $\chi^\delta_l(x; y_\delta) \in L(A)$, and (after adding dummy variables to ensure all formulas in $\overline{\chi}$ have the same parameter variables) there are $m_\delta < \omega$ and a sequence of increasing functions $\overline{\nu}_\delta = \langle \nu_{\delta,l} : l < k_\delta\rangle$ from ${}^{m_\delta}\lambda$ such that

$$\varphi(x; a) \vdash \bigvee_{l<k_\delta} \chi^\delta_l(x; b_\delta, \ldots, b_{\delta+m_\delta-1}) \leftrightarrow \neg\chi^\delta_l(x; b_{\delta+\nu_{\delta,l}(0)}, \ldots, b_{\delta+\nu_{\delta,l}(m_\delta-1)}).$$

Let $u_\delta = \{i : i < m_\delta\} \cup \{\nu_{\delta,l}(i) : i < m_\delta, l < k_\delta\}$. Let n_δ be the least natural number such that $|u_\delta| < n_\delta$.

By the pigeonhole principle and the regularity of λ, there is a subset of limit ordinals $X \subseteq \lambda$ of size λ, $n, m < \omega$ and $\overline{\chi} = \langle \chi_l : l < k\rangle$ so that $\delta \in X$ implies $n_\delta = n$, $k_\delta = k$, $m_\delta = m$, and $\overline{\chi}^\delta = \overline{\chi}$. Further refining X, we may assume $\delta < \delta'$ from X implies $\delta + i < \delta'$ for all $i \in u_\delta$. Let $Y = \{\delta + i : \delta \in X, i \in u_\delta\} \subseteq \lambda$. Let $\langle \alpha_i : i < \lambda\rangle$ be an increasing enumeration of a subset of λ containing Y so that $\langle \alpha_{(2n)\cdot i} : i < \lambda\rangle$ enumerates X (which is possible by the choice of n). Then if $\delta = \alpha_{(2n)\cdot j} \in X$, we can find for each $l < k$ an increasing function $\eta_{\delta,l} \in {}^n(2n)$ so that

$$\delta + \nu_{\delta,l}(i) = \alpha_{(2n)\cdot j+\eta_{\delta,l}(i)},$$

for all $i < m$ (we do not place any constraints on $\eta_{\delta,l}(i)$ for $m \leq i < n$ other than the requirement that $\eta_{\delta,l}$ is an increasing function—note that $\alpha_{(2n)\cdot j+\eta_{\delta,l}(i)} < \alpha_{(2n)(j+1)}$, which is the next ordinal in X after δ). Write $\overline{\eta}_\delta$ for this sequence of functions. By one last application of the pigeonhole principle, we can find $X' \subseteq X$ of size λ and $\overline{\eta}$ so that $\delta \in X'$ implies $\overline{\eta} = \overline{\eta_\delta}$ and let $\langle \alpha'_i : i < \lambda\rangle$ be an increasing enumeration of $\{\alpha_{i+j} : \alpha_i \in X', j < 2n\}$. Write $\overline{b}' = \langle b'_i : i < \lambda\rangle$ for the subsequence of $\overline{b}$ defined by $b'_i = b_{\alpha'_i}$. Note that if δ is divisible by $2n$, then $\alpha'_\delta \in X'$.

Unraveling definitions, we see that

$$\varphi(x; a) \vdash \bigvee_{l<k} \chi_l(x; b'_\delta, \ldots, b'_{\delta+m-1}) \leftrightarrow \neg\chi_l(x; b'_{\delta+\eta_l(0)}, \ldots, b'_{\delta+\eta_l(m-1)}),$$

for all $\delta < \lambda$ divisible by $2n$. Because $m < n$, by adding dummy variables to each χ_l, we obtain formulas ψ_l so that

$$\varphi(x; a) \vdash \bigvee_{l<k} \psi_l(x; b'_\delta, \ldots, b'_{\delta+n-1}) \leftrightarrow \neg\psi_l(x; b'_{\delta+\eta_l(0)}, \ldots, b'_{\delta+\eta_l(n-1)}),$$

as desired. $\square$

Remark 2.4. The proof shows, in fact, that any sequence witnessing that $\varphi(x; a)$ λ-shreds over A gives rise to a sequence $\bar{b}$ as in (2) by restricting to a subsequence.

Corollary 2.5. *Assume $\lambda = cf(\lambda) > |T| + |A|$ and $\varphi(x; a)$ λ-shreds over A. Then there is an A-indiscernible sequence $\langle b_\alpha : \alpha < \lambda\rangle$ and $m < \omega$ so that*

- *$\langle (b_{m\cdot\alpha}, b_{m\cdot\alpha+1}, \dots, b_{m\cdot\alpha+m-1}) : \alpha < \lambda\rangle$ is Aa-indiscernible.*
- *$\langle b_\alpha : \alpha < \lambda\rangle$ witnesses that $\varphi(x; a)$ λ-shreds over A and, additionally, for every $c \in \varphi(\mathbb{M}, a)$ and $\alpha < \lambda$, the finite sequence $(b_{m\cdot\alpha}, b_{m\cdot\alpha+1}, \dots, b_{m\cdot\alpha+m-1})$ is not Ac-indiscernible.*

Proof. Suppose $\varphi(x; a)$ λ-shreds over A. By Lemma 2.3, there is an A-indiscernible sequence $\langle c_\alpha : \alpha < \lambda\rangle$, a number $n < \omega$, a sequence of $L(A)$-formulas $\overline{\psi} = \langle \psi_l(x; y_0, \dots, y_{n-1}) : l < k\rangle$, and a sequence $\overline{\eta} = \langle \eta_l : l < k\rangle$ with each $\eta_l \in {}^{n}(2n)$ an increasing function, such that, for every $\delta < \lambda$ divisible by $2n$,

$$\varphi(x; a) \vdash \bigvee_{l<k} \left[\psi_l(x; c_\delta, \dots, c_{\delta+n-1}) \leftrightarrow \neg\psi_l(x; c_{\delta+\eta_l(0)}, \dots, c_{\delta+\eta_l(n-1)})\right].$$

Let $m = 2n$ and extract an Aa-indiscernible sequence $\langle (b_{m\cdot\alpha}, b_{m\cdot\alpha+1}, \dots, b_{m\cdot\alpha+m-1}) : \alpha < \lambda\rangle$ from $\langle (c_{m\cdot\alpha}, c_{m\cdot\alpha+1}, \dots, c_{m\cdot\alpha+m-1}) : \alpha < \lambda\rangle$. Then for all $\delta < \lambda$ divisible by $2n$,

$$\varphi(x; a) \vdash \bigvee_{l<k} \left[\psi_l(x; b_\delta, \dots, b_{\delta+n-1}) \leftrightarrow \neg\psi_l(x; b_{\delta+\eta_l(0)}, \dots, b_{\delta+\eta_l(n-1)})\right]$$

and $\langle b_\alpha : \alpha < \lambda\rangle$ is an A-indiscernible sequence, so we are done. □

From Lemma 2.3, we obtain a variant of shredding that is somewhat more cumbersome and less natural, but will be useful in the arguments below.

Definition 2.6. For an infinite cardinal λ, we say $\varphi(x; a)$ *explicitly λ-shreds* over A if there are n, $\bar{b}$, $\overline{\eta}$, and $\overline{\psi}$ satisfying:

(1) $\bar{b} = \langle b_\alpha : \alpha < \lambda\rangle$ is an A-indiscernible sequence.
(2) $\overline{\eta} = \langle \eta_l : l < k\rangle$ is a finite sequence of increasing functions in ${}^{n}(2n)$.
(3) $\overline{\psi} = \langle \psi_l(x; y_0, \dots, y_{n-1}; a'_l) : l < k\rangle$ is a sequence of formulas with $a'_l \in A$.
(4) For every $\delta < \lambda$ divisible by $2n$, we have

$$\varphi(x; a) \vdash \bigvee_{l<k} \left[\psi_l(x; b_\delta, \dots, b_{\delta+n-1}, a'_l) \leftrightarrow \neg\psi_l(x; b_{\delta+\eta_l(0)}, \dots, b_{\delta+\eta_l(n-1)}, a'_l)\right].$$

We will often say that the tuple $(\bar{b}, n, \overline{\eta}, \overline{\psi})$ witnesses that $\varphi(x; a)$ explicitly λ-shreds over A. We say $\varphi(x; a)$ *explicitly shreds* over A if it explicitly λ-shreds over A for some λ. As before, we will say that a type p over $B \supseteq A$ explicitly shreds over A if it implies some formula that does, and it explicitly shreds over A with a built-in witness if the witnessing A-indiscernible sequence $\bar{b}$ may be chosen to be contained in B.

The point of introducing this definition is that explicit shredding is a notion that lends itself to compactness arguments, as in the following easy lemma:

Lemma 2.7. *The following are equivalent:*

(1) The formula $\varphi(x; a)$ shreds over A.

(2) The formula $\varphi(x;a)$ explicitly shreds over A.
(3) The formula $\varphi(x;a)$ explicitly $\aleph_0$-shreds over A.
(4) The formula $\varphi(x;a)$ explicitly λ-shreds over A for all infinite cardinals λ.

Proof. (1) $\Longrightarrow$ (2) is Lemma 2.3, and (2) $\Longrightarrow$ (3) is immediate, by restricting the witnessing indiscernible sequence to an initial segment of length ω. (4) $\Longrightarrow$ (1) is also immediate, taking any $\lambda \geq (|A|+|T|)^+$, since explicit shredding implies shredding.

(3) $\Longrightarrow$ (4) Let λ be any uncountable cardinal and suppose $\varphi(x;a)$ explicitly $\aleph_0$-shreds, witnessed by $(\overline{b}, n, \overline{\eta}, \overline{\psi})$, where $\overline{b} = \langle b_i : i < \omega\rangle$. Define $b'_i = (b_{2n\cdot i}, \dots, b_{2n\cdot i+2n-1})$ for all $i<\omega$. The sequence $\langle b'_i : i<\omega\rangle$ is also A-indiscernible and, without loss of generality, by (the proof of) Corollary 2.5, we may assume further that it is Aa-indiscernible. Then applying compactness, we can stretch it to $\overline{b}' = \langle b'_i : i < \lambda\rangle$ with $b'_i = (b_{2n\cdot i}, \dots, b_{2n\cdot i+2n-1})$ for all $i<\lambda$. Then the sequence $\langle b_i : i<\lambda\rangle$ is A-indiscernible and, together with n, $\overline{\eta}$, and $\overline{\psi}$ witnesses that $\varphi(x;a)$ explicitly λ-shreds. This shows (4). $\square$

Lemma 2.8. *Suppose A is a set of parameters and $B \subseteq A$. The following are equivalent:*

(1) $\varphi(x;a)$ shreds over A.
(2) There is an A-indiscernible sequence $\overline{b} = \langle b_i : i<\lambda\rangle$ for $\lambda = (|A|+|T|)^+$ such that for no $c \in \varphi(\mathbb{M};a)$ and for no $\alpha<\lambda$ is $\overline{b}_{\geq\alpha}$ indiscernible over Bc.
(3) $\varphi(x;a)$ explicitly shreds over A witnessed by a tuple $(\overline{b}, n, \overline{\eta}, \overline{\psi})$, where the formulas $\overline{\psi}$ have no parameters (i.e. are over the empty set).

Proof. (2) $\Longrightarrow$ (1) is clear by the definition of shredding, since in particular (2) entails that for no $c \in \varphi(\mathbb{M};a)$ and $\alpha<\lambda$ is $\overline{b}_{\geq\alpha}$ indiscernible over Ac.

(3) $\Longrightarrow$ (2) since, if $\overline{b} = \langle b_\alpha : \alpha<\lambda\rangle$, then, for all $\delta<\lambda$ divisible by $2n$, we have the implication

$$\varphi(x;a) \vdash \bigvee_{l<k} \left[\psi_l(x;b_\delta,\dots,b_{\delta+n-1}) \leftrightarrow \neg\psi_l(x;b_{\delta+\eta_l(0)},\dots,b_{\delta+\eta_l(n-1)})\right],$$

which implies that no end segment of $\overline{b}$ can be indiscernible over a realization of $\varphi(x;a)$ (with no additional parameters). *A fortiori*, no end segment of $\overline{b}$ can be indiscernible over a set consisting of B and a realization of $\varphi(x;a)$.

To prove (1) $\Longrightarrow$ (3), we know, by Lemma 2.7, $\varphi(x;a)$ explicitly shreds over A, witnessed by the tuple $(\overline{b}, n, \overline{\eta}, \overline{\psi})$. Let c be a tuple enumerating the parameters occurring in the $\overline{\psi}$ and let $\overline{d}$ be the sequence $\overline{d} = \langle d_\alpha : \alpha<\lambda\rangle = \langle (b_\alpha, c) : \alpha<\lambda\rangle$, which is A-indiscernible since $\overline{b}$ was assumed to be A-indiscernible and c comes from A. Then it is easily seen that by merely adding dummy variables to the formulas $\overline{\psi}$, we get $\overline{\psi}' = \langle \psi'_l : l<k\rangle$ such that for every $\delta<\lambda$ divisible by $2n$, we have

$$\varphi(x;a) \vdash \bigvee_{l<k} \left[\psi'_l(x;d_\delta,\dots,d_{\delta+n-1}) \leftrightarrow \neg\psi'_l(x;d_{\delta+\eta_l(0)},\dots,d_{\delta+\eta_l(n-1)})\right].$$

Then $\varphi(x;a)$ explicitly shreds, witnessed by the tuple $(\overline{d}, n, \overline{\eta}, \overline{\psi}')$, where the formulas $\overline{\psi}'$ have no parameters. $\square$

The direction (1) $\Longrightarrow$ (2) of Lemma 2.8 gives base monotonicity for shredding:

Corollary 2.9. *Suppose $B \subseteq A$ and $\varphi(x;a)$ shreds over A, then $\varphi(x;a)$ shreds over B.*

Proposition 2.10. *Suppose κ is a regular cardinal and $m<\omega$. The following are equivalent:*

(1) There is an increasing sequence $\overline{A} = \langle A_i : i \le \kappa \rangle$ with $A_\kappa = \bigcup_{i<\kappa} A_i$ and $p \in S^m(A_\kappa)$ such that p (explicitly) shreds over A_i for all $i < \kappa$.

(2) There is an increasing continuous sequence of models $\overline{M} = \langle M_i : i \le \kappa \rangle$ with $M_\kappa = \bigcup_{i<\kappa} M_i$ and some $p \in S^m(M_\kappa)$ such that $p \restriction M_{i+1}$ shreds over M_i with a built-in witness.

Proof. The direction (2) $\Longrightarrow$ (1) is immediate by Lemma 2.7, taking $A_i = M_i$ for all $i \le \kappa$.

(1) $\Longrightarrow$ (2): for each $i < \kappa$, fix a formula $\varphi_i(x; a_i) \in p$ that explicitly shreds over A_i, witnessed by $(\overline{b}_i, n_i, \overline{\eta}_i, \overline{\psi}_i)$. By Lemma 2.8, we may assume that $\overline{b}_i$ and $\overline{\psi}_i$ have been chosen so that the formulas in $\overline{\psi}_i$ have no parameters. By the regularity of κ, after replacing the sequence with a subsequence, we may assume $\varphi_i(x; a_i) \in p \restriction A_{i+1}$. Moreover, without loss of generality, we may assume $\overline{b}_i = \langle b_{i,j} : j < \omega \rangle$ for all $i < \kappa$.

Our assumption that $\overline{\psi}_i$ contains no parameters entails that $\varphi_i(x; a_i)$ explicitly shreds over any subset of A_i and, in particular, that $\varphi_i(x; a_i)$ shreds over $a_{<i}$. Therefore we may replace A_i by $a_{<i}$ and p by $p \restriction a_{<\kappa}$ and, hence, without loss of generality, the sequence $\langle A_i : i < \kappa \rangle$ is increasing and continuous.

Let $\overline{\lambda} = \langle \lambda_i : i < \kappa \rangle$ be an increasing and continuous sequence of cardinals $\ge |T|$ with $\lambda_i \ge |A_i|$ and λ_{i+1} regular for all $i < \kappa$. Denote $\lim_{i<\kappa} \lambda_i$ by μ. Let $\overline{y} = \langle y_j : j < \mu \rangle$ be a sequence of variables of length μ and denote by $\overline{y}_i$ the restriction $\langle y_j : j < \lambda_i \rangle$ to the first λ_i variables.

Let $\Gamma(\overline{y}, \overline{z}_i : i < \kappa)$ be a partial type over A_κ such that the variables $\overline{z}_i = \langle z_{i,j} : j < \lambda_{i+1} \rangle$ have length λ_{i+1}, and which naturally expresses the following, for all $i < \kappa$:

(1) The sequence $\overline{y}_i$ enumerates a model containing A_i.

(2) The sequence $\overline{z}_i$ is indiscernible over $\overline{y}_i$, realizes the same EM-type over A_i as $\overline{b}_i$, and is contained in $\overline{y}_{i+1}$.

(3) The formula $\varphi_i(x; a_i)$ explicitly shreds over $\overline{y}_i$, witnessed by $(\overline{z}_i, n_i, \overline{\eta}_i, \overline{\psi}_i)$.

It suffices to show that this partial type is consistent, as to conclude we may take any complete type over the union of models realizing the $\overline{y}_i$ containing $\{\varphi(x; a_i) : i < \kappa\}$. By compactness, it suffices to show this for κ finite. By induction on $\kappa < \omega$, we will show that we can find models and sequences satisfying the conditions in the partial type above. Suppose this has been shown for $\kappa = l$. By induction, we know there are models $\langle M_j : j < l \rangle$ and sequences $\langle \overline{c}_j : j < l \rangle$ satisfying the requirements. Choose an arbitrary model M of size λ_l containing $A_l M_{l-1} \overline{c}_{l-1}$. Extract an M-indiscernible sequence $\overline{b}_l'$ from $\overline{b}_l$. Then $\overline{b}_l' \equiv_{A_l} \overline{b}_l$ so there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A_l)$ with $\sigma(\overline{b}_l') = \overline{b}_l$. For each $j < l$, define $M_j' = \sigma(M_j)$ and $\overline{c}_j' = \sigma(\overline{c}_j)$, and then put $M_l' = \sigma(M)$.

Finally, let $m = n_l$ and consider the sequence $\langle (b_{l,2m\cdot i}, \ldots, b_{l,2m\cdot i+2m-1}) : i < \omega \rangle$. Let $\overline{b}_l'' = \langle (b''_{2m\cdot i}, \ldots, b''_{2m\cdot i+2m-1}) : i < \lambda_{l+1} \rangle$ be an $M_l' a_l$-indiscernible sequence realizing the same EM-type over $M_{l-1} A_l a_l$ as $\langle (b_{l,2m\cdot i}, \ldots, b_{l,2m\cdot i+2m-1}) : i < \omega \rangle$. Then defining $\overline{c}_l' = \langle b_i'' : i < \lambda_{l+1} \rangle$, we have that $\overline{c}_l'$ is an M_l'-indiscernible sequence and $\varphi_l(x; a_l)$ explicitly shreds over M_l', witnessed by $(\overline{c}_l', n_l, \overline{\eta}_l, \overline{\psi}_l)$. It follows that $\langle M_j' : j < l+1 \rangle$ and $\langle \overline{c}_j' : j < l+1 \rangle$ satisfy the requirements, completing the induction and the proof. $\square$

Remark 2.11. Note that, in the course of the proof Proposition 2.10, we were able to replace each A_i with $a_{<i}$, in which case we clearly have $|A_i| + \aleph_0 = |i| + \aleph_0$ (in fact, for finite i we have $|A_i| = l(a_0)i$ and for infinite i we have $|A_i| = |i|$).

It follows, then, that if κ is a regular cardinal and $\kappa^m_{\mathrm{shred}}(T) \ge \kappa^+$, then we can find a witness of the form $\langle M_i : i \le \kappa \rangle$ and $p \in S^m(M_\kappa)$ with $|M_0|$ an arbitrary regular cardinal $\ge |T|$, $\langle |M_i| : i < \kappa \rangle$ an increasing and continuous sequence of cardinals, and with $|M_{i+1}|$ a regular cardinal for all $i < \kappa$.

2.2. Shredding and classification theory

Here we establish some preliminary connections between the concepts of shredding and unshreddable theories with NIP and simplicity.

Definition 2.12. Recall that the formula $\varphi(x;y)$ has the *independence property* if for every n, there are $a_0, \dots, a_{n-1}$ and tuples b_w for every $w \subseteq \{0, \dots, n-1\}$ so that

$$\models \varphi(a_i, b_w) \iff i \in w.$$

A theory is said to have the independence property if some formula does modulo T, otherwise T is NIP.

Equivalently, the formula $\varphi(x;y)$ has the independence property if there is an indiscernible sequence $\langle a_i : i < \omega \rangle$ and b so that $\models \varphi(a_i, b)$ if and only if i is even (see, e.g., [13, Lemma 2.7]).

Proposition 2.13. *If $\lambda = cf(\lambda) > |T| + |A|$ and some consistent formula $\varphi(x;a)$ λ-shreds over A, then T has the independence property.*

Proof. Suppose $\varphi(x;a)$ λ-shreds over A. Then by Lemma 2.7, it explicitly λ-shreds so we may fix k, n, $\overline{\psi}$, $\overline{\eta}$, and $\overline{b} = \langle b_\alpha : \alpha < \lambda \rangle$ as in the definition of explicit shredding. Let c be an arbitrary element of $\varphi(\mathbb{M}; a)$. By the pigeonhole principle, there is a subset $X \subseteq \lambda$ of size λ, $l < k$, and $t \in \{0, 1\}$ so that

$$\models \psi_l(c; b_{\omega\cdot\alpha}, \dots, b_{\omega\cdot\alpha+n-1}, a'_l)^t \wedge \psi_l(c; b_{\omega\cdot\alpha+\eta_l(0)}, \dots, b_{\omega\cdot\alpha+\eta_l(n-1)}, a'_l)^{1-t}$$

for all $\alpha \in X$. Let $\langle \alpha_i : i < \lambda \rangle$ be an increasing enumeration of X. For $i < \lambda$ even, we define $d_i = (b_{\omega\cdot\alpha_i}, \dots, b_{\omega\cdot\alpha_i+n-1})$ and for $i < \lambda$ odd, we define $d_i = (b_{\omega\cdot\alpha_i+\eta_l(0)}, \dots, b_{\omega\cdot\alpha_i+\eta_l(n-1)})$. Then $\langle d_i : i < \lambda \rangle$ is an A-indiscernible sequence, by the A-indiscernibility of $\overline{b}$, and we have

$$c \models \{\psi_l(x, d_i, a'_l)^t : i < \lambda \text{ even}\} \cup \{\psi_l(x; d_i, a'_l)^{1-t} : i < \lambda \text{ odd}\},$$

which shows $\chi(x, z; y) = \psi_l(x, y, z)$ has the independence property. □

Recall that a formula $\varphi(x; a_0)$ *divides* over a set A if there is an A-indiscernible sequence $\langle a_i : i < \omega \rangle$ such that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent. A formula $\varphi(x; b)$ *forks* over A if $\varphi(x; b) \vdash \bigvee_{i<\kappa} \psi(x; a_i)$ where each $\psi_i(x; a_i)$ divides over A. A type divides or forks over A if it implies a formula that respectively divides or forks over A. A theory is called *simple* if there is a cardinal κ such that, whenever p is a type (in finitely many variables) over A, there is $B \subseteq A$ over which p does not fork with $|B| < \kappa$. The least such cardinal κ is called $\kappa(T)$ and the least such regular cardinal is called $\kappa_{\mathbf{r}}(T)$.

Proposition 2.14. *If $\varphi(x;a)$ shreds over A then $\varphi(x;a)$ forks over A.*

Proof. Suppose $\lambda = (|T| + |A|)^+$ and, by Lemma 2.7, we know $\varphi(x;a)$ explicitly λ-shreds over A. Hence, there is an A-indiscernible sequence $\overline{b} = \langle b_i : i < \lambda \rangle$ such that there is a sequence of $L(A)$-formulas $\langle \psi_l(x; y_0, \dots, y_{n-1}) : l < k \rangle$ and a sequence $\langle \eta_l : l < k \rangle$ with the property that

$$(*) \quad \varphi(x;a) \vdash \bigvee_{l<k} \psi_l(x; b_\delta, \dots, b_{\delta+n-1}) \leftrightarrow \neg\psi_l(x; b_{\delta+\eta_l(0)}, \dots, b_{\delta+\eta_l(n-1)})$$

for all $\delta < \lambda$ divisible by $2n$. Given $\alpha < \lambda$, let $\overline{b}_\alpha = \langle b_{\omega\cdot\alpha+i} : i < \omega \rangle$. By the proof of Corollary 2.5, we can moreover assume that $\langle \overline{b}_\alpha : \alpha < \lambda \rangle$ is an Aa-indiscernible sequence. We will choose $(a_\alpha)_{\alpha<\lambda}$ so that

Sh:1192

(1) For all $\alpha < \lambda$, $a_\alpha \models \mathrm{tp}(a/A\overline{b}_{<\alpha})$.
(2) For all $\alpha < \lambda$, $\overline{b}_\alpha$ is an $a_\alpha A$-indiscernible sequence.

Given $(a_\beta)_{\beta<\alpha}$, to choose a_α, first apply Ramsey and compactness to extract from $\overline{b}_\alpha$ a sequence $\overline{b}^*_\alpha = \langle b^*_{\omega\cdot\alpha+i} : i < \omega\rangle$ which is $Aa\overline{b}_{<\alpha}$-indiscernible. Then as $\overline{b}_\alpha \equiv_{A\overline{b}_{<\alpha}} \overline{b}^*_\alpha$, we can choose a_α so that $a_\alpha\overline{b}_\alpha \equiv_{A\overline{b}_{<\alpha}} a\overline{b}^*_\alpha$. The sequence $(a_\alpha)_{\alpha<\lambda}$ satisfies both (1) and (2) by construction. By Ramsey, compactness, and automorphism, we may moreover assume the sequence $\langle(a_\alpha, \overline{b}_\alpha) : \alpha < \lambda\rangle$ is an A-indiscernible sequence.

By the finite Ramsey theorem, there is n_* so that $n_* \to (2n)^n_{2^k}$. Let $\Lambda = \{\nu \in {}^{2n}(n_*) : \nu \text{ increasing}\}$ and for $\nu \in \Lambda$, let $\overline{b}_{\alpha,\nu} = (b_{\omega\cdot\alpha+\nu(i)})_{i<2n}$. Let $\varphi'(x;\overline{b}_{\alpha,\nu})$ (suppressing parameters from A) denote the formula

$$\bigwedge_{l<k} \psi_l(x; b_{\omega\cdot\alpha+\nu(0)}, \dots, b_{\omega\cdot\alpha+\nu(n-1)}) \leftrightarrow \psi_l(x; b_{\omega\cdot\alpha+\nu(\eta_l(0))}, \dots, b_{\omega\cdot\alpha+\nu(\eta_l(n-1))}).$$

Let $\varphi_*(x; a_\alpha, \overline{b}_{\alpha,\nu})$ denote the formula $\varphi(x; a_\alpha) \wedge \varphi'(x; \overline{b}_{\alpha,\nu})$.

Claim 1: $\varphi(x; a_0) \vdash \bigvee_{\nu\in\Lambda} \varphi_*(x; a_0, \overline{b}_{0,\nu})$.

Proof of claim. This proof is purely combinatorial and will not make use of (1), (2), or (*). Let c be any tuple with $\mathbb{M} \models \varphi(c; a_0)$. Given any increasing $\xi \in {}^n(n_*)$, define $\chi(\xi) = \{l < k : \mathbb{M} \models \psi_l(c; b_{\xi(0)}, \dots, b_{\xi(n-1)}, a'_l)\}$. This defines a coloring with 2^k possible colors. As $n_* \to (2n)^n_{2^k}$, there is $\nu \in \Lambda$ so that ν is an increasing enumeration of a homogeneous subset of n_* of size $2n$. For each $l < k$, by homogeneity, both $(\nu(0), \dots, \nu(n-1))$ and $(\nu(\eta_l(0)), \dots, \nu(\eta_l(n-1)))$ take on the same value with respect to the coloring χ, hence

$$\mathbb{M} \models \bigwedge_{l<k} \psi_l(c; b_{\nu(0)}, \dots, b_{\nu(n-1)}) \leftrightarrow \psi_l(c; b_{\nu(\eta_l(0))}, \dots, b_{\nu(\eta_l(n-1))}).$$

This shows $\mathbb{M} \models \varphi_*(x; a_0, \overline{b}_{0,\nu})$, proving the claim. □

Claim 2: For each $\nu \in \Lambda$, $\varphi_*(x; a_0, \overline{b}_{0,\nu})$ divides over A.

Proof of claim. Let $\nu_* = (0, \dots, 2n-1)$. We will first show that $\varphi_*(x; a_0, \overline{b}_{0,\nu_*})$ divides over A. By (∗),

$$\varphi(x; a) \vdash \neg \bigwedge_{l<k} \psi_l(x; b_{\omega\cdot\alpha}, \dots, b_{\omega\cdot\alpha+n-1}) \leftrightarrow \psi_l(x; b_{\omega\cdot\alpha+\eta_l(0)}, \dots, b_{\omega\cdot\alpha+\eta_l(n-1)}),$$

and therefore $\varphi(x; a) \vdash \neg\varphi'(x; \overline{b}_{\alpha,\nu_*})$ for all $\alpha < \lambda$. For all α, we have $a_\alpha \equiv_{A\overline{b}_{<\alpha}} a$, so if $\beta < \alpha$, then $\varphi(x; a_\alpha) \vdash \neg\varphi'(x; \overline{b}_{\beta,\nu_*})$. Therefore, when $\beta < \alpha$, we have

$$\varphi_*(x; a_\alpha, \overline{b}_{\alpha,\nu_*}) \vdash \neg\varphi_*(x; a_\beta, \overline{b}_{\beta,\nu_*}),$$

from which it follows that $\{\varphi_*(x; a_\alpha, \overline{b}_{\alpha,\nu_*}) : \alpha < \lambda\}$ is 2-inconsistent. Since $\langle(a_\alpha, \overline{b}_{\alpha,\nu_*}) : \alpha < \lambda\rangle$ is an A-indiscernible sequence, we have shown $\varphi_*(x; a_0, \overline{b}_{0,\nu_*})$ divides over A.

Finally, as $\overline{b}_0$ is an Aa_0-indiscernible sequence, we have $\overline{b}_{0,\nu} \equiv_{Aa_0} \overline{b}_{0,\nu_*}$ for all $\nu \in \Lambda$. It follows that $\varphi_*(x; a_0, \overline{b}_{0,\nu})$ divides over A for all $\nu \in \Lambda$. This proves the claim and therefore proves the proposition, by Claim 1. □

As a corollary, we obtain the following:

Proposition 2.15. *If T is simple, then $\kappa_{\mathrm{shred}}(T) \leq \kappa_{\mathbf{r}}(T)$.*

Proof. Suppose not, so $\kappa_{\mathbf{r}}(T) < \kappa_{\text{shred}}(T)$. Let $\kappa = \text{cf}(\kappa) \geq \kappa_{\mathbf{r}}(T)$ with $\kappa < \kappa_{\text{shred}}(T)$. Then we have the following:

- $\langle M_i : i \leq \kappa \rangle$ is an increasing sequence of models of T.
- $p(x) = \{\varphi(x; a_i) : i < \kappa\}$ is a consistent partial type.
- $\varphi(x; a_i)$ shreds over M_i.
- $a_i \in M_{i+1}$.

Then by Proposition 2.14, p forks over M_i for all $i < \kappa$. Let $M_\kappa = \bigcup_{i<\kappa} M_i$. As T is simple, there is subset $A \subseteq M_\kappa$ with $|A| < \kappa_{\mathbf{r}}(T)$ such that p does not fork over A. As κ is regular, there is some $i < \kappa$ so that $A \subseteq M_i$, from which it follows that p does not fork over M_i as well, a contradiction to the definition of $\kappa_{\mathbf{r}}(T)$. $\square$

Corollary 2.16. *The class of unshreddable theories contains the NIP and simple theories.*

Proof. This follows immediately from Proposition 2.13 and Proposition 2.15. $\square$

3. Respect and exact saturation

3.1. Respect

For the entirety of this subsection, we fix a singular cardinal μ. Writing $\text{cf}(\mu) = \kappa$, we will assume there is an increasing and continuous sequence of cardinals $\overline{\lambda} = \langle \lambda_i : i \leq \kappa \rangle$ such that $\lambda_0 > \kappa$, λ_{i+1} is regular for all $i < \kappa$, and $\lambda_\kappa = \mu$. We will assume we have fixed for each $i < \kappa$ a sequence $\overline{a}_i = \langle a_{i,j} : j < \lambda_{i+1} \rangle$, which is $\overline{a}_{<i}$-indiscernible. Additionally, we will assume that T is a theory with $\kappa^1_{\text{shred}}(T) \leq \kappa$.

Definition 3.1. Suppose $i < \kappa$ and A is a set of parameters.

(1) We say that A *respects* $\overline{a}_i$ when for any finite subset $C \subseteq A$, there is $\alpha < \lambda_{i+1}$ such that $\overline{a}_{i,\geq\alpha}$ is C-indiscernible.
(2) We say $p \in S^{<\omega}(A)$ *respects* $\overline{a}_i$ when, for every $c \models p$, the set Ac respects $\overline{a}_i$.

Remark 3.2. In Definition 3.1(1), by the regularity of λ_{i+1}, we could have equivalently asked for the existence of such an $\alpha < \lambda_{i+1}$ for any $C \subseteq A$ with $|C| < \lambda_{i+1}$, since there are fewer than λ_{i+1} finite subsets of any such C.

Definition 3.3. We define $\mathbb{K}$ to be the class of $\overline{A}$ such that:

(1) $\overline{A} = \langle A_i : i \leq \kappa \rangle$ is increasing continuous.
(2) $|A_i| = \lambda_i$ for all $i < \kappa$.
(3) $\overline{a}_i \subseteq A_{i+1}$ for all $i < \kappa$.
(4) A_i respects $\overline{a}_i$ for all $i < \kappa$, i.e. there is some $\alpha < \lambda_{i+1}$ such that $\overline{a}_{i,\geq\alpha}$ is A_i-indiscernible, using Remark 3.2.

Given $\overline{A}, \overline{B} \in \mathbb{K}$, we say $\overline{A} \leq_{\mathbb{K}} \overline{B}$ if $A_j \subseteq B_j$ for all $j < \kappa$. We say $\overline{A} \leq_{\mathbb{K},i} \overline{B}$ if $A_j \subseteq B_j$ for all j satisfying $i \leq j < \kappa$ and $\overline{A} \leq_{\mathbb{K},*} \overline{B}$ if $\overline{A} \leq_{\mathbb{K},i} \overline{B}$ for some $i < \kappa$. We may omit the $\mathbb{K}$ subscript when it is clear from context.

Lemma 3.4. *Suppose p is a partial 1-type over A_κ with $|dom(p)| \leq \lambda_i$ for some $i < \kappa$. Then there are i' with $i \leq i' < \kappa$ and $p' \supseteq p$ with $|dom(p')| \leq \lambda_{i'}$ such that, if q is a type over A_κ extending p', then q does not shred over $A_{i'}$.*

Proof. Suppose not. Then we will construct an increasing sequence of types $\langle p_j : j < \kappa\rangle$ extending p and an increasing sequence of ordinals $\langle i_j : j < \kappa\rangle$ such that $|\mathrm{dom}(p_j)| = \lambda_{i_j}$ and p_j shreds over A_{i_j} for all $j < \kappa$. To begin, we set $i_0 = i$ and use our assumption to find some $p_0 \supseteq p$ such that p_0 shreds over A_{i_0}. We may assume $\mathrm{dom}(p_0)$ contains A_{i_0} and has cardinality λ_{i_0}. Given any $\langle p_j : j < \alpha\rangle$ and $\langle i_j : j < \alpha\rangle$ for $\alpha \geq 1$, we put $p' = \bigcup_{j<\alpha} p_j$ and $i' = \sup_{j<\alpha} i_j$ (here we make use of the fact that κ is regular). Then $|\mathrm{dom}(p')| = \lambda_{i'}$ and p' extends p. Let $i_\alpha = i' + 1$. As $i_\alpha \geq i$, by hypothesis, there is some type $p_\alpha \supseteq p'$ such that p_α shreds over $A_{i'+1}$. As this will be witnessed by a single formula, we may assume $\mathrm{dom}(p_\alpha)$ contains A_{i_α} and $|\mathrm{dom}(p_\alpha)| = \lambda_{i_\alpha}$, completing the induction.

Let $p_* = \bigcup_{j<\kappa} p_j$. Then, by construction, we have p_* shreds over A_{i_j} for all $j < \kappa$. By Proposition 2.10, this contradicts $\kappa^1_{\mathrm{shred}}(T) \leq \kappa$. □

Lemma 3.5. *If $\overline{A} \in \mathbb{K}$ and p is a 1-type over A_κ with $|\mathrm{dom}(p)| < \mu$, then there is $\overline{A}' \in \mathbb{K}$ such that $\overline{A} \leq_{\mathbb{K}} \overline{A}'$ and some $c \in A'_\kappa$ realizes $p(x)$.*

Proof. By Lemma 3.4 and the choice of μ, we may extend p to a type p' such that, for some $i < \kappa$, $|\mathrm{dom}(p')| \leq \lambda_i$ and no type extending p' over A_κ shreds over A_i, and hence does not shred over $A_{i'}$ for any $i' \geq i$ by base monotonicity. Without loss of generality, we may assume $p = p'$.

By induction on $j \in [i, \kappa]$, we will define types $p_j \in S^1(A_j)$ so that

(1) The types p_j are increasing with j.
(2) For all $j \in [i, \kappa)$, $p_j \cup p$ is consistent.
(3) For all $j \in [i, \kappa)$, if $c \models p_{j+1}$, then for some $\alpha < \lambda_{j+1}$, $\overline{a}_{j,\geq\alpha}$ is $A_j c$-indiscernible.

Let $p_i \in S^1(A_i)$ be any type consistent with p. Given p_j, we note that $p \cup p_j$ extends p and therefore does not explicitly shred over A_j. Because $|p \cup p_j| < \lambda_{j+1}$, by compactness and the fact that A_j respects $\overline{a}_j$, there is a realization $c \models p \cup p_j$ and $\alpha < \lambda_{j+1}$ such that $\overline{a}_{j,\geq\alpha}$ is $A_j c$-indiscernible. We put $p_{j+1} = \mathrm{tp}(c/A_{j+1})$. Finally, given $\langle p_j : j \in [i, \delta)\rangle$ for δ limit $> i$, we set $p_\delta = \bigcup_{j\in[i,\delta)} p_j$.

Define $p_\kappa = \bigcup_{j\in[i,\kappa)} p_j$. Let c realize p_κ and define $\overline{A}_*$ by $A^*_j = A_j$ for all $j < i+1$ and $A^*_j = A_j c$ for all $j \geq i+1$. For all $j \in [i, \kappa)$, as c realizes p_{j+1}, we know there is $\alpha < \lambda_{j+1}$ such that $\overline{a}_{j,\geq\alpha}$ is cA_j-indiscernible. It follows that $\overline{A}_* \in \mathbb{K}$, completing the proof. □

3.2. A one variable theorem

Theorem 3.6. *For all m, we have $\kappa^m_{\mathrm{shred}}(T) = \kappa^1_{\mathrm{shred}}(T)$.*

Proof. The inequality $\kappa^m_{\mathrm{shred}}(T) \geq \kappa^1_{\mathrm{shred}}(T)$ is clear, so it suffices to show $\kappa^1_{\mathrm{shred}}(T) \geq \kappa^m_{\mathrm{shred}}(T)$. Suppose $\kappa \geq \kappa^1_{\mathrm{shred}}(T)$ is a regular cardinal, $\langle \lambda_i : i < \kappa\rangle$ is an increasing continuous sequence of cardinals with $\lambda_0 > \kappa + |T|$ and λ_{i+1} regular for all $i < \kappa$. Let $\mu = \sup_{i<\kappa} \lambda_i$. Note $\mu > \kappa$.

We will prove by induction on m that, if $\kappa < \kappa^m_{\mathrm{shred}}(T)$, there is an increasing and continuous sequence of sets $\langle B_i : i \leq \kappa\rangle$ and $q(y) \in S_1(B_\kappa)$ such that $q \restriction B_{i+1}$ shreds over B_i. This contradictions our assumption that $\kappa \geq \kappa^1_{\mathrm{shred}}(T)$, by Proposition 2.10.

When $m = 1$, we immediately have a contradiction since $\kappa^1_{\mathrm{shred}}(T) \leq \kappa < \kappa^1_{\mathrm{shred}}(T)$.

Suppose it has been proven for m and suppose $\langle A_i : i \leq \kappa\rangle$ is an increasing continuous sequence of models with $|A_i| = \lambda_i$ and $p(x_0, \ldots, x_m) \in S^{m+1}(A_\kappa)$ is a type such that $p \restriction A_{i+1}$ shreds over A_i with a built-in witness $\bar{b}_i$, witnessed by the formula $\varphi_i(x_0, \ldots, x_m; a_i) \in p \upharpoonright A_{i+1}$. Then because $\bar{b}_i$ is A_i-indiscernible, we have $\langle A_i : i \leq \kappa \rangle \in \mathbb{K}$ in the notation of Subsection 3.1 with the $\bar{b}_i$ playing the role of $\bar{a}_i$.

Let $p'(x_0, \ldots, x_m) = \{\varphi(x_0, \ldots, x_m; a_i) : i < \kappa\}$ and let $p''(x_m)$ be defined by

$$\begin{aligned} p''(x_m) &= (\exists x_0, \ldots, x_{m-1}) \bigwedge p'(x_0, \ldots, x_m) \\ &= \{(\exists x_0, \ldots, x_{m-1}) \bigwedge_{\varphi \in w} \varphi(x_0, \ldots, x_{m-1}) : w \subseteq p' \text{ finite}\}. \end{aligned}$$

Note that $|p''| = \kappa < \mu$. By Lemma 3.5, there is $\overline{B} = \langle B_i : i \leq \kappa \rangle \in \mathbb{K}$ such $\overline{A} \leq_{\mathbb{K}} \overline{B}$ and such that p'' is realized by some $c \in B_\kappa$. By the definition of $\mathbb{K}$, for each $i < \kappa$, there is some $\alpha_i < \lambda_{i+1}$ such that $\bar{b}_{i, \geq \alpha_i}$ is B_i-indiscernible. Let i_* be minimal such that $c \in B_{i_*}$ and let $q(x_0, \ldots, x_{m-1}) = p'(x_0, \ldots, x_{m-1}, c)$. Let $q' \in S(B_\kappa)$ be any completion of q. Then for all $i \geq i_*$, we have that $q' \upharpoonright B_{i+1}$ shreds over B_i with the built-in witness $\bar{b}_{i, \geq \alpha_i}$. Reindexing by setting $B'_i = B_{i_*+i}$ and $a_{i,j} = b_{i, \alpha_i + j}$ for all $i < \kappa$ and $j < \lambda_{i+1}$, we may apply the induction hypothesis to complete the proof. $\square$

3.3. Exact saturation

As in Subsection 3.1, we fix a singular cardinal μ. Writing $\mathrm{cf}(\mu) = \kappa$, we will assume there is an increasing and continuous sequence of cardinals $\overline{\lambda} = \langle \lambda_i : i \leq \kappa \rangle$ such that $\lambda_0 > \kappa$, λ_{i+1} is regular for all $i < \kappa$, and $\lambda_\kappa = \mu$.

We write I to denote $\{(i, \alpha) : i < \kappa, \alpha < \lambda_{i+1}\}$ ordered lexicographically. We write $I_{i, \geq \beta} = \{(j, \alpha) : j = i \text{ and } \alpha \geq \beta\}$ and we write I_i for $I_{i, \geq 0}$. We also fix an indiscernible sequence $\bar{a} = \langle a_t : t \in I \rangle$. We similarly write $\bar{a}_{i, \geq \beta}$ for $\langle a_t : t \in I_{i, \geq \beta} \rangle$ and $\bar{a}_i$ for $\langle a_t : t \in I_i \rangle$. If $i < \kappa$, and $\alpha < \beta < \lambda_{i+1}$, we write $\bar{a}_{i, \alpha, \beta}$ for the sequence $\langle a_{j, \gamma} : j = i, \gamma \in [\alpha, \beta) \rangle$. Note that, in particular, we have $\bar{a}_i$ is $\bar{a}_{<i}$-indiscernible. In this subsection, we will write $\mathbb{K}$ to refer to the class of $\overline{A}$ as in Definition 3.3 with respect to the sequences $\bar{a}_i$ described above.

Additionally, we will assume that T is a theory with $\kappa^1_{\text{shred}}(T) \leq \kappa = \mathrm{cf}(\mu)$ and with the independence property witnessed by the formula $\varphi(x; y)$ along the sequence $\langle a_i : i \in I \rangle$—that is, for all $X \subseteq I$, we have that $\{\varphi(x; a_i)^{(\text{if} i \in X)} : i \in I\}$ is consistent.

We will construct a model containing $\langle a_i : i \in I \rangle$ that is μ-saturated but every finite tuple from this model has the property that there are intervals from our fixed indiscernible sequence $\langle a_i : i \in I \rangle$ which are indiscernible over it. Because we assume T has the independence property, witnessed along this indiscernible sequence, it will follow that $\{\varphi(x; a_i) : i \text{ even}\} \cup \{\neg \varphi(x; a_i) : i \text{ odd}\}$ is an omitted type, which means that the model produced by our construction is not μ^+-saturated. Our proof pursues the same strategy as the construction of an exactly satured model of a simple theory from [7, Theorem 3.3], but with $\kappa_{\text{shred}}(T) < \infty$ replacing the assumption of simplicity.

In order to organize the construction, we will use the following combinatorial principle:

Definition 3.7. Suppose κ is an uncountable cardinal. For a club C, we write $\mathrm{Lim}(C)$ for the set $\{\alpha \in C : \sup(C \cap \alpha) = \alpha\}$. We write $\square_\kappa$ for the following assertion: there is a sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+) \rangle$ such that

(1) $C_\alpha \subseteq \alpha$ is club.
(2) If $\beta \in \mathrm{Lim}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$.
(3) If $\mathrm{cf}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.

We call such a sequence a *square sequence* (for κ).

The following remark was noted in [7, Remark 3.2] —it will play a similar role in our deduction of the main theorem.

Remark 3.8. Suppose $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ is a square sequence and $C'_\alpha = \mathrm{Lim}(C_\alpha)$. Then we have the following:

(1) If $C'_\alpha \neq \emptyset$ and $\sup(C'_\alpha) \neq \alpha$ then C'_α has a last element and $\mathrm{cf}(\alpha) = \omega$. If $C'_\alpha = \emptyset$ then $\mathrm{cf}(\alpha) = \omega$.
(2) For all $\beta \in C'_\alpha$, $C'_\beta = C'_\alpha \cap \beta$.
(3) If $\mathrm{cf}(\alpha) < \kappa$, then $|C'_\alpha| < \kappa$.

The following is the main theorem of the section. The proof follows [7, Theorem 3.3].

Theorem 3.9. *If T has the independence property and $\kappa_{\mathrm{shred}}(T) < \infty$, then T has an exactly μ-saturated model for any singular $\mu > |T|$ of cofinality $\kappa \geq \kappa_{\mathrm{shred}}(T)$ such that $\square_\mu$ and $2^\mu = \mu^+$.*

Proof. Let $\langle C_\alpha : \alpha \in \mathrm{Lim}(\mu^+)\rangle$ be a sequence as in Remark 3.8. Note that, for all $\alpha \in \mathrm{Lim}(\mu^+)$, we have that $|C_\alpha| < \mu$ by condition (3) of Remark 3.8, as $\alpha < \mu^+$ and hence $\mathrm{cf}(\alpha) < \mu$, as μ is singular. Partition μ^+ into $\{S_\alpha : \alpha < \mu^+\}$ so that each S_α has size μ^+. By induction, we will construct a sequence of pairs $\langle(\overline{A}_\alpha, p_\alpha) : \alpha < \mu^+\rangle$ such that

(1) $\overline{A}_\alpha = \langle A_{\alpha,i} : i < \kappa\rangle \in \mathbb{K}$.
(2) $\overline{p}_\alpha = \langle p_{\alpha,\beta} : \beta \in S_\alpha \setminus \alpha\rangle$ is an enumeration of all complete 1-types over subsets of $\bigcup_i A_{\alpha,i}$ of size $< \mu$ (using $|T| < \mu$ and $2^\mu = \mu^+$).
(3) If $\beta < \alpha$, then $\overline{A}_\beta \leq_* \overline{A}_\alpha$.
(4) If $\alpha \in S_\gamma$ and $\gamma < \alpha$, then $\overline{A}_{\alpha+1}$ contains a realization of $p_{\gamma,\alpha}$.
(5) If α is a limit, then for any $i < \kappa$ such that $|C_\alpha| < \lambda_i$ and $\beta \in C_\alpha$, then we have that $\overline{A}_\beta \leq_i \overline{A}_\alpha$.

At stage 0, we define $\overline{A}_0$ to be the minimal sequence in $\mathbb{K}$—that is, $A_{0,i} = \bigcup \overline{a}_{<i}$ for all $i < \kappa$. For the successor case, use Lemma 3.5.

Now we handle the limit cases.

Case 1: $\sup(C_\alpha) = \alpha$. Let $i_0 = \min\{i < \kappa : |C_\alpha| < \lambda_i\}$ which is necessarily a successor ordinal. For $i < i_0$, we define $A_{\alpha,i} = \overline{a}_{<i}$ and for $i \geq i_0$ successor, we let $A_{\alpha,i} = \bigcup_{\beta\in C_\alpha} A_{\beta,i}$. Note that $|A_{\beta,i}| \leq \lambda_i$ for all $i < \kappa$, and for i limit we define $A_{\alpha,i}$ by continuity, setting

$$A_{\alpha,i} = \bigcup_{\substack{j<i \\ j \text{ successor}}} A_{\alpha,j}.$$

Note that it follows, then, that for i limit, we also have $A_{\alpha,i} = \bigcup_{\beta\in C_\alpha} A_{\beta,i}$.

We have to check (1), (3), and (5). First we show that $\overline{A}_\alpha \in \mathbb{K}$. The only thing to check is that $i \geq i_0$ implies $A_{\alpha,i}$ respects $\overline{a}_i$. Now if $w \subseteq A_{\alpha,i}$ is a finite set, for each $e \in w$, there is some $\beta_e \in C_\alpha$ so that $e \in A_{\beta_e,i}$. Let $\beta = \max\{\beta_e : e \in w\}$. Then $C_\alpha \cap \beta = C_\beta$. By (5), the fact that $|C_\beta| < \lambda_{i_0}$, and induction, we have $\beta_e < \beta$ implies $\beta_e \in C_\beta$ and $\overline{A}_{\beta_e} \leq_{i_0} \overline{A}_\beta$ so $A_{\beta_e,i} \subseteq A_{\beta,i}$. It follows that $w \subseteq A_{\beta,i}$. As $\overline{A}_\beta \in \mathbb{K}$, we know $A_{\beta,i}$ respects $\overline{a}_i$, so there is some $\delta < \lambda_{i+1}$ such that $\overline{a}_{i,\geq\delta}$ is w-indiscernible. As $w \subseteq A_{\alpha,i}$ is arbitrary, this shows $A_{\alpha,i}$ respects $\overline{a}_i$ and, therefore, $\overline{A}_\alpha \in \mathbb{K}$. Next, if $\beta < \alpha$, then, because $\sup(C_\alpha) = \alpha$, there is $\beta' \in C_\alpha$ such that $\beta < \beta'$. By induction, $\overline{A}_\beta \leq_* \overline{A}_{\beta'}$ and, by construction, $\overline{A}_{\beta'} \leq_{i_0} \overline{A}_\alpha$, from which it follows that $\overline{A}_\beta \leq_* \overline{A}_\alpha$, which shows (3). Finally (5) is by construction.

Case 2: $\sup(C_\alpha) < \alpha$. We know in this case C_α has a maximum element γ and $\mathrm{cf}(\alpha) = \omega$. Choose an increasing cofinal sequence $\langle\beta_n : n < \omega\rangle$ in α with $\beta_0 = \gamma$. Then, by induction, we may choose an increasing sequence of successor ordinals $\langle i_n : n < \omega \rangle$ so that $\overline{A}_{\beta_n} \leq_{i_n} \overline{A}_{\beta_{n+1}}$. Setting $i_{-1} = 0$ and $i = \sup\{i_n : n < \omega\}$, we define $\overline{A}_\alpha$ as follows: for successor $j \in [i_{n-1}, i_n)$, we put $A_{\alpha,j} = A_{\beta_n,j}$ and for successor $j \geq i$, we put $A_{\alpha,j} = \bigcup_{n<\omega} A_{\beta_n,j}$. For limit ordinals j, $A_{\alpha,j}$ is defined by continuity. It is easy to see that this satisfies (1) and (3), so we check (5).

First, observe that $\overline{A}_\gamma \leq \overline{A}_\alpha$. To see this, it suffices to show by induction on n, that if $j \geq i_{n-1}$, then $A_{\gamma,j} \subseteq A_{\beta_n,j}$. For $n = 0$ this is by definition. Assuming it is true for n, we can consider an arbitrary $j > i_n$. Then by choice of i_n, $\overline{A}_{\beta_n} \leq_{i_n} \overline{A}_{\beta_{n+1}}$ so $A_{\beta_n,j} \subseteq A_{\beta_{n+1},j}$. As the sequence $\langle i_n : n < \omega \rangle$ is increasing, we have also $j > i_{n-1}$ so, by the inductive hypothesis, $A_{\gamma,j} \subseteq A_{\beta_n,j}$ so, by transitivity, $A_{\gamma,j} \subseteq A_{\beta_{n+1},j}$ as desired.

Now suppose $i < \kappa$, $|C_\alpha| < \lambda_i$, and $\beta \in C_\alpha$. Then $\beta \leq \gamma$ and as $\overline{A}_\gamma \leq \overline{A}_\alpha$ we have in particular that $\overline{A}_\gamma \leq_i \overline{A}_\alpha$, so we may assume $\beta < \gamma$. Then $\beta \in C_\alpha \cap \gamma = C_\gamma$ and $|C_\gamma| = |C_\alpha \cap \gamma| < \lambda_i$ so it follows by induction that $\overline{A}_\beta \leq_i \overline{A}_\gamma \leq \overline{A}_\alpha$ so $\overline{A}_\beta \leq_i \overline{A}_\alpha$.

To conclude, we define a model M by

$$M = \bigcup_{\substack{\alpha<\mu^+ \\ i<\kappa}} A_{\alpha,i}.$$

By (4), the model M is μ-saturated. Moreover M is not μ^+-saturated, as the partial type

$$\{\varphi(x; a_{i,\alpha}) : i < \kappa, \alpha \text{ even}\} \cup \{\neg\varphi(x; a_{i,\alpha}) : i < \kappa, \alpha \text{ odd}\}$$

is omitted by (1). □

Question 3.10. *Suppose T is NTP$_2$ and has the independence property, and assume μ is a singular cardinal such that* $\mathrm{cf}(\mu) > |T|$, $2^\mu = \mu^+$, *and* $\square_\mu$. *Does T have an exactly μ-saturated model?*

4. Examples

4.1. Standard examples for the SOP$_n$ hierarchy

Recall the definition of the SOP$_n$ hierarchy:

Definition 4.1. Suppose $n \geq 3$. The theory T has the *nth strong order property* (SOP$_n$) if there is a formula $\varphi(x; y)$ and a sequence of tuples $\langle a_i : i < \omega \rangle$ so that

(1) $\models \varphi(a_i; a_j)$ if and only if $i < j$.
(2) $\{\varphi(x_i, x_{i+1}) : i < n - 1\} \cup \{\varphi(x_{n-1}, x_0)\}$ is inconsistent.

If T does not have SOP$_n$, we say T is NSOP$_n$.

Note that $\mathrm{SOP}_{n+1} \implies \mathrm{SOP}_n$ for all $n \geq 3$ [12, Claim 2.6].

By a directed graph we mean a set with a binary relation that is asymmetric and irreflexive. Given a natural number $n \geq 3$, we let $L_n = \{R_1(x, y)\} \cup \{S_l(x, y) : 1 \leq l < n\}$ be a language with n binary relations. The theory T_n^0 is the L_n-theory of directed graphs with no cycle of length $\leq n$, where $R_1(x, y)$ is the (asymmetric) edge relation and $S_l(x, y)$ means that there is no directed path in the graph R_1 of length $\leq l$ from x to y. More precisely, T_n^0 consists of the following axioms:

- $R_1(x, y)$ is an irreflexive asymmetric relation:

$$(\forall x, y)[R_1(x, y) \to \neg R_1(y, x)].$$

- There are no directed loops of length $\leq n$. That is, for all k with $1 \leq k \leq n$, we have

$$\neg(\exists z_0, \ldots, z_{k-1}) \left[\bigwedge_{i<k-1} R_1(z_i, z_{i+1}) \wedge R_1(z_{k-1}, z_0) \right].$$

- The relation $S_l(x, y)$ implies that there is no directed path of positive length $\leq l$ from x to y:

$$(\forall x, y) \left[S_l(x, y) \to \neg(\exists z_0, \ldots, z_l) \left[z_0 = x \wedge z_l = y \wedge \bigwedge_{i<l} R_1(z_i, z_{i+1}) \vee z_i = z_{i+1} \right]\right].$$

- Paths satisfy the triangle inequality: if $l + l' < n$, then

$$(\forall x, y, z) \left[\neg S_l(x, y) \wedge \neg S_{l'}(y, z) \to \neg S_{l+l'}(x, z)\right],$$

  and, because there are no loops of size $\leq n$, for all $1 \leq l < n' \leq n$

$$(\forall x, y, z) \left[\neg S_l(x, y) \to S_{n'-l}(y, x)\right].$$

This is a universal theory and the model completion of T_n^0 is denoted T_n—it eliminates quantifiers. Note that $R_1(x, y)$ is equivalent to $\neg S_1(x, y)$. We will write $R_l(x, y)$ for $\neg S_l(x, y)$, which indicates there is a directed path of length $\leq l$ from x to y. We will write $\mathbb{M}_n \models T_n$ for the monster model of T_n. The existence of the model completion is proved in [12, Claim 2.8(3)], where it is also shown that T_n is SOP_n and NSOP_{n+1}.

Proposition 4.2. *If $n \geq 4$, then $\kappa_{\mathrm{shred}}(T_n) = \infty$.*

Proof. Let κ be an arbitrary infinite regular cardinal. Define a directed graph G with domain $\{b_{i,\alpha} : i < \kappa, \alpha < \omega\} \cup \{a_{i,j} : i < \kappa, j < 2\}$ and interpret the edge relation R_1 in G by

$$R_1^G = \{(a_{i,0}, b_{i,\alpha}) : i < \kappa, \alpha < \omega \text{ even}\} \cup \{(b_{i,\alpha}, \alpha_{i,1}) : i < \kappa, \alpha < \omega \text{ odd}\},$$

and then interpret S_l^G and hence R_l^G for $1 \leq l < n$ according to the axioms. This clearly defines a model of T_n^0 so there is an L_n-embedding of G into the monster model $\mathbb{M}_n \models T_n$. Therefore, we may identify G with an L_n-substructure of $\mathbb{M}$. Define $A_i = \overline{a}_{\leq i}\overline{b}_{\leq i}$, for all $i < \kappa$.

Let $\varphi(x; y, z) = R_1(x, y) \wedge R_1(z, x)$ and define a partial type p by $p = \{\varphi(x; a_{i,0}, a_{i,1}) : i < \kappa\}$. It is clear from the construction of G that any vertex satisfying this collection of formulas would not create a cycle, hence in particular, it will not create a cycle of length $\leq n$ and, therefore, p is a consistent set of formulas.

Fix $i < \kappa$. By quantifier-elimination, we have $\overline{b}_{i+1} = (b_{i+1,\alpha})_{\alpha<\omega}$ is A_i-indiscernible. Let c realize $\varphi(x; a_{i+1,0}, a_{i+1,1})$. Then we have

(1) $R_2(c, b_{i+1,\alpha})$ for $\alpha < \omega$ even.
(2) $R_2(b_{i+1,\alpha}, c)$ for $\alpha < \omega$ odd.
(3) $\{R_2(x, b_{i+1,\alpha}), R_2(b_{i+1,\alpha}, x)\}$ is inconsistent for all α, because $n \geq 4$.

It follows that no end-segment of $\overline{b}_{i+1}$ can be c-indiscernible, and therefore cannot be $A_i c$-indiscernible. In fact, $\varphi(x; a_{i+1,0}, a_{i+1,1}) \vdash R_2(x; b_{i+1,\alpha}) \leftrightarrow \neg R_2(x; b_{i+1,\alpha+1})$ for all even $\alpha < \omega$, which shows that $\varphi(x; a_{i+1,0}, a_{i+1,1})$ explicitly shreds over A_i. It follows that $\kappa_{\mathrm{shred}}(T) > \kappa$ and, as κ was arbitrary, we have $\kappa_{\mathrm{shred}}(T) = \infty$. □

Now we analyze T_3:

Lemma 4.3. *In T_3, if $\overline{b} = \langle b_i : i < \lambda \rangle$ is indiscernible over A, then for any tuple a, if c is a tuple disjoint from Aa, then there is $c' \equiv_{Aa} c$ so that $\overline{b}$ is Ac'-indiscernible.*

Proof. Note that T_3 eliminates quantifiers in the language containing only R_1, since $R_2(x, y)$ is definable by the formula $x \neq y \wedge \neg R_1(y, x)$. For simplicity, we will write R for R_1. Because algebraic closure in T_3 is trivial, by replacing c by something with the same type over Aa, we may assume c is disjoint from $Aa\overline{b}$. Define a model $M \models T_3^0$ as follows with underlying set $Aa\overline{b}c$ by defining

$$R^M = R^{\mathbb{M}_3} \restriction Aa\overline{b} \cup R^{\mathbb{M}_3} \restriction Aac.$$

We claim that $M \models T_3^0$. To see this, suppose not and there are distinct $d_0, d_1, d_2 \in M$ so that $R^M(d_0, d_1)$, $R^M(d_1, d_2)$, and $R^M(d_2, d_0)$. Since $\mathbb{M}_3$ has no directed cycles of length 3, it is impossible for d_0, d_1, d_2 to be all contained in $Aa\overline{b}$ or all contained in Aac. Therefore, without loss of generality, $d_0 \in Aa\overline{b} \setminus Aac$. But then since $R^M(d_2, d_0)$ and $R^M(d_0, d_1)$, we have $d_1, d_2 \in Aa\overline{b}$, by the definition of R^M, a contradiction. This shows M has no directed cycle of length 3 so $M \models T_3^0$.

Embed M into $\mathbb{M}_3$ over $Aa\overline{b}$ and let c' be the image of c. By quantifier elimination, we have $c' \equiv_{Aa} c$ and, because c' is disjoint from $Aa\overline{b}$, we have $\overline{b}$ is Ac'-indiscernible. □

Proposition 4.4. $\kappa_{\text{shred}}(T_3) = \aleph_0$.

Proof. By Theorem 3.6, it suffices to show $\kappa^1_{\text{shred}}(T_3) \leq \aleph_0$, and, in fact, we will show there is no shredding chain in a single free variable of length 2. Towards contradiction, suppose A is a set of parameters, $\varphi_0(x; a_0)$ shreds over A witnessed by $\overline{b}_0$, $\varphi_1(x; a_1)$ shreds over Aa_0 witnessed by $\overline{b}_1$, and $\{\varphi_0(x; a_0), \varphi_1(x; a_1)\}$ is consistent, with x a single free variable. Because $\varphi_0(x; a_0)$ has no realization c such that $\overline{b}_0$ is indiscernible over Ac, it follows by Lemma 4.3 that any realization of $\varphi_0(x; a_0)$ is contained in Aa_0. Then let $c \models \{\varphi_0(x; a_0), \varphi_1(x; a_1)\}$. Because c is an element of Aa_0, it follows that $\overline{b}_1$ is Aa_0c-indiscernible, contradicting the fact that $\overline{b}_1$ witnesses that $\varphi_1(x; a_1)$ shreds over Aa_0. This completes the proof. □

4.2. NSOP$_1$ and unshreddability

There is a theory of independence for NSOP$_1$ theories that indicates this class of theories may be considered quite close to the class of simple theories (see, e.g., [6]). In the next two examples, however, we show that unshreddability is independent of NSOP$_1$ and, in particular, that within the class of NSOP$_1$ theories, it is still possible that $\kappa_{\text{shred}}(T) = \infty$. Recall the definition of SOP$_1$:

Definition 4.5. A formula $\varphi(x; y)$ is said to have SOP$_1$ if there is a tree of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ satisfying the following:

(1) For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent.
(2) For all $\eta \perp \nu$ in $2^{<\omega}$, if $(\eta \wedge \nu) \frown 0 \trianglelefteq \eta$ and $(\eta \wedge \nu) \frown 1 = \nu$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.

A theory T is said to have SOP$_1$ if some $\varphi(x; y)$ has SOP$_1$ modulo T, otherwise T is NSOP$_1$.

First, we describe an NSOP$_1$ example of a theory T_1^* with $\kappa_{\text{shred}}(T_1^*) = \aleph_0$. This theory was studied in detail in [6, Subsection 9.2]. The language L_1 consists of unary predicates F and O, a binary relation E, and a binary function eval. The theory T_1 consists of the following axioms:

(1) F and O partition the universe.
(2) $E \subseteq O^2$ is an equivalence relation.

(3) eval : $F \times O \to O$ is a selector function:

(a) $(\forall x \in F)(\forall y \in O)\,[E(y, \mathrm{eval}(x, y))]$.
(b) $(\forall x \in F)(\forall y, z \in O)\,[E(y, z) \to \mathrm{eval}(x, y) = \mathrm{eval}(x, z)]$.

It was shown in [6, Subsection 9.2] that T_1 has a model-completion T_1^*, which is the theory of the Fraïssé limit of finite models of T_1, which is $\aleph_0$-categorical with elimination of quantifiers. It is additionally shown that algebraic closure and definable closure of a set coincide with the structure generated by the set, and that the theory is non-simple NSOP$_1$.

Example 4.6. Suppose $c \in O$ and $\langle b_i : i < \lambda\rangle$ is an indiscernible sequence such that $b_i \in F$ for all $i < \lambda$ and $\mathrm{eval}(b_i, c) = c$ if i is even and the $\mathrm{eval}(b_i, c)$ pairwise distinct and different from c for i odd. Then the formula $E(x; c)$ implies $\mathrm{eval}(b_i, x) = \mathrm{eval}(b_i, c)$ for all i, thus $E(x, c)$ shreds over $\emptyset$, since for any even $\alpha < \lambda$ and d with $\models E(d, c)$, $\mathrm{eval}(b_\alpha, d) = \mathrm{eval}(b_{\alpha+2}, d)$ and $\mathrm{eval}(b_\alpha, d) \neq \mathrm{eval}(b_{\alpha+1}, d)$.

We show that, in a sense made precise by the following lemma, all instances of shredding in the theory T_1^* resemble the previous example.

Lemma 4.7. *Suppose $\varphi(x; a)$ is a non-algebraic formula with $l(x) = 1$ that shreds over A. Then there is some $c \in Aa$, E-equivalent to no element of A, such that $\varphi(x; a) \vdash E(x, c)$. In particular $\varphi(x; a) \vdash x \in O$.*

Proof. Suppose $\varphi(x; a)$ is a non-algebraic formula with $l(x) = 1$ and $\models \varphi(f; a)$ with $\neg E(f, c)$ for every $c \in Aa$ in an E-equivalence class disjoint from A. We must show $\varphi(x; a)$ does not shred over A. As a formula shreds over A only if it shreds over some finite subset of A (i.e. the parameters appearing in the formulas witnessing that it explicitly shreds over A), by Lemma 2.8, we may assume A is finite and that a enumerates the structure generated by A and a.

Fix an A-indiscernible sequence $\overline{b} = (b_i)_{i<\lambda}$ for $\lambda = (|T| + |A|)^+$ where $b_i = (b_{i,0}, \dots, b_{i,n-1})$ for all $i < \lambda$. Additionally, as a is a finite tuple, we can find some ordinal α such that, for each $j < n$, either there is an equivalence class represented by an element of a such that $b_{i,j}$ is in this equivalence class for all $i \geq \alpha$, or $b_{i,j}$ is not equivalent to any element of a for all $i \geq \alpha$; and additionally, either there is an element of a such that $b_{i,j}$ is equal to this element of a $i \geq \alpha$, or $b_{i,j}$ is not equivalent to any element of a for all $j \geq \alpha$ (in other words, we may find some α such that $\overline{b}_{\geq\alpha}$ is Aa-indiscernible in the stable reduct (F, O, E), where we forget the function eval).

Let $B = \langle a\overline{b}_{\geq\alpha}\rangle$ and $C = \langle af\rangle$ be the structures generated by $a\overline{b}_{\geq\alpha}$ and af in $\mathbb{M}$, respectively. By the assumption that $\varphi(x; a)$ is not algebraic, we may assume $f \notin B$. By our assumption that $\neg E(f, c)$ for every $c \in Aa$ in an E-equivalence class disjoint from A, we have the following three cases:

Case 1: $f \in O$ and f is not E-equivalent in $\mathbb{M}$ to any element of $\langle a\rangle$.

In this case, we define a structure D whose underlying set is $B \cup C = B \cup [f]_E^C$, where $[f]_E^C$ denotes the E-class of f in C. We interpret $F^D = F^B$ and $O^D = O^B \cup [f]_E^C$, then we interpret E^D to extend E^B with $[f]_E^C$ forming a new equivalence class (thus also extending E^C). Then we define eval^D to extend eval^B and eval^C (which agree on their common domain) and set $\mathrm{eval}(g, f) = f$ for all $g \in F^B \setminus F^C$.

Case 2: $f \in O$ and f is E-equivalent to some $c \in O^{\langle A\rangle}$.

In this case, we define a structure D whose underlying set is $B \cup C = B \cup \{f\}$. We interpret $F^D = F^B$ and $O^D = O^B \cup \{f\}$, then we interpret E^D to extend E^B and E^C with $[c]_E^D = [c]_E^B \cup \{f\}$. Then we define eval^D to extend eval^B by setting $\mathrm{eval}^D(g, f) = \mathrm{eval}^B(g, c)$ for all $g \in F^B$. Note that this extends eval^C as well.

Case 3: $f \in F$.

Sh:1192

In this case, as before, we define a structure D whose underlying set is $B \cup C \cup \{*_x : x \in (O^B/E) \setminus (O^C/E)\}$ where each $*_x$ is a new formal element indexed by an equivalence class of B which is not represented by any element of $\langle a \rangle$. We interpret $F^D = F^B \cup F^C = F^B \cup \{f\}$ and $O^D = O^B \cup O^C \cup \{*_x : x \in (O^B/E) \setminus (O^C/E)\}$, then we interpret E^D to be the equivalence relation generated by $E^B \cup E^C$ (which extends both E^B and E^C) and the condition that $*_x$ is in the equivalence class x for each $x \in (O^B/E) \setminus (O^C/E)$. Then to define eval^D, extending eval^B and eval^C, we must define $\mathrm{eval}^D(f, c)$ for all $c \in F^B$ which are not E^D-equivalent an element of C. For any such c, we define $\mathrm{eval}^D(f, c) = *_{[c]_E}$.

In each case, D extends B and C and one can check and A-indiscernibility of $\bar{b}$ that, in D, $\bar{b}_{\geq \alpha}$ is quantifier-free indiscernible over C. We may embed D into $\mathbb{M}$ over B, and then the image f' of f along this embedding satisfies $\models \varphi(f'; a)$ and $\bar{b}_{\geq \alpha}$ is Af' indiscernible. This shows that $\varphi(x; a)$ does not shred over A. □

Proposition 4.8. *The theory T_1^* is a non-simple NSOP$_1$ theory with $\kappa_{\mathrm{shred}}(T_1^*) = \aleph_0$.*

Proof. By Theorem 3.6, it suffices to show $\kappa^1_{\mathrm{shred}}(T_1^*) = \aleph_0$. Note that if $\varphi_0(x; a_0)$ shreds over A and $\varphi_1(x; a_1)$ shreds over Aa_0 with $l(x) = 1$ and both φ_0 and φ_1 are non-algebraic, then by Lemma 4.7, we must have both that $\varphi_0(x; a_0)$ implies that x is in an equivalence class represented by an element of a_0 and $\varphi_1(x; a_1)$ implies x is in an equivalence class of an element of a_1 not represented by an element of Aa_0. This implies $\{\varphi_0(x; a_0), \varphi_1(x; a_1)\}$ is inconsistent.

Now suppose $\varphi_i(x; a_i)$ are formulas with $l(x) = 1$ for $i = 0, 1, 2$, such that $\varphi_i(x; a_i)$ shreds over $Aa_{<i}$ for $i = 0, 1, 2$ and $\{\varphi_i(x; a_i) : i < 2\}$ is consistent. Then, by the first paragraph, one of $\varphi_0(x; a_0)$ and $\varphi_1(x; a_1)$ must be algebraic. Hence if $f \models \{\varphi_i(x; a_i) : i < 2\}$, then $f \in \mathrm{acl}(Aa_0a_1)$. But any Aa_0a_1-indiscernible sequence is automatically $\mathrm{acl}(Aa_0a_1)$-indiscernible and therefore Aa_0a_1f-indiscernible. It follows that $\varphi_2(x; a_2)$ cannot shred over Aa_0a_1, a contradiction. Therefore $\kappa^1_{\mathrm{shred}}(T) = \aleph_0$. □

The following theory is a variation on the generic theory of selector functions T_1^* considered above. The language L for our example consists of unary predicates F, O_0, O_1, and O, binary relations E, R_0, and R_1, and a binary function eval. The theory T consists of the following axioms:

(1) F, O_0, and O_1 partition the universe and $O = O_0 \cup O_1$.
(2) $E \subseteq O^2$ is an equivalence relation.
(3) $\mathrm{eval} : F \times O \to O_0$ is a selector function:

   (a) $(\forall x \in F)(\forall y \in O)\,[E(y, \mathrm{eval}(x, y))]$.
   (b) $(\forall x \in F)(\forall y, z \in O)\,[E(y, z) \to \mathrm{eval}(x, y) = \mathrm{eval}(x, z)]$.

(4) The relations R_0, R_1 satisfy:

   (a) $R_0 \subseteq O_0 \times O_1$.
   (b) $R_1 \subseteq F \times O_1$.
   (c) $(\forall x \in F)(\forall z \in O_1)\,[R_0(\mathrm{eval}(x, z), z) \leftrightarrow R_1(x, z)]$.

Define $\mathbb{K}$ to be the class of finite models of T.

Lemma 4.9. *The class $\mathbb{K}$ is a Fraïssé class. Moreover, it is uniformly locally finite.*

Proof. HP is clear as the axioms of T are universal. The argument for JEP is identical to that for SAP, so we show SAP. Suppose $A, B, C \in \mathbb{K}$ where $A \subseteq B, C$ and $B \cap C = A$. It suffices to define a L-structure

Sh:1192

with domain $D = B \cup C$, extending both B and C. First, note that if F^B is non-empty, then every E^B-class intersects O_0^B, but if $F^B = \emptyset$, it is possible that there are E^B-equivalence classes disjoint from O_0^B. In this latter case, we can extend B to B' so that each equivalence class contains an element of O_0: Let $(K_i)_{i<l}$ list the E^B-classes K of B such that $O_0^B \cap K = \emptyset$. Let B' be the L-structure with underlying set $B \cup \{*_i : i < l\}$ where the $*_i$ are new formal elements. Consider B' as an L-structure via the following interpretations: for the unary predicates, interpret $F^{B'} = F^B = \emptyset$, $O_0^{B'} = O_0^B \cup \{*_i : i < l\}$, $O_1^{B'} = O_1^B$, and $O^{B'} = O_0^{B'} \cup O_1^{B'}$. Let $R_0^{B'} = R_0^B$, $R_1^{B'} = R_1^B$, and let $E^{B'}$ be the equivalence relation generated by $E^B \cup \{(b, *_i) : i < l, b \in K_i\}$. As $F^B = F^{B'} = \emptyset$, we can only define $\mathrm{eval}^{B'} : F^{B'} \times O^{B'} \to O_0^{B'}$ to be the empty function. It is clear that B' is in $\mathbb{K}$, extends B, and every equivalence class not represented by an element of A contains an element of O_0. By a symmetric argument, we may also extend C to C' so that every E^C-class not represented by an element of A contains an element of $O_0^{C'}$. Replacing B and C by B' and C' respectively, we may assume that all classes of B and C are either represented by an element of A or by an element of O_0^B or O_0^C respectively.

Now we describe the construction of D. Interpret O_0^D, O_1^D, and F^D by $O_i^D = O_i^B \cup O_i^C$ for $i = 0, 1$, $O^D = O_0^D \cup O_1^D$, and $F^D = F^B \cup F^C$. Let E^D be the equivalence relation generated by $E^B \cup E^C$. It follows that if $b \in B$, $c \in C$ and $(b, c) \in E^D$, then there is some $a \in A$ so that $(a, b) \in E^B$ and $(a, c) \in E^C$ and, moreover, (O^D, E^D) extends both (O^B, E^B) and (O^C, E^C) as equivalence relations. Put $R_0^D = R_0^B \cup R_0^C$.

Next we define the interpretation eval^D. Let $\{a_i : i < k_0\}$ enumerate a collection of representatives for the E^A-classes in A. Then let $\{b_i : i < k_1\}$ and $\{c_i : i < k_2\}$ enumerate representatives for the E^B- and E^C-classes of elements not represented by an element of A, respectively. By the remarks above, we may assume each b_i and c_i are in O_0^D. Then every element of O^D is equivalent to a unique element of

$$X = \{a_i : i < k_0\} \cup \{b_i : i < k_1\} \cup \{c_i : i < k_2\}.$$

Suppose $d \in X$. If $f \in F^A$, define $\mathrm{eval}^D(f, d) = \mathrm{eval}^B(f, d)$ if $d \in B$ and $\mathrm{eval}^D(f, d) = \mathrm{eval}^C(f, d)$ if $d \in C$, which is well-defined as A is a substructure of both B and C. If $f \in F^B \setminus F^A$, define $\mathrm{eval}^D(f, d) = \mathrm{eval}^B(f, d)$ if $d \in B$ and $\mathrm{eval}^D(f, d) = d$ otherwise. Likewise, if $f \in F^C \setminus F^A$, put $\mathrm{eval}^D(f, d) = \mathrm{eval}^C(f, d)$ if $d \in C$ and $\mathrm{eval}^C(f, c) = c$ otherwise. This defines eval on $F^D \times X$. More generally, if $f \in F^D$ and $e \in O^D$, define $\mathrm{eval}^D(f, e) = \mathrm{eval}^D(f, d)$ for the unique $d \in X$ equivalent to e.

To complete the construction, we must describe the interpretation of R_1^D. Put

$$R_1^D = R_1^B \cup R_1^C \cup \{(f, d) \in F^D \times O_1^D : (\mathrm{eval}^D(f, d), d) \in R_0^D\}.$$

We check that this defines an extension of B and C. If $b \in F^B$, $b' \in O_1^B$, and $(\mathrm{eval}^D(b, b'), b') \in R_0^D$, then $(\mathrm{eval}^D(b, b'), b') \in R_0^B$ and $\mathrm{eval}^D(b, b') = \mathrm{eval}^B(b, b')$ so $(\mathrm{eval}^B(b, b'), b') \in R_0^B$ and therefore $R_1^B(b, b')$. This shows $R_1^D \restriction B = R_1^B$. Likewise $R_1^D \restriction C = R_1^C$. Therefore D extends B and C.

Now to conclude we must show $D \in \mathbb{K}$. It is clear that D satisfies axioms (1)-(3), so we are left with checking (4). Suppose $(f, d) \in F^D \times O_1^D \setminus (F^B \times O_1^B \cup F^C \times O_1^C)$ and $d' = \mathrm{eval}^D(f, d)$. Then, by definition, if $(d, d') \in R_0^D$, then $(f, d) \in R_1^D$. On the other hand, if $(f, d) \in R_1^D$ then, because $(f, d) \notin R_1^B \cup R_1^C$, we must have $(d, d') \in R_0^D$, again by the definition of R_1^D. It is clear that if $(f, d) \in F^B \times O_1^B \cup F^C \times O_1^C$ then $(f, d) \in R_0^D$ if and only if $(f, d) \in R_1^D$ because D extends B and C which are in $\mathbb{K}$. Therefore D satisfies axiom (4) which shows $D \in \mathbb{K}$. This shows $\mathbb{K}$ has the amalgamation property.

Finally, note that a structure in $\mathbb{K}$ generated by k elements is obtained by applying $\leq k$ functions of the form $\mathrm{eval}(f, -)$ to $\leq k$ elements in O, so has cardinality $\leq k^2 + k$. This shows $\mathbb{K}$ is uniformly locally finite. $\square$

Corollary 4.10. *T has a model completion T^* which is the theory of the Fraïssé limit of $\mathbb{K}$. The theory T^* eliminates quantifiers and is $\aleph_0$-categorical.*

We will write $\mathbb{M} \models T^*$ for a monster model of T^*. We will now show that T^* is NSOP$_1$ by appealing to the following criterion:

Fact 4.11. [3, Proposition 5.8] Assume there is an Aut($\mathbb{M}$)-invariant ternary relation $\mathop{\smile\!\!\!\!\!|}$ on small subsets of $\mathbb{M}$ satisfying the following properties, for an arbitrary $M \prec \mathbb{M}$ and arbitrary tuples from $\mathbb{M}$:

(1) Strong finite character: if $a \not\mathop{\smile\!\!\!\!\!|}_M b$, then there is a formula $\varphi(x, b, m) \in \mathrm{tp}(a/Mb)$ such that for any $a' \models \varphi(x, b, m)$, $a' \not\mathop{\smile\!\!\!\!\!|}_M b$.
(2) Existence over models: $a \mathop{\smile\!\!\!\!\!|}_M M$.
(3) Monotonicity: if $aa' \mathop{\smile\!\!\!\!\!|}_M bb'$, then $a \mathop{\smile\!\!\!\!\!|}_M b$.
(4) Symmetry: if $a \mathop{\smile\!\!\!\!\!|}_M b$, then $b \mathop{\smile\!\!\!\!\!|}_M a$.
(5) The independence theorem: if $a \mathop{\smile\!\!\!\!\!|}_M b$, $a' \mathop{\smile\!\!\!\!\!|}_M c$, $b \mathop{\smile\!\!\!\!\!|}_M c$ and $a \equiv_M a'$, then there exists a'' with $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$, and $a'' \mathop{\smile\!\!\!\!\!|}_M bc$.

Then T is NSOP$_1$.

Definition 4.12. Define a ternary relation $\mathop{\smile\!\!\!\!\!|}^*$ on small subsets of $\mathbb{M}$ by: $a \mathop{\smile\!\!\!\!\!|}^*_C b$ if and only if

(1) $\mathrm{dcl}(aC)/E \cap \mathrm{dcl}(bC)/E \subseteq \mathrm{dcl}(C)/E$,
(2) $\mathrm{dcl}(aC) \cap \mathrm{dcl}(bC) \subseteq \mathrm{dcl}(C)$,

where $X/E = \{[x]_E : x \in X\}$ denotes the collection of E-classes represented by an element of X.

Lemma 4.13. *The relation $\mathop{\smile\!\!\!\!\!|}^*$ satisfies the independence theorem over models: if $M \models T^*$, $a \equiv_M a'$, and, additionally, $a \mathop{\smile\!\!\!\!\!|}^*_M B$, $a' \mathop{\smile\!\!\!\!\!|}^*_M C$ and $B \mathop{\smile\!\!\!\!\!|}^*_M C$ then there is a'' with $a'' \equiv_{MB} a$, $a'' \equiv_{MC} a'$, and $a'' \mathop{\smile\!\!\!\!\!|}^*_M BC$.*

Proof. Without loss of generality, we may assume that $M \subseteq B, C$, and that B and C are definably closed. Write $a = (d_0, \dots, d_{k-1}, e_0, \dots, e_{l-1}, f_0, \dots, f_{m-1})$ with $d_i \in F$, $e_j \in O_0$, $f_k \in O_1$, and likewise $a' = (d'_0, \dots, d'_{k-1}, e'_0, \dots, e'_{l-1}, f'_0, \dots, f'_{m-1})$. Fix an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$ with $\sigma(a) = a'$. Let $U = \{u_g : g \in \mathrm{dcl}(aB) \setminus B\}$ and $V = \{v_g : g \in \mathrm{dcl}(a'C) \setminus C\}$ denote collection of new formal elements with $u_g = v_{\sigma(g)}$ for all $g \in \langle aM\rangle \setminus B$. Let, then, a_* be defined as follows:

$$\begin{aligned} a_* &= (u_{d_0}, \dots, u_{d_{k-1}}, u_{e_0}, \dots, u_{e_{l-1}}, u_{f_0}, \dots, u_{f_{m-1}}) \\ &= (v_{d'_0}, \dots, v_{d'_{k-1}}, v_{e'_0}, \dots, v_{e'_{l-1}}, v_{f'_0}, \dots, v_{f'_{m-1}}). \end{aligned}$$

We will construct by hand an L-structure D extending $\langle BC\rangle$ with domain $UV\langle BC\rangle$ in which $a_* \equiv_B a$, $a_* \equiv_C a'$ and $a_* \mathop{\smile\!\!\!\!\!|}^*_M BC$.

There is a bijection $\iota_0 : \mathrm{dcl}(aB) \to BU$ given by $\iota_0(b) = b$ for all $b \in B$ and $\iota_0(g) = u_g$ for all $g \in \mathrm{dcl}(aB) \setminus B$. Likewise, we have a bijection $\iota_1 : \mathrm{dcl}(a'C) \to CV$ given by $\iota_1(c) = c$ for all $c \in C$ and $\iota_1(g) = v_g$ for all $g \in \mathrm{dcl}(a'C) \setminus C$. The union of the images of these functions is the domain of the structure D to be constructed and their intersection is $\iota_0(\langle aM\rangle) = \iota_1(\langle a'M\rangle)$. Consider BU and CV as L-structures by pushing forward the structure on $\mathrm{dcl}(aB)$ and $\mathrm{dcl}(a'C)$ along ι_0 and ι_1, respectively. Note that $\iota_0|_{\langle aM\rangle} = (\iota_1 \circ \sigma)|_{\langle aM\rangle}$.

We are left to show that we can define an L-structure on $UV\langle BC\rangle$ extending that of BU, CV, and $\langle BC\rangle$ in such a way as to obtain a model of T. To begin, interpret the predicates by $O_i^D = O_i^{BU} \cup O_i^{CV} \cup O_i^{\langle BC\rangle}$ for $i = 0, 1$, $O^D = O_0^D \cup O_1^D$, $F^D = F^{BU} \cup F^{CV} \cup F^{\langle BC\rangle}$, and $R_0^D = R_0^{BU} \cup R_0^{CV} \cup R_0^{\langle BC\rangle}$. Let E^D be defined to be the equivalence relation generated by E^{BU}, E^{CV}, and $E^{\langle BC\rangle}$. The interpretation of the predicates defines extensions of the given structures since if g is an element of $\iota_0(\langle aM\rangle) = \iota_1(\langle a'M\rangle)$ then $\iota_0^{-1}(g)$ is in the predicate O if and only if $\iota_1^{-1}(g)$ is as well, and, moreover, it is easy to check that our assumptions on a, a', B, C entail that no pair of inequivalent elements in BU, CV, or $\langle BC\rangle$ become equivalent in D.

Next we define the function eval^D extending $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV} \cup \mathrm{eval}^{\langle BC\rangle}$. We first claim that $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV} \cup \mathrm{eval}^{\langle BC\rangle}$ is a function. The intersection of the domains of the first two functions is $\iota_0(\langle aM\rangle) = \iota_1(\langle aM\rangle)$. If b, b' are in this intersection, we must show

$$\mathrm{eval}^{BU}(b, b') = c \iff \mathrm{eval}^{CV}(b, b') = c.$$

Choose $b_0, b_0', c_0 \in \langle aM\rangle$ and $b_1, b_1', c_1 \in \langle a'M\rangle$ with $\iota_i(b_i, b_i', c_i) = (b, b', c)$ for $i = 0, 1$. Then since $\iota_0 = \iota_1 \circ \sigma$ on $\langle aM\rangle$, we have

$$\begin{aligned} \mathbb{M} \models \mathrm{eval}(b_0, b_0') = c_0 &\iff \mathbb{M} \models \mathrm{eval}(\sigma(b_0), \sigma(b_0')) = \sigma(c_0) \\ &\iff \mathbb{M} \models \mathrm{eval}(b_1, b_1') = c_1. \end{aligned}$$

Since eval^{BU} and eval^{CV} are defined by pushing forward the structure on $\langle aB\rangle$ and $\langle a'C\rangle$ along ι_0 and ι_1, respectively, this shows that $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV}$ defines a function. Now the intersection of $\langle BC\rangle$ with $BU \cup CV$ is BC and, by construction, all 3 functions agree on this set. So the union defines a function.

Note that because BU, CV, and $\langle BC\rangle$ all contain a model M and therefore have non-empty F-sort, every E^D class is represented by an element of O_0^D. Choose a complete set of E^D-class representatives $\{d_i : i < \alpha\}$ so that if d_i represents an E^D-class that meets M then $d_i \in M$ and $d_i \in O_0$. If $e \in O^D$ is E^D-equivalent to some e' and (f, e') is in the domain of $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV} \cup \mathrm{eval}^{\langle BC\rangle}$, define $\mathrm{eval}^D(f, e)$ to be the value that this function takes on (f, e'). On the other hand, if $f \in F^D \setminus (F^{BU} \cup F^{CV} \cup F^{\langle BC\rangle})$ or e is not E^D-equivalent to any element on which $\mathrm{eval}^D(f, -)$ has already been defined, put $\mathrm{eval}^D(f, e) = d_i$ for the unique d_i which is E^D-equivalent to e. This now defines eval^D on all of $F^D \times O^D$ and, by construction, $\mathrm{eval}^D(f, -)$ is a selector function for E^D for all $f \in F^D$.

To conclude, we must interpret R_1 on D. In order to build a structure that satisfies axiom (4), we are forced to interpret

$$R_1^D = \{(f, b) \in F \times O_1 : (\mathrm{eval}^D(f, b), b) \in R_0^D\}.$$

In order to ensure that D is an extension of BU, CV, and $\langle BC\rangle$, we have show that for all $X \in \{BU, CV, \langle BC\rangle\}$, $R_1^D \restriction X = R_1^X$. Suppose we have $f, a, b \in X$ with $\mathrm{eval}^X(f, b) = a$. Then because X is a model of T, we have $R_0^X(a, b) \iff R_1^X(f, b)$ and, by construction, $R_0^X(a, b) \iff R_0^D(a, b)$. By definition, $R_1^D(f, b) \iff R_0^D(a, b)$. This shows $R_1^D(f, b) \iff R_1^X(f, b)$, hence $R_1^D \restriction X = R_1^X$.

We have already argued that BU and CV are substructures of D - it follows that every E^D-class represented by an element of a_* can only be equivalent to an element of B or C if it is equivalent to an element of M. Moreover, our construction has guaranteed that $\langle a_*M\rangle^D \cap \langle BC\rangle \subseteq BU \cap \langle BC\rangle^D \subseteq B$ and, by similar reasoning, $\langle a_*M\rangle \cap \langle BC\rangle \subseteq C$. This implies $\langle a_*M\rangle^D \cap \langle BC\rangle B \cap C \subseteq M$, so $a_* \mathop{\smile\!\!\!\!\!\!|}\,^*_M BC$. Embedding D into $\mathbb{M}$ over $\langle BC\rangle$, we conclude. □

Corollary 4.14. *The theory T^* is NSOP$_1$.*

Proof. The relation $\mathop{\smile\!\!\!\!\!\!|}\,^*$ is easily seen to satisfy properties (1) through (4) from Fact 4.11 and the independence theorem is established in Lemma 4.13. This implies T^* is NSOP$_1$. □

Remark 4.15. One may additionally show that $\mathop{\smile\!\!\!\!\!\!|}\,^* = \mathop{\smile\!\!\!\!\!\!|}\,^K$ over models. As we won't need Kim-independence in what follows, we omit the proof.

Proposition 4.16. $\kappa_{\mathrm{shred}}(T^*) = \infty$.

Proof. Let κ be an arbitrary regular cardinal. Inductively, we may choose a sequence of elements $\langle a_i : i < \kappa\rangle$ and a sequence of sequences $\langle \overline{b}_i : i < \kappa\rangle$ so that

(1) For all $i < \kappa$, $a_i \in O_0$.
(2) For all $i < \kappa$, $\overline{b}_i = \langle b_{i,j} : j < \omega\rangle$ is an $a_{<i}\overline{b}_{<i}$-indiscernible sequence of elements of O_1 in the same E-class as a_i, with $R_0(a_i, b_{i,j})$ if and only if j is even.

Let $p(x) = \{\mathrm{eval}(x; a_i) = a_i : i < \kappa\}$ and fix some $i < \kappa$. Because each $b_{i,j}$ is E-equivalent to a_i and $\mathrm{eval}(x, -)$ is a selector function, $\mathrm{eval}(x, a_i) = a_i$ implies $\mathrm{eval}(x; b_{i,j}) = a_i$. It follows from axiom 4(c) of T that $\mathrm{eval}(x, a_i) = a_i$ implies $R_0(a_i, b_{i,j}) \leftrightarrow R_1(x, b_{i,j})$ for all j. Therefore, $\mathrm{eval}(x, a_i) = a_i \vdash R_1(x; b_{i,j})$ if j is even and $\mathrm{eval}(x, a_i) = a_i \vdash \neg R_1(x; b_{i,j})$ if j is odd. This shows $\mathrm{eval}(x; a_i) = a_i \in p \restriction a_{<i+1}$ explicitly shreds over $a_{<i}$. Since κ is arbitrary, we conclude $\kappa_{\mathrm{shred}}(T^*) = \infty$. □

4.3. An NTP$_2$ example

In this subsection, we describe an NTP$_2$ example with $\kappa_{\mathrm{shred}}(T) = \infty$. Recall the definition of NTP$_2$ theories:

Definition 4.17. A formula $\varphi(x; y)$ has *the tree property of the second kind* (TP$_2$) if there is an array of tuples $(a_{i,j})_{i,j<\omega}$ and $k < \omega$ satisfying the following:

(1) For all $f : \omega \to \omega$, $\{\varphi(x; a_{i,f(i)}) : i < \omega\}$ is consistent.
(2) For all $i < \omega$, $\{\varphi(x; a_{i,j}) : j < \omega\}$ is k-inconsistent.

A theory is said to have TP$_2$ if some formula has TP$_2$ modulo T and is otherwise called NTP$_2$.

The class of NTP$_2$ contains both the NIP and simple theories, so it is natural to ask if NTP$_2$ implies $\kappa_{\mathrm{shred}}(T) < \infty$ but we show this is not the case.

The following fact will be useful in checking that the theory we construct is NTP$_2$:

Fact 4.18.

1 If T has TP$_2$, there is a formula $\varphi(x; y)$ witnessing this with $l(x) = 1$ [2, Corollary 2.9].
2 If $\varphi(x; y)$ has TP$_2$, then this will be witnessed with respect to an array of parameters $(a_{i,j})_{i,j<\omega}$ that is *mutually indiscernible*—that is, $\overline{a}_i$ is $\overline{a}_{\neq i}$-indiscernible for all $i < \omega$ [2, Lemma 2.2].

Let L a language consisting of two binary relations $R, \trianglelefteq$, and a binary function $\wedge$ and the sublanguage consisting of just $\trianglelefteq$ and $\wedge$ is L_{tr}. The class $\mathbb{K}$ will consist of finite L-structures $(A, \trianglelefteq^A, \wedge^A, R^A)$ so that $(A, \trianglelefteq^A, \wedge^A)$ is a meet-tree where $\wedge^A$ is the meet function, and R^A is a graph on A. Denote the class of finite $\wedge$-trees $(A, \trianglelefteq^A, \wedge^A)$ by $\mathbb{K}_0$. This is a Fraïssé class with the strong amalgamation property (SAP) and the theory T_{tr} of its Fraïssé limit is dp-minimal [13, Exercise 2.50, Example 4.28], which means given a mutually indiscernible array $(a_{i,j})_{i<2,j<\omega}$ and element c, there is some $i < 2$ such that $\overline{a}_i$ is c-indiscernible.

Lemma 4.19. *The class $\mathbb{K}$ is a Fraïssé class. Moreover, the reduct of the Fraïssé limit of $\mathbb{K}$ to L_{tr} is the Fraïssé limit of $\mathbb{K}_0$.*

Proof. HP is clear and JEP will follow from a similar argument to SAP, so we will prove SAP. Fix $\tilde{A}, \tilde{B}_0, \tilde{B}_1 \in \mathbb{K}$ such that $\tilde{A}$ is an L-substructure of both $\tilde{B}_0$ and $\tilde{B}_1$ and $\tilde{B}_0 \cap \tilde{B}_1 = \tilde{A}$. Let $A = \tilde{A} \restriction L_{\mathrm{tr}}$ and $B_i = \tilde{B}_i \restriction L_{\mathrm{tr}}$

for $i = 0, 1$. By SAP in $\mathbb{K}_0$, there is $D \in \mathbb{F}$ extending both B_0 and B_1. We may expand D to an L-structure $\tilde{D}$ by setting $R^{\tilde{D}} = R^{\tilde{B}_0} \cup R^{\tilde{B}_1}$. This establishes SAP for $\mathbb{K}$.

Next, suppose $A, B \in \mathbb{K}_0$ and $\pi : A \to B$ is an L_{tr}-embedding. If $\tilde{A} \in \mathbb{K}$ is an expansion of A, then we can expand B to the L-structure $\tilde{B}$ in which $R^{\tilde{B}} = \{(\pi(a), \pi(a')) : (a, a') \in R^{\tilde{A}}\}$. Clearly we have $\tilde{B} \in \mathbb{K}$ and π is also an L-embedding so by [9, Lemma 2.8], the reduct of the Fraïssé limit of $\mathbb{K}$ is the Fraïssé limit of $\mathbb{K}_0$. $\square$

By Lemma 4.19, we know that $\mathbb{K}$ has a Fraïssé limit which is an ω-categorical expansion of T_{tr} by a (random) graph. Let T denote its theory and let $\mathbb{M}$ and $\mathbb{M}_{\mathrm{tr}}$ denote the monster models of T and T_{tr} respectively.

Lemma 4.20. *Suppose we are given an L-indiscernible sequence $I = \langle a_i : i \in \mathbb{Z}\rangle$ and an element b so that I is L_{tr}-indiscernible over b. Then there is $b' \equiv^L_{a_0} b$ so that I is L-indiscernible over b'.*

Proof. Let $\sigma \in \mathrm{Aut}_{L_{\mathrm{tr}}}(\mathbb{M}/b)$ be an automorphism so that $\sigma(a_i) = a_{i+1}$. Let B denote the L-structure generated by $\langle a_i : i \in \mathbb{Z}\rangle$ and let A_0 be the L-structure generated by $a_0 b$. Now expand the L_{tr}-structure $\langle b(a_i)_{i\in\mathbb{Z}}\rangle_{L_{\mathrm{tr}}}$ to an L-structure M by setting

$$R^M = R^B \cup \bigcup_{i\in\mathbb{Z}} \sigma^i(R^{A_0}).$$

Claim 1: If $i \in \mathbb{Z}$ and $c, d \in B \cap \sigma^i(A_0)$, then $(c, d) \in R^B$ if and only if $(c, d) \in \sigma^i(R^{A_0})$.

Proof of claim. This is clear if $i = 0$, since $R^B = R^{\mathbb{M}} \restriction B$ and $R^{A_0} = R^{\mathbb{M}} \restriction A_0$. In general, if $c, d \in B \cap \sigma^i(A_0)$, there are L_{tr}-terms t, t', s, s' so that

$$c = t(a_{<i}, a_i, a_{>i}) = t'(b, a_i)$$
$$d = s(a_{<i}, a_i, a_{>i}) = s'(b, a_i).$$

By indiscernibility, it follows that if $\sigma^i(c', d') = (c, d)$, then we have

$$c' = t(a_{<0}, a_0, a_{>0}) = t'(b, a_0)$$
$$d' = s(a_{<0}, a_0, a_{>0}) = s'(b, a_0),$$

and we know that $(c', d') \in R^B$ if and only if $(c', d') \in R^{A_0}$, by the $i = 0$ case. By indiscernibility, $(c', d') \in R^B$ if and only if $(c, d) \in R^B$ and hence $(c, d) \in R^B$ if and only if $(c, d) \in \sigma^i(R^{A_0})$. $\square$

Claim 2: If $i > 0$ and $c, d \in A_0 \cap \sigma^i(A_0)$ then $(c, d) \in R^{A_0}$ if and only if $(c, d) \in \sigma^i(R^{A_0})$.

Proof of claim. As in the proof of the previous claim, there are L_{tr}-terms t, t', s, and s' so that we have the following equalities:

$$c = t(a_0, b) = t'(a_i, b)$$
$$d = s(a_0, b) = s'(a_i, b).$$

Then by L_{tr}-indiscernibility over b, we have also $t(a_0, b) = t'(a_{i+1}, b)$ and $t(a_1, b) = t'(a_{i+1}, b)$, hence $t(a_0, b) = t(a_1, b)$. Likewise, we have $s(a_0, b) = s(a_1, b)$. In particular, this shows $\sigma(c, d) = (c, d)$ so the claim follows. $\square$

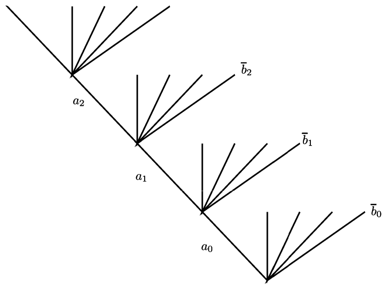

Fig. 1. Illustration of the choice of a_i and $\bar{b}_i$.

Now, by Claim 1, it follows that for all $c, d \in B$, we have $(c, d) \in R^M$ if and only if $(c, d) \in R^B$, so M extends B. Likewise, by Claim 2, M extends A_0 and σ^i induces an L-isomorphism of A_0 and the structure generated by ba_i in M, for all $i \in \mathbb{Z}$. Embed M into $\mathbb{M}$ over B and let b' be the image of b under this embedding. Then by quantifier-elimination, $a_0 b \equiv a_i b'$ for all $i \in \mathbb{Z}$. After applying Ramsey, compactness, and an automorphism, we can find $b'' \equiv_{a_0} b'$ so that I is L-indiscernible over b'', completing the proof. $\square$

Corollary 4.21. *The theory T is NTP$_2$ (and is, in fact, inp-minimal).*

Proof. If T has TP$_2$, then, by Fact 4.18 and compactness, there is an L-formula $\varphi(x; y)$ with $l(x) = 1$ that witnesses TP$_2$ with respect to the mutually indiscernible array $(a_{i,j})_{i<\omega, j\in\mathbb{Z}}$. Let $b \models \{\varphi(x; a_{i,0}) : i < \omega\}$. As T_{tr} is dp-minimal, there is a row $i = 0$ or $i = 1$ so that $\langle a_{i,j} : j \in \mathbb{Z}\rangle$ is b-indiscernible in the language L_{tr}. By Lemma 4.20, there is $b' \equiv^L_{a_{i,0}} b$ such that $\langle a_{i,j} : j \in \mathbb{Z}\rangle$ is b'-indiscernible in the language L. Then $b' \models \{\varphi(x; a_{i,j}) : j \in \mathbb{Z}\}$, contradicting the row-wise inconsistency required for TP$_2$. $\square$

Proposition 4.22. $\kappa_{\mathrm{shred}}(T) = \infty$.

Proof. Let κ be an arbitrary regular cardinal. Inductively, we may choose a sequence of elements $\langle a_i : i < \kappa\rangle$ and a sequence of sequences $\langle \bar{b}_i : i < \kappa\rangle$ so that

(1) For all $i < \kappa$, $\bar{b}_i = \langle b_{i,j} : j < \omega\rangle$ is an $a_{<i}\bar{b}_{<i}$-indiscernible sequence of pairwise incomparable elements, incomparable with a_i, with $b_{i,j} \wedge b_{i,j'} = a_i \wedge b_{i,j}$ for all $j \neq j'$ and $R(a_i \wedge b_{i,j}, b_{i,j})$ if and only if j is even.
(2) If $i < i' < \kappa$, then $a_i \lhd a_{i'} \wedge b_{i',j}$ for all j.

There is no problem continuing the induction, since T is the generic $\wedge$-tree with a random graph (Fig. 1).

Let $p(x) = \{x \trianglerighteq a_i : i < \kappa\}$. Notice that if $x \trianglerighteq a_i$, then $x \wedge b_{i,j} = a_i \wedge b_{i,j}$ and hence $x \trianglerighteq a_i \vdash R(x \wedge b_{i,j}, b_{i,j})$ if and only if j is even. It follows that the formula $x \trianglerighteq a_i$ explicitly shreds over $a_{<i}$. As κ is arbitrary, $\kappa_{\mathrm{shred}}(T) = \infty$. $\square$

5. A criterion for singular compactness

In this section, we give a sufficient condition for having singular compactness, which is the negation of exact saturation (Definition 5.1 below). If $\Delta(x, y)$ is a set of formulas over $\emptyset$, then a (partial) Δ-*type* is a consistent set of instances of formulas from Δ. We may refer to a $\{\varphi\}$-type as a φ-type. It is important to note that by a φ-*type* we mean a consistent set of *positive* instances of φ, and do not include instances of $\neg\varphi$.

Sh:1192

Definition 5.1. Suppose that T is a complete first order theory and Δ is a set of formulas over $\emptyset$. Say that T has*singular compactness* for Δ if whenever $M \models T$ is μ-saturated for a singular cardinal $\mu > |T|$ then M is μ^+, Δ-saturated: for every Δ-type p over a set $A \subseteq M$ with $|A| \leq \mu$, p is realized in M.

Condition 5.2. For every formula $\varphi(x, y)$ (perhaps in a fixed set of formulas Δ) there is some formula $\theta_\varphi(x, z)$ such that for any finite φ-type $r(x)$ over $M \models T$ and every finite set $A \subseteq M^x$ of realizations of r there is some $b \in M^z$ such that $\theta_\varphi(A, b)$ holds (i.e., $M \models \theta_\varphi(a, b)$ for all $a \in A$) and $\theta_\varphi(x, b) \vdash r(x)$.

Lemma 5.3. *Suppose that T is a complete first order theory and that Condition 5.2 holds for $\Delta(x, y)$. Then T has singular compactness for Δ.*

Proof. Let p be a Δ-type over a set A with $|A| = \mu$ and suppose $A \subseteq M$, a μ-saturated model of T. Write $A = \bigcup_{i<\kappa} A_i$ with $|A_i| < \mu$, $\kappa < \mu$. For each $i < \kappa$ find $b_i \in M$ such that $b_i \models p|_{A_i}$ (exists by μ-saturation).

By compactness and Condition 5.2, for each $\varphi \in \Delta$ find $e_i^\varphi \in \mathbb{M}^z$ such that $\theta_\varphi(b_j, e_i^\varphi)$ holds for all $j \geq i$ and $\theta_\varphi(x, e_i^\varphi) \vdash \operatorname{tp}_\varphi^+(b_i/A_i)$, the (positive) φ-type of b_i over A_i. By μ-saturation, find $d_i^\varphi \in M$ such that $d_i^\varphi \equiv_{A_i \cup \{b_i : i<\kappa\}} e_i^\varphi$. Then $\{\theta_\varphi(x, d_i^\varphi) : i < \kappa, \varphi \in \Delta\}$ is a type and hence realized in M. $\square$

When does Condition 5.2 hold? If T is complicated enough, e.g., $T = PA$ or $T = ZFC$, then it holds since given $\varphi(x, y)$, we can choose $\theta_\varphi(x, z) = x \in z$. Indeed, this condition implies that the theory cannot be too tame.

Proposition 5.4. *Assume T has infinite models. If Condition 5.2 holds for every formula with one variable x then T has TP_2, and has SOP_n for all n.*

Proof. We start by showing that T has TP_2. Let $\varphi(x, z)$ be $\theta_{x \neq y}(x, z)$. Let $\psi(x, w) = \theta_{\neg\varphi}(x, w)$. We will show that $\xi(x, zw) = \varphi(x, z) \wedge \psi(x, w)$ witnesses TP_2. Let $\{a_i : i < \omega\}$ be some infinite set in $\mathbb{M}$. Suppose that $\mathcal{F}$ is an arbitrary family of pairwise disjoint subsets of ω. It is enough to find some $b_s \in M^{zw}$ for every $s \in \mathcal{F}$ such that $\xi(a_i, b_s)$ holds whenever $i \in s$, and $\{\xi(x, b_s), \xi(x, b_t)\}$ is inconsistent for all $s \neq t$ from $\mathcal{F}$ (see [5, Lemma 2.19]). By compactness we may assume that $\mathcal{F}$ is finite and consists of finite sets and replace ω by some $n < \omega$.

By choice of $\varphi(x, z)$ there are c_s for $s \in \mathcal{F}$ such that $\varphi(a_i, c_s)$ holds iff $i \in s$: take the finite type $r_s = \{x \neq a_i : i \notin s\}$ and $A_s = \{a_i : i \in s\}$ and apply Condition 5.2. This already shows that T has the independence property so is not NIP.

We can similarly choose d_s by applying Condition 5.2 for φ and taking $r_s = \{\neg\varphi(x, c_t) : t \neq s, t \in \mathcal{F}\}$ and $A_s = \{a_i : i \in s\}$. Then obviously $\xi(a_i, c_s d_s)$ holds if $i \in s$. Also, as $\psi(x, d_s) \vdash \neg\varphi(x, c_t)$ for $t \neq s$, we are done.

Next we show that T has SOP_n for all $n < \omega$.

Let $\varphi_0(x, y_0) = \theta_{\neq}(x, y_0)$, $\varphi_1(x, y_1) = \theta_{\varphi_0}(x, y_1)$ and in general $\varphi_{n+1}(x, y_{n+1}) = \theta_{\varphi_n}(x, y_{n+1})$. Fix some n with $3 \leq n < \omega$. Let $\chi_n(y_0, \ldots, y_{n-1}, x_0; z_0, \ldots, z_{n-1}; x_0')$ with $|z_i| = |y_i|$ say that

$$(\forall x)[\varphi_{i+1}(x, y_{i+1}) \to \varphi_i(x, z_i)]$$

for all $i < n - 1$ and $\varphi_{n-1}(x_0, y_{n-1}) \wedge \neg\varphi_0(x_0', y_0)$. We will show that $\chi = \chi_n$ witnesses SOP_n for all $n \geq 3$.

Let $\langle a_t : t < \omega \rangle$ be some infinite sequence in $\mathbb{M}$. For $t < \omega, i < n$, let $b_t^i \in \mathbb{M}^{y_i}$ be such that $\varphi_i(a_s, b_t^i)$ holds iff $s \leq t$ (i.e., witnessing that φ_i has the order property) and $(\forall x)[\varphi_{i+1}(x, b_t^{i+1}) \to \varphi_i(x, b_{t'}^i)]$ for all $t' \geq t$. We may find such b_t^i's by induction on $i < n$ using Condition 5.2 and compactness as above. For $k < \omega$, let $\bar{b}_k = b_k^0 \ldots b_k^{n-1} a_k$. We have that for $k, l < \omega$, $\mathbb{M} \models \chi(\bar{b}_k, \bar{b}_l)$ if and only if $k < l$. However, it is impossible that $\{\chi(\bar{x}_k, \bar{x}_{k+1}) : k < n - 1\} \cup \{\chi(\bar{x}_{n-1}, \bar{x}_0)\}$ is consistent, since if it were realized by $\bar{c}_k = c_k^0 \ldots c_k^{n-1} d_k$

Sh:1192

for $k < n$, then $\varphi_{n-1}\left(d_0, c_0^{n-1}\right) \Rightarrow \varphi_{n-2}\left(d_0, c_1^{n-2}\right) \Rightarrow \cdots \Rightarrow \varphi_0\left(d_0, c_{n-1}^0\right)$ but as $\chi\left(\bar{c}_{n-1}, \bar{c}_0\right)$ holds, we have that $\neg\varphi_0\left(d_0, c_{n-1}^0\right)$ holds as well which is a contradiction. □

We give an example where this criterion holds.

Example 5.5. Let $L = \{P_i : i < 3\} \cup \{R_{0,1}, R_{0,2}, R_{1,2}\}$ where the P_is are unary predicates and the $R_{i,j}$s are binary relation symbols. Let $T^{\forall}$ say that $\langle P_i : i < 3\rangle$ are disjoint and their union covers the universe, that $R_{i,j} \subseteq P_i \times P_j$ and that:

⊘ If $R_{1,2}(b, c)$ then $(\forall x)\left[R_{0,1}(x, b) \rightarrow R_{0,2}(x, c)\right]$.

Claim 5.6. $T^{\forall}$ is universal, it has the amalgamation property (AP) and the joint embedding property (JEP).

Proof. The fact that $T^{\forall}$ is universal is clear.

JEP: suppose that $M_1, M_2 \models T^{\forall}$ are disjoint. Let M be the following structure. As a set it is $M_1 \cup M_2$. For every relation symbol $Q \in L$, let $Q^M = Q^{M_1} \cup Q^{M_2}$.

AP: suppose that $M_0, M_1, M_2 \models T^{\forall}$ and $M_0 \subseteq M_1, M_2$ and $M_0 = M_1 \cap M_2$. Let M be the following structure. Its universe is just the union of the universes of M_1, M_2. For $i < 3$, $P_i^M = P_i^{M_1} \cup P_i^{M_2}$. $R_{0,1}^M = R_{0,1}^{M_1} \cup R_{0,1}^{M_2}$ and similarly define $R_{1,2}^M = R_{1,2}^{M_1} \cup R_{1,2}^{M_2}$. Let

$$\begin{aligned} R_{0,2}^M = {} & R_{0,2}^{M_1} \cup R_{0,2}^{M_2} \\ & \cup \left\{(a, b) : a \in P_0^{M_1} \backslash M_0, b \in P_2^{M_2} \backslash M_0\right\} \\ & \cup \left\{(a, b) : a \in P_0^{M_2} \backslash M_0, b \in P_2^{M_1} \backslash M_0\right\}. \end{aligned}$$

Let us check that ⊘ holds. Suppose that $M \models R_{1,2}(b, c)$. Then we may assume that $b, c \in M_1$ (for M_2 it is the same argument). Suppose that $M \models R_{0,1}(a, b)$. Then if $a \in M_1$ then $M_1 \models R_{0,2}(a, c)$. Otherwise $a \in M_2$ and $b \in M_0$. If $c \in M_0$ as well, then $M_2 \models R_{1,2}(b, c) \wedge R_{0,1}(a, b)$ so $M_2 \models R_{0,2}(a, c)$ and we are done. Otherwise $c \in M_1 \backslash M_0$, in which case $R_{0,2}^M(a, c)$ holds by choice of $R_{0,2}^M$. □

Corollary 5.7. *$T^{\forall}$ has a model completion T which has quantifier elimination.*

Proposition 5.8. *T is NSOP$_4$ and has SOP$_3$.*

Proof. We start by showing that T is NSOP$_4$. Suppose that $\langle a_i : i < \omega\rangle$ is an indiscernible sequence in some model $M \models T$ which witnesses SOP$_4$. Let A_i be a_i as a set. Let $M_0 = A_2$, $M_0' = A_3$, $M_1 = A_1A_2$, $M_2 = A_2A_3$ and $M_3 = A_3A_4$ with the induced structure from M. So all are models of $T^{\forall}$. Let M' be the amalgam of M_1, M_2 over M_0 as defined in the proof of Claim 5.6, and similarly let M'' be the amalgam of M_2, M_3 over M_0'. Note that both M' and M'' contain M_2 as a substructure and that the universe of M' is $A_1A_2A_3$ and of M'' is $A_2A_3A_4$, but neither are necessarily substructures of M.

Now we can amalgamate M' and M'' over M_2. Moreover,

- Any structure N whose universe is $A_1A_2A_3A_4$ which contains both M', M'' as substructures and satisfies $T^{\forall}$ except perhaps ⊘, and such that $N \restriction A_1A_4 \models T^{\forall}$ will be a model of $T^{\forall}$ (i.e., ⊘ just follows).

To see this, suppose that $N \models R_{1,2}(b, c) \wedge R_{0,1}(a, b)$. We have to show that $N \models R_{0,2}(a, c)$. Note that for every $x \in N$, if $x \in A_i \cap A_j$ for distinct $i, j \in \{1, \ldots, 4\}$, $x \in \bigcap_{i=1}^4 A_i$ by indiscernibility.

If a, b, c all belong to either $A_1A_2A_3$, $A_2A_3A_4$ or A_1A_4 then this is clear, so assume this is not the case.

Suppose that $b,c \in A_1A_2A_3$, $a \in A_4$ (so $a \notin A_1A_2A_3$) and $b \notin A_1$. Then if $b \in A_2 \backslash A_3$ then $M'' \models \neg R_{0,1}(a,b)$ — contradiction, so $b \in A_3$. Then it must be that $c \in A_1 \backslash A_2$ and $b \in A_3 \backslash A_2$ so $M' \models \neg R_{1,2}(b,c)$ — contradiction.

If $b,c \in A_1A_2A_3$, $a \in A_4$ and $b \in A_1$ then $c \notin A_1$. If $c \in A_2 \backslash A_3$ then $M'' \models R_{0,2}(a,c)$ so we are done. Else, $c \in A_3 \setminus A_2$, so since $b \notin A_2$, $M' \models \neg R_{1,2}(b,c)$ — contradiction.

Suppose that $b \in A_1$ and $c \in A_4$. Then $a \in A_2A_3$. If $a \in A_2 \backslash A_3$ then $M'' \models R_{0,2}(a,c)$ so we are done. Otherwise, $a \in A_3 \backslash A_2$, so $M' \models \neg R_{0,1}(a,b)$ — contradiction.

The case where $b \in A_4$ and $c \in A_1$ is done similarly.

By symmetry, this covers all the cases so the bullet is proved.

Let $\sigma : A_1A_4 \to A_1A_4$ be a bijection such that $\sigma(a_1) = a_4$ and $\sigma(a_4) = a_1$ as tuples (hence $\sigma^2 = \mathrm{id}$). Let N_0 be an amalgam of M' and M'' over M_2 with domain $A_1A_2A_3A_4$. Now define N to be a structure with the same underlying set and the same interpretation of the unary predicates, but with each $R_{i,j}$ interpreted as follows:

$$R_{i,j}^N = \left(R_{i,j}^{N_0} \setminus (A_1A_4)^2\right) \cup \{(a,b) \in A_1A_4 : M \models R_{i,j}(\sigma(a), \sigma(b))\}.$$

By indiscernibility, if a,b are either both in A_1 or both in A_4, then $(a,b) \in R_{i,j}^N$ if and only if $(a,b) \in R_{i,j}^M$. Then it is clear that N has underlying set $A_1A_2A_3A_4$ and extends both M' and M'', hence it satisfies the conditions in the bullet point above. This shows $N \models T^\forall$, and hence there is some $N' \models T$ containing N.

But then, if $\varphi(x,y)$ is any quantifier-free formula with $M \models \varphi(a_1,a_2)$, then $N' \models \varphi(a_1,a_2) \wedge \varphi(a_2,a_3) \wedge \varphi(a_3,a_4) \wedge \varphi(a_4,a_1)$. By quantifier elimination, T is NSOP$_4$.

Next we show that T has SOP$_3$. For this we will use the following criterion.

Fact 5.9. [12, Claim 2.19] For a theory T, having SOP$_3$ is equivalent to finding two formulas $\varphi(x,y), \psi(x,y)$ and a sequence $\langle a_i, b_i : i < \omega \rangle$ in some $M \models T$ such that

- For all $i < j$, $M \models \neg \exists x (\varphi(x, a_j) \wedge \psi(x, a_i))$.
- If $i \leq j$ then $M \models \varphi(b_j, a_i)$ and if $j < i$ then $M \models \psi(b_j, a_i)$.

(The definition in [12] additionally requires that $\{\varphi(x;y), \psi(x;y)\}$ is inconsistent, but this added condition is unnecessary: given φ and ψ as above, one can replace φ by $\varphi' = \varphi(x;y) \wedge \neg\psi(x;y)$ and then φ' and ψ will witness the above conditions).

Let $\varphi(x,y') = R_{0,1}(x,y')$ and $\psi(x,y'') = \neg R_{0,2}(x,y'')$. Let $\langle a_i', a_i'', b_i : i < \omega \rangle$ be a sequence such that $R_{1,2}\left(a_i', a_j''\right)$ iff $i > j$, $R_{0,1}(b_j, a_i')$ whenever $i \leq j$ and $\neg R_{0,2}(b_j, a_i'')$ whenever $i > j$. This sequence exists in some model $M \models T$ as we can define a model of $T^\forall$ which contains exactly those elements. Now letting $a_i = (a_i', a_i'')$, the first bullet follows from $\oslash$ and the second bullet by the choice of a_i', a_i'' and b_i. $\square$

Corollary 5.10. *There is a theory T with NSOP$_4$ having SOP$_3$ such that Condition 5.2 holds with $\Delta = \{R_{0,2}(x,y)\}$ and θ_φ from there being $R_{0,1}$. Thus T has Δ-singular compactness by Lemma 5.3.*

Proof. We only need to show that Condition 5.2 holds. Suppose that $M \models T$ and r is some finite Δ-type. Let $A \subseteq M$ be a finite set of realizations. Now the definition of T, we may find some $b \in M$ with $R_{1,2}(b,c)$ whenever $R_{0,2}(x,c) \in r$ and $R_{0,1}(a,b)$ for all $a \in A$. This suffices. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Sh:1192

References

[1] John T. Baldwin, Rami Grossberg, Saharon Shelah, Transfering saturation, the finite cover property, and stability, J. Symb. Log. 64 (2) (1999) 678–684.
[2] Artem Chernikov, Theories without the tree property of the second kind, Ann. Pure Appl. Log. 165 (2) (2014) 695–723.
[3] Artem Chernikov, Nicholas Ramsey, On model-theoretic tree properties, J. Math. Log. (2016) 1650009.
[4] Mirna Džamonja, Saharon Shelah, On $\triangleleft^*$-maximality, Ann. Pure Appl. Log. 125 (1) (2004) 119–158.
[5] Itay Kaplan, Elad Levi, Pierre Simon, Some remarks on dp-minimal groups, in: Groups, Modules, and Model Theory-Surveys and Recent Developments, Springer, 2017, pp. 359–372.
[6] Itay Kaplan, Nicholas Ramsey, On Kim-independence, J. Eur. Math. Soc. (2017), arXiv:1702.03894.
[7] Itay Kaplan, Saharon Shelah, Pierre Simon, Exact saturation in simple and NIP theories, J. Math. Log. 17 (1) (2017) 18, 1750001.
[8] Malliaris Maryanthe, Saharon Shelah, Model-theoretic applications of cofinality spectrum problems, Isr. J. Math. 220 (2) (2017) 947–1014.
[9] Nicholas Ramsey, Invariants related to the tree property, preprint arXiv:1511.06453, 2015.
[10] Saharon Shelah, Simple unstable theories, Ann. Math. Log. 19 (3) (1980) 177–203.
[11] Saharon Shelah, Classification Theory: and The Number of Non-Isomorphic Models, Elsevier, 1990.
[12] Saharon Shelah, Toward classifying unstable theories, Ann. Pure Appl. Log. 80 (3) (1996) 229–255.
[13] Pierre Simon, A Guide to NIP Theories, Cambridge University Press, 2015.