MANY FORCING AXIOMS FOR ALL REGULAR UNCOUNTABLE CARDINALS

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ABSTRACT. A central theme in set theory is to find universes with extreme, well-understood behaviour. The case we are interested in is assuming GCH and has a strong forcing axiom of higher order than usual. Instead of "for every suitable forcing notion for λ" we shall say "for every such family of forcing notions, depending on stationary $S \subseteq \lambda$, for some such stationary set we have. . . ". Such notions of forcing are important for Abelian group theory, but this application is delayed for a sequel.

1. INTRODUCTION

One of the original motivations for the work presented in this paper is to show the consistency of the failure of singular compactness for properties such as being Whitehead ($\mathrm{Ext}(G, \mathbb{Z}) = 0$). Under $V = L$, for example, singular compactness for being Whitehead holds, but this is because $V = L$ implies that Whitehead is equivalent to free [Sh:44], and singular compactness for free groups is a ZFC theorem [Sh:52]; see [Ekl80, EFS90, Hod81, Sh:266].

Those problems are closely related to so-called uniformisation principles (see [EM02]), and in many cases even equivalent them. The first work along this line is [Sh:64], where it is proved that the GCH is consistent with diamond holding on some stationary subset of ω_1, while it fails for others, indeed on the latter, some uniformisation principle (usually derived from MA) holds. For more see [Sh:587, Sh:667].

In this paper we present an axiom which guarantees, among other consequences, instances of uniformisation. What we term the "task axiom" for a regular cardinal λ ensures that mutually competing principles, such as diamonds and uniformisation, all hold on (necessarily disjoint) stationary sets.

A feature of this axiom is that it entails Π_4, rather than Π_2 statements: to satisfy a "task", we need not only a sufficiently generic filter for an appropriate notion of forcing, but also many filters for subsequent notions of forcing which are determined by the first filter. The two examples we present in this paper are exact diamonds (Proposition 4.9) and uniformisation (Proposition 4.12). For example, for the former, the first notion of forcing adds a stationary set and a diamond sequence on that set; subsequent notions of forcing ensure that this diamond sequence is exact. The technical work is then to show that the set added remains stationary after adding the subsequent generics. Interleaved, there is work done toward satisfying other tasks. As we said, these tasks are fulfilled on disjoint sets (modulo the nonstationary ideal), and so should not interfere with each other. The

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technical expression of this live-and-let-live approach is a notion of closure of a notion of forcing on a given fat set, related to the notions of S-completeness defined in [Sh:64, Sh:587, Sh:667].

1.1. **The contents of the paper.** Tasks (Definition 3.3) are defined in Section 3; the task axiom (Definition 4.8) is defined in Section 4. Beforehand, we develop the tools required to formulate tasks and to work with them. Throughout, we fix a regular uncountable cardinal λ, and assume that the GCH holds below λ.

To start, we define two notions of completeness:

- explicitly S-closed notions of forcing (Definition 2.1);
- S-sparse names for subsets of λ (Definition 2.9).

The former attaches ordinals to conditions, and states that increasing sequences of conditions with limit points in S have upper bounds. The latter is similar, except that we now associate to conditions initial segments of a subset of λ in the generic extension, and state that increasing sequences of conditions which determine a closed set disjoint from the subset in question have upper bounds.

We then turn to develop machinery that will help us work with forcing iterations. The general situation is a $<\lambda$-support iteration $\langle \mathbb{P}_\zeta, \mathbb{Q}_\zeta \rangle_{\zeta<\xi}$, one of which we will use to show consistency of the task axiom. In the intended application, some of the notions of forcing $\mathbb{Q}_\zeta$ add new stationary sets W_ζ, on which we will want to fulfill some "task"; that is done by some of the subsequent notions of forcing $\mathbb{Q}_\eta$ appearing (cofinally) later in the iteration. The typical task will state the existence of a stationary set S and possibly some associated object (for example a diamond sequence), for which some $\Pi_2(\mathcal{H}_\lambda)$ fact holds (for example, the diamon sequence is exact).

For simplicity of notation, we will set $W_\eta = \varnothing$ for the notions $\mathbb{Q}_\eta$ which do not start a new task. Thus, for many of our definitions and lemmas, we do not need to differentiate between the two kinds of $\mathbb{Q}_\zeta$'s. We will ensure that the sets W_ζ are pairwise disjoint (modulo clubs).

In Section 2 we show (Corollary 2.32) how to construct, for each $\zeta \leqslant \xi$, a sparse $\mathbb{P}_\zeta$-name for the least upper bound of the sets $W_{\zeta'}$ (for $\zeta' < \zeta$), again modulo clubs. These names, and how they cohere with each other, will be one of the main tools we use in the analysis of tasks, ensuring that they can be adequately fulfilled.

In Section 3 we define tasks. To motivate following definitions, before we define the task axiom, we give a prototypical example of an iteration, of length λ^+, attempting to fulfill tasks. We formulate the example of exact diamonds and show that these exist in the generic extension (Proposition 3.8).

To formulate the task axiom, we need to bear in mind that some tasks may be too ambitious; an attempt to fulfill them, along with other tasks, will result in failure, for example, in a hoped-for stationary set not being stationary anymore. Thus we need to define "correctness conditions" for tasks (Definition 4.5). It is only reasonable to expect that a task be fulfilled if a typical forcing iteration preserves this correctness condition; this is the notion of niceness for tasks (Definition 4.6), with which we can then state the task axiom.

Unfortunately, the iteration of length λ^+ given in Section 3 may be too short to expose those tasks which are not nice, i.e., to witness the failure of the correctness condition after some iteration. The consistency of the task axiom is then proved by a longer iteration (still of length $< \lambda^{++}$), with sufficient closure. At steps of cofinality λ^+, we need auxiliary notions of forcing to ensure that the sequence of sets W_ζ have upper bounds; we develop this machinery (Proposition 4.3) in the beginning of Section 4.

Finally, as a second example, we show that the task axiom implies instances of uniformisation (Proposition 4.12).

We remark that the construction can be modified to ensure the preservation of suitable large cardinals; we do not pursue this in this paper.

1.2. **Notation and terminology.** We list some of the notation that we use. We follow the Israeli convention for extension in notions of forcing; $p \leqslant q$ means that q extends p. Complete embeddings $\mathbb{P} \lessdot \mathbb{Q}$ of forcing notions will always be accompanied with a restriction function $p \mapsto p \upharpoonright \mathbb{P}$ from $\mathbb{Q}$ to $\mathbb{P}$. For typographical convenience, for a notion of forcing $\mathbb{P}$ and a definable set or class X, we let $X(\mathbb{P})$ denote the interpretation of X in the Boolean valued model $V^{\mathbb{P}}$. So for example $V(\mathbb{P}) = V^{\mathbb{P}}$, $\mathcal{P}(\lambda)(\mathbb{P}) = (\mathcal{P}(\lambda))^{V^{\mathbb{P}}}$ is the collection of names $a \in V^{\mathbb{P}}$ such that $\Vdash_{\mathbb{P}} a \subseteq \lambda$, etc. When we say that a statement φ holds in $V(\mathbb{P})$, we mean that every condition forces φ.

For a binary string σ (a function from an ordinal into 2), we let $|\sigma| = \operatorname{dom}\sigma$ denote the length of σ. We will at times be imprecise and identify sets with characteristic functions; that is, we identify $\sigma\colon \alpha \to 2$ with $\{\beta < \alpha : \sigma(\beta) = 1\}$, when there is no danger that we forget $\alpha = |\sigma|$. we write $\sigma \preccurlyeq \tau$ to indicate that τ extends σ, that is, $\sigma = \tau \upharpoonright |\sigma|$.

$\operatorname{Cof}(\theta)$ denotes the class of ordinals of cofinality θ; similarly we use $\operatorname{Cof}(\leqslant \theta)$ etc. For a cardinal λ, we let $\operatorname{Cof}_\lambda(\theta)$ denote $\lambda \cap \operatorname{Cof}(\theta)$.

We let $\mathcal{H}_\chi$ denote the collection of sets whose transitive closure has size $< \chi$.

1.3. **The underlying hypothesis.** Throughout this paper, λ denotes a regular uncountable cardinal, and we assume that the GCH holds below λ.

1.4. **Approachable ordinals.** In this section we isolate a tool that will allow us to make use of closure conditions on fat sets. The technique we use was introduced in [Sh:108], where it is used to show that there are many "approachable" ordinals below λ (so a large subset of a given fat subset of λ will be in $\check{I}[\lambda]$). Recall that a subset S of λ is *fat* iff for every regular $\theta < \lambda$, for every club C of λ, $S \cap C$ contains a closed subset of order-type $\theta + 1$. If S is fat then for all regular $\theta < \lambda$, $S \cap \operatorname{Cof}(\theta)$ is stationary in λ.

For the following lemma, recall that a λ*-filtration of the universe* is an increasing and continuous sequence $\bar{N} = \langle N_\gamma \rangle_{\gamma < \lambda}$ such that for some large χ, for all $\gamma < \lambda$,

(i) N_γ is an elementary submodel of $\mathcal{H}_\chi$;
(ii) $|N_\gamma| < \lambda$;
(iii) $\gamma \subseteq N_\gamma$; and
(iv) $\bar{N} \upharpoonright (\gamma + 1) \in N_{\gamma+1}$.

Lemma 1.1. Suppose that $S \subseteq \lambda$ is fat. For every regular cardinal $\theta < \lambda$ and every set X with $|X| < \lambda$ there is a λ-filtration $\bar{N}$ such that $X \subseteq N_0$, and a set $D \subset \lambda$ such that:

(i) $\operatorname{otp}(D) = \theta + 1$;
(ii) The closure $\overline{D}$ of D is a subset of S;
(iii) For all $\gamma \in D$, $N_\gamma \cap \lambda = \gamma$;
(iv) For all $\gamma \in D$ other than $\max D$, $D \cap \gamma \in N_{\gamma+1}$.

Note that the set X needn't be a subset of λ; we will have $X \in \mathcal{H}_\chi$ for some large χ, of which the N_γ would be elementary submodels.

Proof. There are two cases. The easier one is when λ is not the successor of a singular cardinal, that is, that it is either the successor of a regular cardinal, or inaccessible. In that case we can build the filtration $\langle N_\gamma \rangle$ inductively, requiring that for all $\gamma < \lambda$, $[\gamma]^{<\theta} \subset N_{\gamma+1}$. This we can do because $|\gamma|^{<\theta} < \lambda$, as in either case there is a regular cardinal $\kappa < \lambda$ such that $|\gamma|, \theta \leqslant \kappa$, and for all regular $\kappa < \lambda$, $\kappa^{<\kappa} = \kappa$. We then let $E = \{\gamma < \lambda : N_\gamma \cap \lambda = \gamma\}$; this is a club of λ, so $S \cap E$ contains a closed subset D of order-type $\theta + 1$. If $\gamma \in D$ and $\gamma \neq \max D$ then $|D \cap \gamma| < \theta$, so is an element of $N_{\gamma+1}$.

We now suppose that $\lambda = \mu^+$ where μ is singular; let $\kappa = \mathrm{cf}(\mu)$. In this case we may have $|\gamma|^{<\theta} = \lambda$, so we need a finer approach. We know that $\theta < \mu$.

Let $\langle \mu_\xi \rangle_{\xi<\kappa}$ be a sequence of cardinals increasing to μ. For all $\alpha \in [\mu, \lambda)$ fix a partition $\{A^\alpha_\xi : \xi < \kappa\}$ of α such that $|A^\alpha_\xi| = \mu_\xi$. We build $\langle N_\gamma \rangle_{\gamma<\lambda}$ such that $|N_\gamma| = \mu$ and ensure that all bounded subsets of μ are in N_0. We also put the map $(\alpha, \xi) \mapsto A^\alpha_\xi$ into N_0. We define E as above.

We may assume that $\theta > \kappa$. By the GCH below λ, we know that $(2^\theta)^+ < \lambda$, and so we can find a closed set $D^* \subset S \cap E$ of size $(2^\theta)^+$. By the Erdös-Rado theorem we can find $D \subset D^*$ of order-type $\theta + 1$ (in fact of size θ^+) such that for all $\gamma < \delta$ from D, $\gamma \in A^\delta_{\xi^*}$ for some fixed $\xi^* < \kappa$. The set D may not be closed but its closure is a subset of D^*, and so of $S \cap E$. For all $\gamma \in D$, $D \cap \gamma \subseteq A^\gamma_{\xi^*}$. Since $A^\gamma_{\xi^*} \in N_{\gamma+1}$, and this set is bijective with μ_{ξ^*}, every subset of $A^\gamma_{\xi^*}$ is in $N_{\gamma+1}$. So $D \cap \gamma \in N_{\gamma+1}$. □

2. Closed notions of forcing and sparse names

2.1. Explicit S-closure.

Definition 2.1. Let $S \subseteq \lambda$. A notion of forcing $\mathbb{P}$ is *explicitly S-closed* if there is a function $\delta \colon \mathbb{P} \to \lambda$ satisfying:

(i) $p \leqslant q$ implies $\delta(p) \leqslant \delta(q)$;
(ii) For every $\gamma < \lambda$, the collection of conditions p with $\delta(p) \geqslant \gamma$ is dense in $\mathbb{P}$;
(iii) whenever $\bar{p} = \langle p_i \rangle_{i<i^*}$ is an increasing sequence of conditions from $\mathbb{P}$, with $i < j < i^*$ implying $\delta(p_i) < \delta(p_j)$, and $\alpha = \sup_{i<i^*} \delta(p_i) \in S$, then $\bar{p}$ has an upper bound p^* in $\mathbb{P}$ with $\delta(p^*) = \alpha$.

We call such an upper bound p^* an *exact upper bound* of $\bar{p}$.

Lemma 2.2. If S and S' are equivalent modulo the club filter, then a notion of forcing $\mathbb{P}$ is explicitly S-closed if and only if it is explicitly S'-closed. □

Proof. If $\mathbb{P}$ is explicitly S-closed, witnessed by δ, and C is a club, then by replacing $\delta(p)$ by $\sup(C \cap \delta(p))$, we may assume that δ takes values in C. □

Proposition 2.3. *Suppose that $S \subseteq \lambda$ is fat, and that $\mathbb{P}$ is explicitly S-closed. Then $\mathbb{P}$ is $<\lambda$-distributive, and S is fat in $V(\mathbb{P})$.*

Proof. Let $\theta < \lambda$ be a regular cardinal, and let $\{U_i : i < \theta\}$ be a family of dense open subsets of $\mathbb{P}$; let $p_0 \in \mathbb{P}$. Let $\bar{N} = \langle N_\gamma \rangle_{\gamma<\lambda}$ and D be given by Lemma 1.1, with $\mathbb{P}, p_0, \langle U_i \rangle \in N_0$.

We enumerate D as $\langle \gamma_i \rangle_{i \leqslant \theta}$ and define an increasing sequence $\bar{p} = \langle p_i \rangle_{i \leqslant \theta}$ (starting from p_0) of conditions from $\mathbb{P}$. We fix a well-ordering $\leqslant^*_{\mathbb{P}}$ of $\mathbb{P}$ which is an element of N_0, and define the sequence of conditions as follows:

(a) Given p_i for $i < \theta$, p_{i+1} is the $\leqslant^*_{\mathbb{P}}$-least $p \in \mathbb{P}$ extending p_i such that $\delta(p) \geqslant \gamma_i$ and $p \in U_i$.
(b) For limit $i \leqslant \theta$, p_i is the $\leqslant^*_{\mathbb{P}}$-least upper bound of $\bar{p} \upharpoonright i$.

Of course we need to argue that this construction is possible, which means showing that at a limit step $i \leqslant \theta$, $\bar{p} \upharpoonright i$ has an upper bound. First we observe that if the construction has been performed up to and including step $i+1$, then the sequence $\langle p_j \rangle_{j \leqslant i+1}$ is definable from $\mathbb{P}, p_0, \langle U_i \rangle, \leqslant^*_{\mathbb{P}}$ and the sequence $\langle \gamma_j \rangle_{j \leqslant i}$. These parameters are all in N_{γ_i+1}, so $p_{i+1} \in N_{\gamma_i+1}$. It follows that $\delta(p_{i+1}) < \gamma_{i+1}$.

Suppose now that $i \leqslant \theta$ is a limit ordinal and that the sequence $\langle p_j \rangle_{j<i}$ has been successfully defined. Then as $\gamma_j \leqslant \delta(p_{j+1}) < \gamma_{j+1}$ for all $j < i$, we have $\sup_{j<i} \delta(p_j) = \sup_{j<i} \gamma_j$ is a limit point of D, and hence is in S; so the closure assumption says that $\bar{p} \upharpoonright i$ has an upper bound in $\mathbb{P}$, whence p_i is defined and the construction can proceed. Then p_θ is an extension of p_0 in $\bigcap_i U_i$.

To show that S is fat in $V(\mathbb{P})$, again let $\theta < \lambda$ be regular, and let $C \in V(\mathbb{P})$ be a club of λ. We perform a similar construction, this time with p_{i+1} forcing some $\beta_i > \gamma_i$ into C. As $p_{i+1} \in N_{\gamma_i+1}$, we have $\beta_i < \gamma_{i+1}$. The final condition forces that the limit points of D are all in C, and so form a closed set of order-type $\theta + 1$ in $S \cap C$. □

Note that if $S' \subseteq S$ then $\mathbb{P}$ is also explicitly S'-closed, and so if S' is fat, then it is fat in $V(\mathbb{P})$. The argument for the case $\theta = \omega$ shows that if $S' \subseteq S \cap \mathrm{Cof}(\aleph_0)$ is stationary, then it remains stationary in $V(\mathbb{P})$.

We introduce terminology:

Definition 2.4. We say that $\mathbb{P}$ is *explicitly closed outside* S if it is explicitly $(\lambda \backslash S)$-closed.

Strategic closure. We remark that we can generalise Definition 2.1:

Definition 2.5. Let $S \subseteq \lambda$. A notion of forcing $\mathbb{P}$ is *strategically S-closed* if the "completeness" player has a winning strategy in the following game, of length $\leqslant \lambda$: the players alternate choosing conditions forming an increasing sequence $\langle p_i \rangle$ from $\mathbb{P}$ (the incompleteness player chooses first; the completeness player chooses at limit steps), and also choosing ordinals below λ which form an increasing and continuous sequence $\langle \varepsilon_i \rangle$ (so at limit steps there is no choice of ordinal).

- The incompleteness player loses at a limit stage i if $\varepsilon_i \notin S$;
- The completeness player loses at a limit stage i if $\varepsilon_i \in S$ and $\langle p_j \rangle_{j<i}$ does not have an upper bound in $\mathbb{P}$.

If the play lasts λ moves then the completeness player wins.

Note that if $S = \lambda$ then there is no need to choose ordinals; it is then the usual notion of λ-strategic closure. Also note that if S and S' are equivalent modulo the club filter, then $\mathbb{P}$ is strategically S-closed if and only if it is strategically S'-closed.

Lemma 2.6. If $\mathbb{P}$ is explicitly S-closed then it is strategically S-closed. □

Proposition 2.3 holds with the weaker hypothesis that $\mathbb{P}$ is strategically S-closed. The proof is modified so that the sequence $\bar{p}$ is part of a play $(\bar{p}, \bar{\varepsilon})$ in which the completeness player follows her winning strategy; the incompleteness player chooses conditions in the desired dense sets, and plays the ordinal $\varepsilon_i = \gamma_i$. The completeness player's response is inside N_{γ_i+1}, so the ordinal she plays is below γ_{i+1}.

Remark 2.7. If $\mathbb{P}$ is strategically S-closed then there is some explicitly S-closed notion of forcing $\mathbb{P}'$ and a complete projection from $\mathbb{P}'$ onto $\mathbb{P}$: fixing a strategy $\mathfrak{s}$ witnessing strategic closure of $\mathbb{P}$, we let $\mathbb{P}'$ consists of all plays in which the completeness player follows $\mathfrak{s}$ and which have a last move, made by the incompleteness player. This gives an alternative proof of Proposition 2.3 when $\mathbb{P}$ is strategically S-closed.

2.2. **Adding sparse sets.**

Definition 2.8. Let $\mathbb{P}$ be a notion of forcing. An *explicit* $\mathbb{P}$*-name for a subset of* λ is a partial map σ from $\mathbb{P}$ to $2^{<\lambda}$ satisfying:

(i) If $p, q \in \operatorname{dom}\sigma$ and q extends p then $\sigma(q) \geqslant \sigma(p)$;
(ii) $\operatorname{dom}\sigma$ is dense in $\mathbb{P}$; in fact, for every $\gamma < \lambda$, the set of conditions $p \in \operatorname{dom}\sigma$ with $|\sigma(p)| \geqslant \gamma$ is dense in $\mathbb{P}$.[1]

That is, we insist that all bounded initial segments of the set named are determined in V. To avoid confusion, we denote by W_σ the actual $\mathbb{P}$-name of the resulting subset of λ: $(p, \alpha) \in W_\sigma$ if $\alpha < |\sigma(p)|$ and $\sigma(p)(\alpha) = 1$. In other words, if $G \subset \mathbb{P}$ is generic, then

$$W_\sigma[G] = \{\alpha < \lambda : (\exists p \in G \cap \operatorname{dom}\sigma)\ \sigma(p)(\alpha) = 1\}.$$

Of course, if $\mathbb{P}$ is $<\lambda$-distributive then every subset of λ in $V(\mathbb{P})$ has an explicit name. We remark that it is not difficult to extend σ to be defined on all of $\mathbb{P}$, but we will naturally work with dense subsets of orderings in a way that makes this formulation more convenient.

Suppose that σ is an explicit $\mathbb{P}$-name for a subset of λ. For an increasing sequence $\bar{p} = \langle p_i \rangle_{i<i^*}$ of conditions from $\operatorname{dom}\sigma$ (where $i^* < \lambda$), we let

$$\sigma(\bar{p}) = \bigcup_{i<i^*} \sigma(p_i).$$

An *exact sparse upper bound* (with respect to σ) for a sequence $\bar{p}$ is an upper bound p of $\bar{p}$ in $\operatorname{dom}\sigma$ such that $\sigma(p) = \sigma(\bar{p})\hat{}0$.

Definition 2.9. Let σ be an explicit $\mathbb{P}$-name for a subset of λ.

(a) A *choice of sparse upper bounds* is a partial function f, defined on a collection of increasing sequences of condtions from $\operatorname{dom}\sigma$, such that for each $\bar{p} \in \operatorname{dom} f$, $f(\bar{p})$ is an exact sparse upper bound of $\bar{p}$.
(b) Two increasing sequences $\bar{p} = \langle p_i \rangle_{i<i^*}$ and $\bar{q} = \langle q_j \rangle_{j<j^*}$ are *co-final* if for all i there is a j such that $p_i \leqslant q_j$ and vice-versa. We say that f is *canonical* if whenever $\bar{p}$ and $\bar{q}$ are co-final and $f(\bar{p})$ is defined, then $f(\bar{q})$ is defined as well, and $f(\bar{q}) = f(\bar{p})$.
(c) We say that an increasing sequence $\bar{p} = \langle p_i \rangle_{i<i^*}$ is a *sparse sequence* (for σ and f) if for all $i < j < i^*$, $|\sigma(p_i)| < |\sigma(p_j)|$, and for all limit $i < i^*$, $p_i = f(\bar{p} \restriction i)$.

[1]Recall that $|\tau| = \operatorname{dom}\tau$ is the length of the string τ.

A sparse sequence $\bar{p} = \langle p_i \rangle_{i \leqslant i^*}$ determines a closed set disjoint from W_σ, namely the set

$$\{|\sigma(\bar{p} \upharpoonright j)| \,:\, j \leqslant i^* \text{ limit}\}\,;$$

this is a set in V, and p_{i^*} forces that this set is disjoint from W_σ (which note is not in V).

Definition 2.10. Let $S \subseteq \lambda$. An *S-sparse $\mathbb{P}$-name* (for a subset of λ) is a pair (σ, f) consisting of an explicit $\mathbb{P}$-name σ for a subset of λ, and a choice f of sparse upper bounds for σ, satisfying:

- For any sparse sequence $\bar{p}$ for σ and f (of limit length), if $|\sigma(\bar{p})| \in S$, then $f(\bar{p})$ is defined.

Notation 2.11. If (σ, f) is a sparse name then we usually denote f by cb_σ (standing for "canonical bound"). We use σ to also denote the pair $(\sigma, \mathrm{cb}_\sigma)$.

Remark 2.12. Suppose that σ is S-sparse. Any increasing ω-sequence $\langle p_n \rangle$ of conditions from $\operatorname{dom} \sigma$ (with $\sigma(p_n)$ strictly increasing) is, vacuously, sparse for σ, and so if $|\sigma(\bar{p})| \in S$ then $\mathrm{cb}_\sigma(\bar{p})$ is defined.

Lemma 2.13. If there is an S-sparse $\mathbb{P}$-name then $\mathbb{P}$ is strategically S-closed.

Proof. A winning strategy for the completeness player ensures that at limit steps, we use the canonical choice of a sparse upper bound, thus ensuring that the sequence of conditions played by the completeness player is sparse. The ordinal played at successor step i is $|\sigma(p_i)|$. □

Lemma 2.14. If σ is an S-sparse $\mathbb{P}$-name, and S' is equivalent to S modulo the club filter on λ, then there is an S'-sparse $\mathbb{P}$-name σ' such that in $V(\mathbb{P})$, $W_\sigma = W_{\sigma'}$.[2]

Proof. We extend the argument of Lemma 2.2. Suppose that $S \cap C = S' \cap C$ for a club C. Determine that $\operatorname{dom} \sigma' = \operatorname{dom} \sigma$. For $p \in \operatorname{dom} \sigma$, let $\alpha = \sup(C \cap |\sigma(p)|)$. If $\alpha = |\sigma(p)|$ then let $\sigma'(p) = \sigma(p)$. Otherwise let $\sigma'(p) = \sigma(p) \upharpoonright (\alpha + 1)$. Suppose that $\bar{p} = \langle p_i \rangle_{i < i^*}$ is an increasing sequence of conditions with $i < j < i^*$ implying $|\sigma'(p_i)| < |\sigma'(p_j)|$. So $C \cap [|\sigma'(p_i)|\,, |\sigma'(p_j)|) \neq \varnothing$. It follows that if i^* is a limit then $\sigma(\bar{p}) = \sigma'(\bar{p})$. We therefore use the same choice of canonical sparse upper bounds; if $\bar{p}$ is a sparse sequence for σ', then it is a sparse sequence for σ. □

Proposition 2.15. *Suppose that $S \subseteq \lambda$ is fat, and that σ is an S-sparse $\mathbb{P}$-name. Then $\mathbb{P}$ is $<\lambda$-distributive, and $S \backslash W_\sigma$ is fat in $V(\mathbb{P})$.*

And as above, the proof also shows that for all stationary $S' \subseteq S \cap \operatorname{Cof}(\aleph_0)$, $S' \backslash W_\sigma$ is stationary in $V(\mathbb{P})$.

Proof. By Lemma 2.13, we know that $\mathbb{P}$ is $<\lambda$-distributive.

To see that $S \backslash W_\sigma$ is fat, we mimic the proof of Proposition 2.3, ensuring that the sequence of conditions $\bar{p}$ is sparse, with $|\sigma(p_{i+1})| > \gamma_i$. At limit steps we take the canonical sparse upper bound. Again $p_{i+1} \in N_{\gamma_i+1}$ forces some $\beta_i > \gamma_i$ into the club C. The limit points of D are disjoint from W_σ. □

Lemma 2.16. Suppose that $\mathbb{P}$ is explicitly S-closed. Then there is an S-sparse $\mathbb{P}$-name for the empty set.

[2]Recall that this means that every condition forces that $W_\sigma = W_{\sigma'}$.

Proof. Let $\delta\colon \mathbb{P} \to \lambda$ show that $\mathbb{P}$ is explicitly S-closed. For $p \in \mathbb{P}$ we let $\sigma(p) = 0^{\delta(p)+1}$, that is, a string of zeros of length $\delta(p)+1$. By well-ordering all (co-finality equivalence classes of) increasing sequences of conditions, we can make a choice of canonical exact upper bounds. ☐

2.3. **Sparseness and iterations: the successor case.**

Definition 2.17. Suppose that $\mathbb{P} \lessdot \mathbb{R}$, $S \subseteq \lambda$, that ρ is an S-sparse $\mathbb{P}$-name and that τ is an S-sparse $\mathbb{R}$-name. We say that τ *coheres* with ρ (and write $\rho \lessdot \tau$) if:

(i) If $p \in \operatorname{dom}\tau$ then $p \restriction \mathbb{P} \in \operatorname{dom}\rho$ and $|\rho(p\restriction\mathbb{P})| = |\tau(p)|$;[3]

(ii) For an increasing sequence $\bar{p}$ from $\operatorname{dom}\tau$, if $\operatorname{cb}_\tau(\bar{p})$ is defined, then $\operatorname{cb}_\rho(\bar{p}\restriction\mathbb{P})$ is defined and equals $\operatorname{cb}_\tau(\bar{p})\restriction\mathbb{P}$.

Note that if $\rho \lessdot \tau$ and $\bar{p}$ is a sparse sequence for τ, then $\bar{p}\restriction\mathbb{P}$ is a sparse sequence for ρ.

Definition 2.18. Let $S \subseteq \lambda$. Suppose that $\mathbb{P}$ is $<\!\lambda$-distributive; suppose that ρ is an S-sparse $\mathbb{P}$-name; suppose that in $V(\mathbb{P})$, $\mathbb{Q}$ is a notion of forcing and σ is an $S\backslash W_\rho$-sparse $\mathbb{Q}$-name. We define $\tau = \rho \vee \sigma$ as follows:

- $\operatorname{dom}\tau \subseteq \mathbb{P}*\mathbb{Q}$ is the collection of $(p,q) \in \mathbb{P}*\mathbb{Q}$ such that $p \in \operatorname{dom}\rho$, and, letting $\alpha = |\rho(p)|$, there is some string $\pi \in 2^{<\lambda}$ (in V) of length α such that $p \Vdash_{\mathbb{P}} q \in \operatorname{dom}\sigma \;\&\; \sigma(q) = \pi$.
- For $(p,q) \in \operatorname{dom}\tau$ we define $\tau(p,q)$ to be the characteristic function of the union of $\rho(p)$ and $\sigma(q)$. That is, if $p \Vdash \sigma(q) = \pi$ (where $|\pi| = \alpha = |\rho(p)|$) then we declare that $|\tau(p,q)| = \alpha$, and for all $\beta < \alpha$, $\tau(p,q)(\beta) = 1$ if and only if $\rho(p)(\beta) = 1$ or $\pi(\beta) = 1$.
- If $(\bar{p},\bar{q})$ is an increasing sequence of conditions from $\operatorname{dom}\tau$, then we define $\operatorname{cb}_\tau(\bar{p},\bar{q}) = (\operatorname{cb}_\rho(\bar{p}), \operatorname{cb}_\sigma(\bar{q}))$. That is, $\operatorname{cb}_\tau(\bar{p},\bar{q})$ is defined to be (p^*,q^*) if $p^* = \operatorname{cb}_\rho(\bar{p})$ is defined, and p^* forces that $q^* = \operatorname{cb}_\sigma(\bar{q})$.[4]

Lemma 2.19. Suppose that the hypotheses of Definition 2.18 hold: that $S \subseteq \lambda$, $\mathbb{P}$ is $<\!\lambda$-distributive, ρ is an S-sparse $\mathbb{P}$-name, and in $V(\mathbb{P})$, $\mathbb{Q}$ is a notion of forcing and that σ is an $S\backslash W_\rho$-sparse $\mathbb{Q}$-name. Suppose further that $S \cap \operatorname{Cof}(\aleph_0)$ is stationary. Then:

- $\rho \vee \sigma$ is an S-sparse $\mathbb{P}*\mathbb{Q}$-name;
- $\rho \lessdot \rho \vee \sigma$; and
- in $V(\mathbb{P}*\mathbb{Q})$, $W_{\rho\vee\sigma} = W_\rho \cup W_\sigma$.

Proof. Let $\tau = \rho \vee \sigma$.

Let us first show that $\operatorname{dom}\tau$ is dense in $\mathbb{P}*\mathbb{Q}$. Given $(p_0,q_0) \in \mathbb{P}*\mathbb{Q}$, since $S \cap \operatorname{Cof}(\aleph_0)$ is stationary, we find an increasing sequence of ordinals $\langle\gamma_n\rangle$ and a λ-filtration $\bar{N}$ such that $N_{\gamma_n} \cap \lambda = \gamma_n$ for all n, $\mathbb{P},\mathbb{Q},\rho,\sigma,p_0,q_0 \in N_0$, and $\gamma_\omega = \sup_n \gamma_n \in S$. We then define an increasing sequence of conditions $\langle p_n,q_n\rangle \in \mathbb{P}*\mathbb{Q}$ such that $(p_n,q_n) \in N_{\gamma_n}$, $p_n \in \operatorname{dom}\rho$, $|\rho(p_n)| \geqslant \gamma_{n-1}$, and p_n forces that $q_n \in \operatorname{dom}\sigma$ and $\sigma(q_n) = \pi_n$ for some string $\pi_n \in V$ with $\gamma_{n-1} \leqslant |\pi_n|$. Here we use that $\mathbb{P}$ does not add sequences of ordinals of length $<\!\lambda$. Of course, since $(p_n,q_n) \in N_{\gamma_n}$, we have $|\rho(p_n)|, |\pi_n| < \gamma_n$.

[3] Note that we do not require that $\rho(p\restriction\mathbb{P}) = \tau(p)$.

[4] The condition $\operatorname{cb}_\tau(\bar{p},\bar{q})$ is unique once we identify conditions (p,q) and (p,q') such that $p \Vdash q = q'$.

Let $\pi_\omega = \bigcup_n \pi_n$. Since $\gamma_\omega \in S$, $p_\omega = \mathrm{cb}_\rho(\langle p_n \rangle)$ is defined; and p_ω forces that $\langle q_n \rangle$ is increasing in $\mathbb{Q}$ and that $\sigma(\bar{q}) = \pi_\omega$ and so has length γ_ω. Also, as $\rho(p_\omega)(\gamma_\omega) = 0$, p_ω forces that $\gamma_\omega \notin W_\rho$; so p_ω forces that $q_\omega = \mathrm{cb}_\sigma(\bar{q})$ is defined. Note that $p_\omega \Vdash_{\mathbb{P}} \sigma(q_\omega) = \pi_\omega\hat{\ }0$. Then $(p_\omega, q_\omega) \in \operatorname{dom}\tau$ and extends (p_0, q_0). It is now not difficult to see that in $V(\mathbb{P}*\mathbb{Q})$, $W_\tau = W_\rho \cup W_\sigma$.

Next, we observe that if defined, $\mathrm{cb}_\tau(\bar{p}, \bar{q})$ is in $\operatorname{dom}\tau$, and $\tau(\mathrm{cb}_\tau(\bar{p}, \bar{q})) = \tau(\bar{p}, \bar{q})\hat{\ }0$; this is because $\rho(\mathrm{cb}_\rho(\bar{p})) = \rho(\bar{p})\hat{\ }0$ and $\mathrm{cb}_\rho(\bar{p})$ forces that $\sigma(\mathrm{cb}_\sigma(\bar{q})) = \sigma(\bar{q})\hat{\ }0$. We also observe that cb_τ is canonical. Finally, suppose that $(\bar{p}, \bar{q})$ is sparse for τ, with $\alpha = |\tau(\bar{p}, \bar{q})| \in S$. Then $\bar{p}$ is sparse for ρ, and $|\rho(\bar{p})| = \alpha$; let $p^* = \mathrm{cb}_\rho(\bar{p})$. Then, p^* forces that $\bar{q}$ is sparse for σ, and that $|\sigma(\bar{q})| = \alpha \in S \backslash W_\rho$; so forces that $\mathrm{cb}_\sigma(\bar{q})$ is defined, whence $\mathrm{cb}_\tau(\bar{p}, \bar{q})$ is defined. It follows that τ is S-sparse, and that $\rho \lessdot \tau$. $\square$

Lemma 2.16 yields:

Corollary 2.20. *Let $S \subseteq \mathrm{Cof}_\lambda(\aleph_0)$ be stationary. Suppose that ρ is an S-sparse $\mathbb{P}$-name; suppose that in $V(\mathbb{P})$, $\mathbb{Q}$ is a notion of forcing which is explicitly $S \backslash W_\rho$-closed. Then there is an S-sparse $\mathbb{P}*\mathbb{Q}$-name τ for W_ρ, which coheres with ρ.*

2.4. **Sparseness and iterations: the limit case.**

Definition 2.21. Let $\bar{\mathbb{P}} = \langle \mathbb{P}_i \rangle$ be an iteration. A *coherent system of S-sparse names for* $\bar{\mathbb{P}}$ is a sequence $\bar{\tau} = \langle \tau_i \rangle$ such that each τ_i is an S-sparse $\mathbb{P}_i$-name, and for $i < j < |\bar{\mathbb{P}}|$, $\tau_i \lessdot \tau_j$.

Definition 2.22. Let $S \subseteq \lambda$. Suppose that $\theta < \lambda$ is regular, that $\bar{\mathbb{P}} = \langle \mathbb{P}_i \rangle_{i \leqslant \theta}$ is an iteration with full support (a directed system with inverse limits), and that $\bar{\tau} = \langle \tau_i \rangle_{i < \theta}$ is a coherent system of S-sparse names for $\bar{\mathbb{P}} \restriction \theta$. We define $\tau_\theta = \bigvee \bar{\tau}$ as follows. First,

- $p \in \operatorname{dom}\tau_\theta$ if for all $i < \theta$, $p \restriction i \in \operatorname{dom}\tau_i$.

Suppose that $p \in \operatorname{dom}\tau_\theta$. For $i < j < \theta$, because $\tau_i \lessdot \tau_j$, we have $|\tau_i(p \restriction i)| = |\tau_j(p \restriction j)|$.

- For $p \in \operatorname{dom}\tau_\theta$, we let $\tau_\theta(p)$ be the characteristic function of the union of $\tau_i(p \restriction i)$. That is, if $\alpha = |\tau_i(p \restriction i)|$ for all $i < \theta$, then we declare that $|\tau_\theta(p)| = \alpha$, and for $\beta < \alpha$, $\tau_\theta(p)(\beta) = 1$ if and only if there is some $i < \theta$ such that $\tau_i(p \restriction i)(\beta) = 1$.
- For an increasing sequence of conditions $\bar{p}$ from $\operatorname{dom}\tau_\theta$, we let $q = \mathrm{cb}_{\tau_\theta}(\bar{p})$ if for all $i < \theta$, $q \restriction i = \mathrm{cb}_{\tau_i}(\bar{p} \restriction i)$.

Lemma 2.23. Suppose that the hypotheses of Definition 2.22 hold: $S \subseteq \lambda$; $\theta < \lambda$ is regular; $\bar{\mathbb{P}} = \langle \mathbb{P}_i \rangle_{i \leqslant \theta}$ is an iteration with full support, and $\bar{\tau} = \langle \tau_i \rangle_{i < \theta}$ is a coherent system of S-sparse names for $\bar{\mathbb{P}} \restriction \theta$. Suppose further that S is fat.

Then:

- $\tau_\theta = \bigvee \bar{\tau}$ is an S-sparse $\mathbb{P}_\theta$-name;
- for all $i < \theta$, $\tau_i \lessdot \tau_\theta$; and
- in $V(\mathbb{P}_\theta)$, $W_{\tau_\theta} = \bigcup_{i<\theta} W_{\tau_i}$.

Proof. We show that $\operatorname{dom}\tau_\theta$ is dense in $\mathbb{P}_\theta$. Let $r \in \mathbb{P}_\theta$. Obtain a λ-filtration $\bar{N}$ and a set $D = \{\gamma_i : i \leqslant \theta\}$ given by Lemma 1.1, with $S, r, \bar{\mathbb{P}}, \bar{\tau} \in N_0$ (and $\theta \subset N_0$). We build a sequence $\bar{p} = \langle p_i \rangle_{i \leqslant \theta}$ of conditions with the following properties:

(i) $p_i \in \mathbb{P}_i$;

(ii) p_i extends $r \restriction i$;
(iii) If i is a successor then $p_i \in \operatorname{dom} \tau_i$ and $|\tau_i(p_i)| > \gamma_{i-1}$;
(iv) for all $k < i$, $\langle p_j \restriction k \rangle_{j \in (k,i)}$ is a sparse sequence for τ_k.

Note that (iv) implies that for all $j < i < \theta$, $p_i \restriction j \in \operatorname{dom} \tau_j$. We do not however assume that for limit i, $p_i \in \operatorname{dom} \tau_i$.

We start with p_0 being the empty condition. Given p_i we find an extension $p_{i+1} \in \operatorname{dom} \tau_{i+1}$ which also extends $r \restriction (i+1)$. By extending, we can make $|\tau_{i+1}(p_{i+1})| \geqslant \gamma_i$.

Now suppose that $i \leqslant \theta$ is a limit ordinal. As we argued in the proof of Proposition 2.3, for all $j < i$, $p_{j+1} \in N_{\gamma_j+1}$, and so $|\tau_{j+1}(p_{j+1})| < \gamma_{j+1}$. Let $\alpha_i = \sup_{j<i} \gamma_j$; so $\alpha_i \in S$ (recall that the sequence $\langle \gamma_i \rangle$ need not be continuous). Now (iv) above holds for i by induction; for all $k < i$, $|\tau_k(\langle p_j \restriction k \rangle_{j \in (k,i)})| = \alpha_i$. It follows that for all $k < i$, $q_k = q^i_k = \operatorname{cb}_{\tau_k}(\langle p_j \restriction k \rangle_{j \in (k,i)})$ is defined. Further, if $k < k' < i$ then as $\operatorname{cb}_{\tau_k}$ is canonical, $q_k = \operatorname{cb}_{\tau_k}(\langle p_j \restriction k \rangle_{j \in (k',i)})$. By coherence, $q_k = q_{k'} \restriction k$. Since $\mathbb{P}_i$ is the inverse limit of $\bar{\mathbb{P}} \restriction i$, we let $p_i \in \mathbb{P}_i$ be the inverse limit of the sequence $\langle q_k \rangle_{k<i}$, that is, for all $k < i$, $q_k = p_i \restriction k$. Note that (iv) now holds for $i+1$. Also note that for all $k < i$, q_k extends p_k which extends $r \restriction k$; it follows that p_i extends $r \restriction i$. At step θ we get $p_\theta \in \operatorname{dom} \tau_\theta$ and extending r.

If $p, q \in \operatorname{dom} \tau_\theta$ and q extends p, then for all $i < \theta$, $\tau_i(p \restriction i) \preccurlyeq \tau_i(q \restriction i)$, so $\tau_\theta(p) \preccurlyeq \tau_\theta(q)$. In $V(\mathbb{P}_\theta)$, $W_{\tau_\theta} = \bigcup_i W_{\tau_i}$.

Let $\bar{p}$ be an increasing sequence from $\operatorname{dom} \tau_\theta$, and suppose that $p^* = \operatorname{cb}_{\tau_\theta}(\bar{p})$ is defined. By definition, $p^* \in \operatorname{dom} \tau_\theta$. Let $\alpha = |\tau_\theta(\bar{p})|$. For all $i < \theta$, $|\tau_i(\bar{p} \restriction i)| = \alpha$ and $\tau_i(p^* \restriction i)(\alpha) = 0$, so by our definition, $\tau_\theta(p^*) = \tau_\theta(\bar{p})\hat{\ }0$. It is also easy to see that cb_θ is canonical.

Suppose that $\bar{p}$ is a sparse sequence for τ_θ, and that $\alpha = |\tau_\theta(\bar{p})| \in S$. For all $i < \theta$, $\bar{p} \restriction i$ is sparse for τ_i, and so $q_i = \operatorname{cb}_{\tau_i}(\bar{p} \restriction i)$ is defined; and as above, for $i < i' < \theta$, $q_i = q_{i'} \restriction i$. Then the inverse limit of $\langle q_i \rangle$ equals $\operatorname{cb}_{\tau_\theta}(\bar{p})$.

We see that τ_θ is S-sparse and that $\tau_i \lessdot \tau_\theta$ for all $i < \theta$. □

For the next definition and lemma, note that if τ is an S-sparse $\mathbb{P}$-name, and $A \subseteq \operatorname{dom} \tau$ is a dense final segment of $\operatorname{dom} \tau$, then $\tau \restriction A$ is also an S-sparse $\mathbb{P}$-name, and in $V(\mathbb{P})$, $W_\tau = W_{\tau \restriction A}$.

Definition 2.24. Let $S \subseteq \lambda$. Suppose that $\bar{\mathbb{P}} = \langle \mathbb{P}_i \rangle_{i \leqslant \lambda}$ is an iteration with inverse limits below λ and a direct limit at λ; suppose that $\bar{\tau} = \langle \tau_i \rangle_{i<\lambda}$ is a coherent system of S-sparse names for $\bar{\mathbb{P}} \restriction \lambda$. We define $\tau_\lambda = \bigvee \bar{\tau}$ as follows. First,

- we let $\operatorname{dom} \tau_\lambda$ be the collection of $p \in \mathbb{P}_\lambda$ such that for some limit $\alpha < \lambda$, $p \in \mathbb{P}_\alpha$, and for all $\beta < \alpha$, $p \restriction \beta \in \operatorname{dom} \tau_\beta$ and $|\tau_\beta(p \restriction \beta)| = \alpha + 1$.

For all $p \in \operatorname{dom} \tau_\lambda$ there is a unique α witnessing this fact; α is determined by $|\tau_\gamma(p \restriction \gamma)|$ for any $\gamma < \alpha$.

- For $p \in \operatorname{dom} \tau_\lambda$, witnessed by α, we declare that $|\tau_\lambda(p)| = \alpha + 1$ and that for $\beta \leqslant \alpha$, $\tau_\lambda(p)(\beta) = 1$ if and only if $\tau_\gamma(p \restriction \gamma)(\beta) = 1$ for some $\gamma < \beta$. That is, $\tau_\lambda(p) = \nabla_{\gamma<\alpha} \tau_\gamma(p \restriction \gamma)$.
- Suppose that $\bar{p}$ is an increasing sequence from $\operatorname{dom} \tau_\lambda$. Let $\alpha = |\tau_\lambda(\bar{p})|$. We let $\operatorname{cb}_{\tau_\lambda}(\bar{p}) = p^*$ if $p^* \in \mathbb{P}_\alpha$ and for all $\beta < \alpha$, $p^* \restriction \beta$ is $\operatorname{cb}_{\tau_\beta}$ of a tail of $\bar{p} \restriction \beta$.

Lemma 2.25. Suppose that the hypotheses of Definition 2.24 hold: $S \subseteq \lambda$; $\bar{\mathbb{P}} = \langle \mathbb{P}_i \rangle_{i \leqslant \lambda}$ is a $<\lambda$-support iteration; $\bar{\tau}$ is a coherent system of S-sparse names for $\bar{\mathbb{P}} \restriction \lambda$. Suppose further that $S \cap \mathrm{Cof}(\aleph_0)$ is stationary in λ.

Then:

- $\tau_\lambda = \bigvee \bar{\tau}$ is an S-sparse $\mathbb{P}_\lambda$-name;
- In $V(\mathbb{P}_\lambda)$, $W_{\tau_\lambda} = \nabla_{i<\lambda} W_{\tau_i}$; and

letting

$$C_\alpha = \{p \in \operatorname{dom} \tau_\lambda \,:\, |\tau_\lambda(p)| > \alpha + 1\}\,,$$

- for all $\alpha < \lambda$, $\tau_\alpha \lessdot \tau_\lambda \restriction C_\alpha$.

Note that indeed each C_α is a dense final segment of $\operatorname{dom} \tau_\lambda$.

Proof. We show that $\operatorname{dom} \tau_\lambda$ is dense in $\mathbb{P}_\lambda$. Let $p_0 \in \mathbb{P}_\lambda$. Obtain an increasing sequence $\langle \gamma_n \rangle$ with $\gamma_\omega = \sup_n \gamma_n \in S$, and models N_{γ_n} with $N_{\gamma_n} \cap \lambda = \gamma_n$, and all relevant information, including p_0, is in N_{γ_0}. We then define an increasing sequence $\langle p_n \rangle$ with $p_n \in N_{\gamma_n}$ and for some $\alpha_n \in [\gamma_{n-1}, \gamma_n)$, $p_n \in \operatorname{dom} \tau_{\alpha_n}$, and $|\tau_{\alpha_n}(p_n)| \geqslant \alpha_n$. As usual, $|\tau_{\alpha_n}(p_n)| < \gamma_n$. Now for all $\beta < \gamma_\omega$, for all but finitely many n (say for all $n \geqslant n_\beta$), $p_n \restriction \beta \in \operatorname{dom} \tau_\beta$; since $|\tau_\beta(\langle p_n \restriction \beta \rangle_{n \geqslant n_\beta})| = \gamma_\omega \in S$, $q_\beta = \mathrm{cb}_{\tau_\beta} \langle p_n \restriction \beta \rangle_{n \geqslant n_\beta}$ is defined. As above, this value does not change if we take a tail of the sequence, so for $\beta < \alpha < \gamma_\omega$ we have $q_\beta = q_\alpha \restriction \beta$; so the inverse limit of the sequence $\langle q_\beta \rangle$ is in $\operatorname{dom} \tau_\lambda$ (note that $|\tau_\beta(q_\beta)| = \gamma_\omega + 1$).

Now for all $\alpha < \lambda$, $p \restriction \alpha \in \operatorname{dom} \tau_\alpha$ for all $p \in C_\alpha$. Further, by definition of τ_λ, for $p \in C_\alpha$ we have $|\tau_\lambda(p)| = |\tau_\alpha(p \restriction \alpha)|$.

Suppose that $p, q \in \operatorname{dom} \tau_\lambda$ and that q extends p; say $|\tau_\lambda(p)| = \alpha + 1$ and $|\tau_\lambda(q)| = \beta + 1$. For any $\gamma < \alpha, \beta$, we have $p \restriction \gamma \leqslant q \restriction \gamma$ and so $\tau_\gamma(p \restriction \gamma) \preccurlyeq \tau_\gamma(q \restriction \gamma)$; and $\alpha + 1 = |\tau_\gamma(p \restriction \gamma)|$, $\beta + 1 = |\tau_\gamma(q \restriction \gamma)|$. Hence $\alpha \leqslant \beta$.

It is then not difficult to see that $\tau_\lambda(p) \preccurlyeq \tau_\lambda(q)$: to determine the value on $\gamma \leqslant \alpha$ we note that for all $\delta < \gamma$, $\tau_\delta(p \restriction \delta)$ and $\tau_\delta(q \restriction \delta)$ agree on γ. It follows that τ_λ is an explicit $\mathbb{P}_\lambda$-name for $\nabla_{i<\lambda} W_{\tau_i}$, and that each C_α is a final segment of $\operatorname{dom} \tau_\lambda$.

Suppose that $\bar{p}$ is an increasing sequence from $\operatorname{dom} \tau_\lambda$, and let $\alpha = |\tau_\lambda(\bar{p})|$. Suppose that $p^* = \mathrm{cb}_{\tau_\lambda}(\bar{p})$ is defined. Let $\beta < \alpha$. Then $|\tau_\beta(p^* \restriction \beta)| = \alpha + 1$ and $\tau_\beta(p^* \restriction \beta)(\alpha) = 0$. by definition, $p^* \in \operatorname{dom} \tau_\lambda$, and $\tau_\lambda(p^*)(\alpha) = 0$. Also, $\mathrm{cb}_{\tau_\lambda}$ is canonical.

Suppose that $\bar{p} = \langle p_i \rangle_{i<i^*}$ is a sparse sequence for σ_λ. Let $\alpha = |\tau_\lambda(\bar{p})|$, and suppose that $\alpha \in S$. For $\beta < \alpha$ let $i(\beta)$ be the least i such that $|\tau_\lambda(p_i)| > \beta + 1$. Then $\langle p_i \rangle_{i \in [i(\beta), i^*)}$ is a sequence from C_β, and $\langle p_i \restriction \beta \rangle_{i \in [i(\beta), i^*)}$ is sparse for τ_β. As usual, since $\mathbb{P}_\alpha$ is an inverse limit we get $q \in \mathbb{P}_\alpha$ such that for all $\beta < \alpha$, $q \restriction \beta = \mathrm{cb}_{\tau_\beta}(\langle p_i \restriction \beta \rangle_{i \in [i(\beta), i^*)})$; $q = \mathrm{cb}_{\tau_\lambda}(\bar{p})$.

This argument shows that τ_λ is S-sparse and that $\tau_\alpha \lessdot \tau_\lambda \restriction C_\alpha$ for all α. $\square$

Note that the sets C_α have a continuity property: suppose that $\bar{p} = \langle p_i \rangle$ is an increasing sequence (of limit length) from $\operatorname{dom} \tau_\lambda$ and $q = \mathrm{cb}_{\tau_\lambda}(\bar{p})$ is defined. For $\alpha < \lambda$, if for all i, $p_i \notin C_\alpha$, then $q \notin C_\alpha$. Also note that for all $p \in \operatorname{dom} \tau_\lambda$, $p \in C_\alpha$ for fewer than λ many α.

2.5. **Named λ-iterations.**

Definition 2.26. We say that $\langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$, where $\xi \leqslant \lambda^+$, is a *named λ-iteration*, if there is a sequence $\langle \mathbb{Q}_\zeta, \sigma_\zeta \rangle_{\zeta < \xi}$ such that:

(i) $\langle \mathbb{P}_\zeta, \mathbb{Q}_\zeta \rangle$ is a $<\lambda$-support iteration;
(ii) For all $\zeta < \xi$, $\mathbb{P}_\zeta$ is $<\lambda$-distributive;
(iii) For all $\zeta < \xi$, $\sigma_\zeta \in V(\mathbb{P}_\zeta)$ is an explicit $\mathbb{Q}_\zeta$-name for a subset of λ.

Note that we are not requiring that $\mathbb{P}_\xi$ is $<\lambda$-distributive; under further assumptions, this will follow, as we shall shortly see. We require that $\mathbb{P}_\zeta$ is $<\lambda$-distributive for $\zeta < \lambda$ so that λ is regular in $V(\mathbb{P}_\zeta)$, so that the notion of σ_ζ being an explicit $\mathbb{Q}_\zeta$-name for a subset of λ (and later, a sparseness requirement) makes sense. When ξ is a limit ordinal, we also refer to the restriction $\mathbb{P} \upharpoonright \xi$ as a named λ-iteration.

If $\langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$ is a named λ-iteration, then we let, for all $\zeta \leqslant \xi$, $\zeta < \lambda^+$, in $V(\mathbb{P}_\zeta)$, $\mathbf{u}_\zeta \in \mathcal{P}(\lambda)/\mathrm{NS}_\lambda$ be the least upper bound of

$$\{W_{\sigma_\upsilon}/\mathrm{NS}_\lambda \,:\, \upsilon < \zeta\}.$$

Definition 2.27. Let $S \subseteq \lambda$. We say that a named λ-iteration $\langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$ is *S-sparse* if for every $\zeta < \xi$, in $V(\mathbb{P}_\zeta)$, σ_ζ is $S \backslash \mathbf{u}_\zeta$-sparse.[5]

Our aim is to show that when S is fat, if $\langle \mathbb{P}_\zeta \rangle$ is an S-sparse iteration, then for each $\zeta \leqslant \xi$, $\zeta < \lambda^+$ there is an S-sparse $\mathbb{P}_\zeta$-name τ_ζ such that (in $V(\mathbb{P}_\zeta)$) $W_{\tau_\zeta}/\mathrm{NS}_\lambda = \mathbf{u}_\zeta$. However, because we may have $\xi > \lambda$, we will not be able to get a coherent sequence $\bar{\tau}$ of names. We need a weak notion of coherence:

Definition 2.28. Suppose that $\bar{\mathbb{P}} = \langle \mathbb{P}_\zeta \rangle_{\zeta < \xi}$ is an iteration, and that $\bar{\tau} = \langle \tau_\zeta \rangle_{\zeta < \xi}$ is a sequence such that for all $\zeta < \xi$, τ_ζ is an explicit $\mathbb{P}_\zeta$-name for a subset of λ. We say that $\bar{\tau}$ is *weakly coherent* if for $\upsilon \leqslant \zeta < \xi$ there are sets A^ζ_υ satisfying:

(a) Each A^ζ_υ is a dense final segment of $\operatorname{dom} \tau_\zeta$;
(b) For all $\upsilon \leqslant \zeta < \xi$, $\tau_\upsilon \lessdot \tau_\zeta \upharpoonright A^\zeta_\upsilon$;
(c) $A^\zeta_\zeta = \operatorname{dom} \tau_\zeta$;
(d) For $\alpha \leqslant \beta \leqslant \gamma < \xi$, $p \in A^\gamma_\alpha \cap A^\gamma_\beta$ implies $p \upharpoonright \beta \in A^\beta_\alpha$;
(e) For $\alpha \leqslant \beta \leqslant \gamma < \xi$, $p \in A^\gamma_\beta$ and $p \upharpoonright \beta \in A^\beta_\alpha$ implies $p \in A^\gamma_\alpha$;
(f) For all $\zeta < \xi$ and every $p \in \operatorname{dom} \tau_\zeta$, there are $<\lambda$ many $\upsilon < \zeta$ such that $p \in A^\zeta_\upsilon$;
(g) For $\upsilon < \zeta$, if $\bar{p} = \langle p_i \rangle$ is an increasing sequence from $\operatorname{dom} \tau_\zeta$, $p = \mathrm{cb}_{\tau_\zeta}(\bar{p})$ is defined, and for all i, $p_i \notin A^\zeta_\upsilon$, then $p \notin A^\zeta_\upsilon$.

We will show that we can construct a weakly coherent sequence of names as required. We will use more properties of the sequence, which we incorporate into the following definition.

Definition 2.29. Let $\langle \mathbb{P}_\zeta \rangle_{\zeta < \xi}$ be an S-sparse iteration. An *associated sequence* is a sequence $\bar{\tau} = \langle \tau_\zeta \rangle_{\zeta \in [1,\xi)}$ such that:

(1) Each τ_ζ is an S-sparse $\mathbb{P}_\zeta$-name;
(2) In $V(\mathbb{P}_\zeta)$, $W_{\tau_\zeta}/\mathrm{NS}_\lambda = \mathbf{u}_\zeta$;
(3) The sequence $\bar{\tau}$ is weakly coherent;
(4) For $p \in \operatorname{dom} \tau_\zeta$, for all $\upsilon < \zeta$, $p \in A^\zeta_{\upsilon+1}$ if and only if $\upsilon \in \operatorname{supp}(p)$, in which case there is a string $\pi_\upsilon \in 2^{<\lambda}$ of length $|\tau_\zeta(p)|$ such that $p \upharpoonright \upsilon \Vdash_{\mathbb{P}_\upsilon} p(\upsilon) \in \operatorname{dom} \sigma_\zeta \;\&\; \sigma_\upsilon(p(\upsilon)) = \pi_\upsilon$;

We write $\sigma_\upsilon(p(\upsilon))$ for the string π_υ;

[5]We mean that it is $S \backslash U$-sparse for some U such that $U/\mathrm{NS}_\lambda = \mathbf{u}_\zeta$. By Lemma 2.14, up to a slight modification of σ_ζ, the choice of U does not matter.

(5) If $p, q \in \operatorname{dom} \tau_\zeta$, q extends p, and $\upsilon \in \operatorname{supp}(p)$, then $\sigma_\upsilon(q(\upsilon)) \setminus |\tau_\zeta(p)| \subseteq \tau_\zeta(q)$;[6]
(6) If $p \in \operatorname{dom} \tau_\zeta$, $\alpha < |\tau_\zeta(p)|$, and for all $\upsilon \in \operatorname{supp}(p)$, $\sigma_\upsilon(p(\upsilon))(\alpha) = 0$, then $\tau_\zeta(p)(\alpha) = 0$;
(7) If $\bar{p} = \langle p_i \rangle$ is an increasing sequence from $\operatorname{dom} \tau_\zeta$ and $p = \operatorname{cb}_{\tau_\zeta}(\bar{p})$ is defined, then $\operatorname{supp}(p) = \bigcup_i \operatorname{supp}(p_i)$;
(8) If $\bar{p} = \langle p_i \rangle$ is an increasing sequence from $\operatorname{dom} \tau_\zeta$ and $p = \operatorname{cb}_{\tau_\zeta}(\bar{p})$ is defined, then for all $\upsilon \in \operatorname{supp}(p)$, $p \restriction \upsilon$ forces that $p(\upsilon)$ is $\operatorname{cb}_{\sigma_\upsilon}$ of a tail of $\langle p_i(\upsilon) \rangle$.

Notation 2.30. If $\bar{\mathbb{P}}$ is an S-sparse iteration, and $\bar{\tau}$ is an associated sequence, then we write W_ζ for W_{σ_ζ} and U_ζ for W_{τ_ζ}.

Proposition 2.31. *Let $S \subseteq \lambda$ be fat. Suppose that $\langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$ is an S-sparse named λ-iteration, with $\xi < \lambda^+$, and that $\bar{\tau} = \langle \tau_\zeta \rangle_{\zeta \in [1,\xi)}$ is an associated sequence for $\bar{\mathbb{P}} \restriction \xi$.*

Then there is some τ_ξ such that $\bar{\tau}\hat{\ }\tau_\xi$ is an associated sequence for $\bar{\mathbb{P}}$.

In particular, $\mathbb{P}_\xi$ is $<\lambda$-distributive.

Corollary 2.32. *If S is fat, $\xi \leqslant \lambda^+$, and $\bar{\mathbb{P}} = \langle \mathbb{P}_\zeta \rangle_{\zeta < \xi}$ is an S-sparse iteration, then $\bar{\mathbb{P}}$ has an associated sequence $\bar{\tau}$.*

Proof of Proposition 2.31. The definition of τ_ξ is of course by cases.

Case I: $\xi = 1$. We let $\tau_1 = \sigma_0$ (recall that $\mathbb{P}_1 = \mathbb{Q}_0$).

Case II: $\xi = \vartheta + 1$, $\vartheta > 0$. We apply Lemma 2.19 to $\mathbb{P} = \mathbb{P}_\vartheta$, $\rho = \tau_\vartheta$, $\mathbb{Q} = \mathbb{Q}_\vartheta$, and σ being some mild variation of σ_ϑ which is sparsely $S \setminus U_\vartheta$-closed; and let τ_ξ be the τ obtained. For $\zeta \leqslant \vartheta$, we let $A^\xi_\zeta = \{p \in \operatorname{dom} \tau_\xi : p \restriction \vartheta \in A^\vartheta_\zeta\}$; note that $A^\xi_\vartheta = \operatorname{dom} \tau_\xi$.

For (2) of Definition 2.29, note that $\mathbf{u}_\xi = \mathbf{u}_\vartheta \vee W_\vartheta/\mathrm{NS}_\lambda$, and $U_\xi = U_\vartheta \cup W_\vartheta$. (3) is not difficult. For (4), note that for all $p \in \operatorname{dom} \tau_\xi$, $\vartheta \in \operatorname{supp}(p)$ and $p \restriction \vartheta \Vdash p(\vartheta) = \pi$ for some π of length $|\tau_\xi(p)| = |\tau_\vartheta(p \restriction \vartheta)|$. (5) and (6) follow from $\tau_\xi(q) = \tau_\vartheta(p \restriction \vartheta) \cup \sigma_\vartheta(p(\vartheta))$. (7) follows from the fact that $\operatorname{cb}_{\tau_\vartheta}(\bar{p} \restriction \vartheta) = p \restriction \vartheta$, and that $\vartheta \in \operatorname{supp}(q)$ for all $q \in \operatorname{dom} \tau_\xi$. (8) follows similarly, noting that Definition 2.18 implies that it holds for $\upsilon = \vartheta$.

Case III: ξ is a limit ordinal and $\operatorname{cf}(\xi) < \lambda$. For all $\zeta < \xi$, $\mathbb{P}_\zeta$ is $<\lambda$-distributive. Then for all $J \subseteq \zeta$, if $|J| < \lambda$ then

$$A^\zeta_J = \bigcap_{\upsilon \in J} A^\zeta_\upsilon$$

is a dense final segment of $\operatorname{dom} \tau_\zeta$; for all $\upsilon \in J$, $\tau_\upsilon \lessdot \tau_\zeta \restriction A^\zeta_J$. Further, for $J \subseteq \xi$ with $|J| < \lambda$, the sequence

$$\left\langle \tau_\zeta \restriction A^\zeta_{J \cap \zeta} \right\rangle_{\zeta \in J}$$

is coherent; this follows from condition (d) of Definition 2.28.

We fix a closed, unbounded $J \subseteq \zeta$ of order-type $\theta = \operatorname{cf}(\zeta)$, and apply Lemma 2.23 to the sequence $\langle \tau_\zeta \restriction A^\zeta_{J \cap \zeta} \rangle_{\zeta \in J}$; notice that for limit points δ of J, as $\operatorname{cf}(\delta) < \lambda$, $\mathbb{P}_\delta$ is the inverse limit of $\langle \mathbb{P}_\upsilon \rangle_{\upsilon \in J \cap \delta}$. We let τ_ξ be the τ obtained. So for all $\delta \in J$, $\tau_\delta \restriction A^\delta_{J \cap \delta} \lessdot \tau_\xi$.

[6] Again, recall that we identify sets and characteristic functions; so this means: for all α with $|\tau_\zeta(p)| \leqslant \alpha < |\tau_\zeta(q)|$, if $\sigma_\upsilon(q(\upsilon))(\alpha) = 1$ then $\tau_\zeta(q)(\alpha) = 1$.

For (2), we use $U_\xi = \bigcup_{\delta \in J} U_\delta$, and the fact that J is cofinal in ξ. For (3), we notice that for all $\gamma < \xi$, for $p \in \operatorname{dom} \tau_\xi$, there is some $\delta \in J$, $\delta \geqslant \gamma$ such that $p \restriction \delta \in A^\delta_\gamma$ if and only if for all $\delta \in J$ such that $\delta \geqslant \gamma$, $p \restriction \delta \in A^\delta_\gamma$; we use either (d) or (e) of Definition 2.28; in either case we note that for all $\delta < \varepsilon$ from J, for all $p \in \operatorname{dom} \tau_\xi$, $p \restriction \varepsilon \in A^\varepsilon_\delta$. We let $p \in A^\xi_\gamma$ if these equivalent conditions hold. Note that $A^\xi_\delta = \operatorname{dom} \tau_\xi$ for all $\delta \in J$. For (f), we use that $\bar{\tau}$ is an associated sequence, and $|J| < \lambda$. (5) and (6) of Definition 2.29 follow from $\tau_\xi(q) = \bigcup_{\delta \in J} \tau_\delta(p \restriction \delta)$. (7) follows from the fact that if $p = \operatorname{cb}_{\tau_\xi}(\bar{p})$ then for all $\delta \in J$, $p \restriction \delta = \operatorname{cb}_{\tau_\delta}(\bar{p} \restriction \delta)$, and of course $\operatorname{supp}(q) = \bigcup_{\delta \in J} \operatorname{supp}(q \restriction \delta)$ for all $q \in \mathbb{P}_\xi$. (8) follows from (7) and our assumptions on $\bar{\tau}$.

Case IV: $\operatorname{cf}(\xi) = \lambda$. The construction is similar to case (III). We fix $J \subseteq \xi$ closed and unbounded of order-type λ; let $\langle \delta_i \rangle_{i<\lambda}$ be the increasing enumeration of J. We again have that $\langle \tau_\delta \restriction A^\delta_{J \cap \delta} \rangle_{\delta \in J}$ is a coherent system, with $\mathbb{P}_\delta$ being an inverse limit of $\langle \mathbb{P}_\upsilon \rangle_{\upsilon \in J \cap \delta}$ for all limit points δ of J, whereas this time, $\mathbb{P}_\xi$ is the direct limit of $\langle \mathbb{P}_\delta \rangle_{\delta \in J}$. So this time we apply Lemma 2.25 to get τ_ξ. In this case, for $\delta = \delta_i \in J$ we obtain the sets

$$C_\delta = \{p \in \operatorname{dom} \tau_\xi \ : \ |\tau_\xi(p)| > i + 1\}$$

So if $|\tau_\xi(p)| = j + 1$ then $p \in C_{\delta_i} \iff i < j$, and $\tau_\delta \restriction A^\delta_{J \cap \delta} \lessdot \tau_\xi \restriction C_\delta$ for all $\delta \in J$.

For (2), we use the fact that $U_\xi = \nabla U_{\delta_i}$ so $\mathbf{u}_\xi = \sup_{\delta \in J} \mathbf{u}_\delta$. Toward (3), let $\gamma < \xi$, let $p \in \operatorname{dom} \tau_\xi$, and let $|\tau_\xi(p)| = j + 1$. If $\gamma < \delta_j$, then the following are equivalent: $p \restriction \delta \in A^\delta_\gamma$ for some $\delta \in J \cap [\gamma, \delta_j)$; and $p \restriction \delta \in A^\delta_\gamma$ for all $\delta \in J \cap [\gamma, \delta_j)$. We let $p \in A^\xi_\gamma$ if these equivalent conditions hold. If $\gamma \geqslant \delta_j$ then $p \notin A^\xi_\gamma$. The argument is then the same as in case (III), restricting to $\delta < \delta_j$; (f) follows from $j < \lambda$. For (g), letting $j + 1 = |\tau_\xi(p)|$, assuming that $\upsilon < \delta_j$, we fix some $\delta \in J \cap [\gamma, \delta_j)$; for a tail of $\bar{p}$ we have $p_i \in C_\delta$; we use the fact that $\operatorname{cb}_{\tau_\xi}(\bar{p})$ does not change when restricting to this tail.

For (4), if $|\tau_\xi(p)| = j + 1$ then by the construction in Lemma 2.25, $p \in \mathbb{P}_{\delta_j}$, in other words $\operatorname{supp}(p) \subseteq \delta_j$. Also $p \in A^\xi_\upsilon$ implies $\upsilon < \delta_j$. For $\upsilon < \delta_j$, we find $\delta \in J \cap (\upsilon, \delta_j)$, and use (4) for τ_δ.

For (5) and (6) we use the definition of $\tau_\xi(p)$: if $|\tau_\xi(p)| = j+1$ then $\tau_\xi(p)(\beta) = 1$ if and only if $\tau_{\delta_i}(p \restriction \delta_i)(\beta) = 1$ for some $i < \beta$. If $i < \beta$ and for all $\upsilon \in \operatorname{supp}(p \restriction \delta_i)$, $\sigma_\upsilon(p(\upsilon))(\beta) = 0$, then $\tau_{\delta_i}(p \restriction \delta_i)(\beta) = 0$. On the other hand, suppose that q extends p, $\nu \in \operatorname{supp}(p)$, that $\beta \in [|\tau_\xi(p)|, |\tau_\xi(q)|)$, and that $\sigma_\upsilon(q)(\beta) = 1$. Let $j + 1 = |\tau_\xi(p)|$. Since $\upsilon \in \operatorname{supp}(p)$, $\upsilon < \delta_j$. Choose $\delta \in J \cap [\upsilon, \delta_j)$. Then $p \restriction \delta, q \restriction \delta \in \operatorname{dom} \tau_\delta$, so by assumption on $\bar{\tau}$, $\tau_\delta(q \restriction \delta)(\beta) = 1$. Since $\beta \geqslant \delta_j > \delta$, by definition, $\tau_\xi(q)(\beta) = 1$.

Finally, for (7), let $p = \operatorname{cb}_{\tau_\xi}(\bar{p})$; let $j + 1 = |\tau_\xi(p)|$. Then by construction, $\operatorname{supp}(p) = \bigcup_{\delta \in J \cap \delta_j} \operatorname{supp}(\operatorname{cb}_{\tau_\delta}(\bar{p} \restriction \delta))$ (where we actually take a tail of $\bar{p} \restriction \delta$). Again (8) follows. □

2.6. Some more on named iterations.

Lemma 2.33. Let ξ be a limit ordinal with $\operatorname{cf}(\xi) \geqslant \lambda$, suppose that $\langle \mathbb{P}_\zeta \rangle_{\zeta < \xi}$ is a $<\lambda$-support iteration, that each $\mathbb{P}_\zeta$ is $<\lambda$-distributive, and that $\bar{\tau}$ is a weakly coherent, S-sparse sequence for $\bar{\mathbb{P}}$. Then $\mathbb{P}_\xi$ is strategically S-closed.

As discussed above, by the generalisation of Proposition 2.3 to strategic closure, this implies that if S is fat then $\mathbb{P}_\xi$ is $<\lambda$-distributive.

Proof. The strategy for the completeness player is to play sequences $\bar{p} = \langle p_i \rangle$, where $p_i \in \mathbb{P}_{\zeta_i}$, with $\langle \zeta_i \rangle$ increasing and continuous, such that:

- If the completeness player plays p_{i+1}, then $p_{i+1} \in \bigcap_{j \leqslant i} A_{\zeta_j}^{\zeta_{i+1}}$; the ordinal played is $\left|\tau_{\zeta_{i+1}}(p_{i+1})\right|$.
- For limit i, for all $k < i$, $p_i \restriction \zeta_k = \mathrm{cb}_{\tau_{\zeta_k}}(\langle p_j \restriction \zeta_k \rangle_{j \in (k,i)})$. $\square$

Lemma 2.34. Suppose that $\langle \mathbb{P}_\zeta \rangle_{\zeta < \xi}$ is an S-sparse iteration, and that S is fat. For all $\upsilon < \xi$, in $V(\mathbb{P}_\upsilon)$, the iteration $\langle \mathbb{P}_\zeta / \mathbb{P}_\upsilon \rangle_{\zeta \in [\upsilon, \xi)}$ is $S \backslash U_\upsilon$-sparse (witnessed by $\langle \mathbb{Q}_\zeta, \sigma_\zeta \rangle_{\zeta \in [\upsilon,\xi)}$).

Proof. All we need to observe is that as $\mathbb{P}_\upsilon$ is $<\lambda$-distributive, the quotient iteration is $<\lambda$-support. $\square$

Lemma 2.35. Suppose that $\langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$ is an S-sparse iteration, that $\xi < \lambda^+$, that S is fat, and let $\bar{\tau}$ be an associated sequence.

Let $\upsilon < \xi$. Suppose that $\kappa < \lambda$ and that $\{p_i : i < \kappa\} \subseteq \mathbb{P}_\xi$ is a family of conditions such that $p_i \restriction \upsilon = p_j \restriction \upsilon$ for all $i, j < \kappa$. Then there is a collection of conditions $\{q_i : i < \kappa\}$ such that:

(i) Each q_i extends p_i;
(ii) For all i, $q_i \in \operatorname{dom} \tau_\xi$, indeed $q_i \in A_\upsilon^\xi$
(iii) For $i < j < \kappa$, $q_i \restriction \upsilon = q_j \restriction \upsilon$

Note that it follows that for all $i < j < \kappa$, $|\tau_\xi(q_i)| = |\tau_\xi(q_j)|$.

Proof. By induction on κ. First we consider $\kappa = k$ finite. We define a sequence $\langle r_n \rangle$ of conditions such that $\langle r_n \restriction \upsilon \rangle$ is increasing, and as usual $\sup_n |\tau_\upsilon(r_n \restriction \upsilon)| \in S$. We start with $r_i = p_i$ for $i < k$; then at step $n = i \mod k$, we find an extension of $r_{n-1} \restriction \upsilon \vee r_{n-k}$ in A_υ^ξ. At the limit, for $i < k$ we let $q_i = \mathrm{cb}_{\tau_\xi}(\langle r_{kn+i} \rangle_{n<\omega})$. The sequences $\langle r_{kn+i} \restriction \upsilon \rangle$ are co-final and so have the same canonical upper bound.

For infinite $\kappa < \lambda$, let $\theta = \mathrm{cf}(\kappa)$, and let $\langle \alpha_i \rangle_{i \leqslant \theta}$ be a continuous sequence increasing to κ. We build an array of conditions $\langle r_i^j \rangle_{i \leqslant \theta, j < \alpha_i}$ such that:

- For each $j < \kappa$, say $j \in [\alpha_k, \alpha_{k+1})$, the sequence $\langle r_i^j \rangle_{i > k}$ is a sparse sequence for τ_ξ;
- Each r_i^j is in A_υ^ξ;
- For $j < j' < \alpha_i$, $r_i^j \restriction \upsilon = r_i^{j'} \restriction \upsilon$.
- For each $i < \theta$ and $j < \alpha_i$, r_i^j extends p_j.

Let s_i be the common value of $r_i^j \restriction \upsilon$; it follows that $\langle s_i \rangle$ is sparse for τ_υ. At step $k + 1 < \theta$ we apply the inductive hypothesis to the collection of conditions

$$\left\{r_k^j : j < \alpha_k\right\} \cup \left\{s_k \vee p_j : j \in [\alpha_k, \alpha_{k+1})\right\}.$$

As usual everything is happening within a filtration $\bar{N}$ of length $\theta + 1$ with limit points in S, so at limit steps we can take canonical upper bounds. We let $q_j = r_\theta^j$. $\square$

3. Tasks and the task iteration

3.1. Tasks.

Notation 3.1. Let $\mathscr{F}_\lambda$ be the collection of $<\lambda$-distributive notions of forcing. Each $\mathbb{P} \in \mathscr{F}_\lambda$ preserves $\mathcal{H}_\lambda$.

Note that since we assume that the GCH holds below λ, we identify $\mathcal{P}(\lambda)$ with $\mathcal{P}(\mathcal{H}_\lambda)$.

Generically uniform definitions. Let $O \subseteq \lambda$ and let φ be a first-order formula. For every $A \in \mathcal{P}(\lambda)$ we define

$$\mathscr{B}(A) = \mathscr{B}_{O,\varphi}(A) = \{x \in \mathcal{H}_\lambda \,:\, (\mathcal{H}_\lambda; O, A) \models \varphi(x)\}\,.$$

We call this a uniform definition of $\mathscr{B}(A)$ from A; O can be considered as an "oracle", but can also incorporate any parameters from $\mathcal{H}_\lambda$ if we so choose.

Further, for every $\mathbb{P} \in \mathscr{F}_\lambda$ and every $A \in \mathcal{P}(\lambda)(\mathbb{P})$, that is, any $A \in V(\mathbb{P})$ such that $\Vdash_{\mathbb{P}} A \subseteq \lambda$, we let $\mathscr{B}(A)$ be $\mathbb{P}$-name denoting the result of this definition in $V(\mathbb{P})$. Thus, O and φ give us what we call a *generically uniform definition* of subsets of $\mathcal{H}_\lambda$.

We remark that if $\mathbb{P} \lessdot \mathbb{Q}$ and $\mathbb{Q} \in \mathscr{F}_\lambda$ then as $\mathcal{H}_\lambda$ is the same in $V(\mathbb{P})$ and in $V(\mathbb{Q})$, we can naturally consider $\mathcal{P}(\lambda)(\mathbb{P})$ as a subset of $\mathcal{P}(\lambda)(\mathbb{Q})$. Since φ is first-order, the interpretation of $\mathscr{B}(A)$ in $V(\mathbb{P})$ and $V(\mathbb{Q})$ is the same.

Limited genericity.

Definition 3.2. Suppose that $\mathbb{P} \subseteq \mathcal{H}_\lambda$ is a notion of forcing and that $O \in \mathcal{P}(\lambda)$. A filter $G \subseteq \mathbb{P}$ is *O-generic* if it meets every dense subset of $\mathbb{P}$ which is first-order definable in the structure $(\mathcal{H}_\lambda; \mathbb{P}, O)$.

Note that if $\mathbb{P}$ is λ-strategically closed then for any O there is an O-generic $G \subseteq \mathbb{P}$ in V.[7]

If σ is an explicit $\mathbb{P}$-name for a subset of λ, then for any σ-generic filter $G \subseteq \mathbb{P}$, $W_\sigma[G]$ is well-defined and is an element of 2^λ.

Tasks.

Definition 3.3. A λ-*task* $\mathfrak{t}$ consists of three generically uniform definitions $\mathbb{Q}^{\mathfrak{t}}, \sigma^{\mathfrak{t}}$ and $\mathbb{S}^{\mathfrak{t}}$ (say with oracle $O = O^{\mathfrak{t}}$) such that for all $\mathbb{P} \in \mathscr{F}_\lambda$ and all $C \in \mathcal{P}(\lambda)(\mathbb{P})$, in $V(\mathbb{P})$,

(A) $\mathbb{Q}^{\mathfrak{t}}(C)$ is a notion of forcing, and $\sigma^{\mathfrak{t}}(C)$ is a λ-sparse $\mathbb{Q}^{\mathfrak{t}}(C)$-name for a subset of C.[8] We let $W^{\mathfrak{t}}(C) = W_{\sigma^{\mathfrak{t}}(C)}$.

(B) For any $G \subseteq \mathbb{Q}^{\mathfrak{t}}(C)$ which is (O, C)-generic, for any $\mathbb{P}' \in \mathscr{F}_\lambda$ with $\mathbb{P} \lessdot \mathbb{P}'$, for any $A \in \mathcal{P}(\lambda)(\mathbb{P}')$, in $V(\mathbb{P}')$, $\mathbb{S}^{\mathfrak{t}}(C, G, A)$ is a notion of forcing which is explicitly closed outside $W^{\mathfrak{t}}(C)[G]$.[9]

Note that explicit closure or sparse closure of a notion of forcing does not depend on the universe we work in, as it only involves sequences of length $< \lambda$.

Example: exact diamonds.

Definition 3.4. Let $\mu < \lambda$ be a cardinal (possibly finite, but $\geqslant 2$), and let $S \subseteq \lambda$ be stationary. A μ-sequence on S is a sequence $\bar{F} = \langle F_\alpha \rangle_{\alpha \in S}$ such that for each $\alpha \in S$, $F_\alpha \subseteq \mathcal{P}(\alpha)$ and $|F_\alpha| = \mu$.

A μ-sequence $\bar{F}$ *guesses* a set $X \subseteq \lambda$ if for stationarily many $\alpha \in S$, $X \cap \alpha \in F_\alpha$.

We say that $\bar{F}$ is a *μ-diamond sequence* if it guesses every $X \subseteq \lambda$.

[7] Strategic S-closure for a fat S does not seem to suffice.

[8] This means that $\Vdash_{\mathbb{Q}^{\mathfrak{t}}(C)} W_{\sigma^{\mathfrak{t}}(C)} \subseteq C$.

[9] Recall that this means that it is explicitly $\lambda \backslash W^{\mathfrak{t}}(C)[G]$-closed.

We say that it is a μ-*almost diamond sequence* if it guesses all subsets of λ except possibly for a collection of at most λ many subsets X.

We say that $\bar{F}$ is a μ-*exact diamond sequence* if it is a μ-dimaond sequence, and whenever we choose $x_\alpha \in F_\alpha$ for all $\alpha \in S$, the sequence $\langle F_\alpha \backslash \{x_\alpha\}\rangle_{\alpha \in S}$ is not a μ-almost diamond sequence (let alone a μ-diamond sequence).

We define the μ-*exact diamond task* $\mathfrak{t} = \mathfrak{t}_\diamondsuit(\mu)$. The forcing $\mathbb{Q}^\mathfrak{t}(C)$ adds a μ-diamond sequence on a subset of C; the subsequent forcings $\mathbb{S}^\mathfrak{t}$ ensure that it is exact.

(i) For $C \subseteq \lambda$, $p \in \mathbb{Q}^\mathfrak{t}(C)$ if $p = (\sigma(p), \bar{F}(p))$ where:
  - $\sigma(p) \in 2^{<\lambda}$ and $\sigma(p) \subseteq C$;[10]
  - $\bar{F}(p) = \langle F_\alpha(p)\rangle_{\alpha \in \sigma(p)}$ with $F_\alpha(p) \subseteq \mathcal{P}(\alpha)$ and $|F_\alpha(p)| = \mu$;
  - extension is by extending the sequences in both coordinates.

  $\sigma^\mathfrak{t}(C) = \sigma$;

(ii) Fixing C, suppose that $G \subseteq \mathbb{Q}^\mathfrak{t}(C)$ is C-generic; let $W = W^\mathfrak{t}(C)[G]$, and let $\bar{F}$ be the generic μ-sequence of sets.

  For a sequence $\bar{x} = \langle x_\alpha\rangle_{\alpha \in W}$ with $x_\alpha \in F_\alpha$, $\mathbb{S}^\mathfrak{t}(C, G, \bar{x})$ is the notion of forcing consisting of conditions (d, y), where
  - $d, y \in 2^{<\lambda}$ and $|d| = |y| + 1$;
  - d is a closed subset of $|d|$;
  - for all $\alpha \in d \cap W$, $y \cap \alpha \notin F_\alpha \backslash \{x_\alpha\}$;
  - extension is extension in both coordinates.

Observe that indeed $\sigma^\mathfrak{t}(C)$ is λ-sparse: for a canonical sparse upper bound of an increasing sequence $\bar{p} = \langle p_i \rangle$ we choose a condition p^* defined by $\sigma(p^*) = \sigma(\bar{p})\hat{\ }0$, and $\bar{F}(p^*) = \bar{F}(\bar{p}) = \bigcup_i \bar{F}(p_i)$. Of course one of the main points is that $\alpha = |\sigma(\bar{p})|$ is excluded from $\sigma(p^*)$, so we do not need to define $F_\alpha(p^*)$. Note that in fact, $\mathbb{Q}^\mathfrak{t}(C)$ is $<\lambda$-closed. Also observe that $\mathbb{S}^\mathfrak{t}(C, G, \bar{x})$ is explicitly closed outside W, by taking $\delta(d, y) = |y|$.

Remark 3.5. In the example above, what is $\mathbb{S}^\mathfrak{t}(C, G, A)$ if A is not of the form $\bar{x}$ where $x_\alpha \in F_\alpha(C)$? Note that the collection of "legal" A is $\Pi^0_1(\mathcal{H}_\lambda; C, G)$, so the generic definition $\mathbb{S}^\mathfrak{t}(C, G, A)$ can involve a check and output a trivial notion of forcing when A does not have the right form.

3.2. **The task iteration.** We now define the λ-*task iteration*. This will be a λ-sparse iteration $\langle \mathbb{P}_\zeta, \mathbb{Q}_\zeta, \sigma_\zeta\rangle$ of length λ^+. We let $\langle \tau_\zeta\rangle$ be the associated sequence, and as above use the abbreviations $W_\zeta = W_{\sigma_\zeta}$ and $U_\zeta = W_{\tau_\zeta}$. Together with the iteration we will define bookkeeping objects $\mathbf{I}, \mathbf{f}$ such that $\mathbf{I} \subseteq \lambda^+$, and for all $\zeta < \lambda^+$,

(i) If $\zeta \in \mathbf{I}$, then $\mathbf{f}(\zeta) \in V(\mathbb{P}_\zeta)$ is a λ-task $\mathfrak{t}$, and $\mathbb{Q}_\zeta = \mathbb{Q}^\mathfrak{t}(\lambda \backslash U_\zeta)$, $\sigma_\zeta = \sigma^\mathfrak{t}(\lambda \backslash U_\zeta)$;

(ii) If $\zeta \notin \mathbf{I}$, then $\mathbf{f}(\zeta) = (\upsilon, A)$ where:
  - $\upsilon < \zeta$ and $\upsilon \in \mathbf{I}$, and
  - $A \in \mathcal{P}(\lambda)(\mathbb{P}_\zeta)$;

  and $\mathbb{Q}_\zeta = \mathbb{S}^\mathfrak{t}(\lambda \backslash U_\upsilon, G_\upsilon, A)$, where $\mathfrak{t} = \mathbf{f}(\upsilon)$ and $G_\upsilon \in V(\mathbb{P}_{\upsilon+1})$ is the generic for $\mathbb{Q}_\upsilon = \mathbb{Q}^\mathfrak{t}(\lambda \backslash U_\upsilon)$. In this case, as $\mathbb{Q}_\zeta$ is explicitly closed outside W_υ, we let δ_ζ witness this fact, and let σ_ζ be the associated explicit name for the empty set (see Lemma 2.16). Note that $W_\upsilon \subseteq U_\zeta$ modulo clubs, so $\mathbb{Q}_\zeta$ is explictly closed outside U_ζ as required.

[10]As usual, identifying the characteristic function with its underlying set.

Lemma 3.6. $\mathbb{P}_{\lambda^+}$ is strategically λ-closed, and has the λ^+-chain condition.

Proof. Strategic λ-closure follows from Lemma 2.33. For the λ^+-chain condition, note that for all $\zeta < \lambda^+$, $\Vdash_{\mathbb{P}_\zeta} |\mathbb{Q}_\zeta| \leqslant \lambda$, and that the iteration is $<\lambda$-support. $\square$

Corollary 3.7. *$\mathcal{P}(\lambda)(\mathbb{P}_{\lambda^+}) = \bigcup_{\zeta<\lambda^+} \mathcal{P}(\lambda)(\mathbb{P}_\zeta)$; $\mathbb{P}_{\lambda^+}$ is $<\lambda$-distributive; in $V(\mathbb{P}_{\lambda^+})$, no cofinalities are changed and no cardinals are collapsed; if the* GCH *holds in V then it also holds in $V(\mathbb{P}_{\lambda^+})$.*

Corollary 3.7 allows us to define the bookkeeping functions so that:

- For every λ-task $\mathfrak{t} \in V(\mathbb{P}_{\lambda^+})$ there is some $\xi \in \mathbf{I}$ such that $\mathbf{f}(\xi) = \mathfrak{t}$;
- For every $\xi \in \mathbf{I}$, for every $A \in \mathcal{P}(\lambda)(\mathbb{P}_{\lambda^+})$, there are unboundedly many $\zeta > \xi$ such that $\mathbf{f}(\zeta) = (\xi, A)$.

3.3. **Exact diamonds in the extension.**

Proposition 3.8. *In $V(\mathbb{P}_{\lambda^+})$, for every regular $\theta < \lambda$, for every cardinal $\mu \in [2, \lambda)$, there is a stationary set $W \subseteq \mathrm{Cof}_\lambda(\theta)$ and a μ-exact diamond sequence on W.*

To prove this proposition, we let $\mathfrak{t} = \mathfrak{t}_\diamondsuit(\mu, \theta)$ be the obvious modification of the task $\mathfrak{t}_\diamondsuit(\mu)$ above, where the requirement $\sigma(p) \subseteq C$ is replaced by $\sigma(p) \subseteq C \cap \mathrm{Cof}(\theta)$.

We fix some $\upsilon \in \mathbf{I}$ such that $\mathbf{f}(\upsilon) = \mathfrak{t}$. Let $W = W_\upsilon = W^{\mathfrak{t}}(\lambda \backslash U_\upsilon)[G_\upsilon]$ and let $\bar{F} = \bar{F}[G_\upsilon]$.

The easier part is to check exactness.

Lemma 3.9. In $V(\mathbb{P}_{\lambda^+})$, for every $\bar{x} = \langle x_\alpha \rangle_{\alpha \in W}$ for which $x_\alpha \in F_\alpha$ for all $\alpha \in W$, there are λ^+-many $Y \in \mathcal{P}(\lambda)$ which are not guessed by $\langle F_\alpha \backslash \{x_\alpha\} \rangle$.

Proof. Fix a set $T \subset \mathcal{P}(\lambda)(\mathbb{P}_{\lambda^+})$ with $|T| \leqslant \lambda$. Find some ζ sufficiently large so that $T \in V(\mathbb{P}_\zeta)$ and $\mathbf{f}(\zeta) = (\upsilon, \bar{x})$, so $\mathbb{Q}_\zeta = \mathbb{S}^{\mathfrak{t}}(\lambda \backslash U_\upsilon, G_\upsilon, \bar{x})$. This notion of forcing adds a set $Y \in \mathcal{P}(\lambda) \backslash T$ and a club D witnessing that $\langle F_\alpha \backslash \{x_\alpha\} \rangle$ does not guess Y. $\square$

What is trickier is showing that $\bar{F}$ is a diamond sequence in $V(\mathbb{P}_{\lambda^+})$. Certainly it is a diamond sequence in $V(\mathbb{P}_{\upsilon+1})$: But we need to show that it remains a diamond sequence in $V(\mathbb{P}_{\lambda^+})$. The argument to a large degree follows the main argument of [Sh:122].

Let $Z \in 2^\lambda(\mathbb{P}_{\lambda^+})$, and let $C \subseteq \lambda$ in $V(\mathbb{P}_{\lambda^+})$ be a club; find some limit $\xi \in (\upsilon, \lambda^+)$ such that $C, Z \in V(\mathbb{P}_\xi)$.

For the rest of the proof, we work in $V(\mathbb{P}_\upsilon)$. In that universe, U_υ is co-fat, and $\mathbb{P}_\xi/\mathbb{P}_\upsilon$ is the result of the named λ-iteration $(\mathbb{P}_\zeta/\mathbb{P}_\upsilon, \mathbb{Q}_\zeta, \sigma_\zeta)_{\zeta \in [\upsilon, \xi)}$ (Lemma 2.34).

Let

$$Q = \left\{ \zeta \in (\upsilon, \xi) \,:\, \mathbf{f}(\zeta) = (\upsilon, \bar{x}) \text{ where in } V(\mathbb{P}_\zeta),\, \bar{x} \in \prod_{\alpha \in W} F_\alpha \right\}.$$

For $\zeta \in Q$ we write $\bar{x}^\zeta = \left\langle x^\zeta_\alpha \right\rangle_{\alpha \in W}$ where $\mathbf{f}(\zeta) = (\upsilon, \bar{x}^\zeta)$, and we let $(D_\zeta, Y_\zeta) \in V(\mathbb{P}_{\zeta+1})$ be the pair of club and subset of λ which are added by $\mathbb{Q}_\zeta = \mathbb{S}^{\mathfrak{t}}(\lambda \backslash U_\upsilon, G_\upsilon, \bar{x}^\zeta)$.

We note:

- If $\zeta \in (\upsilon, \xi) \backslash Q$, then σ_ζ is W-sparse.

For if $\zeta \in \mathbf{I}$ then σ_ζ is λ-sparse; if $\zeta \notin \mathbf{I}$ then $\mathbb{Q}_\zeta$ is explicitly closed outside W_ϑ for some $\vartheta < \zeta$ distinct from υ; and $W_\vartheta \cap W = \varnothing$ modulo the club filter. So $\mathbb{Q}_\zeta$ is explicitly W-closed; and σ_ζ (a $\mathbb{Q}_\zeta$-name for the empty set) is W-sparse.

We apply Corollary 2.32 to the iteration $\langle \mathbb{P}_\zeta/\mathbb{P}_\upsilon\rangle_{\zeta\in[\upsilon,\xi]}$, with $S = \lambda\backslash U_\upsilon$, obtaining an associated sequence of names $\langle \tau_\zeta\rangle_{\zeta\in(\upsilon,\xi]}$, which is sparse outside U_υ. However we modify the construction at successor steps $\zeta+1$ for $\zeta \in Q$ to determine not only the height $\delta_\zeta(p(\zeta)) = |y^{p(\zeta)}|$ of a condition, but the actual values (d, y) of the condition. So for all $p \in \operatorname{dom}\tau_\xi$, for all $\zeta \in \operatorname{supp}(p)\cap Q$, there is a pair $(d^p_\zeta, y^p_\zeta) \in V(\mathbb{P}_\upsilon)$ (rather than $V(\mathbb{P}_\zeta)$) such that $p\restriction\zeta$ forces that $p(\zeta) = (d^p_\zeta, y^p_\zeta)$. Note that for such p and ζ we have $|d^p_\zeta| = |\tau_\xi(p)|$. For brevity, let $\tau = \tau_\xi$.

There are two cases.

In the first case, we work above a condition $p^* \in \mathbb{P}_\xi/\mathbb{P}_\upsilon$ which forces that for all $\zeta \in Q$, $Z \neq Y_\zeta$. Because $\mathbb{P}_\xi/\mathbb{P}_\upsilon$ is $<\lambda$-distributive, every $p \geqslant p^*$ in $\mathbb{P}_\xi/\mathbb{P}_\upsilon$ has an extension $r \in \operatorname{dom}\tau$ such that for some string $\pi \in 2^{<\lambda}$ with $|\pi| > |\tau(p)|$,

(i) $r \Vdash \pi \prec Z$; and
(ii) for every $\zeta \in Q \cap \operatorname{supp}(p)$, $\pi \perp y^r_\zeta$.

Since U_υ is co-fat, we use our usual technique to get a sparse sequence $\bar{p} = \langle p_i\rangle_{i<\theta}$[11] in $\operatorname{dom}\tau$ such that for limit $i \leqslant \theta$, $\sup_{j<i}\gamma_j \notin U_\upsilon$; and such that each p_{i+1} separates Z from $y^{p_i}_\zeta$ as above, namely: for some π_i of length in $[\gamma_i, \gamma_{i+1})$, we have $p_{i+1} \Vdash \pi_i \prec Z$, and for all $\zeta \in Q \cap \operatorname{supp}(p_i)$, $\pi_i \perp y^{p_{i+1}}_\zeta$. Naturally $\langle \pi_i\rangle$ is increasing. Also, we ensure that p_{i+1} forces that $C \cap [\gamma_i, \gamma_{i+1})$ is nonempty. We may assume that $\upsilon \in \operatorname{supp}(p_0)$.

We are after a condition q which forces that Z is guessed by $\bar{F}$ at some point in C. Informally, we define q to be a modification of the canonical sparse upper bound p_θ of $\bar{p}$; the difference is in $q(\upsilon)$, which recall is a condition in $\mathbb{Q}_\upsilon = \mathbb{Q}^{\mathrm{t}}(\lambda\backslash U_\upsilon)$. Let $\alpha^* = \sup_{i<\theta}\gamma_i$, so $\operatorname{cf}(\alpha^*) = \theta$ and $\alpha^* \notin U_\upsilon$. Formally, by induction on $\zeta \in (\upsilon, \xi]$, we define $q \restriction \zeta$, extending each $p_i \restriction \zeta$, as follows. We decalre that $\operatorname{supp}(q) = \bigcup_{i<\theta}\operatorname{supp}(p_i)$. For $\zeta \in \operatorname{supp}(q)\cap Q$, let $y^*_\zeta = \bigcup_{i<\theta} y^{p_i}_\zeta$ and $d^*_\zeta = \bigcup_{i<\theta} d^{p_i}_\zeta$. Also let $\pi^* = \bigcup_{i<\theta}\pi_i$.

(1) First we define $q(\upsilon)$ by letting $\sigma(q(\upsilon)) = \bigcup_{i<\theta}\sigma(p_i(\upsilon))\hat{\ }1$ and $\bar{F}(q(\upsilon)) = \bigcup_{i<\theta}\bar{F}(p_i(\upsilon))\hat{\ }F_{\alpha^*}$, where $\pi^* \in F_{\alpha^*}$ but for all $\zeta \in \operatorname{supp}(q)\cap Q$, $y^*_\zeta \notin F_{\alpha^*}$. Note that this is a legitimate condition because $\alpha^* \in \operatorname{Cof}_\lambda(\theta)\backslash U_\upsilon$.

Let $\zeta > \upsilon$ be in $\operatorname{supp}(q)$, and suppose that $q\restriction[\upsilon,\zeta)$ has already been defined, and that for all $i < \theta$, $q\restriction[\upsilon,\zeta)$ extends $p_i\restriction[\upsilon,\zeta)$.

(2) Suppose that $\zeta \notin Q$. Since $q\restriction[\upsilon,\zeta)$ extends each $p_i\restriction[\upsilon,\zeta)$, it forces that a tail of $\langle p_i(\zeta)\rangle$ is a sparse sequence for σ_ζ (we use (8) of Definition 2.29). Also, $q \restriction [\upsilon,\zeta)$ forces that $\alpha^* \in W$, so forces that $\langle p_i(\zeta)\rangle$ has an upper bound, which we set to be $q(\zeta)$.
(3) Suppose that $\zeta \in Q$. We declare that $q(\zeta) = (d^*_\zeta\hat{\ }1, y^*_\zeta)$. This is a legitimate condition because $\alpha^* \in W$ and $y^*_\zeta \notin F_{\alpha^*}$.

The condition q forces that $\pi^* = Z \cap \alpha^*$, $\alpha^* \in C \cap W$ and $\pi^* \in F_{\alpha^*}$, finishing the proof in this case.

In the second case, we work above a condition p^* which forces that $Z = Y_\varrho$ for some $\varrho \in Q$. (Note that we still have to work in $V(\mathbb{P}_\xi)$ rather than $V(\mathbb{P}_\varrho)$, as $C \in V(\mathbb{P}_\xi)$ may not be in $V(\mathbb{P}_\varrho)$.) We may assume that $p^* \in \operatorname{dom}\tau$ and $\varrho \in \operatorname{supp}(p^*)$. Let $\alpha = |\tau(p^*)| + \mu$, where recall μ is the size of each F_β. Let $p^*(\varrho) = (d^*, y^*)$. Find strings y^i, for $i < \mu$, of length α, each extending y^*, which are pairwise

[11] Recall that θ is the cofinality on which W concentrates.

incomparable; let p^i agree with p^* except that $p^i(\varrho) = (d, y^i)$, where $|d| = \alpha + 1$ is an extension of d^* by a string of zeros.

Using Lemma 2.35, we follow our usual construction, this time constructing an array $\langle p^i_j \rangle_{i<\mu, j<\theta}$ such that:

(i) For each $i < \mu$, $\langle p^i_j \rangle_{j\leqslant\theta}$ is a sparse sequence for τ;
(ii) For each $i < i' \leqslant \theta$, $p^i_j \restriction \varrho = p^{i'}_j \restriction \varrho$, call the common value r_j;
(iii) $p^i_0 = p^i$;
(iv) For each $i < \mu$ and $j < \theta$, $p^i_j \in A^{\xi}_{\varrho}$;
(v) For each $i, i' < \mu$, $j < \theta$, and $\zeta \in Q \cap \operatorname{supp}(p^i_j)$ other than ϱ, $y_\zeta^{p^i_{j+1}} \perp y_\varrho^{p^{i'}_{j+1}}$.

At step θ, instead of taking the sparse upper bound, we inductively define a condtion $q \in \operatorname{dom} \tau_\varrho$ by induction as above. For $i < \mu$ let $y^i_\varrho = \bigcup_{j<\theta} y_\varrho^{p^i_j}$. Let $\alpha^* = |\tau_\varrho(\bar r)|$, which has cofinality θ and is out of U_υ. We start with setting $q(\upsilon)$ so that $\alpha^* \in \sigma_\upsilon(q(\upsilon))$ and F_{α^*}, according to $q(\upsilon)$, is $\{y^i_\varrho : i < \mu\}$. We then define $q(\zeta)$ for $\zeta \in \operatorname{supp}(q) = \bigcup_j \operatorname{supp}(r_j)$ as above, noting that for all $\zeta \in Q \cap \operatorname{supp}(q)$, for all $i < \mu$, $y^i_\varrho \neq y^q_\zeta$, where of course $y^q_\zeta = \bigcup_{j<\theta} y^{r_j}_\zeta$.

Now, we extend q to a condition $q' \in \operatorname{dom} \tau_\varrho$ such that for some π of length α^*, $q' \Vdash x^\varrho_{\alpha^*} = \pi$. Since q' decides the value of F_{α^*}, necessarily $\pi = y^{i^*}_\varrho$ for some $i^* < \mu$. As above, extend q' to a condition $s \in \operatorname{dom} \tau_\xi$ extending $p^{i^*}_j$ for all $j < \theta$. Defining $s(\varrho) = (d^{i^*}_\varrho, y^{i^*}_\varrho)$ (where of course $d^i_\varrho = (\bigcup_{j<\theta} d_\varrho^{p^i_j})\hat{\ }1)$ is legitimate since q' forces that $y^{i^*}_\varrho \notin F_{\alpha^*} \backslash \{x^\varrho_{\alpha^*}\}$. The condition s forces that $\alpha^* \in C$ and that $Z \cap \alpha^* = y^{i^*}_\varrho \in F_{\alpha^*}$.

4. The task axiom

4.1. **Named iterations beyond λ^+.** Let $\mathscr{U}$ be a family of subsets of λ. We define the notion of forcing $\mathbb{R}(\mathscr{U})$:

- Conditions are pairs $p = (u(p), \sigma(p))$ where $\sigma(p) \in 2^{<\lambda}$ and $u(p) \subseteq \mathscr{U}$ has size $<\lambda$.
- q extends p if $u(p) \subseteq u(q)$, $\sigma(p) \preccurlyeq \sigma(q)$ and for all $U \in u(p)$, $U \cap [|\sigma(p)|, |\sigma(q)|) \subseteq \sigma(q)$.

This notion of forcing is $<\lambda$-closed; σ is an explicit $\mathbb{R}(\mathscr{U})$-name for a subset of λ. In $V(\mathbb{R}(\mathscr{U}))$, W_σ is an upper bound of $\mathscr{U}$ modulo clubs (indeed modulo bounded subsets of λ). We will not need to directly appeal to the following lemma, since in our context we will use the heavier machinery of sparse names.

Lemma 4.1. For every $S \subseteq \lambda$ in V, if for all $U \in \mathscr{U}$, $S \backslash U$ is stationary, then in $V(\mathbb{R}(\mathscr{U}))$, $S \backslash W_\sigma$ is stationary. □

We can now give a general definition of named λ-iterations (of any length):

Definition 4.2. A sequence $\langle \mathbb{P}_\zeta \rangle_{\zeta<\xi}$ is a named λ-iteration if it satisfies conditions (i)-(iii) of Definition 2.26, together with:

(iv) For $\zeta < \xi$ such that $\operatorname{cf}(\zeta) > \lambda$, $\mathbb{Q}_\zeta = \mathbb{R}(\{W_{\sigma_\upsilon} : \upsilon < \zeta\})$ and σ_ζ is the corresponding explicit name.

Using this definition, as above, for such an iteration we can define a sequence $\langle \mathbf{u}_\zeta \rangle_{\zeta<\xi, \operatorname{cf}(\zeta)\leqslant\lambda}$ with $\mathbf{u}_\zeta \in \mathcal{P}(\lambda)/\operatorname{NS}_\lambda(\mathbb{P}_\zeta)$. This will be an increasing sequence in

$\mathcal{P}(\lambda)/\mathrm{NS}_\lambda$. If $\zeta = \upsilon + 1$ and $\mathrm{cf}(\upsilon) > \lambda$ then we let $\mathbf{u}_\zeta = W_{\sigma_\upsilon}/\mathrm{NS}_\lambda$. Otherwise we can take least upper bounds as above. Definition 2.27 now makes sense for named λ-iterations of length beyond λ^+ as well. Similarly we can generalise Definition 2.29; the only change is that τ_ζ is defined only for $\zeta \in [1, \xi) \cap \mathrm{Cof}(\leqslant \lambda)$. We can then extend Proposition 2.31:

Proposition 4.3. *Let $S \subseteq \lambda$ be fat. Suppose that $\langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$ is an S-sparse iteration, and that $\bar\tau = \langle \tau_\zeta \rangle_{\zeta \in [1,\xi) \cap \mathrm{Cof}(\leqslant\lambda)}$ is an associated sequence for $\bar{\mathbb{P}} \restriction \xi$. Suppose that $\mathrm{cf}(\xi) \leqslant \lambda$.*

Then there is a name τ_ξ such that $\bar\tau\hat{\ }\tau_\xi$ is an associated sequence for $\bar{\mathbb{P}}$.

Proof. The proof of Proposition 2.31 holds in all previous cases, so we assume that $\xi = \vartheta + 1$ where $\mathrm{cf}(\vartheta) > \lambda$. In this case, by Lemma 2.33, $\mathbb{P}_\vartheta$ is $<\lambda$-distributive.

For brevity, let $I = \xi \cap \mathrm{Cof}(\leqslant \lambda)$. Also for brevity, let $\mathbb{R} = \mathbb{Q}_\vartheta = \mathbb{R}(\{W_{\sigma_\zeta} : \zeta < \xi\})$. Since $\mathbb{P}_\vartheta$ is $<\lambda$-distributive, the set of conditions $(p, q) \in \mathbb{P}_\vartheta * \mathbb{R} = \mathbb{P}_\xi$ such that for some $v \subseteq I$ of size $<\lambda$ and some string $\pi \in 2^{<\lambda}$,

$$p \Vdash_{\mathbb{P}_\vartheta} \sigma(q) = \pi \ \ \& \ \ u(q) = \{W_{\sigma_\zeta} : \zeta \in v\}$$

is dense in $\mathbb{P}_\vartheta * \mathbb{R}$. We restrict ourselves to this dense subset, and write $v(q), \sigma(q)$ for the corresponding v and π.

Let $\langle A^\beta_\alpha \rangle$ witness the weak coherence of $\bar\tau$. We let $(p, q) \in \mathrm{dom}\, \tau_\xi$ if:

(i) $p \in \mathbb{P}_{\sup v(q)}$;
(ii) For all δ, $\delta \in v(q) \iff \delta + 1 \in v(q)$;
(iii) for all $\delta \in v(q)$, $p \restriction \delta \in \mathrm{dom}\, \tau_\delta$ and $|\tau_\delta(p \restriction \delta)| = |\sigma(q)|$; and
(iv) for all $\delta \in v(q)$, for all $\varepsilon < \delta$ in I, $p \restriction \delta \in A^\delta_\varepsilon$ if and only if $\varepsilon \in v(q)$.

For $(p, q) \in \mathrm{dom}\, \tau_\xi$ we let $\tau_\xi(p, q) = \sigma(q)$. If $(\bar p, \bar q) = \langle (p_i, q_i) \rangle_{i<i^*}$ is an increasing sequence from $\mathrm{dom}\, \tau_\xi$, then we let $\mathrm{cb}_{\tau_\xi}(\bar p, \bar q) = (p^*, q^*)$ if:

(v) $(p^*, q^*) \in \mathrm{dom}\, \tau_\xi$;
(vi) $v(q^*) = \bigcup_{i<i^*} v(q_i)$;
(vii) $\sigma(q^*) = \bigcup_{i<i^*} \sigma(q_i)\hat{\ }0$;
(viii) For all $\delta \in v(q^*)$, $p^* \restriction \delta$ is $\mathrm{cb}_{\tau_\delta}$ of a tail of $\bar p \restriction \delta$.

We show that $\mathrm{dom}\, \tau_\xi$ is dense in $\mathbb{P}_\xi$. Given $(p_0, q_0) \in \mathbb{P}_\xi$, we obtain a λ-filtration $\bar N$ and sequence γ_n with limit $\gamma_\omega \in S$ as usual. We define an increasing sequence $\langle p_n, q_n \rangle_{n<\omega}$ with $(p_n, q_n) \in N_{\gamma_n}$. We require that:

- $|\sigma(q_n)| > \gamma_{n-1}$;
- $N_{\gamma_{n-1}} \cap I \subseteq v(q_n)$;
- $p_n \in \mathbb{P}_{\zeta_n}$ for some $\zeta_n \in N_{\gamma_n}$, $\zeta_n > \sup v(q_n)$;
- $|\tau_{\zeta_n}(p_n)| > \gamma_{n-1}$; and
- for all $\delta \in v(q_n)$, $p_n \in A^{\zeta_n}_\delta$.

Note that having size $<\lambda$, $v(q_n) \subset N_{\gamma_n}$.

Let $v^* = \bigcup_n v(q_n) = N_{\gamma_\omega} \cap I$, which is cofinal in $N_{\gamma_\omega} \cap \xi$; let $\zeta^* = \sup v^*$. Also let $\pi^* = \bigcup_n \sigma(q_n)$, which has length γ_ω. Since $|v^*| < \lambda$, $\mathbb{P}_{\zeta^*}$ is the inverse limit of $\langle \mathbb{P}_\delta \rangle_{\delta \in v^*}$. For $\delta \in v^*$, a tail of $\langle p_n \restriction \delta \rangle$ is in $\mathrm{dom}\, \tau_\delta$; we let $r_\delta = \mathrm{cb}_{\tau_\delta}(\langle p_n \restriction \delta \rangle)$ be its canonical upper bound. By cohernece, $r_\delta \restriction \varepsilon = r_\varepsilon$ for $\varepsilon < \delta$ in v^*, so we obtain an inverse limit $r \in \mathbb{P}_{\zeta^*}$; this condition extends each p_n. Define $(r, s) \in \mathbb{P}_\vartheta * \mathbb{R}$ by letting $\sigma(s) = \pi^*\hat{\ }1$, and $v(s) = v^*$. We claim that $(r, s) \in \mathrm{dom}\, \tau_\xi$. Requirements (i) – (iii) and one direction of (iv) are clear; for the other direction of (iv), we suppose that $\delta \in v^*$, $\varepsilon < \delta$, and $r_\delta = r \restriction \delta \in A^\delta_\varepsilon$; we need to show that $\varepsilon \in v^*$. By the continuity property (g) of Definition 2.28, for large enough n, $p_n \restriction \delta \in A^\delta_\varepsilon$. Fixing some large n, we know (by (f)) that ε is among the fewer than λ-many $\varepsilon' < \delta$ such that $p_n \restriction \delta \in A^\delta_{\varepsilon'}$. Since $\langle A^\beta_\alpha \rangle, I, p_n \in N_{\gamma_n}$ it follows that each such ε' is in N_{γ_n}, so $\varepsilon \in I \cap N_{\gamma_n} \subseteq \upsilon(q_{n+1}) \subset \upsilon^*$.

To show that (r,s) extends (p_n, q_n), we need to show that if $\delta \in \upsilon(q_n)$ and $\alpha \in (|\sigma(q_n)|, \gamma_\omega]$, then r forces that if $\alpha \in W_{\sigma_\delta}$, then $\sigma(s)(\alpha) = 1$. But we may assume that $\alpha < \gamma_\omega$; if $m > n$ is sufficiently large so that $|\sigma(q_m)|, |\tau_\delta(p_m \restriction \delta)| > \alpha$, then p_m forces what's required, as it forces that q_m extends q_n.

We check that the requirements of Definition 2.29 hold. We first need to check that τ_ξ is S-sparse. Suppose that $(\bar{p}, \bar{q}) = \langle p_i, q_i \rangle_{i<i^*}$ is a sparse sequence for τ_ζ, with $\alpha^* = |\tau_\zeta(\bar{p}, \bar{q})| \in S$. Let $\upsilon^* = \bigcup_{i<i^*} \upsilon(q_i)$; let $\pi^* = \bigcup_{i<i^*} \sigma(q_i)$. Let $\zeta^* = \sup \upsilon^*$. For all $\delta \in \upsilon^*$, a tail of $\bar{p} \restriction \delta$ is in $\operatorname{dom} \tau_\delta$ and is a sparse sequence for τ_δ; we let r_δ be the canonical upper bound; we let r be the inverse limit of $\langle r_\delta \rangle$; we define s by $\sigma(s) = \pi^{*\frown}0$ and $\upsilon(s) = \upsilon^*$. It is not difficult to see that $(r,s) \in \operatorname{dom} \tau_\xi$ and $(r,s) = \operatorname{cb}_{\tau_\xi}(\bar{p}, \bar{q})$, except that we need to show that (r,s) extends each (p_i, q_i). The only potentially contentious point is the legitimacy of setting $\sigma(s)(\alpha^*) = 0$. Let $\delta \in \upsilon(q_i)$. Then $\delta + 1 \in \upsilon(q_i) \subseteq \upsilon(s)$, whence $p_i \restriction \delta + 1 \in \operatorname{dom} \tau_{\delta+1}$. By (4) for $\tau_{\delta+1}$, $\delta \in \operatorname{supp}(p_i)$, and since $\tau_{\delta+1}(r_{\delta+1})(\alpha^*) = 0$, by (5) (applied to $q = r_{\delta+1}$ and $p = p_i \restriction \delta + 1$), $\sigma_\delta(r)(\alpha^*) = 0$, so δ and α^* are not an obstacle to (r,s) extending (p_i, q_i).

(2) follows from the fact that we defined $\mathbf{u}_\xi = W_{\sigma_\vartheta}/\mathrm{NS}_\lambda$.

For (3), for $\delta \in I$ and $(p,q) \in \operatorname{dom} \tau_\xi$, we let $(p,q) \in A^\xi_\delta$ if and only if $\delta \in \upsilon(q)$. The requirements of Definition 2.28 follow from our definitions.

For (4), let $\zeta \in \operatorname{supp}(p)$; since $p \in \mathbb{P}_{\sup \upsilon(q)}$, $\zeta < \sup \upsilon(q)$; let $\delta \in \upsilon(q)$, $\delta > \zeta$. Then $p \restriction \delta \in \operatorname{dom} \tau_\delta$, and so $p \restriction \delta \in A^\delta_{\zeta+1}$; it follows that $\zeta + 1 \in \upsilon(q)$. On the other hand we note that $\vartheta \in \operatorname{supp}(p,q)$.

(6) is immediate: if for all $\delta \in \operatorname{supp}(p,q)$, $\sigma_\delta(p,q)(\alpha) = 0$, then $\tau_\xi(p,q)(\alpha) = \sigma_\vartheta(q)(\alpha) = 0$.

For (5), suppose that $\zeta \in \operatorname{supp}(p,q)$, and that (p', q') extends (p,q). We may assume that $\zeta < \vartheta$, so $\zeta \in \upsilon(q)$. Then $\sigma_\zeta(p'(\zeta)) \backslash |\sigma_\zeta(p(\zeta))| \subseteq \sigma_\vartheta(q') = \tau(p', q')$ because p' forces that q' extends q.

Finally, (7) and (8) follow from our definitions. □

Quotients of named iterations. Let S be fat, and let $\langle \mathbb{P}_\zeta \rangle_{\zeta<\xi}$ be an S-sparse iteration. Let $\upsilon < \xi$. If $\operatorname{cf}(\upsilon) \leqslant \lambda$ then the usual argument shows that the quotient iteration $\langle \mathbb{P}_\zeta / \mathbb{P}_\upsilon \rangle_{\zeta \in [\upsilon, \xi)}$ is, in $V(\mathbb{P}_\upsilon)$, an $S \backslash U_\upsilon$-sparse iteration.

What happens though if $\operatorname{cf}(\upsilon) > \lambda$? In this case, the iteration $\langle \mathbb{P}_\zeta / \mathbb{P}_\upsilon \rangle$ is a one-step extension $\mathbb{R}(\mathscr{U}) * \langle \mathbb{P}_\zeta / \mathbb{P}_{\upsilon+1} \rangle_{\zeta \in (\upsilon, \xi)}$, where $\mathscr{U} = \{W_\zeta : \zeta < \upsilon\}$. The second step, as mentioned, is a $S \backslash U_{\upsilon+1}$-sparse iteration, where $U_{\upsilon+1} = W_\upsilon$ comes from the generic for $\mathbb{R}(\mathscr{U})$; we know that $S \backslash U_{\upsilon+1}$ is fat in $V(\mathbb{P}_{\upsilon+1})$.

4.2. Nice tasks and the task axiom.

Definition 4.4. Let $\mathfrak{t}$ be a λ-task, let $S \subseteq \lambda$ be fat. An S-sparse iteration $\langle \mathbb{P}_\zeta \rangle_{\zeta<\xi}$ is $\mathfrak{t}$-*friendly* if $\mathbb{Q}_0 = \mathbb{Q}^{\mathfrak{t}}(S)$, and letting G_0 be the generic for $\mathbb{Q}_0$ and $W = W^{\mathfrak{t}}(S)[G_0] = W_0$, for all $\zeta \in [1, \xi)$ with $\operatorname{cf}(\zeta) \leqslant \lambda$, in $V(\mathbb{P}_\zeta)$,

(i) either σ_ζ is W-sparse; or
(ii) $\mathbb{Q}_\zeta = \mathbb{S}^{\mathfrak{t}}(S, G_0, A)$ for some A.

For example, for the task iteration above, for every $\upsilon \in \mathbf{I}$, in $V(\mathbb{P}_\upsilon)$, the quotient iteration $\langle \mathbb{P}_\zeta/\mathbb{P}_\upsilon \rangle$ is $\mathbf{f}(\upsilon)$-friendly (with $S = \lambda \backslash U_\upsilon$).

Definition 4.5. Let $\mathfrak{t}$ be a λ-task. A *correctness condition* for $\mathfrak{t}$ is a Π^1_1 sentence ψ such that for all fat sets C, $\mathbb{Q}^{\mathfrak{t}}(C)$ forces that $(\mathcal{H}_\lambda; O^{\mathfrak{t}}, C, G) \models \psi$ (where $O^{\mathfrak{t}}$ is the oracle for $\mathfrak{t}$).

For example, for $\mathfrak{t} = \mathfrak{t}_\diamondsuit(\mu)$ (or $\mathfrak{t}_\diamondsuit(\mu, \theta)$) as above, the correctness condition is that the resulting sequence $\langle F_\alpha \rangle_{\alpha \in W}$ is a μ-diamond sequence.[12]

Definition 4.6. Let $\mathfrak{t}$ be a λ-task and let ψ be a correctness condition for $\mathfrak{t}$. We say that the pair $(\mathfrak{t}, \psi)$ is *nice* if for any $<\lambda$-distributive $\mathbb{R}$, in $V(\mathbb{R})$, for any fat S, for any S-sparse iteration $\langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$ with $\xi < \lambda^{++}$ which is $\mathfrak{t}$-friendly, in $V(\mathbb{R} * \mathbb{P}_\xi)$, $(\mathcal{H}_\lambda; O, S, G_0) \models \psi$ (where as above G_0 is the generic for $\mathbb{Q}_0$).

Definition 4.7. Let $\mathfrak{t}$ be a λ-task and let ψ be a correctness condition for $\mathfrak{t}$. We say that the pair $(\mathfrak{t}, \psi)$ is *satisfied* if:

(1) There is a fat set C and a filter $G \subset \mathbb{Q}^{\mathfrak{t}}(C)$ which is $(O^{\mathfrak{t}}, C)$-generic;
(2) $(\mathcal{H}_\lambda; O^{\mathfrak{t}}, C, G) \models \psi$; and
(3) For all $A \in \mathcal{P}(\lambda)$, there is a filter $H_A \subset \mathbb{S}^{\mathfrak{t}}(C, G, A)$ which is $(O^{\mathfrak{t}}, C, G, A)$-generic.

We can now define the λ-task axiom.

Definition 4.8. The λ-task axiom TaskAx_λ states:

Every nice pair $(\mathfrak{t}, \psi)$ is satisfied.

Again we emphasise that throughout, we assume that the GCH holds below λ.

Diamonds and the task axiom. As mentioned above, for $\mathfrak{t} = \mathfrak{t}_\diamondsuit(\mu, \theta)$, the natural correctness condition ψ for $\mathfrak{t}$ is that $\bar{F} = \langle F_\alpha \rangle_{\alpha \in W}$ is a μ-diamond sequence. Further, the argument proving Proposition 3.8 shows that $(\mathfrak{t}, \psi)$ is nice. We modify the interpretation of the extra oracle A in the definition of $\mathbb{S}^{\mathfrak{t}}(C, G, A)$, so that A not only codes the sequence $\bar{x} = \langle x_\alpha \rangle_{\alpha \in W}$ but also codes a family of λ-many $Z \in \mathcal{P}(\lambda)$, the result being that the generic Y satisfies $Y \neq Z$ for all Z coded by A. The result is:

Proposition 4.9. TaskAx_λ *implies that for all $\mu < \lambda$ and all regular $\theta < \lambda$, there is an exact μ-diamond sequence concentrating on* $\mathrm{Cof}_\lambda(\theta)$.

4.3. **Consistency of the task axiom.** We remind the reader that as usual, λ is regular and uncountable, and that the GCH holds below λ. In this subsection we prove:

Proposition 4.10. *There is a notion of forcing $\mathbb{P}$ which is $<\lambda$-distributive and has the λ^+-chain condition (and so preserves cardinals and cofinalities), preserves the* GCH *(if it holds in V), such that the task axiom* TaskAx_λ *holds in* $V(\mathbb{P})$.

Proof. Toward constructing $\mathbb{P}$, we define a λ-sparse iteration $\langle \mathbb{P}_\zeta \rangle_{\zeta < \lambda^{++}}$; $\mathbb{P}$ will be a proper initial segment of this iteration. As above we have book-keeping devices $\mathbf{I}$ and $\mathbf{f}$, except that for $\upsilon \in \mathbf{I}$, $\mathbf{f}(\upsilon)$ is a pair $(\mathfrak{t}, \psi)$ consisting of a task and a correctness condtion for that task. Also, of course, $\mathbf{f}$ is only defined on $\lambda^{++} \cap \mathrm{Cof}(\leqslant \lambda)$, as $\mathbb{Q}_\zeta$

[12]Note that if $G \subset \mathbb{Q}^{\mathfrak{t}}(C)$ is merely C-generic, then for this task it is not the case that ψ holds in $(\mathcal{H}_\lambda; C, G)$; for ψ to hold we need a filter G fully generic over V.

is prescribed when $\mathrm{cf}(\zeta) = \lambda^+$. Also, we will repeat tasks when earlier attempts have resulted in failure. Suppose that $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$, and that $\zeta > \upsilon$. We say that $\mathbf{f}(\upsilon)$ *has failed by stage* ζ if there is some condition in $\mathbb{P}_\zeta$ which forces that in $V(\mathbb{P}_\zeta)$, $(\mathcal{H}_\lambda; O, \lambda \backslash U_\upsilon, G_\upsilon) \models \neg\psi$. We keep our books so that:

- For every pair $(\mathfrak{t}, \psi) \in V(\mathbb{P}_{\lambda^{++}})$, either
  - There is some $\upsilon \in \mathbf{I}$ such that $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$ and for all $\zeta > \upsilon$, $\mathbf{f}(\upsilon)$ does not fail by stage ζ; or
  - There are unboundedly many $\upsilon \in \mathbf{I}$ such that $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$, and if $\upsilon < \upsilon'$ are in $\mathbf{I}$ and $\mathbf{f}(\upsilon) = \mathbf{f}(\upsilon')$, then $\mathbf{f}(\upsilon)$ has failed by stage υ'.
- In the first case, with υ last such that $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$, for all $A \in \mathcal{P}(\lambda)(\mathbb{P}_{\lambda^{++}})$, there are unboundedly many $\zeta < \lambda^{++}$ such that $\mathbf{f}(\zeta) = (\upsilon, A)$.

Then by a standard closure argument, we can find some $\xi^* < \lambda^{++}$ such that:

(i) $\mathrm{cf}(\xi^*) = \lambda^+$;
(ii) For every pair $(\mathfrak{t}, \psi) \in V(\mathbb{P}_{\xi^*})$, if there is a last $\upsilon \in \mathbf{I}$ such that $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$, then this υ is smaller than ξ^*;
(iii) If $\upsilon < \xi^*$ is last with $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$, then for all $A \in \mathcal{P}(\lambda)(\mathbb{P}_{\xi^*})$ there are unboundedly many $\zeta < \xi^*$ such that $\mathbf{f}(\zeta) = (\upsilon, A)$.

We let $\mathbb{P} = \mathbb{P}_{\xi^*}$. Suppose that $\mathfrak{t} \in V(\mathbb{P})$, that ψ is a correctness condition for $\mathfrak{t}$, and that $(\mathfrak{t}, \psi)$ is nice in $V(\mathbb{P})$. Then there is a last $\upsilon \in \mathbf{I}$ such that $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$. For otherwise, there is some $\upsilon > \xi^*$ such that $\mathbf{f}(\upsilon) = (\mathfrak{t}, \psi)$ and some $\xi > \upsilon$ such that $\mathbf{f}(\upsilon)$ has failed by stage ξ. Then the iteration $\langle \mathbb{P}_\zeta / \mathbb{P}_\upsilon \rangle_{\zeta \in [\upsilon^*, \xi)}$ witnesses (in $V(\mathbb{P}_\upsilon)$, which is an extension of $V(\mathbb{P}_{\xi^*})$ by $\mathbb{P}_\upsilon / \mathbb{P}_{\xi^*}$, which as a quotient of a $<\lambda$-distributive notion of forcing is also $<\lambda$-distributive), that $(\mathfrak{t}, \psi)$ is not nice.

Taking this last υ, we have $\upsilon < \xi^*$; we know that ψ holds in $V(\mathbb{P}_{\xi^*})$; and our bookkeeping ensures that $(\mathfrak{t}, \psi)$ is satisfied in $V(\mathbb{P}_\xi)$. □

Iterating with set Easton support, we get:

Corollary 4.11. *Assuming the* GCH*, there is a class forcing extension preserving all cofinalities and cardinals, and* GCH*, in which* TaskAx_λ *holds for every regular uncountable cardinal* λ.

4.4. **Uniformisation.** Let $W \subseteq \lambda$. Recall that a *ladder system* on W is a sequence $\bar{E} = \langle E_\alpha \rangle_{\alpha \in W}$ such that each E_α is an unbounded subset of α of order-type $\mathrm{cf}(\alpha)$. A *2-colouring* of a ladder system $\bar{E}$ is a sequence $\bar{c} = \langle c_\alpha \rangle_{\alpha \in W}$ such that for all $\alpha \in W$, $c_\alpha \colon E_\alpha \to 2$. A function $g \colon \lambda \to 2$ *uniformises* the 2-colouring $\bar{c}$ if for every $\alpha \in W$, $g \restriction E_\alpha =^* c_\alpha$, meaning that $\{\gamma \in E_\alpha : g(\gamma) \neq c_\alpha(\gamma)\}$ is bounded below α.

A ladder system $\bar{E}$ on W *has uniformisation* if every 2-colouring of $\bar{e}$ is uniformised by some g.

Since we are assuming the GCH below λ, there are some restrictions on what kind of uniformisation we can deduce from the task axiom. If $\lambda = \mu^+$ and $\theta = \mathrm{cf}(\mu)$ then no ladder system on any stationary $W \subseteq \mathrm{Cof}_\lambda(\neq \theta)$ can have uniformisation, as $\diamondsuit_S$ holds [Gre76, Sh:108, Sh:922]. Also, if $\lambda = \mu^+$ and μ is singular, then for no stationary $W \subseteq \lambda$ is it the case that every ladder system on W has uniformisation [Sh:667].

Proposition 4.12. *Suppose that* λ *is inaccessible and* $\theta < \lambda$ *is regular; or that* $\lambda = \mu^+$ *and* $\theta = \mathrm{cf}(\mu)$*. Then* TaskAx_λ *implies that for any ladder system* $\bar{E}$ *on* $\mathrm{Cof}_\lambda(\theta)$ *there is some stationary* $W \subseteq \mathrm{Cof}_\lambda(\theta)$ *such that the restriction* $\bar{E} \restriction W$ *has uniformisation.*

Proof. Let $\bar{E}$ be a ladder system on $\mathrm{Cof}_\lambda(\theta)$. Define the following task $\mathfrak{t} = \mathfrak{t}_{\mathrm{UP}}(\theta)$:

(i) For $C \subseteq \lambda$, $p \in \mathbb{Q}^{\mathfrak{t}}(C)$ if $p \in 2^{<\lambda}$, $p \subset C \cap \mathrm{Cof}(\theta)$, and the restriction of $\bar{E}$ to p has uniformisation.
We let $\sigma^{\mathfrak{t}}(C)(p) = p$.

(ii) If $G \subseteq \mathbb{Q}^{\mathfrak{t}}(C)$ is C-generic, then we let $W = W[G] = W_{\sigma^{\mathfrak{t}}(C)}[G]$.
If $\bar{c}$ is a 2-colouring of $\bar{E} \restriction W$, then $\mathbb{S}^{\mathfrak{t}}(C, G, \bar{c})$ consists of conditions $q \in 2^{<\lambda}$ which are an initial segment of a uniformising function: for all $\alpha \in W$ with $\alpha \leqslant |q|$, $q \restriction E_\alpha =^* c_\alpha$.

First, we verify Definition 3.3. It is clear that $\mathbb{S}^{\mathfrak{t}}(G, \bar{c})$ is explicitly closed outside W; take $\delta(q) = |q|$. To see that $\sigma^{\mathfrak{t}}(C)$ is λ-sparse (using $\mathrm{cb}_\sigma(\bar{p}) = (\bigcup \bar{p})\hat{\ }0$), suppose that $\bar{p} = \langle p_i \rangle_{i<i^*}$ is sparse for σ. Let $p^* = (\bigcup \bar{p})\hat{\ }0$. We need to argue that the restriction of $\bar{E}$ to p^* has uniformisation. This follows from the closed set disjoint from p^* determined by the sequence $\bar{p}$: let $\gamma_i = |p_i|$. Every $\alpha \in p^*$ is in (γ_i, γ_{i+1}) for some i, the salient point being that $\alpha > \gamma_i$. So if $\bar{c}$ is a 2-colouring of $\bar{E} \restriction p^*$, then as each $p_i \in \mathbb{Q}^{\mathfrak{t}}(C)$, we let $g_i \colon \gamma_i \to 2$ uniformise $\bar{c} \restriction p_i$; define $g \colon |p^*| \to 2$ by letting g agree with g_i on $[\gamma_i, \gamma_{i+1})$.

Let ψ be the correctness condition which states that $W = W[G]$ is stationary. We argue that this is forced by $\mathbb{Q}^{\mathfrak{t}}(C)$. To see this, let $D \in V(\mathbb{Q}^{\mathfrak{t}}(C))$ be a club; and suppose that C is fat. Obtain a filtration $\bar{N}$ and a sequence $\langle \gamma_i \rangle_{i<\theta}$ (with limits in C) as usual. Define an increasing sparse sequence $\bar{p}$ of length θ as usual, so any upper bound forces that $\alpha^* = \sup_i \gamma_i$ is in D; it is, of course, also in $C \cap \mathrm{Cof}(\theta)$. We argue that $p^* = (\bigcup \bar{p})\hat{\ }1$ is a valid condition in $\mathbb{Q}^{\mathfrak{t}}(C)$, namely, that it has uniformisation. Let $\bar{c}$ be a 2-colouring of $\bar{E} \restriction p^*$. As above, for each $i < \theta$ let g_i uniformise the restriction of $\bar{c}$ to p_i. We define $g \colon \alpha^* \to 2$ uniformising $\bar{c}$, as follows: for $\beta \in [\gamma_i, \gamma_{i+1})$, we let $g(\beta) = g_i(\beta)$, except that if $\beta \in E_{\alpha^*}$ we set $g(\beta) = c_{\alpha^*}(\beta)$. That g uniformises $\bar{c}$ follows from the fact that the changes from g_i are bounded: let $\beta \in p^*$, $\beta < \alpha^*$; so $\beta \in (\gamma_i, \gamma_{i+1})$ for some i. Since $\mathrm{cf}(\beta) = \theta$, $E_{\alpha^*} \cap \beta$ is bounded below β, so $g \restriction E_\beta =^* g_i \restriction E_\beta =^* f_\beta$.

Suppose that the pair $(\mathfrak{t}, \psi)$ is satisfied, say by C and G; then $W = W[G] \subset \mathrm{Cof}_\lambda(\theta)$ is stationary. The ladder system $\bar{E} \restriction W$ has uniformisation: let $\bar{c}$ be a 2-colouring of $\bar{E} \restriction W$; let $H = G_{\bar{c}} \subseteq \mathbb{S}^{\mathfrak{t}}(C, G, \bar{c})$ be $(C, G, \bar{c})$-generic. Since each $p \in \mathbb{Q}^{\mathfrak{t}}(C)$ has uniformisation, $g_H = \bigcup H$ is a function with domain λ (for all $\gamma < \lambda$, the conditions in $\mathbb{S}^{\mathfrak{t}}$ with domain $\geqslant \gamma$ are dense), and it uniformises $\bar{c}$.

It remains to show that $(\mathfrak{t}, \psi)$ is nice. The argument follows [Sh:64, Sh:186].

After passing, possibly, to a generic extension, suppose that $S \subseteq \lambda$ is fat, and that $\bar{\mathbb{P}} = \langle \mathbb{P}_\zeta \rangle_{\zeta \leqslant \xi}$ is S-sparse and $\mathfrak{t}$-friendly. We need to check that $W = W[G_0]$ is stationary in $V(\mathbb{P}_\xi)$. Let $D \in V(\mathbb{P}_\xi)$ be a club. By extending by one step, we may assume that $\mathrm{cf}(\xi) \leqslant \lambda$.

Let $\bar{\tau}$ be an associated sequence for $\bar{\mathbb{P}}$. Let Q be the collection of $\zeta < \xi$ for which $\mathbb{Q}_\zeta = \mathbb{S}^{\mathfrak{t}}(S, G_0, \bar{c})$ for some appropriate $\bar{c} \in V(\mathbb{P}_\zeta)$; we write $\bar{c}^\zeta$ for $\bar{c}$. By assumption, for $\zeta \in \xi \backslash Q$, if $\mathrm{cf}(\zeta) \leqslant \lambda$ then σ_ζ is W-sparse; if $\mathrm{cf}(\zeta) > \lambda$ then we know that $\mathbb{Q}_\zeta$ is $<\lambda$-closed. We further modify the construction of $\bar{\tau}$ to ensure that for all $\varsigma \leqslant \xi$ with $\mathrm{cf}(\varsigma) \leqslant \lambda$ (so τ_ς is defined), for all $p \in \mathrm{dom}\, \tau_\varsigma$ and all $\zeta \in Q \cap \mathrm{supp}(p)$, $p \in A^\varsigma_\zeta$ and there is some string $\pi = \pi(p, \zeta)$ (in V) such that $p \restriction \zeta$ forces that $p(\zeta) = \pi$, and the length of π is $|\tau_\varsigma(p)| - 1$.

Elaborating only a little on our standard construction, we obtain a filtration $\bar{N} = \langle N_i \rangle_{i \leqslant \theta}$ such that letting $\gamma_i = N_i \cap \lambda$, we have:

- All objects above are elements of N_0, including an initial condition $p_{-1} \in \mathbb{P}_\xi$;
- $\theta \subset N_0$, and if $\lambda = \mu^+$ then $\mu \subseteq N_0$;
- For limit $i \leqslant \theta$, $\gamma_i \in S$;
- For successor $i < \theta$, $N_i^{<\theta} \subseteq N_i$.

We can do this since either λ is inaccessible, or $\lambda = \mu^+$ and $\mu^{<\theta} = \mu$. Note that the sequence $\langle \gamma_i \rangle$ is continuous. Let $\alpha^* = \gamma_\theta$. What is pertinent is that for successor $i < \theta$, $\operatorname{cf}(\gamma_i) \geqslant \theta$, so $E_{\alpha*} \cap \gamma_i$ is bounded below γ_i. Note that for limit $i < \theta$, $N_i^{<\theta} \subset N_{i+1}$.

We use the trees-of-conditions method to obtain a sparse sequence $\bar{p}$ and an upper bound q forcing $\alpha^* \in W \cap D$; to do this, we need to take care of all possible choices for $p_i(\zeta) \restriction E_{\alpha*}$ for $\zeta \in Q \cap \operatorname{supp} q$.

For $i \leqslant \theta$ we define:

- an ordinal δ_i;
- sets $u_i \subset Q$;
- functions $m_i \colon u_i \to i$.

Let T_i be the collection of all sequences $\bar{f} = \langle f_\zeta \rangle_{\zeta \in v}$, where $v \subseteq u_i$ is an initial segment of u_i and for all $\zeta \in v$, $f_\zeta \colon E_{\alpha*} \cap \gamma_i \to 2$. We also define:

- for each $\bar{f} \in T_i$, a condition $p_i(\bar{f}) \in \mathbb{P}_\xi$.

For a proper initial segment v of u_i, let $\varsigma(v) = \min(u_i \backslash v)$; let $\varsigma(u_i) = \xi$. For $\bar{f} \in T_i$ defined on v, we write $v(\bar{f}) = v$ and $\varsigma(\bar{f}) = \varsigma(v(\bar{f}))$. We say that $\bar{f}$ is *maximal* (for T_i) if $\operatorname{dom} \bar{f} = u_i$, i.e., if $\varsigma(\bar{f}) = \xi$. If $\bar{f} \in T_i$ and $\zeta \leqslant \xi$ then we write $\bar{f} \restriction \zeta$ for $\bar{f} \restriction (v \cap \zeta)$. Note that whether maximal or not, $\operatorname{cf}(\varsigma(\bar{f})) \leqslant \lambda$ (no $\zeta \in Q$ has cofinality λ^+), so in any case, $\tau_{\varsigma(\bar{f})}$ is defined; we write $\tau_v = \tau_{\varsigma(v)}$ and $\tau_{\bar{f}} = \tau_{\varsigma(\bar{f})} = \tau_{v(\bar{f})}$.

We ensure that the objects defined have the following properties:

(1) u_i, m_i, T_i and the map $\bar{f} \mapsto p_i(\bar{f})$ are all in N_{i+1} (and in fact for successor i, they will be in N_i);
(2) $|u_i| < \lambda$; if $\lambda = \mu^+$ then $|u_i| < \mu$;
(3) $u_i \subseteq u_j$ if $i < j$, and $u_j = \bigcup_{i<j} u_i$ for limit i;
(4) if $i < j$ then $m_i = m_j \restriction u_i$;
(5) $\delta_i = \gamma_i$ for limit i; for successor i, $\gamma_{i-1} < \delta_i < \gamma_i$ and $\sup(E_{\alpha*} \cap \gamma_i) < \delta_i$;
(6) For all $\bar{f} \in T_i$, $p_i(\bar{f}) \in \mathbb{P}_{\varsigma(\bar{f})}$, in fact $p_i(\bar{f}) \in \operatorname{dom} \tau_{\bar{f}}$, and $|\tau_{\bar{f}}(p_i(\bar{f})| = \delta_i + 1$;
(7) For all $\bar{f} \in T_i$ and all $\zeta \in v(\bar{f})$, $p_i(\bar{f}) \restriction \mathbb{P}_\zeta = p_i(\bar{f} \restriction \zeta)$;
(8) For $\bar{f} \in T_i$, $v(\bar{f}) \subset \operatorname{supp}(p_i(\bar{f}))$; we write $\pi_i(\bar{f}, \zeta)$ for the string $\pi(p_i(\bar{f}), \zeta)$ mentioned above.
(9) For all $\bar{f} \in T_i$ and $\zeta \in v(\bar{f})$, for all $\beta \in E_{\alpha*} \cap [\gamma_{m_i(\zeta)}, \gamma_i)$, we have $\pi_i(\bar{f}, \zeta)(\beta) = f_\zeta(\beta)$.
(10) For all $\bar{f} \in T_i$, $Q \cap \operatorname{supp}(p_i(\bar{f})) \subseteq u_\theta$;
(11) For all maximal $\bar{f} \in T_{i+1}$, $p_{i+1}(\bar{f})$ forces that $D \cap [\gamma_i, \gamma_{i+1})$ is nonempty;
(12) For all $j < i$, for all $\bar{f} \in T_i$, $p_i(\bar{f})$ extends $p_j(\bar{f}[j])$, where $\bar{f}[j] \in T_j$ is the sequence $\langle f_\zeta \restriction (E_\alpha^* \cap \gamma_j) \rangle_{\zeta \in u_j \cap v(\bar{f})}$;
(13) For all $j < i$ and $\bar{f} \in T_i$, if $\varsigma(\bar{f}) = \varsigma(\bar{f}[j])$, then the sequence $\langle p_k(\bar{f}[k] \rangle_{k \in [j,i)}$ is sparse for $\tau_{\bar{f}}$;
(14) $0 \in \operatorname{supp}(p_i(\bar{f}))$ for all $\bar{f} \in T_i$.

Let us show how to construct such objects. We start with $u_0 = \varnothing$; for $\bar f$ being the empty sequence (the only element of T_0) we let $p_0(\bar f)$ be some extension of p_{-1} (the initial condition we started with) in $\operatorname{dom} \tau_\xi$; note that $\varsigma(\bar f) = \xi$. We let $\delta_0 = |\tau_\xi(p_0(\bar f))| - 1$. We can ensure that $0 \in \operatorname{supp}(p_0(\bar f))$.

Suppose that $i \leqslant \theta$ is a limit, and that all objects indexed by $j < i$ have been defined, and satisfy the properties above, except of course for (10). As required, we define $u_i = \bigcup_{j<i} u_j$, $m_i = \bigcup_{j<i} m_j$, and $\delta_i = \gamma_i$. Let $\bar f \in T_i$ and let $v = v(\bar f)$. There is some $j < i$ such that $\varsigma(\bar f) = \varsigma(\bar f[j])$ (either $\varsigma(\bar f) \in u_j$, or is ξ); by (13), we can let $p_i(\bar f) = \mathrm{cb}_{\tau_{\bar f}}(\langle p_k[\bar f[k]]\rangle_{k\in[j,i)})$. To define the sequence of objects up to i, we need the sequence $\langle E_{\alpha*} \cap \gamma_j\rangle_{j<i}$; as each $E_{\alpha*} \cap \gamma_i$ has size $< \theta$, and $i < \theta$, this sequence is in N_{i+1}. Now (7) for i follows from the canonical choices of bounds being, well, canonical, and $\tau_\zeta \lessdot \tau_{\bar f} \restriction A_\zeta^{\varsigma(\bar f)}$. (9) follows from $\pi_i(\bar f, \zeta) = \bigcup_{k\in[j,i)} \pi_j(\bar f[k], \zeta)$ (as $p_i(\bar f)$ extends $p_j(\bar f[j])$), where $j < i$ is any such that $\zeta \in u_j$.

Suppose that $i < \theta$ is a successor ordinal, and that all objects have been defined for $j \leqslant i-1$. First, we define u_i. This is done to make progress towards (10):

- If λ is inaccessible, then we can let $u_i = \bigcup_{\bar f\in T_{i-1}} Q \cap \operatorname{supp}(p_{i-1}(\bar f))$.
- Otherwise, $\lambda = \mu^+$. If μ is a limit cardinal, let $\langle \mu_j\rangle_{j<\theta}$ be a sequence of cardinals increasing to μ. In this case $|T_{i-1}| < \mu$, so we just ensure that for all $\bar f \in T_{i-1}$, "μ_i-much" of $Q \cap \operatorname{supp}(p_{i-1}(\bar f))$ is added to u_i (note that $|\operatorname{supp}(p_{i-1}(\bar f))|$ is likely μ).
- If $\mu = \nu^+$ is a successor, then we will likely have $|T_{i-1}| = \mu$ (even though $|u_{i-1}| \leqslant \nu$), as $\theta = \mu$ and $(<\theta)^\nu = \mu$. We then add "i-much" of $Q \cap \operatorname{supp}(p_{i-1}(\bar f))$ for "i-many" $\bar f \in T_{i-1}$.

We define m_i to extend m_{i-1} by letting $m_i(\zeta) = i-1$ for all $\zeta \in u_i \backslash u_{i-1}$. Note that u_i and γ_i determine T_i.

To define $p_i(\bar f)$ for $\bar f \in T_i$ we perform a transfinite construction. We work in N_i. By our analysis immediately above, there is a regular cardinal $\kappa < \lambda$ with $\kappa \geqslant |T_i|$. Let $\bar M$ be a filtration (with all relevant objects in M_0); we obtain a sequence $\langle \varepsilon_\ell\rangle_{\ell<\kappa}$ with the usual properties: its limit points are in S, $M_{\varepsilon_\ell} \cap \lambda = \varepsilon_\ell$, and for $\ell < \kappa$ we have $\langle \varepsilon_{\ell'}\rangle_{\ell'<\ell} \in M_{\varepsilon_\ell+1}$. Since κ may be larger than θ, we cannot require that the sequence is continuous.

Let $\langle \bar f_\ell\rangle_{\ell<\kappa}$ be a list of all maximal $\bar f \in T_i$, where each such $\bar f$ appears unboundedly often. This list is in M_0. We also ensure that $\sup(E_{\alpha*} \cap \gamma_i) < \varepsilon_0$ (recall that $E_{\alpha*} \cap \gamma_i$ is bounded below γ_i for successor i).

For $\ell \leqslant \kappa$ and $\bar f \in T_i$ we define conditions $r_\ell(\bar f)$, satisfying the following:

(i) $r_0(\bar f) = p_{i-1}(\bar f[i-1])$;
(ii) Either $r_\ell(\bar f) = r_0(\bar f)$, or $v(\bar f) \subset \operatorname{supp}(r_\ell(\bar f))$ and $r_\ell(\bar f) \in \operatorname{dom} \tau_{\bar f}$, in fact $r_\ell(\bar f) \in A_\zeta^{\varsigma(\bar f)}$ for all $\zeta \in v(\bar f)$;

If $r_\ell(\bar f) \neq r_0(\bar f)$ then we write $\eta_\ell(\bar f) = |\tau_{\bar f}(r_\ell(\bar f))|$, and for $\zeta \in v(\bar f)$, we let $\pi_\ell(\bar f, \zeta) = \pi(r_\ell(\bar f), \zeta)$;

(iii) If $\zeta \in v(\bar f)$ then $r_\ell(\bar f \restriction \zeta)$ extends $r_\ell(\bar f) \restriction \mathbb{P}_\zeta$;
(iv) For $\ell < \kappa$, the map $\bar f \mapsto r_\ell(\bar f)$ is in $M_{\varepsilon_\ell+1}$;
(v) For $\ell' < \ell$, $r_\ell(\bar f)$ extends $r_{\ell'}(\bar f)$;

We say that $r_\ell(\bar f)$ is *new* if $\ell > 0$ and for all $\ell' < \ell$, $r_{\ell'}(\bar f) \neq r_\ell(\bar f)$;

(vi) If $r_\ell(\bar f)$ is new then $\eta_\ell(\bar f) \geqslant \varepsilon_{<\ell} := \sup_{\ell'<\ell} \varepsilon_{\ell'}$;

(vii) If $r_\ell(\bar f)$ is new then for all $\zeta \in v(\bar f)$, $r_\ell(\bar f \restriction \zeta)$ is new, and equals $r_\ell(\bar f) \restriction \mathbb{P}_\zeta$;

(viii) If $r_\ell(\bar f) \neq r_0(\bar f)$ then for all $\zeta \in v(\bar f)$, $\pi_\ell(\bar f, \zeta)$ extends $f_\zeta \restriction [\gamma_{i-1}, \gamma_i)$.

(ix) For all limit $\ell < \kappa$, the subsequence

$$\left\langle r_{\ell'}(\bar f) \,:\, r_{\ell'}(\bar f) \text{ is new } \right\rangle$$

is sparse for $\tau_{\bar f}$ (but note that it may not be cofinal in $\langle r_{\ell'}(\bar f)\rangle_{\ell'<\ell}$, in which case the latter sequence is eventually constant.)

For $\ell = 0$ we follow (iii). At limit $\ell \leqslant \kappa$ we let $r_\ell(\bar f)$ be either the eventually constant value of $r_{\ell'}(\bar f)$ for $\ell' < \ell$, if such exists; otherwise, we let $r_\ell(\bar f)$ be the $\tau_{\bar f}$-canonical upper bound of the sparse subsequence of new $r_{\ell'}(\bar f)$. Note that $r_\ell(\bar f)$ is new if and only if the second case holds, in which case, by (vii), for all $\zeta \in v(\bar f)$, the second case holds for defining $r_\ell(\bar f \restriction \zeta)$, and it equals $r_\ell(\bar f) \restriction \mathbb{P}_\zeta$ by the coherence $\tau_\zeta \lessdot \tau_{\bar f} \restriction A_\zeta^{\varsigma(\bar f)}$.

for the successor case, suppose that $r_\ell(\bar f)$ have been defined for all $\bar f$. We now consider $\bar f_\ell$ in steps:

(1) First, obtain the condition s_0 which is the "sum" of the sequence $\left\langle r_\ell(\bar f_\ell \restriction v)\right\rangle$ for v an initial segment of u_i; for $\zeta \in \operatorname{supp}(s_0) = \bigcup_{v \leqslant u_i} \operatorname{supp}(r_\ell(\bar f_\ell \restriction v))$, having defined $s_0 \restriction \zeta$ extending each $r_\ell(\bar f_\ell \restriction v) \restriction \mathbb{P}_\zeta$, we define $s_0(\zeta)$ to be $r_\ell(\bar f_\ell \restriction \zeta+1)(\zeta)$.

Note that for all $\zeta \in u_i \cap \operatorname{supp}(s_0)$, $s_0 \restriction \zeta$ forces that $s_0(\zeta)$ is $\pi(s_0, \zeta) = \pi(r_\ell(\bar f_\ell \restriction \zeta+1), \zeta)$, and either $|\pi(s_0,\zeta)| \leqslant \gamma_{i-1}$, or $\pi(s_0,\zeta)$ extends $f_\zeta \restriction [\gamma_{i-1}, \gamma_i)$ (where f_ζ) of course comes from the sequence $\bar f_\ell$).

(2) Extend s_0 to a condition $s_1 \in \mathbb{P}_\xi$ by setting $\operatorname{supp}(s_1) = \operatorname{supp}(s_0) \cup u_i$; for each $\zeta \in u_i$ such that $|\pi(s_0,\zeta)| \leqslant \gamma_{i-1}$, or $\zeta \notin \operatorname{supp}(s_0)$, we set $s_1(\zeta)$ to be some string (in V) which extends $\pi(s_0,\zeta)$ if defined, and which extends $f_\zeta \restriction [\gamma_{i-1}, \gamma_i)$.

(3) Extend s_1 to a condition s_2 in $\bigcap_{\zeta \in u_i} A_\zeta^\xi$, and also ensure that s_2 forces some $\beta \in [\gamma_{i-1}, \gamma_i)$ into D, where recall D is the club in $V(\mathbb{P}_\xi)$ we want to get to meet W at α^*. Also ensure that $|\tau_\xi(s_2)| > \varepsilon_{\ell-1}$. Further, by extending, we can ensure that for all initial segments v of u_i, $s_2 \restriction \mathbb{P}_{\varsigma(v)}$ properly extends $s_1 \restriction \mathbb{P}_\zeta$, and so properly extends $r_\ell(\bar f_\ell \restriction v)$.

(4) Set $r_{\ell+1}(\bar f_\ell) = s_2$, and for all $\zeta \in u_i$, set $r_{\ell+1}(\bar f_\ell \restriction \zeta) = s_2 \restriction \mathbb{P}_\zeta$. For $\bar g \in T_i$ which is not an initial segment of $\bar f_\ell$, set $r_{\ell+1}(\bar g) = r_\ell(\bar g)$.

Note that we ensured that $r_{\ell+1}(\bar g)$ is new if and only if $\bar g$ is an initial segment of $\bar f_\ell$. Also note that for $\zeta < \zeta' \in u_i$, $r_{\ell+1}(\bar f \restriction \zeta') \in A_\zeta^{\zeta'}$ follows from Definition 2.28(d).

This completes the construction of all $r_\ell(\bar f)$; we let $p_i(\bar f) = r_\kappa(\bar f)$. Since each $\bar f$ is tended to unboundedly many times, we see that each $r_\kappa(\bar f)$ is new; so for all $\bar f$, $\eta_\kappa \bar f = \varepsilon_{<\kappa} + 1$, which we set to be $\delta_i + 1$. This completes the construction of all $p_i(\bar f)$ for $i \leqslant \theta$.

We want to find some maximal $\bar f \in T_\theta$ and some condition q, extending $p_i(\bar f)$ for all $i < \theta$ (and as usual, not extending $p_\theta(\bar f)$), forcing that $\alpha^* \in W$; such a condition also forces that $\alpha^* \in D$. To this end, call a sequence $\bar f \in T_\theta$ *acceptable* if there is some condition $q \in \mathbb{P}_{\varsigma(\bar f)}$, extending $p_i(\bar f)$ for all $i < \theta$, which forces that $\alpha^* \in W$. So we want to show that some maximal $\bar f \in T_\theta$ is acceptable.

To do that, we show by induction on initial segments υ of u_θ, that some $\bar{f}$ with $\upsilon(\bar{f}) = \upsilon$ is acceptable. To do that, we show:

(1) The empty sequence is acceptable; and
(2) For all initial segments $w < u$ of u_θ, if $\bar{g} \in T_\theta$ with $\upsilon(\bar{g}) = w$ is acceptable, witnessed by some q, then $\bar{g}$ can be extended to an acceptable $\bar{f}$ with $\upsilon(\bar{f}) = u$, witnessed by some r extending q.

Let us show (1), that the empty sequence $\langle\rangle$ is acceptable. We have $0 \in \operatorname{supp}(p_\theta(\langle\rangle))$, and $p_\theta(0)$ is a sequence of length $\alpha^* + 1$ ending with 0. We let $q(0)$ agree with $p_\theta(\langle\rangle)(0)$, except that we change the last 0 to a 1, i.e., we say $\alpha^* \in q(0)$. This condition extends $p_i(\langle\rangle)$ for all $i < \theta$. Let $\upsilon = \varsigma(\langle\rangle) = \min u_\theta$; since $Q \cap \operatorname{supp}(p_\theta(\langle\rangle)) \subseteq u_\theta$, we have $Q \cap \operatorname{supp}(p_\theta(\langle\rangle)) = \varnothing$. Thus, we can define the condition $q \in \mathbb{P}_\upsilon$ by setting $\operatorname{supp}(q) = \operatorname{supp}(p_\theta(\langle\rangle))$, and by defining $q \restriction \zeta$ by induction on $\zeta \leqslant \upsilon$. This has already been done for $\zeta = 0$. If $\zeta > 0$ is in $\operatorname{supp}(q)$ and $q \restriction \zeta$ has already been defined, then as $\zeta \notin Q$, we know that $\mathbb{Q}_\zeta$ is explicitly W-closed; $p_i(\langle\rangle)(\zeta)$ is defined for a final segment of ζ, with height δ_i; since $q \restriction \zeta$ forces that $\alpha^* \in W$, it forces that there is some upper bound for $\langle p_i(\langle\rangle)\rangle$, which we set to be $q(\zeta)$.

Now we tend to (2), which is proved by induction on (the order-type of) u. Suppose this has been proved for all initial segments u' of u. There are two cases, depending on the order-type of u. First, suppose that u has a greatest element υ. Let $\varrho = \varsigma(u)$. By induction, it suffices to show (2) for $w = u \cap \upsilon$. Suppose that $\upsilon(\bar{g}) = w$ and that $\bar{g}$ is acceptable, witnessed by some q.

We define $r \restriction \mathbb{P}_\zeta$ by induction on $\zeta \in [\upsilon, \varrho]$. We let $r \restriction \mathbb{P}_\upsilon$ be some extension of q which decides the value of $c^\upsilon_{\alpha^*}$ (where recall $\mathbb{Q}_\upsilon = \mathbb{S}^{\mathfrak{t}}(S, G_0, \bar{c}^\upsilon)$), say it forces that $c^\upsilon_{\alpha^*} = f_\upsilon$ (where $f_\upsilon \in V$); we let $\bar{f} = \bar{g}\hat{\ } f_\upsilon$. We also let $r(\upsilon) = \pi_\theta(\bar{f}, \upsilon)$; then $r \restriction \upsilon + 1$ is a condition since $\pi_\theta(\bar{f}, \upsilon)$ agrees with f_υ from $\gamma_{m_\theta(\upsilon)}$ onwards.

We then repeat the argument for the empty sequence; we set $\operatorname{supp}(r)$ to agree with $\operatorname{supp}(p_\theta(\bar{f}))$ on (υ, ϱ), and note that there it is disjoint from Q; for $\zeta \in (\upsilon, \varrho) \cap \operatorname{supp}(r)$ we let $r(\zeta)$ be an upper bound of $\langle p_i(\bar{f})(\zeta)\rangle_{i \in [i^*, \theta)}$, which is forced by $r \restriction \zeta$ to exist since it forces that $\alpha^* \in W$.

Finally, suppose that the order-type of u is a limit; let $\kappa < \lambda$ be regular and let $\langle w_i \rangle_{i \leqslant \kappa}$ be an increasing and continuous sequence of initial segments of u with $w_\kappa = u$, with $w_0 = w$ being the initial segmen we start with; let $\bar{g}$ be acceptable, witnessed by some q_0, with $\upsilon(\bar{g}) = w_0$.

As expected, we work with a filtration $\bar{N}$ and a sequence $\langle \varepsilon_j \rangle_{j < \kappa}$ with limit points in S, $N_{\varepsilon_j} \cap \lambda = \varepsilon_j$, with N_0 containing all pertinent objects. We mimic the construction of Lemma 2.23. We define sequences $\langle q_j \rangle_{j \leqslant \kappa}$ of conditions and $\bar{f}_j$ satisfying:

(a) $\langle q_\ell, \bar{f}_\ell \rangle_{\ell < j}, \in M_{\varepsilon_j + 1}$;
(b) $\upsilon(\bar{f}_j) = w_j$, and $q_j \in \mathbb{P}_{\varsigma(w_j)}$ witnesses that $\bar{f}_j$ is acceptable;
(c) If $\ell < j$ then q_j extends q_ℓ and $\bar{f}_\ell = \bar{f}_j \restriction w_\ell$;
(d) For successor $j < \kappa$, $q_j \in \operatorname{dom} \tau_{\varsigma(w_j)}$, indeed $q_j \in A^{\varsigma(w_j)}_\zeta$ for all $\zeta \in w_j$; and $|\tau_{\varsigma(w_j)}| > \varepsilon_j$;
(e) For all $\ell < j$, $\langle q_k \restriction \mathbb{P}_{\varsigma(w_\ell)} \rangle_{k \in (\ell, j)}$ is sparse for $\tau_{\varsigma(w_\ell)}$.

At successor steps we apply the induction hypothesis from w_j to w_{j+1}, and then extend to a condition in dom $\tau_{\varsigma(w_{j+1})}$ as required; at limit steps we take canonical sparse upper bounds and then an inverse limit. This completes the proof. $\square$

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