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INFINITE STABLE GRAPHS WITH LARGE CHROMATIC NUMBER

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Abstract. We prove that if $G = (V, E)$ is an ω-stable (respectively, superstable) graph with $\chi(G) > \aleph_0$ (respectively, $2^{\aleph_0}$) then G contains all the finite subgraphs of the shift graph $\mathrm{Sh}_n(\omega)$ for some n. We prove a variant of this theorem for graphs interpretable in stationary stable theories. Furthermore, if G is ω-stable with $\mathrm{U}(G) \leq 2$ we prove that $n \leq 2$ suffices.

1. Introduction

The chromatic number $\chi(G)$ of a graph $G = (V, E)$ is the minimal cardinal $\varkappa$ for which the exists a vertex coloring with $\varkappa$ colors. There is a long history of structure theorems deriving from large chromatic number assumptions. For example if $\chi(G) \geq \aleph_1$ then G must contain all finite bipartite graphs [EH66, Corollary 5.6] and every sufficiently large odd circuit [EHS74, Theorem 3], [Tho83]. See [Kom11] for more information.

In [Tay71, Problem 1.14], Taylor asked what is the least cardinal $\varkappa$ such that every graph G with $\chi(G) \geq \varkappa$ is elementary equivalent to graphs of arbitrarily large chromatic number. It is clear that such a minimal cardinal exists (see [Tay71, Theorem 1.13]). Taylor noted that necessarily $\varkappa \geq \aleph_1$. Nowadays, Taylor's conjecture is usually phrased in the following way (see [Kom11, Section 3]).

Conjecture (Taylor's conjecture)**.** *For any graph G with $\chi(G) \geq \aleph_1$ and cardinal κ there exists a graph H with $\chi(H) \geq \kappa$ such that G and H share the same finite subgraphs.*

For a cardinal κ the shift graph $\mathrm{Sh}_n(\kappa)$ is the graph whose vertices are increasing n-tuples s of ordinals less than κ, where we put an edge between s and t if for every $1 \leq i \leq n-1$, $s(i) = t(i-1)$ or vice-versa. The shift graphs $\mathrm{Sh}_n(\kappa)$ have large chromatic numbers depending on κ; see Fact 2.6. Erdös-Hajnal-Shelah [EHS74, Problem 2] and Taylor [Tay70, Problem 43, page 508] proposed the following strengthening of the previous conjecture.

Conjecture (Strong Taylor's conjecture)**.** *For any graph G with $\chi(G) \geq \aleph_1$ there exists an $n \in \mathbb{N}$ such that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$.*

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Assuming the strong Taylor's conjecture, if $\chi(G) \geq \aleph_1$ there exists an elementary extension $G \prec \mathcal{G}$ that has $\mathrm{Sh}_n(\beth_{n-1}(\kappa)^+)$ as a subgraph, and thus $\chi(\mathcal{G}) \geq \kappa^+$; see Fact 2.6. So the strong Taylor's conjecture implies Taylor's conjecture. It is known that Taylor's conjecture is consistently false and that a relaxation of Taylor's conjecture is consistently true, namely assuming that $\chi(G) \geq \aleph_2$ [KS05]. The strong Taylor's conjecture was refuted in [HK84, Theorem 4].

Since the (strong) Taylor's conjecture fails in general, one may wonder if it holds for a "tame" class of graphs. Classification theory provides "dividing lines" separating "tame" and "wild" classes of structures (and theories). These dividing lines are usually defined by requiring that a structure omits a certain class of (definable) combinatorial patterns. It is thus not surprising that restricting to such graphs will yield better combinatorial results.

An important instance of this phenomena is when tame=stable. Stable theories, which originated in the work of the third author in the 60s and 70s, is the most extensively studied class. Examples of stable theories include abelian groups, modules, algebraically closed fields, graph theoretic trees, or more generally superflat graphs [PZ78]. Stablility also had an impact in combinatorics, e.g. [MS14] and [CPT20] to name a few.

In this paper we prove variants of the strong Taylor's conjecture for some classes of stable graphs.

Theorem. *Let $G = (V, E)$ be a graph. If*

(1) *G is ω-stable and $\chi(G) > \aleph_0$ or*
(2) *G is superstable and $\chi(G) > 2^{\aleph_0}$ or*
(3) *G is interpretable in a stable structure, in which every type (over any set) is stationary, and $\chi(G) > \beth_2(\aleph_0)$*

then G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \in \mathbb{N}$.

Furthermore, if G is ω-stable with $\chi(G) > \aleph_0$ and $\mathrm{U}(G) \leq 2$ then $n \leq 2$ suffices.

Items (1) and (2) are Corollary 4.5, (3) is Corollary 5.21 and the furthermore is Theorem 6.9.

The following remains open.

Question.

(1) What is the situation with general stable graphs?
(2) Is it enough to assume $\chi(G) > \aleph_0$ in the above theorem?
(3) What about other tameness assumptions, e.g. NIP or simplicity?

2. Notation and preliminaries

We use fairly standard model theoretic terminology and notation, see for example [TZ12]. We use small latin letters a, b, c for tuples and capital letters A, B, C for sets. We also employ the standard model theoretic abuse of notation and write $a \in A$ even for tuples when the length of the tuple is immaterial or understood from context. When we write $a \equiv_A b$ we mean that $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$.

For any two sets A and J, let $A^{\underline{J}}$ be the set of injective functions from J to A (where the notation is taken from the falling factorial notation), and if $(A, <)$ and $(J, <)$ are both linearly ordered sets, let $(A^{\underline{J}})_<$ be the subset of $A^{\underline{J}}$ consisting of strictly increasing functions. If we want to emphasize the order on J we will write $(A^{\underline{(J,<)}})_<$. For an ordinal γ, we set $A^{\underline{<\gamma}} := \bigcup_{\alpha<\gamma} A^{\underline{\alpha}}$. Throughout this paper, we interchangeably use sequence notation and function notation for elements of $A^{\underline{J}}$, e.g. for $f \in A^{\underline{J}}$, $f(i) = f_i$. For any sequence η we denote by $\mathrm{Range}(\eta)$ the underlying set of the sequence (i.e. its image). If $(A, <^A)$ and $(B, <^B)$ are linearly ordered sets, then the most significant coordinate of the lexicographic order on $A \times B$ is the first one.

By a *graph* we mean a pair $G = (V, E)$ where $E \subseteq V^2$ is symmetric and irreflexive. A *graph homomorphism* between $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a map $f : V_1 \to V_2$ such that $f(e) \in E_2$ for every $e \in E_1$. If f is injective we will say that f embeds G_1 into G_2 a subgraph. If in addition we require that $f(e) \in E_2$ if and only if $e \in E_1$ we will say that f embeds G_1 into G_2 as an induced subgraph.

Definition 2.1. Let $G = (V, E)$ be a graph.

(1) For a cardinal $\varkappa$, a *vertex coloring* (or just coloring) of size $\varkappa$ is a function $c : V \to \varkappa$ such that $x \mathrel{E} y$ implies $c(x) \neq c(y)$ for all $x, y \in V$.
(2) The *chromatic number* $\chi(G)$ is the minimal cardinality of a vertex coloring of G.

Remark 2.2. Note that for a graph $G = (V, E)$ with $|V| \geq 2$, $\chi(G) = 1$ if and only if $|E| = \emptyset$.

Here are some useful easy and well known properties of the chromatic number function of graphs (we provide proofs for the convenience of the reader).

Lemma 2.3. *Let $G = (V, E)$ be a graph.*

(1) *If $V = \bigcup_{i \in I} V_i$ then $\chi(G) \leq \sum_{i \in I} \chi(V_i, E \restriction V_i)$.*
(2) *If $E = \bigcup_{i \in I} E_i$ (with the E_i being symmetric) then $\chi(G) \leq \prod_{i \in I} \chi(V, E_i)$.*
(3) *If $\varphi : H \to G$ is a graph homomorphism then $\chi(H) \leq \chi(G)$.*
(4) *If $\varphi : (H, E^H) \to (G, E^G)$ is a surjective graph homomorphism with $e \in E^H \iff \varphi(e) \in E^G$ then $\chi(H) = \chi(G)$.*

Proof. (1) Let $c_i : V_i \to \varkappa_i$ be a coloring of $(V_i, E \restriction V_i)$. Define a coloring $c : V \to \bigcup\{\varkappa_i \times \{i\} : i \in I\}$ by choosing for any $v \in V$ an $i_v \in I$ such that $v \in V_{i_v}$ and setting $c(v) = (c_{i_v}(v), i_v)$.

(2) Let $c_i : V_i \to \varkappa_i$ be a coloring of (V, E_i). Define a coloring $c : V \to \prod_{i \in I} \varkappa_i$ by $c(v)(i) = c_i(v)$.

(3) Write $G = (V^G, E^G)$ and $H = (V^H, E^H)$ and let $c : V^G \to \varkappa$ be a coloring of G. Define a coloring $c' : V^H \to \varkappa$ of H by $c'(v) = c(f(v))$.

(4) Let $c : V^H \to \varkappa$ be a coloring. We define a coloring $c' : V^G \to \varkappa$ by choosing for any element $v \in V^G$ an element $w \in \varphi^{-1}(v)$ and setting $c'(v) = c(w)$. It is a legal coloring since if $v_1 \mathrel{E^G} v_2$ then $w_1 \mathrel{E^H} w_2$ for any $w_1 \in \varphi^{-1}(v_1)$ and $w_2 \in \varphi^{-1}(v_2)$. □

We will mainly be interested with the following so called "Shift Graphs", first defined by Erdös-Hajnal in [EH68].

Example 2.4 (Shift Graph)**.** For any finite number $1 \leq r$ and any linearly ordered set $(A, <)$, let $\mathrm{Sh}_r(A)$, or $\mathrm{Sh}_r(A, <)$ if we want to emphasize the order, (the shift graph on A) be the following graph: its set of vertices is the set $(A^{\underline{r}})_<$ of increasing r-tuples, $s_0, \ldots, s_{r-1}$, and we put an edge between s and t if for every $1 \leq i \leq r-1$, $s(i) = t(i-1)$, or vice-versa. It is an easy exercise to show that $\mathrm{Sh}_r(A)$ is a connected graph. If $r = 1$ this gives K_A, the complete graph on A.

Example 2.5 (Symmetric Shift Graph)**.** Let $1 \leq r$ be any natural number and A any set. The *symmetric shift graph* $\mathrm{Sh}_r^{sym}(A)$ is defined similarly as the shift graph but with set of vertices $A^{\underline{r}}$ (set of distinct r-tuples). Note that $\mathrm{Sh}_r(A)$ is an induced subgraph of $\mathrm{Sh}_r^{sym}(A)$ (and that for $r = 1$ they are both the complete graph on A). Recall that $\beth_0(\varkappa) := \varkappa$ and $\beth_{k+1}(\varkappa) := 2^{\beth_k(\varkappa)}$.

Fact 2.6 ([EH68, Proof of Theorem 2])**.** Let $2 \leq r < \omega$ be a natural number and $\varkappa$ be an infinite cardinal,

$$\chi\left(\mathrm{Sh}_r^{sym}(\beth_{r-1}(\varkappa))\right) \leq \varkappa$$

and

$$\chi\left(\mathrm{Sh}_r(\beth_{r-1}(\varkappa)^+)\right) \geq \varkappa^+.$$

Proof. We first show that $\chi\left(\mathrm{Sh}_r^{sym}(\beth_{r-1}(\varkappa))\right) \leq \varkappa$. The proof is by induction on $r \geq 2$. Suppose $r = 2$. Let $<$ be the lexicographical order on $2^\varkappa$. Let $Y_1 = \left((2^\varkappa)^{\underline{2}}\right)_<$ be the set of increasing pairs, let Y_2 be the complement. By Lemma 2.3(1) it is enough to show that $\chi\left(\mathrm{Sh}_2^{sym}(2^\varkappa) \restriction Y_1\right) \leq \varkappa$, $\chi\left(\mathrm{Sh}_2^{sym}(2^\varkappa) \restriction Y_2\right) \leq \varkappa$. The proofs for Y_1 and Y_2 are similar so we prove it just for Y_1.

Given $(x, y) \in Y_1$, let $c(x, y) = \min\{i < \varkappa : x(i) \neq y(i)\}$. Suppose that $x < y < z \in 2^\varkappa$ are such that $c(x, y) = c(y, z)$. Then $x \wedge y = y \wedge z$ (where $x \wedge y = x \restriction c(x, y)$). As $x < y$ it must be that $x(c(x, y)) = 0$ and $y(c(x, y)) = 1$, but then there is no room for $z(c(x, y))$ — contradiction.

Now suppose that the claim is true for r and $\varkappa$. By induction, there is a coloring $d : \mathrm{Sh}_r^{sym}(\beth_r(\varkappa)) \to 2^\varkappa$. Let $\psi : \mathrm{Sh}_{r+1}^{sym}(\beth_r(\varkappa)) \to \mathrm{Sh}_2^{sym}(2^\varkappa)$ be the following homomorphism. Given $u = (u_0, \dots, u_r) \in \mathrm{Sh}_{r+1}^{sym}(\beth_r(\varkappa))$, let $\psi(u) = (d(u_0, \dots, u_{r-1}), d(u_1, \dots, u_r))$. Note that by choice of d,

$$d(u_0, \dots, u_{r-1}) \neq d(u_1, \dots, u_r).$$

In addition, if u and v are connected in $\mathrm{Sh}_{r+1}^{sym}(\beth_r(\varkappa))$, then easily $\psi(u)$ and $\psi(v)$ are distinct (because if not, then $\psi(u)_0 = \psi(v)_0 = \psi(u)_1$ contradiction) and connected in $\mathrm{Sh}_2^{sym}(2^\varkappa)$. Hence we are done by Lemma 2.3(3).

As for the second inequality, assume towards a contradiction that there exists a coloring $c : \mathrm{Sh}_r(\beth_{r-1}(\varkappa)^+) \to \varkappa$ be a coloring. The coloring c induces a coloring on $[\beth_{r-1}(\varkappa)^+]^r$. By Erdös-Rado, there is a subset $U \subseteq \beth_{r-1}(\varkappa)^+$ of cardinality $\varkappa^+$ such that $c \restriction [U]^r$ is constant, i.e. after identifying $[U]^r$ with $(U^{\underline{r}})_<$, every r-tuple of increasing elements from U is colored by the same color. Let $u \in [U]^r$ be any element and let $v \in [U]^r$ be defined by $v(i) = u(i+1)$ for $0 \leq i < r-1$ and $v(r-1)$ is any element in U larger than $v(r-2)$. They are obviously connected by an edge, contradicting the fact that c is a coloring. $\square$

3. Embedding a shift graph

The aim of this section is to present some general assumptions on a graph G that will imply that G contains the finite subgraphs of some shift graph.

3.1. **Reducing injective homomorphisms to homomorphisms.** As a first result we prove the following, probably well known, proposition. By Lemma 2.3(3), if there is a homomorphism $\varphi : H \to G$ then $\chi(H) \leq \chi(G)$. In particular, if H is a shift graph then there are elementary extensions of G with arbitrary large chromatic numbers. Indeed, one may take elementary extensions of the structure (H, G, φ) and apply Fact 2.6.

Fact 3.1 ([ER50, Theorem 1]). Let R be an equivalence relation on $(\omega^{\underline{n}})_<$. Then there exists an infinite subset $N \subseteq \omega$ and $0 \le i_1 < \cdots < i_m \le n-1$ such that for $\bar{a}, \bar{b} \in (N^{\underline{n}})_<$,

$$\bar{a} \mathrel{R} \bar{b} \iff \bigwedge_{j=1}^{m} a_{i_j} = b_{i_j}.$$

Proposition 3.2. *Let $G = (V, E)$ be a graph and assume there exists a homomorphism of graphs $t : \mathrm{Sh}_k(\omega) \to G$. Then there exists $n \le k$, such that*

(†) G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$.

Consequently, if H is a graph that contains all finite subgraphs of $\mathrm{Sh}_k(\omega)$, for some k, and $t : H \to G$ is a homomorphism of graphs, then there exists some $n \le k$ such that G satisfies (†).

Proof. Assume that $t = t(x_0, \ldots, x_{k-1})$. The relation $t(\bar{a}) = t(\bar{b})$ for $\bar{a}, \bar{b} \in (\omega^{\underline{k}})_<$, is an equivalence relation on $(\omega^{\underline{k}})_<$. By Fact 3.1, there exists an infinite subset $N \subseteq \omega$ and $0 \le i_1 < \cdots < i_m \le k-1$ such that for $\bar{a}, \bar{b} \in (N^{\underline{k}})_<$

$$(\dagger\dagger) \qquad t(\bar{a}) = t(\bar{b}) \iff \bigwedge_{j=1}^{m} a_{i_j} = b_{i_j}.$$

Note that $m \ge 1$ since otherwise $t(\bar{a}) = t(\bar{b})$ for any $\bar{a}$ and $\bar{b}$, but this is impossible since there are $\bar{a}, \bar{b} \in (N^{\underline{k}})_<$ that are connected by an edge.

Let $S = \{i_1, \ldots, i_m\}$. There exists a unique set $I \subseteq \{1, \ldots, m\}$ and a unique sequence of natural numbers $\bar{n} = \langle n_j : j \in I \rangle$ such that $S = \bigcup_{j \in I} [i_j, i_j + n_j]$ and each interval $[i_j, i_j + n_j]$ is maximal with respect to containment.

Consider the first-order structure $M = ((N, <), G = (V, E), t : (N^{\underline{k}})_< \to G)$. Since (†) and (††) are elementary properties, replacing M by an elementary extension, we may assume that $(I \times \mathbb{Z}, <_{\mathrm{lex}}) \subseteq (N, <)$.

We define an injective homomorphism $\mathrm{Sh}_{n+1}(\omega) \to G$, where $n = \max_{j \in I}\{n_j\}$. For any $f \in (\omega^{\underline{n+1}})_<$ we associate $\psi_f \in V$. For that we first define $\eta_f \in ((I \times \mathbb{Z})^{\underline{k}})_<$ and then set $\psi_f = t(\eta_f)$. For any $j \in I$ and $0 \le r \le n_j$ we define

$$\eta_f(i_j + r) = (j, f(r)).$$

For $0 \le i \le k-1$ with $i \notin S$, set $\eta_f(i)$ any way we want provided η_f is increasing, which we can since we have copies of $\mathbb{Z}$. Note that the choice of $\eta_f(i)$ for $i \notin S$ does not influence $t(\eta_f)$ by (††).

We check that $f \mapsto \psi_f$ is an injective homomorphism. Injectivity: if $t(\eta_f) = t(\eta_g)$ then by (††), $\eta_f(i) = \eta_g(i)$ for all $i \in S$. In particular for $j \in I$ with $n_j = n$ and for any $0 \le r \le n$, $f(r) = g(r)$, as needed.

Homomorphism: let $f, g \in \mathrm{Sh}_{n+1}(\omega)$ be two vertices connected by an edge and assume without loss of generality that for every $1 \le r \le n$, $f(r) = g(r-1)$ and in case $n = 0$ assume that $f(0) < g(0)$.

Define $\eta'_g \in ((I \times \mathbb{Z})^{\underline{k}})_<$ as follows. For every $j \in I$ and $0 \le r \le n_j$ let $\eta'_g(i_j + r) = \eta_g(i_j + r)$ and if $i_j > 0$ then set

$$\eta'_g(i_j - 1) = (j, f(0)).$$

Note that if $n \ne 0$ then

$$\eta'_g(i_j - 1) = (j, f(0)) < (j, f(1)) = (j, g(0)) = \eta'_g(i_j)$$

and if $n = 0$ then

$$\eta'_g(i_j - 1) = (j, f(0)) < (j, g(0)) = \eta'_g(i_j).$$

Hence η'_g restricted to $S \cup \{i_j - 1 : j \in I, i_j > 0\}$ is increasing. For any other $i \in I$ set $\eta'_g(i)$ any way we want provided η'_g is increasing. Note that $\eta'_g(i) = \eta_g(i)$ for any $i \in S$.

Define $\eta'_f \in ((I \times \mathbb{Z})^{\underline{k}})_<$ by

$$\eta'_f(i) = \eta'_g(i-1)$$

for all $1 \leq i \leq k-1$.

If $i_1 = 0$ define $\eta'_f(0) = (1, f(0))$. Note that then if $n \neq 0$ then $\eta'_f(0) = (1, f(0)) < (1, f(1)) = (1, g(0)) = \eta'_g(0) = \eta'_f(1)$ and if $n = 0$ then $\eta'_f(0) = (1, f(0)) < (1, g(0)) = \eta'_g(0) = \eta'_f(1)$.

Otherwise, i.e. $i_1 \neq 0$, define $\eta'_f(0)$ to be any element smaller than $\eta'_g(0)$.

If we show that $\eta'_f(i) = \eta_f(i)$ for all $i \in S$ then this would imply that $t(\eta'_f) = t(\eta_f)$. Since η'_f and η'_g are connected by an edge and t is a homomorphism it follows that $\psi_f = t(\eta_f) = t(\eta'_f)$ and $\psi_g = t(\eta_g) = t(\eta'_g)$ are connected by an edge, as required.

So we show that $\eta'_f(i) = \eta_f(i)$ for all $i \in S$. Let $j \in I$ and $0 \leq r \leq n_j$. If $r \geq 1$ (and so $n > 0$) then

$$\eta_f(i_j + r) = (j, f(r)) = (j, g(r-1)) = \eta'_g(i_j + r - 1) = \eta'_f(i_j + r).$$

If $r = 0$ and $i_j > 0$ then

$$\eta_f(i_j) = (j, f(0)) = \eta'_g(i_j - 1) = \eta'_f(i_j).$$

Finally, if $r = 0$ and $i_j = 0$ (so $j = 1$) then

$$\eta_f(0) = (1, f(0)) = \eta'_f(0).$$

As for the "consequently" part, consider (H, t, G) as a first order structure. In an elementary extension $(H, t, G) \prec (\mathcal{H}, t, \mathcal{G})$, $\mathcal{H}$ contains $\mathrm{Sh}_n(\omega)$ as a subgraph. Restricting t to $\mathrm{Sh}_n(\omega)$ and applying the above, $\mathcal{G}$ contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \leq k$. As a result, so does G. □

3.2. **Variants of the shift graph.**

Let A and J be two (possibly linearly ordered) sets.

Definition 3.3. For any $\bar{a}, \bar{b} \in A^{\underline{J}}$ (respectively, $(A^{\underline{J}})_<$), let $f_{\bar{a},\bar{b}} = \{(i,j) \in J \times J : a_i = b_j\}$.

Since the tuples $\bar{a}$ and $\bar{b}$ are without repetitions, $f_{\bar{a},\bar{b}}$ is a (possibly empty) injective partial function. If $\bar{a}, \bar{b} \in (A^{\underline{J}})_<$ then $f_{\bar{a},\bar{b}}$ is order-preserving, i.e. for all $i < j \in \mathrm{Dom}(f)$, $f(i) < f(j)$.

Definition 3.4. Let $Id \neq f \subseteq J \times J$ be a partial function. We define a graph E^A_f and a directed graph D^A_f on $A^{\underline{J}}$:

- $\bar{a}\ E^A_f\ \bar{b} \iff f_{\bar{a},\bar{b}} = f \vee f_{\bar{b},\bar{a}} = f$
- $\bar{a}\ D^A_f\ \bar{b} \iff f_{\bar{a},\bar{b}} = f$.

Similarly for $(A^{\underline{J}})_<$. We omit A from E^A_f and D^A_f when it is clear from the context.

Remark 3.5. We required $f \neq Id$ in order to ensure irreflexivity.

A homomomorphism between directed graphs is a map preserving the directed graph relation.

Since the symmetric closure of the relation D_f is exactly E_f, any homomorphism of directed graphs $(A_1^{\underline{J_1}}, D_{f_1}) \to (A_2^{\underline{J_2}}, D_{f_2})$ is also a homomorphism of graphs $(A_1^{\underline{J_1}}, E_{f_1}) \to (A_2^{\underline{J_2}}, E_{f_2})$, and similarly in the ordered case.

Example 3.6. When $J = n$ and $f = \{(i, i-1) : 1 \leq i \leq n-1\}$, $((A^{\underline{n}})_<, E_f)$ is exactly $\mathrm{Sh}_n(A)$.

Definition 3.7. Let $\mathrm{LSh}_n(A) = ((A^{\underline{n}})_<, D_f)$, where $f = \{(i, i-1) : 1 \leq i \leq n-1\}$, and $\mathrm{RSh}_n(A) = ((A^{\underline{n}})_<, D_f)$, where $f = \{(i-1, i) : 1 \leq i \leq n-1\}$.

Lemma 3.8. *Let $(J, <)$ be a finite linearly ordered set and $Id \neq f \subseteq J \times J$ a non-empty partial function. Assume that*

1. *f is order preserving, i.e. for all $i < j \in \mathrm{Dom}(f)$, $f(i) < f(j)$,*
2. *all orbits in f are* increasing, *i.e. for all $i \in \mathrm{Dom}(f)$, $i < f(i)$.*

Then for any countable dense linear order $(Q, <)$ and large enough $k \in \mathbb{N}$ there exists a homomorphism of directed graphs $\varphi : \mathrm{RSh}_k(\omega) \to ((Q^{\underline{J}})_<, D_f)$.

Proof. We may assume that $(Q, <) = (\mathbb{Q}, <)$.[1] Let A be the ordinal ω^ω, seen as a substructure of $\mathbb{Q}$. As J is finite, we may assume that $(J, <)$ is a substructure of $(\mathbb{Q}, <)$.

Claim. There is no harm in replacing J by $\mathrm{Dom}(f) \cup \mathrm{Range}(f)$ and $\mathbb{Q}$ by A.

Proof. Inductively, for every $u \in \mathrm{RSh}_k(\omega)$ choose a dense subset $Q_u \subseteq \mathbb{Q}$ such that for every $u \neq v \in \mathrm{RSh}_k(\omega)$, $Q_u \cap Q_v = \emptyset$ and $Q_u \cap \omega^\omega = \emptyset$.

Now, let $\widehat{J} = \mathrm{Dom}(f) \cup \mathrm{Range}(f)$ and assume we have a homomorphism $\varphi : \mathrm{RSh}_k(\omega) \to ((\omega^\omega)^{\underline{\widehat{J}}})_<, D_f)$. For each $u \in \mathrm{RSh}_k(\omega)$, extending $\varphi(u)$ to an increasing J-tuple of elements from $\mathbb{Q}$ by adding elements from Q_u, defines a map $\varphi' : \mathrm{RSh}_k(\omega) \to ((\mathbb{Q}^{\underline{J}})_<, D_f)$. Since $\widehat{J} = \mathrm{Dom}(f) \cup \mathrm{Range}(f)$ and passing from $\varphi(u)$ to $\varphi'(u)$ adds only new elements, φ' is a homomorphism of directed graphs. $\square$ (claim)

Let $I = \mathrm{Dom}(f) \setminus \mathrm{Range}(f)$. We prove by induction on $|I|$ that for any large enough k there exists a homomorphism $g : \mathrm{RSh}_k(\omega) \to ((A^{\underline{J}})_<, D_f)$, where $J = \mathrm{Dom}(f) \cup \mathrm{Range}(f)$ is non-empty.

For any $\beta \in I$ let n_β be the maximal natural number $n \geq 1$ such that $f^{n-1}(\beta) \in \mathrm{Dom}(f)$. Note that

$$\mathrm{Dom}(f) = \bigcup_{\beta \in I} \{\beta, \ldots, f^{n_\beta - 1}(\beta)\}$$

and that

$$J = \mathrm{Dom}(f) \cup \mathrm{Range}(f) = \bigcup_{\beta \in I} \{\beta, \ldots, f^{n_\beta}(\beta)\}.$$

Let β_0 be the minimal element of I. Note that β_0 is also the minimal element of J.

Claim. There exist $J \subseteq \widetilde{J} \subseteq \mathbb{Q}$ and $f \subseteq \widetilde{f} \subseteq \widetilde{J} \times \widetilde{J}$ such that

- (1), (2) of the lemma hold for $\widetilde{J}$ and $\widetilde{f}$,
- $\widetilde{f} \cap (J \times J) = f$,

[1] Since then the isomorphism to $\mathbb{Q}$ induces an isomorphism between the digraphs.

- $\mathrm{Dom}(\widetilde{f}) \setminus \mathrm{Range}(\widetilde{f}) = I$,
- $\min \widetilde{J} = \beta_0$ and
- $(\star)$ letting $\widetilde{n}_{\beta_0}$ be the maximal natural number $n \geq 1$ such that $\widetilde{f}^{n-1}(\beta_0) \in \mathrm{Dom}(\widetilde{f})$, $f^{\widetilde{n}_{\beta_0}}(\beta_0) = \max \widetilde{J}$.

Proof. If $(\star)$ holds for f and J, we are done. Otherwise, let $i \in \mathrm{Range}(f)$ be minimal such that $f^{n_{\beta_0}}(\beta_0) < i$ and let j be such that $f(j) = i$. Either $j < f^{n_{\beta_0}}(\beta_0)$ or $f^{n_{\beta_0}}(\beta_0) < j$. If the former happens let $f' = f$, so assume it is the latter, i.e. that $f^{n_{\beta_0}}(\beta_0) < j < i$ (so $j \in \mathrm{Dom}(f) \setminus \mathrm{Range}(f)$). Let $f' = f \cup \{(f^{n_{\beta_0}}(\beta_0), y)\}$ for some $j < y \in \mathbb{Q} \setminus J$ which satisfies $y < x$ for all $j < x \in J$ and $J' = J \cup \{y\}$. It is still order preserving and still has increasing orbits. Let n'_{β_0} be as in $(\star)$ with respect to f'.

In either case, we have that $j < (f')^{n'_{\beta_0}}(\beta_0) < i$. Since f' is order preserving we may extend it to an automorphism σ of $\mathbb{Q}$ and thus $i = \sigma(j) < \sigma((f')^{n'_{\beta_0}}(\beta_0))$. By careful adjustments we may assume that $\sigma((f')^{n'_{\beta_0}}(\beta_0)) \notin J'$. Let $f'' = f' \cup \{((f')^{n'_{\beta_0}}(\beta_0), \sigma((f')^{n'_{\beta_0}}(\beta_0)))\}$ and let $J'' = J \cup \{\sigma((f')^{n'_{\beta_0}}(\beta_0))\}$. It is still order preserving and still has increasing orbits.

Note that,

$$|\{i \in \mathrm{Range}(f'') : i > (f'')^{n''_{\beta_0}}(\beta_0)\}| < |\{i \in \mathrm{Range}(f) : i > f^{n_{\beta_0}}(\beta_0)\}|,$$

where n''_{β_0} is defined as in $(\star)$ with respect to f''.

Continue doing this until this set is empty. Let $\widetilde{f}$ be the end function and let $\widetilde{J} = J \cup \mathrm{Dom}(\widetilde{f}) \cup \mathrm{Range}(\widetilde{f})$. $\square$ (claim)

As a consequence of the claim we may assume that $(\star)$ holds for f and J. Indeed, assume we found a homomorphism $\varphi : \mathrm{RSh}_k(\omega) \to ((A^{\underline{\widetilde{J}}})_<, D_{\widetilde{f}})$, for some k. Since the projection map $\pi : ((A^{\underline{\widetilde{J}}})_<, D_{\widetilde{f}}) \to ((A^{\underline{J}})_<, D_f)$ is a graph homomorphism (this uses the second bullet in the claim above), $\pi \circ \varphi$ is the desired map.

Let $J' = J \setminus \{\beta_0, \dots, f^{n_{\beta_0}}(\beta_0)\}$. If $J' = \emptyset$ let g_k be the empty function for all $k \in \mathbb{N}$. Otherwise, by induction there exists $l \in \mathbb{N}$ such that for all $k \geq l$ there is a homomorphism $g_k : \mathrm{RSh}_k(\omega) \to ((A^{\underline{J'}})_<, D_{f \cap (J' \times J')})$. Let $k > \max\{n_\beta + 1 : \beta \in I\} \cup \{l\}$ and set some order isomorphism $\phi : \omega \times (A \cup \{-1\}) \to A$, where -1 is a new element which is smaller than any element of A (recall that $A = \omega^\omega$).

We construct a homomorphism mapping $\mu \in \mathrm{RSh}_k(\omega)$ to $\psi_\mu \in ((A^{\underline{J}})_<, D_f)$. Let $\mu \in \mathrm{RSh}_k(\omega)$. For any $0 \leq h \leq n_{\beta_0}$ we define

$$\psi_\mu(f^h(\beta_0)) = \phi(\mu(h), -1).$$

For any $\beta \in I$, with $\beta \neq \beta_0$, and $0 \leq h \leq n_\beta$ we define

$$\psi_\mu(f^h(\beta)) = \phi(\mu(\tilde{h}), g_k(\mu)(f^h(\beta))),$$

for $0 \leq \tilde{h} \leq n_{\beta_0}$ maximal satisfying $f^h(\beta) > f^{\tilde{h}}(\beta_0)$, which exists by minimality of β_0.

We check that ψ_μ is increasing and that $\mu \mapsto \psi_\mu$ is a homomorphism.

To show that ψ_μ is increasing, suppose $f^{h_1}(\beta_1) < f^{h_2}(\beta_2) \in J$ and go over the different possibilities. Note that we use -1 in the case when $\beta_1 = \beta_0, \beta_2 \neq \beta_0$ and $\widetilde{h}_2 = h_1$.

We show that $\mu \mapsto \psi_\mu$ is a homomorphism. Suppose that $\mu, \nu \in \mathrm{RSh}_k(\omega)$ are such that $\mu(n) = \nu(n+1)$ for all $0 \le n < k-1$. We need to check that $f(i) = j$ if and only if $\psi_\mu(i) = \psi_\nu(j)$.

Assume that $f(i) = j$ (so $i \in \mathrm{Dom}(f)$). Suppose that $i = f^h(\beta_0)$ for some $0 \le h < n_{\beta_0}$, so $j = f^{h+1}(\beta_0)$. Then

$$\psi_\mu(i) = \phi(\mu(h), -1) = \phi(\nu(h+1), -1) = \psi_\nu(f^{h+1}(\beta_0)).$$

Now suppose that $i = f^h(\beta)$ for some $\beta \neq \beta_0$ and $0 \le h < n_\beta$, so $j = f^{h+1}(\beta)$. Let $\widetilde{h}$ be maximal such that $f^{\widetilde{h}}(\beta_0) < f^h(\beta)$. Note that by $(\star)$, $\widetilde{h} < n_{\beta_0}$. It follows that $f^{\widetilde{h}+1}(\beta_0)$ is defined and $f^{\widetilde{h}+1}(\beta_0) < f^{h+1}(\beta)$. On the other hand, it cannot be that $f^{\widetilde{h}+2}(\beta_0) < f^{h+1}(\beta)$ (again, $f^{\widetilde{h}+2}(\beta_0)$ is defined by the claim) for then we would have $f^{\widetilde{h}+1}(\beta_0) < f^h(\beta)$, contradicting the maximality of $\widetilde{h}$. It follows that $\widetilde{h} + 1 = \widetilde{h+1}$. Since g_k is a homomorphism,

$$\psi_\mu(i) = \phi(\mu(\widetilde{h}), g_k(\mu)(f^h(\beta))) = \phi(\nu(\widetilde{h}+1), g_k(\nu)(f^{h+1}(\beta)) = \psi_\nu(j).$$

Now assume that $\psi_\mu(i) = \psi_\nu(j)$. If $i = f^h(\beta_0)$ for some $0 \le h \le n_{\beta_0}$ then $j = f^{h'}(\beta_0)$ for some h' (since $\psi_\mu(j)$ has the form $\phi(-,-1)$) and so $\mu(h) = \nu(h')$. By the choice of the k, $h+1 < k$ and consequently $\mu(h) = \nu(h+1) = \nu(h')$ so $h' = h+1$ and $f(i) = j$ (note that it follows that $h < n_\beta$).

Suppose $i = f^h(\beta)$ for some $\beta \neq \beta_0$ and $0 \le h \le n_\beta$. As this is encoded by ϕ, by the assumption necessarily $j = f^{h'}(\beta')$ for some $\beta' \neq \beta_0$ and $0 \le h' \le n_{\beta'}$. Let $\widetilde{h}$ be maximal such that $f^{\widetilde{h}}(\beta_0) < i$ and $\widetilde{h}'$ maximal such that $f^{\widetilde{h}'}(\beta_0) < j$. So $\psi_\mu(i) = \phi(\mu(\widetilde{h}), g_k(\mu)(i))$ and $\psi_\nu(j) = \phi(\nu(\widetilde{h}'), g_k(\nu)(j))$. It follows that $g_k(\nu)(j) = g_k(\mu)(i)$ and we are done by the choice of g_k since $i, j \in J'$. $\square$

Before continuing to the main proposition, as auxiliary results, we calculate the chromatic number of some (well known) graphs.

Example 3.9 (Symmetric cyclic graph)**.** Let $r > 1$ be a natural number. Let $\mathrm{Cyc}_r^{sym}(A)$ be the graph on $A^{\underline{r}}$ with an edge between $(a_0, \ldots, a_{r-1})$ and $(b_0, \ldots, b_{r-1})$ if $a_0 = b_1, \ldots, a_{r-2} = b_{r-1}, a_{r-1} = b_0$ (or vice-versa). We thus have an injective graph homomorphism $\mathrm{Cyc}_r^{sym}(A) \to \mathrm{Sh}_r^{sym}(A)$ (but not an embedding).

Lemma 3.10. *For every natural number $r > 1$ and any set A,*

$$\chi\left(\mathrm{Cyc}_r^{sym}(A)\right) = \begin{cases} 2 & r \text{ is even} \\ 3 & r \text{ is odd.} \end{cases}$$

Proof. The graph $\mathrm{Cyc}_r^{sym}(A)$ partitions into connected components, each one of them a cycle graph on r vertices. It is well known and easy to see that you need 2 colors to color even cycle graphs and 3 colors to color odd cycle graphs. $\square$

The next two examples are somewhat similar and they both have very small chromatic number. We define them and prove that their chromatic number is 2.

Example 3.11 (Denumerable tuples symmetric shift graph)**.** Let A be an infinite set. The *denumerable tuples symmetric shift graph* $\mathrm{Sh}_\omega^{sym}(A)$ is defined similarly as the symmetric shift graph but with vertices $A^{\underline{\omega}}$. There is an edge between two vertices f and g if $f(n) = g(n+1)$ for all $n < \omega$ (or vice-versa).

Example 3.12 (Glued increasing symmetric shift graphs)**.** Let $\bar{n} = \langle n_i : i < \omega \rangle$ be a strictly increasing sequence of natural numbers. We define the graph $\mathrm{Sh}^{sym}_{\bar{n},u}(A)$, for an infinite set A. The vertices are injective functions $\coprod_{i<\omega}[0, n_i] \to A$. Thus every vertex can be written as $f = \coprod_{i<\omega} f_i$. We will say that there is an edge between two vertices f and g if

$$f_i(m) = g_i(m+1)$$

for every $0 \le m < n_i$, $i < \omega$ or

$$g_i(m) = f_i(m+1)$$

for every $0 \le m < n_i$, $i < \omega$.

Lemma 3.13. *Let X be a set. Suppose $(n, x) \mapsto n + x$ is a free action of $\mathbb{Z}$ on X. We define a graph relation E on X by setting that $x \mathrel{E} y$ if either $1 + x = y$ or $1 + y = x$. Then $\chi(X, E) = 2$.*

Proof. Since (X, E) is a disjoint union of pairwise unconnected $\mathbb{Z}$-paths, the result follows. □

Lemma 3.14. *For any infinite set A and a strictly increasing sequence of natural numbers $\bar{n}$, $\chi\left(\mathrm{Sh}^{sym}_{\omega}(A)\right) = \chi\left(Sh^{sym}_{\bar{n},u}(A)\right) = 2$.*

Proof. The proofs for these two graphs are the same, albeit the definitions are slightly different. We prove for $\mathrm{Sh}^{sym}_{\omega}(A)$ and present the appropriate definitions for $\mathrm{Sh}^{sym}_{\bar{n},u}(A)$ at the end.

Let $X \subseteq A^{\omega}$ be the set of all functions f which are eventually injective, i.e. there exists an n such that $f \restriction [n, \infty)$ is injective.

Fix some element $e \in A$. The integers $\mathbb{Z}$ acts on X by translation: if $z \in \mathbb{Z}$ and $f \in X$ then we define

$$(z + f)(m) = \begin{cases} f(m - z) & 0 \le m - z \\ e & \text{otherwise.} \end{cases}$$

We define an equivalence relation R on X:

$$f \mathrel{R} g \iff \exists n (f \restriction [n, \infty) = g \restriction [n, \infty)).$$

Note that if $f \mathrel{R} g$ and $z \in \mathbb{Z}$ then $z + f \mathrel{R} z + g$, so the $\mathbb{Z}$-action induces an action on X/R. We note that if $z + [f] = [f]$ for $f \in X$ (and $[f]$ being the class of f in X/R) then $z = 0$ by eventual injectivity of f. Or in other words, the $\mathbb{Z}$-action on X/R is free.

Since if $f, g \in \mathrm{Sh}^{sym}_{\omega}(A)$ are connected by an edge then either $[f] = 1 + [g]$ or $[g] = 1 + [f]$, by Lemma 3.13 and Lemma 2.3(3), $\chi\left(\mathrm{Sh}^{sym}_{\omega}(A)\right) = 2$.

For $\mathrm{Sh}^{sym}_{\bar{n},u}(A)$ we define:

Let X be the set of all functions $f : \coprod_{i<\omega} \to [0, n_i]$ satisfying the property that there exists an n such that for all $i < \omega$, $f_i \restriction [n, n_i - n]$ is injective.

For every $z \in \mathbb{Z}$ and $f \in X$ we define for $i < \omega$

$$(z + f)_i(m) = \begin{cases} f_i(m - z) & 0 \le m - z \le n_i \\ e & \text{otherwise.} \end{cases}$$

We define an equivalence relation R on X:

$$f \mathrel{R} g \iff \exists n \forall i < \omega (f_i \restriction [n, n_i - n] = g_i \restriction [n, n_i - n]).$$

□

On the other hand if the glued shift graphs are bounded the picture is different.

Example 3.15 (A sequence of bounded shift graphs)**.** Let n be a natural number, I a set and $\bar{n} = \langle n_i : i \in I \rangle$ a sequence of natural numbers satisfying $0 < n_i \leq n$ for all $i \in I$. We define $\mathrm{Sh}_{\bar{n},b}(A)$ for an infinite linearly ordered set $(A, <)$. The vertices are sequences of functions $f = (f_i)_{i \in I}$, where each $f_i : [0, n_i] \to A$ is order preserving. We will say that there is an edge between two vertices f and g if $f_i(m) = g_i(m+1)$ for every $0 \leq m < n_i$, $i \in I$ (or vice-versa).

Lemma 3.16. *Let $(A, <)$ be an infinite linearly ordered set, $\bar{n} = \langle n_i : i \in I \rangle$ a uniformly bounded sequence of natural numbers with $n_i \geq 1$ and let $n = \max_{i \in I}\{n_i\}$. Then there exists an injective homomorphism $\mathrm{Sh}_{n+1}(A) \to \mathrm{Sh}_{\bar{n},b}(A)$.*

Proof. For any tuple $u \in (A^{\underline{n+1}})_<$ we define a vertex $f_u \in \mathrm{Sh}_{\bar{n},b}(A)$. For every $0 \leq h \leq n_i$, $i \in I$, we set $(f_u)_i(h) = u(h)$. Set $f = (f_i)_{i \in I}$. Note that if $0 \leq h < h' \leq n_i$ then $u(h) < u(h')$ so $(f_u)_i(h) < (f_u)_i(h')$. By the choice of n, $u \mapsto f_u$ is injective as well.

We show that $u \mapsto f_u$ is a homomorphism. Assume that, without loss of generality, $u(h) = v(h+1)$ for every $0 \leq h < n$. For every $i \in I$ and for every $0 \leq h < n_i$

$$(f_u)_i(h) = u(h) = v(h+1) = (f_v)_i(h+1).$$

As needed. $\square$

The following propositions will be the backbone behind the main results.

Proposition 3.17. *Let A be an infinite set, λ a cardinal with $2^\lambda \leq |A|$ and $G = (A^{\underline{\lambda}}, E)$ a graph on $A^{\underline{\lambda}}$. If $\chi(G) \geq \beth_2(\lambda)^+ + \aleph_0$ and*

($\star$) *for all $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in A^{\underline{\lambda}}$ if $\bar{a} \mathrel{E} \bar{b}$ and $f_{\bar{a},\bar{b}} = f_{\bar{c},\bar{d}}$ then $\bar{c} \mathrel{E} \bar{d}$*

then there exists an $n \in \mathbb{N}$ and an injective homomorphism from $\mathrm{Sh}_n(\omega)$ to G.

Proof. Let $F = \{f_{\bar{a},\bar{b}} : \bar{a} \mathrel{E} \bar{b}\}$ be the collection of all functions arising as $f_{\bar{a},\bar{b}}$ for some $\bar{a}$ and $\bar{b}$ sharing an edge. If we set $E_f = \{(\bar{a}, \bar{b}) : f = f_{\bar{a},\bar{b}} \vee f = f_{\bar{b},\bar{a}}\}$ (see also Definition 3.4) then, since by ($\star$), $E = \bigcup_{f \in F} E_f$, then by Lemma 2.3(2)

$$\beth_2(\lambda)^+ + \aleph_0 \leq \chi(G) \leq \prod_{f \in F} \chi\left(V, E_f\right).$$

If λ is infinite then, since $|F| \leq 2^\lambda$, there exists $f \in F$ with $\chi(V, E_f) > 2^\lambda \geq \aleph_0$. If λ is finite then the same conclusion holds since $|F|$ is finite. Replace G by (V, E_f). Note that although now we only have that $\chi(V, E) \geq \aleph_0$, we gained that $\bar{a}$ and $\bar{b}$ are connected by an edge if and only if $f_{\bar{a},\bar{b}} = f$ or $f_{\bar{b},\bar{a}} = f$.

For any $\beta \in \mathrm{Dom}(f) \subseteq \lambda$, we distinguish between four possibilities:

(1) "β is a fixed point": $f(\beta) = \beta$;
(2) "β generates a finite cycle": there exists a natural number $1 < n \in \mathbb{N}$ such that $f^n(\beta) = \beta$ and $f^{n-1}(\beta) \neq \beta$;
(3) "β generates a finite shift": there exists a natural number $0 < n \in \mathbb{N}$ such that $f^n(\beta) \notin \mathrm{Dom}(f)$;
(4) "β generates an infinite shift": the set $\{f^n(\beta) : n < \omega\}$ is infinite.

We first note the following observations, which will allow us to cross out some of the possibilities:

- No $\beta \in \mathrm{Dom}(f)$ generates a finite cycle. Assume there exists $\beta \in \mathrm{Dom}(f)$ and $1 < n < \omega$ such that $f^n(\beta) = \beta$ and $f^{n-1}(\beta) \neq \beta$. Define a homomorphism $G \to \mathrm{Cyc}_n^{sym}(A)$ which maps $\bar{a}$ to $(a_\beta, a_{f(\beta)}, \ldots a_{f^{n-1}(\beta)})$. It is a homomorphism because if there is an edge between $\bar{a}$ and $\bar{b}$ then by definition of f, $a_\beta = b_{f(\beta)}, \ldots, a_{f^{n-1}(\beta)} = b_\beta$. By Lemma 3.10 and Lemma 2.3(3), $\chi(G) \leq \chi(\mathrm{Cyc}_n^{sym}(A)) \leq 3$, contradiction.
- No $\beta \in \mathrm{Dom}(f)$ generates an infinite shift. Assume there exists $\beta < \lambda$ with $\{f^n(\beta) : n < \omega\}$ infinite. Define a homomorphism $G \to \mathrm{Sh}_\omega^{sym}(A)$ which maps $\bar{a}$ to $n \mapsto a_{f^n(\beta)}$. By definition this is a homomorphism of graphs. By Lemma 3.14 and Lemma 2.3(3), $\chi(G) \leq \chi(\mathrm{Sh}_\omega^{sym}(A)) = 2$, contradiction.

Let $I = \mathrm{Dom}(f) \setminus \mathrm{Range}(f)$. For any $\beta \in I$ let n_β be the maximal natural number $n \geq 1$ such that $f^{n-1}(\beta) \in \mathrm{Dom}(f)$. Note that

$$\mathrm{Dom}(f) = \bigcup_{\beta \in I} \{\beta, \ldots, f^{n_\beta - 1}(\beta)\} \cup \{\beta < \lambda : f(\beta) = \beta\}$$

and that

$$\mathrm{Dom}(f) \cup \mathrm{Range}(f) = \bigcup_{\beta \in I} \{\beta, \ldots, f^{n_\beta}(\beta)\} \cup \{\beta < \lambda : f(\beta) = \beta\}.$$

We are thus left with two cases:

Case 1. $I = \emptyset$. Thus f is the identity on $\mathrm{Dom}(f)$. If $\mathrm{Dom}(f) = \lambda$ then G is an anticlique and can thus can be colored by only one color, contradiction. Hence $\mathrm{Dom}(f) \subsetneq \lambda$. Since $\lambda \times \aleph_0 \leq |A|$, we may find $G_0 = \{\bar{a}_i \in G : i < \omega\}$, such that for any $i, j < \omega$, $(\bar{a}_i)_k = (\bar{a}_j)_l \iff k = l \in \mathrm{Dom}(f)$. By the definition of the edge relation G_0 is a complete graph of size $|A|$. In particular we may embed the complete graph on ω as a subgraph.

Case 2. $I \neq \emptyset$.

Claim. There exists a uniform bound on $\{n_\beta : \beta \in I\}$.

Proof. Otherwise, assume there are $\langle \beta_i : i < \omega \rangle$ such that the sequence $\bar{n} := \langle n_{\beta_i} : i < \omega \rangle$ is strictly increasing. We define a homomorphism from G to $\mathrm{Sh}_{\bar{n},u}^{sym}(A)$ similarly as before. Consequently, $\chi(G) \leq 2$ (by using Lemma 3.14, Lemma 2.3(3) and the relevant homomorphisms), contradiction. □ (claim)

Let $n = \max_{\beta \in I}\{n_\beta\}$, $\bar{n} = \langle n_\beta : \beta \in I \rangle$ and let $\phi : \lambda \times \omega \times (\mathrm{Sh}_{\bar{n},b}(\omega) \cup \{0,1\}) \to A$ be an injective function, which exists since $(\aleph_0)^\lambda + \aleph_0 + \lambda \leq |A|$.

We define an injective homomorphism from $\mathrm{Sh}_{\bar{n},b}(\omega)$ into G. For every function $\mu = (\mu_\beta)_{\beta \in I}$, where $\mu_\beta : [0, n_\beta] \to \omega$ is order preserving, we associate an injective function $\psi_\mu : \lambda \to A$ as follows. For every $\beta \in I$ and $h \in [0, n_\beta]$ we define

$$\psi_\mu(f^h(\beta)) = \phi(\beta, \mu_\beta(h), 0),$$

note that this is well defined. For every $\alpha \notin \bigcup_{\beta \in I}\{\beta, \ldots, f^{n_\beta}(\beta)\}$ such that $f(\alpha) = \alpha$ we define

$$\psi_\mu(\alpha) = \phi(\alpha, 0, 1)$$

and otherwise we define

$$\psi_\mu(\alpha) = \phi(\alpha, 0, \mu).$$

We claim that the map $\mu \mapsto \psi_\mu$ is an injective homomorphism.

Injectivity: Let $\mu, \nu \in \mathrm{Sh}_{\bar{n},b}(\omega)$ with $\psi_\mu = \psi_\nu$. Let $\beta \in I$ and $h \in [0, n_\beta]$. Since $\psi_\mu(f^h(\beta)) = \psi_\nu(f^h(\beta))$ and ϕ is injective, $\mu_\beta(h) = \nu_\beta(h)$.

Homomorphism: Assume that μ and ν are connected by an edge, i.e. without loss of generality for every $\beta \in I$ and $h \in [0, n_\beta)$, $\mu_\beta(h) = \nu_\beta(h+1)$. We need to show that for every $i, j < \lambda$, $\psi_\mu(i) = \psi_\nu(j)$ if and only $f(i) = j$.

Assume that $f(i) = j$. In particular, $i \in \mathrm{Dom}(f)$. If $i = \beta = f(\beta) = j$ then $\psi_\mu(\beta) = \phi(\beta, 0, 1) = \psi_\nu(\beta)$. Otherwise, $i = f^h(\beta)$ for some $\beta \in I$ and $h \in [0, n_\beta)$. Thus

$$\psi_\mu(i) = \psi_\mu(f^h(\beta)) = \phi(\beta, \mu_\beta(h), 0) =$$
$$\phi(\beta, \nu_\beta(h+1), 0) = \psi_\nu(f^{h+1}(\beta)) = \psi_\nu(j).$$

Assume that $\psi_\mu(i) = \psi_\nu(j) = e$. Since $\mu \neq \nu$, by the injectivity of ϕ we have only two possibilities: either $e = \phi(\cdots, 1)$ or $e = \phi(\cdots, 0)$. If the former happens, necessarily $f(i) = i$, $f(j) = j$ and $i = j$.

Otherwise, $i = f^h(\beta)$ and $j = f^{h'}(\beta')$ for some $\beta, \beta' \in I$, $h \in [0, n_\beta]$ and $h' \in [0, n_{\beta'}]$. Also, since

$$\phi(\beta, \mu_\beta(h), 0) = \phi(\beta', \nu_{\beta'}(h'), 0),$$

$\beta = \beta'$ and $\mu_\beta(h) = \nu_\beta(h')$. If $h \in [0, n_\beta)$ then $\mu_\beta(h) = \nu_\beta(h+1)$ since μ and ν are connected by an edge, so since ν_β is injective $h' = h+1$. Hence $f(i) = j$. Otherwise, $h = n_\beta$. If $h' > 0$ then since $\mu_\beta(h) = \nu_\beta(h') = \mu_\beta(h'-1)$ we get a contradiction to the injectivity of μ_β. Consequently it must be that $h' = 0$ and thus

$$\mu_\beta(n_\beta) = \nu_\beta(0) < \nu_\beta(1) = \mu_\beta(0) < \mu_\beta(n_\beta),$$

contradiction.

Applying Lemma 3.16 we may conclude that there exists an injective homomorphism from $\mathrm{Sh}_{n+1}(\omega)$ into $\mathrm{Sh}_{\bar{n},b}(\omega)$, and thus into G as well. □

Proposition 3.18. *Let $(A, <)$ be an infinite linearly ordered set, $m < \omega$ and $G = ((A^{\underline{m}})_<, E)$ a graph on $(A^{\underline{m}})_<$. Assume $\chi(G) \geq \aleph_0$ and that for all $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in (A^{\underline{m}})_<$ if $\bar{a}\ E\ \bar{b}$ and $f_{\bar{a},\bar{b}} = f_{\bar{c},\bar{d}}$ then $\bar{c}\ E\ \bar{d}$. Then there exists $n < \omega$ such that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$.*

Proof. As was done in the proof of Proposition 3.17, letting $F = \{f_{\bar{a},\bar{b}} : \bar{a}\ E\ \bar{b}\}$, since $|F| < \aleph_0$, we may assume that $E = E_f$ for some $f \in F$ (see Definition 3.4).

Since the tuples are increasing, f is necessarily an order preserving function ($i < j \in \mathrm{Dom}(f) \implies f(i) < f(j)$). Thus, as m is finite, for any $\beta \in \mathrm{Dom}(f) \subseteq m$ with $f(\beta) \neq \beta$, "β generates a finite shift" (in the context of Proposition 3.17), i.e. there exists a natural number $0 < n \in \mathbb{N}$ such that $f^n(\beta) \notin \mathrm{Dom}(f)$.

Let $I = \mathrm{Dom}(f) \setminus \mathrm{Range}(f)$. For any $\beta \in I$ let n_β be the maximal natural number $n \geq 1$ such that $f^{n-1}(\beta) \in \mathrm{Dom}(f)$. Note that

$$\mathrm{Dom}(f) = \bigcup_{\beta \in I} \{\beta, \ldots, f^{n_\beta - 1}(\beta)\} \cup \{\beta < m : f(\beta) = \beta\}$$

and that

$$\mathrm{Dom}(f) \cup \mathrm{Range}(f) = \bigcup_{\beta \in I} \{\beta, \ldots, f^{n_\beta}(\beta)\} \cup \{\beta < m : f(\beta) = \beta\}.$$

As in the proof of Proposition 3.17(Case 1) we can disregard the case $I = \emptyset$, because then in this case G contains any finite complete graph. Say that $\beta \in I$ is *increasing* if $\beta < f(\beta)$ and *decreasing* otherwise (equivalently, $f(\beta) < \beta$). Also, as f is order preserving and the tuples are increasing, we may find a partition $m = J_1 \cup \cdots \cup J_N$ satisfying that

- each of the J_i are convex and $J_1 < \cdots < J_N$;
- if $\beta \in J_i \cap \mathrm{Dom}(f)$ then $f(\beta) \in J_i$;
- if $\beta \in I \cap J_i$ is increasing then every $\beta' \in I \cap J_i$ is increasing;
- if $\beta \in I \cap J_i$ is decreasing then every $\beta' \in I \cap J_i$ is decreasing and
- if for $\beta \in J_i$, $f(\beta) = \beta$ then for every $\beta' \in (\mathrm{Dom}(f) \cup \mathrm{Range}(f)) \cap J_i$, $\beta = \beta'$.

For every $1 \leq i \leq N$, set $f_i = f \cap (J_i \times J_i)$.

For any $1 \leq i \leq N$ if J_i is of increasing type, by applying Lemma 3.8, with $(Q,<) = (\mathbb{Q},<)$, there is a homomorphism $g_{i,k} : \mathrm{RSh}_k(\omega) \to ((\mathbb{Q}^{\underline{J_i}})_<, D_{f_i})$ for any large enough k.

For any $1 \leq i \leq N$ if J_i is of decreasing type, by applying Lemma 3.8 to $(J_i,<^*)$ (the reverse order on J_i) with $(Q,<) = (\mathbb{Q},<^*)$ (the reverse order on $\mathbb{Q}$) there is a homomorphism $g^*_{i,k} : \mathrm{RSh}_k(\omega) \to ((\mathbb{Q}^{\underline{(J_i,<^*)}})_{<^*}, D_{f_i})$ for any large enough k. Since the identity function is an isomorphism of directed graphs

$$((\mathbb{Q}^{\underline{(J_i,<^*)}})_{<^*}, D_{f_i}) \cong ((\mathbb{Q}^{\underline{J_i}})_<, D_{f_i}),$$

we may compose and get a homomorphism $g_{i,k} : \mathrm{RSh}_k(\omega) \to ((\mathbb{Q}^{\underline{J_i}})_<, D_{f_i})$.

Let k be large enough so that $g_{i,k}$ are defined for all i and set $g_i = g_{i,k}$.

For any $1 \leq i \leq N$, if J_i is of constant type fix some embedding $g_i : (J_i,<) \to (\mathbb{Q},<)$.

Let $(A,<) \prec (\mathcal{A},<)$ be a sufficiently saturated extension with $(\mathcal{A},<)$ containing $(Q,<) = (\{1,\ldots,N\} \times \mathbb{Q} \times (\mathrm{RSh}_k(\omega) \cup \{0\}), <_{\mathrm{lex}})$ as a substructure, where we may choose any linear order on $\mathrm{RSh}_k(\omega) \cup \{0\}$. Note that the inclusion $(Q,<) \subseteq (\mathcal{A},<)$ induces an injective homomorphism

$$((Q^{\underline{m}})_<, D_f) \to ((\mathcal{A}^{\underline{m}})_<, D_f).$$

We will now construct a homomorphism $\mathrm{RSh}_k(\omega) \to ((Q^{\underline{m}})_<, D_f)$.

Let $\mu \in \mathrm{RSh}_k(\omega)$. We define $\psi_\mu \in (Q^{\underline{m}})_<$ as follows. If $\alpha \in J_i$, with J_i increasing or decreasing then

$$\psi_\mu(\alpha) = (i, g_i(\mu)(\alpha), 0).$$

If $\alpha \in J_i$, with J_i of constant type, and $f(\alpha) = \alpha$ then

$$\psi_\mu(\alpha) = (i, g_i(\alpha), 0).$$

If $\alpha \in J_i$, with J_i of constant type, and $f(\alpha) \neq \alpha$ then

$$\psi_\mu(\alpha) = (i, g_i(\alpha), \mu).$$

Since $J_1 < \cdots < J_N$ then by definition, ψ_μ is increasing. We claim that $\mu \mapsto \psi_\mu$ is a homomorphism.

Assume that $\mu, \nu \in \mathrm{RSh}_k(\omega)$ are such that $\mu(h) = \nu(h+1)$ for $0 \leq h < k-1$. We will show that for every $\alpha, \beta < m$: $f(\alpha) = \beta$ if and only if $\psi_\mu(\alpha) = \psi_\nu(\beta)$.

If $f(\alpha) = \beta$ then $\alpha, \beta \in J_i$ for some $1 \leq i \leq N$. If J_i is not of constant type then since g_i is a homomorphism, $g_i(\mu)(\alpha) = g_i(\nu)(\beta)$, so

$$\psi_\mu(\alpha) = (i, g_i(\mu)(\alpha), 0) = (i, g_i(\nu)(\beta), 0) = \psi_\nu(\beta).$$

If J_i is of constant type then $\beta = f(\alpha) = \alpha$ and

$$\psi_\mu(\alpha) = (i, g_i(\alpha), 0) = (i, g_i(\beta), 0) = \psi_\nu(\beta).$$

Now assume that $\psi_\mu(\alpha) = \psi_\nu(\beta)$. By definition, $\alpha, \beta \in J_i$ for some $1 \leq i \leq N$. If J_i is of constant type then since $\mu \neq \nu$ (and both are not 0) by the definition of ψ, $f(\alpha) = \alpha$, $f(\beta) = \beta$ and $g_i(\alpha) = g_i(\beta)$. Consequently, $\alpha = \beta$ and as a result $\alpha = \beta = f(\alpha)$.

If J_i is not of constant type then $g_i(\mu)(\alpha) = g_i(\nu)(\beta)$. By the fact that g_i is a homomorphism, $f(\alpha) = \beta$.

We have constructed a homomorphism $\mathrm{RSh}_k(\omega) \to ((Q^{\underline{m}})_<, D_f) \subseteq ((\mathcal{A}^{\underline{m}})_<, D_f)$, which also gives a graph homomorphism $\mathrm{Sh}_k(\omega) \to ((\mathcal{A}^{\underline{m}})_<, E_f)$. By Proposition 3.2, $((\mathcal{A}^{\underline{m}})_<, E_f)$ contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \leq k$ and since $((A^{\underline{m}})_<, E_f) \prec ((\mathcal{A}^{\underline{m}})_<, E_f)$ and $E_f \subseteq E$ so does G. $\square$

4. Superstable and ω-stable Graphs

We use the main result of the previous section in order to prove the strong form of Taylor's conjecture for ω-graphs and a suitable variant for superstable graphs.

The following result is somewhat reminiscent (in flavor) of [ER50, Theorem III]. It is a local version of the the well known fact that, in stable theories, every indiscernible sequence is an indiscernible set [TZ12, Lemma 9.1.1] (it is possibly known, but we could not find a reference).

Generalizing the notation from Definition 3.3, for two tuples, possibly of different length, $\bar{a}$ and $\bar{b}$, we denote $f_{\bar{a},\bar{b}} = \{(i,j) : a_i = b_j\}$.

Recall that for a set of formulas Δ, a Δ-*indiscernible sequence* is a sequence of elements that are indiscernible only with respect to formulas from Δ. For a formula $\varphi(x_0, \ldots, x_{n-1})$ let $\Delta_\varphi := \{\varphi(x_{\pi(0)}, \ldots, x_{\pi(n-1)}) : \pi \text{ is a function from } n \text{ to } n\}$.

Proposition 4.1. *Let T be a complete theory and $\varphi(x, y)$ a partitioned stable formula, with x and y possibly of different lengths. Let I be a Δ_φ-indiscernible[2] sequence indexed by an infinite linearly ordered set $(Q, <)$.*

If $\bar{a}, \bar{c} \in (I^{\underline{|x|}})_<$ and $\bar{b}, \bar{d} \in (I^{\underline{|y|}})_<$ are increasing tuples then

$(\star)$ $\varphi(\bar{a}, \bar{b}) \wedge f_{\bar{a},\bar{b}} = f_{\bar{c},\bar{d}} \implies \varphi(\bar{c}, \bar{d})$.

Remark 4.2. In particular, if there exist $\bar{a} \in (I^{\underline{|x|}})_<$ and $\bar{b} \in (I^{\underline{|y|}})_<$ disjoint increasing tuples such that $\varphi(\bar{a}, \bar{b})$ holds then for every disjoint increasing tuples $\bar{c} \in (I^{\underline{|x|}})_<$ and $\bar{d} \in (I^{\underline{|y|}})_<$, $\varphi(\bar{c}, \bar{d})$ holds.

Proof. Let I' be an indiscernible sequence with the same EM-type as I. Suppose $(\star)$ is not true as witnessed by $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, then let $\bar{a}', \bar{b}', \bar{c}', \bar{d}'$ in I' be such that $\bar{a}'\bar{b}'$ as the same order type as $\bar{a}\bar{b}$, and $\bar{c}'\bar{d}'$ has the same order type as $\bar{c}\bar{d}$. It follows by the choice of Δ_φ that $(\star)$ is not true for $\bar{a}', \bar{b}', \bar{c}', \bar{d}'$. We may thus assume that I is an indiscernible sequence. Similarly, we may assume that $(Q, <)$ is $(\mathbb{Q}, <)$. Also, we endow I with the order induced by Q, i.e. we write $a_i < a_j$ but mean $i < j \in Q$.

We prove by induction on $n < \omega$ that for any set A, stable formula $\varphi(x, y)$ over A and an A-indiscernible sequence I indexed by $(\mathbb{Q}, <)$, $(\star)$ holds for $\bar{a}, \bar{c} \in (I^{\underline{|x|}})_<$ and $\bar{b}, \bar{d} \in (I^{\underline{|y|}})_<$ with $|\mathrm{Dom}(f_{\bar{a},\bar{b}})| \leq n$.

Assume that $n = 0$. Let $\varphi(x, y)$, $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ and I be as in the induction hypothesis. For simplicity assume that $A = \emptyset$. Since $n = 0$, $\bar{a}$ and $\bar{b}$ are disjoint (it follows that $\bar{c}$

[2]For partitioned formulas, we define Δ_φ as above forgetting the partition.

and $\bar{d}$ are disjoint as well). By applying an automorphism we may assume that $\bar{c} = \bar{a}$ (which exists since the index set of I is $\mathbb{Q}$). Let $X = \{\bar{c} \in (I \setminus \bar{a})^{\underline{|y|}} : \varphi(\bar{a}, \bar{c}) \text{ holds}\}$. Note that $\bar{b} \in X$.

By stability the φ-type $\mathrm{tp}_\varphi(\bar{a}/I \setminus \bar{a})$ is definable, i.e. there is some formula $\psi(y, \bar{e})$ with $\bar{e} \in I \setminus \bar{a}$ such that

$$\bar{c} \in X \iff \bar{c} \models \psi(y, \bar{e}).$$

Let $\bar{h} \in I \setminus \bar{a}\bar{e}$ have the same order type as $\bar{e}$ over $\bar{a}$, which exists by density. Let σ be an automorphism of $(I, <)$ which fixes $\bar{a}$ and maps $\bar{e}$ to $\bar{h}$. By indiscernibility it follows that $\sigma(X) = X$.

We claim that X is definable over both $\bar{e}$ and $\bar{h}$ in the structure $(I \setminus \bar{a}, <)$. Indeed if $\bar{c}_1, \bar{c}_2$ have the same order type over $\bar{e}$ then $\bar{c}_1 \in X \iff \bar{c}_2 \in X$. Since there only finitely many order types over $\bar{e}$, this shows the claim for $\bar{e}$. As $\sigma(X) = X$, we have it also for $\bar{h}$.

As DLO eliminates imaginaries [TZ12, Exercise 8.4.3], X has a code $\ulcorner X \urcorner \in \mathrm{dcl}(\bar{e}) \cap \mathrm{dcl}(\bar{h})$. As dcl is trivial in DLO and $\bar{e}$ and $\bar{h}$ are disjoint, X is definable over $\emptyset$ (in the structure $(I \setminus \bar{a}, <)$). Since $\bar{b} \in X$, it follows $((I \setminus \bar{a})^{\underline{|y|}})_< \subseteq X$. This proves the first part.

Now assume that $n > 0$ and let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be as above with $|\mathrm{Dom}(f_{\bar{a},\bar{b}})| = n > 0$. Assume that $\varphi(\bar{a}, \bar{b})$ holds and $f_{\bar{a},\bar{b}} = f_{\bar{c},\bar{d}}$. By applying an automorphism, we may assume that $\bar{c} = \bar{a}$. Let i be the maximal element of $\mathrm{Dom}(f)$.

Note that $I_{<a_i}$ is indiscernible over $I_{\geq a_i}$. Consider the formula $\psi(u, v) = \varphi(u a_{\geq i}, v b_{\geq f(i)})$ (recall $b_{f(i)} = a_i$). Applying the induction hypothesis to $\psi(u, v)$ we conclude that $\varphi(\bar{a}, d_{<f(i)} b_{\geq f(i)})$ and so also $\varphi(\bar{a}, d_{\leq f(i)} b_{>f(i)})$ (because $d_{f(i)} = a_i = b_{f(i)}$). Now note that $I_{>a_i}$ is indiscernible over $I_{\leq a_i}$ and we consider the formula $\theta(u, v) = \varphi(a_{\leq i} u, d_{\leq f(i)} v)$ (recall that $d_{f(i)} = a_i$). Applying the base of the induction hypothesis to $\theta(u, v)$, we conclude that $\varphi(\bar{a}, \bar{d})$, as required. □

Definition 4.3. Let $\mathcal{L}$ be a first order language and T a complete $\mathcal{L}$-theory with infinite models and let Δ be a set of formulas. An *EM_Δ-Model* of T is a model which is generated by a Δ-indiscernible sequence, i.e. a model $M \models T$ with a Δ-indiscernible sequence I such that for every $b \in M$ there exist a term $t(x_0, \ldots, x_{n-1})$ and elements $a_0 < \cdots < a_{n-1} \in I$ with $b = t(a_0, \ldots, a_{n-1})$. If Δ is the set of all formulas we omit Δ from the notation.

For a binary relation E, let $\Delta(E)$ be the collection of formulas of the form $E(t(x), t(y))$, where t is a term.

Theorem 4.4. *Let $\mathcal{L} = \{E, \ldots\}$ be a first order language with E a binary relation. Let T an $\mathcal{L}$-theory specifying that E is a symmetric and irreflexive stable relation. Let $G = (V; E, \ldots) \models T$ be an $EM_{\Delta(E)}$-model. If $\chi(V, E) \geq (|T| + \aleph_0)^+$ then there exists a natural number n such that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$.*

Proof. There is not harm in assuming that $|T|$ is infinite, so $(|T| + \aleph_0)^+ = |T|^+$. Let $(A, <)$ be a linearly ordered set, $I = \langle r_i : i \in A \rangle$ an indiscernible sequence and $\{t_\alpha\}_{\alpha < |T|}$ a set of terms satisfying that $V = \bigcup_{\alpha < |T|} t_\alpha(I)$, where $t_\alpha(I)$ is the image of the map $I \mapsto V$ given by substituting increasing tuples in t_α.

By Lemma 2.3(1), $|T|^+ \leq \chi(G) \leq \sum_{\alpha < |T|} \chi(t_\alpha(I), E \restriction t_\alpha(I))$. Since ever successor cardinal is regular, there exists an α such that $\chi(t_\alpha(I), E \restriction t_\alpha(I)) \geq |T|^+$. We may thus assume that $V = t(I)$ for some term $t = t(\bar{x}) = t(x_0, \ldots, x_{n-1})$.

The map $t : (I^{\underline{n}})_< \to t(I)$ induces a graph on $(I^{\underline{n}})_<$ by specifying that $\bar{a}\ \widetilde{E}\ \bar{b}$ if and only if $t(\bar{a})\ E\ t(\bar{b})$. By Lemma 2.3(4), $\chi((I^{\underline{n}})_<, \widetilde{E}) \geq |T|^+$ as well.

Since the edge relation $E(v, u)$ is stable, so is $E(t(\bar{x}), t(\bar{y}))$. As a result, Proposition 4.1 allows us to apply Proposition 3.18. Hence $(I^{\underline{n}})_<$ contains all finite subgraphs of $\mathrm{Sh}_k(\omega)$ for some k. We may now conclude by applying the consequently part of Proposition 3.2. $\square$

Corollary 4.5. *Let $G = (V, E)$ be a graph. If*

- *G is superstable and $\chi(G) > 2^{\aleph_0}$ or*
- *G is ω-stable and $\chi(G) > \aleph_0$*

then G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \in \mathbb{N}$.

Proof. Suppose G is superstable and $\chi(G) > 2^{\aleph_0}$. By [She, Claim 16.2(2B.c)] or [Mar01, page 345], [Mar99, Theorem 3.B] there exists $\{E\} \subseteq \mathcal{L}$ of cardinality $2^{\aleph_0}$ and an $\mathcal{L}$-saturated EM-model $\mathcal{G}$ such that $\mathrm{Th}(\mathcal{G}) \restriction \{E\} = \mathrm{Th}(G)$. Since $\mathcal{G}$ is saturated, we may embed G as an elementary substructure of $\mathcal{G}$. Since $\chi(\mathcal{G}) > 2^{\aleph_0}$, by Theorem 4.4 all finite subgraphs of $\mathrm{Sh}_n(\omega)$ are contained in $\mathcal{G}$ for some $n \in \mathbb{N}$. The result now follows since $G \prec \mathcal{G}$, as graphs.

For ω-stable graphs we may use [Mar01, Theorem C] to find an $\mathcal{L}$-saturated EM-model in a countable language. $\square$

5. Stationary stable graphs

The crucial part of the proof of Theorem 4.4 was the existence of a saturated EM-model. It is a natural question to ask whether the technique from the previous section can be generalized to any stable graph, i.e. is the following true:

> There exists a cardinal κ such that for every stable graph with $\chi(G) \geq \kappa$ there exists a saturated EM-model $\mathcal{G}$, in an expansion $\mathcal{L} \supseteq \{E\}$ with $|\mathcal{L}| < \kappa$, such that $G \prec \mathcal{G} \restriction \{E\}$.

However, Mariou has shown in [Mar99, Theorem 3.A] that if a stable theory T has a κ^+-saturated EM-model in an expansion $\mathcal{L}$ with $|\mathcal{L}| \leq \kappa$ then T is superstable. As a result, a positive result would imply superstability. For general stable graphs a different approach is needed.

A connected notion to that of EM-models is the that of representations of structures from [CS16]. We will need a variation on the theme.

Definition 5.1 (The free algebra)**.** Suppose A is a pure set. Let $\mathcal{M}_{\mu,\kappa}(A)$ be the (non first order) structure whose vocabulary is $\mathcal{L}_{\mu,\kappa} = \{F_{\alpha,\beta} : \alpha < \mu, \beta < \kappa\}$, where each $F_{\alpha,\beta}$ is a β-ary function symbol for all $\alpha < \mu$ (note that we allow infinite arity). The universe of $\mathcal{M}_{\mu,\kappa}(A)$ is $\bigcup_{\gamma \in \mathrm{Ord}} \mathcal{M}_{\mu,\kappa,\gamma}(A)$. Where

- $\mathcal{M}_{\mu,\kappa,0}(A) = A$,
- for limit γ, $\mathcal{M}_{\mu,\kappa,\gamma}(A) = \bigcup_{\gamma' < \gamma} \mathcal{M}_{\mu,\kappa,\gamma'}(A)$,
- and for successor

$$\mathcal{M}_{\mu,\kappa,\gamma+1} = \mathcal{M}_\gamma \cup \{F_{\alpha,\beta}(\bar{b}) : \bar{b} \in (\mathcal{M}_{\mu,\kappa,\gamma})^\beta,\ \alpha < \mu,\ \beta < \kappa\}.$$

We treat $F_{\alpha,\beta}(\bar{b})$ as a new formal object.

For a cardinal κ, let $\mathrm{reg}(\kappa)$ be κ^+ if κ is singular and κ otherwise.

Fact 5.2 ([CS16, Remark 2.3])**.** Let A and $\mathcal{M}_{\mu,\kappa}(A)$ be as before. $\mathcal{M}_{\mu,\kappa}(A)$ is a set whose cardinality is at most $(|A|+\mu)^{<\mathrm{reg}(\kappa)}$ (though defined as a class).

Remark 5.3. Fixing a set of variables $X = \{x_i : i < \mathrm{reg}(\kappa)\}$, the set of terms in $\mathcal{L}_{\mu,\kappa}$ in X can be identified with $\mathcal{M}_{\mu,\kappa}(X)$. It follows from Fact 5.2 that their number is bounded by $(\mathrm{reg}(\kappa)+\mu)^{<\mathrm{reg}(\kappa)}$.

For any permutation π of A we denote by $\widehat{\pi}$ the induced automorphism of $\mathcal{M}_{\mu,\kappa}(A)$.

Definition 5.4. Let M be a structure. A *homogeneous representation* of M in $M_{\mu,\kappa}(A)$ is a function $\Phi : M \to \mathcal{M}_{\mu,\kappa}(A)$ satisfying

(1) For every term $t(\bar{x})$, where $\bar{x}$ is tuple of length $\beta < \kappa$ containing the variables of t, if $t(\bar{a}) \in \mathrm{Im}(\Phi)$ for some $\bar{a} \in A^{\underline{\beta}}$ then $t(\bar{b}) \in \mathrm{Im}(\Phi)$ for all $\bar{b} \in A^{\underline{\beta}}$;

(2) For any two finite sequences $\bar{a}, \bar{b} \in M^{\underline{n}}$, if there exists a permutation π of A such that $\widehat{\pi}(\Phi(a_i)) = \Phi(b_i)$, for all $i < n$, then
$$\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^M(\bar{b}).$$

We say that Φ is a *skeletal homogeneous representation* if it is an injective partial function satisfying (1) and (2) on its domain and that $\mathrm{dcl}(\mathrm{Dom}(\Phi)) = M$.

Remark 5.5. Representations were originally defined in [CS16, Definition 2.1] and the definition was that of a function $\Phi : M \to \mathcal{M}_{\mu,\kappa}(A)$ satisfying that
$$\mathrm{qftp}(\Phi(\bar{a})) = \mathrm{qftp}(\Phi(\bar{b})) \implies \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^M(\bar{b}).$$
Since every permutation of A lifts to an automorphism of the free algebra, the antecedent in condition (2) implies that $\Phi(\bar{a})$ and $\Phi(\bar{b})$ have the same quantifier-free type. As a result, every representation satisfies condition (2) of a homogeneous representation.

Proposition 5.6. *Let M be a structure in a countable language and $\varkappa$ a regular cardinal. Assume there exists a skeletal homogeneous representation $\Phi : \mathrm{Dom}(\Phi) \to \mathcal{M}_{\mu,\kappa}(A)$ of M, where A is a pure set, κ and μ are infinite, and that*

(1) $(\mathrm{reg}(\kappa)+\mu)^{<\mathrm{reg}(\kappa)} < \varkappa$,
(2) $\beth_2(\lambda) < \varkappa$ *for all* $\lambda < \mathrm{reg}(\kappa)$,
(3) $2^{<\mathrm{reg}(\kappa)} \leq |A|$.

For every graph $G = (V, E)$ that is $\emptyset$-interpretable in M with $\chi(G) \geq \varkappa$ there exists an $n \in \mathbb{N}$ such that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$.

Proof. Since G is interpretable in M there exist $r \in \mathbb{N}$, a definable subset $V_0 \subseteq M^r$ and an interpretation $g : V_0 \to V$ (see [Hod93, Section 5.3]). By definition, g is surjective and $G_0 = (V_0, g^{-1}(E))$ is a definable graph. Thus g is a surjective homomorphism and by Lemma 2.3(4) $\chi(G) = \chi(G_0)$. Note that if G_0 contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ then by Proposition 3.2 so does G (maybe for a different n). Consequently, we may assume that the graph $G = (V, E)$ is $\emptyset$-definable in M^r.

Let $D = \mathrm{Dom}(\Phi)$ and let $\langle f_i(\bar{v}_i) : i < \omega\rangle$ be an enumeration of all $\emptyset$-definable functions to M. Let
$$\mathcal{U} = \{F_{i,|\bar{v}_i|}(b_0, \ldots, b_{|\bar{v}_i|-1}) : b_0, \ldots, b_{|\bar{v}_i|-1} \in \mathrm{Im}(\Phi),\, i < \omega\} \subseteq \mathcal{M}_{\mu,\kappa}(A).$$

Define a surjective map $\Psi_0 : \mathcal{U} \to M$ by mapping $F_{i,|\bar{v}_i|}(b_0, \dots, b_{|\bar{v}_i|-1})$ to

$$f_i(\Phi^{-1}(b_0), \dots, \Phi^{-1}(b_{|\bar{v}_i|-1})).$$

Note that Φ is injective so this is well defined.

Let $\Psi_1 = (\Psi_0)^r : (\mathcal{U})^r \to M^r$, $\mathcal{V} = \{a \in (\mathcal{U})^r : \Psi_1(a) \in V\}$ and $\Psi = \Psi_1 \restriction \mathcal{V} : \mathcal{V} \to V$. Let $\mathcal{E} = \Psi^{-1}(E)$, hence $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph and note that $\chi(G) = \chi(\mathcal{G})$ by Lemma 2.3(4). Let $\nu = (\mathrm{reg}(\kappa) + \mu)^{<\mathrm{reg}(\kappa)}$.

Let $X = \{x_i : i < \mathrm{reg}(\kappa)\}$ be a set of variables as in Remark 5.3. Let $\mathfrak{t}_0$ be the set of pairs $(t, \bar{x})$, where $\bar{x}$ is a sequence of variables from X of length $< \mathrm{reg}(\kappa)$ and t is a term in $\mathcal{L}_{\mu,\kappa}$ with variables contained in $\bar{x}$. Let $\mathfrak{t}_1$ be the subset of $\mathfrak{t}_0$ consisting of pairs of the form $(F_{i,|\bar{v}_i|}(t_0, \dots, t_{|\bar{v}_i|-1}), \bar{x})$, where $i < \omega$, and $(t_0, \bar{x}), \dots, (t_{|\bar{v}_i|-1}, \bar{x}) \in \mathfrak{t}_0$. Let $\mathfrak{t} = \{((s_0, \bar{x}_0), \dots, (s_{r-1}, \bar{x}_{r-1})) \in (\mathfrak{t}_1)^r : \bar{x}_0 = \dots = \bar{x}_{r-1}\}$. We may enumerate $\mathfrak{t} = \{\bar{s}_i(\bar{x}_i) : i < \nu\}$, where for ease of notation we write $(\bar{s}, \bar{x})$ as $\bar{s}(\bar{x})$.

Since $\mathcal{V}$ is covered by the union of $\{\bar{s}_i(\bar{a}) : \bar{a} \in A^{\underline{|\bar{x}_i|}}\}_{i<\nu}$, $\mathcal{V} = \bigcup_{i<\nu} \mathcal{V}_i$, where $\mathcal{V}_i = \{\bar{s}_i(\bar{a}) : \bar{a} \in A^{\underline{|\bar{x}_i|}}\} \cap \mathcal{V}$.

By Lemma 2.3(1), assumption (1) and since $\varkappa$ is regular, there exists some $i < \nu$ with $\chi(\mathcal{G}_i) \geq \varkappa$, where $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E} \restriction \mathcal{V}_i \times \mathcal{V}_i)$.

Set $\bar{s} = \bar{s}_i$ and $\bar{x} = \bar{x}_i$. Assume, for simplicity, that

$$\bar{s}(\bar{x}) = (F_{0,k_0}(t_{0,0}(\bar{x}), \dots, t_{0,k_0-1}(\bar{x})), \dots, F_{r-1,k_{r-1}}(t_{r-1,0}(\bar{x}), \dots, t_{r-1,k_r-1}(\bar{x}))).$$

Claim. $\bar{s}$ defines a surjective function $A^{\underline{|\bar{x}|}} \to \mathcal{V}_i$.

Proof. Since $\mathcal{G}_i$ is non-empty, there exists $\bar{a} \in A^{\underline{|\bar{x}|}}$ such that $\bar{s}(\bar{a}) \in \mathcal{V}$. Let $\bar{b} \in A^{\underline{|\bar{x}|}}$. By Definition 5.4(1), $F_{i,k_i}(t_{i,0}(\bar{b}), \dots, t_{i,k_i-1}(\bar{b})) \in \mathcal{U}$ for all $i < r$. Note that $|\bar{x}| < |A|$ by assumption (3) and so there exists a permutation π of A mapping $\bar{a}$ to $\bar{b}$, and let $\widehat{\pi}$ be induced automorphism of $\mathcal{M}_{\mu,\kappa}$. Thus $\widehat{\pi}(\bar{s}(\bar{a})) = \bar{s}(\bar{b})$. Since V is $\emptyset$-definable and $\Psi_1(\bar{s}(\bar{a})) \in V$, Definition 5.4(2) gives that $\mathrm{tp}^M(\Psi_1(\bar{s}(\bar{b})) = \mathrm{tp}^M(\Psi_1(\bar{s}(\bar{a}))$ and hence $\Psi_1(\bar{s}(\bar{b}) \in V$ as well. Consequently, $\bar{s}$ defines a function. Surjectivity is straightforward. $\square$ (claim)

Let $R = \bar{s}^{-1}(\mathcal{E} \restriction \mathcal{V}_i \times \mathcal{V}_i)$ be the edge relation $\bar{s}$ induces on $A^{\underline{|\bar{x}|}}$.

By assumptions $(2, 3)$, in order to apply Proposition 3.17, we are left to verify assumption $(\star)$ of Proposition 3.17.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in A^{\underline{|\bar{x}|}}$ satisfying $\bar{a}\, R\, \bar{b}$ and $f_{\bar{a},\bar{b}} = f_{\bar{c},\bar{d}}$. The latter condition implies that the coordinate-wise map sending $\bar{a}\bar{b}$ to $\bar{c}\bar{d}$ is well defined and injective. Since $|\bar{x}| < |A|$, we may find a permutation π of A which maps $\bar{a}\bar{b}$ to $\bar{c}\bar{d}$. This permutation lifts to an automorphism $\widehat{\pi}$ of the free algebra, with $\widehat{\pi}(\bar{s}(\bar{a})) = \bar{s}(\bar{c})$ and $\widehat{\pi}(\bar{s}(\bar{b})) = \bar{s}(\bar{d})$.

Thus $\widehat{\pi}(t_{i,j}(\bar{a})) = t_{i,j}(\bar{c})$ and $\widehat{\pi}(t_{i,j}(\bar{b})) = t_{i,j}(\bar{d})$, for $i < r$ and $j < k_i$. By Definition 5.4(2),

$$\mathrm{tp}^M((\Phi^{-1}(t_{i,j}(\bar{a})))_{i<r,j<k_i}, (\Phi^{-1}(t_{i,j}(\bar{b})))_{i<r,j<k_i}) =$$

$$\mathrm{tp}^M((\Phi^{-1}(t_{i,j}(\bar{c})))_{i<r,j<k_i}, (\Phi^{-1}(t_{i,j}(\bar{d})))_{i<r,j<k_i}),$$

and consequently

$$\mathrm{tp}^M(\Psi(\bar{s}(\bar{a})), \Psi(\bar{s}(\bar{b}))) = \mathrm{tp}^M(\Psi(\bar{s}(\bar{c})), \Psi(\bar{s}(\bar{d}))).$$

Since $\bar{a}\ R\ \bar{b}$, $\bar{s}(\bar{a})\ \mathcal{E}\ \bar{s}(\bar{b})$ and so $\Psi(\bar{s}(\bar{a}))\ E\ \Psi(\bar{s}(\bar{b}))$. As this is specified by the type of the pair,

$$\Psi(\bar{s}(\bar{c}))\ E\ \Psi(\bar{s}(\bar{d}))$$

as well. As a result, $\bar{s}(\bar{c})\ \mathcal{E}\ \bar{s}(\bar{d})$ and $\bar{c}\ R\ \bar{d}$.

By Proposition 3.17, there exists $m \in \mathbb{N}$ and an injective homomorphism from $\mathrm{Sh}_m(\omega)$ to $A^{\underline{|\bar{x}|}}$. By composing with $\bar{s}$ and Ψ and applying Proposition 3.2, there exists $n \leq m$ such that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$. □

If T is a countable ω-stable theory and $M \models T$, one may find an injective representation, in the sense of Remark 5.5, of M in $\mathcal{M}_{\aleph_0,\aleph_0}(A)$, for some set A; see [CS16, Theorem 4.4]. Similarly superstable theories may be represented in $\mathcal{M}_{2^{|T|},\aleph_0}(A)$, for some set A, see [She19, Theorem 2.1]. However, we may not apply the previous proposition to these representations since they may not homogeneous. We build such homogeneous representations for stable theories in which every type is stationary.

Definition 5.7. We say that a theory T is *stationary* if all types (over any set) are stationary.

Remark 5.8. Rothmaler studies stationarity of modules in [Rot83], e.g. he gives a complete description of stationary abelian groups in [Rot83, Theorem 4(ii)].

Recall that a formula $\varphi(x,d)$ is *almost over* A if there exists an equivalence relation with finitely many classes $E(x,x')$ over A such that $\forall x \forall x'(E(x,x') \to (\varphi(x,d) \leftrightarrow \varphi(x',d)))$.

Fact 5.9 ([Rot83, Lemma 2, Theorem 1])**.** Let T be a stable theory. The following are equivalent:

(1) T is stationary;
(2) for any A, every formula which is almost over A is over A;
(3) all 1-types over (over any set) are stationary.

Proposition 5.10. *Let T be a complete stationary stable theory in a language $\mathcal{L}$. For every sublanguage $\mathcal{L}_0 \subseteq \mathcal{L}$ there is some $\mathcal{L}_0 \subseteq \mathcal{L}' \subseteq \mathcal{L}$ with $|L'| = |\mathcal{L}_0| + \aleph_0$ such that $T \restriction \mathcal{L}'$ is stationary.*

Proof. By Fact 5.9, T is stationary if and only if for every equivalence relation with finitely many classes $E(x,x')$ over A, every class of E is definable over A.

Claim. For every $\psi(x,x',z)$ and $n < \omega$ there are finitely many formulas $\theta_i(x,z)$ $(i < k)$ such that

(†) for any z-tuple c such that $\psi(x,x',c)$ defines an equivalence relation with $\leq n$ classes, and for any x'-tuple d there is some $i < k$ such that $\psi(x,d,c)$ is equivalent to $\theta_i(x,c)$.

Proof. Note that (†) is a first order sentence.

Suppose not and fix $\psi(x,x',z)$ and $n < \omega$. This means that for every finite collection of formulas $\theta_i(x,z)$ $(i < k)$ there are some c and d witnessing the failure of (†). Let $\Gamma(x',z)$ be

$$\{\psi(-,-,z) \text{ defines an equivalence relation with } \leq n \text{ classes}\} \cup$$
$$\{\exists x \neg(\psi(x,x',z) \leftrightarrow \theta(x,z)) :\ \theta(x,z) \text{ any formula}\}.$$

By assumption, Γ is consistent. Let $(d,c) \models \Gamma(x',z)$. Then $\psi(x,d,c)$ is almost over c but not over c, contradiction. □ (claim)

Now, let $\mathcal{L}_0 \subseteq \mathcal{L}$ be a sublanguage. We construct an increasing sequence of languages $\mathcal{L}_m$ as follows. The language $\mathcal{L}_0$ is given. Assume we have constructed $\mathcal{L}_m$. For any $\psi(x, x', z)$ in the language $\mathcal{L}_m$ and $n < \omega$ let $\{\theta_{\psi,n,i}(x, z)\}_{i<k_{\psi,n}}$ be a finite set of formulas satisfying (†) (such a set exists by the claim). Let $\mathcal{L}_{m+1} = \mathcal{L}_m \cup \{$the symbols in the formula $\theta_{\psi,n,i} : \psi \in \mathcal{L}_m,\, n < \omega, i < k_{\psi,n}\}$. Now set $\mathcal{L}' = \bigcup_{m<\omega} \mathcal{L}_m$. It follows that $T \restriction \mathcal{L}'$ is stationary by Fact 5.9(2). $\square$

We leave the proof of the following easy lemma to the reader.

Lemma 5.11. *If T is a complete stationary stable theory then* $\mathrm{dcl}(A) = \mathrm{acl}(A)$ *for any set A.*

Remark 5.12. If T is a stable theory, then, since for any A every type over $\mathrm{acl}^{eq}(A)$ is stationary, if T is eliminates imaginaries and has no algebraicity (i.e. $\mathrm{acl}(A) = \mathrm{dcl}(A)$ for any A) then T is stationary. However, as the theory of the infinite set shows, the other direction is not true (it does not eliminate imaginaries).

We will need the following lemma, which is a consequence of [She78, Lemma III.3.10], but for the convenience of the reader we give a direct proof. Recall that for a stable theory T, $\kappa(T)$ is the least cardinal κ such that for all B and type $p \in S(B)$ there exists $A \subseteq B$ with $|A| < \kappa$ such that p does not fork over A ([She78, Definition III.3.1]).

For any infinite indiscernible sequence I and a set A, let $\lim(I/A)$ be the limit type of I in A (it is denoted by $\mathrm{Av}(I, A)$ in [She78]), i.e.

$$\lim(I/A) = \{\varphi(x, c) : c \in A,\ \ \varphi(a, c) \text{ holds for cofinitely many } a \in I\}.$$

It is a consistent complete type over A by stability.

Lemma 5.13. *Let T be a stationary stable theory, M a model and $\lambda > \kappa(T)$ a cardinal. If for every non-algebraic type $q \in S(C)$ with $|C| < \kappa(T)$ and $C \subseteq M$ there is a C-independent set of realizations of q in M of cardinality λ, then M is λ-saturated.*

Proof. Let $p \in S(A)$ be a complete type with $|A| < \lambda$. If p is algebraic then it is realized, so we may assume that p is non-algebraic. Let $C \subseteq A$ with $|C| < \kappa(T)$ be such that p does not fork over C. By assumption, we may find a C-independent set I of realizations of $p|C$ in M (so indiscernible over C by stationarity). By [She78, Lemma III.1.10(2)], $\lim(I/A) = p$. By [She78, Corollary III.3.5(1)], there is $I_0 \subseteq I$ with $I \setminus I_0$ indiscernible over A and $|I_0| \leq \kappa(T) + |A| < \lambda$. In particular, $|I \setminus I_0| \geq \aleph_0$ and thus for every $c \in I \setminus I_0$, $p = \mathrm{tp}(c/A)$. $\square$

We fix the following notation for the rest of the section. Let T be a complete stable theory with infinite models, and let $\mathbb{U}$ be a monster model. Let $\kappa = \kappa_r(T)$, i.e. $\kappa = \kappa(T)^+$ if $\kappa(T)$ is singular or $\kappa(T)$ if not (for the sake of the following, one can also take $\kappa = |T|^+$) and let $\mu = \mu^{<\kappa}$ be a cardinal, with $\mu > \kappa$, such that T is μ-stable, e.g. if $\mu \geq 2^{|T|}$ (see [She78, Lemma III.3.6]), and thus there exists a saturated model of cardinality μ [She78, Theorem III.3.12]. Fix some partition $\mu = \uplus_{i<\kappa} U_i$ to sets each of cardinality μ. From now on we also assume that T is stationary.

Definition 5.14. Let I be any set. We define $OB(I)$ to be the collection of triples

$$\mathbf{a} := \left(i_{\mathbf{a}}, \{U_j^{\mathbf{a}}\}_{j<i_{\mathbf{a}}}, B^{\mathbf{a}}\right)$$

satisfying:

(1) $i_{\mathbf{a}} \leq \kappa$;
(2) $U_j^{\mathbf{a}} \subseteq U_j$ for all $j < i_{\mathbf{a}}$, and we set $U_{<j}^{\mathbf{a}} := \bigcup_{k<j} U_k^{\mathbf{a}}$;
(3) $B^{\mathbf{a}} = \langle b_{\alpha,\eta}^{\mathbf{a}} \in \mathbb{U} : \alpha \in U_j^{\mathbf{a}}, \eta \in I^{\underline{j}},\ j < i_{\mathbf{a}}\rangle$ are such that:
  (a) $B^{\mathbf{a}}$ is with no repetitions;
  (b) $B_j^{\mathbf{a}} := \{b_{\alpha,\eta}^{\mathbf{a}} : \alpha \in U_j^{\mathbf{a}}, \eta \in I^{\underline{j}}\}$ is independent over $B_{<j}^{\mathbf{a}} = \bigcup_{k<j} B_k^{\mathbf{a}}$, for every $j < i_{\mathbf{a}}$.
  For ease, we denote for $j < i_{\mathbf{a}}$, $W_j^{\mathbf{a}} := \{(\alpha,\eta) : \alpha \in U_j^{\mathbf{a}}, \eta \in I^{\underline{j}}\}$, and likewise $W_{<j}^{\mathbf{a}}$.

As usual, when $\mathbf{a}$ is clear from the context we omit it.

Note that any permutation π of the set I induces a permutation $\widehat{\pi}$ of $I^{\underline{\gamma}}$, for any γ.

Definition 5.15. Let I be a set and $OB(I)$ as above.

(1) We say that $\mathbf{a} \in OB(I)$ is *homogeneous* if for any permutation π of I, the set of pairs

$$\pi[\mathbf{a}] := \{(b_{\alpha,\eta}^{\mathbf{a}}, b_{\alpha,\widehat{\pi}(\eta)}^{\mathbf{a}}) : (\alpha,\eta) \in W_{<i_{\mathbf{a}}}^{\mathbf{a}}\}$$

is an elementary embedding.

(2) We say that $\mathbf{a} \in OB(I)$ is *full* if for every $j < i_{\mathbf{a}}$ and $\eta \in I^{\underline{j}}$, any non-algebraic type p over $B_{<j}^{\mathbf{a}}$ which does not fork over

$$\{b_{\alpha,\nu}^{\mathbf{a}} \in B_{<j}^{\mathbf{a}} : \mathrm{Range}(\nu) \subseteq \mathrm{Range}(\eta)\}$$

is realized by $b_{\alpha,\eta}^{\mathbf{a}}$ for some $\alpha \in U_j^{\mathbf{a}}$.

Lemma 5.16. *Let I be any set and $\mathbf{a} \in OB(I)$ with $i_{\mathbf{a}} = \kappa$. For every $C \subseteq B_{<\kappa}^{\mathbf{a}}$ with $|C| < \kappa(T) \leq \kappa$ there exist some $j < \kappa$ and $\eta \in I^{\underline{j}}$ satisfying*

$$C \subseteq \{b_{\alpha,\nu} \in B_{<j}^{\mathbf{a}} : \mathrm{Range}(\nu) \subseteq \mathrm{Range}(\eta)\}.$$

Proof. If C is empty then it is easy so assume that $C \neq \emptyset$. Since κ is regular, there exist $\tilde{j} < \kappa$ with $C \subseteq B_{<\tilde{j}}^{\mathbf{a}}$. Let $J = \bigcup_{b_{\alpha,\nu} \in C} \mathrm{Range}(\nu)$. Let $j = \max\{\tilde{j}, |J|\} < \kappa$ and $\eta \in I^{\underline{j}}$ with $\mathrm{Range}(\eta) = J$. □

Proposition 5.17. *Let I be any set with $|I| \geq \mu$. If $\mathbf{a} \in OB(I)$ is full and $i_{\mathbf{a}} = \kappa$ then $M := \mathbb{U} \restriction \mathrm{dcl}(B_{<\kappa}^{\mathbf{a}})$ is a saturated elementary substructure of $\mathbb{U}$ of cardinality $|I|^{<\kappa}$.*

Proof. To show that it is an elementary substructure we use Tarski-Vaught. Let $\varphi(x,b)$ be a consistent formula with $b \in \mathrm{dcl}(B_{<\kappa}^{\mathbf{a}})$. There is no harm in assuming that $b \in B_{<\kappa}^{\mathbf{a}}$. If $\varphi(x,b)$ is algebraic then by Lemma 5.11, any realization is already in $\mathrm{dcl}(B_{<\kappa}^{\mathbf{a}})$. Otherwise, let p be any non-algebraic complete type over b containing $\varphi(x,b)$. Let $j < \kappa$ and $\eta \in I^{\underline{j}}$ be given by Lemma 5.16 for $C = \{b\}$. By stationarity, there is a unique non-forking extension $\mathfrak{p}$ of p to $B_{<j}^{\mathbf{a}}$, which is necessarily non-algebraic as well. By fullness, we may realize $\mathfrak{p}$ by some element $b_{\alpha,\eta}$ for $\alpha \in U_j^{\mathbf{a}}$. In particular $b_{\alpha,\eta} \in B_{<\kappa}^{\mathbf{a}}$ realizes $\varphi(x,b)$.

To show $|I|^{<\kappa}$-saturation we apply Lemma 5.13 (recall $\mu > \kappa$). Let $q \in S(C)$ be a non-algebraic type with $C \subseteq \mathrm{dcl}(B_{<\kappa}^{\mathbf{a}})$ and $|C| < \kappa(T)$. There is no harm to take

$C \subseteq B^{\mathbf{a}}_{<\kappa}$. Let $j < \kappa$ and $\eta \in I^{\underline{j}}$ be as supplied by Lemma 5.16 with respect to C. Let

$$\Delta := \{\nu \in \bigcup_{j \le k < \kappa} I^{\underline{k}} : \mathrm{Range}(\eta) \subseteq \mathrm{Range}(\nu)\}.$$

By assumption of fullness, for every $\nu \in \Delta$ there is some $\alpha_\nu \in U^{\mathbf{a}}_{|\nu|}$, such that $b_{\alpha_\nu,\nu} \models q|B^{\mathbf{a}}_{<|\nu|}$, and so satisfies q as well. The set of realizations of q, $\{b_{a_\nu,\nu} : \nu \in \Delta\}$, is independent over C by the definition of $OB(I)$. Indeed, by Definition 5.14(3b), for any $k \ge j$ and $\nu \in I^{\underline{k}} \cap \Delta$, $b_{\alpha_\nu,\nu} \mathop{\smile\!\!\!\!\!\mid}_{B^{\mathbf{a}}_{<k}} \{b_{\alpha_\rho,\rho} : \rho \in I^{\underline{k}} \cap \Delta,\ \rho \ne \nu\}$ and by the choice of α_ν, $b_{\alpha_\nu,\nu} \mathop{\smile\!\!\!\!\!\mid}_C B^{\mathbf{a}}_{<k}$. Thus by transitivity

$$b_{\alpha_\nu,\nu} \mathop{\smile\!\!\!\!\!\mid}_{C} B^{\mathbf{a}}_{<k}\{b_{\alpha_\rho,\rho} : \rho \in I^{\underline{k}} \cap \Delta,\ \rho \ne \nu\}.$$

Consequently, after choosing a well ordering $<^*$ of $\{b_{\alpha_\nu,\nu} : \nu \in \Delta\}$ satisfying that if $k_2 > k_1 \ge j$, $b_{\alpha_{\nu_1},\nu_1} \in I^{\underline{k_1}} \cap \Delta$ and $b_{\alpha_{\nu_2},\nu_2} \in I^{\underline{k_2}} \cap \Delta$ then $b_{\alpha_{\nu_1},\nu_1} <^* b_{\alpha_{\nu_2},\nu_2}$ we may conclude by induction.

We note that $|\Delta| = |I|^{<\kappa}$ and thus by Lemma 5.13, $\mathrm{dcl}(B^{\mathbf{a}}_{<\kappa})$ is $|I|^{<\kappa}$-saturated. Furthermore, as $|I| \ge \mu$ and the $U^{\mathbf{a}}_j$'s are non-empty for every $j < \kappa$ (by fullness), it follows that $|I|^{<\kappa} = |B^{\mathbf{a}}_{<\kappa}| = |\mathrm{dcl}(B^{\mathbf{a}}_{<\kappa})|$ and so $\mathrm{dcl}(B^{\mathbf{a}}_{<\kappa})$ is saturated. □

We define the following partial order on the elements of $OB(I)$.

Definition 5.18. Let I be a set. For $\mathbf{a}, \mathbf{b} \in OB(I)$ we say that $\mathbf{a} \le \mathbf{b}$ if

(1) $i_{\mathbf{a}} \le i_{\mathbf{b}}$;
(2) for $j < i_{\mathbf{a}}$ we have $U^{\mathbf{a}}_j = U^{\mathbf{b}}_j$ and $b^{\mathbf{a}}_{\alpha,\eta} = b^{\mathbf{b}}_{\alpha,\eta}$, for $(\alpha, \eta) \in W^{\mathbf{a}}_j$.

Proposition 5.19. *Let I be a set with $|I| \ge \kappa$. Then there exists a full homogeneous $\mathbf{a} \in OB(I)$ with $i_{\mathbf{a}} = \kappa$.*

Proof. We choose full homogeneous $\mathbf{a}_j \in OB(I)$ by induction on $j \le \kappa$ such that $\mathbf{a}_j \in OB(I)$ with $i_{\mathbf{a}_j} = j$ and such that $k_1 \le k_2 < j$ implies $\mathbf{a}_{k_1} \le \mathbf{a}_{k_2}$.

For $j = 0$ choose $\mathbf{a}_0 = (0, \emptyset, \emptyset)$ and note that the conditions hold trivially.

Let $j \le \kappa$ be a limit ordinal, set $i_{\mathbf{a}_j} = j$, $U^{\mathbf{a}_j}_k := U^{\mathbf{a}_{k+1}}_k$ for $k < j$ and $B^{\mathbf{a}_j} := \bigcup_{k<j} B^{\mathbf{a}_k}$, $\mathbf{a}_j$ has the desired properties.

Let $j < \kappa$ and assume that $\mathbf{a} := \mathbf{a}_j$ is full and homogeneous with $i_{\mathbf{a}_j} = j$. We will construct a full homogeneous $\mathbf{b} \in OB(I)$ with $i_{\mathbf{b}} = j + 1$ such that $\mathbf{a} \le \mathbf{b}$. Then we can set $\mathbf{a}_{j+1} := \mathbf{b}$.

Let

$$\mathcal{P} := \{(p, C, \eta) : p \in S(C) \text{ non-algebraic}, \eta \in I^{\underline{j}},$$
$$C \subseteq \{b_{\alpha,\nu} \in B^{\mathbf{a}}_{<j} : \mathrm{Range}(\nu) \subseteq \mathrm{Range}(\eta)\} \text{ of cardinality } < \kappa\}.$$

We define the following equivalence relation on $\mathcal{P}$:

$$(p_1, C_1, \eta_1)\ E\ (p_2, C_2, \eta_2)$$

if and only if for some permutation π of I

$$\pi\langle \mathbf{a}\rangle (p_1, C_1, \eta_1) = (p_2, C_2, \eta_2),$$

which means $\widehat{\pi}(\eta_1) = \eta_2$, $\pi[\mathbf{a}]$ (as an elementary mapping) maps C_1 onto C_2 and p_1 onto p_2. Since $|I| \ge \kappa > j$, for every $\eta_1, \eta_2 \in I^{\underline{j}}$ there is a permutation π of I mapping η_1 to η_2. As $\pi\langle \mathbf{a}\rangle$ is a permutation of $\mathcal{P}$ and there are at most μ inequivalent triples whose last coordinate is η_1, E has at most μ equivalence classes (because $\mu^{<\kappa} = \mu$).

Since $|U_j| = \mu$ we may find some $U'_j \subseteq U_j$ that enumerates the different classes, i.e let $\langle X_\alpha : \alpha \in U'_j \subseteq U_j \rangle$ list $\mathcal{P}/\,E$. Since $\pi\langle \mathbf{a} \rangle$ is a permutation of $\mathcal{P}$, $\pi\langle \mathbf{a} \rangle \restriction X_\alpha$, for $\alpha \in U'_j$, is also a permutation.

For any $\eta \in I^{\underline{j}}$ and $\alpha \in U'_j$ we claim that there exists a unique (p, C) such that $(p, C, \eta) \in X_\alpha$. To show uniqueness, note that if $\widehat{\pi}$ fixes η then $\pi[\mathbf{a}]$ fixes C. For existence, let $\nu \in I^{\underline{j}}$, p and C be such that $(p, C, \nu) \in X_\alpha$. Since $|I| > j$ we may find a permutation π of I mapping ν to η. Now note that $(\pi[\mathbf{a}](p), \pi[\mathbf{a}](C), \widehat{\pi}(\nu)) \in X_\alpha$.

For any $\eta \in I^{\underline{j}}$ and $\alpha \in U'_j$ we name the unique pair by $(p_{\alpha,\eta}, C_{\alpha,\eta})$. By fixing some well order on $\{(\alpha, \eta) : \alpha \in U'_j, \eta \in I^{\underline{j}}\}$, we may inductively find $\langle b_{\alpha,\eta} : \alpha \in U'_j, \eta \in I^{\underline{j}} \rangle$ such that $b_{\alpha,\eta}$ realizes the unique non-forking extension of $p_{\alpha,\eta}$ to $B^{\mathbf{a}}_{<j} \cup \{b_{\alpha',\eta'} : (\alpha', \eta') < (\alpha, \eta)\}$. In particular $\mathrm{tp}(b_{\alpha,\eta}/B^{\mathbf{a}}_{<j})$ does not fork over $C_{\alpha,\eta}$ and $\langle b_{\alpha,\eta} : \alpha \in U'_j, \eta \in I^{\underline{j}} \rangle$ is independent over $B^{\mathbf{a}}_{<j}$.

We may now define $\mathbf{b} \in OB(I)$ by $i_{\mathbf{b}} = j + 1$, $\mathbf{a} \leq \mathbf{b}$, $U^{\mathbf{b}}_j = U'_j$ and $b^{\mathbf{b}}_{\alpha,\eta} = b_{\alpha,\eta}$ for $\alpha \in U'_j, \eta \in I^{\underline{j}}$. Note that since the types $p_{\alpha,\eta}$ are not algebraic, it follows that $B^{\mathbf{b}}$ is without repetitions. It remains to check that $\mathbf{b}$ is full and homogeneous.

$\mathbf{b}$ is full: by the induction hypothesis, it is enough to consider $\eta \in I^{\underline{j}}$ and a non-algebraic type p over $B^{\mathbf{b}}_{<j}$ which does not fork over

$$\{b_{\alpha,\nu} \in B^{\mathbf{b}}_{<j} : \mathrm{Range}(\nu) \subseteq \mathrm{Range}(\eta)\}.$$

Let C be a subset with $|C| < \kappa$ such that p does not fork over C. Consider the triple $(p|C, C, \eta)$ and let $\alpha \in U^{\mathbf{b}}_j$ be such that $(p|C, C, \eta) \in X_\alpha$. By uniquness, $(p|C, C) = (p_{\alpha,\eta}, C_{\alpha,\eta})$ (in the above notation), and thus $b^{\mathbf{b}}_{\alpha,\eta}$ satisfies the unique non-forking extension of $p|C$ to $B^{\mathbf{b}}_{<j}$ which is equal to p.

$\mathbf{b}$ is homogeneous: let π be a permutation of I. To show that $\pi[\mathbf{b}]$ is elementary, we show by induction on $k \geq 0$ that for $(\alpha_1, \eta_1), \dots, (\alpha_k, \eta_k) \in W^{\mathbf{b}}_j$

$$b_{\alpha_1,\eta_1} \dots b_{\alpha_k,\eta_k} B^{\mathbf{b}}_{<j} \equiv b_{\alpha_1,\widehat{\pi}(\eta_1)} \dots b_{\alpha_k,\widehat{\pi}(\eta_k)} \pi[\mathbf{b}](B^{\mathbf{b}}_{<j}).$$

For $k = 0$, it follows since $\pi[\mathbf{a}]$ is elementary and since $B^{\mathbf{b}}_{<j} = B^{\mathbf{a}}_{<j}$.

For $k = 1$: since $\pi[\mathbf{b}]$ is an elementary map mapping $p_{\alpha,\eta}$ onto $p_{\alpha,\widehat{\pi}(\eta)}$, by stationarity it also maps $p_{\alpha,\eta}|B^{\mathbf{b}}_{<j}$ onto $p_{\alpha,\widehat{\pi}(\eta)}|B^{\mathbf{b}}_{<j}$ and so $b_{\alpha,\eta} B^{\mathbf{b}}_{<j} \equiv b_{\alpha,\widehat{\pi}(\eta)} \pi[\mathbf{b}](B^{\mathbf{b}}_{<j})$.

The induction step: let $(\alpha_1, \eta_1), \dots, (\alpha_k, \eta_k) \in W^{\mathbf{b}}_j$ with $k \geq 2$. By the induction hypothesis

$$b_{\alpha_1,\eta_1} \dots b_{\alpha_{k-1},\eta_{k-1}} B^{\mathbf{b}}_{<j} \equiv b_{\alpha_1,\widehat{\pi}(\eta_1)} \dots b_{\alpha_{k-1},\widehat{\pi}(\eta_{k-1})} \pi[\mathbf{b}](B^{\mathbf{b}}_{<j})$$

and $b_{\alpha_k,\eta_k} B^{\mathbf{b}}_{<j} \equiv b_{\alpha_k,\widehat{\pi}(\eta_k)} \pi[\mathbf{b}](B^{\mathbf{b}}_{<j})$.

Moreover, since the elements are independent,

$$b_{\alpha_k,\eta_k} \mathop{\downarrow}_{B^{\mathbf{b}}_{<j}} b_{\alpha_1,\eta_1} \dots b_{\alpha_{k-1},\eta_{k-2}} \text{ and}$$

$$b_{\alpha_k,\widehat{\pi}(\eta_k)} \mathop{\downarrow}_{\pi[\mathbf{b}](B^{\mathbf{b}}_{<j})} b_{\alpha_1,\widehat{\pi}(\eta_1)} \dots b_{\alpha_{k-1},\widehat{\pi}(\eta_{k-2})}$$

and consequently by stationarity (see also Lemma 6.8)

$$b_{\alpha_1,\eta_1} \dots b_{\alpha_k,\eta_k} B^{\mathbf{b}}_{<j} \equiv b_{\alpha_1,\widehat{\pi}(\eta_1)} \dots b_{\alpha_k,\widehat{\pi}(\eta_k)} \pi[\mathbf{b}](B^{\mathbf{b}}_{<j}).$$

As required. $\square$

Theorem 5.20. *Let T be a complete stationary stable theory. Let $\kappa = \kappa_r(T)$ and $M \models T$ be a saturated model of cardinality $\geq \mu = \mu^{<\kappa}$ such that $\mu \geq 2^{|T|}$ and $\kappa < \mu$. Let I be any set such that $|I|^{<\kappa} = |M|$.*

Then there exists a skeletal homogeneous representation of M in $\mathcal{M}_{\mu,\kappa}(I)$. In fact, the representation will be in $\mathcal{M}_{\mu,\kappa,1}(I)$.

Proof. Let I be any set such that $|I|^{<\kappa} = |M| \geq \mu$ and let $\mathbf{a} \in OB(I)$ be a full homogeneous object with $i_{\mathbf{a}} = \kappa$ as supplied by Proposition 5.19. By Proposition 5.17, $\mathrm{dcl}(B^{\mathbf{a}}_{<\kappa}) \models T$ is saturated of cardinality $|I|^{<\kappa} = |M|$. In particular M is isomorphic to $\mathrm{dcl}(B^{\mathbf{a}}_{<\kappa})$. Without loss of generality we assume $M = \mathrm{dcl}(B^{\mathbf{a}}_{<\kappa})$. We define a function

$$\Phi : B^{\mathbf{a}}_{<\kappa} \to \mathcal{M}_{\mu,\kappa,1}(I)$$

by $\Phi(b_{\alpha,\eta}) = F_{\alpha,j}(\eta)$ for the unique $j < \kappa$ such that $\eta \in I^{\underline{j}}$ and $\alpha \in U^{\mathbf{a}}_j$.

The map Φ is injective. If $F_{\alpha,j}(\eta) = F_{\beta,k}(\nu)$ then, since it is a free algebra, $\alpha = \beta, j = k$ and $\eta = \nu$ so $b_{\alpha,\eta} = b_{\beta,\nu}$.

The map Φ is a homogeneous representation. Indeed, to show condition (1), let $t(\bar{x})$ be any term with $|\bar{x}| = \beta < \kappa$ and let $\bar{a} \in I^{\underline{\beta}}$ with $t(\bar{a}) \in \mathrm{Im}(\Phi)$. Thus there exist $j < \kappa$ and $\alpha \in U^{\mathbf{a}}_j$ such that $t(\bar{a}) = F_{\alpha,j}(\eta_{\bar{a}})$ for some $\eta_{\bar{a}} \in I^{\underline{j}}$. Thus for any $\bar{b} \in I^{\underline{\beta}}$ there is some $\eta_{\bar{b}} \in I^{\underline{j}}$ with $t(\bar{b}) = F_{\alpha,j}(\eta_{\bar{b}})$. In particular, $t(\bar{b}) \in \mathrm{Im}(\Phi)$, as needed.

For condition (2), let $b_{\alpha_1,\eta_1} \dots b_{\alpha_k,\eta_k} \in (B^{\mathbf{a}}_{<\kappa})^{\underline{k}}$ and let π be a permutation of I. Since $\mathbf{a}$ is homogeneous $\mathrm{tp}(b_{\alpha_1,\eta_1} \dots b_{\alpha_k,\eta_k}) = \mathrm{tp}(b_{\alpha_1,\widehat{\pi}(\eta_1)} \dots b_{\alpha_k,\widehat{\pi}(\eta_k)})$, as needed. □

Corollary 5.21. *Let $G = (V, E)$ be a graph that is interpretable (possibly with parameters) in a stationary stable structure. If $\chi(G) > \beth_2(\aleph_0)$ then there exists an $n \in \mathbb{N}$ such that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$.*

Proof. Assume G is interpretable in a stationary stable structure N over some finite set of a parameters $A \subseteq N$ and let $T = Th(N)$. Since adding constants to the language preserves stationarity, we may assume that G is interpretable in N over $\emptyset$. Since the interpretation only uses a finite fragment of the language, by applying Proposition 5.10 we may assume that $|T| = \aleph_0$.

Let $\mu = 2^{\aleph_0}$ and $\kappa = \kappa(T)$. Note that $\kappa(T) \leq \aleph_1$ ([She78, Corollary III.3.3]) which implies $\kappa(T) = \kappa_r(T)$ and $\mu^{<\kappa} = \mu$. Let I be any set satisfying $|I| \geq \max\{\mu, |N|\}$ (which implies $|I|^{<\kappa} \geq \max\{\mu, |N|\}$) and let $M \models T$ be a saturated elementary extension of N of cardinality $|I|^{<\kappa}$ (exists by [She78, Lemma III.3.6 and Theorem III.3.12]).

By Theorem 5.20 and Proposition 5.6, there exists $n \in \mathbb{N}$ such that $(G(M), E(M))$, the realizations in M of the interpretation of G, contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ and since $N \prec M$ the result follows. □

A natural question is whether every stable structure is interpretable in a stationary stable structure. We thank Hrushovski for the following argument.

Proposition 5.22. *Let T be a stationary stable theory. Then T does not interpret an infinite p-root closed field, where p is a prime number different from $\mathrm{char}(F)$. In particular, ACF is not interpretable in any stationary stable theory.*

Proof. Let $M \models T$ and assume towards a contradiction that it interprets an infinite field F, that is p-root closed for $p \neq \mathrm{char}(F)$.

For ease of writing, we assume that F is an $\emptyset$-definable field in M^{eq}. So there exists an $\emptyset$-definable set $D \subseteq M^n$, for some $n < \omega$, and a definable (in M^{eq}) surjective map $\pi : D \to F$.

Let $1 \neq \zeta_p \in F$ be a p-th root of unity (such exists since $p \neq \mathrm{char}(F)$) and for every $a_0, \dots, a_{p-1} \in D$, let $\sigma_{a_0,\dots,a_{p-1}} = \Sigma_{i=0}^{p-1} \pi(a_i)\zeta_p^i$. Note that $\zeta_p \sigma_{a_0,\dots,a_{p-1}} = \sigma_{a_{p-1},a_0,\dots,a_{p-2}}$.

For any $a_0, \dots, a_{p-1} \in D$, $(\pi(x) = \pi(y) \wedge \pi(x)^p = \sigma_{a_0,\dots,a_{p-1}}) \vee (\pi(x)^p \neq \sigma_{a_0,\dots,a_{p-1}} \wedge \pi(y)^p \neq \sigma_{a_0,\dots,a_{p-1}})$ defines a finite equivalence relation on D over $\zeta_p, a_0, \dots, a_{p-1}$ (definable in M). By Fact 5.9 each of the equivalence classes are definable over $\zeta_p, a_0, \dots, a_{p-1}$.

By compactness, there is a definable (over ζ_p) function $f : D^p \to F$ satisfying

$$f(a_0, \dots, a_{p-1})^p = \sigma_{a_0,\dots,a_{p-1}}.$$

Let $\mathfrak{p}$ be a non-algebraic global type on F (exists since F is infinite), and since π is surjective, we may find a global type $\mathfrak{q}$ on D with $\pi_* \mathfrak{q} = \mathfrak{p}$. After naming parameters, we may assume that both $\mathfrak{p}$ and $\mathfrak{q}$ are $\emptyset$-definable.

Let $(a_0, \dots, a_{p-1}) \models \mathfrak{q}^{(p)}|\zeta_p$ (where $\mathfrak{q}^{(p)} = \mathfrak{q} \otimes \mathfrak{q}^{(p-1)}$). Since $\zeta_p f(a_0, \dots, a_{p-1})^p = f(a_{p-1}, a_0, \dots, a_{p-2})^p$, then letting $\omega = \frac{f(a_{p-1},a_0,\dots,a_{p-2})}{f(a_0,\dots,a_{p-1})} \in F$ we have that $\omega^p = \zeta_p$. Since $(a_{\tau(0)}, \dots, a_{\tau(p-1)}) \models \mathfrak{q}^{(p)}|\zeta_p$, for any permutation τ on $\{0, \dots, p-1\}$,

$$\omega f(a_0, \dots, a_{p-1}) = f(a_{p-1}, a_0, \dots, a_{p-2})$$

$$\omega f(a_{p-1}, a_0, \dots, a_{p-2}) = f(a_{p-2}, a_{p-1}, a_0, \dots, a_{p-3})$$

$$\vdots$$

$$\omega f(a_1, \dots, a_{p-1}, a_0) = f(a_0, \dots, a_{p-1}).$$

Thus $f(a_0, \dots, a_{p-1}) = \omega^p f(a_0, \dots, a_{p-1}) = \zeta_p f(a_0, \dots, a_{p-1})$. This implies that $\sigma_{a_0,\dots,a_{p-1}} = 0$, contradicting the non-algebraicity of $\mathfrak{p}$. $\square$

Remark 5.23. Since every first order infinite structure (in a finite language) is bi-interpretable with a graph [Hod93, Theorem 5.5.1], it follows that there is a stable graph that is not interpretable in any stationary stable structure.

6. Quantitative bounds

The following section is joint work with Elad Levi.

The aim of this section is to prove that if $G = (V, E)$ is an ω-stable graph with uncountable chromatic number and the U-rank of G, $\mathrm{U}(G)$, is at most 2 then it contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \leq 2$. For the definition of U-rank see [TZ12, Definition 8.6.1]. Throughout, we will use Lascar's equality when the U-rank is finite, see [TZ12, Exercise 8.6.5].

For certain parts of the argument we will need the following assumption.

Assumption $\diamondsuit$**.** G is a saturated ω-stable structure that eliminates imaginaries in a countable language with $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$. Let $p(x) \in S(\emptyset)$ be a non-algebraic type of finite U-rank and let $E \subseteq p(G)^2$ be a type-definable subset such that $G_p = (p(G), E)$ is a graph with $\chi(G_p) \geq \aleph_1$.

Note that assumption $\diamondsuit$ implies that every type over $\emptyset$ is stationary.

Assume $\diamondsuit$ and let $E_{alg} = \{(a,b) \in E : a \in \operatorname{acl}(b) \wedge b \in \operatorname{acl}(a)\}$ be the set of interalgebraic pairs belonging to E. Note that if $\mathrm{U}(a) = \mathrm{U}(b)$ then $a \in \operatorname{acl}(b)$ if and only if $b \in \operatorname{acl}(a)$. Indeed, by Lascar's equality

$$\mathrm{U}(a/b) + \mathrm{U}(b) = \mathrm{U}(ab) = \mathrm{U}(a) + \mathrm{U}(b/a)$$

and for any type q, $\mathrm{U}(q) = 0$ if and only if it is algebraic, see [TZ12, Exercise 8.6.1]. Let $E_{nalg} = E \setminus E_{alg}$, it is definable by a countable type.

Lemma 6.1. *Assume* $\diamondsuit$.

(1) $\chi(p(G), E_{nalg}) \geq \aleph_1$.
(2) *If there exist* $a, b \in G_p$ *with* $a\ E\ b$ *and* $a \mathop{\smile\!\!\!\!\!\!\mid}\, b$ *then any Morley sequence based on* p *forms an infinite complete graph.*
(3) *If* $\mathrm{U}(p) = 1$ *then every Morley sequence based on* p *forms an infinite complete graph.*

Proof. (1) By interalgebraicity, every connected component of $(p(G), E_{alg})$ is countable and consequently $\chi(p(G), E_{alg}) \leq \aleph_0$. By Lemma 2.3(2), $\chi(p(G), E_{nalg}) \geq \aleph_1$.

(2) Assume there exist $a, b \in G_p$ with $a\ E\ b$ and $a \mathop{\smile\!\!\!\!\!\!\mid}\, b$. Since every type over $\emptyset$ is stationary it follows that every Morley sequence based on p forms an infinite complete graph.

(3) Assume $\mathrm{U}(p) = 1$. Since $\chi(p(G), E_{nalg}) \geq \aleph_1$, there must exist some $a, b \in p(G)$ with $a\ E_{nalg}\ b$. If $a \not\mathop{\smile\!\!\!\!\!\!\mid}\, b$ then $\mathrm{U}(a/b) < \mathrm{U}(a) = 1$, which implies that $a \in \operatorname{acl}(b)$ (so $b \in \operatorname{acl}(a)$), contradiction. Thus $a \mathop{\smile\!\!\!\!\!\!\mid}\, b$ and we may use (2). $\square$

Definition 6.2. We say that a stationary type $\operatorname{tp}(a/A)$ is *pseudo-one-based* if $\operatorname{Cb}(a/A) \subseteq \operatorname{acl}^{eq}(a)$.

Remark 6.3. Compare with the last paragraph of page 105 in [Pil96].

We give some examples of pseudo-one-based types.

Lemma 6.4.

(1) *In a one-based theory every stationary type (over any base) is pseudo-one-based.*
(2) *Let* M *be a stable structure. If* $\mathrm{U}(a) = \mathrm{U}(b) = 1$ *and* $a \not\mathop{\smile\!\!\!\!\!\!\mid}\, b$ *then* $\operatorname{tp}(a/\operatorname{acl}^{eq}(b))$ *is pseudo-one-based.*
(3) *Let* M *be a stable structure and* $a, b \in M$ *non interalgebraic, with* $\mathrm{U}(a) = \mathrm{U}(b) = 2$. *Let* X *and* Y *be infinite mutually indiscernible sets with* $a \in X$ *and* $b \in Y$. *If* $a \not\mathop{\smile\!\!\!\!\!\!\mid}\, b$ *then* $\operatorname{tp}(a/\operatorname{acl}^{eq}(b))$ *is pseudo-one-based.*

Proof. (1) A stable theory is one-based if for all a, B, $\operatorname{Cb}(a/\operatorname{acl}^{eq}(B)) \subseteq \operatorname{acl}^{eq}(a)$. Note that since $\operatorname{tp}(a/A)$ is stationary, $\operatorname{Cb}(a/\operatorname{acl}^{eq}(A)) = \operatorname{Cb}(a/A)$. The result follows.

(2) Since $a \not\mathop{\smile\!\!\!\!\!\!\mid}\, b$, by U-rank considerations as before, a and b are interalgebraic. So $\operatorname{Cb}(a/\operatorname{acl}^{eq}(b)) \subseteq \operatorname{acl}^{eq}(b) \subseteq \operatorname{acl}^{eq}(a)$.

(3) We note that for any $b \neq b' \in Y$, $a \mathop{\smile\!\!\!\!\!\!\mid}_{b'} b$. Indeed, otherwise $\mathrm{U}(a/bb') < \mathrm{U}(a/b') < \mathrm{U}(a)$ and since $\mathrm{U}(a) = 2$, $a \in \operatorname{acl}(bb')$, contradicting the mutual indiscernibility of X, Y. Similarly, $a \mathop{\smile\!\!\!\!\!\!\mid}_b b'$. Consequently, setting $e := \operatorname{Cb}(a/\operatorname{acl}^{eq}(bb'))$, $e \subseteq \operatorname{acl}^{eq}(b) \cap \operatorname{acl}^{eq}(b')$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_e bb'$. Similarly we get that $b \mathop{\smile\!\!\!\!\!\!\mid}_a b'$ and by the properties of forking $e \mathop{\smile\!\!\!\!\!\!\mid}_a e$ and hence $e \in \operatorname{acl}^{eq}(a)$. Finally, since $a \mathop{\smile\!\!\!\!\!\!\mid}_e b$ (and $e \in \operatorname{acl}^{eq}(b)$), $\operatorname{Cb}(a/\operatorname{acl}^{eq}(b)) \subseteq \operatorname{acl}^{eq}(e) \subseteq \operatorname{acl}^{eq}(a)$, as needed. $\square$

Abundance of pseudo-one-based types will be a key tool in our proofs. The above shows that this can be achieved in one-based theories and U-rank 1 types. For U-rank 2 we observe the following:

Lemma 6.5. *Assume $\diamondsuit$ and that $\mathrm{U}(p) = 2$. Then either we can embed an infinite complete graph into G_p or there exists a type-definable symmetric irreflexive relation $E_0 \subseteq E_{nalg}$ such that*

($\dagger$) *for every $(a, b) \in E_0$, $\mathrm{tp}(a/\operatorname{acl}(b))$ is pseudo-one-based. Moreover, if $F \subseteq E$ is a symmetric irreflexive type-definable relation with $\chi(G_p, F) \geq \aleph_1$ then $F \cap E_0 \neq \emptyset$.*

Proof. Assume that we cannot embed an infinite complete graph into G_p, in particular by Lemma 6.1, for every $a, b \models p$ with $a\ E\ b$, $a \not\perp b$.

Let E_0 be the set of pairs $(a, b) \in E$ such that there exists a complete bipartite subgraph $K_{X,Y}$ of G_p such that X and Y are infinite mutually indiscernible sets with $a \in X$ and $b \in Y$. Easily, E_0 is type-definable by a countable type. Since, by [EH66, Corollary 5.6], G_p contains $K_{n,n}$ (the complete bipartite graph on n vertices) for every $n < \omega$, E_0 non-empty. Clearly, $E_0 \subseteq E_{nalg}$. By Lemma 6.4(3), for every $(a, b) \in E_0$, $\mathrm{tp}(a/\operatorname{acl}(b))$ is pseudo-one-based.

For the moreover part, if $\chi(G_p, F) \geq \aleph_1$ then by [EH66, Corollary 5.6] we may embed $K_{n,n}$ into it for any $n < \omega$. Thus by saturation necessarily $F \cap E_0 \neq \emptyset$. $\square$

The following is the key proposition of the proof and where pseudo-one-based types show their usefulness.

Proposition 6.6. *Assume $\diamondsuit$ and that $E_0 \subseteq E_{nalg}$ is a type-definable symmetric irreflexive relation satisfying ($\dagger$) from Lemma 6.5.*

(1) *For any $(a, b) \in E_0$ there is a finite tuple e such that $a \perp_e b$ and $\mathrm{tp}(a/e)$ is stationary.*

(2) *Let Ψ be the collection of all pairs of formulas $(\varphi(u, x), \psi(u, x))$ satisfying*
   (a) *$\varphi(u, a)$ and $\psi(u, a)$ are algebraic formulas each isolating a complete type over some (any) $a \models p$;*
   (b) *there exist $a, b \models p$ and e such that $(a, b) \in E_0$, $a \perp_e b$, $\varphi(e, a)$, $\psi(e, b)$ and $\mathrm{tp}(a/e)$ is stationary.*

   For any $(\varphi, \psi) \in \Psi$ let
$$E_{\varphi,\psi} = \{(a, b) \in E_{nalg} : \exists e\, (\varphi(e, a) \wedge \psi(e, b))\}.$$
   Then either ($*$) we can embed an infinite complete graph into $(p(G), E_{nalg})$ or there exists $(\varphi(u, x), \psi(u, x)) \in \Psi$ such that $(**)_{\varphi,\psi}$: $p \vdash \forall u(\varphi(u, x) \to \neg\psi(u, x))$ and $\chi(G_{p,\varphi,\psi}) \geq \aleph_1$, where $G_{p,\varphi,\psi} := (p(G), E_{\{\varphi,\psi\}})$ and $E_{\{\varphi,\psi\}} = E_{\varphi,\psi} \vee E_{\psi,\varphi}$.

(3) *Assume $\neg(*)$ and let $(\varphi, \psi) \in \Psi$. There exists $q_\varphi \in S(\emptyset)$ such that for any $a \models p$ and $e \models \varphi(u, a)$, $e \models q_\varphi$. Similarly, there exists $q_\psi \in S(\emptyset)$ such that for any $a \models p$ and $e \models \psi(u, a)$, $e \models q_\psi$. Furthermore, $q := q_\varphi = q_\psi$ and $0 < \mathrm{U}(q) < \mathrm{U}(p)$.*

(4) *Assume $\neg(*)$ and $(**)_{\varphi,\psi}$ and let $q = q_\varphi = q_\psi$. The type-definable relation $e_1\ R\ e_2$ given by*
$$(\exists a \models p)\, ((\varphi(e_1, a) \wedge \psi(e_2, a)) \vee (\varphi(e_2, a) \wedge \psi(e_1, a)))\,,$$
   defines a graph H_q on realizations of q with $\chi(H_q) \geq \aleph_1$.

Proof. (1, 2) We start by showing that

$$E_0 \subseteq \bigcup_{(\varphi,\psi)\in\Psi} E_{\{\varphi,\psi\}}.$$

Let $(a,b) \in E_0$. Since $\mathrm{tp}(a/\operatorname{acl}(b))$ is pseudo-one-based, $\mathrm{Cb}(a/\operatorname{acl}(b)) \subseteq \operatorname{acl}(a) \cap \operatorname{acl}(b)$. By [TZ12, Exercise 8.4.7], there is a finite tuple e such that $\operatorname{dcl}(e) = \mathrm{Cb}(a/\operatorname{acl}(b))$. This proves (1). We choose $\varphi(u,a)$ to be a formula isolating $\mathrm{tp}(e/a)$ and $\psi(u,b)$ to be a formula isolating $\mathrm{tp}(e/b)$. Hence $(a,b) \in E_{\{\varphi,\psi\}}$ and $(\varphi,\psi) \in \Psi$.

Since E_0 is type-definable, by saturation there exists a finite subset $\Psi_0 \subseteq \Psi$ such that

$$E_0 \subseteq \bigcup_{(\varphi,\psi)\in\Psi_0} E_{\{\varphi,\psi\}}.$$

Assume that we cannot embed an infinite complete graph into $(p(G), E_{nalg})$.

Claim. For any $(\varphi,\psi) \in \Psi$, $p \vdash \forall u(\varphi(u,x) \to \neg\psi(u,x))$.

Proof. Choose any $(\varphi,\psi) \in \Psi$. By part (b) of the definition of Ψ there are $(a,b) \in E_0$ and e be such that $\varphi(e,a)$, $\psi(e,b)$, $a \mathop{\smile\!\!\!\!\!\!\;|}_e b$ and $\mathrm{tp}(a/e)$ stationary. Assume, toward a contradiction that there is some e' with $\varphi(e',b)$ and $\psi(e',b)$. Since by part (a) of the definition of Ψ both $\varphi(u,b)$ and $\psi(u,b)$ isolate a complete type over b and are mutually consistent, they must be equivalent (i.e. define the same definable set). So $\varphi(e,a)$ and $\varphi(e,b)$ hold.

Let σ be an automorphism satisfying $\sigma(a) = b$. Applying to the formulas above we get that $\varphi(e,b)$ and $\varphi(\sigma(e),b)$ hold. Since $\varphi(u,b)$ isolates a complete type over b there exists an automorphism τ fixing b and mapping $\sigma(e)$ to e. Combining, $\tau\circ\sigma$ fixes e and maps a to b, i.e. $a \equiv_e b$.

By assumption $\mathrm{tp}(a/e)$ is stationary and $b \models \mathrm{tp}(a/e)|ea$ so we may construct a Morley sequence over e starting with a, b. Since $a\ E_0\ b$ we get an infinite complete graph, contradicting our assumption. □ (claim)

Note that each $E_{\{\varphi,\psi\}}$ defines a graph relation. Let

$$\theta(x,y) = \bigvee_{(\varphi,\psi)\in\Psi_0} \exists e\,(\varphi(e,x) \wedge \psi(e,y)) \vee \exists e\,(\varphi(e,y) \wedge \psi(e,x))\,.$$

Set $E_1 = \{(a,b) \in E_{nalg} : \theta(a,b)\}$ and $E_2 = \{(a,b) \in E_{nalg} : \neg\theta(a,b)\}$. Obviously, $E_{nalg} = E_1 \cup E_2$ and both E_1 and E_2 are symmetric. If $\chi(p(G), E_2) \geq \aleph_1$ then by (†) from Lemma 6.5 there exists $(a,b) \in E_0 \cap E_2$, contradicting the choice of Ψ_0. Thus, by Lemma 2.3(2) and Lemma 6.1(1), $\chi(p(G), E_1) \geq \aleph_1$. Again by Lemma 2.3(2), there is some $(\varphi,\psi) \in \Psi_0$ such that $\chi(G_{p,\varphi,\psi}) \geq \aleph_1$. This proves (2).

(3) Let $q_\varphi = \mathrm{tp}(e_1)$ for some $e_1 \models \varphi(u,a)$ and some $a \models p$ and let $q_\psi = \mathrm{tp}(e_2)$ for some $e_2 \models \psi(u,b)$ and some $b \models p$. Since p is a complete type and φ and ψ each isolate a complete type it follows that q_φ and q_ψ do not depend on a, b, e_1 or e_2.

As $(\varphi,\psi) \in \Psi$, there exist $(a,b) \in E_{nalg}$ and e such that $\varphi(e,a) \wedge \psi(e,b)$ and $a \mathop{\smile\!\!\!\!\!\!\;|}_e b$. Consequently, $e \models q_\varphi$ and $e \models q_\psi$ and hence $q_\varphi = q_\psi$. Since $e \in \operatorname{acl}(a)$,

$$\mathrm{U}(a/e) + \mathrm{U}(e) = \mathrm{U}(a).$$

If $\mathrm{U}(e) = 0$ then $a \mathop{\smile\!\!\!\!\!\!\;|} e$ so by transitivity of forking $a \mathop{\smile\!\!\!\!\!\!\;|} b$, but then we may embed an infinite complete graph as in Lemma 6.1, which contradicts $\neg(*)$. If

$\mathrm{U}(a/e) = 0$ then $a \in \mathrm{acl}(e) \subseteq \mathrm{acl}(b)$ so a and b are interalgebraic, contradiction (see above Lemma 6.1).

(4) Note that R defines a graph, i.e. it is irreflexive by (2) and $(**)_{\varphi,\psi}$. Let n be $|\varphi(G,a)|$ and m be $|\psi(G,a)|$ for some (any) $a \models p$. For any $a \models p$ choose enumerations $\varphi(G,a) = \{e_i(a) : i < n\}$ and $\psi(G,a) = \{e'_i(a) : i < m\}$.

For $i < n$ and $j < m$ let $H_{i,j} = \{(e_i(a), e'_j(a)) : a \models p\}$.

We define an edge relation on $H_{i,j} = \{(e_i(a), e'_j(a)) : a \models p\}$ (for $i < n, j < m$) as follows: $(e_i(a), e'_j(a))$ is connected to $(e_i(b), e'_j(b))$ if and only if $e_i(b) = e'_j(a)$ or $e_i(a) = e'_j(b)$. Note that $e_i(a) \neq e'_j(a)$ for all $a \models p$ and $i < n$, $j < m$ by (2) $(**)_{\varphi,\psi}$, hence this relation is irreflexive.

Claim. There exist $i_0 < n$ and $j_0 < m$ such that $\chi(H_{i_0,j_0}) \geq \aleph_1$.

Proof. Assume that for all $i < n$, $j < m$, $H_{i,j}$ is countably colorable, say by the coloring function $c_{i,j} : H_{i,j} \to \aleph_0$. We claim that this entails that $G_{p,\varphi,\psi}$ is countably colorable, which would give a contradiction to choice of (φ, ψ).

We define a coloring $c : G_{p,\varphi,\psi} \to (\aleph_0)^{n\times m}$ by $c(a)(i,j) = c_{i,j}(e_i(a), e'_j(a))$. The contradiction will follow if we show that this is a legal coloring. Let $(a,b) \in E_{\varphi,\psi}$ ($(a,b) \in E_{\psi,\varphi}$ is similar). Thus there exists some $e \models \varphi(u,a) \wedge \psi(u,b)$. Consequently, $e = e_i(a) = e'_j(b)$ for some $i < n, j < m$, so

$$c_{i,j}(e_i(a), e'_j(a)) \neq c_{i,j}(e_i(b), e'_j(b)),$$

and $c(a) \neq c(b)$. $\square$ (claim)

Now, assume that $\chi(H_q) \leq \aleph_0$ and let $c : q(G) \to \aleph_0$ be a coloring. We define $f : H_{i_0,j_0} \to \aleph_0 \times \aleph_0$ by $f(e_{i_0}(a), e'_{j_0}(a)) = \big(c(e_{i_0}(a)), c(e'_{j_0}(a))\big)$. This gives a legal coloring of H_{i_0,j_0} using countably many colors, and we reach a contradiction: assume without loss of generality that $(e_{i_0}(a), e'_{j_0}(a)), (e_{i_0}(b), e'_{j_0}(b)) \in H_{i_0,j_0}$ with $e'_{j_0}(a) = e_{i_0}(b)$. Since $e_{i_0}(a)\ R\ e'_{j_0}(a)$, $c(e_{i_0}(a)) \neq c(e'_{j_0}(a)) = c(e_{i_0}(b))$ and thus $f(e_{i_0}(a), e'_{j_0}(a)) \neq f(e_{i_0}(b), e'_{j_0}(b))$, as needed. $\square$

Remark 6.7. We remark that if $\mathrm{tp}(a/e)$ is stationary and $e' \models \mathrm{tp}(e/a)$ then $\mathrm{tp}(a/e')$ is also stationary.

The procedure outlined in the items of Proposition 6.6 supplies, under some assumptions, a graph, with uncountable chromatic number, concentrated on a type of lower U-rank than the one we started with. This hints that some induction procedure may be possible (at least for one-based theories). We will not pursue this further now. For now we concentrate on graphs of at most U-rank 2.

The following is an easy exercise in stability theory.

Lemma 6.8. *Let T be a stable theory. Assume that $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, $A' \mathop{\smile\!\!\!\!\!\!\mid}_C B'$, $B \equiv_C B'$, $A \equiv_C A'$ and $\mathrm{tp}(A/C)$ stationary. Then $AB \equiv_C A'B'$.*

Theorem 6.9. *Let $G = (V,E)$ be an ω-stable graph with $\chi(G) \geq \aleph_1$. If $\mathrm{U}(G) \leq 2$ then G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \leq 2$.*

Proof. We may assume that G is saturated (in particular $\aleph_1$-saturated) and we may also work in G^{eq}. Fix some countable $G_0 \prec G$ and add constants for it (so every type over $\emptyset$ is stationary). By ω-stability and Lemma 2.3(1) there is some type $p \in S_1(\emptyset)$ such that $\chi(G_p) \geq \aleph_1$ with $G_p = (p(G), E \restriction p(G))$. We are now in the situation of Assumption $\diamondsuit$ (with E there being $E \restriction p(G)$).

If $\mathrm{U}(p) = 0$ then p is algebraic (even realized), contradicting $\chi(G_p) \geq \aleph_1$. If $\mathrm{U}(p) = 1$ then we may embed an infinite complete graph by Lemma 6.1(3).

We may thus assume that $\mathrm{U}(p) = 2$ and that G does not contain an infinite complete graph.

Let $E_0 \subseteq E$ be the type-definable set from Lemma 6.5 and φ, $\psi, G_{p,\varphi,\psi}$, q, R and H_q be as supplied by Proposition 6.6 with respect to E_0 and p (so necessarily $\mathrm{U}(q) = 1$). Noting that Assumption $\diamondsuit$ is true for (H_q, R), we may apply Lemma 6.1(3) and thus there exists a Morley sequence $\langle e_i : i < \omega \rangle$ such that $e_i \mathrel{R} e_j$ for all $i \neq j$.

Since $e_0 \mathrel{R} e_1$ there is some $a_{0,1} \models p$ such that, without loss of generality, $\varphi(e_0, a_{0,1}) \wedge \psi(e_1, a_{0,1})$. For any $i < j < \omega$ let $a_{i,j}$ be such that $a_{i,j} e_i e_j \equiv a_{0,1} e_0 e_1$.

Note that for every $i < j < \omega$, $a_{i,j} \in \mathrm{acl}(e_i, e_j)$. Indeed, by Lascar's equality $\mathrm{U}(a_{i,j}/e_i e_j) + \mathrm{U}(e_i/e_j) = \mathrm{U}(a_{i,j} e_i/e_j)$ and since $e_i \in \mathrm{acl}(a_{i,j})$ the right hand side is also equal to $\mathrm{U}(a_{i,j}/e_j)$. Now we note that $\mathrm{U}(a_{i,j}/e_j) + \mathrm{U}(e_j) = \mathrm{U}(a_{i,j} e_j)$, but as before $e_j \in \mathrm{acl}(a_{i,j})$ so the right hand side is equal to 2 and since $\mathrm{U}(e_j) = 1$ we conclude that $\mathrm{U}(a_{i,j}/e_j) = 1$. As $e_i \mathop{\smile\!\!\!\!\!\mid} e_j$ we have that $\mathrm{U}(e_i/e_j) = 1$ as well so we combine everything and get that $\mathrm{U}(a_{i,j}/e_i e_j) = 0$.

Define a map $f : \mathrm{Sh}_2(\omega) \to G_{p,\varphi,\psi}$ by $(i,j) \mapsto a_{i,j}$. We claim that this is an injective graph homomorphism.

For $(i,j) \neq (i',j') \in \mathrm{Sh}_2(\omega)$, we claim that $a_{i,j}$ and $a_{i',j'}$ are not interalgebraic and in particular f is injective. Indeed, assume $\mathrm{acl}(a_{i,j}) = \mathrm{acl}(a_{i',j'})$. Since $(i,j) \neq (i',j')$, $|\{i,j,i',j'\}| \geq 3$, and we assume that $i \neq j, i', j'$ (the other cases are similar). Then we get

$$e_i \in \mathrm{acl}(a_{i,j}) = \mathrm{acl}(a_{i',j'}) \subseteq \mathrm{acl}(e_{i'}, e_{j'}),$$

contradicting indiscernibility.

f is a graph homomorphism: Let $(i,j), (j,k) \in \mathrm{Sh}_2(\omega)$, so that $i < j < k < \omega$. As $a_{i,j}$ and $a_{j,k}$ are not interalgebraic, necessarily $a_{i,j} \mathop{\smile\!\!\!\!\!\mid}_{e_j} a_{j,k}$ for otherwise

$$\mathrm{U}(a_{i,j}/e_j a_{j,k}) < \mathrm{U}(a_{i,j}/e_j) = 1$$

and then $a_{i,j} \in \mathrm{acl}(e_j a_{j,k}) \subseteq \mathrm{acl}(a_{j,k})$.

By the choice of (φ, ψ) in Proposition 6.6, we may find $a, b \models p$ and $e \models q$ with $a \mathrel{E} b$, $a \mathop{\smile\!\!\!\!\!\mid}_e b$, $\mathrm{tp}(a/e)$ stationary, $\varphi(e,a)$ and $\psi(e,b)$. By applying an automorphsim mapping e_j to e we may assume $e_j = e$. Let σ be an automorphism mapping a to $a_{j,k}$, thus $ae \equiv a_{j,k}\sigma(e)$ and $\sigma(e) \models \varphi(u, a_{j,k})$. As $\varphi(u, a_{j,k})$ isolates a complete type, there is an automorphism mapping $\sigma(e)$ to e but fixing $a_{j,k}$, and after applying this automorphism we conclude that $a \equiv_e a_{j,k}$ and similarly $b \equiv_e a_{i,j}$. Since $\mathrm{tp}(a/e)$ is stationary and $a_{j,k} \mathop{\smile\!\!\!\!\!\mid}_e a_{i,j}$, by Lemma 6.8, $ab \equiv a_{j,k} a_{i,j}$ so $a_{i,j} \mathrel{E} a_{j,k}$ as well. $\square$

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