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Forcing axioms for λ-complete μ^+-c.c.

Saharon Shelah[1,2,*]

[1] Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
[2] Department of Mathematics, Hill Center – Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ, 08854-8019, United States of America

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We consider forcing axioms for suitable families of μ-complete μ^+-c.c. forcing notions. We show that some form of the condition "p_1, p_2 have a $\leq_{\mathbb{Q}}$ -lub in $\mathbb{Q}$" is necessary. We also show some versions are really stronger than others.

1 Introduction

1.1 Is "well met" necessary in some forcing axiom?

We investigate the relationships between some forcing axioms related to pressing down functions for μ^+-c.c., mainly from [14]. This in particular is to answer Kolesnikov's question of having $\mathbb{P}$ satisfying one condition but with no $\mathbb{P}'$ equivalent to $\mathbb{P}$ satisfying another. A side issue is clarifying a point in [1] (a rephrasing is $(2)^{\varepsilon}_{c,D}$ from Definition 1.3). We intend to continue this considering related axioms in [6].

We justify the "well met, having lub" in some forcing axioms, e.g., condition (c) in $(*^1_{\mu,\mathbb{Q}})$.

In [9] such a forcing axiom was proved consistent, for a forcing notion satisfying (for $\mu^{<\mu} = \mu$; we may write "$\mathbb{Q}$ satisfies $*^1_\mu$" instead of $(*^1_{\mu,\mathbb{Q}})$, similarly below):

$(*^1_{\mu,\mathbb{Q}})$ $\mathbb{Q}$ is a forcing notion such that:
- (a) $(<\mu)$-complete, i.e., any increasing sequence of length $<\mu$ has an upper bound;
- (b) μ^+-regressive-c.c.: if $p_\alpha \in \mathbb{Q}$ for $\alpha < \mu^+$ then for some club E of μ^+ and pressing down function f on E we have $[\delta_1 \in E \wedge \delta_2 \in E \wedge (f(\delta_1) = f(\delta_2)) \wedge (\mathrm{cf}(\delta_1) = \mu = \mathrm{cf}(\delta_2)) \Longrightarrow p_{\delta_1}, p_{\delta_2}$ are compatible];
- (c) if $p_1, p_2 \in \mathbb{Q}$ are compatible then p_1, p_2 have a lub.

An easily stated version which is still enough is:

$(*^2_{\mu,\mathbb{Q}})$ $\mathbb{Q}$ is a forcing notion satisfying clause (a) and
- $(b)'$ if $p_\alpha \in \mathbb{Q}$ for $\alpha < \mu^+$ then for some $(E, \bar{q}, f)$ we have
  - i. E a club of μ^+;
  - ii. $\bar{q} = \langle q_\alpha : \alpha < \mu^+\rangle$;
  - iii. $p_\alpha \leq_{\mathbb{Q}} q_\alpha$;
  - iv. f is a pressing down function on E;
  - v. if $\delta_1 \in E \wedge \delta_2 \in E \wedge \mathrm{cf}(\delta_1) = \mu = \mathrm{cf}(\delta_2) \wedge f(\delta_1) = f(\delta_2)$ then $q_{\delta_1}, q_{\delta_2}$ has a lub.

An obvious fact used is

* E-mail: shelah@math.huji.ac.il

⊞ Assume $\mathbb{Q}$ is a forcing notion, $\varepsilon < \mu$ a limit ordinal, $\bar{p}_\ell = \langle p_{\ell,\alpha} : \alpha < \varepsilon \rangle$ is $\leq_{\mathbb{Q}}$-increasing for $\ell = 1, 2$ and for every $\alpha < \varepsilon$ the condition $p_\alpha \in \mathbb{Q}$ is a $\leq_{\mathbb{Q}}$-lub of $p_{1,\alpha}, p_{2,\alpha}$ (i.e., $\bigwedge_{\ell=1}^{2} p_{\ell,\alpha} \leq_{\mathbb{Q}} p_\alpha$ and $(\forall q)(p_{1,\alpha} \leq_{\mathbb{Q}} q \wedge p_{2,\alpha} \leq_{\mathbb{Q}} q \Longrightarrow p_\alpha \leq_{\mathbb{Q}} q)$). Then $\langle p_\alpha : \alpha < \varepsilon \rangle$ is $\leq_{\mathbb{Q}}$-increasing, hence if $\{p_\alpha : \alpha < \varepsilon\}$ has an upper bound then so does $\{p_{1,\alpha}, p_{2,\alpha} : \alpha < \varepsilon\}$.

Now [2] mainly deals with consistency results for singular μ, but on the way has (with a complete proof of the iteration theorem) suggested a condition weaker than the one in [9] and even the one in [10] and is stronger than the one in [14, 1.7(1)], using a trivial strategy and $\varepsilon = \omega$. Using Definition 1.2, the condition from [14] is $(2)^{\varepsilon}_{c,D}$, where ε is a limit ordinal $< \mu$, and the condition from [2] is

$(*^3_{\mu,\mathbb{Q}})$ $\mathbb{Q}$ a forcing notion such that

- (a) as above,
- (b) as above,
- (c) if, for every $n < \omega$ we have $p_n \leq p_{n+1}$, $q_n \leq q_{n+1}$ and p_n, q_n are compatible then the set $\{p_n, q_n : n < \omega\}$ has a common upper bound (here this is clause $(3)_{b,\omega}$ of Definition 1.2).

Our main results are Conclusions 2.9 & 2.10, Theorem 3.1 & Conclusion 4.10.

The immediate reason for this paper is that the statement in Baldwin, Kolesnikov and Shelah [1, 3.6] is misquoting [10, 4.12]. We shall show below that the statement is inconsistent because as stated it totally waives the condition "every two compatible members of $\mathbb{P}$ have a lub". Also, it is stated that in [10, 4.12] this was claimed, but quoting only [9]. In Shelah and Spinas [16] we consider another strengthening of the axioms.

More fully, [10, 4.12] omits the condition above, but demands the existence of lub's of some pairs of conditions so that it holds in the cases it is actually used. So, in that case the proof of [9] works, and see more in [14, Definition 1.1] which gives an even weaker condition called $(*^{\varepsilon}_{\mu})$.

Concerning $(*^1_{\mu,\mathbb{Q}})$, the preservation of a related condition was proved independently by Baumgartner, who instead of (b) used a somewhat stronger condition $(b)^+$, which says that $\mathbb{Q}$ is the union of μ sets of pairwise compatible elements with lub; this is represented in Kunen and Tall [4]; see the history in the end of [9] and see more in [14]. We thank Mirna Džamonja for drawing our attention to the problem and Ashutosh Kumar and Shimoni Garti for various corrections and the referee for helpful suggestions.

1.2 Are some versions of axioms equivalent?

To phrase our problem see the Definition below.

Kolesnikov asked:

Question 1.1 Is there a forcing notion $\mathbb{P}$ satisfying $(1)_a$, $(2)_b$, $(3)_{b,\omega}$ but not equivalent to a forcing notion $\mathbb{P}'$ satisfying $(1)_a$, $(2)_b$, $(3)_a$?

Definition 1.2 Consider the following conditions on a forcing notion $\mathbb{P}$ for a fixed $\mu = \mu^{<\mu}$:
Completeness:

$(1)_a$ increasing chains of length $< \mu$ have a lub.
$(1)_{a,<\vartheta} = (1)_{a,\vartheta}$ increasing chains of length $< \vartheta$ have a lub.
$(1)_{a,\leq\vartheta}$ increasing chains of length $\leq \vartheta$ have a lub.
$(1)_{a,=\vartheta}$ increasing chains of length ϑ have a lub.
$(1)_b$ increasing chains of length $< \mu$ have a ub.
$(1)_{b,<\vartheta} = (1)_{b,\vartheta}$ increasing chains of length $< \vartheta$ have an ub.
$(1)_{b,\leq\vartheta}$ increasing chains of length $\leq \vartheta$ have an ub.
$(1)_{b,=\vartheta}$ increasing chains of length ϑ have an ub.
$(1)_c$ $\mathbb{P}$ is strategically α-complete for every $\alpha < \mu$; cf. Definition 1.11.
$(1)_{c,\alpha}$ $\mathbb{P}$ is strategically α-complete; where here $\alpha \leq \mu$.

$(1)^+_c$ There is a "stronger" order $<_{\rm st}$ on $\mathbb{P}$ which means:

i. $p_1 <_{\rm st} p_2 \Longrightarrow p_1 <_{\mathbb{P}} p_2$;
ii. $p_1 \le_{\mathbb{P}} p_2 <_{\rm st} p_3 \le_{\mathbb{P}} p_4 \Longrightarrow p_1 <_{\rm st} p_4$;
iii. any $<_{\rm st}$-increasing chain of length $< \mu$ has a $\le_{\mathbb{P}}$-ub (hence a $<_{\rm st}$-ub);
iv. for every p there is q satisfying $p <_{\rm st} q$.

$(1)_{d,<\vartheta} = (1)_{d,\vartheta}$ any increasing continuous chain of length $< \vartheta$ has a lub.
$(1)_{d,=\vartheta}$ any increasing continuous chain of length ϑ has a lub.

Strong μ^+*-c.c.*: For a stationary $S \subseteq S^{\mu^+}_\mu$, the default value being $S^{\mu^+}_\mu$ (cf. Notation 1.10); we may write $(2)_x[S]$ when S is neither the default value nor clear from the context.

$(2)_a$ Given a sequence $\langle p_i : i < \mu^+\rangle$ of members of $\mathbb{P}$ there are a club C of μ^+ and a regressive function $\mathbf{h}$ on $C \cap S$ such that $\alpha, \beta \in C \cap S \wedge h(\alpha) = h(\beta) \Longrightarrow p_\alpha, p_\beta$ have a lub.
$(2)_b$ Like $(2)_a$ but demanding just that p_α, p_β have an ub.
$(2)^+_{a,\vartheta}$ If $p_\alpha \in \mathbb{P}$ for $\alpha < \mu^+$ then we can find a club E of μ^+ and a regressive $\mathbf{h} : S \cap E \to \mu^+$ such that: if $i(*) < 1 + \vartheta$, $\delta_i \in S \cap E$ for $i < i(*)$ and $\mathbf{h}{\restriction}\{\delta_i : i < i(*)\}$ is constant then $\{p_{\delta_i} : i < i(*)\}$ has a lub.
$(2)^+_{b,\vartheta}$ Like $(2)^+_{a,\vartheta}$ but in the end the set has a ub.
$(2)^*_{a,\vartheta}$ If $p_\alpha \in \mathbb{P}$ for $\alpha < \mu^+$ then we can find $\bar{q}, E, \mathbf{h}$ such that

i. $\bar{q} = \langle q_\alpha : \alpha < \mu^+\rangle$;
ii. $p_\alpha \le_{\mathbb{P}} q_\alpha$;
iii. E a club of μ^+;
iv. h is a regressive function on $S \cap E$;
v. if $\mathscr{U} \subseteq S \cap E$ has cardinality $< 1 + \vartheta$ and $\mathbf{h}{\restriction}\mathscr{U}$ is constant, then $\{q_\delta : \delta \in \mathscr{U}\}$ has a lub.

$(2)^*_{b,\vartheta}$ Like $(2)^*_{a,\vartheta}$ but in the end the set has a ub (note that this is equivalent to $(2)^+_{b,\vartheta}$.
For $\varepsilon < \mu$ a limit ordinal, e.g., ω:
$(3)_a$ Any two compatible $p_1, p_2 \in \mathbb{P}$ have a lub.
$(3)_{b,\varepsilon}$ If $\langle p_{\ell,\zeta} : \zeta < \varepsilon\rangle$ is increasing for $\ell = 1, 2$ and $p_{1,\zeta}, p_{2,\zeta}$ are compatible for every $\zeta < \varepsilon$ then $\{p_{\ell,\zeta} : \ell \in \{1, 2\}, \zeta < \varepsilon\}$ has an upper bound; recall $\boxplus$ of § 1.1.
$(3)_{b,\vartheta,\varepsilon}$ If (a) then (b) where:

(a) i. $p_{\zeta,i} \in \mathbb{P}$ for $\zeta < \varepsilon$ and $i < i_* < \vartheta$;
ii. if $i < i_*$ then the sequence $\langle p_{\zeta,i} : \zeta < \varepsilon\rangle$ is $<_{\rm st}$-increasing ; (usually $<_{\rm st}$ is from $(1)^+_c$);
iii. for each $\zeta < \varepsilon$ the set $\{p_{\zeta,i} : i < i_*\}$ has a common upper bound;

(b) the set $\{p_{\zeta,i} : \zeta < \varepsilon, i < i_*\}$ has a common upper bound.

$(3)_{a,\vartheta,\varepsilon}$ Like $(3)_{b,\vartheta,\varepsilon}$ but in iii. we have lub.

Definition 1.3 Assume first that D is a normal filter on μ^+ to which $S^{\mu^+}_\mu$ belongs (we may omit D when it is (the club filter on μ^+) + $S^{\mu^+}_\mu$—cf. Definition 1.12; also we may omit D if clear from the context). We may write S instead of D when D is (the club filter on μ^+) + S. Second, $2 \le \vartheta \le \mu$, and we may omit ϑ when $\vartheta = 2$; we may write $= \vartheta$ or $\le \vartheta$ instead of ϑ^+ or (essentially equivalent) $\vartheta + 1$. Third, assume $\mathbb{P}$ is a forcing notion and $\varepsilon < \mu$ is an ordinal; a limit ordinal if not said otherwise. Writing $< \xi$ instead of ε means "for every limit ordinal $< \xi$". Note that $(2)^\varepsilon_{c,D}$ is equal to $*^\varepsilon_{\mu,D}$ of [14].

Then we define the following conditions on $\mathbb{P}$:

$(2)^\varepsilon_{c,\vartheta,D} = (2)_{c,\vartheta,D,\varepsilon}$ In the following game the COM player has a winning strategy:

(a) a play lasts ε-moves;
(b) in the ζ-th move a triple $(\bar{p}_\zeta, \mathbf{h}_\zeta, S_\zeta)$ is chosen such that:
(α) $\bar{p}_\zeta = \langle p_{\zeta,\alpha} : \alpha \in S_\zeta\rangle$;
(β) $p_{\zeta,\alpha} \in \mathbb{P}$;
(γ) $S_\zeta \in D$;

(δ) $S_\zeta \subseteq \cap\{S_\xi : \xi < \zeta\}$;
(ε) if $\alpha \in S_\zeta$ then $\langle p_{\xi,\alpha} : \xi \leq \zeta\rangle$ is a $\leq_{\mathbb{P}}$-increasing sequence
(ζ) $\mathbf{h}_\zeta$ is a pressing down function on S_ζ.

(c) COM chooses[1] $(\bar{p}_\zeta, \mathbf{h}_\zeta)$ when $1+\zeta$ is even, INC chooses it when $1+\zeta$ is odd.

(d) COM wins a play when it always could have made a legal move, and in the end there is $S_\varepsilon \in D$ included in $\bigcap_{\zeta<\varepsilon} S_\zeta$ such that:
if $i_* < \vartheta$ and $\alpha_i \in S_\varepsilon$ for $i < i_*$ and for each $i < i_*$ we have $\bigwedge_{\zeta<\varepsilon} \mathbf{h}_\zeta(\alpha_i) = \mathbf{h}_\zeta(\alpha_0)$ then the set $\{p_{\alpha_i,\zeta} : \zeta < \varepsilon, i < i_*\}$ has an ub.

$(2)^\varepsilon_{d,\vartheta,D}$ is defined as above replacing clause $(b)(\varepsilon)$ by:

$(\varepsilon)'$ if $\alpha \in S_\zeta$ then $\langle p_{\xi,\alpha} : \xi \leq \zeta\rangle$ is $\leq_{\mathbb{P}}$-increasing continuous.

Remark 1.4

1. So for a forcing notion $\mathbb{Q}$, $(2)^\varepsilon_{c,D}$ for ε the limit is $*^\varepsilon_D[\mathbb{Q}]$ is the same as in [14, Th.0.7]. Also "$\mathbb{Q}$ satisfies $(1)_b + (2)^2_{b,2,D} + (3)_a$" means $(*^1_{\mu,\mathbb{Q}})$ from § 1.1. Also "$\mathbb{Q}$ satisfies $(1)_c + (2)^1_{a,2}$" means $(*^2_{\mu,\mathbb{Q}})$ from § 1.1.
2. Note that "$\mathbb{P}$ satisfies $(2)^\varepsilon_{c,D}$" implies a weak version of strategic completeness (see $(1)_{b,\vartheta}$ for $\vartheta = |\varepsilon|^+$).

Definition 1.5

1. For suitable x, y, z, (but we may omit, e.g., $(3)_z$) let $\mathrm{Ax}_{\lambda,\mu}((1)_x, (2)_y, (3)_z)$ mean: if (μ is as in Definition 1.2), $\mathbb{P}$ is a forcing notion satisfying those conditions and $\mathscr{I}_i \subseteq \mathbb{P}$ is dense open for $i < i(*) < \lambda$ then some directed $\mathbf{G} \subseteq \mathbb{P}$ meets every $\mathscr{I}_i$.
2. We may omit λ if $\lambda = 2^\mu \geq \mu^+$; we may more generally write $\mathrm{Ax}_{\lambda,\mu}(K)$ for K being a property of the forcing notion.
3. For an ordinal[2] $\varepsilon < \mu$ being a limit ordinal if not said otherwise, let $\mathrm{Ax}^\varepsilon_{\lambda,\mu}$ mean: $\mathrm{Ax}_{\lambda,\mu}((1)_c + (2)^\varepsilon_c)$; we may omit λ if $\lambda = 2^\mu \geq \mu^+$.

See for more on axioms Roslanowski and Shelah [5], parallel to forcing and [13] and references therein. In § 2 if we replace C_δ by a stationary, co-stationary subset of δ; we can iterate the appropriate μ^+-c.c. $(<\mu)$-complete forcing notion. Earlier we have wondered (for answers on this question cf. Discussion 1.7(2)):

Question 1.6 Assume $\mu = \mu^{<\mu}$.

1. In [9], can the demand "well met" be omitted?
2. Is there an example $\mathbb{P}$ where $(1)_c + (2)^\vartheta_c$ holds but $(1)_c + (2)^\partial_c$ fails for any $\partial \in \mathrm{Reg}\backslash\{\vartheta\}$ where $\vartheta = \mathrm{cf}(\vartheta) < \mu$, $\mathrm{cf}(\partial) = \partial < \mu$? The case $\partial = \aleph_0 < \vartheta$ is natural.
3. Do we have an example for $\mathrm{Ax}((1)_b + (2)_b + (3)_a)$ but not $\mathrm{Ax}^\varepsilon_\mu$ with, e.g., $\varepsilon = \omega$?

Discussion 1.7

1. Note: if we have $(3)_a$ = called well met then we have $(2)_a \equiv (2)_b$. If in addition to $(3)_a + (2)_b$ we have $(1)_b$ then we have $(2)^\varepsilon_c$ for every ε. Hence 1.6(2) may be the true question.
2. In § 2 (cf. Conclusion 2.9) we shall show that the demand "well met" cannot be omitted in [9]; in other words, the statement $\mathrm{Ax}_\mu((1)_a, (2)_b)$ is inconsistent.
   In § 3 for $\vartheta, \partial < \mu$ regular not equal we get the consistency of $\mathrm{Ax}_\mu((1)_c + (2)^+_{a,=\vartheta})$ but not $\mathrm{Ax}_\mu((1)_c + (2)^+_{a,\partial})$ (cf. Conclusion 3.14), but this does not answer Question 1.6(2). In § 4 we answer Question 1.6(2).
3. Suppose we consider a forcing notion as in § 2, i.e., for § 3 use $\vartheta = 1$, but as in Definition 4.3, for $\alpha \in C_\delta \cap S^{\mu^+}_\vartheta$ no uniformization is demanded. This makes $\mathrm{Ax}^\vartheta_\mu$ holds for this forcing notion, but $*^\partial_\mu$ fail, so all seems fine.

[1] Why $1+\zeta$ not, e.g., $\zeta+1$? First, we like the INC to have the first move so that if $\mathbb{P}$ satisfies the condition and $p \in \mathbb{P}$ then $\mathbb{P}{\restriction}\{q : p \leq_{\mathbb{P}} q\}$ satisfies the condition. Second, we like the player COM to move in limit stages, as this is a weaker demand.
[2] Really omitting $(1)_b$ does not make a real difference but is natural.

4. Below, in fact for $\langle C_\delta, \mathbf{f}_\delta : \delta \in S\rangle$, we may force also the C_δ (in $\mathbb{Q}$ in § 2); we may not ask that C_δ is closed in δ and let $\bar\alpha^*_\delta = \langle \alpha^*_{\delta,\xi} : \xi < \mu\rangle$ list C_δ in increasing order so with limit δ, but generically we can have $\alpha^*_{\delta_1,\zeta} = \alpha^*_{\delta_2,\zeta}$, $\mathbf{f}_{\delta_1}(\alpha^*_{\delta_1,\zeta}) \neq \mathbf{f}_{\delta_2}(\alpha^*_{\delta_2,\zeta})$ for $*^1_\mu$, i.e., anyhow seems reasonable.

Observation 1.8 *Assume $\mu = \mu^{<\mu}$ and $\varepsilon < \mu$ limit.*

1. *If the forcing notion $\mathbb{Q}$ satisfies the conditions $(1)_{b,|\varepsilon|^+}$, $(3)_a$ and $(2)_b$, here equivalently $(2)_a$ then $\mathbb{Q}$ satisfies $(2)^\varepsilon_c$ from Definition 1.3.*
2. *If $\mathbb{P}$ satisfies $(3)_a$ then $\mathbb{P}$ satisfies $(3)_{a,\varepsilon}$.*
3. *If $\mathbb{P}$ satisfies $(1)_{b,|\varepsilon|^+} + (2)^+_{a,2}$ then $\mathbb{P}$ satisfies $(2)^\varepsilon_c$.*
4. *For any $\mathbb{P}$ we have: $(1)_a \Longrightarrow (1)_b \Longrightarrow (1)^+_c \Longrightarrow (1)_c$ and $(1)_a \Longrightarrow (1)_{d,\mu} \Longrightarrow (1)_c$. Similarly $(1)_{a,\vartheta} \Longrightarrow (1)_{b,\vartheta} \Longrightarrow (1)_{c,\vartheta}$ and $(1)_{a,=\vartheta} \Longrightarrow (1)_{b,=\vartheta}$ and $(1)_{a,\vartheta} \Longrightarrow (1)_{d,\vartheta}$ and $(1)_{a,=\vartheta} \Longrightarrow (1)_{d,=\vartheta}$.*
5. *For any $\mathbb{P}$ we have $(2)^+_{a,\vartheta} \Longrightarrow (2)^*_{a,\vartheta} \Longrightarrow (2)^+_{b,\vartheta}$.*
6. *If $\mathbb{P}$ satisfies $(2)^\varepsilon_{c,D}$ then forcing with $\mathbb{Q}$ adds no new sequence of ordinals of length $\leq \varepsilon$.*

P r o o f . Just read the definitions carefully. E.g., for (3) recall $\boxplus$ of § 1.1. $\square$

Claim 1.9

1. $\mathrm{Ax}^\varepsilon_\mu$, *i.e.,* $\mathrm{Ax}_\mu((1)_c + (2)^\varepsilon_c)$ *is equivalent to the axiom in [14].*
2. $\mathrm{Ax}_\mu((1)_b, (2)_a, (3)_a)$ *is the axiom from [9]. If ϑ, σ are regular cardinals $< \mu$ and $\mathrm{Ax}^\vartheta_\mu$ does not imply Ax^σ_μ then $\mathrm{Ax}_\mu((1)_b, (2)_a, (3)_a)$ so the axiom from [9], does not imply Ax^σ_μ.*

P r o o f . Easy, too. $\square$

For works on forcing for uniformizing cf. [8], [15], [12, Ch.VIII], and on ZFC results cf. [3], [12, AP, § 2].

1.3 Preliminaries

Notation 1.10 1. For regular $\vartheta < \lambda$ let $S^\lambda_\vartheta = \{\delta < \lambda : \delta \text{ has cofinality } \vartheta\}$.
2. We may write $\vartheta(+)$ instead of ϑ^+ in subscripts.

Definition 1.11

1. We say that a forcing notion $\mathbb{P}$ is strategically α-complete when for each $p \in \mathbb{P}$ in the following game $\Game_\alpha(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy.
   A play lasts α moves; in the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Longrightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$ and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq_{\mathbb{P}} q_\beta$.
   The player COM wins a play if it has a legal move for every $\beta < \alpha$.
2. We say that a forcing notion $\mathbb{P}$ is $(< \lambda)$-strategically complete when it is α-strategically complete for every $\alpha < \lambda$.

Definition 1.12 For a filter D on a set I:

(a) $D^+ = \{A \subseteq I : I\backslash A \notin D\}$;
(b) for $S \in D^+$ let $D + S = \{A \subseteq I : A \cup (I\backslash S) \in D\}$.

Theorem 1.13 *Assume $\mu = \mu^{<\mu}$ and D is a normal filter on μ^+ to which $S^{\mu^+}_\mu$ belongs; note that in $\mathbf{V}^{\mathbb{P}}$ we interpret D as the normal filter on μ^+ it generates. Assume further that $2 \leq \vartheta \leq \mu$. Then each of the following properties listed in (B) of forcing notions is preserved by $(< \mu)$-support iteration, which means clause (A) is satisfied:*

(A) If $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \lg(\mathbf{q}), \beta < \lg(\mathbf{q})\rangle$ is a $(< \mu)$-support iteration and for each $\beta < \lg(\mathbf{q})$ we have $\Vdash_{\mathbb{P}_\beta}$ "($\underset{\sim}{\mathbb{Q}}_\beta$ satisfies the property Pr*" then the forcing notion $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\lg(\mathbf{q})}$ satisfies the property* Pr.

(B) The property Pr *of forcing notion $\mathbb{Q}$ is one of the following (where $\varepsilon < \mu$ is a limit ordinal):*
  (a) the property $(1)_c + (2)^\varepsilon_{c,D}$,

(b) *the property* $(1)_{c,\vartheta}$,
(c) *the property* $(1)^+_{c,\vartheta}$,
(d) *the property* $(1)_c + (2)^{\varepsilon}_{c,\vartheta,D}$,
(e) *the property* $(1)_c + (2)^{\varepsilon}_{d,\vartheta,D}$.

Proof. Cases (b) & (c) are well known. Case (a) holds by [14]. Case (d): cf. Shelah and Spinas [16]. Case (e): Similarly. □

2 On μ^+-regressive-c.c.; an example

We shall show that in [9], we have to use some form of the well met condition. First, we shall concentrate on the case μ is not strongly inaccessible.

Hypothesis 2.1

1. $\mu = \mu^{<\mu} > \aleph_0$.
2. $S \subseteq S^{\mu^+}_{\mu} = \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mu\}$ is stationary, the main case is $S = S^{\mu^+}_{\mu}$.

Definition 2.2 $\bar{C}$ is an S-club system when $\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ a club of δ of order type μ.

Definition 2.3

1. We say $(\mathscr{W}, \bar{\mathbf{f}})$ is an $(S, \bar{C}, \kappa)$-parameter or just a $(\bar{C}, \kappa)$-parameter when:
   (a) $S \subseteq S^{\mu^+}_{\mu}$ is stationary; cf. Hypothesis 2.1(2),
   (b) $\bar{C}$ is an S-club-system so we may omit S,
   (c) $\kappa \le \mu$ is ≥ 2, if $\kappa = 2$ we may omit κ and write $\bar{C}$,
   (d) $\mathscr{W} \subseteq \mu$; if $\mathscr{W} = \mu$ we may omit $\mathscr{W}$,
   (e) $\bar{\mathbf{f}} = \langle \mathbf{f}_\delta : \delta \in S\rangle$,
   (f) $\mathbf{f}_\delta : C_\delta \to \kappa$.
2. For $(\mathscr{W}, \bar{\mathbf{f}})$ an $(S, \bar{C}, \kappa)$-parameter we define a forcing notion $\mathbb{Q} = \mathbb{Q}_{(\mathscr{W},\bar{\mathbf{f}},\bar{C})}$ as follows:
   (A) $p \in \mathbb{Q}$ iff p consists of
      (a) $v \in [S]^{<\mu}$;
      (b) h is a function with domain v;
      (c) if $\delta \in v$ then $h(\delta)$ is a non-empty bounded subset of μ closed in its supremum;
      (d) if $\delta_1, \delta_2 \in v$ and $\alpha \in C_{\delta_1} \cap C_{\delta_2}$ and $\mathrm{otp}(\alpha \cap C_{\delta_\ell}) \in h(\delta_\ell)$ and $\mathrm{otp}(C_{\delta_\ell} \cap \alpha) \in \mathscr{W}$ for $\ell = 1, 2$ then $\mathbf{f}_{\delta_1}(\alpha) = \mathbf{f}_{\delta_2}(\alpha)$;
      (e) if $\delta_1 \ne \delta_2 \in v$ and $\beta \in C_{\delta_1} \cap C_{\delta_2}$ then for $\ell = 1, 2$ there is $\beta_\ell \in h_p(\delta_\ell)$ satisfying $\mathrm{otp}(C_{\delta_\ell} \cap \beta) \le \beta_\ell$.
   (B) $p \le_{\mathbb{Q}} q$ iff
      (a) $v_p \subseteq v_q$;
      (b) $\delta \in v_p \Longrightarrow h_p(\delta) \trianglelefteq h_q(\delta)$.
3. If $\mathscr{W} = \mu$ we may omit it.

Definition 2.4 Let $(\mathscr{W}, \bar{\mathbf{f}})$ be a $(\bar{C}, \kappa)$-parameter and let $\mathbb{Q} = \mathbb{Q}_{\mathscr{W},\bar{\mathbf{f}},\bar{C}}$.

1. For $p \in \mathbb{Q}$ let g_p be the function
   (a) with domain
   $$\{\alpha : \text{some } \delta \text{ witnesses } \alpha \in \mathrm{Dom}(\mathbf{h}_p) \text{ which means } \delta \in v_p, \alpha \in C_\delta, \\ \mathrm{otp}(C_\delta \cap \alpha) \in h_p(\delta) \text{ and } \mathrm{otp}(C_\delta \cap \alpha) \in \mathscr{W}\};$$
   (b) for $\alpha \in \mathrm{Dom}(g_p)$ we have:
   $$g_p(\alpha) = \mathbf{f}_\delta(\alpha) \text{ for every witness } \delta \text{ for } \alpha \in \mathrm{dom}(g_p).$$

2. Let $\underset{\sim}{g}$ be the $\mathbb{Q}$-name for $\cup\{g_p : p \in \underset{\sim}{\mathbf{G}}\}$.
3. Let $\underset{\sim}{E}_\delta = \underset{\sim}{E}_\delta[\mathbb{Q}]$ be the $\mathbb{Q}$-name for $\cup\{h_p(\delta) : p \in \underset{\sim}{\mathbf{G}}, \delta \in v_p\}$ and let $\underset{\sim}{\mathscr{W}}_\delta = \{\alpha \in \underset{\sim}{E}_\delta : \mathrm{otp}(C_\delta \cap \alpha) \in \mathscr{W}\}$.

Claim 2.5 *Assume $(\mathscr{W}, \bar{\mathbf{f}})$ is an $(S, \bar{C}, \kappa)$-parameter and $\mathbb{Q} = \mathbb{Q}_{(\mathscr{W}, \bar{\mathbf{f}}, \bar{C})}$ lub.*

1. $\mathbb{Q}$ is $(< \mu)$-complete, moreover any $\leq_{\mathbb{Q}}$-increasing sequence of length $< \mu$ has a $\leq_{\mathbb{Q}}$-lub that is $(1)_a$.
2. If $\delta \in S$ and $\alpha < \mu$ then the following subsets of $\mathbb{Q}$ are dense and for i., ii. also open:
   i. $\mathscr{I}_\delta = \{p \in \mathbb{Q} : \delta \in v_p\}$;
   ii. $\mathscr{I}_{\delta,\alpha} = \{p \in \mathscr{I}_\delta : \alpha < \sup(h_p(\delta))\}$;
   iii. $\mathscr{I}^*_\alpha = \{p \in \mathbb{Q}$: if $\delta \in v_p$ then $\alpha < \sup(h_p(\delta))$ and $h_p(\delta)$ has a last member$\}$.
3. For every $\delta \in S$, the function $\underset{\sim}{g}$ almost extends $\mathbf{f}_\delta$, i.e., $\Vdash_{\mathbb{Q}} \underset{\sim}{g} \supseteq \mathbf{f}_\delta{\restriction}\{\alpha \in C_\delta : \mathrm{otp}(\alpha \cap C_\delta) \in \underset{\sim}{\mathscr{W}}_\delta\}$, recalling $\mathscr{W}_\delta = \mathscr{W} \cap \underset{\sim}{E}_\delta$. Also $\underset{\sim}{E}_\delta$ is a club of μ and if $\mathscr{W} = \mu$ then $\underset{\sim}{\mathscr{W}}_\delta$ is a club of μ".

P r o o f. (1): Straightforward, see clause (A)(e) of Definition 2.3(2) in particular.
(2), (3): Also easy. □

Claim 2.6 *Let $(\mathscr{W}, \bar{\mathbf{f}})$, $(S, \bar{C}, \kappa)$, $\mathbb{Q}$ be as above. Then $\mathbb{Q}$ satisfies clause $(2)_b$ of Definition 1.2, i.e.:*

$(*^0_\mu)$ If $\bar{p} = \langle p_\alpha : \alpha \in S\rangle$ and $\alpha \in S \Longrightarrow p_\alpha \in \mathbb{Q}$ then there is a club E of μ^+ and pressing down function $f : S \cap E \to \mu^+$, i.e. $f(\delta) < \delta$, such that: $(\delta_1 \neq \delta_2 \in S \cap E) \wedge f(\delta_1) = f(\delta_2) \Longrightarrow p_{\delta_1}, p_{\delta_2}$ are compatible.

P r o o f. First, by Claim 2.5(1)(2), we choose $\langle q_\alpha : \alpha \in S\rangle$ such that, for every $\alpha \in S$:

$\odot_1$ (a) $p_\alpha \leq q_\alpha$;
(b) if $\delta \in v_{q_\alpha}$ but $\delta > \alpha$ then $\mathrm{otp}(C_\delta \cap \alpha) < \sup(h_{q_\alpha}(\delta))$;
(c) $\alpha \in v_{q_\alpha}$;
(d) $h_{q_\alpha}(\alpha)$ has a last element.

Second, choose a club E of μ^+ such that $\alpha \in S \cap E \Longrightarrow \sup(v_{q_\alpha}) < \min((E\backslash(\alpha + 1))$.
Third, choose a regressive function $\mathbf{h}$ with domain $E \cap S$ such that:

$\odot_2$ If $\delta(1) = \delta_1 < \delta_2 = \delta(2)$ are from $E \cap S$ and $\mathbf{h}(\delta_1) = \mathbf{h}(\delta_2)$ and $\langle \alpha_{\ell,i} : i < \mathrm{otp}(v_{q_{\delta(\ell)}})\rangle$ lists $v_{q_{\delta(\ell)}}$ in increasing order for $\ell = 1, 2$ then for some j_*:
(a) $\mathrm{otp}(v_{q_{\delta(1)}}) = \mathrm{otp}(v_{q_{\delta(2)}})$ call it $i(*)$;
(b) $j_* < i(*)$ and $\alpha_{1,j_*} = \delta_1, \alpha_{2,j_*} = \delta_2$;
(c) if $j < j_*$ then $\alpha_{1,j} = \alpha_{2,j}$;
(d) if $j > j_*$ but $j < i(*)$ then $C_{\alpha_{1,j}} \cap \delta_1 = C_{\alpha_{2,j}} \cap \delta_2$;
(e) $h_{q_{\delta(1)}}(\alpha_{1,i}) = h_{q_{\delta(2)}}(\alpha_{2,i})$ for $i < i(*)$;
(f) if $\varepsilon \in h_{q_{\delta(1)}}(\delta_1)$ then the ε-th member of C_{δ_1} is equal to the ε-th member of C_{δ_2}.

Now it suffices to prove:

$\odot_3$ If $\delta_1 \neq \delta_2 \in S \cap E$ and $\mathbf{h}(\delta_1) = \mathbf{h}(\delta_2)$ then $q_{\delta_1}, q_{\delta_2}$ are compatible in $\mathbb{Q}$,

Why? Define q as follows:

i. $v_q = v_{q_{\delta(1)}} \cup v_{q_{\delta(2)}}$;
ii. $h_q(\delta) = h_{q_{\delta(\ell)}}(\delta)$ if $\ell \in \{1, 2\}$ and $\delta \in v_q\backslash\{\delta_\ell\}$;
iii. $h_q(\delta_\ell) = h_{q_{\delta(\ell)}}(\delta_\ell) \cup \{\beta_\ell\}$ where $\beta_\ell < \mu$, $\beta_\ell > \max\{h_{q_{\delta(1)}}(\delta_1) \cup h_{q_{\delta(2)}}(\delta_2)\}$ and $\beta_\ell > \sup\{\mathrm{otp}(\alpha \cap C_{\delta_\ell}) : \alpha \in C_{\delta_1} \cap C_{\delta_2}\}$.

First, q is well defined because in ii., if $h_q(\alpha)$ is defined in two ways, then $\alpha < \delta_1$ and they are equal because of $\odot_2$.

Second, why $q \in \mathbb{Q}$? We have to check clauses (a)-(e) of Definition 2.3(2)(A). Now clauses (a), (b), and (c) are obvious. For clause (d), assume $\gamma_1, \gamma_2 \in v_q$, and $\alpha \in C_{\gamma_1} \cap C_{\gamma_2}$ and $\mathrm{otp}(C_{\gamma_\ell} \cap \alpha) \in h_q(\gamma_\ell) \cap \mathscr{W}$ for $\ell = 1, 2$.

Sh:1036

If $\gamma_1, \gamma_2 \in v_{q_{\delta(1)}}$ then use $q_{\delta(1)} \in \mathbb{Q}$, and similarly if $\gamma_1, \gamma_2 \in v_{q_{\delta(2)}}$ then use $q_{\delta(2)} \in \mathbb{Q}$. So without loss of generality $\gamma_1 \in v_{q_{\delta(1)}} \setminus v_{q_{\delta(2)}}$ and $\gamma_2 \in v_{q_{\delta(2)}} \setminus v_{q_{\delta(1)}}$, so necessarily $\gamma_1 \geq \delta(1)$, $\gamma_2 \geq \delta_2$ and $\alpha \in C_{\gamma_1} \cap C_{\gamma_2} \subseteq \delta_1 \cap \delta_2$ (using the choice of $\bar{C}$ and E); using the notation of $\odot_2$ let $i(\ell)$ be such that $\gamma_\ell = \alpha_{\ell,i(\ell)}$ so $i(\ell) \in [j(*), i(*))$ for $i(\ell) = 1, 2$. Now we get the result by applying clause (d) for $q_{\delta(2)} \in \mathbb{Q}$ for $\gamma_1, \gamma_2, \alpha_{2,i(1)}\alpha_{2,i(1)}, \alpha_{2,i(2)} = \gamma_2$ recalling $\odot(d), (e)$, noting that in the case $(\gamma_1, \gamma_2) = (\delta_1, \delta_2)$ necessarily $i_1 \neq \beta_1 \wedge i_2 \neq \beta_2$ (as $\beta_1, \beta_2 < \mu$ were chosen large enough) so $\mathrm{otp}(C_{\delta(1)} \cap \alpha) = \mathrm{otp}(C_{\delta(2)} \cap \alpha) \in h_{p_{\delta(1)}}(\alpha) = h_{p_{\delta(2)}}(\alpha)$ and if $i(1) = i(2)$ then by the choice of $\mathbf{h}$.

We are left with clause (e) which is proved similarly, recalling iii. above.

It is easy to check that $q \in \mathbb{Q}$ and $q_{\delta_1} \leq q$, $q_{\delta_2} \leq q$, so $\odot_3$ holds indeed. □

Theorem 2.7 *If (A) then (B) where*

(A) $\mu, S, \bar{C}, \kappa, \vartheta$ *satisfy*
- *(a)* $\mu = \mu^{<\mu} > \aleph_0$;
- *(b)* $S = S^{\mu+}_{\mu}$;
- *(c)* $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *is an S-club system and for* $\delta \in S$ *we let* $\eta_\delta \in {}^{\mu}\delta$ *list* C_δ *in increasing order;*
- *(d)* F *is a function from* $\mathscr{F}_\mu$ *to* κ *where* $\mathscr{F}_\mu = \{f : f$ *is a function from some* $u \in [\mu^+]^{<\mu}$ *to* $\mu\}$*; the default case is* $F(f) = f(\max(\mathrm{dom}(f))$ *when well defined and zero otherwise;*
- *(e)* $\bar{a} = \langle a_{\delta,\alpha} : \delta \in S, \alpha < \mu\rangle$ *where* $a_{\delta,\alpha} \subseteq \eta_\delta(\alpha) + 1$*; the default value of* $a_{\delta,\alpha}$ *is* $\{\eta_\delta(\alpha)\}$*;*
- *(f) either* μ *is a (strongly) inaccessible cardinal, and* $\vartheta < \kappa = \mu$ or $\kappa = 2$, $\vartheta < \mu = 2^\vartheta$*;*

(B) we can find $\bar{\mathbf{c}}$ *satisfying*
- *(a)* $\bar{\mathbf{c}} = \langle \mathbf{c}_\delta : \delta \in S\rangle$*;*
- *(b)* $\mathbf{c}_\delta$ *is a function from* C_δ *to* κ*;*
- *(c) if f is a function from* μ^+ *to* κ*, then for stationarily many* $\delta \in S$*, for stationarily many* $\varepsilon \in C_\delta$ *we have:* $\kappa = 2 \Longrightarrow \mathbf{c}_\delta(\alpha) = F(f{\restriction}a_{\delta,\alpha})$ *and* $\kappa = \mu \Longrightarrow \mathbf{c}_\delta(\alpha) \neq F(f{\restriction}a_{\delta,\alpha})$.

Discussion 2.8 Cf. [12, AP.3.9, p. 990]. But there, only the case $\mu = \aleph_1$, $\kappa = 2$ is really proved, the case μ an accessible cardinal and $\kappa = 2$ is stated to be similar. In the case where μ is inaccessibe, $\kappa = 2$, the statement consistently fails as said in [12, 3.8(1)]; cf. [8, 11 15]. So by a request we give here a full proof.

P r o o f . Why? Let λ be big enough (e.g., $(2^{\mu^+})^+$), and M^* be an expansion of $(\mathscr{H}(\lambda), \in)$ by Skolem functions (so countably many; essentially, if we expand just by a definable well ordering it suffices).

Suppose toward contradiction that clause (A) holds but clause (B) fails. It is known that there is a function G from $\{A : A \subseteq \mu^+, |A| < \mu\}$ to μ such that $G(A) = G(B)$ implies that A, B have the same order type and their intersection is an initial segment of both (e.g., if $h_\alpha : \alpha \to \mu$ is one-to-one for $\alpha < \mu$, we let $G_0(A) =_{\mathrm{df}} \{(\mathrm{otp}(A \cap \alpha), \mathrm{otp}(A \cap \beta), h_\beta(\alpha)) : \alpha \in A$ and $\beta \in A\}$. Now G_0 is as required except that $\mathrm{Rang}(G_0) \not\subseteq \mu$ but $|\mathrm{Rang}(G_0)| \leq \mu$ so we can correct this by renaming).

We shall now define for any $\mathbf{p} \in \mathscr{H}(\lambda)$ a sequence $\langle \mathbf{c}^{\mathbf{p}}_\delta : \delta \in S\rangle$ where $\mathbf{c}^{\mathbf{p}}_\delta : \mu \to \mathscr{H}(\mu)$, which we shall use later.

For every $\delta \in S, i < \mu$, let $N^{\mathbf{p}}_{\delta,i}$ be the minimal submodel of M^* (so closed under the Skolem functions) including $\{\delta, i, \mathbf{p}\}$ such that its intersection with μ is an ordinal so $N^{\mathbf{p}}_{\delta,i}$ has cardinality $< \mu$ and

$(*)_1$ let
- (a) $\pi^{\mathbf{p}}_{\delta,\alpha}$ be the Mostowski collapse mapping from $N^{\mathbf{p}}_{\delta,\alpha}$;
- (b) $\mathbf{c}^{\mathbf{p}}_\delta$ is a function from μ into $\mathscr{H}(\mu)$;
- (c) for $\alpha < \mu$ we let $\mathbf{c}^{\mathbf{p}}_\delta(\alpha) =_{\mathrm{df}} \langle(\pi^{\mathbf{p}}_{\delta,\alpha}"((N^{\mathbf{p}}_{\delta,\alpha}, \mathbf{p}, \delta, \alpha), G(N^{\mathbf{p}}_{\delta,\alpha} \cap \mu^+)\rangle$ which belongs to $\mathscr{H}(\mu)$.

Note that $(N^{\mathbf{p}}_{\delta,i}, \mathbf{p}, i, \delta)$ is $N^{\mathbf{p}}_{\delta,i}$ expanded by three individual constants. Now recall that toward contradiction we are assuming that clause (B) of the theorem fails. This means that

$(*)_2$ for every sequence $\bar{\mathbf{c}} = \langle \mathbf{c}_\delta : \delta \in S\rangle$ where $\mathbf{c}_\delta$ is a function from C_δ to κ there is an $h_{\mathbf{c}} : \mu^+ \to \kappa$ such that: For a closed unbounded subset E of μ^+ for every $\delta \in S \cap E$, for a closed unbounded set of $\alpha \in C_\delta$ we have $\mathbf{c}_\delta(\alpha) = F(h_{\bar{\mathbf{c}}}{\restriction}a_{\delta,\alpha})$; note that in the case $\kappa = 2$, replacing non-equal by equal makes no difference!

Sh:1036

Now

$(*)_3$ in $(*)_2$ we can replace κ by the set $\mathscr{H}(\mu)$, by changing F.

[Why? If $\kappa = \mu$ this is obvious as μ and $\mathscr{H}(\mu)$ have the same cardinality. So we can assume $\kappa = 2$, and we can replace $\mathscr{H}(\mu)$ by ${}^{\vartheta}2$ because the latter has cardinality μ. For $\varepsilon < \vartheta$ and h any function into ${}^{\vartheta}2$, let $h^{[\varepsilon]}$ be defined by $h^{[\varepsilon]}(\alpha) = (h(\alpha))(\varepsilon)$ for $\alpha \in \mathrm{Dom}(h)$. Define the function F^* by: $F^*(h) = \langle F(h^{[\varepsilon]}) : \varepsilon < \vartheta\rangle$ so $F^*(h) \in {}^{\vartheta}2$. We shall prove that replacing F by F^*, the statement $(*)_3$ holds. So assume we are given $\langle \mathbf{c}_\delta : \delta \in S\rangle$ where $\mathbf{c}_\delta \in {}^{(C_\delta)}({}^{\vartheta}2)$, i.e., $\mathbf{c}_\delta : C_\delta \to {}^{\vartheta}2$; then for $\varepsilon < \vartheta$ the function $\mathbf{c}_\delta^{[\varepsilon]} \in {}^{\mu}2$ is well defined for each $\delta \in S$. Now for each $\varepsilon < \vartheta$, we can apply $(*)_2$ so we can choose $h^{(\varepsilon)} : \mu^+ \to 2$ such that for some club E of μ^+ for every $\delta \in S \cap E$ for a club of $\alpha < C_\delta$ we have

$$\mathbf{c}^{[\varepsilon]}(\alpha) = F(h^{(\varepsilon)} \restriction a_{\delta,\alpha}).$$

Define $h : \mu^+ \to {}^{\vartheta}2$ by $h(\alpha) = \langle h^{(\varepsilon)}(\alpha) : \varepsilon < \vartheta\rangle$, it is as required. So $(*)_3$ holds indeed.]

Now we shall define by induction on $\varepsilon < \vartheta$, $\mathbf{p}(\varepsilon) \in \mathscr{H}(\lambda)$, and $h_\varepsilon : \mu^+ \to \mathscr{H}(\mu)$.

Arriving to ε, let $\mathbf{p}(\varepsilon) = (\langle (h_\zeta, \mathbf{p}(\zeta), \bar{N}_\zeta) : \zeta < \varepsilon\rangle, \bar{C}, F, \bar{a}, G)$ where $\bar{N}_\zeta = \langle N^{\mathbf{p}}(\zeta)_{\delta,i} : \delta \in S, i < \mu\rangle$; see before $(*)_1$. Also let $\mathbf{c}_\delta^{\mathbf{p}(\varepsilon)} : \mu \to \mathscr{H}(\mu)$ be as we have defined above (in $(*)_1$), so by $(*)_3$

$(*)_4$ there are $h_\varepsilon, W^\varepsilon, \bar{W}_\varepsilon$ such that:
- (a) $h_\varepsilon : \mu^+ \to \mathscr{H}(\mu)$;
- (b) $W^\varepsilon \subseteq \mu^+$ is a closed unbounded subset of μ^+;
- (c) $\bar{W}_\varepsilon = \langle W_\delta^\varepsilon : \delta \in W \cap S\rangle$;
- (d) for every $\delta \in W^\varepsilon \cap S$, W_δ^ε is a closed unbounded subset of μ;
- (e) for $\alpha \in W_\delta^\varepsilon$, $\delta \in W^n \cap S$ we have: $\mathbf{c}_\delta^{\mathbf{p}(\varepsilon)}(\alpha) = F^*(h_\varepsilon \restriction a_{\delta,\alpha})$.

Now

$(*)_5$ let
- (a) let $W = \bigcap_{\varepsilon<\vartheta} W^\varepsilon$,
- (b) for $\delta \in W \cap S$ let $W_\delta = \bigcap_{\varepsilon<\vartheta} W_\delta^\varepsilon$.

Clearly W is a closed unbounded subset of μ^+, and W_δ is a closed unbounded subset of μ for $\delta \in W \cap S$. So for every $\delta \in W \cap S$, we can choose $\alpha(\delta) \in W_\delta$; hence by the Fodor lemma, for some $\alpha(*) < \mu^+$ and $\nu, \bar{b}$ the set $S_* = \{\delta \in W \cap S : \alpha(\delta) = \alpha(*), \eta_\delta \restriction (\xi + 1) = \nu, \langle a_{\delta,i} : i \le \alpha(*)\rangle = \bar{b}\}$ is stationary. As $\mu = \mu^{<\mu}$ holds there are δ_1, δ_2 and $\xi < \mu$ such that:

$(*)_6$
- (A) $\delta_1 < \delta_2$ are from S_*;
- (B) $\xi \in W_{\delta_\ell}$ for $\ell = 1, 2$;
- (C) $\eta_{\delta_1}(\xi) = \eta_{\delta_2}(\xi)$;
- (D) $\eta_{\delta_1} \restriction (\xi + 1) = \eta_{\delta_2} \restriction (\xi + 1)$;
- (E) $\langle a_{\delta_1,\alpha} : \alpha \le \alpha(*)\rangle = \langle a_{\delta_2,\alpha} : \alpha \le \alpha(*)\rangle$.

So clearly we can assume

$(*)_7$ there are no $\delta_1^\dagger, \delta_2^\dagger$ satisfying (A)-(E) such that $\delta_1^\dagger \le \delta_1$, $\delta_2^\dagger \le \delta_2$ and $(\delta_1^\dagger, \delta_2^\dagger) \ne (\delta_1, \delta_2)$.

Now as $\delta_1 < \delta_2$, for some $\alpha > \xi$, $\eta_{\delta_1}(\alpha) \ne \eta_{\delta_2}(\alpha)$, and there is a minimal such α; but as $\eta_{\delta_1}, \eta_{\delta_2}$ are increasing and continuous, clearly α is a successor ordinal.

Let $\upsilon = \{\zeta < \mu : \eta_{\delta_1} \restriction \zeta = \eta_{\delta_2} \restriction \zeta, \eta_{\delta_1}(\zeta) = \eta_{\delta_2}(\zeta)$ and $\zeta \in W_{\delta_1} \cap W_{\delta_2}\}$. This set is non-empty (as ξ belongs to it), is closed (as $W_{\delta_1}, W_{\delta_2}$ are closed and η_{δ_ℓ} are increasing continuous) and is bounded in μ (by the beginning of this paragraph). Together we know that there is a maximal $\zeta \in \upsilon$.

So

$(*)_8$ $\mathbf{c}_{\delta_1}^{\mathbf{p}(\varepsilon)}(\zeta) = \mathbf{c}_{\delta_2}^{\mathbf{p}(\varepsilon)}(\zeta)$ for every $\varepsilon < \vartheta$.

[Why? As both are equal to $F^*(h_\varepsilon \restriction a_{\delta_\ell,\zeta})$.]

Sh:1036

Fix a non-zero $\varepsilon < \vartheta$ for a while. Looking at the definition of $\mathbf{c}_\delta^{\mathbf{p}(\varepsilon)}(\zeta)$ (cf. $(*)_1$) we see that $N_{\delta_1,\zeta}^{\mathbf{p}(\varepsilon)}$ is isomorphic to $N_{\delta_2,\zeta}^{\mathbf{p}(\varepsilon)}$, and let the isomorphism be called g_ε. Note that the isomorphism is unique (as $\in$ in those models is transitive and well founded) and maps $\bar{C}, F, \bar{a}$ to themselves.

By the definition of $\mathbf{c}_\delta^{\mathbf{p}(\varepsilon)}(\zeta)$, clearly

$(*)_9$ (a) $g_\varepsilon(\mathbf{p}(\varepsilon)) = \mathbf{p}(\varepsilon)$ hence $g_\varepsilon((\bar{C}, F, \bar{a}, G)) = (\bar{C}, F, \bar{a}, G)$;
(b) $g_\varepsilon(\delta_1) = \delta_2, g_\varepsilon(\zeta) = \zeta, g_\varepsilon(\varepsilon) = \varepsilon$;
(c) $g_\varepsilon(\eta_{\delta_1}) = \eta_{\delta_2}$;
(c) $g_\varepsilon(W^\xi) = W^\xi$ and $g_\varepsilon(W_{\delta_1}^\xi) = W_{\delta_2}^\xi$ for every $\xi < \varepsilon$;
(e) $g_\varepsilon(N_{\delta_1,\zeta}^{\mathbf{p}(\xi)}) = g_\varepsilon(N_{\delta_2,\zeta}^{\mathbf{p}(\xi)}) \in N_{\delta_2,\zeta}^{\mathbf{p}(\varepsilon)}$ for every $\xi < \varepsilon$.

[Why? Look at the definition of $\mathbf{p}(\varepsilon)$]

For $\xi < \varepsilon$, as $N_{\delta_\ell,\zeta}^{\mathbf{p}(\xi)}$ is of cardinality $< \mu$, its intersection with μ is an ordinal and it belongs to $N_{\delta_\ell,\zeta}^{\mathbf{p}(\varepsilon)}$, it is also included in it, hence $g_\varepsilon \restriction N_{\delta_1,\zeta}^{\mathbf{p}(\xi)}$ is an isomorphism from $N_{\delta_1,\zeta}^{\mathbf{p}(\xi)}$ onto $N_{\delta_2,\zeta}^{\mathbf{p}(\xi)}$; hence (by the uniqueness of g_ε and $(*)_9(b)$):

$(*)_{10}$ $g_\varepsilon \supseteq g_\xi$ for $\xi < \varepsilon$.

We now stop fixing ε. For $\ell = 1, 2$ (recalling $\vartheta < \mu$ in both cases), let $N_\ell = \bigcup_{\varepsilon<\vartheta} N_{\delta_\ell,\zeta}^{\mathbf{p}(\varepsilon)}$ and $g = \bigcup_{\varepsilon<\vartheta} g_\varepsilon$; so g is an isomorphism from N_1 to N_2. By the definition of $\mathbf{c}_{\delta_\ell}^{\mathbf{p}(\varepsilon)}(\zeta)$, clearly the second coordinates are the same, thus:

$(*)_{11}$ $G(N_{\delta_1,\zeta}^{\mathbf{p}(\varepsilon)} \cap \mu^+) = G(N_{\delta_2,\zeta}^{\mathbf{p}(\varepsilon)} \cap \mu^+)$,

Hence those sets have their intersection an initial segment of both; as this holds for every $\varepsilon < \vartheta$, clearly $N_1 \cap \mu^+, N_2 \cap \mu^+$ have their intersection an initial segment of both (as usual, we are not strictly distinguishing between a model and its universe), hence (recalling the choice of the $N_{\delta,i}^{\mathbf{p}}$-s), g is the identity on $N_1 \cap N_2 \cap \mu^+$.

Note that clearly $\delta_1 \notin N_2$ as $g(\delta_1) = \delta_2 \neq \delta_1$, hence $\delta_2 \notin N_1$. Now

$(*)_{12}$ (a) Letting $\delta_\ell^* =_{\rm df} \mathrm{Min}(\mu^+ \cap N_\ell \setminus (N_1 \cap N_2))$, we have: $\delta_\ell^* \leq \delta_\ell$, is a limit ordinal
(b) $g(\delta_1^*) = \delta_2^*$ and so
(c) $\mathrm{cf}(\delta_1^*) = \mathrm{cf}(\delta_2^*)$.
(d) $\mathrm{cf}(\delta_\ell^*) = \mu$.

Why? Clauses (a), (b) are obvious and clause (c) follows. Clause (d) (that is $\mathrm{cf}(\delta_\ell^*) = \mu$) holds as otherwise for some regular cardinal $\sigma < \mu$ we have $\mathrm{cf}(\delta_1^*) = \sigma$, and as $\delta_1^* \in N_1$ for some $\boldsymbol{\zeta} < \vartheta$, $\delta_1 \in N_{\delta_1,\zeta}^{\mathbf{p}(\varepsilon)}$, hence there is $\{\beta_\iota : \boldsymbol{\iota} < \sigma\} \in \delta_1^* \cap N_{\delta_1,\zeta}^{\mathbf{p}(\varepsilon)}$ cofinal in δ_1^*. As $\sigma < \mu$ necessarily it is included in $N_{\delta_1,\zeta}^{\mathbf{p}(\varepsilon)}$, without loss of generality β_ι is increasing with ι. By the choice of δ_1^*, if $\iota < \sigma$ then $\beta_\iota \in N_1 \cap N_2$, hence $g(\beta_\iota) = \beta_\iota$; let $\beta^* = \min(N_{\delta_2,\zeta}^{\mathbf{p}(\varepsilon)} \setminus \bigcup_\iota \beta_\iota)$, so $\beta^* \in N_{\delta_2,\zeta}^{\mathbf{p}(\varepsilon)} \subseteq N_{\delta_2,\zeta}^{\mathbf{p}(\varepsilon+1)}$, so $\delta_1^* = \sup\{\beta_\iota : \iota < \sigma\} = \sup(\beta^* \cap N_{\delta_2,\zeta}^{\mathbf{p}(\varepsilon)}) \in N_2$, contradiction. So we have proved $(*)_{12}$.]

Now for $\ell = 1, 2$ let $\alpha_\ell =_{\rm df} N_\ell \cap \mu$, (this intersection is an initial segment of μ) and $\beta_\ell =_{\rm df} \sup(N_\ell \cap \delta_\ell^*)$ hence $\beta_1 = \beta_2$ (by δ_ℓ^* definition) and call it β. As $\mathrm{cf}(\delta_\ell^*) = \mu$ clearly $\delta_\ell^* \geq \mu$, and so clearly by g's existence $\alpha_1 = \alpha_2$ and call it $\alpha_* = \alpha(*)$, (also as $\mu \in N_1 \cap N_2 \cap \mu^+$, necessarily $N_1 \cap \mu = N_2 \cap \mu$).

As $\eta_{\delta_1^*}$ is a one to one function (being increasing) from μ, clearly

$(*)_{13}$ for every $\alpha < \mu$ we have $\eta_{\delta_1^*}(\alpha) \in N_1 \iff \alpha < \alpha(*)$.

Also $N_1 \models$ "$\langle \eta_{\delta_1^*}(\alpha) : \alpha < \mu \rangle$" is unbounded below δ_1^* (remember $N_1 \prec M^*$ as $N_{\delta_1,\zeta}^{\mathbf{p}(\varepsilon)} \prec M^*$ for each ε).

So clearly $\beta = \beta_1 = \sup\{\eta_{\delta_1^*}(\alpha) : \alpha < \alpha_*\}$; but $\eta_{\delta_1^*}$ is increasing continuous and α_* is a limit ordinal (being $N_\ell \cap \mu$), hence $\beta = \eta_{\delta_1^*}(\alpha_*)$.

For the same reasons $\beta = \eta_{\delta_2^*}(\alpha_*)$.

Similarly $\eta_{\delta_1^*} \restriction \alpha_* = \eta_{\delta_2^*} \restriction \alpha_*$ because $g(\eta_{\delta_1^*}) = \eta_{\delta_2^*}$, and $\alpha_* \in W_{\delta_\ell^*}^\varepsilon$ for each $\varepsilon < \vartheta (\ell = 1, 2)$ as $N_\ell \models$ "$W_{\delta_\ell^*}^\varepsilon$ is a closed unbounded subset of μ". For similar reasons $\delta_\ell^* \in W_\varepsilon$ for each $\varepsilon < \vartheta$: recall $W_\varepsilon \in N_{\delta_\ell,\zeta}^{\mathbf{p}(\varepsilon+1)}$ and so $W_\varepsilon \in N_\ell$ hence $W_\varepsilon \in N_1 \cap N_2$, and as $N_1, N_2 \prec M^*$, M^* has Skolem functions, clearly $N_1 \cap N_2 \prec M^*$, so W_ε is an unbounded subset of $N_1 \cap N_2 \cap \mu^+$. So in N_ℓ, W_ε is unbounded in $\delta_\ell^* = \mathrm{Min}[(\mu^+ \cap N_\ell) \setminus (N_1 \cap N_2)]$, hence $N_\ell \models$ "$\delta_\ell^* \in W_\varepsilon$" hence $\delta_\ell^* \in W_\varepsilon$.

We can conclude that $\delta_1^*, \delta_2^*, \beta$ satisfy the requirements (A)-(E) of $(*)_6$ on δ_1, δ_2, ξ. Hence by $(*)_7$ we have $\delta_1 = \delta_1^*, \delta_2 = \delta_2^*$. But, $\zeta \in N^{\mathbf{p}(\varepsilon)}_{\delta_\ell,\zeta} \subseteq N_\ell$ hence $\zeta < \mu \cap N_1 \cap N_2$ hence $\zeta < \alpha$, so clause $(*)_8$ contradicts the choice of ζ, so we get a contradiction, thus finishing the proof of the theorem. $\square$

Conclusion 2.9 *The condition "have least upper bound" cannot be omitted in*[3] *[9]. That is:*

$\boxplus$ *There are $\mathbb{Q}$ and $\mathscr{I}_\alpha (\alpha < \mu^+)$ such that:*

- *(a) $\mathbb{Q}$ is a forcing notion, $(< \mu)$-complete; in fact every $\le_{\mathbb{Q}}$-increasing sequence of length $< \mu$ has a lub, i.e., satisfies $(1)_a$;*
- *(b) $\mathbb{Q}$ satisfies $(2)_b$, equivalently $*^1_{\mu,\mathbb{Q}}(b)$; cf. Claim 2.6;*
- *(c) each $\mathscr{I}_\alpha$ is a dense open subset of $\mathbb{Q}$;*
- *(d) no directed $\mathbf{G} \subseteq \mathbb{Q}$ meets every $\mathscr{I}_\alpha, \alpha < \mu^+$.*

Proof. Let $\kappa = 2$ and $\bar{C}$ be an S-club system. If μ is a successor or just not strongly inaccessible, choose $\bar{\mathbf{f}}$ and $\bar{\mathscr{I}} = \langle \mathscr{I}_\delta, \mathscr{I}_{\delta,i} : \delta \in S, i < \mu\rangle$ as in Claims 2.7 & 2.5(2) resp., so $\mathbb{Q} = \mathbb{Q}_{(\mathscr{W},\bar{\mathbf{f}},\bar{C})}$ from Definition 2.3(2). So $\mathbb{Q}$ satisfies clause (a) by Claim 2.5(1), satisfies clause (b) by Claim 2.6 and satisfies clauses (c),(d) by the choice of $\bar{\mathbf{f}}$ and $\bar{\mathscr{I}}$. We are left with the case μ is strongly inaccessible, then we use Theorem 2.7 for the case $\kappa = \mu$ instead of the case $\kappa = 2$. $\square$

In Conclusion 2.9 above we get a failure when we waive in [9] the "well met condition".

Conclusion 2.10 *In Conclusion 2.9, we may replace (a) by $(a)'$ and add (e) where:*

- *$(a)'$ $\mathbb{Q}$ is a forcing notion strategically $(< \mu)$-complete (i.e., $(1)_c$), in fact some partial order $\le_{\rm st}$ witnesses it in a strong way (i.e., $(1)_c^+$),*
- *(e) (well met) $(3)_a$ holds, that is if $p, q \in \mathbb{Q}$ are compatible then they have a lub, (so in clause (a)' above we get $(2)_a$).*

Proof. We use a variant of the forcing from Definition 2.3(2) but in clause (A)(c) there we demand $h_p(\delta)$ has a last element (so is closed) and we repeat the proof for Definition 2.4. Actually similarly to the proof of Conclusion 2.9; cf. 3.1 in particular. In details, this forcing notion satisfies clause $(a)'$ by Claim 3.8(1),(2) below; clause (b), i.e., $(2)_b$, by Claim 3.8(5) below. As for clauses (c),(d) we choose $\bar{\mathbf{f}}$ by Theorem 2.7. $\square$

Remark 2.11 (1) In Claims 2.6 & 2.5 we can moreover find $\langle \mathscr{I}_\varepsilon : \varepsilon < \mu\rangle$ such that $\mathscr{I} = \bigcup\limits_{\varepsilon<\mu} \mathscr{I}_\varepsilon \subseteq \mathbb{Q}$ is dense and $p, q \in \mathscr{I}_\varepsilon \Longrightarrow p, q$ are compatible (as in [4]).

Why? Let $\mathscr{I} = \{p \in \mathbb{Q}$: if $\alpha_1 < \alpha_2$ belongs to v_p then the set $h_p(\alpha_1)$ has a last member and there is an $\alpha \in C_{\alpha_2} \backslash \alpha_1$ such that $\mathrm{otp}(\alpha \cap C_{\alpha_2}) \in h_p(\alpha_2)\}$. By Claim 2.5(2) we have $\mathscr{I}$ is a dense subset of $\mathbb{Q}$.

For $p \in \mathscr{I}$ let

1. $u_p = \{\alpha : \alpha \in v_p$ *or* for some $\beta \in v_p$ we have $\alpha \in C_\beta$ and $\mathrm{otp}(\alpha \cap C_\beta) \le \max(h_p(\beta))$ (implied by $\mathrm{otp}(\alpha \cap C_\beta) \in h_p(\beta)$ for some $\beta \in v_p$)$\}$;
2. $\mathbf{E}_1 = \{(p_1, p_2) : p_1, p_2 \in \mathscr{I}$ and $\mathrm{otp}(u_{p_1}) = \mathrm{otp}(u_{p_2})$ and the order preserving function g from u_{p_1} onto u_{p_2} maps v_{p_1} onto v_{p_2}, $C_\alpha \cap u_{p_1}$ onto $C_{h(\alpha)} \cap u_{p_2}$ for $\alpha \in v_p$ and maps $h_{p_1}(\alpha)$ to $h_{p_2}(g(\alpha))$ for $\alpha \in v_p\}$.

So $\mathbf{E}_1$ is an equivalence relation on $\mathscr{I}$ with $\le \mu$ classes: it is known that there is an equivalence relation $\mathbf{E}_2$ on $[\mu^+]^{<\mu}$ with μ equivalence classes such that $u_1 \mathbf{E}_2 u_2 \Longrightarrow u_1 \cap u_2 \trianglelefteq u_\ell$.

Easily the equivalence relation $\{(p_1, p_2) : p_1 \mathbf{E}_1 p_2$ and $u_{p_1} \mathbf{E}_2 u_{p_2}\}$ on $\mathscr{I}$ is as required.

[Why? Assume $p_1 \mathbf{E}_2 p_2$ and $\alpha_\ell \in v_{p_\ell}$ and $\alpha_2 \in v_{p_2}, \gamma \in C_{\alpha_1} \cap C_{\alpha_2}$ and $\mathrm{otp}(\gamma \cap C_{\alpha_\ell}) \in h_{p_\ell}(\alpha_\ell)$ for $\ell = 1, 2$. But then $\gamma \in u_{p_1} \cap u_{p_2}$ and $\gamma \in \mathrm{dom}(g_{p_1}) \cap \mathrm{dom}(g_{p_2})$, hence necessarily $\mathrm{otp}(\gamma \cap C_{\alpha_1}) = \mathrm{otp}(\gamma \cap C_{\alpha_2})$ and $g_{p_1}(\gamma) = g_{p_2}(\gamma)$. Let $v = v_{p_1} \cup v_{p_2}$ and choose $\langle \gamma_\alpha : \alpha \in v\rangle$ such that $\gamma_\alpha \in C_\alpha$ and $\delta \in v \Longrightarrow \gamma_\alpha > sup(C_\delta \cap v)$. Define $p \in \mathbb{Q}$ by:

$(*)_8$ (a) $v_p = v$;

(b) $u_p = u_{p_1} \cup u_{p_2} \cup \{\gamma_\alpha : \alpha \in v\}$;

[3] And in the related works.

(c) $h_p(\alpha) = h_{p_\ell}(\alpha) \cup \{\gamma_\alpha\}$, when $\alpha \in v_{p_\ell}$;
(d) $g_p = g_{p_1} \cup g_{p_2} \cup \{(\gamma_\alpha, \mathbf{f}(\gamma_\alpha) : \alpha \in v\}$.

We can easily check that p is well defined (that is in clause (c) if $\alpha \in v_{p_1} \cup v_{p_2}$ then the two definitions agree; similarly in clause (d).]

(2) Note that for the forcing notion $\mathbb{Q}$ from Conclusion 2.10, every $\leq_{\mathbb{Q}}$-increasing continuous sequence of length $< \mu$ has a lub.

3 Forcing axiom: non-equivalence

We use Definitions 1.2 & 1.3 freely; this section is dedicated to proving the following theorem:

Theorem 3.1 *Assume $\vartheta + \aleph_0 < \mu = \mu^{<\mu}$ and $2 \leq \vartheta < \mu$ and $\mathbb{Q}$ is adding μ^+ many μ-Cohen. Then in $\mathbf{V}^{\mathbb{Q}}$ we have:*

$\boxplus_{\mu,\varepsilon}$ *For some $\mathbb{P}$*
(a) (α) $\mathbb{P}$ is a forcing notion;
(β) $\mathbb{P}$ satisfies $(2)^\varepsilon_c$ from Definition 1.3;
(γ) $\mathbb{P}$ has cardinality μ^+;
(δ) $\mathbb{P}$ is strategically μ-complete (i.e., satisfies $(1)_{c,\mu}$ or even $(1)^+_c$);
(ε) we have $(2)^+_{a,\mu}$;
(ζ) if $p, q \in \mathbb{P}$ are compatible then they have a lub, i.e., $(3)_a$ holds;
(η) $(2)^\varepsilon_c$ holds for every limit $\varepsilon < \mu$;
(b) (α) $\mathbb{P}$ is not equivalent to any forcing notion satisfying $(1)_c + (2)^+_{a,\vartheta(+)}$;
(β) moreover, there is a sequence $\bar{\mathscr{I}} = \langle \mathscr{I}_\alpha : \alpha < \mu^+\rangle$ of dense open subsets of $\mathbb{P}$ such that: if $\mathbb{R}$ is a forcing notion satisfying the conditions from $(b)(\alpha)$ above, then $\Vdash_{\mathbb{R}}$ "there is no directed $\mathbf{G} \subseteq \mathbb{P}$ which meets $\mathscr{I}_\alpha$ for $\alpha < \mu^+$".

Remark 3.2 Hence the relevant forcing axioms are not equivalent!

P r o o f. By Claims 3.8, 3.12 & 3.13 below.
In details: Let $\bar{\mathbf{f}}$ be from Claim 3.12(1), (i.e., after the preliminary forcing $\mathbb{Q}$, in $\mathbf{V}^{\mathbb{Q}}$) and $\mathbb{P} = \mathbb{P}_{\bar{\mathbf{f}},\vartheta}$, as defined in Definition 3.6.
Clause $(a)(\alpha)$ $\mathbb{P}$ a forcing notion, holds by Definition 3.6, i.e., the first statement of Claim 3.8(1).
Clause $(a)(\beta)$, i.e., for every limit ordinal $\varepsilon < \mu$ the statement $(2)^\varepsilon_c$ holds by Claim 3.8(5)
Clause $(a)(\gamma)$, "$\mathbb{P}$ of cardinality μ^+", holds by Claim 3.8(1).
Clause $(a)(\delta)$, $(1)^+_c$ and so $\mathbb{P}$ is strategically μ-complete, by Claim 3.8(1),(2);
Clause $(a)(\varepsilon)$, means $(2)^+_a$ which holds by Claim 3.8(6).
Clause $(a)(\zeta)$, "if p, q are compatible then they have a lub", holds by Claim 3.8(3).
Clause $(b)(\alpha)$, "$\mathbb{P}$ not equivalent to a forcing satisfying $(1)_b + (2)^+_{b,\vartheta}$" holds, by Clause $(b)(\beta)$.
Clause $(b)(\beta)$, "$\mathbb{R}$ satisfies $(1)_b + (2)^+_{a,\vartheta(+)}$", this holds by Claim 3.13(2) because it assumption holds by Claim 3.12. $\square$

Conclusion 3.3 *If $\vartheta = \mathrm{cf}(\vartheta) < \mu = \mu^{<\mu}$ then $\mathrm{Ax}_\mu((1)_c + (2)^+_{a,\vartheta}))$ does not imply $\mathrm{Ax}^\vartheta_\mu$ and even $\mathrm{Ax}_{\mu^{++},\mu}((1)_c + (2)^\vartheta_c)$ from Definition 1.5(3).*

P r o o f. Let $\lambda = \lambda^{<\lambda}$, $\mathbb{Q}$, $\mathbb{P}$ as in Theorem 3.1(b)(α) and $\mathbf{V}_1 = \mathbf{V}^{\mathbb{Q}}$. In $\mathbf{V}_1$ we can find a forcing notion $\mathbb{R}$ which forces $\mathrm{Ax}_\mu((1)_c + (2)^+_{a,\vartheta(+)})$ and satisfies those conditions, we know such $\mathbb{R}$ exists because $(< \mu)$-support iterations preserve the property $(1)_c + (2)^+_{a,\vartheta(+)}$); cf. 1.13. Now also in the universe $\mathbf{V}^{\mathbb{R}}_1$ the forcing notion $\mathbb{P}$ satisfies the conditions in $\mathrm{Ax}^\vartheta_\mu$ from Definition 1.5.

So by clause $(b)(\beta)$ of Theorem 3.1, in $\mathbf{V}^{\mathbb{R}}_1$ the axiom $\mathrm{Ax}^\vartheta_\mu$ fail as exemplified by $\mathbb{P}$ because of Hypothesis 4.1(a), so we are done proving the conclusion. $\square$

For this section (clearly if $\mu = \mu^{<\mu} > \aleph_0$ then there are such objects)

Hypothesis 3.4

1. $\mu = \mu^{<\mu} > \vartheta \geq 2$ and $\mu > \aleph_0$
2. $S = S^{\mu^+}_\mu = \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mu\}$ or S just a stationary subset of $S^{\mu^+}_\mu$.
3. $\bar{C}$ is an S-club sytem; cf. Definition 2.2.
4. $\bar{\mathbf{f}}$ is as in Definition 3.6 but $\mathbf{f}_\delta : C_\delta \to \vartheta$.

Discussion 3.5

1. A major difference between the forcing in Definition 3.6 below and the one in Definition 2.3(2) above is that:
   (A) there the generic gives a function $\underset{\sim}{g}$ from λ to κ such that for every $\delta \in S$ for "most" $\alpha \in C_\delta$ we have $\underset{\sim}{g}(\alpha) = \mathbf{f}_\delta(\alpha)$;
   (B) here the generic gives a function $\underset{\sim}{g}$ such that for every $\delta \in S$ for "most" $\alpha \in C_\delta$ we have $\mathbf{f}_\delta(\alpha) \in \underset{\sim}{g}(\alpha)$.
2. See more in Remark 3.7(2).
3. Also here g_p is part of the condition instead being defined, a minor change.
4. In addition $h_p(\delta)$ is here a subset of C_δ instead of a subset of μ.

Definition 3.6 For $\bar{\mathbf{f}}$ an $(S, \bar{C}, \vartheta)$-parameter (cf. Definition 2.3), we define a forcing notion $\mathbb{P} = \mathbb{P}_{\bar{\mathbf{f}},\vartheta}$ as follows (but abusing our notation we may omit ϑ):

(A) $p \in \mathbb{P}$ iff p consists of (so $u_p = u$, etc.):
  (a) $u \in [\mu^+]^{<\mu}$;
  (b) $g : u \to [\mu]^{<\vartheta}$ (one can use $g : u \to \vartheta$ when $\vartheta = \mathrm{cf}(\vartheta) \geq \aleph_0$ because $\bigwedge\limits_\delta \mathrm{Rang}(\mathbf{f}_\delta) \subseteq \vartheta$);
  (c) $v \subseteq S$ of cardinality $< \mu$;
  (d) h a function with domain v;
  (e) if $\delta \in v$ then
  $(\alpha)h(\delta)$ is a closed bounded non-empty subset of C_δ;
  $(\beta)h(\delta) \subseteq u$;
  (γ)if $\beta \in h(\delta)$ then $\beta \in u$ and $\mathbf{f}_\delta(\beta) \in g(\beta)$.

(B) $p \leq q$, i.e., $\mathbb{P}_{\bar{\mathbf{f}}} \models$ "$p \leq q$" iff
  (a) $u_p \subseteq u_q$ and $g_p \subseteq g_q$;
  (b) $v_p \subseteq v_q$;
  (c) if $\delta \in v_p$ then $h_p(\delta)$ is an initial segment of $h_q(\delta)$;n
  (d) if $\delta \in v_p$ and $\alpha \in h_q(\delta) \setminus h_p(\delta)$ (hence $h_q(\delta) \neq h_p(\delta)$), then $u_p \cap C_\delta \subseteq \alpha$;

(C) we define $<_{\mathrm{st}} = <^{\mathbb{P}}_{\mathrm{st}}$, the strong order by: $p <_{\mathrm{st}} q$ iff $p \leq q$ and
  (e) if $\delta \in v_p$ and $h_p(\delta) \neq h_q(\delta)$ then $\sup(h_q(\delta)) > \sup(\cup\{\delta \cap C_\gamma : \gamma \in v_p \backslash \{\delta\}\})$;

(D) let $\underset{\sim}{g} = \{g_p : p \in \underset{\sim}{\mathbf{G}}\}$ and $\underset{\sim}{h} = \{h_p : p \in \underset{\sim}{\mathbf{G}}\}$.

Remark 3.7

1. In Definition 3.6 we may choose $\bar{\mathbf{f}}$ such that $\mathbf{f}_\delta$ is a function to $\kappa = \mu$ instead of to $\kappa = \vartheta$ the forcing is defined similarly. It has similar properties but it seems that the case $\kappa = \vartheta$ is enough for us.
2. If in clause $(A)(e)(\alpha)$ of Definition 3.6 we would have demanded only "$h(\delta)$ is only closed in its supremum but if $\alpha = \sup(h(\delta)) \notin h(\delta)$ then $\{\mathbf{f}_\delta(\alpha) : \delta \in v,\ \alpha \in C_\delta\}$" then we get an equivalent forcing, we lose some nice properties but gain others. Mainly we gain in having more cases of having a lub, in particular for an increasing sequence which has an upper bound, really any set of $< \mathrm{cf}(\vartheta)$ members which has an upper bound; but we lose for Δ-systems, i.e., Claim 3.8(6). Also we have to be more careful in Claim 3.9. We shall use the "closed in its supremum" version also in § 4.

Claim 3.8 *Let $\bar{\mathbf{f}}$ be an $(S, \bar{C}, \vartheta)$-parameter as in 2.1, so S is a stationary subset of $S^{\mu^+}_{\mu}$.*

1. *$\mathbb{P}_{\bar{\mathbf{f}}}$ is a forcing notion of cardinality μ^+, also $<_{\rm st}$ is a partial order $\subseteq <_{\mathbb{P}}$ and $p_1 \le p_2 <_{\rm st} p_3 \le p_4 \Longrightarrow p_1 <_{\rm st} p_4$ and $(\forall p)(\exists q)(p <_{\rm st} q)$.*
2. *Any $<_{\rm st}$-increasing sequence in $\mathbb{P}_{\bar{\mathbf{f}}}$ of length $< \mu$ has an upper bound (this is a strong/no memory version of strategic μ-completeness), i.e., $<_{\rm st}$ exemplifies $(1)^+_c$.*
3. *If $p_1, p_2 \in \mathbb{P}_{\bar{\mathbf{f}}}$ are compatible then they have a lub.*
4. *The set $\{p_i : i < i(*)\}$ has a $\le$-lub in $\mathbb{P}_{\bar{\mathbf{f}}}$ when $\bigwedge_{i,j<i(*)}$ (p_i, p_j are compatible) and $i(*)$ is finite or $i(*) < \mu$ and for every δ, the set $\{h_{p_i}(\delta) : i < i(*)$ satisfies $\delta \in v_{p_i}\}$ is finite or at least has a maximal member. Note this set is linearly ordered by being an initial segment.*

4A. *The set $\{p_i : i < i(*)\}$ has an ub when $i(*) < \mu$ and $\{p_i : i < i(*)\}$ is a set of pairwise compatible members of $\mathbb{P}_{\bar{\mathbf{f}}}$ and $i(*)$ is finite or $i(*) < \vartheta$ or at least $i(*) < \mu$ and for every limit ordinal α the following set has cardinality $< \vartheta$:*
   (a) $\{\delta \in \bigcup_i v_{p_i} : \alpha = \sup\{h_{p_i}(\delta) + 1 : i < i(*)$ and $\delta \in v_{p_i}\}\}$.

5. *The forcing notion $\mathbb{P}_{\bar{\mathbf{f}}}$ satisfies $(2)^{\varepsilon}_c$ for $\varepsilon < \mu$.*
6. *$\mathbb{P}_{\bar{\mathbf{f}}}$ satisfies clauses $(2)_a, (2)^+_{a,\partial}$ of Definition 1.2 when $\partial \le \mu$.*

P r o o f.

1. Recall that $\mu = \mu^{<\mu}$ hence $\mu^+ = (\mu^+)^{<\mu}$ and easily $|\mathbb{P}| = \mu^+$. Also the statements on $<_{\rm st}$ are obvious. What about $\mathbb{P}_{\bar{\mathbf{f}}}$ being a quasi order? Assume that $p_1 \le p_2 \le p_3$ and we shall prove that $p_1 \le p_3$; clauses (a), (b), (c) of Definition 3.6(B) are immediate and we shall elaborate on clause (d). So assume $\delta \in v_{p_1}$ and $\alpha \in h_{p_3}(\delta) \setminus h_{p_1}(\delta)$ and we should prove that $u_{p_1} \cap h_{p_1}(\delta) \subseteq \alpha$. First assume $\alpha \in h_{p_2}(\delta)$, then $p_1 \le p_2$ implies $u_{p_1} \cap C_\delta \subseteq \alpha$ as required. Second assume $\alpha \notin h_{p_2}(\delta)$ then $p_2 \le p_3$ implies $u_{p_2} \cap h_{p_2}(\delta) \subseteq \alpha$ but $u_{p_1} \subseteq u_{p_2}$ so we are done.
2. Let $\gamma < \mu$ be a limit ordinal and $\bar{p} = \langle p_i : i < \gamma\rangle$ be a $<_{\rm st}$-increasing sequence of members of $\mathbb{P}_{\bar{\mathbf{f}}}$.
   Let
   $(*)_1$ (a) $v_* = \bigcup_i \{v_{p_i} : i < \gamma\}$;
   (b) let $\mathbf{i} : v_* \to \gamma$ be $\mathbf{i}(\delta) = \min\{i < \gamma : \delta \in v_{p_i}\}$;
   (c) let $v^*_2 = \{\delta \in v_*$: the sequence $\langle h_{p_i}(\delta) : i \in [\mathbf{i}(\delta), \gamma)\rangle$ is not eventually constant$\}$;
   (d) for $\delta \in v^*_2$ let $\zeta_\delta = \sup(\cup\{h_{p_i}(\delta) : i \in [\mathbf{i}(\delta), \gamma)\})$;
   (e) let $v^*_1 = v_* \backslash v^*_2$.
   We try naturally to define $p = (u_p, v_p, g_p, h_p)$ almost as $\bigcup_{i<\gamma} p_i$, i.e.,
   $(*)_2$ (a) $v_p = v_* := \cup\{v_{p_i} : i < \gamma\}$;
   (b) $u_p = \cup\{u_{p_i} : i < \gamma\} \cup \{\zeta_\delta : \delta \in v^*_2\}$;
   (c) $g_p = \cup\{g_{p_i} : i < \gamma\} \cup \{\langle \zeta_\delta, \{\mathbf{f}_\delta(\zeta_\delta)\}\rangle : \delta \in v^*_2\}$;
   (d) h_p is a function with domain v_p such that
   (α) if $\delta \in v^*_1$ then $h_p(\delta) = p_i(\delta)$ for $i < \delta$ large enough;
   (β) if $\delta \in v^*_2$ then $h_p(\delta) = \cup\{h_{p_i}(\delta) : i \in [\mathbf{i}(\delta), \gamma)\} \cup \{\zeta_\delta\}$.
   The point is to check that $p \in \mathbb{P}$, because $i < \gamma \Longrightarrow p_i \le p$ is immediate:
   i. $u_p \in [\mu^+]^{<\mu}$ because $u_{p_i} \in [\mu^+]^{<\mu}$ and $\gamma < \mu = {\rm cf}(\mu)$ and $|v^*_2| \le \Sigma\{|v_{p_i}| : i < \gamma\} < \mu$;
   ii. $v_p \in [S]^{<\mu}$ because $v_{p_i} \in [S]^{<\mu}$ and $\gamma < \mu = {\rm cf}(\mu)$;
   iii. h_p is a function with domain v_p such that $\delta \in v_p \Longrightarrow h_p(\delta)$ is a bounded closed subset of C_δ (check the two cases);
   iv. g_p is a function from u_p to ϑ as each g_{p_i} is a function from u_{p_i} to λ and $\bar{p}$ is $<_{\rm st}$-increasing and:
   $(*)$ if $\delta \in v^*_2$ then $\zeta_\delta \notin \bigcup_i u_{p_i}$.
   [Why? This holds by Definition 3.6(B)(d) applied to $p_i \le p_j$ for $i < j < \gamma$.]
   $(**)$ if $\delta_1 \ne \delta_2 \in v^*_2$ then $\zeta_{\delta_1} \ne \zeta_{\delta_2}$ and $\zeta_{\delta_1} \ne C_{\delta_2}$.
   [Why? Cf. Definition 3.6(C)(e)].

3. Assume $p_1, p_2 \in \mathbb{P}$ have a common upper bound.
   $(*)_1$ We define $p \in \mathbb{P}$ as follows:
   (a) $v_p = v_{p_1} \cup v_{p_2}$;
   (b) $u_p = u_{p_1} \cup u_{p_2}$;
   (c) $g_p = g_{p_1} \cup g_{p_2}$;
   (d) h_p is the function with domain v_p and for $\delta \in v_p$ we have
   i. if $\delta \in v_{p_1} \setminus v_{p_2}$ then $h_p(\delta) = h_{p_1}(\delta)$;
   ii. if $\delta \in v_{p_2} \setminus v_{p_1}$ then $h_p(\delta) = h_{p_2}(\delta)$;
   iii. if $\delta \in v_{p_1} \cap v_{p_2}$ then $h_p(\delta) = h_{p_1}(\delta) \cup h_{p_2}(\delta)$.

   Now indeed

   $(*)_2$ $p \in \mathbb{P}$.

   Also

   $(*)_3$ $p_\ell \leq p$ for $\ell = 1, 2$.

   [Why? E.g., for Definition 3.6(B)(d), let $\delta \in v_p$ and $\alpha \in h_p(\delta) \setminus h_{p_\ell}(\delta)$. By the choice of p, necessarily $\alpha \in h_{p_{3-\ell}}(\delta) \setminus h_{p_\ell}(\delta)$. Let q be a common upper bound of p_1, p_2, exist by our present assumption; so clearly $\alpha \in h_q(\delta) \setminus h_{p_\ell}(\delta)$ hence $u_{p_\ell} \cap C_\delta \subseteq \alpha$ as promised.]

   $(*)_4$ if q is a common upper bound of p_1, p_2 then $p \leq q$.

   Why? E.g., for Definition 3.6(B)(d), assume $\delta \in v_p$ and $\alpha \in h_q(\delta) \setminus h_p(\delta)$ we should prove that $u_p \cap C_\delta \subseteq \alpha$. Now for $\ell = 1, 2$ we have $p_\ell \leq q$, $\delta \in v_{p_\ell}$ and $\alpha \in h_q(\delta)) \setminus h_{p_\ell}(\delta))$ hence $u_{p_\ell} \cap C_\delta \subseteq \alpha$. So clearly

   $$u_p \cap C_\delta = (u_{p_1} \cup u_{p_2}) = (u_{p_1} \cap C_\delta) \cup (u_{p_2} \cap C_\delta) \subseteq \alpha.$$

   So we are done.
4. The proof is similar.

4A. Similar to the proof of part (2).

5. The statement $(2)^\varepsilon_c$ holds by parts (2) & (3).
6. For $(2)_a$ by the proof of Claim 2.6, i.e., defining **h** as there, recalling part (3).
   For $(2)_{a,\partial}$ for $\partial \leq \mu$ choose **h** as above, using part (4) instead of part (3). □

Claim 3.9

1. $\mathscr{I}_{\bar{\mathbf{f}},\alpha}$ is a dense open subset of $\mathbb{P}_{\bar{\mathbf{f}}}$ where:
(a) $\mathscr{I}_{\bar{\mathbf{f}},\alpha} = \{p \in \mathbb{P}_{\bar{\mathbf{f}}} : \alpha \in u_p$ and $\alpha \in S \Longrightarrow \alpha \in v_p\}$.

2. If $\delta \in S$ and $\alpha \in C_\delta$ then $\mathscr{I}_{\delta,\alpha}$ is a dense open subset of $\mathbb{P}_{\bar{\mathbf{f}}}$ where: $\mathscr{I}_{\delta,\alpha} = \{p \in \mathbb{P}_{\bar{\mathbf{f}}} : \delta \in v_p$ and $h_p(\delta) \nsubseteq \alpha\}$.

Proof.

1. Assume $p \in \mathbb{P}_{\bar{\mathbf{f}}}$ and we shall find $q \in \mathscr{I}_{\bar{\mathbf{f}},\alpha}$ such that $p \leq q$. Note that α is fixed.
   Case 1. If $(\alpha \notin S \vee \alpha \in v_p)$ and $\alpha \in u_p$: Let $q = p$.
   Case 2. If $(\alpha \notin S \vee \alpha \in v_p)$ and $\alpha \notin u_p$: Define q by:
   (a) $u_q = u_p \cup \{\alpha\}$;
   (b) $v_q = v_p$;
   (c) $g_q = g_p \cup \{\langle \alpha, \{0\}\rangle\}$;
   (d) $h_q = h_p$.

   Now check that $q \in \mathbb{P} \wedge \alpha \in u_q$. Also $p \leq q$ is clear, e.g., Definition 3.6(B)(d) holds because $\delta \in v_p \Longrightarrow h_p(\delta) = h_q(\delta)$.
   Case 3. $\alpha \in S$ and for simplicity $\alpha \notin v_p$: Let $\beta \in C_\alpha$ be such that $\delta \in v_p \setminus \{\alpha\} \Longrightarrow \beta > \sup(C_\delta \cap \alpha)$ and $\sup(u_p \cap \alpha) < \beta$ and define $q \in \mathbb{P}_{\bar{\mathbf{f}}}$ by:
   i. $u_q = u_p \cup \{\beta\}$,
   ii. $v_q = v_p \cup \{\alpha\}$,
   iii. $g_q = g_p \cup \{(\beta, \{\mathbf{f}_\alpha(\beta)\})\}$,

iv. for $\delta \in v_q$ we define $h_q(\delta)$ as:
(a) $h_p(\delta)$ when $\delta \neq \alpha$,
(b) $\{\beta\}$ when $\delta = \alpha \notin v_q$,
(c) $h_p(\delta) \cup \{\beta\}$ when $\delta = \alpha \in v_p$.

Clearly $p \leq q \in \mathscr{I}_{\bar{\mathbf{f}},\alpha}$.

2. Similarly. □

Definition 3.10

1. We say that $\bar{\mathbf{f}}$ is (κ, ∂)-generic enough when $(A) \Longrightarrow (B)$ and recall $\bar{\mathbf{f}} = \langle \mathbf{f}_\delta : \delta \in S\rangle, \mathbf{f}_\delta : C_\delta \to \vartheta$ where ∂ is a regular cardinality $< \mu$ and $\kappa \in [\vartheta, \mu)$ (and recall ϑ is a cardinal $[2, \mu)$ and $\langle \alpha_{\delta,i} : i < \mu\rangle$ list C_δ in increasing order):
   (A) (a) E is a club of μ^+;
   (b) $\langle \alpha_{\delta,\zeta} : \zeta < \mu\rangle$ is an increasing continuous sequence of the members of C_δ for $\delta \in E \cap S$;
   (c) h_ζ is a pressing down function from $E \cap S$ for $\zeta < \mu$;
   (B) we can find $\xi < \mu$ of cofinality ∂ and a sequence $\langle \delta_i : i < \kappa\rangle$ of ordinals from $E \cap S$ such that:
   i. if $\zeta < \xi$ then $h_\zeta \restriction \{\delta_i : i < \kappa\}$ is constant;
   ii. $\langle \alpha_{\delta_i,\zeta} : \zeta < \xi\rangle$ does not depend on $i < \kappa$ hence also $\alpha = \alpha_{\delta_i,\xi}$ by continuity;
   iii. the set $\{\mathbf{f}_{\delta_i}(\alpha) : i < \kappa\}$ is equal to ϑ where α is from ii.
2. We say that $\bar{\mathbf{f}}$ is weakly (κ, ∂)-generic enough when as above except that in (B)iii. we demand just that the set has cardinality ϑ.

Remark 3.11

1. This is used when we demand that any $< \vartheta$ has an ub inside the proof of Claim 3.13.
2. For $\vartheta = 2$ as Claim 3.8(2) does not apply, we shall in Claim 3.13 need a stronger version—with the game; cf. § 4.
3. In Definition 3.10 we may add:
   iv. $\{\alpha \in C_{\delta_i} : \alpha < \alpha_{\delta_i,\zeta}\}$ for some $\zeta < \xi$ does not depend on i;
   v. the $\mathbf{f}_{\delta_i}$'s agree on this set.

Now in Claims 3.12 & 3.13 we shall arrive at the main point.

Claim 3.12

1. *For ∂ as in Definition 3.10 assume $\mathbb{Q}$ is the forcing notion for adding μ^+ many μ-Cohens. Then in $\mathbf{V}^{\mathbb{Q}}$, there is an $(S, \bar{C}, \mu)$-parameter $\bar{\mathbf{f}}$ which is (κ, ∂)-generic enough (in the sense of Definition 3.10) for our cardinals $\vartheta \in [2, \mu)$ and regular $\partial \in [\aleph_0, \mu)$;*
2. *If $\diamondsuit_S$ then there is $\bar{\mathbf{f}}$ as above.*

Proof.

1. Now (modulo equivalence, so without loss of generality) $\mathbb{Q}$ can be described as follows:
   $(*)_1$ (a) $p \in \mathbb{Q}$ iff p is a function, $\mathrm{dom}(p) \in [S]^{<\mu}$ and for every $\delta \in \mathrm{dom}(p)$, $p(\delta)$ is a function from some strict initial segment of C_δ into ϑ recalling $C_\delta \subseteq \delta$ is a club of δ of order type μ;
   (b) $\mathbb{Q} \models$ "$p \leq q$" iff $\alpha \in \mathrm{dom}(p) \Longrightarrow (\alpha \in \mathrm{dom}(q)) \wedge (p(\alpha) \trianglelefteq q(\alpha))$;
   (c) let $\underset{\sim}{\mathbf{f}}_\delta$ for $\delta \in S$ be $\cup\{p(\delta) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}$ satisfies $\delta \in \mathrm{dom}(p)\}$.

   It suffices to prove $\Vdash_{\mathbb{Q}}$ "$\langle \underset{\sim}{\mathbf{f}}_\delta : \delta \in S\rangle$ is as required".

   So assume

   $(*)_2$ $p_* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{h}_\zeta$ is a pressing down function on S for $\zeta < \mu$ and $\langle \underset{\sim}{\alpha}_{\delta,\zeta} : \zeta < \mu\rangle$ is increasing continuous sequence of members of C_δ for $\delta \in S$".

   It suffices to find a condition q above p_* forcing that there are $\langle \delta_i : i < \kappa\rangle$ and ξ as in clause (B) of Definition 3.10. For each $\delta \in S$ we choose $(p_{\delta,\varepsilon}, \xi_{\delta,\varepsilon}, \bar{\alpha}_{\delta,\varepsilon})$ by induction on $\varepsilon < \partial$ such that:

   $(*)^3_{\delta,\varepsilon}$ (a) $p_{\delta,\varepsilon} \in \mathbb{Q}$ is above p_*;

Sh:1036

(b) $\varepsilon(1) < \varepsilon \Longrightarrow p_{\delta,\varepsilon(1)} \leq_{\mathbb{Q}} p_{\delta,\varepsilon}$;
(c) $\delta \in \mathrm{dom}(p_{\delta,\varepsilon})$;
(d) $\xi_{\delta,\varepsilon} = \mathrm{otp}(\mathrm{dom}(p_{\delta,\varepsilon}(\delta)))$;
(e) if $\varepsilon = \varepsilon(1) + 1$ then
i. $p_{\delta,\varepsilon}$ forces a value $h^*_\zeta(\delta)$ to $\underset{\sim}{h}_\zeta(\delta)$ for $\zeta < \xi_{\delta,\varepsilon(1)}$;
ii. $p_{\delta,\varepsilon}$ forces a value $\bar{\alpha}_{\delta,\varepsilon(1)}$ to $\langle \underset{\sim}{\alpha}_{\delta,\zeta} : \zeta \leq \xi_{\delta,\varepsilon(1)} + 1\rangle$;
iii. $\xi_{\delta,\varepsilon} > \xi_{\delta,\varepsilon(1)}$ and $\mathrm{rang}(\bar{\alpha}_{\delta,\varepsilon(1)}) \subseteq \mathrm{dom}(p(\delta))$.

There is no problem to carry out the induction. Let $\xi_\delta = \cup\{\xi_{\delta,\varepsilon} : \varepsilon < \partial\} < \mu$, $\alpha^*_\delta = \sup\{\mathrm{dom}(p_{\delta,\varepsilon}(\delta)) : \varepsilon < \partial\}$, $p_\delta = \cup\{p_{\delta,\varepsilon} : \varepsilon < \partial\}$.

Now we can define a pressing down function h on S such that:

$(*)_4$ if $\delta_1, \delta_2 \in S$ and $h(\delta_1) = h(\delta_2)$, $\varepsilon < \partial$ then:
(a) $\bar{\alpha}_{\delta_1,\varepsilon} = \bar{\alpha}_{\delta_2,\varepsilon}$;
(b) for every $\alpha \in \mathrm{Rang}(\bar{\alpha}_{\delta_1,\varepsilon})$ we have
i. $(C_{\delta_1} \cap \alpha) = (C_{\delta_2} \cap \alpha)$,
ii. $p_{\delta_1}(\delta_1){\restriction}(C_{\delta_1} \cap \alpha) = p_{\delta_2}(\delta_2){\restriction}(C_{\delta_2} \cap \alpha)$;
(c) $h^*_\varepsilon(\delta_1) = h^*_\varepsilon(\delta_2)$ so $\xi_{\delta_1} = \xi_{\delta_2}$ and $p_{\delta_1,\varepsilon}{\restriction}\delta_1 = p_{\delta_2,\varepsilon}{\restriction}\delta_2$.

Next choose an increasing sequence $\langle \delta_i : i < \kappa\rangle$ of members of S such that h is constant on $\{\delta_i : i < \kappa\}$ and $i < j \Longrightarrow \mathrm{dom}(p_{\delta_i}) \subseteq \delta_j$.

Define $q \in \mathbb{Q}$:

$(*)_5$ (a) $\mathrm{dom}(q) = \cup\{\mathrm{dom}(p_{\delta_i,\varepsilon} : i < \kappa, \varepsilon < \kappa\}$;
(b) if $i < \kappa$ then $q(\delta_i) = \cup\{p_{\delta_i,\varepsilon}(\delta_i) : \varepsilon < \partial\} \cup \{\langle \alpha^*_\delta, i\rangle\}$ where $j = i$ if $i < \vartheta$ and $j = 0$ otherwise;
(c) if $\delta \in \mathrm{dom}(q) \setminus \{\delta_i : i < \kappa\}$ then $q(\alpha) = \cup\{p_{\delta_i,\varepsilon}(\alpha) : \alpha \in \mathrm{dom}(p_{\delta_i,\varepsilon})\}$.

2. Also easy. □

Claim 3.13

1. *There are dense sets $\mathscr{I}_\alpha \subseteq \mathbb{P} = \mathbb{P}_{\bar{\mathbf{f}}}$ for $\alpha < \mu^+$, such that if $\mathbf{G} \subseteq \mathbb{P}$ is directed and meets every $\mathscr{I}_\alpha$, then $\mathbf{G}$ is ϑ^+-directed and even $(< \mu)$-direccted.*
2. *If $\bar{\mathbf{f}}$ is weakly (ϑ, ∂)-generic enough and the forcing notion $\mathbb{R}$ satisfies $(1)_c + (2)^+_{a,\vartheta(+)}$ (cf. Theorem 1.13) then in $\mathbf{V}^{\mathbb{R}}$ there is no $(< \mu)$-directed $\mathbf{G} \subseteq \mathbb{P} = \mathbb{P}_{\bar{\mathbf{f}}}$ meeting all the sets from Claim 3.9.*
3. *Also there is no such $\mathbb{R}$ satisfying $(2)^\varepsilon_{c,\vartheta,D}$ when $\varepsilon < \mu$ is a limit ordinal*

P r o o f.

1. Let $\mathscr{S} = \{\bar{p} : \bar{p}$ is a directed sequence of conditions in $\mathbb{P}$ of limit length $< \mu\}$. Since $\mu^{<\mu} = \mu$ and $|\mathbb{P}| = \mu^+$ it follows that $|\mathscr{S}| \leq \mu^+$. For each $\bar{p} = \langle p_i : i < i_*\rangle \in \mathscr{S}$, let $\mathscr{I}_{\bar{p}} = \{q \in \mathbb{P} : q$ is either incompatible with p_i for some $i < i_*$ or $p_i \leq q$, for every $i < i_* < \mu\}$. Since $\mathbb{P}$ is μ-strategically complete (by Claim 3.8(1),(2)), the set $\mathscr{I}_{\bar{p}}$ is dense and open. Let $\mathbf{G}$ meet $\mathscr{I}_{\bar{p}}$, for every $\bar{p} \in \mathscr{S}$. Then $\mathbf{G}$ is ϑ^+-directed.
2. Towards contradiction, assume $p_* \Vdash_{\mathbb{R}}$ "$\underset{\sim}{\mathbf{H}} \subseteq \mathbb{P}$ is $(< \mu)$-directed, meeting all the sets from Claim 3.9". Using $(1)_{c,\mu}$, fix a winning strategy **st** for COM, the completeness player in the game $\Game_\mu(p^*, \mathbb{R})$ (cf. Definition 1.11(1)), choose $(E_\zeta, \bar{q}_\zeta, \bar{r}_\zeta, \bar{\mathbf{h}}_\zeta, \bar{p}_\zeta, \bar{\alpha}_\zeta)$ by induction on $\zeta < \mu$ such that:

$(*)$ (a) $\bar{q}_\zeta = \langle q_{\zeta,\delta} : \delta \in E_\zeta\rangle$ and $\bar{r}_\zeta = \langle r_{\zeta,\delta} : \delta \in E_\zeta\rangle$;
(b) $p_* \leq q_{\zeta,\delta} \leq r_{\zeta,\delta}$ are from $\mathbb{R}$;
(c) $\langle (q_{\xi,\delta}, r_{\xi,\delta}) : \xi \leq \zeta\rangle$ is an initial segment of a play of $\Game_\mu(p^*, \mathbb{R})$ in which the player COM uses **st**;
(d) $E_\zeta \subseteq \mu^+$ is a club;
(e) $\mathbf{h}_\zeta$ is a regressive function on $S \cap E_\zeta$;
(f) if $\mathscr{U} \subseteq E_\zeta \cap S$, $|\mathscr{U}| < \vartheta$ and $\mathbf{h}_\zeta{\restriction}\mathscr{U}$ is constant, then $\{r_{\zeta,\delta} : \delta \in \mathscr{U}\}$ has a lub in $\mathbb{R}$;
(g) $\bar{p}_\zeta = \langle p_{\zeta,\delta} : \delta \in E_\zeta\rangle$;
(h) $r_{\zeta,\delta} \Vdash_{\mathbb{R}}$ "$p_{\zeta,\delta} \in \underset{\sim}{\mathbf{H}}$ is above $p_{\xi,\delta}$ for $\xi < \zeta$";
(i) $\bar{\alpha}_\zeta = \langle \alpha_{\delta,\zeta} : \delta \in S \cap E_\zeta\rangle$;
(j) $\alpha_{\delta,\zeta}$ is a member of $h_{p_{\zeta,\delta}}(\delta)$ above $\mathrm{dom}(h_{p_{\xi,\delta}}(\delta))$ for every $\xi < \varepsilon$.

For clauses (e)+(f), we use condition $(2)^+_{a,\vartheta}$.
Since $\bar{\mathbf{f}}$ is (ϑ,ϑ)-generic enough, we can find $\langle \delta_i : i < \vartheta\rangle$ and ξ as in Definition 3.10 and let $\langle \zeta_i : i < \vartheta\rangle$ be increasing with limit ξ.
By clause (f), for each $j < \vartheta$, the set $\{r_{\zeta_j,\delta_i} : i < j\}$ has a lub $r^*_j \in \mathbb{R}$—so necessarily $j_1 < j_2 < \vartheta \Longrightarrow r^*_{j_1} \leq r^*_{j_2}$. Hence the sequence $\langle r^*_j : j < \vartheta\rangle$ has an upper bound r_* (by $(1)_{b,=\vartheta}$). So $r_* \Vdash_{\mathbb{R}} \{p_{\zeta_i,\delta_j} : i < j < \vartheta\} \subseteq \underset{\sim}{\mathbf{H}}$.
As $r_* \Vdash_{\mathbb{R}} \underset{\sim}{\mathbf{H}}$ is $< \vartheta^+$-directed, we can find some $p \in \mathbb{P}$, $r_{**} \geq r_*$ such that $r_{**} \Vdash_{\mathbb{R}} p \in \underset{\sim}{\mathbf{H}}$ is an upper bound for $\{p_{\zeta_i,\delta_j} : i < j < \vartheta\}$.
So, on one hand, $g_p(\alpha_{\delta_0,\xi})$ is a subset of μ of cardinality $< \vartheta$—by the definition of $\mathbb{P}$. On the other hand, $i < \vartheta \Longrightarrow \alpha_{\xi,\delta_i} = \alpha_{\xi,\delta_0}$ and $\mathbf{f}_{\delta_i}(\alpha_{\delta_i,\xi}) \in g_p(\alpha_{\delta_i,\xi})$. But by Definition 3.10(B)iii. this is impossible. $\square$

Conclusion 3.14 *If $\lambda = \lambda^{<\lambda} > \mu = \mu^{<\mu} > \aleph_0$ and $\vartheta \neq \partial$, $\partial = \mathrm{cf}(\partial) < \mu$ (and recall $2 \leq \vartheta \leq \mu$) then for some forcing notion $\mathbb{R}$ we have:*

(a) $\mathbb{R}$ satisfies $(1)_c + (2)^+_{a,=\vartheta}$, of cardinality λ (so adds no new sequences of length $< \mu$, collapses no cardinality, changes no cofinality and the only possible change in cardinal arithmetic is making $2^\mu = \lambda$)

(b) in $\mathbf{V}^{\mathbb{R}}$ we have $\mathrm{Ax}_{\lambda,\mu}((1)_c + (2)^+_{a,\vartheta(+)})$;

(c) in $\mathbf{V}^{\mathbb{R}}$ the axiom $\mathrm{Ax}((1)_c + (2)^+_{a,\partial})$ fails.

4 Separating $\mathrm{Ax}^\vartheta_\mu$, Ax^∂_μ for regular ϑ, ∂

Recall that $\mathrm{Ax}^\vartheta_{\mu,D}$ is $\mathrm{Ax}_\mu((1)_c + (2)^\vartheta_{c,D})$, we usually omit D and μ is understood from the context.

Hypothesis 4.1

1. $\mu = \mu^{<\mu}$.
2. $S \subseteq S^{\mu^+}_\mu$ stationary.
3. $\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ a closed unbounded subset of δ of order type μ, listed by $\langle \alpha^*_{\delta,\zeta} : \zeta < \mu\rangle$ in increasing order.
4. $\bar{\mathbf{f}}$ as in Definition 4.2.
5. $\Theta \subseteq \mathrm{Reg} \cap \mu^+$, let $S^{\mu^+}_\Theta = \{\delta < \mu^+ : \mathrm{cf}(\delta) \in \Theta\}$.
6. $2 \leq \vartheta < \mu$ but our main interest is $\vartheta = 2$.

Definition 4.2 We say $\bar{\mathbf{f}}$ is a $(\bar{C},\vartheta)$-parameter (or uniformization problem) when $\bar{\mathbf{f}} = \langle \mathbf{f}_\delta : \delta \in S\rangle$, $\mathbf{f}_\delta : C_\delta \to \vartheta$.

Definition 4.3

1. We define $\mathbb{P}^1_{\bar{\mathbf{f}}}$ and $<_{\mathrm{st}}$ as in Definition 3.6 but we change clause $(A)(e)$ by:
   (e)' if $\delta \in v_p$ then
   - (α) $h_p(\delta)$ is a bounded subset of C_δ closed only in its supremum,
   - (β) $h_p(\delta) \subseteq u_p$,
   - (γ) if $\beta \in h_p(\delta)$ so $\delta \in v_p$ then $\mathrm{cf}(\beta) \in \Theta \Longrightarrow \mathbf{f}_\delta(\beta) \in g_p(\beta)$ (so really only $g_p{\restriction}(u_p \cap S^{\mu^+}_\Theta)$ matters),
   - (δ) if $\beta \in h_p(\delta)$ and $\mathrm{cf}(\beta) \notin S^{\mu^+}_\Theta$ then $g_p(\beta) = \varnothing$.
2. We define $\mathscr{I}^1_{\bar{\mathbf{f}},\alpha} \subseteq \mathbb{P}^1_{\bar{\mathbf{f}}}$ as in Definition 3.9.

Claim 4.4 $\mathbb{P}^1_{\bar{\mathbf{f}}}$ *satisfies*

(a) any increasing sequence of length $\delta < \mu$, $\mathrm{cf}(\delta) \notin \Theta$ has a lub, i.e., $(1)_{a,=\partial}$ for $\partial \notin \Theta$;

(b) a set of pairwise compatible conditions of cardinality $< \min(\Theta \cup \{\Theta\})$ has a lub—the union, i.e., $(1)_{a,<\min(\Theta)}$ holds.

P r o o f. Easy. $\square$

Claim 4.5 $\mathbb{P}^1_{\bar{\mathbf{f}}}$ *satisfies:*

(a) *we have* $(1)^+_c$, *i.e.,*

(α) $<_{\rm st}$ *is a partial order and* $p_1 \le p_2 <_{\rm st} p_3 < p_4 \Longrightarrow p_1 <_{\rm st} p_4$;

(β) *any* $<_{\rm st}$*-increasing chain of length* $< \mu$ *has an ub;*

(b) (α) *we have* $(3)_a$, *i.e., if* $p, q \in \mathbb{P}^1_{\bar{\mathbf{f}}}$ *are compatible then they have a lub;*

(β) $\{p_i : i < i(*)\}$ *has a lub when* $i(*) < \mu$ *and* $\{p_i : i < i(*)\}$ *is a set of pairwise compatible conditions and for each* $\delta \in S$, *the set* $\{h_{p_i}(\delta) : i < i(*)$ *and* $\delta \in v_{p_i}\}$ *is finite; note that this set is linearly ordered by being an initial segment;*

(γ) $\{p_i : i < i(*)\}$ *has a ub when* $i(*) < \mu$ *and* $\{p_i : i < i(*)\}$ *is a set of pairwise compatible conditions and if* $\mathrm{cf}(\alpha) \in \Theta$ *then* $|w_{p,\alpha}| < \vartheta$ *where* $w_{p,\alpha} = \{\delta : \delta \in \bigcup_i v_{p_i}$ *and* $\alpha = \sup\{\sup(g_{p_i}(\delta)) + 1 : i < i(*)$ *and* $\delta \in v_{p_i}\}\}$.

(c) (α) $(2)_a$ *holds;*

(β) $(2)^\partial_c$ *that is* $*^\partial_\mu$ *holds if* $\partial < \mu$ *is regular and* $\vartheta \ge 2 \vee \partial \notin \Theta$;

(d) $(3)_{b,\varepsilon}$ *holds if* $\kappa = \mathrm{cf}(\varepsilon) \in \mu \backslash \Theta$ *so is regular.*

P r o o f . Like for Claim 3.8, e.g.,
Clause (a): As in 3.8(1),(2).
Clause (b): Should be clear.
Clause (c): If $\vartheta \ge 2$ we use $(3)_a$, i.e., the parallel of 3.8(3). If $\vartheta = 1$ and $\partial \notin \Theta$ use clause (d).
Clause (d): Just recall $(e)(\gamma)$ of Definition 4.3. □

Claim 4.6 $\mathscr{I}_{\bar{\mathbf{f}},\alpha}$ *is a dense open subset of* $\mathbb{P}^1_{\bar{\mathbf{f}}}$ *where*

1. $\mathscr{I}_{\bar{\mathbf{f}},\alpha} = \{p \in \mathbb{P}_{\bar{\mathbf{f}}} : \alpha \in u_p$ *and* $\alpha \in S \Longrightarrow \alpha \in v_p\}$.

P r o o f . Should be clear. □

Definition 4.7 For $(\mu, \vartheta, \partial, D, \bar{\mathbf{f}})$ as in clause (A) below we define a game $\eth_{\rm gn}(\bar{\mathbf{f}}, \vartheta, \partial, D)$ in clause (B) below where:

(A) (a) $\mu = \mu^{<\mu} > \partial = \mathrm{cf}(\partial) \ge \aleph_0$ and

(b) $S \subseteq S^{\mu^+}_\mu$, $\bar{C} = \langle C_\delta : \delta \in S\rangle$ a club sytem,

(c) D is a normal filter on μ^+ to which S belongs,

(d) $\bar{\mathbf{f}} = \langle \mathbf{f}_\delta : \delta \in S\rangle$, $\mathbf{f}_\delta$ is a function from C_δ to ϑ.

(B) (a) a play lasts ∂ moves,

(b) in the ζ-th move, the players choose $S^\ell_\zeta \in D$ such that $S^2_\zeta \subseteq S^1_\zeta \subseteq S \wedge (\forall \xi < \zeta)(S^1_\zeta \subseteq S^2_\xi)$ and $\bar{\alpha}^\ell = \langle \alpha^\ell_{\zeta,\delta} : \delta \in S^\ell_\zeta\rangle$, $\alpha^\ell_{\zeta,\delta} \subseteq C_\delta$, $\alpha^2_{\zeta,\delta} > \alpha^1_{\zeta,\delta} > \sup\{\alpha^2_{\xi,\delta} : \xi < \delta\}$ and $\mathbf{h}^\ell_\zeta$ pressing down functions on S^ℓ_ζ,

(c) in the ζ-th move, the anti-generic player chooses $S^1_\zeta, \bar{\alpha}^1_\zeta, \mathbf{h}^1_\zeta$ and then the generic player chooses $S^2_\zeta, \bar{\alpha}^2, \mathbf{h}^2_\zeta$,

(d) in the end of the play the generic player wins when for some $\delta_1 < \delta_2$ from $\cap\{S^2_\zeta : \zeta < \partial\}$ we have $\sup\{\alpha^\ell_{\zeta,\delta_1} : \zeta < \partial, \ell = 1, 2\} = \sup\{\alpha^\ell_{\zeta,\delta_2} : \zeta < \partial, \ell = 1, 2\}$, call it α and $\mathbf{f}_{\delta_1}(\alpha) \ne \mathbf{f}_{\delta_2}(\alpha)$, $\bigwedge_{k<\partial} h^\ell_k(\delta_1) = h^\ell_k(\delta_2)$.

Theorem 4.8 *If* $\sigma \in \Theta$, $\vartheta = 2$ *and* $\bar{\mathbf{f}}$ *is such that in the game* $\eth_{\rm gn}(\bar{\mathbf{f}}, \vartheta, \sigma, D)$ *from Definition 4.7 the generic player wins or just does not lose, (so D is a normal filter on* μ^+, $S^{\mu^+}_\mu \in D$*) then :*

(a) $\mathbb{P}^1_{\bar{\mathbf{f}}}$ *fails* Ax^σ_μ.

(b) *no forcing satisfying* $*^\sigma_{\mu,D}$ *adds a generic to* $\mathbb{P}^1_{\bar{\mathbf{f}}}$, *moreover*

(c) *no forcing satisfying* $*^\sigma_{\mu,D}$ *adds a* $(< \mu)$*-directed or just* $< (\sigma^+)$*-directed* $\mathbf{G} \subseteq \mathbb{P}^1_{\bar{\mathbf{f}}}$ *meeting* $\mathscr{I}_{\bar{\mathbf{f}},\alpha}$ *for every* $\alpha < \mu^+$ *(defined in 3.9).*

P r o o f . As in the proof of Claim 3.13(1), e.g., *Clause (c)*:

In the proof of Claim 3.13(1), we replace **st** by a winning strategy of the completeness player in the game for $(2)^\sigma_{d,D}$ (cf. Definition 1.3) and toward contradiction assume $\bar{\mathbf{f}}$ is an $(S, \bar{C}, \vartheta)$-parameter, $p_* \in \mathbb{P}^1_{\bar{\mathbf{f}}}$ and $p_* \Vdash$ "$\underset{\sim}{\mathbf{H}} \subseteq \mathbb{P}^1_{\bar{\mathbf{f}}}$ is a $(< \sigma^+)$-directed and meets every $\mathscr{I}_{\bar{\mathbf{f}},\alpha}, \alpha < \mu^+$".

Now for $\zeta < \sigma$ let $\mathbf{Y}_\zeta$ be the set of $(\bar{q}_\zeta, \bar{r}_\zeta, \mathbf{h}_\zeta, E_\zeta, \bar{p}_\zeta, \bar{\alpha}_\zeta)$ such that:

$\boxplus$ (a) $\langle \bar{q}_\xi, \bar{r}_\xi, h_\xi : \xi \leq \zeta\rangle$ is an initial segment of a play of the game from Definition 1.3 in which the player COM uses the strategy **st**;
(b) so $\bar{q}_\zeta = \langle q_{\zeta,\delta} : \delta \in S_\zeta\rangle$, $\bar{r}_\zeta = \langle r_{\zeta,\delta} : \delta \in S_\zeta\rangle$, $S_\zeta \in D$ and $S_\zeta \subseteq \{S_\xi$: for $\xi < \zeta\}$;
(c) $\bar{p}_\zeta = \langle p_{\zeta,\delta} : \delta \in S_\zeta\rangle$ and $p_{\zeta,\delta} \in \mathbb{P}^1_{\bar{\mathbf{f}}}$;
(d) $r_{\zeta,\delta} \Vdash_{\mathbb{R}}$ "$p_{\zeta,\delta} \in \underset{\sim}{\mathbf{H}}$";
(e) $\delta \in v_{p_{\zeta,\delta}}$;
(f) $\langle \sup(\mathrm{dom}(h_{p_{\xi,\delta}})) : \xi \leq \zeta\rangle$ is strictly increasing.

Now we use the definition of the game $\partial_{\mathrm{gn}}(\bar{\mathbf{f}}, \vartheta, \sigma, D)$ to finish as in Definition 3.10. □

The above theorem helps for further problems:

Claim 4.9

1. *If a forcing notion $\mathbb{P}$ satisfies $(1)_b + (2)_a$ and $\sigma \in \mathrm{Reg} \cap \mu$ then $\mathbb{P}$ satisfies $(2)^\sigma_c$.*
2. *If $\mathbb{Q}$ is adding μ^+, μ-Cohen $\langle \eta_\alpha : \alpha < \mu^+\rangle$, $\eta_\alpha \in {}^\mu\vartheta$ and $\vartheta \leq \mu$, $\aleph_1 \leq \sigma = \mathrm{cf}(\sigma) < \mu$, D is a normal filter on μ^+ such that $S^{\mu^+}_\mu \in D$ then $\Vdash_{\mathbb{Q}}$ "$\langle \underset{\sim}{\eta}_\alpha : \alpha < \mu^+\rangle$ is a $(\bar{C}, \mu)$-parameter and is (ϑ, σ)-generic enough and also the generic player wins in the game $\partial_{\mathrm{gn}}(\underset{\sim}{\bar{\eta}}, 2, \sigma, D)$", pedantically replacing D by the normal filter it generates.*
   Explain Claim 3.9(2).

Conclusion 4.10 *Assume $\aleph_0 \leq \sigma = \mathrm{cf}(\sigma) < \mu = \mu^{<\mu}$ and $\mathbb{Q}$ is the forcing notion of adding μ^+, μ-Cohens.*

1. *In $\mathbf{V}^{\mathbb{Q}}$, there is a forcing notion $\mathbb{P}$ satisfying $(1)^+_c, (2)^\vartheta_c$ for $\vartheta \in \mathrm{Reg} \cap \mu \backslash \{\sigma\}$ but not $(2)^\sigma_c$.*
2. *Moreover in $\mathbf{V}^{\mathbb{Q}}$, if $\mathbb{R}$ is a forcing notion satisfying $(1)_b, (2)^\sigma_c$ then it adds no generic to $\mathbb{P}$, in fact $|\mathbb{P}| = \mu^+$ and we should demand "$\mathbf{G} \subseteq \mathbb{P}$ is σ^+-directed, $\mathbf{G} \cap \mathscr{I}_\alpha \neq \varnothing$ for $\alpha < \mu^+$" for some dense $\mathscr{I}_\alpha \subseteq \mathbb{P}$ for $\alpha < \mu^+$.*
3. *So for some $(< \mu)$-complete μ^+-c.c. forcing notion (satisfying $(1)_b + (2)^\sigma_c$), in $(\mathbf{V}^{\mathbb{Q}})^{\mathbb{P}}$ we have Ax^σ_μ but no $\mathbf{G} \subseteq \mathbb{P}$ as above.*

P r o o f . In $\mathbf{V}^{\mathbb{Q}}$ let $\bar{\mathbf{f}}$ be from Claim 4.9(2), $\mathbb{P}$ be $\mathbb{P}^1_{\bar{\mathbf{f}}}$ from Definition 4.3.

Now (1) follows from (2). For (2) use Theorem 4.8 and Claims 4.4, 4.5 & 4.6. For part (3) use the forcing from [14, 1.1-1.18]. □

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