COMPACTNESS OF THE QUANTIFIER ON "COMPLETE EMBEDDING OF BA'S" SH482

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Abstract. We try to build, provably in ZFC, for a first order T a model in which any isomorphism between two Boolean algebras is definable (i.e. by a first order formula with parameters in the model). The problem, compared to [Shee], is with pseudo-finite Boolean algebras. A side benefit is that we do not use Skolem function (which do not matter for proving compactness of logics but still their elimination is of interest). Let λ be 2^μ if regular and its successor otherwise. Model theoretically we investigate notions of bigness of types, usually those are ideals of the set of formulas in a model, definable in appropriate sense. We build a model of cardinality λ^+ by a sequence of models M_α of cardinally λ for $\alpha < \lambda^+$, each M_α equips with a sequence $\langle (M_{\alpha,i}, a_{\alpha,i}, \Omega_{\alpha,i}) : i \in S_\alpha \subseteq \lambda \rangle$, with $M_{\alpha,i}$ being of cardinality $< \lambda, \prec$-increasing continuous with $i, \Omega_{\alpha,i}$ a bigness notion defined using parameters from $M_{\alpha,i}$ and $a_{\alpha,i}$ realized in $M_{\alpha,i+1}$ over $M_{\alpha,i}$ a $\Omega_{\alpha,i}$-big type. As α increase, not only M_α increase, but this extra structure increasing modulo a club of λ, this is why we have insisted on λ being regular.

This can be considered as a way to omit types of cardinality λ, which in general is hard. The fact that λ is not too much larger than μ help us to guarantee that any possible automorphism of structures be defined in $M = \cup\{M_\alpha : \alpha < \lambda^+\}$ by approximations of cardinality μ and so we can enumerate them all.

The bigness notions involved has to relate to the kind of structures we are interested in interpreting in M, e.g. for linear orders being dense and for pseudo finite Boolean algebras, subsets of $\mathscr{P}(n)$ of large cardinality for n pseudo finite. During the construction for each α, some $b_\alpha \in M_{\alpha+1}$ realizes a big type over M_α for an appropriate bigness notion. We have to guarantee that the bigness notions used in the horizontal direction (that is $\alpha < \lambda^+$) and the bigness notions used in the vertical direction (that is for $i < \lambda$) do not interact. This will mean we have to prove that enough pairs of bigness notions are so called orthogonal: if $p_\ell(x_\ell) \in \mathbf{S}(M)$ is a Ω_ℓ-big for $\ell = 1, 2$ then we can find $p(x_0, x_1) \in \mathbf{S}^2(M)$ extending both such that it says that "x_ℓ is Ω_ℓ-big over $M + x_{1-\ell}$" for $\ell = 1, 2$.

Date: September 13, 2016.

2010 *Mathematics Subject Classification.* Primary: 03C80, 03C30; Secondary: 03C35.

Key words and phrases. model theory, generalized quantifiers, compact logics, constructing models, Boolean Algebras.

The author thanks Alice Leonhardt for the beautiful typing. First version: January 1993; May, 1993; corrected May 7, 1993; next correction Nov 1993, Jan 2002; next correction introduced Nov 2002; next Dec 2002, LAST May 2003. Publication 482.

§ 0. Introduction

We continue here the attempt of extracting and strengthening the purely combinatorial content of building "many complicated models" in [She78a, §2,Ch.VIII] = [She90, §2,Ch.VIII] (i.e., many models for unsuperstable theories), as done in [Shei, §2], [Shea, §1,§2] (and [Shea, §3]); (so also [She83a]) or [Sheg]) and more specifically later see [Shee] and history there.

More specifically, our main theory is

Theorem 0.1. *For every first order theory T and $\lambda = (2^\kappa)^+, \kappa \geq |T|$, <u>then</u> for some $T_1 \supseteq T$ there is a κ^+-saturated model M of T_1 of cardinality λ^+ which satisfies:*

- *if B_1, B_2 are Boolean algebras (first order with parameters) definable in M and π an isomorphism from B_1, B_2 then π is definable in M.*

In [Shee] *we succeed to get "complicated models" by omitting small (e.g. countable) types, so that building a model of size λ^+ by a sequence of λ^+ approximation each of size λ, λ regular we suffice to guess countable elementary submodels, e.g. when $\lambda = \lambda^{\aleph_0}$. This works, e.g. for atomless Boolean Algebras. But we have been stuck on the problem of automorphism of pseudo finite Boolean algebras. Here we use a different approach, building a model of size λ^+, λ is, e.g. $(2^\kappa)^+$; so we can enumerate all subsets of size κ, and instead of guessing automorphism on $\mathfrak{B}_\alpha$ we try to make the model code them by a subset of size κ, so we can enumerate them. See general construction in §4, our specific construction in §5.*

The model $\mathfrak{B}_{\alpha+1}$ is build over $\mathfrak{B}_\alpha$ as an increasing *sequence of length λ of approximations, each a type $p^\alpha_i(\langle \bar{x}_j : j < i\rangle)$ over $\mathfrak{B}_\alpha$ of cardinality $< \lambda$ for $i < \lambda$, restricted by being "big" in appropriate sense. Bigness notions are defined in §1, bigness notion of general type are investigated in §2 and more specific ones in §3.*

But how do we omit types? Generally we do not know how to omit types of cardinality λ (as we know for the case $\lambda = \aleph_0$), <u>but</u> we know how to omit types of some special forms: we represent $\mathfrak{B}_\alpha$ as $\langle \bar{a}^\alpha_i : i < \lambda\rangle$, and demand that for a club of $\alpha < \lambda$, in $\mathfrak{B}_{\alpha+1}$ the type $\mathrm{tp}(\bar{a}^\alpha_i, \bigcup\limits_{j<i}(\bar{a}^\alpha_j \cup \bar{x}^\alpha_j), \mathfrak{B}_{\alpha+1})$ is big in appropriate sense.

To a large extent here we continue [She78b], [She78c] *rather than* [She83b]*. In the later we use a general omitting types theorem for λ^+, quite powerful but it depends on $(\mathrm{Dl})_\lambda$ (hence necessarily $\lambda = \lambda^{<\lambda}$). In* [She78b], [She78c] *we use a special way to omit types: we build a model of cardinality λ^+, by an increasing chain of $\mathfrak{B}_\alpha$ for $\alpha < \lambda$, and the omitting of types in stage α has the following form: the type is represented by a stationary $S \subseteq \lambda$ and $\langle\langle a^\beta_i : i < \omega\rangle : \beta \in S\rangle$ with $a^\beta_i \in \mathfrak{B}_\alpha$ and we "promise" that for every $\beta \geq \alpha$ and finite $A \subseteq \mathfrak{B}$, $\{\beta \in S : \langle a^\beta_i : i < \omega\rangle$ is not discernible over $A\}$ is not stationary. Such properties are preserved in any limit stage, even of small cofinality, the problematic case. For wider framework we use "bigness of types" as in* [She83b]*, but here the restriction of bigness act in two ways: "horizontally", building $\mathfrak{B}_\alpha$ by a sequence of Ω_α-big types over $\mathfrak{B}_\alpha$, and "vertically", preserving: for $\alpha < \beta < \lambda^+$ and any finite $A \subseteq \mathfrak{B}_\alpha$, for a club of $i < \lambda$, the element a^α_i realizes a Γ^*_i-big type over A. To be able to do it we need the so called "orthogonality". See more in* [Shek] *and see history in* [Shee, §0].

This paper was supposed to be Ch.XI to the book "Non-structure" and probably will be if it materializes, it has been circulated and lectured on since 1993.

Our main results are on models of T which is first order complete, first order thoery, usually coding enough set theory, see clause (B) of 1.1. This is enough for proving compactness of first order logic extended by suitable second order quantifiers. Probably we can get all those results for any (first order complete) T, i.e. as in 1.1(A), using reducts of global bigness notions but this is delayed.

The intentions were [Sheh] *(revising* [She86a]*) for Ch.I, and* [Shej] *for Ch.II and* [Shei] *for Ch.III and* [Sheb] *for Ch.IV and* [Shel] *for Ch.V and* [Shea] *for Ch.VI and* [Shec] *for Ch.VII and* [Sheg]*, a revision of* [She85]*, for Ch.VIII, and* [Shed]*, for the appendix and* [She04]*,* [Shee]*,* [Shef] *and* [Shek]*, for Ch. IX, X, XI, XII respectively. References like* [Shed, 3.7=Lc2] *means that c2 is the label of 3.7 in* [Shed]*, will only help the author if changes in the paper* [Shed] *will change the number.*

Notation 0.2. 1) Let $A+\bar a$ be $A \cup \mathrm{Rang}(\bar a)$, similarly $A+a$, $A+\bar a+b+C$.
2) Let $\mathscr{L}$ denote a logic, τ a vocabulary, $\mathscr{L}(\tau)$ the language i.e. the set of $\mathscr{L}$-formulas in the vocabulary τ, τ_M is the vocabulary of the model M; $\mathscr{L}(\tau, A)$ means we add all members of A as individual constants to the vocabulary τ.
3) Let $\mathbb{L}$ be first order logic.
4) $\mathscr{L}(\dot{\mathbf{Q}})$ means we add to to the logic $\mathscr{L}$ the quantifier $\dot{\mathbf{Q}}$.
5) Let T denote a theory, first order if not said otherwise, usually complete, $\mathfrak{C}_T$ is a monster for T.
6) For a theory $T, \tau(T) = \tau_T$ is its vocabulary, $\mathscr{L}(T) = \mathscr{L}(\tau(T))$ the corresponding language (first order for $\mathbb{L}(T) = \mathbb{L}(\tau_T)$)
7) If $A \subseteq M$, $M \models T$ then:

- (a) $T[A] = \{\varphi(\bar a) : \bar a \in {}^{\omega>}A, \quad M \models \varphi[\bar a]\}$
- (b) so $\tau(T[A]) = \tau(T) \cup A$
- (c) $\mathrm{acl}(A, M) = \{b \in M : \mathrm{tp}(b, A, M)$ is algebraic, i.e. some formula in it is realized by only finitely many elements$\}$
- (d) $\bar x_{[I]} = \langle x_s : s \in I\rangle$ and for a formula $\varphi(\bar x_{[I]}, \bar a), \bar a$ a sequence from M let $\varphi(M, \bar a) = \{\bar b \in {}^I M : M \models \varphi[\bar b, \bar a]\}$.

8) We say $\mathfrak{p}$ is a type definition over N <u>when</u> one of the following occurs (with $\bar x = \langle x_i : i < \alpha\rangle$)

- (a) $\mathfrak{p}$ is an ultrafilter on ${}^\alpha N$, and if $N \subseteq A \subseteq M, N \prec M$ then $\mathfrak{p}^A = \{\varphi(\bar x, \bar a) : \bar a \in {}^{\omega>}A$ and $\{\bar b \in {}^\alpha N : M \models \varphi[\bar b, \bar a]\} \in \mathfrak{p}\}$, so $\mathfrak{p}^A \in \mathbf{S}^\alpha(A)$
- (b) $\mathfrak{p}$ is a function from $\{\langle \varphi(\bar x, \bar y), q(\bar y)\rangle : \varphi$ a formula, $q(\bar y)$ a complete type over $N\}$ to $\{$truth,false$\}$ and if $N \subseteq A \subseteq M$ and $N \prec M$ <u>then</u> $\mathfrak{p}^A := \{\varphi(\bar x, \bar a) : \bar a \in {}^{\omega>}A$ and $\mathfrak{p}(\langle \varphi(\bar x, \bar y), \mathrm{tp}(\bar a, N, M)\rangle) = $ truth$\}$ so again $\mathfrak{p}^A \in \mathbf{S}^\alpha(A)$.

8A) Above we say $\mathfrak{p}$ is of kind $\biguplus_{\mathrm{fs}}$ or of kind $\biguplus_{\mathrm{nsp}}$ respectively. In the first section we use an arbitrary compact logic $\mathscr{L}$ but the reader may concentrate on $\mathbb{L}$, first order logic.

§ 1. Bigness notions: basic definitions and properties.

For a complete first order theory T, a ℓ-bigness notion Γ (ℓ for local) is a scheme defining for every model M of T, an ideal of the (Boolean Algebra consisting of the) formulas $\varphi(\bar{x}_\Gamma, \bar{a})$, $\bar{a} \subseteq M$ up to equivalence.

We are interested in such ideals preserved by elementary embedding. Such notions play crucial role in our construction of models. This section is soft - just giving definitions and easy applications. If this section is too abstract, the reader can read parallelly §2, §3 which deal with examples of bigness notions.

The reader may concentrate on the case $\mathfrak{k} = (\mathbf{K}, \leq)$ is the class of models of a complete first order theory T with Skolem functions, $\leq = \prec, \mathfrak{C} = \mathfrak{C}_T$ a monster model and on the case of simple ℓ-bigness notion.

Context 1.1. Let τ be a vocabulary, $\mathfrak{k} = (\mathbf{K}_\mathfrak{k}, \leq_\mathfrak{k}) = (\mathbf{K}, \leq)$ a class of τ-models so M means $M \in \mathbf{K}_\mathfrak{k}$ (usually the class of models of a fixed first order theory, $M \leq_\mathfrak{k} N$ iff $M \prec N$), $\mathscr{L}$ a logic. Always satisfaction of $\mathscr{L}$-formulas is preserved by extensions and the pair $(\mathfrak{k}, \mathscr{L})$ is compact; see below. Usually $(\mathfrak{k}, \mathscr{L})$ is one of the following:

(A) (a) T is complete first order, $\mathbf{K}$ the class of models of T i.e. $\mathbf{K} = \mathrm{Mod}(T)$ (of course τ-models $= \tau(T)$-models) and $M \leq_\mathfrak{k} N \Leftrightarrow M \prec N$
- (b) $\mathscr{L} = \mathbb{L}$ first order logic
- (c) $\mathfrak{C} = \mathfrak{C}_T$ is a monster model
- (d) $\mathbf{S}^\alpha(A, M, \mathfrak{k}) = \mathbf{S}^\alpha_\mathfrak{k}(A, M) = \{p : p$ a complete type over A in $M\}$

(B) like (A), but we may denote T by T^* and it has a model $\mathfrak{C}^*$, an expansion of $(\mathscr{H}(\chi^*), \in, <^*)$ where χ^* is strong limit cardinal, $<^*$ a well ordering of $\mathscr{H}(\chi)$, see [Shee, 2.1]. Then $\mathfrak{C}$ denote a "monster" model of T^* and $\dot{e}$ denote the membership inside it, i.e. $\dot{e}^{\mathfrak{C}^*} = \in \restriction \mathscr{H}(\chi)$. We call such T "of set theory character". Saying "n" we mean a true natural number but also its interpretation in $\mathfrak{C}$, we use $\dot{n}, \dot{m}, \dot{k}$ for members of $\mathbb{N}^\mathfrak{C}$. So many times it is better to deal with "sets" not classes.

(C) (a) T is a universal first order theory with amalgamation, $\mathbf{K} = \mathrm{Mod}(T), \leq_\mathfrak{k}$ is being a submodel (so $\mathfrak{k} = (\mathrm{Mod}_T, \subseteq)$)
- (b) $\mathscr{L} = \mathbb{L}$
- (c) $\mathbf{S}^\alpha(A, M, \mathbf{K}) = \{\mathrm{tp}_{\mathrm{qf}}(\bar{a}, A, N) : M \leq_\mathfrak{k} N, N \models T, \bar{a} \in {}^\alpha N\}$

(D) (a) T is a universal first order theory, $\mathbf{K}$ is the family of existentially closed models of $T, \leq_\mathfrak{k}$ is being a submodel,
- (b) let $\mathscr{L}(\tau) = \Sigma(\tau) = \{\varphi : \varphi$ an existential first order formula in the vocabulary $\tau\}$
- (c) $\mathbf{S}^\alpha_\mathfrak{k}(A, M) = \{\mathrm{tp}_{\Sigma(\tau)}(\bar{a}, N, N) : M \leq_\mathfrak{k} N, N \models T, \bar{a} \in {}^\alpha N\}$.

Definition 1.2. 1) For $A \subseteq M, \bar{a}$ a sequence from a model M let

(a) $\mathrm{tp}_\mathscr{L}(\bar{a}, A, M) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^{\omega>}A, \ell g(\bar{x}) = \ell g(\bar{a}), M \models \varphi[\bar{a}, \bar{b}]$ and $\varphi \in \mathscr{L}(\tau_M)\}$

(b) $\mathbf{S}^\alpha_\mathscr{L}(A, M) = \{\mathrm{tp}_\mathscr{L}(\bar{a}, A, N) : M \leq_\mathfrak{k} N$ and $\bar{a} \in {}^\alpha N\} \subseteq \{p : p$ is a set of $\mathscr{L}(\tau_M)$-formulas with free variables among $\bar{x} = \langle x_i : i < \alpha \rangle$ and parameters from A; if $\mathscr{L} = \mathbb{L}$ we may omit it; the length of $\bar{a}$, i.e. of $\bar{x}$ is not necessarily finite. Writing $p \in \mathbf{S}_\mathscr{L}(A, M)$ means for some α clear from the context.

2) $(\mathfrak{k}, \mathscr{L})$ is $\mathscr{L}$-compact means:

- (a) $\mathfrak{k}$ has amalgamation
- (b) if Δ is a set of $\mathscr{L}$-formulas with parameters from $M \in \mathbf{K}$ and is finitely satisfiable in M or just any finite subset is realized in some N satisfying $M \leq_{\mathfrak{k}} N$, <u>then</u> it is realized in some N, $M \leq_{\mathfrak{k}} N \in \mathbf{K}$.

3) We let $\mathfrak{C} = \mathfrak{C}_T$ or $\mathfrak{C}_{\mathfrak{k}}$ be a monster.

Discussion 1.3. We may consider several further general contexts:

- (A) The class of models of T, a complete countable theory in $\mathscr{L} = \mathbb{L}(\mathbf{Q})$ (with the quantifier $\mathbf{Q}x$ interpreted as $(\exists^{\geq \lambda} x)$, usually $\lambda = \aleph_1$), with elimination of quantifiers for simplicity, $M \leq_{\mathfrak{k}} N$ if $M \prec N$ and $M \models \neg \mathbf{Q}x\varphi(x, \bar{a}) \Rightarrow \varphi(N, \bar{a}) \subseteq M$; so here we do not use a monster model
- (B) T is first order complete, $D \subseteq D(T) := \{\text{tp}(\bar{a}, \emptyset, M) : M \models T, \bar{a} \in {}^{\omega >}M\}$, and D is good (see [She71]), $\mathfrak{C}$ a monster model, i.e. $(D, \bar{\kappa})$-sequence-homogeneous model, $K_{\mathfrak{k}} = \{M : M$ a model of T such that every finite $\bar{a} \subseteq M$ realizes a type from $D\}$ and $\leq_{\mathfrak{k}} = \prec$; in this case as well as in (C),(D),(E) we do not have compactness as in 1.2(2) hence it is natural to use global bigness notions
- (C) $\mathbf{K}$ is a universal class (i.e. $M \in \mathfrak{k}$ iff for every $\bar{a} \in {}^{\omega >}M, M \restriction c\ell_M(\bar{a}) \in \mathbf{K}$), the relation $\leq_{\mathfrak{k}}$ is being a submodel (i.e. locally finite models of a universal theory),

$$\mathbf{S}_{\mathscr{L}}(A, M, \mathfrak{k}) = \{\text{tp}_{\mathscr{L}}(\bar{a}, A, N) : M \leq_{\mathfrak{k}} N \in \mathbf{K}, \bar{a} \subseteq N\}$$

$$\mathbf{S}^{\alpha}_{\mathscr{L}}(A, M, \mathfrak{k}) = \{\text{tp}_{\mathscr{L}}(\bar{a}, A, N) : M \leq_{\mathfrak{k}} N \in \mathbf{K}, \bar{a} \in {}^{\alpha}N\}.$$

E.g. the class of locally finite groups (or existentially closed ones), see [She17]

- (D) For some class $\mathbf{K}' \subseteq \mathbf{K}$ and partial order $\leq'$ on $\mathbf{K}'$ the union of a $\leq'$-directed system of models from $\mathbf{K}'$ (were $\mathbf{K}', \leq'$ closed under isomorphism satisfying natural conditions)
- (E) Abstract elementary class (amalgamation is not demanded).

The contexts from 1.3 will not be used, but we may remark on them or give examples. We now introduce a major notion: local bigness.

Definition 1.4. 1) We call Γ a l-bigness (=local bigness) notion for $(\mathfrak{k}, \mathscr{L})$ (with set of parameters $A_\Gamma \subseteq M_\Gamma \in \mathbf{K}$; for simplicity, we usually restrict ourselves to $\{M \in \mathbf{K} : M_\Gamma \leq_{\mathfrak{k}} M\}$ or $\mathfrak{C}$ is a monster model for T, $A_\Gamma \subseteq \mathfrak{C}$ is “small”) <u>if</u> (it gives a sequence $\bar{x} = \bar{x}_\Gamma$ of variables of length $\alpha(\Gamma)$, in the usual case singleton x or at least finite and):

- (a) for every $M \in \mathbf{K}$ (such that $M_\Gamma \leq_{\mathfrak{k}} M$), $\Gamma^-_M = \Gamma^-(M)$ is a subset of the family of formulas $\varphi(\bar{x}, \bar{a})$, $\varphi \in \mathscr{L}(\tau)$, $\bar{a} \subseteq M$, and $\Gamma_M = \Gamma^+_M = \Gamma^+(M)$ is the complement of $\Gamma^-(M)$ inside this family
- (b) Γ^-_M is preserved by automorphisms of M over A_Γ
- (c) Γ^-_M is a proper ideal of formulas, i.e.
  - (α) if $M \models (\forall \bar{x})[\varphi(\bar{x}, \bar{a}) \rightarrow \psi(\bar{x}, \bar{b})]$ and $\psi(\bar{x}, \bar{b}) \in \Gamma^-_M$ <u>then</u> $\varphi(\bar{x}, \bar{b}) \in \Gamma^-_M$

(β) if $\varphi_1(\bar x,\bar a_1),\ \varphi_2(\bar x,\bar a_2) \in \Gamma^-_M$ then $\varphi_1(\bar x,\bar a_1) \vee \varphi_2(\bar x,\bar a_2) \in \Gamma^-_M$

(γ) $\Gamma^+_M \neq \emptyset$.

2) Assume $\bar x_\Gamma$ is finite then Γ is called non-trivial if, when $M_\Gamma \le_{\mathfrak{k}} M \in \mathbf{K}$.

($*$) $\bar x = \bar a \in \Gamma^-_M$.

3) We call members of Γ^-_M "Γ-small in M", members of Γ^+_M, "Γ-big in M". We may write $M \models (\mathbf{Q}^\Gamma \bar x)\varphi(\bar x,\bar a)$ for "$\varphi(\bar x,\bar a)$ is Γ-big in M" and $M \models (\mathbf{Q}_\Gamma \bar x)\varphi(\bar x,\bar a)$ for "$\varphi(\bar x,\bar a)$ is Γ-small in M", (in this notation, $\Gamma_1 \perp \Gamma_2$ defined below essentially means that $\mathbf{Q}^{\Gamma_1},\mathbf{Q}^{\Gamma_2}$ commute).
4) A Γ-big type $p(\bar x)$ is a set of formulas $\psi(\bar x,\bar a)$, any finite conjunction of which is Γ-big.

Example 1.5. Let χ be an infinite cardinal, and let $\exists^{\ge\chi}\bar x\varphi(\bar x,\bar y)$ be the formula which says "at least χ pairwise disjoint sequences $\bar x$ satisfy $\varphi(\bar x,\bar y)$" and $\mathscr{L} = \mathbb{L}(\exists^{\ge\chi})$. Consider a theory T in $\mathscr{L}(\tau)$, without loss of generality every formula is equivalent to a predicate, and $T' = T \cap \mathbb{L}(\tau)$ (so not exactly in the context 1.1(1) for T). This naturally defines a local notion Γ_T of bigness for T': for M a model of $T, \varphi(x,\bar a)$ is Γ-big iff $M \models R_\varphi(\bar a)$ where $(\forall \bar y)(\exists^{\ge\chi}\bar x)(\varphi(x,\bar y) \equiv R(\bar y)) \in T$.

Convention 1.6. 1) We will, abusing notation, first define bigness notions and only then prove they are bigness notions.
2) As we shall deal here only with invariant bigness notions [see Definition 1.7(2) below] we may "forget" this adjective.

Definition 1.7. Let Γ be a local bigness notion for $(\mathfrak{k},\mathscr{L})$.
1) We say that Γ is weakly invariant if for every $\varphi(\bar x,\bar y) \in \mathscr{L}(\tau), \bar a \in M, M \le_{\mathfrak{k}} N$ (models in $\mathbf{K}$, so $M_\Gamma \le_{\mathfrak{k}} M$ or $A_\Gamma \subseteq M \le_{\mathfrak{k}} \mathfrak{C}_{\mathfrak{k}}$) we have: $\varphi(\bar x,\bar a)$ is Γ-big in M iff $\varphi(\bar x,\bar a)$ is Γ-big in N.
2) We say Γ is invariant if $\mathrm{tp}_{\mathscr{L}}(\bar a, A_\Gamma, M)$ and $\varphi(\bar x,\bar y)$ determine whether $\varphi(\bar x,\bar a)$ is Γ-big (in M).
3) We say that Γ is λ-strong [or λ-co strong] if for every $M \in \mathbf{K}$ and $\varphi(\bar x,\bar a)$ which is Γ-big [or Γ-small] in M there is $\tau_\varphi \subseteq \tau_{\mathfrak{k}}, |\tau_\varphi| < \lambda$ such that: $\varphi(\bar x,\bar a')$ is Γ-big [or Γ-small] in M' whenever $\bar a' \subseteq M' \in \mathbf{K}$, $\bar a'$ realized in M' the $\mathscr{L}(\tau_\varphi)$-type which $\bar a$ realize over A_Γ in $M \restriction \tau_\varphi$.
4) We say Γ is very λ-strong if for every $\varphi = \varphi(\bar x,\bar y) \in \mathscr{L}(\tau_{\mathfrak{k}})$ there is $\tau_\varphi \subseteq \tau_{\mathbf{K}}, |\tau_\varphi| < \lambda$ such that: for every $\varphi(\bar x,\bar a), \bar a \subseteq M \in \mathbf{K}$ the type $\mathrm{tp}_{\mathscr{L}}(\tau_\varphi)(\bar a, A_\Gamma, M \restriction \tau_\varphi)$ determine if $\varphi(\bar x,\bar a)$ is Γ-small or Γ-big in M.

Definition 1.8. Let Γ be a local invariant bigness notion.
1) We say that Γ is λ-simple [or λ-co-simple] if for every $M \in \mathbf{K}, \bar a \subseteq M$, and φ such that $\varphi(\bar x,\bar a)$ is Γ-big [or Γ-small] in M there is $q \subseteq \mathrm{tp}_{\mathscr{L}}(\bar a, A_\Gamma, M), |q| < \lambda$ such that:

$$[\bar a' \in M' \in \mathbf{K}\ \&\ \bar a' \text{ realizes } q \text{ in } M' \Rightarrow \varphi(\bar x,\bar a') \text{ is } \Gamma\text{-big [or } \Gamma\text{-small] in } M'].$$

If $\lambda = \aleph_0$ we may omit it.
2) Γ is very λ-simple when for every $\varphi(\bar x,\bar y)$ there is a set Δ of $< \lambda$ formulas of the form $\psi(\bar y;\bar a) \in \mathscr{L}(\tau, A_\Gamma)$, with $\bar a \subseteq A_\Gamma$ such that: if $M \in \mathbf{K}$ and

$$\Delta \cap \mathrm{tp}_{\mathscr{L}}(\bar b^1, A_\Gamma, M) = \Delta \cap \mathrm{tp}_{\mathscr{L}}(\bar b^2, A_\Gamma, M)$$

then: $\varphi(\bar{x}, \bar{b}^1)$ is Γ-big iff $\varphi(\bar{x}, \bar{b}^2)$ is Γ-big. If $\lambda = \aleph_0$ we may omit it.
3) We say Γ is uniformly λ-simple [or λ-co-simple] when for any $\varphi(\bar{x}, \bar{y})$ with $\lg(\bar{x}) = \alpha(\Gamma)$ there is a type $q_\varphi(\bar{y})$ over A_Γ such that: for any relevant $M, \bar{a}$ we have $\varphi(\bar{x}, \bar{a})$ is Γ-big iff $\bar{a}$ realizes q_φ [or: $\varphi(\bar{x}, \bar{a})$ is Γ-small iff $\bar{a}$ realizes q_φ].

Claim 1.9. *Let Γ be a local bigness notion.*
1) If $\lambda > |A_\Gamma| + |\mathscr{L}|$, $\mathscr{L}$ has finite occurrence[1] *number then*

(a) *Γ is λ-strong if and only if Γ is λ-simple*
(b) *Γ is λ-co-strong if and only if Γ is λ-co-simple*
(c) *Γ is very λ-strong if and only if Γ is very λ-simple iff Γ is uniformly λ-simple and uniformly λ-co-simple.*

2)

(a) *If Γ is very λ-simple then Γ is λ-simple and λ-co-simple*
(b) *If Γ is very λ-strong then Γ is λ-strong and λ-co-strong*
c) *If Γ is λ-simple then Γ is λ-strong*
(d) *If Γ is λ-co-simple then Γ is λ-co-strong*
(e) *If Γ is very λ-simple then Γ is very λ-strong*
(f) *If $\lambda_1 < \lambda_2$ and Γ is λ_1-strong then Γ is λ_2-strong; similarly for "λ_ℓ-co-strong", "very λ_ℓ-strong", "λ_ℓ-simpleq", "λ_ℓ-co-simple", "very λ_ℓ-simple"*
(g) *if $\lambda > |T| + \aleph_0$ then the corresponding strong and simple properties are equal.*

3) If Γ is λ-simple and co-λ-simple then Γ is very simple (using the logic $\mathscr{L}$ being compact).
4) If $A \subseteq M$, p a Γ-big type over A in M then we can find a Γ-big $q \in \mathbf{S}_{\mathscr{L}}(A, M)$ extending p using the logic $\mathscr{L}$ being compact.
5) Parallel of parts (1), (2), (3) hold for global bigness notion defined below.
6) If $\lambda > |\mathscr{L}(\tau)| + |A_\Gamma|$ then: Γ is very λ-strong, very λ-simple.
7) If Γ is uniformly λ-simple then Γ is co-simple. If Γ is uniformly λ-co-simple then Γ is λ-simple.

Proof. By the definitions (and compactness when demanded).
We may also use a relative of "local bigness" called "global bigness". $\square_{1.9}$

Definition 1.10. 1) We say Γ is a g.(=global) bigness notion for $(\mathfrak{k}, \mathscr{L})$ (with set parameters $A_\Gamma \subseteq M_\Gamma \in \mathbf{K}$) if (it gives a sequence $\bar{x} = \bar{x}_\Gamma$ of variables and):

(a) for every $M \in \mathbf{K}$ (satisfying $M_\Gamma \leq_{\mathfrak{k}} M$ as usual), Γ_M is a family of types $p(\bar{x})$ such that for some A, $A_\Gamma \subseteq A \subseteq M$ and $p(\bar{x}) \in \mathbf{S}^{\alpha(\Gamma)}_{\mathscr{L}}(A, M)$
(b) local character: if $M \in \mathbf{K}, A_\Gamma \subseteq A \subseteq M$, $p(\bar{x}) \in \mathbf{S}_{\mathscr{L}}(A, M)$ then $p(\bar{x}) \in \Gamma_M \Leftrightarrow (\forall$ finite $B \subseteq A)[p(\bar{x}) \restriction (A_\Gamma \cup B) \in \Gamma_M]$
(c) the extension property: if $A_\Gamma \subseteq A \subseteq B \subseteq M$, $p \in \mathbf{S}^{\alpha(\Gamma)}_{\mathscr{L}}(A, M)$ is in Γ_M then some extension $q \in \mathbf{S}_{\mathscr{L}}(B, M)$ of p is in Γ_M

[1] this means that every $\varphi \in \mathscr{L}(\tau)$ for some finite $\tau' \subseteq \tau$; note $|\mathscr{L}|$ is the number of sentences up to renaming the predicates and function symbols, so $|\mathscr{L}(\tau)| \leq |\tau| + |\mathscr{L}| + \aleph_0$.

(d) existence: if $A_\Gamma \subseteq A \subseteq M$ then there is a Γ-big $p \in \mathbf{S}^{\alpha(\Gamma)}(A, M)$. [We can close the family under restriction thus allowing "$p \in \mathbf{S}^{\alpha(\Gamma)}(A, M)$ is Γ-big" though $A_\Gamma \not\subseteq A$.]

2) We define "Γ, a g.bigness notion is weakly invariant/invariant" as in definition 1.7(1),(2) above.
3) A g.bigness notion Γ is λ-strong [or λ-co-strong] if for every Γ-big [or Γ-small] $p = p(\bar{x}, \bar{a}) = \mathrm{tp}_{\mathscr{L}}(\bar{b}, \bar{a} \cup A_\Gamma, M) \in \mathbf{S}^{\alpha(\Gamma)}_{\mathscr{L}}(\bar{a} \cup A_\Gamma M)$, for some $\tau_p \subseteq \tau$ of cardinality $< \lambda$, we have: if $M \leq_{\mathfrak{k}} N$ and $\mathrm{tp}_{\mathscr{L}\restriction\tau_p}(\bar{b}\hat{\ }\bar{a}, A_\Gamma, M) = \mathrm{tp}_{\mathscr{L}\restriction\tau_p}(\bar{b}'\hat{\ }\bar{a}', A_\Gamma, N)$ then $\mathrm{tp}_{\mathscr{L}}(\bar{b}', \bar{a}', N)$ is Γ-big [or Γ-small].
4) We say Γ is very λ-strong if for any $n < \omega$ for some τ_n for any n and $p(\bar{x}) \in \mathbf{S}^{\alpha(\Gamma)}(A_\Gamma, M), q(\bar{y}) \in \mathbf{S}^n(A_\Gamma, M)$ with $\ell g(\bar{y}) = n$, there is $\tau_{p,q} \subseteq \tau(\mathbf{K})$ of cardinality $< \lambda$ such that: if for $\ell = 1, 2$ we have $M \leq_{\mathfrak{k}} N_\ell, \bar{a}_\ell, \bar{b}_\ell \in N_\ell$ realizes q, p respectively in N_ℓ and $\mathrm{tp}_{\mathscr{L}(\tau_{p,q})}(\bar{a}_1\hat{\ }\bar{b}_1, A_\Gamma, N_1) = \mathrm{tp}_{\mathscr{L}(\tau_{p,q})}(\bar{a}_2\hat{\ }\bar{b}_2, A_\Gamma, N_2)$ then $\mathrm{tp}_{\mathscr{L}}(\bar{b}_1, \bar{a}_1 \cup A_\Gamma, N)$ is Γ- big iff $\mathrm{tp}_{\mathscr{L}}(\bar{b}_2, \bar{a}_2 \cup A_\Gamma, N_2)$ is Γ-big.
5) A g.bigness notion Γ is very λ-simple if for every m there is a set Δ of $< \lambda$ formulas $\psi(\bar{y})$ (with parameters from A_Γ) such that: if $\bar{a} \in {}^m M$, "$\mathrm{tp}_{\mathscr{L}}(\bar{b}, \bar{a} \cup A_\Gamma, M)$ is Γ-big" depend just on $\mathrm{tp}_\Delta(\bar{b}\hat{\ }\bar{a}, A_\Gamma, M)$.
6) A g.bigness notion Γ is λ-simple [or λ-co-simple] if $\mathrm{tp}_{\mathscr{L}}(\bar{b}, \bar{a} \cup A_\Gamma, M) \in \mathbf{S}^\alpha_{\mathscr{L}}(\bar{a} \cup A_\Gamma, M)$ is Γ-big [is Γ-small] in M implies that for some $q \subseteq \mathrm{tp}_{\mathscr{L}}(\bar{b}\hat{\ }\bar{a}, A_\Gamma, M)$ of cardinality $< \lambda$ we have $\bar{b}\hat{\ }\bar{a}' \in M' \in \mathbf{K}$ realizes $q \Rightarrow \mathrm{tp}_{\mathscr{L}}(\bar{b}', \bar{a}' \cup A_\Gamma, M')$ is Γ-big [or Γ-small] in M' (for example inconsistent).
7) We say Γ is a semi-g. bigness notion if above we omit the local character.

Claim 1.11. *1) Every ℓ-bigness notion is a g-bigness notion (when we restrict ourselves to complete $\mathscr{L}$-types over sets which includes A_Γ; we do not always bother to make the distinction).*
2) If an ℓ-bigness notion Γ is λ-strong/co-λ-strong/λ-simple/ co-λ-simple/weakly invariant/ invariant then as a g-bigness notion it satisfies the corresponding property. If $\lambda > |\mathscr{L}(\tau_{\mathfrak{k}})|$ this holds also for ?

Proof. Easy.
We may like to compose (or iterate) g.bigness notions. $\square_{1.11}$

Definition 1.12. 1) If $\bar{\Gamma} = \langle \Gamma^i = \Gamma_i : i < \alpha \rangle$ is a sequence of g-bigness notion (with $\bar{x}_{\Gamma_i}$ pairwise disjoint for notational simplicity), we consider $\bar{\Gamma}$ also as a g-bigness notion by: $\bar{x}_{\bar{\Gamma}} = \langle \bar{x}_{\Gamma_i} : i < \alpha \rangle$ (formally - their concatenation), $A_\Gamma = \cup\{A_{\Gamma_i} : i < \alpha\}$ and: for $p \in \mathbf{S}^{\alpha(\bar{\Gamma})}_{\mathscr{L}}(A, M), A_\Gamma \subseteq A$ we have: p is $\bar{\Gamma}$-big if and only if $p = p(\ldots, \bar{x}_{\Gamma_i}, \ldots)_{i<\alpha}$, and whenever $M \leq_{\mathfrak{k}} N \in \mathbf{K}, \langle \bar{a}_i : i < \alpha \rangle$ realizes p we have $\mathrm{tp}_{\mathscr{L}}(\bar{a}_i, A \cup \bigcup_{j<i} \bar{a}_j, M) \in \Gamma^i_N$ for each $i < \alpha$.
2) Similarly when $A_{\Gamma_i} \subseteq A_\Gamma \cup \{\bar{x}_{\Gamma_j} : j < i\}$ defined naturally.

Claim 1.13. *1) If $\bar{\Gamma}$ is a sequence of invariant g-bigness notions then $\bar{\Gamma}$ is itself an invariant g-bigness notion.*
2) If $\bar{\Gamma}$ is a sequence of [very] λ-[co-]strong g-bigness notions and $\ell g(\bar{\Gamma}) < \mathrm{cf}(\lambda)$ then $\bar{\Gamma}$ is a [very] λ-[co-]strong g-bigness notion.
3) If $\bar{\Gamma}$ is a sequence of [very] λ-simple g-bigness notions and $\ell g(\bar{\Gamma}) < \mathrm{cf}(\lambda)$ then $\bar{\Gamma}$ is a [very] λ-[co-]simple g.bigness notion.

Proof. Easy. $\square_{1.13}$

Remark 1.14. 1) What about $\bar{\Gamma} = \langle \Gamma_t : t \in I \rangle, I$ a linear order which is not a well ordering? If the Γ_t's are very simple, this is O.K.
2) If $A_{\Gamma_t} \backslash A_\Gamma$ is finite for every $t \in T$, there are no restrictions on I.
3) Why the asymmetry in Claim 1.13(3), i.e. why we omit "λ-co-simple"? As in Definition 1.12(1) is not symmetric, we use $\bar{\Gamma}$-big, not $\bar{\Gamma}$-small.

Now we turn to the central relation here between bigness notions here: orthogonality.

Definition 1.15. 1) Let Γ_1, Γ_2 be two g.bigness notions for $(\mathfrak{k}, \mathscr{L})$, for the sequences of variables $\bar{x}^1, \bar{x}^2$ respectively (maybe infinite). We say that Γ_1, Γ_2 are orthogonal (or say Γ_1 is orthogonal to Γ_2, or say $\Gamma_1 \perp \Gamma_2$) <u>when</u>:
for any model $M \in \mathbf{K}$, $A \subseteq M$, and sequences $\bar{a}^1, \bar{a}^2 \in M$ of length $\ell g(\bar{x}^1), \ell g(\bar{x}^2)$ respectively such that $\text{tp}_{\mathscr{L}}(\bar{a}^l, A, M)$ is Γ_l-big for $l = 1, 2$, <u>there are</u> an $\le_{\mathfrak{k}}$-extension N of M, and sequences $\bar{b}^1, \bar{b}^2 \in N$ of length $\ell g(\bar{x}^1), \ell g(\bar{x}^2)$ respectively such that for $l = 1, 2$ the sequence $\bar{b}^l$ realizes $\text{tp}_{\mathscr{L}}(\bar{a}^l, A, N)$ and $\text{tp}_{\mathscr{L}}(\bar{b}^l, A \cup \bar{b}^{3-l}, N)$ is Γ_l-big. Similarly "for T".
2) In part (1) we say Γ_1, Γ_2 are nicely orthogonal or we say Γ_1 is nicely orthogonal to Γ_2, or we write $\Gamma_1 \perp_n \Gamma_2$, <u>when</u>:
adding to the assumption $A_{\Gamma_1} \cup A_{\Gamma_2} \subseteq A = \text{acl}_M A$ we can add to the conclusion $\text{acl}_M(A \cup \bar{b}^1) \cap \text{acl}_M(A \cup \bar{b}^2) = A$ (acl stands for algebraic closure, i.e. $\text{acl}_M(A) = \{b \in M$: for some $\bar{a} \subseteq A \subseteq M$ and $\varphi(y, \bar{x})$ we have $M \models \varphi[b, \bar{a}]$ and $M \models (\exists^{<n} y)\varphi(y, \bar{a})$ for some finite $n\}$).

Remark 1.16. 1) If Γ_l has parameters in M_{Γ_l} and some $M^* \in \mathbf{K}$ such that $M_{\Gamma_1} \le_{\mathfrak{k}} M^*$ and $M_{\Gamma_2} \le_{\mathfrak{k}} M^*$ is given, in 1.15 by $M (\in \mathbf{K})$ we mean any $\le_{\mathfrak{k}}$-extension of M^*, otherwise we look at any $\le_{\mathfrak{k}}$-extensions of $M_{\Gamma_1}, M_{\Gamma_2}$.
2) Under context 1.1, orthogonal is equivalent to nicely orthogonal and every Γ is nice, see below Definitions 1.17 and Claim 1.18(4).
3) In fact also in 1.15(2) we are demanding $A_{\Gamma_1} \cup A_{\Gamma_2} \subseteq A$. However in 1.10(1) we can weaken $A_\Gamma \subseteq A \subseteq M$ to $A_\Gamma \cup A \subseteq M$, and in 1.10(1)(b) replace $p(\bar{x}) \restriction B$, etc. with minor changes.

Definition 1.17. Γ is **nice** when: if $p \in \mathbf{S}_{\mathscr{L}}(A, M)$ is in Γ_M, and α an ordinal, $A = \text{acl}_M A$ <u>then</u> in some $N, M \le_{\mathfrak{k}} N \in \mathbf{K}$ we can find $\bar{a}^i \subseteq N$ for $i < \alpha$ such that:

- (a) $\bar{a}^i$ realizes p (in N)
- (b) $\text{tp}(\bar{a}^i, A \cup \bigcup_{j<i} \bar{a}^j) \in \Gamma_N$
- (c) $\text{acl}_N(A \cup \bar{a}^i) \setminus \text{acl}(A)$ are pairwise disjoint (for $i < \alpha$).

We now give some basic properties of those notions.

Claim 1.18. *1) If $\bar{\Gamma}^\ell = \langle \Gamma^\ell_i : i < \alpha_\ell \rangle$ $(\ell = 1, 2)$ are two sequences of g-bigness notions and $(\forall i < \alpha_1)(\forall j < \alpha_2)[\Gamma^1_i \perp \Gamma^2_j]$ <u>then</u> $\bar{\Gamma}^1 \perp \bar{\Gamma}^2$ (on such g-bigness notions, see definition 1.12).*
2) If Γ_1, Γ_2 are nicely orthogonal g-bigness notion <u>then</u> they are orthogonal.
3) If Γ_1, Γ_2 are orthogonal g-bigness notion and each Γ_ℓ is invariant and at least one is nice <u>then</u> Γ_1, Γ_2 are nicely orthogonal.
4) If Γ is an invariant g-bigness notion <u>then</u> Γ is nice.

5) If during a proof of the orthogonality of Γ_1, Γ_2 *we are given* Γ_ℓ*-big* $p_\ell \in \mathbf{S}^{\alpha(\Gamma_\alpha)}_{\mathscr{L}}(A, M^*)$ *for* $\ell = 1, 2$ *we can replace* M^* *by any* N^* *such that* $M^* \le_{\mathfrak{k}} N^*$ *and* A, p_1, p_2 *by* A', $A \subseteq A' \subseteq N^*$ *and* p'_1, p'_2 *respectively such that* $p_\ell{}' \in \mathbf{S}^{\alpha(\Gamma_\alpha)}_{\mathscr{L}}(A, M')$ *extend* p_ℓ *and is* Γ_ℓ*-big.*
6) If $\bar\Gamma = \langle \Gamma_i : i < \alpha \rangle$, *each* Γ_i *is an invariant g-bigness notion and nice* <u>then</u> *so is* $\bar\Gamma$.
7) Being orthogonal is a symmetric relation.

Proof. E.g.
3) Say Γ_1 is nice. Let p^ℓ for $\ell = 1, 2$ be a complete Γ_ℓ-big type over A in M, $A_{\Gamma_1} \cup A_{\Gamma_2} \subseteq A \subseteq M$. Find $N \in \mathbf{K}, M \le_{\mathfrak{k}} N$ and $\langle \bar a^1_i : i < \lambda^+ \rangle$ (where λ is the supremum on the number of $\mathscr{L}$-formulas over a set of cardinality $\le |A| + \aleph_0$), such that: $\bar a^1_i \subseteq N$ realizes p^1 and $\langle \mathrm{acl}_N(A \cup \bar a^1_i) \setminus \mathrm{acl}_N(A) : i < \lambda^+ \rangle$ are pairwise disjoint and $\mathrm{tp}(\bar a^1_i, A \cup \bigcup\limits_{j<i} \bar a^1_j, N)$ is Γ_1-big (possible as Γ_1 is nice). Choose by induction on $i \le \lambda^+$ a type $p^2_i \in \mathbf{S}^{\alpha(\Gamma_2)}_{\mathscr{L}}(A \cup \bigcup\limits_{j<i} \bar a^1_j, N)$ such that:

- (α) p^2_i is Γ_2-big
- (β) if $\bar a^2$ realizes p^2_i in N', $N \le_{\mathfrak{k}} N'$ then $\mathrm{tp}_{\mathscr{L}}(\bar a^1_i, (A \cup \bigcup\limits_{j<i} \bar a^1_j) \cup \bar a^2, N')$ is Γ_1-big
- (γ) p^2_i is increasing continuously
- (δ) $p^2_0 = p^2$.

[Why possible? For $i = 0$, trivial: for $i = j+1$ we can take care of clauses $(\alpha)+(\beta)$ as $\Gamma_1 \perp \Gamma_2$; for i limit use Definition 1.10(1)(b)).
Without loss of generality some $\bar b \subseteq N$ realizes $p^2_{\lambda^+}$. Choose i such that $\mathrm{acl}_N(A \cup \bar a^1_i) \setminus \mathrm{acl}_M(A)$ is disjoint to $\mathrm{acl}_N(A \cup \bar b)$. Now $N, \bar a^1_i, \bar b$ are as required.
4) Clearly to prove that Γ is nice, it suffice to prove:

- $(*)$ for Γ-big, $p \in \mathbf{S}^{\alpha(\Gamma)}_{\mathscr{L}}(A, M)$, and $A \subseteq B \subseteq M$ we can find $\bar a$, N such that $M \le_{\mathfrak{k}} N$, $\bar a \subseteq N$, $\mathrm{tp}_{\mathscr{L}}(\bar a, B, M)$ is a Γ-big type extending p and $\mathrm{acl}_N(A + \bar a) \cap \mathrm{acl}_N(B) = \mathrm{acl}(A)$.

(just use it repeatedly).
To prove $(*)$, let λ be as in the proof of part (3) above, without loss of generality $A = \mathrm{acl}_M(A)$; by the compactness (and basic properties of algebraicity) we can find M', $M \le_{\mathfrak{k}} M'$ and for $i < \lambda^+$, elementary mapping f_i (for M') such that $\mathrm{Dom}(f_i) = B$, $f_i \restriction A = \mathrm{id}$

$$[i < j < \lambda^+ \Rightarrow \mathrm{acl}_M(f_i(B)) \cap \mathrm{acl}_M(f_j(B)) = \mathrm{acl}(A)]$$

[Why? We should consider

$$\{x_b \ne c : b \subseteq B \setminus A, c \in M\} \cup \{\varphi(x_{b_1} \dots, x_{b_n}, \bar a) : \bar a \subseteq A, M \models \varphi[b_1, \dots, b_n, \bar a]$$

it is finitely satisfiable in M by the definition of algebraic and the finite $\triangle$-system lemma). Then we can find $N, \bar a$ such that $M' \le_{\mathfrak{k}} N$, $\mathrm{tp}_{\mathscr{L}}(\bar a, \bigcup\limits_{i<\lambda^+} f_i(B), N)$ extend p and is Γ-big. Clearly for some $i < \lambda^+$, $\mathrm{acl}_N(f_i(B)) \cap \mathrm{acl}_N(A + \bar a) = \mathrm{acl}_N(A)$, and by invariance we are done.] $\square_{1.18}$

Now we consider a quite general scheme for defining bigness notion; as an example see 3.1. We are interested in the case where for a given first order theory $\mathfrak{s}, \mathbf{\Gamma}_{\mathfrak{s}}$ is a $\mathfrak{s}$-bigness notion scheme or co-$\mathfrak{s}$-bigness notion scheme, this is quite a general way to define bigness notions. But for this we need

Definition 1.19. Let $\mathfrak{s}$ be a first order theory.
1) We say $\mathbf{\Gamma}$ is a $\mathfrak{s}$-bigness notion scheme <u>when</u> $\mathbf{\Gamma}$ is a sentence $\psi_{\mathbf{\Gamma}}$ in the vocabulary $\tau_{\mathfrak{s}} \cup \{P\}$, in some logic (possibly infinitary) P an $\ell g(\bar{x}_{\mathbf{\Gamma}})$-place predicate not from $\tau_{\mathfrak{s}}$ and for every $M_* \in \mathbf{K}$ and interpretation $\bar{\varphi}$ of $\mathfrak{s}$ in M_*, see Definition 1.19(2) a formula $\vartheta(\bar{x}_\Gamma, \bar{b})$ is Γ we have $\Gamma = \mathbf{\Gamma}[\bar{\varphi}]$ is an invariant bigness notion, where:

- (a) if $M_* \le_{\mathfrak{k}} M$ then $\Gamma^+_M = \mathbf{\Gamma}[\bar{\varphi}]^+_M$ is the set of formulas $\vartheta(M, \bar{b})) \models \psi_{\mathbf{\Gamma}}$, on $M^{[\bar{\varphi}]}$, see 1.19(2)
- (b) $A_{\mathbf{\Gamma}[\bar{\varphi}]}$ is M_* or just the elements of M_* appearing in $\bar{\varphi}$.

1A) A co-pre $\mathfrak{s}$-bigness notion scheme[2] Γ is a sentence (in possibly infinitary logic) called $\psi_{\mathbf{\Gamma}}$ in the vocabulary $\tau(\mathfrak{s}) \cup \{P\}, P$ has arity $\ell g(\bar{x}_{\mathbf{\Gamma}})$ but is not in $\tau(\mathfrak{s})$, ($\bar{x}_{\mathbf{\Gamma}}$ finite for simplicity-otherwise we should have P_w for every finite $w \subseteq \ell g(\bar{x}_{\mathbf{\Gamma}})$). We may write $\psi_{\mathbf{\Gamma}}(P)$, (treating P as a variable; we shall use P^*, if P is already occupied).
2) We call $\bar{\varphi}$ an interpretation with parameters of $\mathfrak{s}$ in a model $M^* \in \mathbf{K}$ <u>if</u>: $\bar{\varphi} = \langle \varphi_R(\bar{y}_R, \bar{a}_R) : R \in \tau(\mathfrak{s}) \rangle$ where $\varphi_R \in \mathscr{L}(\tau_{\mathbf{K}})$ and $\tau(\mathfrak{s})$ is the vocabulary of $\mathfrak{s}$ including equality[3] (for each sort of $\tau(\mathfrak{s})$), treating also function symbols as predicates, so R is interpreted as $\{\bar{b} : M^* \models \varphi_R(\bar{b}, \bar{a}_R),\ \ell g(\bar{b}) = \ell g(\bar{y}_R)\ (= \text{arity of } R)\}$. The interpreted model has universe $\{b : \varphi(M \models \varphi_=[b, b, \bar{a}_=]\}$, if $\mathfrak{s}$ is multi-sort we have equality for each sort. Of course, we assume that (it holds in the cases we are considering) if $M^* \le_{\mathfrak{k}} M$ and $\bigcup\limits_{R \in \tau} \bar{a}_R \subseteq N \le_{\mathfrak{k}} M$ then N inherits the interpretation. The interpreted model is called $M[\bar{\varphi}]$ or $M^{[\bar{\varphi}]}$ and we demand that it is a model[4] of $\mathfrak{s}$; we demand further (for $\bar{\varphi}$ to be an interpretation of $\mathfrak{s}$ in M) that if $M^{[\bar{\varphi}]}$ is a model of $\mathfrak{s}$ and $M \le_{\mathfrak{k}} N$ then $N^{[\bar{\varphi}]}$ is a model of $\mathfrak{s}$ and $M^{[\bar{\varphi}]} \le_{\mathfrak{k}} N^{[\bar{\varphi}]}$, in the context 1.9(A) we get $M^{[\bar{\varphi}]} \prec N^{[\bar{\varphi}]}$.
3) For a pre-$\mathfrak{s}$-bigness notion scheme $\mathbf{\Gamma} = \psi_{\mathbf{\Gamma}}$ and interpretation $\bar{\varphi}$ of $\mathfrak{s}$ in $M^* \in \mathbf{K}$, we define $\mathbf{\Gamma}[\bar{\varphi}] = \mathbf{\Gamma}[\bar{\varphi}, M^*]$, the $\bar{\varphi}$-derived local bigness notion, as follows: given $M \in \mathbf{K}$ such that $M^* \le_{\mathfrak{k}} M$, $\vartheta(\bar{x}, \bar{b})$ is $\mathbf{\Gamma}[\bar{\varphi}]$-small in M <u>iff</u> for any quite saturated (see below) $N^*, M \prec N^*$ letting $P = \{\bar{a} : N^* \models \vartheta[\bar{a}, \bar{b}]$ and $\bar{a} \subseteq N^*[\bar{\varphi}]\}$ (in the relevant sorts, of course) we have $(N^*[\bar{\varphi}], P) \models \psi_{\mathbf{\Gamma}}$.
3A) The "quite saturated" means:

- (a) if $(\mathfrak{k}, \mathscr{L}) = (\mathrm{Mod}(T), \mathbb{L}), T$ first order, means λ^+-saturated where $\lambda = |T| + |\tau(\mathfrak{s})| + \aleph_0$
- (b) if $\mathbf{K}$ is an a.e.c. with amalgamation (the last follows by compactness if $\mathscr{L}$ is non-trivial), we use λ^+-model-homogeneous universal where $\lambda = \mathrm{L.S.T.}(\mathbf{K}) + |\tau(\mathfrak{s})|$ This is needed for invariance to hold.

[2]In principle we should denote schemes by a different letter, so in the definition we use $\mathbf{\Gamma}$ but usually we do not

[3]We may add: $\varphi_=(x, y, \bar{a}_=) \to x = x$.

[4]We use equivalence classes as elements, equality is interpreted as equivalence relation and we will not take the trouble of dividing by it; alternatively we can have $\mathfrak{s}$ not first order

3B) For a model N of $\mathfrak{s}$ and the identity interpretations we define $\Gamma_N \subseteq \mathscr{P}(N)$ as above.
4) We omit the "pre" <u>if</u> every $\mathbf{\Gamma}[\bar{\varphi}, M^*]$ is a ℓ-bigness notion (usually but not always for our fixed $\mathbf{K}$). [If this holds for some $\mathfrak{s}$, we write $*$.]

Γ is derived from $\mathbf{\Gamma}$ if it is of the form $\mathbf{\Gamma}[\bar{\varphi}, M^*]$ for some $\bar{\varphi}, M^*$.
5) We say a property holds for $\mathbf{\Gamma}$ <u>if</u> it holds for every derived ℓ. bigness notion (so we demand that $\bar{\varphi}$ is an interpretation of $\mathfrak{s}$ in M; see part (3)) in which not every formula is small. We may put the "co-" before big.
6) In parts 0),1) we may replace "pre" by "co-pre" if we replace $\Gamma[\bar{\varphi}]$-big by $\Gamma[\bar{\varphi}]$-small (so there is no real need for both notions). Similarly in part 3),7).
7) We say "Γ is a co-$\mathfrak{s}$-bigness notion, it is of the form $\mathbf{\Gamma}[\bar{\varphi}, M]$, for $\mathbf{\Gamma}$ a co-$\mathfrak{s}$-bigness notion scheme, $\bar{\varphi}$ and interpretation of $\mathfrak{s}$ in M.
8) We can define global $\mathbf{\Gamma}$ parallely.

Observation 1.20. *Assume in Definition 1.19 that $\Gamma^* = \mathbf{\Gamma}[\bar{\varphi}]$ for $\mathbf{K}$, and $\mathbf{\Gamma}$ is a co-[pre]-$\mathfrak{s}$-bigness notion scheme.*
1) It has the parameters from the interpretation $\bar{\varphi}$ i.e. A_Γ is the set of parameters appearing in $\bar{\varphi}$.
2) $\mathbf{\Gamma}[\bar{\varphi}]$ is invariant.

Definition 1.21. 1) A local bigness notion Γ (for $(\mathfrak{k}, \mathscr{L})$) is λ-presentable <u>if</u> $\bar{x} = \bar{x}_\Gamma$, and for each $\varphi(\bar{x}_\Gamma, \bar{y}) \in \mathscr{L}$, for some set of $\mathscr{L}$-formulas Δ_Γ of the form $\vartheta(\ldots, \bar{x}^i, \ldots)_{i \in I_\Gamma} \in \mathscr{L}(\tau(\mathfrak{k}))$ with $\lambda > |I_\Gamma|, |\triangle|$ (we may have parameters) where for $i \in I_\Gamma, \ell g(\bar{x}^i) = \ell g(\bar{x}_\Gamma)$ and for $M \in \mathbf{K}, \bar{b} \subseteq M$ we have $|\Delta_\Gamma| < \lambda$ and: $\varphi(\bar{x}, \bar{b})$ is Γ-big in M <u>if and only if</u> the set $\{\varphi(\bar{x}^i, \bar{b}) : i \in I\} \cup \Delta_\Gamma$ is finitely satisfiable in M.
2) If we omit λ we mean $\lambda = \aleph_0$. Without loss of generality $|I_\Gamma| \leq |\Delta_\Gamma|$.
3) We define similarly a λ-co-presentable bigness notion scheme, i.e. we replace above "Γ-big" by "Γ-small".

Claim 1.22. *1) If Γ is $\mathfrak{s}$-[co-]bigness notion <u>then</u> it is invariant.*
2) If Γ is a λ-presentable local bigness scheme <u>then</u> Γ is invariant very λ-strong uniformly λ-simple (hence co-simple) local bigness notion.
3) If Γ is λ-co-presentable local bigness scheme <u>then</u> Γ is invariant presentable, very λ-strong, uniformly λ-co-simple (hence simple) local bigness notion.
4) For "very λ-strong" we can replace λ by $\aleph_0 + |\tau(\Delta_\Gamma)|$.
5) In parts 2), 3), if $\lambda = \aleph_0$ we get "very simple" = "very co-simple".

Proof. Easy (for the co-simple/simple in parts (2),(3), use 1.9(7). $\square_{1.22}$

Claim 1.23. *1) If $\langle \bar{x}^i : i \in I_\Gamma \rangle$ and a set Δ of $< \lambda$ formulas in $\mathscr{L}(\tau(\mathfrak{k}))$ and the variable $\bar{x}^i$ $(i \in I_\Gamma)$ are given, $\ell g(\bar{x}^i) = \alpha$ for $i \in I_\Gamma$, and we try to define a λ-presentable Γ, i.e. "Γ-big formula in $M \in \mathbf{K}$", i.e. as in 1.21(1) (so $\alpha(\Gamma) = \alpha$), <u>then</u> for some $\mathfrak{s}$ (a set of $< \lambda$ formulas of $\mathscr{L}(\tau(\mathfrak{k}))$ possibly with parameter) Γ is a derived case of the scheme pre $\mathfrak{s}$-bigness notion scheme.*
2) Similarly for co-representable and co-pre $\mathfrak{s}$-bigness notion scheme.

Convention 1.24. If we define $\Gamma^{\bar{x}}_{\bar{a}}$ for every appropriate sequence $\bar{a}$ of parameters (from some $M \in \mathbf{K}$) <u>then</u> we call $\Gamma^{\bar{x}}$ a scheme, and $\Gamma^{\bar{x}}_{\bar{a}}$ an instance of this scheme. Abusing notation we may call $\mathbf{\Gamma}[\bar{\varphi}]$ from 1.19 a case of $\mathbf{\Gamma}$; in definition 1.19 we may write Γ instead of $\mathbf{\Gamma}$.

See more in 5.2 (and 5.8-5.10).

§ 2. General examples of bigness notions

We deal here with example of bigness notions which are general, i.e. no formulas play special roles. Consider a model M, very saturated and $A \subseteq M$, and type $p(\bar{x})$ over A in M and we describe the examples defined below. We say $p(\bar{x})$ is $\Gamma^{\rm tr}$-big always, $\Gamma^{\rm na}$-big if some $\bar{a}$ realizing p is disjoint to $\mathrm{acl}(M)$. This will be helpful in guaranteeing no undesirable algebraicity.

We say $p = p(\langle \bar{x}_i : i < \alpha \rangle)$ is $\Gamma^{\rm ids}_\alpha$-big if some/any sequence $\langle \bar{a}_i : i < \alpha \rangle$ realizing p is indiscernible over A; this is helpful in omitting types of a sort: if we know that $\mathbf{f}$ (say an outside automorphism we would like to "kill") satisfies $f(a) = b \notin \mathrm{acl}(A+a)$ and $\langle b_n : n < \omega \rangle$ is indiscernible over $A + a$, if we guarantee it is still indiscernible over $B + a$, $A \subseteq B$, this helps to "kill" $\mathbf{f}$.

Then we consider $\Gamma^{\rm av}_{\dot{D},\bar{a}}$ where $p(x)$ is $\Gamma^{\rm av}_{\dot{D},\bar{a}}$-big iff every formula $\varphi \in p$ (or finite conjunction of such formulas) is satisfied by a $\dot{D}$-positive set of a_i. This is used, e.g. when adding a very small non-standard natural number.

For each we are interested in its simplicity etc. and in orthogonality. Now for theories T with enough set theory coded in then we can more easily define bigness notion, so we may expand T to such T^+, define there Γ^+ and see what it induce on T, this is promising, but as not presently used, we say little (see 1.13, 1.16). We consider also weakening the local property of g-bigness notion.

Context 2.1. T is first order complete, $\tau = \tau(T)$, $L = \mathbb{L}(\tau)$, and all Γ here are (by 1.18(4)) nice; so $\mathfrak{k} = (\mathrm{Mod}_T, \prec)$, $\leq_{\mathfrak{k}} = \prec$ and $\mathfrak{C}$ a monster for T. So tp will mean $\mathrm{tp}_{\mathbb{L}}$ and $\leq_{\mathfrak{k}}$ is $\prec$ and M, N are models of T.

Definition 2.2. 1) $\Gamma^0 = \Gamma^{\rm tr}$, the trivial bigness notion is defined by: $\varphi(x, \bar{a})$ is $\Gamma^{\rm tr}$-big in M <u>if and only if</u> ($\bar{a} \in M, \varphi \in \mathbb{L}_\tau$ and) $M \models \exists x \varphi(x, \bar{a})$.
2) $\Gamma^1 = \Gamma^{\rm na}$, the non-algebraicity bigness notion is defined by: $\varphi(x, \bar{a})$ is $\Gamma^{\rm na}$-big in M <u>if and only if</u> ($\bar{a} \in M$, $\varphi \in \mathscr{L}$ and): $M \models \exists^{\geq n} x \varphi(x, \bar{a})$ for every natural number n.
3) $\Gamma^1_\alpha = \Gamma^{\rm na}_\alpha$, the α-non-algebraicly bigness notion, is defined by (where $\bar{x} = \bar{x}_{[\alpha]} \langle x_i : i < \alpha \rangle) : \varphi(\bar{x}, \bar{a})$ is Γ^1_α-big in M, a model of T <u>if and only if</u> $\{\varphi(\langle x^k_i : i < \alpha \rangle, \bar{a}) : k < \omega\} \cup \{x^k_i \neq x^m_i : i < \alpha, k < m < \omega\}$ is finitely satisfiable in M.

Claim 2.3. *1) $\Gamma^{\rm tr}$ is a nice, presentable (invariant) ℓ-bigness notion, orthogonal to every (invariant) g-bigness notions Γ (trivially sometimes) and $A_{\Gamma^{\rm tr}} = \emptyset$.*
2) $\Gamma^{\rm na}$ is $\aleph_1$-presentable invariant ℓ.bigness notion orthogonal to every invariant Γ and $A_{\Gamma^{na}_\alpha} = \emptyset$.
3) $\Gamma^{\rm na}_\alpha$ is $(\aleph_1 + |\alpha|^+)$-presentable invariant ℓ-bigness notion orthogonal to every invariant Γ and $A_{\Gamma^{\rm tr}_\alpha} = \emptyset$.

Proof. By the proof of 1.18(4). $\square_{2.3}$

Definition 2.4. $\Gamma^2_\alpha = \Gamma^{\rm ids}_\alpha$ (for $\alpha \geq \omega$), the indiscernibility bigness notion, in the variables $\bar{x} = \langle x_i : i < \alpha \rangle$ (or each x_i replaced by a sequence of length n (possibly infinite) - does not matter) is defined by: $\varphi(\bar{x}, \bar{a})$ is $\Gamma^{\rm ids}_\alpha$-big in a model M of T <u>if and only if</u> $\varphi(\bar{x}, \bar{a}) \cup \{\psi(x_{i_0}, \ldots, x_{i_{n-1}}, \bar{a}) \equiv \psi(x_{j_0}, \ldots, x_{j_{n-1}}, \bar{a}) : \psi \in \mathbb{L}(\tau)$ and $i_0 < i_1 < \cdots < i_n < \alpha$ and $j_0 < j_1 < \cdots < j_n < \alpha\}$ is finitely satisfiable in M.

Claim 2.5. *1) $\Gamma^{\rm ids}_\alpha$ is a $(|\tau| + \aleph_0)^+$-presentable ℓ-bigness notion, $A_{\Gamma^{\rm ids}_\alpha} = \emptyset$.*
2) $\Gamma^{\rm ids}_\alpha$ is orthogonal to any g-bigness notion.

Proof. For part (1), the proof of the (relatively) non-trivial part is contained in the proof of part (2) which is rephrased and proved in 2.6 below. $\square_{2.5}$

Lemma 2.6. *Suppose M is λ-saturated (or just λ-compact), p is a Γ-big $\alpha(\Gamma)$-type over M, Γ any invariant g.bigness notion for T, $|p| < \lambda$, and the set of parameters appearing in p is $\subseteq A$ and $A_\Gamma \subseteq A$. Let $\dot{\mathbf{I}} = \{\bar{a}_i : i < \alpha\} \subseteq M$ be an infinite indiscernible sequence over A, then we can find a Γ-big type $q \in \mathbf{S}^{\alpha(\Gamma)}(A \cup \mathbf{I})$, such that $p \subseteq q$, q is Γ-big, and: if $\bar{a}$ realizes q, then $\dot{\mathbf{I}}$ is an indiscernible sequence over $A \cup \bar{a}$.*

Remark 2.7. We can do this to $\dot{\mathbf{I}}$ whose index-set is any infinite (linearly) ordered set.

Proof. For notational simplicity assume $\alpha = \alpha(\Gamma) < \omega$ (see below). We can replace M by any μ^+-saturated elementary extension for any μ and similarly α can be increased. So without loss of generality $\mu = \beth_{(2^\chi)^+}$, $\chi = \lambda + |T| + |A| + |\alpha(\Gamma)|$ and $\alpha = \mu$. We can extend p to some Γ-big $p_1 \in \mathbf{S}^\alpha(A \cup \dot{\mathbf{I}})$ and assume $\bar{a} \in M$ realizes p_1. Expand M to N by making all elements of $A \cup \bar{a}$ into individual constants, and making the set $R^N = \dot{\mathbf{I}}$ and the order $<^N = \{\langle \bar{a}_i, \bar{a}_j \rangle : i < j < \mu\}$ into relations of M. The fact that $\bar{a}$ realizes over $A \cup \dot{\mathbf{I}} = A \cup R^N$ a Γ-big complete $\lg(\bar{a})$-type, can be expressed by omitting some types (remember the "local character", i.e. Definition 1.10(b)).

By Morley theorem on the Hanf number of omitting types, (see, e.g. [She78a, Ch.VII,§5] = [She90, Ch.VII,§5]), there is a model N', elementarily equivalent to N and omitting all the types over $\emptyset$ that N omits, such that in $R^{N'}$ there is an infinite indiscernible sequence $\dot{\mathbf{J}}$ (even in the vocabulary of N'). As Γ has local character (see 1.10 clause (b)) necessarily $\bar{a}$ realizes a Γ-big complete $\ell g(\bar{a})$-type over $A \cup \dot{\mathbf{J}}$ in $N' \restriction \tau_{\mathfrak{k}}$. Now we can compute q from $\mathrm{tp}(\bar{a}, A \cup \dot{\mathbf{J}})$ in the $\tau_{\mathbf{K}}$-reduct of N'. If $\alpha(\Gamma) \geq \omega$ then in N we have also the partial function defined by $F^N_\zeta, F^N_\epsilon(a_{i,0}) = a_{i,\zeta}$ for $\zeta < \alpha(\Gamma)$ and $<^N = \{(a_{i,\zeta}, a_{j,0}) : i < j < \mu\}$. $\square_{2.6}$

Remark 2.8. We can generalize this to other cases where we have a generalization of Ramsey theorem (for 2.6 if Γ is very simple) or Erdös-Rado theorem with enough colours (for 2.6 in the general case) [Shei, §2].

Definition 2.9. 1) We define $\Gamma = \Gamma^{\mathrm{av}}_{\dot{D},\bar{a}}$ (the averaging ℓ-bigness notion) where $\bar{a} = \langle \bar{a}_\beta : \beta < \alpha \rangle$ is a sequence of sequences from some $M_0 = M_\Gamma$, $\dot{D}$ a filter on α, as follows ($\ell g(\bar{x}) = \ell g(\bar{a}_\beta)$ is constant): $\varphi(\bar{x}, \bar{a})$ is Γ-big in the model M <u>if and only if</u> ($\bar{a} \subseteq M, M_\Gamma \leq_{\mathfrak{k}} M \in \mathbf{K}$ and) $\{\beta < \alpha : M \models \varphi[\bar{a}_\beta, \bar{a}]\} \neq \emptyset \mod \dot{D}$. (So we call $\Gamma^{\mathrm{av}}_{\dot{D},\bar{a}}$ an instance of the scheme $\Gamma_{\dot{D}^{\mathrm{av}}}$.)

2) We say $\dot{D}, \bar{a}$ is non-trivial if for some $\zeta < \ell g(\bar{x})$ and for every finite $u \subseteq \ell g(\bar{a})$, the set $\{\beta < \alpha : a_{\beta,\zeta} \notin \{a_{\gamma,\zeta} : \gamma \in u\}\}$ belongs to $\dot{D}$. If $\bar{a}_i = \langle a_i \rangle$ we may write a_i instead $\bar{a}_i\}$.

Claim 2.10. *If $\bar{a}_i \in M_\Gamma \in \mathbf{K}$ for $i < \alpha$ and $\dot{D}$ a λ-complete filter on α, and $\Gamma = \Gamma^{\mathrm{av}}_{\dot{D},\bar{a}}$ <u>then</u>*

(1) $A_\Gamma = \bigcup_{i<\alpha} \bar{a}_i$ *(but also $\bigcup_{i \in Y} \bar{a}_i$ for $Y \in \dot{D}$ is O.K.)*

(2) *Γ is an invariant, very $\aleph_0$-strong, very $|\alpha|^+$-simple ℓ-bigness notion*

(3) *If $\dot{D}$ is an ultrafilter, then Γ is orthogonal to any uniformly λ-co-simple ℓ-bigness notion*

(4) $\Gamma^{\mathrm{av}}_{\dot{D},\bar{a}}$ *is non-trivial if $\dot{D}$ is non-trivial.*

Proof. For part (2) the proof of the (relatively) non-trivial part is contained in the proof of part (3) which is rephrased and proved in 1.10 below. $\square_{2.10}$

Lemma 2.11. *Suppose*

(a) *p is a Γ-big type over A in M, and $\bar{a} \in {}^{\zeta}M$, $A_\Gamma \subseteq A \subseteq M$*
(b) *$\bar{a}_i \in {}^{\zeta}A$ for each $i < \alpha$, and $\dot{D}$ is a λ-complete filter over I*
(c) *for any formula $\varphi(\bar{x}, \bar{b})$ with parameters from A, if $\models \varphi[\bar{a}, \bar{b}]$ then $\{i \in I : M \models \varphi[\bar{a}_i, \bar{b}]\} \in \dot{D}$, (hence $\mathrm{tp}(\bar{a}, A, M)$ is $\Gamma^{\mathrm{av}}_{\dot{D},\langle \bar{a}_i : i<\alpha\rangle}$-big; if $\dot{D}$ is an ultrafilter this is an equivalent formulation)*
(d) *Γ is a uniformly λ-co-simple notion of ℓ-bigness.*

We can conclude that we can extend p to a Γ-big type $q \in \mathbf{S}^m(A \cup \bar{a})$ such that for any formula $\varphi(\bar{x}, \bar{y}, \bar{b})$: if $\bar{b} \in A$, and $\{i < \alpha : \varphi(\bar{x}, \bar{a}_i, \bar{b}) \in q\} \in \dot{D}$, then $\varphi(\bar{x}, \bar{a}, \bar{b}) \in q$ (i.e. if $\bar{c}$ realizes q then $\mathrm{tp}(\bar{a}, A \cup \bar{c}, M)$ is $\Gamma^{\mathrm{av}}_{\dot{D},\bar{a}}$-big).

Proof. Let $r_{\varphi(\bar{y},\bar{z})}(\bar{z})$ be a type of cardinality $< \lambda$ (over A_Γ) such that: $\varphi(\bar{y}, \bar{b})$ is Γ-small if and only if $\bar{b}$ realizes $r_{\varphi(\bar{y},\bar{z})}(\bar{z})$ (in N whenever $M \leq_{\mathfrak{k}} N, \bar{b} \in N$ of course; exist as Γ is λ-co-simple). Without loss of generality $p \in \mathbf{S}^{\alpha(\Gamma)}(A, M)$, and now we define q, by

$$q = q(\bar{x}, \bar{y}) = p(\bar{x}) \cup q_0(\bar{y}) \cup q_1(\bar{x}, \bar{y}) \cup q_2(\bar{x}, \bar{y})$$

where

$$q_0(\bar{y}) = \{\varphi(\bar{y}, \bar{b}) : \bar{b} \in A, \text{ and } M \vDash \varphi[\bar{a}, \bar{b}]\}$$

$$q_1(\bar{x}, \bar{y}) = \left\{\varphi(\bar{x}, \bar{y}, \bar{b}) : \bar{b} \in A,\ \{i < \alpha : \varphi(\bar{x}, \bar{a}_i, \bar{b}) \in p(\bar{x})\} \in \dot{D}\right\}$$

and

$$q_2(\bar{x}, \bar{y}) = \{\neg\varphi(\bar{x}, \bar{y}, \bar{b}) : \bar{b} \in A \text{ and } \varphi(\bar{x}, \bar{a}, \bar{b}) \text{ is } \Gamma\text{-small}\}.$$

By the hypothesis on $\bar{a}$

$(*)_1$ q extend $q_0(\bar{y}) = \{\varphi(\bar{y}, \bar{b}) : \bar{b} \in A, M \models \varphi[\bar{a}, \bar{b}]\}$ and
$(*)_2$ $p(\bar{x}) \subseteq q_1(\bar{x}, \bar{y}) \subseteq q$

and lastly

$(*)_3$ every finite $q' \subseteq q$ is realized in M.

Why $(*)_3$ holds? As p, q_0 are complete types over $A, \dot{D}$ is a filter and the set of Γ-small formulas form an ideal clearly $p(\bar{x}), q_0(\bar{y}), q_1(\bar{x}, \bar{y}), q_2(\bar{x}, \bar{y})$ are closed under conjunctions hence without loss of generality $q' = \{\varphi(\bar{x}, \bar{b})\} \cup \{\varphi_0(\bar{y}, b_0)\} \cup \{\varphi_1(\bar{x}, \bar{y}, \bar{b}_1)\} \cup \{\varphi_2(\bar{x}, \bar{y}, \bar{b}_2)\}$ where $\varphi(\bar{x}, \bar{b}) \in p(\bar{x}), \varphi_0(\bar{y}, b_0) \in q_0(\bar{y})$ and $\varphi_\ell(\bar{x}, \bar{y}, \bar{b}_\ell) \in q_\ell$ for $\ell = 1, 2$. As $\varphi_2(\bar{x}, \bar{y}, \bar{b}_2) \in q_2$ necessarily $\bar{a}\hat{\ }\bar{b}_2$ realizes a type $r_{\psi(\bar{x};\bar{y}\hat{\ }\bar{z})}(\bar{y}, \bar{z})$

when we let $\varphi_2 = \neg\psi$ which satisfies: if $\bar{a}'\hat{}\bar{b}'$ realizes the type $r_{\psi(\bar{x},\bar{y}\hat{}\bar{z})}(\bar{y},\bar{z})$ then $\psi(\bar{x},\bar{a}',\bar{b})$ is Γ-small.

Now

$$\vartheta(\bar{y},\bar{z}) \in r_{(\bar{x};\bar{y}\hat{}\bar{z})}(\bar{y},\bar{z}) \Rightarrow M \models \vartheta[\bar{a},\bar{b}_2] \Rightarrow I_{\vartheta(\bar{y},\bar{z})} =: \{i < \alpha :\models \vartheta[\bar{a}_i,\bar{b}_2]\} \in \dot{D}.$$

But by assumption (c) as $\dot{D}$ is λ-complete $\mathscr{U}^* = \bigcap\{I_{\vartheta(\bar{y},\bar{z})} : \vartheta(\bar{y},\bar{z}) \in r_{\psi(x,\bar{y}\hat{}\bar{z})}\}$ belongs to $\dot{D}$. Now $\varphi_1(\bar{x},\bar{y},\bar{b}_1) \in q_1(\bar{x},\bar{y})$ hence $\mathscr{U}' =: \{i < \alpha : \varphi_1(\bar{x},\bar{a}_i,\bar{b}_1) \in p(\bar{x})\} \in \dot{D}$. As $\varphi_0(\bar{y},\bar{b}) \in q_0(\bar{y})$ clearly $\mathscr{U} = \{i < \alpha :\models \varphi_0[\bar{a}_i,\bar{b}]\} \neq \emptyset \mod \dot{D}$.

We conclude that $(\mathscr{U} \cap \mathscr{U}^*) \cap \mathscr{U}' \neq \emptyset \mod \dot{D}$ hence there is $i \in \mathscr{U} \cap \mathscr{U}' \cap \mathscr{U}^*$.

Now we can find $N, \bar{c}$ such that $M \le N$ and $\bar{c} \in N$ such that $\mathrm{tp}(\bar{c}, A \cup \bar{a}_i, N)$ is Γ-big and extend $p(\bar{x})$ exists by 1.9(4). We claim that $\bar{c}\hat{}\bar{a}_i$ realizes q' in N. Why? First $N \models \varphi[\bar{c},b]$ as $\varphi(\bar{x},b) \in p(\bar{x}) \subseteq \mathrm{tp}(\bar{c}, A \cup \bar{a}, N)$. Second, $N \models \varphi_0[\bar{a}_i,\bar{b}]$ as $i \in \mathscr{U}$. Third, $N \models \varphi_1[\bar{c},\bar{a}_i,b]$ as $i \in \mathscr{U}'$ which implies $\varphi_1(\bar{x},\bar{a}_i,\bar{b}_1) \in p(\bar{x}) \subseteq \mathrm{tp}(\bar{c},A,N) \subseteq \mathrm{tp}(\bar{c}, A \cup \bar{a}_i, N)$ and $\bar{c}$ realizes this type. Fourth, $N \models \varphi_2[\bar{c},\bar{a}_i,\bar{b}_2]$ which holds as $\bar{a}_i\hat{}\bar{b}_2$ realizes the type $r_{\psi(x,\bar{y}\hat{}z)}(\bar{y},\bar{z})$ hence $\psi(\bar{x},\bar{a}_i,\bar{b}_2)$ is Γ-small but $\mathrm{tp}(\bar{c}, A \cup \bar{a}_i, N)$ is Γ-big in N so necessarily $\neg\psi(\bar{x},a_i,b_2) \notin \mathrm{tp}(\bar{c}, A \cup \bar{a}_i, N)$ hence $\neg\psi(x,\bar{a}_i,\bar{b}_2) \in \mathrm{tp}(\bar{c}, A \cup \bar{a}_i, \bar{b}_2)$ so $N \models \neg\psi[\bar{c},\bar{a}_i,\bar{b}_2]$ but $\varphi_2(\bar{x},\bar{y},\bar{z}) = \neg\psi(\bar{x},\bar{y},\bar{z})$ so we are done. So $\bar{c}\hat{}\bar{a}_i$ really realizes q', so q' is realized as promised in $(*)_3$.

So q is indeed finitely satisfiable (in M) but $q \ge p_0(\bar{y}) \in \mathbf{S}^\zeta(A,M)$ hence even $q(\bar{x},\bar{a})$ is finitely satisfiable so let $\bar{c}^*$ realizes $q(\bar{x},a)$ in N^* where $M \le N^*$ and let $q^* = \mathrm{tp}(\bar{c}, A \cup \bar{a}, N^*)$. By our choice $q_2(\bar{x},\bar{a}) \cup q_1(\bar{x},\bar{a}) \subseteq q^*$ so clearly $q(\bar{x},\bar{a})$ is also Γ-big. Obviously it extends p, and satisfies the conclusion of the lemma. $\square_{2.11}$

Claim 2.12. *In 2.11, if $\dot{D}$ is an ultrafilter* <u>then</u> *the conclusion of 2.11 is valid even if we (seemingly) weaken the demand (c) to*

(c)$^-$ $\mathrm{tp}(\bar{a},A,M)$ *is* $\Gamma^{\mathrm{av}}_{\dot{D},\langle \bar{a}_i : i<\alpha\rangle}$*-big.*

Proof. Should be clear. $\square_{2.12}$

Observation 2.13. *1) If Γ_1, Γ_2 are local bigness notions,* <u>then</u> *the following are equivalent:*

(a) $\Gamma_1 \perp \Gamma_2$

(b) *for any Γ_1-big, $p_1 = \mathrm{tp}(\bar{a}^1,A,M)$ and Γ_2-big $p_2 = \mathrm{tp}(\bar{a}^2,A,M)$ the following set of formulas is finitely satisfiable (in M)*

$$p_1(\bar{x}^1)\,\&\quad \begin{array}{l}\cup p_2(\bar{x}^2) \cup \{\neg\varphi(\bar{x}^1,\bar{x}^2,\bar{b}) : \bar{b} \subseteq A, \varphi(\bar{x}^1,\bar{a}^2,\bar{b} \textit{ is } \Gamma_1\textit{-small}\} \\ \cup\{\neg\varphi(\bar{x}^1,\bar{x}^2,\bar{b}) : b \subseteq A, \varphi(\bar{a}^1,\bar{x}^2,\bar{b}) \textit{ is } \Gamma_2\textit{-small}\}\end{array} \tag{2.1}$$

(c) *Assume $M_{\Gamma^\ell} \le N, A_{\Gamma^1} \cup A_{\Gamma^2} \subseteq A \subseteq N, N$ is $|A|^+$-saturated, $\bar{b}_\ell \in N, p_\ell = \mathrm{tp}(\bar{b}_\ell,A,N) \in \Gamma^\ell_N$. We can find $\bar{b}'_1,\bar{b}'_2 \in N$ such that $\bar{b}'_\ell$ realizes p_ℓ $(= \mathrm{tp}(\bar{b}_\ell,A,N))$ and $\mathrm{tp}(\bar{b}'_\ell, A \cup \{\bar{b}'_{3-\ell}\}, N)$ is Γ_ℓ-big for $\ell = 1,2$*

(c)$'$ *there are $N \supseteq A_* \supseteq A_{\Gamma_1} \cup A_{\Gamma_2}$ such that N is $|A_*|^+$-saturated such that if $B \subseteq N$ is finite and $A = A_* \cup B$ then the second sentence in cluase (c) holds.*

2) If Γ_1, Γ_2 are co-simple local bigness notions <u>then</u> *we can add:*

(d) *if p_1 is Γ^1-big, p_2 is Γ^2-big, $\vartheta_1(\bar{x}^1) \in p_1, \vartheta_2(\bar{x}^2) \in p_2$, $\vartheta_1(\bar{x}^1)$ witness $\varphi_1(\bar{x}^2;\bar{x}^1)$ is Γ^2-small, $\vartheta_2(\bar{x}^2)$ witness $\varphi_2(\bar{x}^1;\bar{x}^2)$ is Γ^1-small then $\{\vartheta_1(\bar{x}^1),\vartheta_2(\bar{x}^2),\neg\varphi_1(\bar{x}^2;\bar{x}^1),\neg\varphi_2(\bar{x}^1;\bar{x}^2)\}$ is consistent; of course $\vartheta_1,\vartheta_2,\varphi_1,\varphi_2$ may have parameters. We can guarantee $\varphi_\ell(\bar{x}^{3-\ell};\bar{a}'_\ell)$ is Γ^ℓ-small for any $\bar{a}'_\ell$.*

Proof. Easy. $\square_{2.13}$

Definition 2.14. Let $\tau_1 \subseteq \tau_2$ be vocabularies, T_ℓ a complete theory in $\mathbb{L}(\tau_\ell)$, $T_1 \subseteq T_2$.
1) If Γ^2 is a local bigness notion for T_2, we define $\Gamma^1 = \Gamma^2 \restriction \tau_1$ by: for N_1 a model of $T_1, \Gamma^1_{N_1} = \{\varphi(\bar{x}_{\Gamma^2},\bar{a}) : \bar{a} \subseteq N_1$ and for some $N_2 \models T_2, N_1 \prec N_2 \restriction \tau_1$ we have $\varphi(\bar{x}_{\Gamma^2},\bar{a}) \in \Gamma^2_{N_2}\}$.
2) If Γ^2 is a global bigness notion for T_2, $\Gamma^1 = \Gamma^2 \restriction \tau_1$ is defined by: p, a complete type over A in N_1 is Γ_1-big (in N_1) if for some $N_2 \models T_2$, $N_1 \prec N_2 \restriction \tau_1$ and p can be extended to a complete type over A in N_2 which is Γ^2-big.

Remark 2.15. Note that above $\Gamma^2 \restriction \tau_1$ is not a priori a bigness notion.

Claim 2.16. *Let τ_1,τ_2,T_1,T_2 be as in Definition 2.14.*
1) The two parts of Definition 2.14 are compatible.
2) In Definition 2.14(1) if Γ^2 is a local bigness notion for T_2 then Γ_1 is a local bigness notion for T_1 (and is invariant). Similarly (see Definition 1.10(7)) for global bigness notion.
3) $\Gamma^{\rm tr},\Gamma^{\rm na},\Gamma^{\rm ids}_\alpha,\Gamma^{\rm av}_{\dot{D},\bar{a}}$, commute with the restriction operation.
4) In Definition 2.14(1), if Γ^2 is a λ-strong/ co-λ-strong/very λ-strong/co-simple local bigness notion for T_2 then Γ^1 is a λ-strong/very λ-strong/co-simple local bigness notion for T_1.
5) Assume Γ'_2,Γ''_2 are global (or local) bigness notions for T_2 and $\Gamma'_1 = \Gamma'_2 \restriction \tau_1, \Gamma''_1 = \Gamma''_2 \restriction \tau_2$. If Γ'_1,Γ''_2 are orthogonal then Γ'_1,Γ''_1 are orthogonal.
6) $\mathbf{\Gamma}$ is a co-$\mathfrak{s}$-bigness notion scheme, $\bar{\varphi}$ an interpretation of $\mathfrak{s}$ in a model M_2 of T_2, $\Gamma^2 = \mathbf{\Gamma}[\bar{\varphi},M_2]$ (see Definition 1.19) and $\Gamma^1 = \Gamma^2 \restriction \tau_1$ the relevant formulas φ_i belong to $\mathbb{L}(\tau(T_1))$, then $\Gamma^1 = \mathbf{\Gamma}[\bar{\varphi},M_2 \restriction \tau_1]$. Similarly for $\lambda-[$co$-]$ representable.
7) If in Definition 2.14, Γ^2 is a κ-weakly global bigness notion then so is Γ^1 (see Definition 2.18 below).

Remark 2.17. Why in 2.16(4) we have "co-simple" and not simple? The point is that in Definition 2.14 we have:

- $\varphi(\bar{x}_{\Gamma_2},\bar{a})$ is Γ_1-big in $M_1 \models T_1$ iff there is $M_2 \models T_2$ such that $M_1 \prec M_2{\restriction}\tau(T_1)$ and $\varphi(\bar{x}_{\Gamma_2},\bar{a})$ is Γ_2-big in M_2.

So for Γ_1-small we have to say "for every M_2 ...".

Proof. Straightforward, e.g.
4) For notational simplicity let $A_\Gamma = \emptyset$.

Case 1: Γ_1 is λ-strong.
Assume, $\varphi(\bar{x},\bar{a})$ is a Γ_1-big formula in the model M_1 of T_1, so for some model M_2 of T_2 we have $M_1 \prec M_2 \restriction \tau_1$ and $\varphi(\bar{x},\bar{a})$ is Γ_2-big in M_2, hence for some $\tau^*_2 \subseteq \tau_2$ of cardinality $< \lambda$ we have: if M^*_2 is a model of $T_2, \bar{a}^*_2 \in M^*_2$ realizes ${\rm tp}(\bar{a},\emptyset,M_2 \restriction \tau^*_2)$ then $\varphi(\bar{x},\bar{a}^*)$ is Γ_2 big in M^*_2. We shall show that $\tau^*_1 = \tau_1 \cap \tau^*_2$ is as required, so assume that M^*_1 is a model of T_1 and $\bar{a}^*_1 \in M^*_1$ realizes ${\rm tp}(\bar{a},\emptyset,M_1 \restriction \tau^*_1)$. By

Robinson lemma there is a model M_2^{**} of T_2 such that $M_1^* \prec M_2^{**} \restriction \tau_1$ and $\bar{a}_1^*$ realizes $\mathrm{tp}(\bar{a}, \emptyset, M_2^* \restriction \tau_2^*)$ hence $\varphi(\bar{x}, \bar{a}_1^*)$ is Γ_1-big.

Case 2: co λ-strong. The same proof.

Case 3: very λ-strong. Just easier.

Case 4: simple. Like case 5.

Case 5: co-simple (not co-λ-simple!).

Assume $\varphi(\bar{x}, \bar{a})$ is Γ_1-small in M_1, a model of T_1. Let $T' = T_2 \cup \{\vartheta(\bar{y}) : \vartheta(\bar{y}) \in \mathbb{L}(\tau(T_1))$ and $M_1 \models \vartheta[\bar{a}]\} \cup \{\neg\psi(\bar{y}) : \psi \in \mathbb{L}(\tau_2)$, and if $M_2 \models \psi[\bar{b}]$, M_2 a model of T_2 then $\varphi(\bar{x}, \bar{b})$ is Γ_2-small$\}$, clearly this set of (first order) formulas has no model hence is inconsistent, but the second set in the union is closed under conjunctions and also the third (as $\neg\psi_1(\bar{y}) \wedge \neg\psi_2(\bar{y})$ is $\neg(\psi_1(\bar{y}) \vee \psi_2(\bar{y}))$. So for some $\vartheta(\bar{y}) \in \mathrm{tp}(\bar{a}, \emptyset, M_1)$ and $\psi(\bar{y})$ we have $T_2 \cup \{\vartheta(\bar{y}), \neg\psi(\bar{y})\}$ is inconsistent and $[M_2 \models \psi[\bar{b}] \Rightarrow \varphi(\bar{x}, \bar{b})$ is Γ_2-small$]$ for every $\bar{b} \in M_2$, M_2 a model of T_2. So $\vartheta(\bar{y})$ is as required. $\square_{2.16}$

Definition 2.18. Γ is a $(< \kappa)$-weakly global bigness notion for T if: in Definition 1.10(1) we weaken clause (b) to:

$(b)^-$ for $\lambda < \kappa$, the odd player has a winning strategy in the following game: the game lasts $\lambda + 1$ moves, in the α-th move a Γ-big $p_\alpha \in \mathbf{S}^{\alpha(\Gamma)}(A_\alpha, M_\alpha)$ such that $\alpha < \beta \Rightarrow M_\alpha \le M_\beta \,\&\, A_\alpha \subseteq A_\beta \,\&\, p_\alpha = p_\beta \restriction A_\alpha$, the even/odd player choosing for α even/odd. The even player wins if he has no legal move for some $\alpha \le \lambda$. Otherwise the odd player wins.

Let α^*-weak mean $(< \alpha^* + 1)$-weak.

§ 3. Specific examples of bigness notion schemes

We deal with bigness notions for which some formulas have special roles. One family of natural ones are variants of a subsets of a partially ordered sets which are somewhere dense (i.e. for some x for every $y > x$ there is $z > y$ in the set). By looking at intervals of a linear order we can get as a special instance the case of dense linear orders; note this density has a different meaning, those are important for automorphism of ordered field not treated here. Another family of natural ones (considered even earlier) are connected to independence: outside a small set every combination is possible (for this we need the strong independence property); the main example here is a member of an atomic Boolean algebra such that except for a small set of atoms we have total freedom which ones to put inside and which ones to put outside. The main case here is having a pseudo finite set a as a parameter, and inside $\mathfrak{C}$ (see 1.1(B)) we say $x \subseteq a$ is big if $|x|/|a| \geq c_i$ for each $i < \delta$ where $c_i \in (0,1)^{\mathfrak{B}}_{\mathbb{R}}$ is increasing with i. So "finitary" theorems enter like the law of large numbers. We are in particular interested in the case $a = \mathscr{P}(a_1)$, and in particular if for some $\mathbf{q} \in (0,1)^{\mathfrak{B}}_{\mathbb{R}}$ we give to $b \subseteq a$ the weight $\mathbf{q}^{|b|} \times (1-\mathbf{q})^{|a \setminus b|}$.

Here we usually do not mention "in $\mathfrak{C}$" as it is obvious.

Definition 3.1. 1) Let $\mathfrak{t}^{\mathrm{po}}$ be the first order theory such a structure $(A, <, R)$ satisfies $\mathfrak{t}^{\mathrm{po}}$ if and only if $A \neq \emptyset$, $<$ a partial order, R a symmetric two-place relation satisfying $xRy \to \neg(\exists z)(x \leq z \wedge y \leq z)$, to which $({}^{\omega>}2, \triangleleft, \ntriangleleft)$ can be embedded (where if we omit R it means $xRy := \neg(\exists z)[x \leq z \wedge y \leq z)$.
2) Let $\mathfrak{t}^{\mathrm{poe}}$ be as $\mathfrak{t}^{\mathrm{po}}$ adding to the theory $x < y \Rightarrow (\exists z)[x < z \wedge zRy]$ and $\forall x \exists y(x < y)$.
2A) Let $\mathfrak{t}^{\mathrm{pot}}$ be as $\mathfrak{t}^{\mathrm{po}}$ adding to the theory $(\forall x)(\exists y_1, y_2)[x < y_1 \,\&\, x < y_2 \,\&\, y_1 R y_2]$.
3) Let Γ^{po} be the following pre-$\mathfrak{t}^{\mathrm{po}}$-bigness notion scheme $\psi(P)$ (see Definition 1.19): for M a model of $\mathfrak{t}^{\mathrm{po}}$, and $P \subseteq M$: $M \models \psi[P]$ says that the following is finitely satisfiable in (M, P):

$$\begin{aligned} \{P(\bar{x}_\eta) : \eta \in {}^{\omega>}2\} \quad &\cup \{x_\eta < x_\nu : \eta \triangleleft \nu \in {}^{\omega>}2\} \\ &\cup \{x_{\eta \hat{}\langle 0 \rangle} R x_{\eta \hat{}\langle 1 \rangle} : \eta \in {}^{\omega>}2\}. \end{aligned} \tag{3.1}$$

4) Let Γ^{poe} be the following pre-$\mathfrak{t}^{\mathrm{poe}}$-bigness scheme: for M, a model of $\mathfrak{t}^{\mathrm{poe}}$ and $P \subseteq M$: $\psi[P]$ says $(\exists x)(\forall y)(\exists z \in P)[x < y \to y < z]$ (this means somewhere dense).
5) Γ^{pot} is defined like Γ^{poe}.

Remark 3.2. A natural example of 3.1(4) (more exactly, a model of $\mathfrak{t}^{\mathrm{poe}}$) is the set of open intervals of a dense linear order ordered by inverse (strict) inclusion with $(a,b)R(a',b')$ iff $(a,b) \cap (a',b') = \emptyset$.

Claim 3.3. *1) Γ^{po} is a $\mathfrak{t}^{\mathrm{po}}$-bigness notion scheme and is $\aleph_1$-presentable so by 1.22(2), (3) is invariant, very $\aleph_0$-strong co-simple and uniformly $\aleph_1$-simple local bigness notion.*
2) Γ^{poe} is a $\mathfrak{t}^{\mathrm{po}}$-bigness notion scheme and is presentable (so by 1.22 is invariant, very simple). Similarly for Γ^{pot}.
3) For $M \models \mathfrak{t}^{\mathrm{poe}}$ we have $\Gamma^{\mathrm{po}}_M = \Gamma^{\mathrm{poe}}_M$. If $\bar{\varphi}$ is an interpretation of $\mathfrak{t}^{\mathrm{poe}}$ in a model M then $\Gamma^{\mathrm{po}}_M[\bar{\varphi}] = \Gamma^{\mathrm{poe}}_M[\bar{\varphi}]$. Similarly for Γ^{pot}.
4) $\mathfrak{t}^{\mathrm{poe}} \perp \mathfrak{t}^{\mathrm{tr}}$ and $\mathfrak{t}^{\mathrm{tr}} \perp \mathfrak{t}^{\mathrm{po}}$.

Definition 3.4. 1) $\mathfrak{t}^{\rm ind}$ is the theory saying on a model $M, M = (P, Q, R)$, P, Q are disjoint (are two sorts (or if you prefer – two unary predicates)), such that $R \subseteq P \times Q, R$ has the strong independence property (see Definition 3.5 below).
2) We define a pre-$\mathfrak{t}^{\rm ind}$-bigness notion scheme $\Gamma = \Gamma^{\rm ind}$ as follows: $\psi_\Gamma(P^*)$ says $P^* \subseteq Q$ and (P, Q, R, P^*) satisfies: for every n it is Γ_R-n-big which means that for some finite $A \subseteq P, P^*$ has n-independence outside A (A is called a Γ-n-witness), which means: for every pairwise distinct $a_0, \ldots, a_{2n-1} \in P \setminus A$, for some $c \in P^*$ we have $\forall \ell < 2n \Rightarrow [a_\ell Rc]^{\text{if}(\ell<n)}$; so ψ_Γ is not first order because we have said "some finite A" but $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$.

Definition 3.5. 1) Let P, Q be one place predicates, R a two-place predicate and suppose the theory T contains the formula $(\forall xy)[xRy \to P(x) \,\&\, Q(y)]$.

We say R (more exactly P, Q, R) has the strong independence property (for or in the theory T) <u>if</u>:

- (a) P, Q, R as above
- (b) for every $n < \omega$, $M \models T$, and pairwise distinct $a_1, \ldots, a_{2n} \in P^M$ there is $c \in Q^M$ such that: $a_i Rc$ iff $i \leq n$.

2) We say R i.e. (P, Q, R) has comprehension, i.e.

$$\forall x, y \exists z\ [Q(x) \wedge Q(y) \wedge x \neq y \to P(z) \wedge (zRx \equiv \neg zRy)].$$

Example 3.6. The following are examples of theories T implying (P, Q, R), i.e. $(P^{\mathfrak{C}(H)}, Q^{\mathfrak{C}(T)}, R^{\mathfrak{C}(T)})$ has the strong independence property and comprehension.
1) $T =$ true arithmetic, that is the theory of $\mathbb{N} = (\omega, +, \times, 0, 1)$. Let
$P(x) : x$ is prime-
$Q(x) : x > 0$ not divisible by any square of a prime,
$xRy : x$ divides y, and x is prime and $Q(y)$.
2) T as above
$P(x) : x = x$.
$xRy : y$ codes a sequence in which x appears (using a fix coding).
3) $T =$ the first order theory of infinite atomic Boolean algebras
$P(x) : x$ an atom,
$Q(x) : x = x$,
$xRy : x \leq y$

Claim 3.7. *$\Gamma^{\rm ind}$ is a $\mathfrak{t}^{\rm ind}$-bigness notion scheme (hence invariant) and is very $\aleph_1$-strong and $\aleph_1$-co-strong (but not uniformly).*

Proof. Assume $\bar\varphi$ is a $\mathfrak{t}^{\rm ind}$-interpretation, and let $M[\bar\varphi] = (P, Q, R)$. Concerning the $\aleph_1$-co-simple, note that "$\varphi(x, \bar a)$ is Γ_M-small" iff for some m, we have: $\bar a$ realizes the type

$$q_m(\bar y) =: \{ \begin{array}{l} \neg(\exists y_0) \ldots (\exists y_{n-1})(\forall z_0) \ldots (\forall z_{m-1}) \\ [\bigwedge\limits_{\ell<n} \bigwedge\limits_{k<m} (y_\ell \neq z_k \,\&\, \bigwedge\limits_{\ell<k<m} z_\ell \neq z_k \,\&\, \bigwedge\limits_{k<m} P(z_k) \to \\ \bigwedge\limits_{w \subseteq m} (\exists x)(Q(x) \,\&\, \varphi(x, \bar y) \,\&\, \bigwedge\limits_{k<m} \varphi(x, y_k)^{\text{if}(k \in w)})] : n < \omega \}. \end{array} \tag{3.2}$$

We should check $\Gamma_M = (\Gamma^{\rm ind}_{M[\bar\varphi]})_M$ satisfies "Γ_M is a proper ideal" (the other conditions are obvious). So we should check $(\alpha), (\beta), (\gamma)$ of 1.4(1) (c) in order to show that "Γ_M is a proper ideal".

So $(\alpha) + (\gamma)$ obviously hold. How about (β) i.e. $\varphi_1 \vee \varphi_2$? Suppose φ_1 is not $\Gamma_R - n$-big and φ_2 is not $\Gamma_R - m$-big. Let $A \subseteq M$ be finite and we shall show that A cannot be a $\Gamma_R - (n+m)$-witness for $\varphi_1 \vee \varphi_2$ (see Definition 3.4(2)).

As φ_1 is $\Gamma_M - n$-small, A cannot be a witness hence there are $a_0, \dots, a_{2n-1} \in P^M \setminus A$ with no repetition such that

$$\neg(\exists x)\big(\varphi(x,\bar{a}) \wedge \bigwedge_{i<2n} [a_i R x]^{\mathrm{if}(i<n)}\big).$$

Now let $A' = A \cup \{a_0, \dots, a_{2n-1}\}$ it cannot be a $\Gamma_R - m$-witness for φ_2. So there are $b_0, \dots, b_{2m-1} \in P^M \setminus A'$ with no repetition such that

$$\neg(\exists x)\Big(\varphi(x,\bar{a}) \wedge \bigwedge_{i<2n} [b_i R x]^{\mathrm{if}(i<m)}\Big).$$

Clearly $a_0, \dots, a_{2n-1}, b_0, \dots, b_{2m-1} \in P^M \setminus A$ are pairwise distinct and

$$\neg(\exists x)\Big((\varphi_1 \vee \varphi_2) \,\&\, \bigwedge_{i<2n} [a_i R x]^{\mathrm{if}(i<n)} \,\&\, \bigwedge_{i<2n} [b_i R x]^{\mathrm{if}(i<m)}\Big).$$

So A is not a $\Gamma_R - (n+m)$-witness for $\varphi_1 \vee \varphi_2$. $\square_{3.7}$

Definition 3.8. Let T^* be as in 1.1(2)[5], and $M^* \prec \mathfrak{C}$ be a model of T^*. Let a, c_i ($i < \delta$, δ a limit ordinal) be members of M^* such that in M^*:

- (α) a is a “finite set”
- (β) $|a| \geq n$ for every true natural number n
- (γ) c_i is a rational (or even real), $0 < c_i < \frac{1}{n}$ for every true natural number n
- (δ) $2c_i < c_{i+1}$ and $c_i < c_j$ for $i < j < \delta$

1) Let $\bar{c} = \langle c_i : i < \delta \rangle$. We define the local bigness notion $\Gamma = \Gamma^{\mathrm{ms}}_{a,\bar{c}}$ as follows: $\varphi(\bar{x}, \bar{b})$ is Γ-big <u>if and only if</u> $M^* \models$ “$|\{x : x \text{ is a member of } a, \varphi(x, \bar{b})\}|/|a|$ is $> c_i$” for every $i < \delta$.
2) Let $\Gamma^{\mathrm{ms}}_\delta$ be the scheme whose instances are $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$, where a and $\bar{c} = \langle c_i : i < \delta \rangle$ are as above for T^*. Let Γ^{ms} mean $\bigcup_\delta \Gamma^{\mathrm{ms}}_\delta$.
3) We say “the smallness of φ is witnessed by c_i” <u>if</u> the quotient in part (1) is $< c_i$.
4) If $a, \bar{c}$ satisfy $(\alpha) - (\delta)$ we say that $\bar{c}$ is an increasing sequence for a. It is called O.K. for a if also

- (ϵ) $|a| \times c_i > 1$ for every $i < \delta$.

We may say that $\bar{c}$ is O.K. for a or a, $\bar{c}$ are O.K.
5) We say $\bar{c}$ is wide <u>if</u> for every $i < \delta$ and $n < \omega$ we have $c_{i+1}/c_i > n$. We say that a, $\bar{c}$ are wide for M^* or $\bar{c}$ wide for a in M^*) <u>if</u> clauses $(\alpha) - (\delta)$ above and $\bar{c}$ is wide.

Remark 3.9. On $\Gamma^{\mathrm{ms}}_{a,\bar{c}_1}, \Gamma^{\mathrm{ms}}_{a,\bar{c}_2}$ being equivalent see 3.10(4).

---

[5] we need just some schemes

Claim 3.10. *1) If for $T^*, M^*, a, \bar{c}$ are as in definition 3.8 (so clauses $(\alpha) - -(\delta)$ holds), $\underline{\text{then}}$ $\Gamma = \Gamma^{\rm ms}_{a,\bar{c}}$ is a uniformly $|\delta|^+$-simple ℓ-bigness notion (with set of parameters $\{a\} \cup \bar{c}$) hence Γ is co-simple. If in addition $\bar{c}$ is O.K. for a then Γ is not trivial. If $\bar{c}$ is not O.K. then Γ is trivial.*
2) Suppose T^*, M^* are as in Definition 3.8, and for $\ell = 1, 2$ $a^\ell, \bar{c}^\ell = \langle c^\ell_\alpha : \alpha < \delta \rangle$ are as in Definition 3.8(A) (for T^*, M^*), and $\Gamma^\ell = \Gamma^{\rm ms}_{a^\ell, \bar{c}^\ell}$. $\underline{\text{Then}}$ Γ^1, Γ^2 are orthogonal.
3) Let $\dot{D}$ be a filter say on κ; the bigness notion $\Gamma^{\rm ms}_\delta, \Gamma^{\rm av}_{\dot{D}}$ are orthogonal if $\mathrm{cf}(\delta) > \aleph_0$ (not κ!); also $\Gamma^{\rm ms}_{a,\bar{c}}, \Gamma^{\rm av}_{\dot{D}}$ are orthogonal if $\bar{c}$ is wide.
4) If in $M^*, a, \bar{c}^\ell = \langle c^\ell_i : i < \delta_\ell \rangle$ is as in Definition 3.8 for $\ell = 1, 2$ and $(\forall i < \delta_1)(\exists j < \delta_2)[c^1_i < c^2_j]$ and $(\forall j < \delta_2)(\exists i < \delta_1)[c^2_j < c^1_i]$ $\underline{\text{then}}$ $\Gamma^{\rm ms}_{a,\bar{c}^1} = \Gamma^{\rm ms}_{a,\bar{c}^2}$. Hence for $a, \bar{c}^1$ as above if $\mathrm{cf}(\delta) > \aleph_0$ then for some $\bar{c}^2$ wide for $\bar{a}$, we have $\Gamma^{\rm ms}_{a,\bar{c}^1_*} = \Gamma^{\rm ms}_{a,c^2}$.
5) $\Gamma^{\rm ms}_{a,\bar{c}}$ is non-trivial (i.e. no algebraic type is $\Gamma^{\rm ms}_{a,\bar{c}}$-big) $\underline{\text{if}}$ $\bar{c}$ is wide or just O.K. for a.
6) If $a, \bar{c} = \langle c_i : i < \delta \rangle$ is as in Definition 3.8 and $\mathrm{cf}(\delta) > \aleph_0$ or just $\omega\omega$ divides δ $\underline{\text{then}}$ $\langle c_{\omega i} : \omega i < \delta \rangle$ is wide.

Proof. 1) Note that $M^* \models$ "$2c_i < c_{i+1}$". So assume $\varphi(x, \bar{b}) = \varphi_1(x, \bar{b}_1) \vee \varphi_2(x, \bar{b}_2)$ and $\varphi_\ell(x, \bar{b}_\ell)$ is Γ-small for $\ell = 1, 2$. So for $\ell = 1, 2$ for some $i_\ell < \delta$ the formula $\varphi_\ell(x, \bar{b}_\ell)$ being Γ-small is witnessed by i_ℓ so $M^* \models |\{x \dot{E} a : \varphi_\ell(x, \bar{b}_\ell)\}| \le c_{i_\ell} \times |a|$.
Hence

$$
\begin{aligned}
M^* \models |\{x \dot{e} a : \varphi(x, \bar{b})\}| \quad &\le \sum_{\ell=1}^{2} |\{x \dot{e} a : \varphi_\ell(x, \bar{b}_\ell)\}| \\
&\le \sum_{\ell+1}^{2} c_{i_\ell} \times |a| \le 2c_{\max\{i_1, i_2\}} \times |a| \\
&< c_{\max\{i_1, i_2\}+1} \times |a|
\end{aligned}
\tag{3.3}
$$

so $\varphi(x, \bar{b})$ is Γ-small as witnessed by $\max\{i_1, i_2\} + 1 < \delta$. The other facts are even easier.
2) This is really a discrete version of Fubini theorem but we shall elaborate. Without loss of generality $\bar{c}^\ell$ is O.K. for $\ell = 1, 2$.

Let $A_{\Gamma_1} \cup A_{\Gamma_2} \subseteq A \subseteq M^*$ and for $\ell = 1, 2$ let $p_\ell(\bar{x}_{\Gamma_\ell})$ be a Γ_ℓ-big type over $A, p_\ell(\bar{x}_{\Gamma_\ell}) \in \mathbf{S}^{\alpha(\Gamma_\ell)}(A)$.

As each Γ_ℓ is co-simple let $q := p_1(x) \cup p_2(y) \cup \{\psi_1(y, \bar{d}) \to \neg\varphi_1(x, y, \bar{d}) : \bar{d} \subseteq A, \psi_1(y, \bar{z})$ witness $\varphi_1(x; y, \bar{z})$ is Γ_1-small$\} \cup \{\psi_2(x, \bar{d}) \to \neg\varphi_2(y, x, \bar{d}) : \bar{d} \subseteq A$ and $\psi_2(x, \bar{z})$ witness $\varphi_2(y; x, \bar{z})$ is Γ_2-small$\}$.

By 1.12(1)(b) it suffice to prove that this set of formulas is finitely satisfiable in M^*, assume not. So we have (increasing the sequences of parameters from M^* noting $p_\ell(\bar{x}_{\Gamma_\ell})$ is closed under conjunctions) $\vartheta_1(x, \bar{d}^*) \in p_1(x), \vartheta_2(y, \bar{d}^*) \in p_2(y), \psi_{1,k}(y, \bar{d}^*) \longrightarrow \neg\varphi_{1,k}(x, y, \bar{d}^*)$ for $k < k_1$ from the third term in the union with smallness witnessed by $c^1_{\alpha_k}$ (see 3.8(3)) and $\psi_{2,k}(x, \bar{d}^*) \longrightarrow \neg\varphi_{2,k}(y, x, \bar{d}^*)$ for $k < k_2$ from the fourth term in the union with smallness witnessed by $c^2_{\beta_k}$ (see 3.8(3)). Note k_1, k_2 are true natural numbers. Choose $\alpha(*) < \delta_1$ such that $(\forall k < k_1)[\alpha_k + k_1 < \alpha(*)]$, and choose $\beta(*) < \delta_2$ such that $(\forall k < k_2)[\beta_k + k_2 < \beta(*)]$. Let (recalling $\dot{e}$ is membership in M^*'s sense) $Z = \{(x, y) : x \dot{e} a^1 \,\&\, y \dot{e} a^2 \,\&\, \vartheta_1(x, \bar{d}^*) \,\&\, \vartheta_2(y, \bar{d}^*)\}$. So Z is (representable) in M^* (we do not distinguish).

Let (all is M^*'s sense):

$$Z_1 = \{(x,y)\dot{e}Z : (\exists k < k_1)[\psi_{1,k}(y,\bar{d}^*) \,\&\, \varphi_{1,k}(x;y,\bar{d}^*)]\}$$

$$Z_2 = \{(x,y)\dot{e}Z : (\exists k < k_2)[\psi_{2,k}(x,d^*) \,\&\, \varphi_{2,k}(y;x,\bar{d}^*)]\}.$$

So

(a) $Z = Z_1 \cup Z_2$ (by the "assume not" above)
(b) for every $y, |Z_1^{[y]}|/|a^1| \leq c^1_{\alpha(*)}$ where $Z_1^{[y]} = \{x : (x,y)\dot{e}Z_1\}$.

[Why? As a union of k_1 sets, each with $\leq c^1_{\alpha(*)-k_1} \times |a^1| \leq \frac{1}{2^{k_1}} \times c^1_{\alpha(*)} \times |a^1|$ members has $\leq c^1_{\alpha(*)} \times |a^1|$ members.]

And similarly

(c) for every $x, |Z_2^{[x]}|/|a^2| \leq c^2_{\beta(*)}$ where $Z_2^{[x]} = \{y : (x,y)\dot{e}Z_2\}$.

There is in M^* a set X, such that:

$$x\dot{e}X \Rightarrow x\dot{e}a^1 \,\&\, \vartheta_1(x,\bar{d}^*)$$

$$(c^1_{\alpha(*)+2})|a^1| + 1 \geq |X| \geq (c^1_{\alpha(*)+2}) \times |a^1|.$$

There is in M^*'s sense a set Y such that

$$y\dot{e}Y \Rightarrow y\dot{e}a^2 \,\&\, \vartheta_2(y,\bar{d}^*)$$

$$(c^2_{\beta(*)+2})|a^2| + 1 \geq |Y| > (c^2_{\beta(*)+2})|a^2|.$$

By the choice of Z clearly $X \times Y \subseteq Z$. So in M^* (for the fifth line recall $\bar{c}$ is O.K., for the seventh line recall $2c_i < c_{i+1}$)

(3.4)
$$\begin{aligned}
(c^1_{\alpha(*)+2}|a^1|)(c^2_{\beta(*)+2}|a^2|) \quad & \leq |X| \times |Y| = |X \times Y| \leq |Z_1 \cap (X \times Y)| + |Z_2 \cap (X \times Y)| \\
& \leq |X|(c^1_{\alpha(*)}|a^1|) + |Y|(c^2_{\beta(*)}|a^2|) \\
& \leq (c^1_{\alpha(*)}|a^1|)(c^2_{\beta(*)+2}|a^2| + 1) + (c^1_{\alpha(*)+2}|a^1| + 1)(c^2_{\beta(*)}|a^2|) \\
& \leq (c^1_{\alpha(*)}|a^1|)(2c^2_{\beta(*)+2}|a^2|) + (2c^1_{\alpha(*)+2}|a^1|)(c^2_{\beta(*)}|a^2|) \\
& = (2c^1_{\alpha(*)}|a^1|)(c^2_{\beta(*)+2}|a^2|) + (c^1_{\alpha(*)+2}|a^1|)(2c^2_{\beta(*)}|a^2|) \\
& < (\tfrac{1}{2}c^1_{\alpha(*)+2}|a^1|)(c^2_{\beta(*)+2}|a^2|) + (c^1_{\alpha(*)+2}|a^1|)(\tfrac{1}{2}c^2_{\beta(*)+2}|a^2|) \\
= c^1_{\alpha(*)+2}|a^1|c^2_{\beta(*)+2}|a^2| &
\end{aligned}$$

contradiction.

3) Let $\Gamma_1 = \Gamma^{\rm av}_{\dot{D},\langle b^*_\epsilon:\epsilon<\kappa\rangle}, \Gamma_2 = \Gamma^{\rm ms}_{a,\bar{c}}, \bar{c} = \langle c_i : i < \delta\rangle$.

By (4)+(6) below (which does not depend on 3.10(3)) it suffices to prove the second case i.e. prove orthogonality assuming $(\forall n < \omega)(\forall i < \delta)$"$n < c_{i+1}/c_i$" (as if $\mathrm{cf}(\delta) > \omega$, letting $\bar{c}' = \langle c_{\omega \times j} : \omega j < \delta\rangle$ we have $\Gamma^{\rm ms}_{a,\bar{c}}$, $\Gamma^{\rm ms}_{a,\bar{c}'}$ are equal and $\bar{c}'$ satisfies the requirement above).

Given $A \subseteq M^*$ such that $b^*_\epsilon, a, c_j \in A$ (for $\epsilon < \kappa, j < \delta$), and Γ_ℓ-big $p_\ell = \mathrm{tp}(b_\ell, A, M^*)$ (for $\ell = 1,2$), possibly increasing M^* we can find $c \in M^*$ such that $M^* \models$ "c a natural number, $n < c, 2^{(c^n)} < \log_2(c_{i+1}/c_i)$" for every $n < \omega, i < \delta$, and we can find a pseudo-finite set $d \in M^*$, such that: $M^* \models$ "$|d| = c \,\&\, b^*_\epsilon \dot{e} d$" for $\epsilon < \kappa$.

Let $A_0 = A, A_1 = A_0 + d$ and let $p'_2 \in \mathbf{S}(A_1, M^*)$ be a Γ_2-big extension of p_2. We shall show now that (without loss of generality) there are $e^*_\varphi \in M^*$ (for the $\varphi = \varphi(x, \bar{z}_\varphi) \in \mathbb{L}(T^*[A_1])$) such that $M^* \models$ "$e^*_\varphi \subseteq {}^{n(\varphi)}d$" where $n(\varphi) = \ell g(\bar{z}_\varphi)$ and

$$(*)\ \ p^*_2 = p'_2 \cup \{(\forall \bar{z} \dot{e}^n d)[\bar{z} \dot{e} e^*_\varphi \Leftrightarrow \varphi(x, \bar{z})] : \varphi = \varphi(x, \bar{z}) \in \mathbb{L}_\tau(T^*[A_1])\} \text{ is } \Gamma_2\text{-big.}$$

Why $(*)$ holds? As $\Gamma^{\rm ms}_{a,\bar{c}}$ is a local co-simple bigness notion, we can also replace p'_2 by one formula say $\vartheta_2(x, \bar{b}^*)$, and consider only $\varphi_1, \ldots, \varphi_n \in \mathbb{L}(T^*[A_1])$ for some $n < \omega$. We can find $\varphi = \varphi(x, \bar{z})$ such that for parameters from ${}^{n(\varphi)}d$ we get all the instances of $\varphi_1, \ldots, \varphi_n$ by increasing $\bar{z}$, hence without loss of generality we can consider just one $\varphi = \varphi(x, \bar{z}_\varphi)$. Let $n = n(\varphi) = \ell g(\bar{z}_\varphi)$, then $\vartheta(M^*, \bar{b}^*)$ has size $\geq c_\alpha |a|$ (for every $\alpha < \delta$, internal sense), it is divided to $\leq |\mathscr{P}({}^{n(\varphi)}d)|$ parts according to the φ-type over d, so the largest one (internal sense) is as required (or find a right e^*_φ in a $|\delta|^+$-saturated extension of M^*) so really $(*)$ holds.

So we can find $p''_2 \in \mathbf{S}(A_2, M^*)$ extending p^*_2 which is Γ_2-big where $A_2 = A_1 \cup \{e^*_\varphi : \varphi \in \mathbb{L}(T^*[A_1])\}$.

Let $b'_2 \in M^*$ realize p''_2, b'_1 realize a Γ_1-big $p''_1 \in \mathbf{S}(A_2 + b'_2, M^*)$ extending p_1 so clearly $[x \dot{e} d] \in \bar{p}''_1$. They are as required (think or see 3.31(3)).

4), 5), 6) Trivial. $\square_{3.10}$

Remark 3.11. You may wonder whether we can weaken the demand on T, still demanding that a behave like a finite set. Certainly we can, e.g. by using restriction, see Definition 1.13. We can do it in a more finely tuned way, we hope to deal with it elsewhere.

Definition 3.12. 1) Suppose that $T^*, \mathfrak{C}$ are as in 1.1(B), a in $\mathfrak{C}$, a is a pseudo finite set, $\dot{w}$ a function from a to $[0,1]_{\mathbb{R}}$ such that $\mathfrak{C} \models$ "$\sum\limits_{x \dot{e} a} \dot{w}(x) = 1$", and $\bar{c} = \langle c_i : i < \delta \rangle$ (δ a limit ordinal) is an increasing sequence for a. For every $a' \subseteq a$ let $\dot{w}(a')$ be $\sum\limits_{x \dot{e} a'} \dot{w}(x)$, in $\mathfrak{C}$'s sense; if confusion may arise we shall write $\dot{w}(\{x\})$ for $x \dot{e} a$.

We define $\Gamma = \Gamma^{\rm wm}_{a,\dot{w},\bar{c}}$, a local bigness notion, by: $\psi(x, \bar{b})$ is Γ–small <u>if and only if</u> for some $i < \delta, \mathfrak{C} \models$ "$\dot{w}(\{x \dot{e} a : \psi(x, \bar{b})\}) \leq c_i$".

2) Assume above that in $\mathfrak{C}$ we have: $a^+ = \mathscr{P}(a), \mathbf{q} \dot{e} (0,1)_{\mathbb{R}}$. Let $\dot{w}_{\mathbf{q}} = \dot{w}_{a,\mathbf{q}} : a^+ \to [0,1]_{\mathbb{R}}$ be defined by

$$\dot{w}(b) = \mathbf{q}^{|b|}(1 - \mathbf{q})^{|a \setminus b|}.$$

We let $\Gamma^{\rm wmg}_{a,\mathbf{q},\bar{c}} = \Gamma^{\rm wm}_{\mathscr{P}(a)}, \dot{w}_{\mathbf{q},\bar{c}}$. If $\mathbf{q} = c_0$ we write $\Gamma^{\rm wmg}_{a,\bar{c}}$, we always assume $\mathbf{q} \leq 1/2$.

3) We say $\bar{c}$ is O.K. for $(a, \dot{w})$ or $(a, \dot{w}, \bar{c})$ is O.K. <u>if</u> for some $i < \ell g(\bar{c})$ we have $\mathfrak{C} \Vdash$ "$(\forall x \dot{e} a) \dot{w}(x) < c_i$", we normally assume that this holds for $i = 0$.

4) We say $\bar{c}$ is wide for $(a, \dot{w})$ or $(a, \dot{w}, \bar{c})$ is wide <u>if</u> $(\forall i < \ell g(\bar{c}))(\forall n < \omega)[c_{i+1}/c_i > n]$.

Remark 3.13. Remember $\ln(1-x) \sim\ -x$ for $x \in (0, 1/2)$, more exactly for the natural logarithm, $|\ln(1-x) + x| < x^2$ for every $x \in (0, \frac{1}{2})$. Why? Because by the Taylor series, $\ln(1-x) = (x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots)$ and $\frac{x^3}{3} + \frac{x^4}{4} + \ldots < \frac{1}{3}x^3 + \frac{1}{3}x^3(1 + x + x^2 + \ldots) = \frac{1}{3}x^3 + \frac{1}{3}x^3 \frac{1}{1-x} \leq \frac{1}{3}x^3 \frac{1}{1-1/2} = x^3 < \frac{1}{2}x^2)$.

Claim 3.14. *1) For $T^*, M^*, a, \dot{w}, \bar{c}$ as in Definition 3.12(A), then $\Gamma = \Gamma^{\rm wm}_{a,\dot{w},\bar{c}}$ is a uniformly $|\lg(\bar{c})|^+$-simple ℓ-bigness notion (with set of parameters $\{a\} \cup \{\dot{w}\} \cup \bar{c}$ so Γ is co-simple).*
2) Suppose T^*, M^* are as in Definition 3.12(A), and for $\ell = 1,2$ $a^\ell, \dot{w}^\ell, \bar{c}^\ell = \langle c^\ell_\alpha : \alpha < \delta\rangle$ are as in Definition 3.12(A) (for T^*, M^*), and $\Gamma^\ell = \Gamma^{\rm wm}_{a^\ell,w^\ell,\bar{c}^\ell}$. Then Γ^1, Γ^2 are orthogonal.
3) $\Gamma^{\rm wm}_\delta, \Gamma^{\rm av}_{\dot{D}}$ are orthogonal if $\mathrm{cf}(\delta) > \aleph_0$; $\Gamma^{\rm wm}_{a,\dot{w},\bar{c}}$ $\Gamma^{\rm av}_{\dot{D}}$ are orthogonal if $\bar{c}$ is wide or $\omega\omega$ divides δ.
4) If in $M^*, \bar{c}^\ell = \langle c^\ell_i : i < \delta_\ell\rangle$ for $\ell = 1,2$ and $(\forall i < \delta_1)(\exists j < \delta_2)[c^1_i < c^2_j]$ and $(\forall j < \delta_2)(\exists i < \delta_1)[c^2_j < c^1_i]$ then $\Gamma^{\rm wm}_{a,t,\bar{c}^1} = \Gamma^{\rm wm}_{a,\dot{w},\bar{c}^2}$.
5) $\Gamma^{\rm wm}_{a,\dot{w},\bar{c}}$ is not trivial (i.e. if $p(x)$ is $\Gamma^{\rm wm}_{a,\dot{w},\bar{c}}$-big then $p(x)$ is not an algebraic type) if and only if $(a, \dot{w}, \bar{c})$ is O.K.
6) $\Gamma^{\rm ms}_{a,\bar{c}}$ is a special case of $\Gamma^{\rm wm}_{a,\dot{w},\bar{c}}$.

Proof. Like 3.10 except that in part (2) in the case $(a, \dot{w}, \bar{c})$ is not O.K we have to take more care (and this case is not used). $\square_{3.14}$

A "dual" notion is

Definition 3.15. Let T^* be as in 1.1(B)[6], and M^* be a model of T^*. Let a, c_i ($i < \delta$, δ a limit ordinal) be members of M^* such that in M^*:

- (α) a is a "finite" set
- (β) $|a| \geq n$ for every true natural number n
- (γ) c_i is a "rational", $0 < c_i < \frac{1}{n}$ for every true natural number n
- (δ) $c_i > 2c_{i+1}$ and $i < j \Rightarrow c_i > c_j$.

1) Let $\bar{c} = \langle c_i : i < \delta\rangle$. We define the ℓ-bigness notion $\Gamma = \Gamma^{\rm dms}_{a,\bar{c}}$ as follows: $\varphi(\bar{x}, \bar{b})$ is Γ-big if and only if $M^* \models$ "$|\{x : x$ is a member of $a, \varphi(x, \bar{b})\}|/|a|$ is $> c_i$" for some $i < \delta$.
2) Let $\Gamma^{\rm dms}_\delta$ be the scheme whose instances are $\Gamma^{\rm dms}_{a,\bar{c}}$, where a and $\bar{c} = \langle c_i : i < \delta\rangle$ as above for T^*. Let $\Gamma^{\rm dms}$ mean $\bigcup\limits_\delta \Gamma^{\rm dms}_\delta$.
3) We shall say "the bigness of φ is witnessed by c_i" if the quotient above is $> c_i$.
4) $\bar{c}$ is decreasing sequence if: clauses (α) – (δ) above hold. It is d-O.K. for a if $\frac{n}{|a|} < c_i < \frac{1}{n}$ for $n \leq \omega$. It is wide if $c_i/c_{i+1} > n$ for $i < \delta$, $n < \omega$.

Claim 3.16. *1) For $T^*, M^*, a, \bar{c}$ as in Definition 3.15, then $\Gamma = \Gamma^{\rm dms}_{a,\bar{c}}$ is a uniformly $|\delta|^+$-co-simple ℓ-bigness notion (with set of parameters $\{a\} \cup \bar{c}$) hence is simple; Γ is non-trivial if $\bar{c}$ is d-O.K..*
2) Suppose T^*, M^* are as in Definition 3.15, and for $\ell = 1,2$ $a^\ell, \bar{c}^\ell = \langle c^\ell_i : i < \delta\rangle$ are as in Definition 3.15 (for T^*, M^*), and $\Gamma^\ell = \Gamma^{\rm dms}_{a^\ell,\bar{c}^\ell}$. Then Γ^1, Γ^2 are orthogonal.
3) $\Gamma^{\rm dms}_\delta, \Gamma^{\rm av}_{\dot{D}}$ are orthogonal if $\mathrm{cf}(\delta) > |\mathrm{Dom}(\dot{D})|$ for any filter $\dot{D}$.
4) If in $M^*, \bar{c}^\ell = \langle c^\ell_i : i < \delta_\ell\rangle$ for $\ell = 1,2$ and $(\forall i < \delta_1)(\exists j < \delta_2)[c^2_j < c^1_i]$ and $(\forall j < \delta_2)(\exists i < \delta_1)[c^1_i < c^2_j]$ then $\Gamma^{\rm dms}_{a,\bar{c}^1} = \Gamma^{\rm dms}_{a,\bar{c}^2}$.
5) In Definition 3.15, if $\mathrm{cf}(\delta) > \aleph_0$ or just $\omega\omega$ divides δ then $\bar{c}' = \langle c_{\omega i} : \omega i < \delta\rangle$ is wide and $\bar{c}, \bar{c}'$ are like $\bar{c}^1, \bar{c}^2$ in part (4).

[6] we need just some schemes

6) Assume that $T^*, M^*, a^1, \bar c^1$ are as in Definition 3.8, $\Gamma_1 = \Gamma^{ms}_{\bar a^1, \bar c^1}$, and $(T^*, M^*), a^2, \bar c^2$ are as in Definition 3.15 and $\Gamma_2 = \Gamma^{\rm dms}_{\bar a^2, \bar c^2}$. <u>Then</u> Γ_1, Γ_2 are orthogonal.

Proof. 1) Note that $M^* \models$ "$c_i > 2c_{i+1}$". So assume $\varphi(x, \bar b) = \varphi_1(x, \bar b_1) \vee \varphi_2(x, \bar b_2), \varphi_\ell(x, \bar b_\ell)$ is Γ-small for $\ell = 1, 2$. So for every $i < \delta, \ell = 1, 2$ we have $M^* \models$ "$|\{x \dot{e} a : \varphi_\ell(x, \bar b_\ell)\}| \le c_{i+1} \times |a|$" hence $M^* \models$ "$|\{x \dot{e} a : \varphi(x, \bar b)\}|/|a| \le 2c_{i+1} < c_i$; as this holds for each i, clearly $\varphi(x, \bar b)$ is Γ-small. The other parts in the Definition are even easier.
2) This is really a discrete version of Fubini theorem.

Consider, assuming b_ℓ realizes $p_\ell \in \mathbf{S}(A, M^*)$ which is $\Gamma^{\rm dms}_{a_\ell, \bar c^\ell}$-big, $q := p_1(x) \cup p_2(y) \cup \{\neg \varphi_1(x, y, \bar d) : \bar d \subseteq A, \varphi_1(x; b_\ell, \bar d)$ is Γ_1-small$\} \cup \{\neg \varphi_2(y, x, \bar d) : \bar d \subseteq A$ and $\varphi_2(y; b_1, \bar d)$ is Γ_2-small$\}$.

By 1.12(1) it suffice to prove that this set of formulas is finitely satisfiable in M^*, toward contradiction assume not. So we have (increasing the sequences of parameters from M^* and recalling p_ℓ is closed under conjunctions) $\vartheta_1(x, \bar d^*) \in p_1(x), \vartheta_2(y, \bar d^*) \in p_2(y), \neg \varphi_{1,k}(x, y, \bar d^*)$ for $k < k_1$ from the third term in the union and $\neg \varphi_{2,k}(y, x, \bar d^*)$ for $k < k_2$ from the fourth term in the union. Note k_1, k_2 that are true natural numbers. Choose $\alpha(*) < \delta_1$ such that $\vartheta_1(x, \bar d^*)$ being Γ_1-big is witnessed by $c^1_{\alpha(*)}$, and choose $\beta(*) < \delta_2$ such that $\vartheta_2(y, \bar d^*)$ being Γ_2-big is witnessed by $c^2_{\beta(*)}$. Without loss of generality for every $a \in \vartheta_1(M^*, \bar d^*), M^* \models$ "$|\{y : \varphi_{2,k}(y; a, \bar d^*)\}|/|a^2| < c^2_{\beta(*)+k_2+2}$" and for every $b \in \vartheta_2(M^*, \bar d), M^* \models$ "$|\{x : \varphi_{1,k}(y; x, \bar d^*)\}|/|a^1| < c^1_{\alpha(*)+k_1+2}$".

Let (recall $\dot e$ is membership in M^*'s sense) $Z = \{(x, y) : x \dot e a^1 \,\&\, y \dot e a^2 \,\&\, \vartheta_1(x, \bar d^*) \,\&\, \vartheta_2(y, \bar d^*)\}$. So Z is (representable) in M^* (we do not distinguish).

Let

$$Z_1 = \{(x, y) \dot e Z : (\exists k < k_1) \varphi_{1,k}(x; y, \bar d^*)\}$$

$$Z_2 = \{(x, y) \dot e Z : (\exists k < k_2) \varphi_{2,k}(y; x, \bar d^*)\}.$$

So

(a) $Z = Z_1 \cup Z_2$ (by the "assume not" above)
(b) for every $y, |Z_1^{[y]}|/|a_1| \le c^1_{\alpha(*)+2}$ where $Z_1^{[y]} = \{x : (x, y) \in Z_1\}$.

[Why? As a union of k_1 sets each with $< c^1_{\alpha(*)+k_1+2} \times |a^1| \le \frac{1}{2^{k_1}} \times c^1_{\alpha(*)+2} \times |a^1|$ members has $\le c^1_{\alpha(*)+2} \times |a^1|$ members.]

And similarly

(c) for every $x, |Z_2^{[x]}|/|a_2| \le c^2_{\beta(*)+2}$ where $Z_2^{[x]} = \{y : (x, y) \in Z_2\}$.

There is in M^*'s sense a set X, such that:

$$x \dot e X \Rightarrow x \dot e a^1 \,\&\, \vartheta_1(x, \bar d^*)$$

$$(c^1_{\alpha(*)})|a^1| + 1 \ge |X| > (c^1_{\alpha(*)})|a^1|.$$

There is in M^*'s sense a set Y such that

$$y\dot{e}Y \Rightarrow y\dot{e}a^2 \,\&\, \vartheta_2(y,\bar{d}^*)$$

$$(c^2_{\beta(*)})|a^2|+1 \geq |Y| \geq (c^2_{\beta(*)})|a^2|.$$

By the choice of Z clearly $X \times Y \subseteq Z$.

So in M^*

$$\begin{aligned}(c^1_{\alpha(*)}|a^1|)(c^2_{\beta(*)}|a^2|) \; & < |X| \times |Y| = |X \times Y| \\ & \leq |Z_1 \cap (X \times Y)| + |Z_2 \cap (X \times Y)| \leq (c^1_{\alpha(*)+2}|a^1|)|Y| + |X|(c^2_{\beta(*)+2}|a^2|) \\ & \leq c^1_{\alpha(*)+2}|a^1|(c^2_{\beta(*)}|a^2|+1) + (c^1_{\alpha(*)}|a^1| + (c^2_{\beta(*)+2}|a^2|) \\ & \leq (c^1_{\alpha(*)+2}|a^1|)(2c^2_{\beta(*)}|a^2|) + (2c^1_{\alpha(*)}|a^1|)(c^2_{\beta(*)+2}|a^2|) \\ & = (2c^1_{\alpha(*)+2}|a^1|)(c^2_{\beta(*)}|a^2|) + (c^1_{\alpha(*)}|a^1|)(2c^2_{\beta(*)+2}|a^2|) \\ & < (\tfrac{1}{2}c^1_{\alpha(*)}|a^1|)(c^2_{\beta(*)}|a^2|) + (c^1_{\alpha(*)}|a^1|)(\tfrac{1}{2}c^2_{\beta(*)}|a^2|) \\ & = c^1_{\alpha(*)}|a^1|c^2_{\beta(*)}|a^2|\end{aligned}$$

contradiction. 3) Let $\Gamma_1 = \Gamma^{\rm av}_{\dot{D},\langle b^*_\epsilon:\epsilon<\kappa\rangle}$, $\Gamma_2 = \Gamma^{\rm dms}_{a,\bar{c}}$, $\bar{c} = \langle c_i : i < \delta\rangle$.

Let $p_\ell = {\rm tp}(b_\ell, A, M^*)$ be Γ_ℓ-big for $\ell = 1,2$; without loss of generality $b^*_\epsilon, a, c_i \in A$ for $\epsilon < \kappa, i < \delta$; let $\dot{D}_1$ be an ultrafilter on κ extending $\dot{D}$. Now, possibly increasing M^* without loss of generality b_1 realizes ${\rm Av}_{\dot{D}_1}(\langle b^*_\epsilon : \epsilon < \kappa\rangle, A + b_2, M^*)$ so clearly ${\rm tp}(b_1, A + b_2, M^*)$ is Γ_1-big. Now assume $\varphi(x, b_1, \bar{d}) \in {\rm tp}(b_2, A + b_1, M^*)$ hence $\mathscr{U} =: \{\epsilon < \kappa : \varphi(x, b^*_\epsilon, \bar{d}) \in {\rm tp}(b_2, A, M^*)\} \in \dot{D}_1$, clearly for $\epsilon \in \mathscr{U}$ the formula $\varphi(x, b^*_\epsilon, \bar{d})$ is Γ_2-big (belonging to ${\rm tp}(b_2, A, M^*)$ which is Γ_2-big), so for $\epsilon \in \mathscr{U}$, let $i_\epsilon = \min\{i : \varphi(x, b^*_\epsilon, \bar{d})$ is Γ^*_2-big as witnessed by $c_i\}$; it is well defined, so let $i(*) = \sup\{i_\epsilon : \epsilon \in \mathscr{U}\}$; now $i(*) < \delta$ as ${\rm cf}(\delta) > \kappa \geq |\mathscr{U}|$, and so easily $\varphi(x, b_1, \bar{d})$ is Γ_2-big being witnessed by $c_{i(*)}$.

4),5) Easy.

6) We let $A_{\Gamma_\ell} \subseteq A \subseteq M^*$ and $p_\ell = {\rm tp}(b_\ell, A, M)$ is Γ_ℓ-big for $\ell = 1, 2$, without loss of generality M is $|A|^+$-saturated. We can find a $||M^*||^+$-saturated N^* such that $M^* < N^*$. We can find $\langle c^3_n : n < \omega\rangle$ in N^* such that: $N^* \models$ "$c^3_n < c^3_{n+1} < c^2_\alpha$" for $n < \omega, \alpha < \ell g(\bar{c}^2)$ such that if $c \in (0,1)^{M^*}_{\mathbb{R}}, M^* \models$ "$c < c^2_\alpha$" for $\alpha < \ell g(\bar{c}^2)$ then $N^* \models$ "$c < c^3_n$". Now p_2 is $\Gamma^{\rm ms}_{a^2,\bar{c}^3}$-big and $\Gamma^{\rm ms}_{a^2,\bar{c}^3} \perp \Gamma^{\rm ms}_{a^1,\bar{c}^1}$, and so we can find in N^3 elements b'_1, b'_2 realizing p_1, p_2 respectively such that ${\rm tp}(b'_1, A + b'_2, N^*)$ is $\Gamma^{\rm ms}_{a^1,\bar{c}^1}$-big and ${\rm tp}(b_1, A + b_1, N^*)$ is $\Gamma_{a^2,\bar{c}^3}$-big: so b'_1, b'_2 exemplify the desirable result. $\square_{3.16}$

Definition 3.17. Suppose $T^*, \mathfrak{C}, a, \bar{c}$ are as in Definition 3.15, so is a "finite" and $\dot{w}$ is a function from a to $[0,1]_{\mathbb{R}}$ (in $\mathfrak{C}$-s sense such that $\mathfrak{C} \models \sum\{\dot{w}(x) : x \in a\} = 1$).
1) Let $\Gamma = \Gamma^{\rm dws} = \Gamma^{\rm dws}_{a,\dot{w},\bar{c}}$ be the following local bigness notion: $\varphi(x, \bar{b})$ is Γ-small if and only if for every $i < \delta$ we have $\mathfrak{C} \models$ "$c_i > \Sigma\{\dot{w}(x) : \varphi(x, \bar{b}) \,\&\, x \in a\}$.
2) We say $\bar{c}$ is d-O.K. for $a, \dot{w}$ if and only if for $i < \ell g(\bar{c})$ and every true natural number n and $b_1, \ldots, b_n \in a$ we have $M^* \models \dot{w}(b_1) + \ldots + \dot{w}(b_n) < c_i < 1/n$ (this is retained in any elementary extension of M^*).
3) We define $\Gamma^{\rm dms}_{a,\mathbf{q},\bar{c}}$ parallely to 3.12(2).

Claim 3.18. *The parallel of 3.16 holds for $\Gamma^{\rm dws}$.*

Proof. Similar to the proof of 3.16. $\square_{3.18}$

Claim 3.19. *1) Suppose $M^* \models T^*$ and (in M^*) we have: $(a, \bar{c})$ is increasing and O.K., $A \subseteq M^*, \{a\} \cup \bar{c} \subseteq A, c_\beta / c_{\beta+2} \le c_{\beta+n}$ (for every β for some $n = n_\beta$, e.g. $n = 3$) and $p = \mathrm{tp}(b, A, M^*)$ is $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big, and $a = \mathscr{P}(a_1)$. Assume further $A \subseteq A' \subseteq M^*$, and $B^* := \{d : d \in \mathrm{acl}(A'), d \in {}^{M^*} a_1$, but for no a_2 do we have: $a_2 \in \mathrm{acl}(A), M^* \models$ "$d \dot{\in} a_2 \,\&\, a_2 \subseteq a_1 \,\&\, (\forall \alpha < \ell g(\bar{c}))[2^{-|a_2|} \ge c_\alpha]$"$\}$. If $h : B^* \to \{\mathrm{true}, \mathrm{false}\}$ <u>then</u>:*

(α) *$p' = p \cup \{[d \in x]^{\mathrm{if}(h(d))} : d \in B^*\}$ is $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big,*

moreover for some b' realizing p' in some $N, M^* \le_{\mathfrak{k}} N$ we have

(β) *$\mathrm{tp}(b', A', N)$ is $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big*

(γ) *$\mathrm{acl}_N(A + b') \cap \mathrm{acl}_N(A') = \mathrm{acl}_N(A)$.*

2) If in addition $\log_2(1/c_\beta)/\log_2(1/c_{\beta+1}) < c_{\beta+2}$ for $\beta < \delta$ <u>then</u> we can add:

(δ) *if $d \in B^*$, <u>then</u> for no a_2 do we have: $a_2 \in \mathrm{acl}(A + b')$ and $M^* \models$ "$d \dot{\in} a_2 \subseteq a_1 \,\&\, (\forall \alpha < \ell g(\bar{c}))[2^{-|a_2|} \ge c_\alpha]$".*

Proof. 1) The proof of part 2) is similar only slightly harder, so read it.
2) Without loss of generality $A = \mathrm{acl}(A)$ and $A' = \mathrm{acl}(A')$. It suffices to prove $(\alpha) + (\delta)$, meaning:

$\boxtimes$ the following is a $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big type

$$\begin{aligned} q^*(x) := \; & p(x) \cup \{[d \in x]^{\mathrm{if}(h(d))} : d \in B^*\} \cup \{\neg d \in \sigma(x) : d \in B^* \text{ and} \\ & \sigma(-) \text{ a term with parameters from } A \text{ such that} \\ & p(x) \vdash \text{“}\sigma(x) \text{ a finite set } \& \, 2^{-|\sigma(x)|} \ge c_\beta\text{” for every } \beta < \ell g(\bar{c})\}. \end{aligned}$$

Why is $\boxtimes$ enough? As then we can get (β) by the extension property of "$\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big" types, and to get clause (γ) we replace A', h by $\bigcup_{i<\lambda} A'_i, \bigcup_{i<\lambda} h_i$ where $\lambda > |T| + |A'| + \aleph_0, \mathbf{f}_i$ is an elementary mapping with domain $\mathrm{acl}(A')$ in N where $M^* \le N, \mathbf{f}_i \restriction \mathrm{acl}(A) = \mathrm{id}_{\mathrm{acl}(A)}$ for $i < \lambda$ and $\langle \mathbf{f}_i(\mathrm{acl}(A')) \setminus \mathrm{acl}(A) : i < \lambda \rangle$ are pairwise disjoint, and $A'_i = \mathbf{f}_i(A')$ and $h_i = h \circ \mathbf{f}_i^{-1}$.

Now for simplicity assume $\mathrm{Rang}(\mathbf{f}_i) \subseteq M^*$. Now we apply $\boxtimes$ with $A'' = \cup\{A_i : i < \lambda\}, h' = \cup\{h_i : i < \lambda\}$ instead A, h, easily the desired conclusion follows; so indeed it is enough to prove $\boxtimes$. (We suppress below parameters from A.)

Let $\mathbf{\Sigma} = \{\sigma(x) : \sigma$ a term with parameters from A such that $M^* \models$ "$(\forall x)[|2^{-|\sigma(x)|}| \ge c_\beta \,\&\, \sigma(x)$ a finite subset of $a_1]$" for every $\beta < \delta\}$. Note that if $\sigma_1(x), \sigma_2(x) \in \mathbf{\Sigma}$ and $\sigma(x) := \sigma_1(x) \cup \sigma_2(x)$, in M^*'s sense, then $\sigma(x) \in \mathbf{\Sigma}$ as $\beta < \delta \Rightarrow 2^{-|\sigma(x)|} = 2^{-|\sigma_1(x)|} \cdot 2^{-\sigma_2(x)} \ge c_{\beta+n} \times c_{\beta+2} \ge c_\beta$ when $n = n_\beta$ is as in the assumption (see 3.19(1)). So if the type q^* is not $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big then for some $\vartheta(x) \in p$, $k < \omega$, $\beta(*) < \delta$, $\sigma(x) \in \mathbf{\Sigma}$ and distinct $d_0, \dots, d_{k-1} \in B^*$, the (finite) type

$$q = \{\vartheta(x)\} \cup \{[d_\ell \dot{\in} x]^{h(d_\ell)} : \ell < k\} \cup \{\neg[d_\ell \dot{\in} \sigma(x)] : \ell < k\}$$

is $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-small, say as witnessed by $c_{\beta(*)}$ where $\beta(*) < \ell g(\bar{c})$, that is $c_{\beta(*)}$ is a witness for the conjunction of q so for some $\psi(x_0, \dots, x_{k-1})$ with parameters from A we have $M^* \models \psi[d_0, \dots, d_{k-1}]$ and ψ state the $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-smallness of q by $c_{\beta(*)}$ i.e.

(∗) $\mathfrak{C} \models (\forall y_0, \ldots, y_{k-1})[\psi(y_0, \ldots, y_{k-1}) \rightarrow |\{x \in a : \vartheta(x) \,\&\, (\forall \ell < k)[y_\ell \in x]^{\mathrm{if}(h(d_\ell))} \,\&\, (\forall \ell < k) \neg d_\ell \in \sigma(x)\}|/|a| \leq c_{\beta(*)}]$.

Without loss of generality

$$\mathfrak{C} \models (\forall y_0, \ldots, y_{k-1})[\psi(y_0, \ldots, y_{k-1}) \rightarrow \bigwedge_{\ell < m < k} y_\ell \neq y_m \,\&\, \bigwedge_{\ell < k} y_\ell \dot{e} a_1].$$

Let $\breve{f}^* \in \mathfrak{C}$ be a maximal family of pairwise disjoint k-tuples satisfying ψ, it exists - see 1.1(B) and without loss of generality $\breve{f}^* \in A'$; let $\mathfrak{C} \models$ "$\dot{n} := |\breve{f}^*|$".

Clearly

$$\begin{aligned} M^* \models \text{``} \quad & (\forall y_0 \ldots y_{k-1})[\psi(y_0, \ldots, y_{k-1}) \rightarrow (\exists z_0, \ldots, z_{k-1}) \\ & (\langle z_0, \ldots, z_{k-1} \rangle \in \breve{f}^* \,\&\, \{y_0, \ldots, y_{k-1}\} \cap \{z_0, \ldots, z_{k-1}\} \neq \emptyset)]. \end{aligned}$$

Hence for some $m < k$ and $k(*) < k$ we have $d_{k(*)} \in d^*_m$ where for each $\ell < k$ we let $d^*_\ell := \{d'_\ell : \bar{d}' \dot{e} \breve{f}^*\}$ (in M^*'s sense). So $d^*_m \in \mathrm{acl}(A)$ and $M^* \models$ "$|d^*_m| = \dot{n}$", but as $d_{k(*)} \in d^*_m \,\&\, d_{k(*)} \in B^*$, by the definition of B^* we have $2^{-\dot{n}} = 2^{-|d^*_m|} < c_{\gamma(*)}$ for some $\gamma(*) < \lg(\bar{c})$. Without loss of generality $\gamma(*) = \beta(*) + 1$.

Choose $\alpha(*) < \delta$ such that $c_{\beta(*)}/c_{\beta(*)+2} < c_{\alpha(*)}$.

Let $a^* := \{x \in a : \vartheta(x)\}$, so as $\vartheta(x)$ is $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big clearly $|a^*|/|a| \geq c_\alpha$ for each α. Also by (∗) above and the choice of $\breve{f}$ we have $|\{(\bar{y}, x) : \bar{y} \dot{e} \breve{f}^*, x \dot{e} a^*$ and $(\forall \ell < k)[y_\ell \dot{e} x]^{\mathrm{if}(h(d_\ell))}$ and $(\forall \ell < k)[\neg d_\ell \dot{e} \sigma(x)]\}| \leq \dot{n} \times |a| \times c_{\beta(*)}$; let us define $a' =: \{x \dot{e} a : |\{\bar{y} \dot{e} \breve{f}^* : (\forall \ell < k)[y_\ell \dot{e} x]^{h(d_\ell)}$ and $(\forall \ell < k)[\neg d_\ell \dot{e} \sigma(x)]\}|/\dot{n} \geq c_{\beta(*)+2}\}$, by the previous inequality $|a^* \cap a'| \leq (\dot{n} \times |a| \times c_{\beta(*)})/(\dot{n} \times c_{\beta(*)+2}) = |a| \times (c_{\beta(*)}/c_{\beta(*)+2}) \leq |a| \times c_{\alpha(*)} < \frac{1}{2}|a^*|$. Hence $|a \setminus a'| \geq |a^* \setminus a'| \geq (1/2)|a^*|$.

Now we shall show that $a \setminus a' \subseteq b$ where

$$b = \{x \dot{e} a : |\{\bar{y} \dot{e} \breve{f}^* : \neg(\forall \ell < k)[y_\ell \dot{e} x]^{(\mathrm{if}(h(d_\ell))}\}|/\dot{n} < c_{\beta(*)+3}\}.$$

This holds as if $x \dot{e} a \setminus a'$ then

(i) $|\{\bar{y} \dot{e} \breve{f}^* : (\forall \ell < k)[y_\ell \dot{e} x]^{\mathrm{if}(h(d_\ell))}$ and $(\forall \ell < k)[\neg d_\ell \dot{e} \sigma(x)\}|/\dot{n} < c_{\beta(*)+2}$ (as $x \dot{e} a \setminus a'$)

and

(ii) $|\{\bar{y} \dot{e} \breve{f}^* : (\exists \ell < k)[y_\ell \dot{e} \sigma(x)]\}|/\dot{n} \leq |\sigma(x)|/\dot{n} \leq (\log_2(1/c_{\beta(*)}))/\dot{n} < \log_2(1/c_{\beta(*)})/(\log_2(1/c_{\gamma(*)})) = \log_2((1/c_{\beta(*)})/\log_2(1/c_{\beta(*)+1}) \leq c_{\beta(*)+2}$.

[Why? The first inequality as $\breve{f}$ is a set of pairwise disjoint k types for the second inequality recall that $2^{-\sigma(x)} \geq c_\beta$ holds by the choice of Σ and for the third, $2^{-\dot{n}} < c_{\gamma(*)}$ was noted above the fourth (equally holds as $\gamma(*) = \beta(*) + 1$ and for the last an assumption of part (2).]

(iii) $2c_{\beta(*)+2} \leq c_{\beta(*)+3}$.

By the previous paragraph $|a \setminus a'| \geq (1/2)|a^*|$ hence $M^* \models$ "$|b| \geq (1/2)|a^*|$.

Now for the "random variable" $x \dot{e} a$ the events $(\forall \ell < k)[d'_\ell \dot{e} x]^{\mathrm{if}(h(d'_\ell))}$ for $\langle d'_\ell : \ell < k \rangle \in \breve{f}^*$ each has probability 2^{-k} and they are independent and their number is $\dot{n}$, so the probability that only $\leq c_{\beta(*)+3} \times \dot{n}$ of them occur for x is sufficiently small by the law of large numbers, which mean (see e.g [Spe87, pg.29] recall $\mathbf{e}$ is the basis of natural logarithm) that the probability is, for some $n < \omega$

$$\begin{aligned} \leq \mathbf{e}^{-2(2^{-k}\dot{n} - c_{\beta(*)+3})^{2}/\dot{n}} \quad & \leq \mathbf{e}^{[-2^{-(2k+2)}\dot{n}]} \\ & < 2^{-2^{-(2k+3)}\dot{n}} \\ & \leq 2^{-2^{-(2k+3)}|d^*_m|} \\ & = (2^{-|d^*_n|})^{2^{-(2k+3)}} \\ & < (c_{\gamma(*)})^{2^{-(2k+3)}} \leq c_{\gamma(*)+n} \end{aligned}$$

(last inequality as for every $\gamma < \ell g(\bar{c})$ for some n we have $c_\gamma \leq (c_{\gamma(*)})^{2^{-(2k+3)}}$ hence $c_\gamma \leq c_{\gamma+n} \times c_{\gamma+n}$ so for some n depending on k and $\gamma(*)$, we are O.K.). So $|b| \leq |a| \times \text{prob}(x \notin b) < |a| \times c_{\gamma(*)+n} < \frac{1}{4} \times c_{\gamma(*)+n+2} \times |a| \leq \frac{1}{2} \times |a^*|$.

Together we get a contradiction. $\square_{3.19}$

We could have said something on $\breve{f}^*$

Observation 3.20. *Assume $M^* \models T^*, A \subseteq A' \subseteq M \prec \mathfrak{C}$, $\dot{n} \in A = \text{acl}(A), k < \omega$, for $\ell < k$ we have $d_\ell \in A'$, $a_1 \in A$ and $M^* \models$ "a_1 is finite, $d_\ell \dot{e} a_1$" and $M^* \models$ "$\dot{n}$ a natural number > 0" and for every $a_2 \in A$ and $\ell < k$ we have $\mathfrak{C} \models$ "$d_\ell \dot{e} a_2 \subseteq a_1 \rightarrow |a_2| \geq \dot{n}$". Then we can find $\breve{f} \in \mathfrak{C}$ such that for every $\varphi(\bar{x}) \in \text{tp}(\langle d_0, \ldots, d_{k-1}\rangle, A, \mathfrak{C})$ we have:*

- $(*)$ *$\mathfrak{C} \models$ "$\breve{f}$ is a set of $\dot{n}$ pairwise disjoint k-tuples from a_1, each satisfying $\varphi(\bar{x})$".*

Proof. The properties of $\breve{f}$ can be represented as realizing a k-type, so as $\text{tp}(\langle d_0, \ldots, d_{k-1}\rangle, A, \mathfrak{C})$ is closed under finite conjunction, it is enough to find $f \in \mathfrak{C}$ satisfying $(*)$, for one given $\varphi(\bar{x}) \in \text{tp}(\langle d_0, \ldots, d_{k-1}\rangle, A, \mathfrak{C})$.

In $\mathfrak{C}$ there is a set $\breve{f}^*$ which is (in $\mathfrak{C}$'s sense) a maximal set of pairwise disjoint k-types $\subseteq a_1$ satisfying $\varphi(\bar{x})$. As T^* has Skolem functions without loss of generality $\breve{f}^* \in \text{acl}(A) = A$. If $\mathfrak{C} \models |\breve{f}^*| \geq \dot{n}$ we are done, so assume not; by $\breve{f}^*$'s maximality $\mathfrak{C} \models$ "some d_ℓ appear in one of the k-types from $\breve{f}^*$, say as the $\ell(*)$-th member of this sequence", so d_ℓ satisfies $d_\ell \in \breve{f}^*_{\ell(*)} = \{y : y$ is the $\ell(*)$-th member of some k-type from $\breve{f}^*\}$, but $\breve{f}^*_{\ell(*)} \in \text{acl}(A) = A, \models$ "$|\breve{f}^*_{\ell(*)}| = |\breve{f}^*| < \dot{n}$", contradicting an assumption. $\square_{3.20}$

Claim 3.21. *Suppose $A \subseteq \mathfrak{C}$, $\bar{c}^\ell \cup \{a^\ell, \dot{w}^\ell\} \subseteq A = \text{acl}(A)$, $p_\ell = \text{tp}(b_\ell, A, \mathfrak{C}) \in \mathbf{S}(A, \mathfrak{C})$ is $\Gamma^{\text{wm}}_{a^\ell, \dot{w}^\ell, \bar{c}^\ell}$-big, for $\ell = 1, 2$, $\dot{w}^1$ constant, $A^2 = \text{acl}(A + b_2), B^* = \{d \in A^2 : d \in Ma^1$ but for every $a_2 in A$ such that $d \in a_2$ we have $(\exists \alpha)(2^{-|d_2|} \leq c^1_\alpha)\}$ and $h : B^* \rightarrow \{\text{true}, \text{false}\}$.*

Then we can find b'_1 such that:

- (α) *b'_1 realizes p_1*
- (β) *$\text{acl}(A + b'_1) \cap \text{acl}(A + b_2) = \text{acl}(A) (= A)$*
- (γ) *$\text{tp}(b'_1, A + b_2)$ is $\Gamma^{\text{wm}}_{a^1, \dot{w}^1, \bar{c}^1}$-big*
- (δ) *$\text{tp}(b_2, A + b'_1)$ is $\Gamma^{\text{wm}}_{a^2, \dot{w}^2 \bar{c}^2}$-big*
- (ϵ) *for $d \in B^*, \mathfrak{C} \models [d \dot{e} b'_1]^{h(d)}$*
- (ζ) *every $d \in B^*$ is still "large" over $\text{acl}(A + b)$ (as in the definition of B^*).*

Remark 3.22. Note that the extra constraints in clause (ζ) are on b'_1 only. If $|\mathrm{Dom}(h)| = 1$ simpler better bound suffice, otherwise we use Δ-system. This claim is used in fourth case Stage D proof of 5.2 below. Even if we would have failed, in 5.2, stage D, we can use the weak diamond argument.

Proof. We can repeat proof of 3.10(2) + 3.19. By "the local character of the demand" (really a variant of 1.12) we can replace (ϵ) by

(∗) $(\forall \ell < k)[d_\ell \in b'_1]^{h(d_\ell)}$ for some $k < \omega$ and $d_0, \ldots, d_{k-1} \in B^*$.

$\square_{3.21}$

Definition 3.23. For $M^*, a, \bar{c}$ as in Definition 3.8 and κ we define $\Gamma = \Gamma^{\mathrm{ms},\kappa}_{a,\bar{c}}$ as follows. We let $\bar{x}_\Gamma = \langle x_i : i < \kappa \rangle$, and $p(\bar{x}_\Gamma)$ is Γ-big <u>if and only if</u> for any finite $q \subseteq p$ and finite $w \subseteq \kappa$ such that $[x_i$ appears in $q \Rightarrow i \in w]$ we have

$$M^* \models \text{“}|\{\langle d_i : i \in w \rangle : M^* \models \bigwedge q[\langle d_i : i \in w \rangle] \,\&\, \bigwedge_{i \in w} d_i \in a\}|/|a|^{|w|} \geq c_\alpha\text{”}$$

for each $\alpha < \ell g(\bar{c})$.

Remark 3.24. We can develop this (as in [She83b]) and used it in the stage D of the proof of 5.2. Also we can define a parallel of Γ_{ind} from Definition 3.4 for κ.

Claim 3.25. *Assume $a \in \mathfrak{C}$ is pseudo finite such that $|a| \geq n$ for $n < \omega$ and $\mathfrak{C}$ satisfies: $\mathbf{p} \in (0,1)_{\mathbb{R}}, \bar{c} = \langle c_i : i < \delta \rangle, c_i \in (0,1)_{\mathbb{R}}, \bigwedge_{i<j} 2c_i < c_j$, and let $\dot{w}_{\mathbf{p}} : \mathscr{P}(a) \to [0,1)$ be as in Def. 3.12(2).*

1) <u>Then</u> $\mathscr{P}(a), \dot{w}_{\mathbf{p}}, \bar{c}$ satisfy the requirements in Definition 3.12(1) on a, $\dot{w}$, $\bar{c}$. The bigness notion $\Gamma^{\mathrm{wmg}}_{a,\mathbf{q},\bar{c}}$ is non-trivial <u>if</u> for each $n < \omega$ we have $\mathbf{q}^{|a|} < 1/n$ and $(1-\mathbf{q})^{|a|} < c_\alpha$ for some $\alpha < \lg(\bar{c})$, equivalently for every $n < \omega, \ln(1/\mathbf{q}) > n/|a|$ and $\ln(1/(1-\mathbf{q})) > \ln(1)c_\alpha)/|a|$.
2) If the type p is $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar{c}}$-big, <u>then</u> for no $i < \lg(\bar{c})$ and b do we have, $\mathfrak{C} \models$ "$b \subseteq a, |b| \leq \log(1-c_i)/\log(1-\mathbf{p})$" and "$[x \cap b \neq \emptyset]$"$' \in p$.
3) If the type p is $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar{c}}$-big and $\bigvee_{i<\lg(\bar{c})} \mathbf{p} < c_i$ <u>then</u> for no $d \in \mathfrak{C}$ do we have, "$[d \in x]$" $\in p$.

Proof. 1) Check.
2) As $|b| \leq \frac{\ln(1-c_i)}{\ln(1-\mathbf{p})}$ and as $\ln(1-\mathbf{p}) < 0$ clearly $|b|\ln(1-\mathbf{p}) \geq \ln(1-c_i)$ hence $\ln((1-\mathbf{p})^{|b|}) \geq \ln(1-c_i)$ hence $(1-\mathbf{p})^{|b|} \geq (1-c_i)$. So clearly $\dot{w}_{\mathbf{p}}(\{x \in \mathscr{P}(a) : x \cap b = \emptyset\}) = (1-\mathbf{p})^{|b|} \geq 1 - c_i$ hence $\dot{w}_{\mathbf{p}}(\{x \in \mathscr{P}(a) : x \cap b \neq \emptyset\}) = 1 - \dot{w}_{\mathbf{p}}(\{x \in \mathscr{P}(a) : x \cap b = \emptyset\}) \leq c_i$, hence "$[x \cap b \neq \emptyset]$" $\in p$ gives easy contradiction.
3) Because if $\mathbf{p} < c_i$ then $1 - c_{i+3} > 1 - \mathbf{q}$ hence $\ln(1-c_{i+3}) > \ln(1-\mathbf{q})$ hence as $\ln(1-\mathbf{q})$ is negative $|\{d\}| = 1 < \log(1-c_{i+3})/\log(1-\mathbf{p})$ and apply part (2). $\square_{3.25}$

Claim 3.26. *1) Assume p, a type over $A \subseteq \mathfrak{C}$, is $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar{c}}$-big and $a, \mathbf{p}$ (hence $\dot{w}_{\mathbf{p}}$), $\bar{c}$ as in 3.12(2), $\{a, \mathbf{p}\} \cup \bar{c} \subseteq A = \mathrm{acl}(A)$. Suppose $\bar{c}'$ is wide for a and $e^* \in a$ and $\mathrm{tp}(e^*, A)$ is $\Gamma^{\mathrm{ms}}_{a,\langle c'_i : i < \lg(\bar{c}')\rangle}$-big where $c'_0 \geq \ln(c_0)/(|a| \times \ln(1-\mathbf{p}))$ and assume $i < \lg(\bar{c}) \Rightarrow \ln(1/c_i) < c_{i+1}/c_i$. <u>Then</u> $p(x \cup \{e^*\}) \cup \{\neg(e^* \in x)\}$ is $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar{c}}$-big.*
2) If $A = \mathrm{acl}(A), \{a, \mathbf{p}, d^*, e^*\} \in A, \bar{c}^1, \bar{c}^2 \subseteq A, \mathrm{tp}(d^*, A)$ is $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar{c}^1}$-big and $\mathrm{tp}(e^*, A)$ is $\Gamma^{\mathrm{ms}}_{a,\bar{c}^2}$-big and (∗) below holds <u>then</u> we can find d', e' such that:

(α) $\mathrm{tp}(e', A + d')$ *is* $\Gamma^{\mathrm{ms}}_{a,\bar c^2}$*-big extending* $\mathrm{tp}(e^*, A)$

(β) $e' \in d'$

(γ) $\mathrm{tp}(d' \setminus \{e'\}, A + e')$ *is* $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar c^1}$*-big*

(δ) $\mathrm{tp}(d', A + e')$ *nicely extend* $\mathrm{tp}(d^*, A)$

provided that

(∗) (i) $\bar c^\ell = \langle c^\ell_i : i < \delta^\ell\rangle$ *are as in Definition 3.12 (in particularly, O.K. and wide) for* $\mathscr{P}(a)$ *(if* $\ell = 1$*) or for* a *(if* $\ell = 2$*)*[7]

(ii) $\mathbf{p} \in (0,1)_{\mathbb{R}}$*, and "*$\mathbf{p} < 1/n$*" for each real natural number* n

(iii) $\mathbf{e}!^{-\frac18 \mathbf{p}^2 \times c^2_{j_2} \times |a|} < \frac12 c^1_{j_1}$ *for every* $j_1 < \delta_1$*,* $j_2 < \delta_2$[8] *(or at least for some* (j_1, j_2)*) (by the monotonicity) hence we can omit the* $\frac12$*)*

(iv) $c^2_i \le \mathbf{p} \times c^2_{i+1}$.

Proof. 1) Without loss of generality $p \in \mathbf{S}(A, \mathfrak{C})$. As $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar c}$ is co-simple, p closed under finite conjunctions, clearly if the conclusion fails then for some $i < \lg(\bar c)$ and $\varphi(\bar x) \in p$ (suppressing parameters from A) we have:

(∗) $\varphi(x \cup \{e^*\}) \cup \{\neg(e^* \in x)\}$ is $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar c}$-small as witnessed by c_i.

Similarly there is $\psi(y) \in \mathrm{tp}(e^*, A)$ (suppressing parameters from A) such that for every $e \in \psi(\mathfrak{C})$, we have $e \in a$ and $\varphi(x \cup \{e\}) \cup \{\neg(e \in x)\}$ is $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar c}$-small as witnessed by c_i.

Let $b^1 =: \{x \in \mathscr{P}(a) : \varphi(x)\}$ (in $\mathfrak{C}$), and so as $\varphi(x) \in p$ clearly

$(*)_0$ $\dot w_{\mathbf{p}}(b^1) \ge c_j$ for every $j < \lg(\bar c)$.

Let $b^2 =: \{y \in a : \psi(y)\}$ (in $\mathfrak{C}$), now there is b^3 such that $\mathfrak{C} \models$ "$b^3 \subseteq b^2 \,\&\, |b^3| \sim \frac{\ln c_i}{\ln(1-\mathbf{p})}$" (possible by the assumption on e^* as $c'_j \ge c'_0 \ge \ln(c_0)/(|a| \times \ln(1-\mathbf{p})) \ge \ln(c_i)/(|a| \times \ln(1-\mathbf{p})))$; pedantically we should say $\frac{\ln c_i}{\ln(1-\mathbf{q})} \le |b^3| < \frac{\ln c_1}{\ln(1-\mathbf{q})} + 1$ and complicate the computations a little (in $((*)_1)$. Note (as $\ln(1-\mathbf{p}) \sim -\mathbf{p}$ and $0 < \mathbf{q} < 1/2$), that $\left(\frac{\mathbf{q}}{1-\mathbf{q}}\right) \times \frac{-1}{\ln(1-\mathbf{q})}$ is in the interval $(\frac12, 2)$, and as $\ln(1/c_i) < c_{i+1}/c_i$ holds by an assumption we have

$(*)_1$ $\left(\frac{\mathbf{q}}{1-\mathbf{q}}\right)|b^3| \le \left(\frac{\mathbf{q}}{1-\mathbf{q}}\right) \times \frac{\ln c_i}{\ln(1-\mathbf{q})} = \left(\frac{\mathbf{q}}{(1-\mathbf{q})}\right) \times \frac{-1}{\ln(1-\mathbf{q})} \times \ln(1/c_i) \le 2 \times \ln(1/c_i) \le 2 \times (c_{i+1}/c_i) = (2c_{i+1}/c_i) \le \frac{c_{i+2}}{c_i}$.

Now in $\mathfrak{C}$, by the choice of $\dot w_{\mathbf{p}}$ and of b_3 we have:

$(*)_2$ $\dot w_{\mathbf{p}}(\{x \subseteq a : x \cap b^3 = \emptyset\}) = (1-\mathbf{p})^{|b^3|} = \mathbf{e}!^{|b_3| \times \ln(1-\mathbf{q})} \le c_i$

and for every $e \in b^3$ (as $e \in b^2$ i.e. $\models \psi[e]$ and the choice of ψ)

$(*)_3$ $\dot w_{\mathbf{p}}(\{x \setminus \{e\} : x \in b^1,\ e \in x\}) \le c_i$

and by the choice of $\dot w_{\mathbf{p}}$ we have

$(*)_4$ $\dot w_{\mathbf{p}}(\{x : x \in b^1 \text{ and } e \in x\}) = \left(\frac{\mathbf{p}}{1-\mathbf{p}}\right) \dot w_{\mathbf{p}}(\{x \setminus \{e\} : x \in b^1 \text{ and } e \in x\})$

---

[7] This is an over-kill but suffice

[8] if $\delta_1 = \delta_2$ then the situation is simpler; $\mathbf{e}!$ is the bases of natural logarithm

hence (use logic, logic, $(*)_4, (*)_2{+}(*)_3, (*)_1$, requirement on $\bar{c}$ and $(*)_0$ respectively):

$(*)_5$ $\dot{w}_{\mathbf{p}}(b^1) = w_{\mathbf{p}}\big(\{x : x \in b^1 \text{ and } x \cap b^3 = \emptyset\} \cup \bigcup_{e \in b^3} \{x : x \in b^1 \text{ and } e \in x\}\big) \leq \dot{w}_{\mathbf{p}}(\{x : x \in b^1 \text{ and } x \cap b^3 = \emptyset\}) + \sum_{e \in b^3} \dot{w}_{\mathbf{p}}(\{x : x \in b^1 \text{ and } e \in x\}) = \dot{w}_{\mathbf{p}}(\{x : x \in b^1 \text{ and } x \cap b^3 = \emptyset\}) + \sum_{e \in b^3} \big(\frac{\mathbf{p}}{1-\mathbf{p}}\big) \dot{w}_{\mathbf{p}}(\{x \setminus \{e\} : x \in b^1 \text{ and } e \in x\}) \leq c_i + |b^3| \times \frac{\mathbf{p}}{1-\mathbf{p}} \times c_i \leq c_i + c_{i+2} < c_{i+3} < \dot{w}_{\mathbf{p}}(b^1)$

Contradiction.

2) Let $\dot{w}$ be constantly $\frac{1}{|a|}$ on a.

Note that we can ignore the "nicely", i.e. clause (δ) (Why? As then we let $\lambda = (|T| + |A| + \aleph_0)^+$, and by induction on $\zeta < \lambda$ we choose d_ζ such that:

- (a) $p_\zeta = \text{tp}(d_\zeta, A \cup \{d_\xi : \xi < \zeta\})$ is $\Gamma^{\text{wmg}}_{a,\mathbf{p},\bar{c}^1}$-big, increasing with ζ
- (b) p_ζ nicely extend $\text{tp}(d^*, A)$
- (c) $e^* \in d_\zeta$
- (d) $\text{tp}(d_\zeta \setminus \{e^*\}, A \cup \{d_\xi : \xi < \zeta\} + e^*)$ is $\Gamma^{\text{wmg}}_{a,\mathbf{p},\bar{c}^1}$-big
- (e) $\text{tp}(e^*, A + \{d_\xi : \xi < \zeta\})$ is $\Gamma^{\text{ms}}_{a,\bar{c}^2}$-big.

If we succeed then for some $\zeta, \text{tp}(d_\zeta, A + e^*)$ nicely extend $\text{tp}(d^*, A)$ and we are done. For each ζ we can choose p_ζ satisfying (a) + (b) as $\Gamma^{\text{wmg}}_{a,\mathbf{p},\bar{c}^1}$ is nice, and then d_ζ by the claim (without the "nicely").

Let $p_1 = \text{tp}(d^*, A), p_2 = \text{tp}(e^*, A)$; as the bigness notions are uniformly ∞-simple, if the conclusion fail then by 1.12(2) we can find $i_\ell < \lg(\bar{c}_\ell), \varphi_\ell(x) \in p_\ell$ for $\ell = 1, 2$ (suppressing parameters from A) such that in $\mathfrak{C}, \varphi_1(x_1) \to x_1 \in \mathscr{P}(a), \varphi_2(x_2) \to x_2 \in a$ and:

$(*)_0$ $\varphi_1(x_1) \,\&\, \varphi_2(x_2) \,\&\, x_2 \in x_1 \to \psi_1(x_1 \setminus \{x_2\}, x_2) \vee \psi_2(x_2, x_1)$

where for every $e \in a$ the formula $\psi_1(x_1 \setminus \{e\}, e)$ is small for $\Gamma^{\text{wmg}}_{a,\mathbf{p},\bar{c}^1}$ as witnessed by $c^1_{i_1}$, and for every $d \in \mathbf{q}(a)$ the formula $\psi_2(x_2, d)$ is small for $\Gamma^{\text{ms}}_{a,\bar{c}^2}$ as witnessed by $c^2_{i_2}$ (and we suppress parameters from A).

Let $j_\ell \geq i_\ell + 5$ (and $j_\ell < \delta_\ell$), and the pair (j_1, j_2) is as in $(*)(iii)$ from 3.26(2) and trivially $c^\ell_{i_\ell} \leq \frac{1}{32}\mathbf{q} \times c^\ell_{j_1}$.

Choose non-empty $b_1 \subseteq \{x \in \mathscr{P}(a) : \varphi_1(x)\}$ (in $\mathfrak{C}$) such that:

$(*)_{1,1}$ $c^1_{j_1} \leq \dot{w}_{\mathbf{p}}(b_1)$, and $d \in b_1 \Rightarrow \dot{w}_{\mathbf{p}}(b_1 \setminus \{d\}) < c^1_{j_1}$.

[Why b_1 exists? Choose $b_1 \subseteq \{x \in \mathscr{P}(a) : \varphi_1(x)\}$ such that $c^1_{j_1} \leq \dot{w}_{\mathbf{p}}(b_1)$, now there is at least one: $\{x \in \mathscr{P}(a) : \varphi_1(x)\}$, and $\emptyset$ fails this, and $d \in b_1 \Rightarrow |b_1/\{d\}| < |b_1| \,\&\, \dot{w}_{\mathbf{q}}(b_1/\{d\}) < \dot{w}_{\mathbf{q}}(b_1)$ hence there is such b_1 of minimal cardinality and it is as required].

Similarly choose $b_2 \subseteq \{y \in a : \varphi_2(y)\}$ in $\mathfrak{C}$, such that:

$(*)_{1,2}$ $c^2_{j_2} \leq \dot{w}(b_2)$ and $e \in b_2 \Rightarrow \dot{w}(b'_2 \setminus \{e\}) < c^2_{j_2}$

So (as $d \in b_1 \Rightarrow \cdot\mathbf{q}(\{d\}) < c^1_0, e \in b_2 \Rightarrow \dot{w}(\{e\}) < c^2_0$) easily

$(*)_{1,*}$ $c^1_{j_1} \leq \dot{w}_{\mathbf{q}}(b_1) < 2c^1_{j_1}$ and $c^2_{j_2} \leq \dot{w}(b_2) < 2c^2_{j_2}$ and recall $|b_2|/|a| = \dot{w}(b_2)$.

Now in $\mathfrak{C}$ by the definition of $\dot{w}_{\mathbf{p}}$ we have

$(*)_2$ $\dot{w}_{\mathbf{p}}(\{d \in \mathscr{P}(a) : \dot{w}(d \cap b_2) < (\mathbf{p}/2)|b_2|/|a|\}) \le \mathbf{e}^{-\frac{1}{2}(\mathbf{p}/2)^2|b_2|} \le \mathbf{t}^{-\frac{1}{8}\mathbf{p}^2 \times c^2_{j_2} \times |a|}$

[Why? the second equality (i.e. the left side) is just noting $c^2_{j_2} \le \dot{w}(b_2) = \sum\limits_{e \in b_2} \dot{w}(e) = |b_2| \times (1/|a|)$. For the first inequality, $\dot{w}_{\mathbf{p}}$ of the set is just the probability of d satisfying $|d \cap b_2|/|a| = \dot{w}(d \cap b_2) \le (\mathbf{p}/2)|b_2|/|a|$, where d is gotten by throwing a coin for each $e \in a$ to decide whether $e \in d$ with the probability of yes being $\mathbf{p}$; so the $e \in a \setminus b_2$ are irrelevant and we apply the law of large numbers see e.g. [Spe87, p.29].]

For every $e \in b_2$ (as $x_\ell \in b_\ell \Rightarrow \varphi_\ell(x_\ell)$ and by the choice of i_1 and ψ_1)

$(*)_3$ $\dot{w}_{\mathbf{p}}(\{x \setminus \{e\} : x \in b_1, e \in x, \psi_1(x \setminus \{e\}, e)\}) \le c^1_{i_1}$

and by the choice of $\dot{w}_{\mathbf{p}}$ we have

$(*)_4$ $\dot{w}_{\mathbf{p}}(\{x : x \in b_1 \text{ and } e \in x \text{ and } \psi_1(x \setminus \{e\}, e)\}) = \frac{\mathbf{p}}{1-\mathbf{p}} \dot{w}_{\mathbf{p}} d(\{x \setminus \{e\} : x \in b_1,\ e \in x,\ \psi_1(x \setminus \{e\}, e)\})$

By the choice of ψ_2, for every $d \in b_1$

$(*)_5$ $\dot{w}(\{y \in a : \psi_2(y; d)\}) \le c^2_{i_2}$.

Let $b^*_1 = \{d \in \mathscr{P}(a) : \dot{w}(d \cap b_2) = |d \cap b_2|/|a| < (\mathbf{p}/2) \times |b_2|/|a|\}$. Note that by $(*)_2$ and (iii) of $(*)$

$(*)_6$ $\dot{w}_{\mathbf{p}}(b^*_1) \le \frac{1}{2} c^1_{j_1}$

Now (in $\mathfrak{C}$) on the one hand:

$$\begin{aligned}
\oplus_1\ & \Sigma\{\dot{w}_{\mathbf{p}}(d) \times \dot{w}(e) : d \in b_1,\ e \in b_2,\ e \in d\} \ge \\
& \sum_{d \in b_1 \setminus b^*_1} (\Sigma\{\dot{w}_{\mathbf{p}}(d) \times \dot{w}(e) : e \in b_2, e \in d\}) = \\
& \sum_{d \in b_1 \setminus b^*_1} \dot{w}_{\mathbf{p}}(d) \times \dot{w}(d \cap b_2) \ge \sum_{d \in b_1 \setminus b^*_1} \dot{w}_{\mathbf{p}}(d) \times ((\mathbf{p}/2)|b_2|/|a|) \ge \\
& (\mathbf{p}/2) \times c^2_{j_2} \times \sum_{d \in b_1 \setminus b^*_1} \dot{w}_{\mathbf{p}}(d) = (\mathbf{p}/2) \times c^2_{j_2} \times [\dot{w}_{\mathbf{p}}(b_1) - \dot{w}_{\mathbf{p}}(b^*_1)] \ge \\
& (\mathbf{p}/2) \times c^2_{j_2} \times [\dot{w}_{\mathbf{p}}(b_1) - \tfrac{1}{2} c^1_{j_1}] \ge \\
& (\mathbf{p}/2) \times c^2_{j_2} \times [c^1_{j_1} - \tfrac{1}{2} c^1_{j_1}] \ge \\
& (\mathbf{p}/2) \times c^2_{j_2} \times \tfrac{1}{2} c^1_{j_1} = \tfrac{1}{4}\mathbf{p} \times c^1_{j_1} \times c^2_{j_2}.
\end{aligned}$$

But on the other hand

$$\begin{aligned}
\oplus_2\ & \Sigma\{\dot{w}_{\mathbf{p}}(d) \times \dot{w}(e) : d \in b_1,\ e \in b_2,\ e \in d\} \le \\
& \Sigma\{\dot{w}_{\mathbf{p}}(d) \times \dot{w}(e) : d \in b_1,\ e \in b_2,\ e \in d \text{ and } \psi_1(d \setminus \{e\}, e)\} + \\
& \Sigma\{\dot{w}_{\mathbf{p}}(d) \times \dot{w}(e) : d \in b_1,\ e \in b_2,\ e \in d \text{ and } \psi_2(e, d)\}
\end{aligned}$$

$$= \sum_{e \in b_2} \dot{w}(e) \times \dot{w}_{\mathbf{p}}(\{d : d \in b_1,\ e \in d \text{ and } \psi_1(d \setminus \{e\}, e)\})$$
$$+ \sum_{d \in b_1} \dot{w}_{\mathbf{p}}(d) \times \dot{w}(\{e : e \in b_2,\ e \in d \text{ and } \psi_2(e, d)\}) \le$$
$$\sum_{e \in b_2} \dot{w}(e) \times \tfrac{\mathbf{p}}{1-\mathbf{p}} \times \dot{w}_p(\{d \setminus \{e\} : d \in b_1,\ e \in d \text{ and } \psi_1(d \setminus \{e\}, e)\})$$
$$+ \sum_{d \in b_1} \dot{w}_{\mathbf{p}}(d) \times c^2_{i_2} \le$$
$$\tfrac{\mathbf{p}}{1-\mathbf{p}} \sum_{e \in b_2} \dot{w}(e) \times c^1_{i_1} + \dot{w}_{\mathbf{p}}(b_1) \times c^2_{i_2} =$$
$$\tfrac{\mathbf{p}}{1-\mathbf{p}} \times c^1_{i_1} \times \sum_{e \in b_2} \dot{w}(e) + c^2_{i_2} \times \dot{w}_{\mathbf{p}}(b_1) = \tfrac{\mathbf{p}}{1-\mathbf{p}} \times c^1_{i_1} \times \dot{w}(b_2) + c^2_{i_2} \times \dot{w}_{\mathbf{p}}(b_1)$$
$$\le c^1_{i_1} \times (\tfrac{\mathbf{p}}{1-\mathbf{p}} \times 2 \times c^2_{j_2}) + c^1_{j_1} \times 2 \times c^2_{i_2} < \tfrac{1}{8} \times \mathbf{p} \times c^1_{j_1} \times 2 \times c^2_{j_2} + \tfrac{1}{8} \times c^1_{j_1} \times \mathbf{p} \times c^2_{j_2} = \tfrac{1}{4} \times \mathbf{p} \times c^1_{j_1} \times c^2_{j_2}.$$

Now $(\oplus)_1 + (\oplus)_2$ gives contradiction. $\square_{3.26}$

Remark 3.27. If in 3.26 we agree to have $\delta^1 = \delta^2$, we can weaken (iii) of $(*)$ demanding $j_2 < j_1$.

Claim 3.28. *Assume $a \in \mathfrak{C}$ is pseudo finite and infinite, if $\mathfrak{C} \models$ "$c \in (0,1)_{\mathbb{R}}$ and $\frac{n}{|a|} < c < \frac{1}{n}$" for $n < \omega$ and $\delta, \delta_1, \delta_2$ are limit ordinals <u>then</u>*
1) If $n \times \ln(|a|)/|a| < c$ for $n < \omega$ <u>then</u> we can find $\mathbf{p}, \bar{c} = \langle c_i : i < \delta \rangle$ and $\bar{c}' = \langle c'_i : i < \delta \rangle$ such that:

$(*)$ $a, \mathbf{p}, \bar{c}, \bar{c}'$ are as in 3.26(1), $\lg(\bar{c}) = \delta = \lg(\bar{c'})$, $\mathbf{p} = c = c_0$.

2) We can find $\mathbf{p}, \bar{c}^1, \bar{c}^2$ as in $(*)$ of 3.26(2), and $\bar{c}^1$, $\bar{c}^2$ wide, $\lg(\bar{c}^\ell) = \delta_\ell$ and $\mathbf{p}, c^1_i, c^2_j \ge c$.
3) In part (2), moreover if $\delta_1 = \omega = \delta_2$, we can choose (in $\mathfrak{C}$) any $\mathbf{p} \in (\sqrt[n]{c}, 1/n)$, for every $n < \omega$, and choose (for each n) $c^2_m = 8 \times c \times \mathbf{p}^{-m-2}$ and $c^1_0 = \frac{1}{3} \times \mathbf{e}!^{-c \times p^d \times |a|}$ when $\mathfrak{C} \models$ "$d > n \,\&\, d \in \mathbb{N}$", d small enough, and $c^1_m = \sqrt[m]{c^1_0}$.

Proof. By compactness without loss of generality $\delta = \delta_1 = \delta_2 = \omega$.
1) In $\mathfrak{C}$, first choose $\mathbf{p} = c$; recall that the function $x \ln(1/x)$ is strictly increasing for $x \in (0, 1/\mathbf{e}!)$ (as the derivative is $-(\ln x) - 1$ which is positive), has values in $(0, 1/\mathbf{e})$ and $\lim_{x \to 0}(x \ln(1/x)) = 0$; the same is true for $\sqrt{x}$ except having values in $(0, \sqrt{1/\mathbf{e}!})$. Choose by induction on n: $c_0 = \mathbf{p}$, $c_{n+1} = \sqrt{c_n}$; clearly $(\forall m < \omega)[c_n < 1/m]$ and $2c_n < c_{n+1}$ and even $m < \omega \Rightarrow m \times c_n < c_{n+1}$ (by induction on n), also $c_0 = \mathbf{p} > m/|a|$ for $m < \omega$.
Also

$(*)_1$ $\ln(1/c_n) < c_{n+1}/c_n$.

[Why? As this means $c_n \ln(1/c_n) < c_{n+1}$, but $x \in (0, 10^{-3}) \Rightarrow x \ln(1/x) < \sqrt{x}$ as $\sqrt{x} - x \ln(1/x)$ is increasing in this domain and $0 = \lim_{x \to 0)} (\sqrt{x} - x \ln(1/x))$, so it holds.]
Let

$(*)_2$ $c'_0 =: \ln(c_0)/(|a| \times \ln(1 - \mathbf{p}))$
$(*)_3$ $(\forall m < \omega)\big[c'_0 \ge \frac{m}{|a|}\big]$.

[Why? As $(\forall m < \omega)[\ln(c_0)/\ln(1-\mathbf{p}) > m]$ because $(\ln c_0)/\ln(1-\mathbf{p}) \sim \frac{1}{\mathbf{p}} \times \ln(1/c_0) = \frac{1}{c}\ln(1/c) > m \times 1 = m$].

Lastly

$(*)_4$ $(\forall n < \omega)[c'_0 < 1/n]$.

[Why? Clearly $c'_0 = \ln(c_0)/(|a| \times \ln(1-\mathbf{p})) \sim \frac{1}{\mathbf{p}} \times \ln(\frac{1}{\mathbf{p}})/|a|$ so the requirement mean $\frac{1}{\mathbf{p}} \ln \frac{1}{\mathbf{p}} < |a|/m$ for $m < \omega$. Now $c = \mathbf{p}$ and by assumption $n \times \ln(|a|)/|a| < c$ which mean $\frac{1}{c} < \frac{1}{n} \times \frac{|a|}{\ln(|a|)}$ hence $\frac{1}{\mathbf{q}} \ln \frac{1}{\mathbf{q}} = \frac{1}{c} \ln(\frac{1}{c}) < (\frac{1}{n} \times \frac{|a|}{\ln(|a|)}) \times \ln(\frac{|a|}{\ln(|a|)} \times \frac{1}{n}) < \frac{1}{n} \times \frac{|a|}{\ln(|a|)} \times \ln(|a|) = \frac{|a|}{n}$ as required.]

We have finished as there is no problem to define c'_n for $n \in [1,\omega)$ by induction on n as before e.g. as $\sqrt[n]{c'_o}$.

2) Note that $\mathbf{e}!^{-x \times |a|}$ is decreasing with x, and

$$(\forall x)(\bigwedge_n x > \frac{n}{|a|} \Leftrightarrow \bigwedge_n \mathbf{e}!^{-x \times |a|} < \frac{1}{n}).$$

First choose $\mathbf{p} \in (0,1)_{\mathbb{R}}$ such that

$$\bigwedge_n [c < \mathbf{p}^n \,\&\, \mathbf{p} < \frac{1}{n}]$$

(so clause (ii) of $(*)$ of 3.26(2) holds). Second choose $c^2_m = 8 \times c \times \mathbf{p}^{-m-2}$ (so clause (iv) of $(*)$ of 3.26(2)) hold and $\bar{c}^2 = \langle c^2_n : n < \omega \rangle$ is wide and O.K. which give half of (i) of (*) of 3.26(2)).

Third choose c^1_0 such that

$$n < \omega \Rightarrow \frac{n}{|\mathscr{P}(a)|} < c^1_0 < \frac{1}{n}$$

and

$$\bigwedge_n [\mathbf{e}^{-\frac{1}{8}\mathbf{p}^2 \times c^2_n \times |a|} < \frac{1}{2} c^1_0]$$

equivalently

$$\bigwedge_n [\mathbf{e}^{-c \times \mathbf{p}^{-n} \times |a|} < \frac{1}{2} c^1_0].$$

For this to be possible we need $\bigwedge_n [\mathbf{e}!^{-c \times \mathbf{p}^{-n} \times |a|} < 1/n]$ equivalently $\bigwedge_n [\mathbf{e}!^{-8c \times \mathbf{p}^n \times |a|} < \mathbf{e}!^{-n}]$ hence equivalently $\bigwedge_n [8c \times \mathbf{p}^{-n} > \frac{n}{|a|}]$ i.e. $\bigwedge_n [\frac{|a|}{n} \times c > \mathbf{p}^n.]$, now as $\frac{|a|}{m} \times c > 1, 1 > \mathbf{p}$ this holds.] This guarantee clause (iii) of $(*)$ of 3.26(2) if we shall have $c^1_0 \le c^1_n$ for $n < \omega$.

Lastly choose e.g. $c^1_n = \sqrt[n]{c^1_0}$ so clearly $\bar{c}^1 = \langle c^1_n : n < \omega \rangle$ is wide and O.K. which give the second fall of (i) of (*) of 3.26(2).

3) We are left with proving the "moreover".

As as $\bigwedge_n [\frac{n}{|a|} < c < \frac{1}{n}]$, clearly $\bigwedge_n \sqrt[n]{c} < \frac{1}{n}$ so there is $\mathbf{p}$ satisfying $\bigwedge_n \mathbf{p} \in [\sqrt[n]{c}, \frac{1}{n}]$, so $\bigwedge_n [c < \mathbf{q}^n \,\&\, p < \frac{1}{n}]$ as required in the proof of part (2).

Now we continue as in part (2).

Now check the requirement in $(*)$ of 3.26(2). $\square_{3.28}$

The claim we shall mostly use in this context is

Claim 3.29. *Assume that*

(a) $\mathfrak{C}$ *is as in 1.1(2)*
(b) a *is pseudo finite (in* $\mathfrak{C}$ *so* $|a| > n$ *for* $n < \omega$ *of course)*
(c) δ_1, δ_2 *limit ordinals*
(d) $\mathfrak{C} \models \frac{n}{|a|} < c < \frac{1}{n}$ *for every true natural number* n.

Then we can find $\mathbf{p}, \bar{c}^1, \bar{c}^2$ *such that:*

(A) *the triple* $(a, \mathbf{p}, \bar{c}^1)$ *and the pair* $(a, \bar{c}^2)$ *are is as in 3.12 (so as in* $(*)(i)$ *of 3.26(2))*
(B) *if* $\{\mathbf{p}, c\}, \bar{c}^1, \bar{c}^2 \subseteq A = \mathrm{acl}(A)$ *and* $\mathrm{tp}(d^*, A)$ *is* $\Gamma^{\mathrm{wmg}}_{a,\mathbf{p},\bar{c}^1}$*-big and* $\mathrm{tp}(e^*, A)$ *is* $\Gamma^{\mathrm{ms}}_{a,\bar{c}^2}$*-big then we can find* d', e' *(*$\in \mathfrak{C}$ *of course) such that*
   (i) $\mathrm{tp}(e', A + d')$ *is* $\Gamma^{\mathrm{ms}}_{a,\bar{c}^2}$*-big extending* $\mathrm{tp}(e^*, A)$
   (ii) $e' \in d'$
   (iii) $\mathrm{tp}(d' \setminus \{e'\}, A + e')$ *is* $\Gamma^{\mathrm{mg}}_{a,\mathbf{p},\bar{c}^1}$*-big*
   (iv) $\mathrm{tp}(d', A + e')$ *nicely extend* $\mathrm{tp}(d^*, A)$
(C) $n/|a| < \mathbf{p} < c^1_0 < 1/n$ *for* $n < \omega$
(D) $c^2_i < c$.

Proof. By compactness without loss of generality $\delta_1 = \delta_2 = \omega$. Work in $\mathfrak{C}$. First, we find a non-standard integer $\dot{n}$ small enough, i.e.

$(*)_1$ $\mathfrak{C} \models$ "$n < \dot{n} \,\&\, 2^{\mathbf{n}} < c \times |a|$" for $n < \omega$.

We let $\mathbf{p} = 1/\dot{n}, c^2_i = \mathbf{p}^{-i} 2^{\dot{n}}/|a|$ and lastly $c^1_i = (\ln \dot{n})^i/\dot{n}$. Now clauses (A), (C), (D) are immediate. For clause (B) we have to check the demand $(*)$ in 3.26(2). There clause (i) holds by clause (A), clause (ii) holds by clause (C). Clause (iv) holds by the choice of c^2_i and as for clause (iii) for $i, j < \omega$ we have

$$\mathbf{e}^{-\frac{1}{8}\mathbf{p}^2 \times c^2_j \times |a|} = \mathbf{e}^{-\frac{1}{8}\mathbf{p}^{2-j} 2^{\mathbf{n}}} \le \mathbf{e}^{-2^{\mathbf{n}/2}} < 1/2\mathbf{n} < \frac{1}{2} c^1_0 \le \frac{1}{2} c^1_i.$$

$\square_{3.29}$

Definition 3.30. Let T be a complete first order theory, $\mathfrak{p} = \mathfrak{p}(\bar{x})$ a type definition (see 0.2(8), say with parameters in $A \subseteq M^*$, see more in [Shee]. Then we let $\Gamma = \Gamma^{\mathfrak{p}}$ be the following bigness notion: if $A \subseteq M, \bar{a} \subseteq M$ then: $\varphi(\bar{x}, \bar{a})$ is Γ-big iff $\varphi(\bar{x}, \bar{a}) \in \mathfrak{p}^M$.

Claim 3.31. *1) For T, $\mathfrak{p}, M^*$ as in Definition 3.30, $\Gamma^{\mathfrak{p}}$ is a ℓ-bigness notion.*
2) For T^* as in 1.1(2), if Γ is an instance of $\Gamma^{\rm ms}$,or $\Gamma^{\rm wm}$, or any Γ such that for some $a \in A_\Gamma$, a pseudo finite set and every Γ-big $p \in \mathbf{S}(A_\Gamma, M^*)$ we have $[x_i \in a] \in p$ (for $i < \lg(\bar{x}_\Gamma)$), $\mathfrak{p}$ is a type definition not increasing "finite" sets (for example $\Gamma^{\mathfrak{p}}_{\rm uf}$ see [Shee, 2.11=L2.6]) <u>then</u> $\Gamma \perp \Gamma^{\mathfrak{p}}$.
3) Assume p_2 has a unique extension in $\mathbf{S}(A', M^*)$ which is necessarily Γ_2-big so $\Gamma_1 \perp \Gamma_2$ <u>when</u>:

- $(a)_\ell$ *Γ_1, Γ_2 are bigness notions,*
- $(b)_\ell$ *$p_\ell(\bar{x}^\ell) \in \mathbf{S}^{\lg(\bar{x}^\ell)}(A, M^*)$ is Γ_ℓ-big (for $\ell = 1, 2$),*
- (c) *there is $d \in A, M^* \models$ "d finite", $[\bar{x}^1 \subseteq d] \in p_1$, and for every $\varphi(\bar{x}^1, \bar{x}^2)$ with parameters from A from some $e_\varphi \in A$ we have $(\forall \bar{z}^1)[\bar{z}^1 \subseteq d \to \bar{z}^1 \in e_\varphi \equiv \varphi(\bar{x}^2, \bar{z}^1)] \in p_2$ and $A \subseteq A' \subseteq A \cup d(M^*)$ where $d(M^*) = \{c \in M^* : M^* \models$ "$c \dot{e} d$"$\}$.*

Proof. Straightforward. $\square_{3.31}$

Definition 3.32. We say that a bigness notion Γ for models of T^* is orthogonal to pseudo finite if: for any Γ-big $p(\bar{x})$ and pseudo finite $d \in \mathfrak{C}$, there is an extension $q(\bar{x})$ of $p(\bar{x})$ which is Γ-big and satisfies clause (c) of 3.31(2).

Claim 3.33. *(For T^* as in 1.1(B)). If Γ_1, Γ_2 are bigness notion, $\Gamma_1 \perp \Gamma_2$ if:*

- (a) *Γ_1 is orthogonal to pseudo finite*
- (b) *if $\Gamma(\bar{x})$ is Γ_2-big in $M^* \supseteq A_{\Gamma_2}$ then for some sequence $\langle d_i : i < \lg(\bar{x}) \rangle$ of pseudo finite sets we have $P(\bar{x}_{\Gamma_2}) \cup \{x_i \in d_i : i < \lg(\bar{x}_\Gamma)\}$ is Γ_2-big (we say: pseudo finitary).*

The following generalizes $\Gamma^{\rm na}$.

Definition 3.34. Let T^* be as in 1.1(B) and M^* be a model of T^*. Let $a, \dot{\mathbf{d}}, c_i (i < \delta, \delta$ is limit ordinal) be members of M^* such that (in M^*):

- (a) a is a set
- (b) $\dot{\mathbf{d}}$ is a distance function on a in the sense of M^*, i.e. for $b_1, b_2, b_3 \dot{e}^{M^*} a$ we have
  - $\dot{\mathbf{d}}(b_1, b_2)$ is a non-negative real which is positive <u>if and only if</u> $b_1 \neq b_2$,
  - $\dot{\mathbf{d}}(b_1, b_2) = \mathbf{d}(b_2, b_1)$,
  - $\mathbf{d}(b_1, b_3) \leq \mathbf{d}(b_1, b_2) + \mathbf{d}(b_2, b_3)$
- (c) c_i a positive real
- (d) $2c_i \leq c_{i+1}$ and $i < j \Rightarrow c_i < c_j$
- (e) for every $i < \delta$ and n there are $b_0, \ldots, b_{n-1} \dot{e} a$ such that $\ell < k < n \Rightarrow \dot{\mathbf{d}}(b_\ell, b_k) \geq c_i$.

Let $\bar{c} = \langle c_i : i < \delta \rangle$. We define the ℓ-bigness notion $\Gamma^{\rm mt}_{a, \dot{\mathbf{d}}, \bar{c}}$ (mt for metric) as follows: $\varphi(x, \bar{b})$ is Γ-big <u>if and only if</u> in M^* there are n members of a satisfying $\varphi(-, \bar{b})$ pairwise of distance $\geq c_i$, that is:

$$\psi = \psi_{\varphi(\bar{x},\bar{b}),n} := (\exists x_1, \dots x_n)[\bigwedge_{\ell=1}^{n} \varphi(x_\ell, \bar{b}) \wedge \bigwedge_{\ell<m} \dot{\mathbf{d}}(x_\ell, x_m) \geq c_i \wedge \bigwedge_{\ell=1}^{n} x_\ell \dot{e} a]$$

for every $i < \delta, n < \omega$. Let $\Gamma^{\rm mt}_\delta$ be the corresponding bigness scheme for T^*. We may omit a if it is defined as $\{x : (\exists y)[\dot{\mathbf{d}}(x,y)$ well defined$\}$; we may write "dis" instead of $\dot{\mathbf{d}}$.

Claim 3.35. *Assume $T^*, M^*, a, \dot{\mathbf{d}}, \bar{c} = \langle c_i : i < \delta\rangle$ are as in 3.34.*
1) $\Gamma^{\rm mt}_{a,\dot{\mathbf{d}},\bar{c}}$ is an invariant ℓ-bigness notion, co-simple, $\aleph_1$-presentable and orthogonal to every invariant Γ in particular to $\Gamma^{\rm ms}, \Gamma^{\rm wm}$ (and to $\Gamma^{\rm na}, \Gamma^{\rm ids}$ of course).
2) Suppose $A \subseteq B \subseteq \mathfrak{C}, A \subseteq A' \subseteq M^*, M^*$ a κ-saturated model of T^*. We can find elementary mappings $\mathbf{f}_n$ (for $n < \omega$) such that:

- *(i)* $\mathrm{Dom}(\mathbf{f}_n) = B, \mathbf{f}_n \restriction A = \mathrm{id}_A$
- *(ii) if $\Gamma = \Gamma^{\rm mt}_{a,\dot{\mathbf{d}},\bar{c}}$ for some a, $\dot{\mathbf{d}}, \bar{c}$ from A and $b \in B, \mathrm{tp}(b, \mathrm{acl}(A), M^*)$ is Γ-big <u>then</u> for any $n < \omega, \mathrm{tp}(\mathbf{f}_n(b), A' \cup \bigcup_{\ell \neq n} A_\ell, \mathfrak{C})$ is Γ-big.*

Remark 3.36. 1) Compare with 5.8-5.11 below.
2) In 3.35(2) we can replace w by any Γ.

Proof. 1) Left to the reader e.g. use 2.3(2) (note: if $\mathrm{tp}(d, A)$ is $\Gamma^{\rm mt}_{a,\dot{\mathbf{d}},\bar{c}}$-large then for every λ we can find $\dot{\mathbf{d}}_\alpha$ such that each $\dot{\mathbf{d}}_\alpha$ realizes $\mathrm{tp}(a, A), N^*)$ (for $\alpha < \lambda$) and N^* such that $M^* \prec N^*$ and $\alpha < \beta < \lambda \,\&\, i < \lg(\bar{c}) \Rightarrow N^* \models$ "$\dot{\mathbf{d}}(d_\alpha, d_\beta) \geq c_i \wedge d_\alpha \dot{e} a$".
2) For one Γ, b this should be clear by the definition of $\Gamma^{\rm mt}_{a,\dot{\mathbf{d}},\bar{c}}$. Generally use compactness; in more detail assume to show that it is enough to prove, for any $n < \omega$ that if

- $(*)_1$ for $\ell < m, a_\ell, \dot{\mathbf{d}}_\ell, \bar{c}^\ell$ are from A and as in Definition 3.34, $b_\ell \in B$ and $\mathrm{tp}(b_\ell, A)$ is $\Gamma^{\rm mt}_{a_\ell,\dot{\mathbf{d}}_\ell,\bar{c}^\ell}$ big clearly
- $(*)_2$ it is enough to prove
  - for every $m, m(*)$ and find elementary mapping $\mathbf{f}_n$ for $n < \omega$ such that $\mathrm{Dom}(\mathbf{f}_n) = B, \mathbf{f}_n \restriction A = \mathrm{id}_A$ and for each $\ell < m$, and $n_1 < n_2 < \omega$ and $i < \lg(\bar{c}_\ell)$ we have $\dot{\mathbf{d}}_\ell(\mathbf{f}_{n_1}(b_\ell), \mathbf{f}_{n_2}(b_2)) \geq c_{\ell,i}$.
    So assume that
  - from $(*)_2$ fail, so for some $m, m(*)$ and ψ we have (supervising parameters from A)
- $(*)_3$ (a) $M^* \models \psi[b_0, \dots, b_{m-1}]$
  (b) $M^* \models (\exists y_{\ell,k}, \dots)_{\ell<m,k<n(*)} (\forall x_0, \dots, x_{m-1})[\psi(x_0, \dots, x_{m-1}) \rightarrow \bigvee_{\ell<m} \bigvee_{k<n(*)} \dot{\mathbf{d}}_\ell(x_\ell, y_{\ell,k}) < c_{\ell,i_\ell}]$.

Hence (as we are dealing with models of T^*)

- $(*)_4$ we can find $b_{k,\ell} \in \mathrm{dcl}(A)$ for $\ell < m, k < n(*)$ such that

$$M^* \models (\forall x_0, \dots, x_{m-1})[\psi(x_0, \dots, x_{m-1}) \rightarrow \bigvee_{\ell<m} \bigvee_{k<n(*)} \dot{\mathbf{d}}_\ell(x_2, yb_{\ell,k}) < c_{\ell,i_\ell}].$$

Substituting b_ℓ for x_ℓ we get that for some $\ell < m, k < m(*) M^* \models$ "$\dot{\mathbf{d}}_\ell(b_i, b_{\ell,k}) < c_{\ell,i_\ell}$", hence clearly $\mathrm{tp}(b_\ell, A, M^*) \vdash \dot{\mathbf{d}}_\ell(x, b_{\ell,k}) < c_{i,i_\ell}$, a contradiction to the assumption "$\mathrm{tp}(b_\ell, A, M^*)$ is $\Gamma_{a_\ell, \dot{\mathbf{d}}_\ell, \bar{c}_\ell}$-big". $\square_{3.35}$

Trivial but useful in the proof of 5.6 is:

Observation 3.37. *Let M be a model of T^* from 1.1(B). Let $A = \mathrm{acl}(A) \subseteq M$, $M \restriction A \prec M, e \in M, \{a, \dot{\mathbf{d}}\} \cup \mathrm{Rang}(\bar{c}) \subseteq A$. Now $\mathrm{tp}(e, A, M)$ is $\Gamma^{\mathrm{mt}}_{a, \dot{\mathbf{d}}, \bar{c}}$-big <u>if and only if</u> for every $e' \in A, i < \lg(\bar{c})$ we have $M \models$ "$e' \in a \rightarrow \mathbf{d}(e', e) \geq c_i$".*

Proof. Easy as in the proof of 3.35(2) (see more in 5.8-5.13). $\square_{3.37}$

Definition 3.38. Let T^* be as in 1.1(B) and $\mathfrak{C} \models$ "a an infinite set".
1) We define a local bigness notion scheme Γ^{ms}_a:

- $\varphi(x, \bar{b})$ is Γ^{tms}_a-big <u>iff</u> $\mathfrak{C} \models$ "$\{x \subseteq a : \varphi(x, \bar{b})\}$ is not a null subset of $\mathscr{P}(a)$".

[Note: $\mathfrak{C}$ "think" itself a model of set theory, hence for $A \subseteq \mathscr{P}(a)$ we can define its outer Lebesgue measure identifying $\mathscr{P}(a)$ with ${}^a 2$.]
2) We define a local bigness notion scheme Γ^{inf}_a by:

- $\varphi(x, \bar{b})$ is Γ^{inf}_a-big if $\mathfrak{C} \models$ "$\{x \dot{e} a : \varphi(x, \bar{b})\}$ is infinite".

Claim 3.39. *Assume $\mathfrak{C} \models$ "a, a_1 are infinite, a_2 is finite".*
1) Γ^{tms}_a is a simple invariant ℓ-bigness notion.
2) Γ^{tms}_a is orthogonal to $\Gamma^{\mathrm{tms}}_{a_1}$ and to $\Gamma^{\mathrm{wm}}_{a_2, \dot{w}, \bar{c}}$ if $(a_2, \dot{w}, \bar{c})$ are as in Definition 3.12.
3) Γ^{inf}_a is an invariant ℓ-bigness notion, uniformly $\aleph_1$-simple (hence co-simple) orthogonal to any invariant local and even global bigness notion.

Proof. 1) Easy.
2) Use Fubini theorem.
3) Easy. $\square_{3.39}$

§ 4. General Construction for T

In this section we think on building $\mathfrak{B}_\alpha$ a model of T of cardinality λ by induction on $\alpha < \lambda^+$ representing $\mathfrak{B}_\alpha$ as the increasing union of A^α_j $(j < \lambda)$ and having special $\bar{a}^\alpha_i \subseteq A^\alpha_{i+1}$, we better demand λ is regular uncountable. On the one hand constructing $\mathfrak{B}_{\alpha+1}$ we do it by approximations which are types over $\mathfrak{B}_\alpha$ of cardinality $< \lambda$, restricting ourselves to appropriate Ω^α-big types. On the other hand for $\beta < \alpha$ we demand that for "many" i, $A^\beta_i \subseteq A^\alpha_i$ and $\mathrm{tp}(\bar{a}^\beta_i, A^\alpha_i, \mathfrak{B}_\alpha)$ is a Γ^α_i-big type. To be able to carry this we need the orthogonality of the Ω's with the Γ's. We look at $\langle A^\alpha_j : j < \lambda\rangle$ as increasing vertically and at $\langle \mathfrak{B}_\alpha : \alpha < \lambda^+\rangle$ as increasing horizontally.

Context 4.1. 1) T a complete first order theory (usually as in 1.1(B)), $\mathfrak{C}$ a monster for T.
2) λ a regular cardinal $\geq |T|$ (and $\chi > \lambda$).
3) $\boldsymbol{\Upsilon}_{\mathrm{hor}}$ (hor -short for horizontal) is a set of $\leq \lambda^+$ global-bigness notions and schemes of g-bigness notions for T such that $\Gamma^{\mathrm{tr}} \in \boldsymbol{\Upsilon}_{\mathrm{hor}}$ or $\Gamma^{\mathrm{na}} \in \boldsymbol{\Upsilon}_{\mathrm{hor}}$ and if such scheme has κ parameters then $\lambda^\kappa \leq \lambda^+$ (or we do not use all instances of the scheme).
4) $\boldsymbol{\Upsilon}^{\mathrm{ver}}$ (ver - short for vertical) is a set of g-bigness notions and schemes of g-bigness notions for T such that: if $\Gamma \in \boldsymbol{\Upsilon}^{\mathrm{ver}}$ is a scheme with κ parameters then $\lambda^\kappa = \lambda$.
5) We assume: if $N \models T$, Γ_1 is an instance of $\boldsymbol{\Upsilon}_{\mathrm{hor}}$ for N, Γ_2 is an instance of $\boldsymbol{\Upsilon}^{\mathrm{ver}}$ for N <u>then</u> $\Gamma_1 \perp \Gamma_2$ (at least for those actually used), in fact nicely orthogonal (used only in niceness of (D)(7) below, in the present context is not an extra assumption by 1.18(3),(4)).
6) For a given model N of T, an instance of $\boldsymbol{\Upsilon}_{\mathrm{hor}}$ for N mean $\Gamma \in \boldsymbol{\Upsilon}_{\mathrm{hor}}$ or Γ a case of a scheme $\Gamma \in \boldsymbol{\Upsilon}_{\mathrm{hor}}$ with parameters from N, similarly for $\boldsymbol{\Upsilon}^{\mathrm{ver}}$.
7) Let for $\boldsymbol{\Upsilon}$ as above $\Gamma \in c\ell_N(\boldsymbol{\Upsilon})$ means $\Gamma = \langle \Gamma_i : i < \alpha\rangle$ for some $\alpha < \lambda$, each Γ_i an instance of $\boldsymbol{\Upsilon}$ for N, see 1.12- 1.18.

Discussion 4.2. If λ is a regular uncountable cardinal $S^{\lambda^+}_\lambda := \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\} \in \check{I}[\lambda^+]$, see [Shed, 3.4=Lcdl.1] or at least some stationary $S \subseteq S^{\lambda^+}_\lambda$ is in $\check{I}[\lambda]$, things are nicer. Assuming there are λ^+ almost disjoint stationary subsets of λ (a very weak assumption, see [She86b] or [Shed, 4.1=Ld4] and Gitik-Shelah [GS97]), simplifies (can use $u^\alpha_i = \{i\}$) but till now was not really necessary. Then below $\bar{S} = \langle S_\alpha : \alpha < \lambda^+\rangle$, $S_\alpha \subseteq \lambda$ is stationary, $\beta < \alpha \Rightarrow |S_\beta \cap S_\alpha| < \lambda$.

We shall describe a construction of a model of T of cardinality λ^+ by an increasing continuous sequence $\langle \mathfrak{B}_\alpha : \alpha < \lambda^+\rangle$ of λ^+ approximations: models of T of cardinality λ, and for $\alpha = \beta + 1$, $\mathfrak{B}_\alpha$ is constructed in λ steps; in step $i < \lambda$, we have already constructed a type p^α_i over some $A^\alpha_i \subseteq \mathfrak{B}_\beta$ of cardinality $< \lambda$ (stipulating $\mathfrak{B}_0$ is empty so A^0_i is empty), and a g-bigness notion, Ω^α_i, such that p^α_i is Ω^α_i-big and $p^\alpha_i, \Omega^\alpha_i$ are increasing with i. We described the construction by assigning some persona called *contractor* to perform various tasks. Each contractor may play the major role for some $\alpha < \lambda^+$, so it is called "the contractor at α", but it is also assigned some i's for every α. For each $\alpha < \lambda^+$ a contractor ζ_α plays the major role, in particular chooses a set of permissible sequences $\langle \Omega^\alpha_i : i < \lambda\rangle$ (see below) and possibly a linear ordering $<^\alpha_i$ of i with $j < i \Rightarrow <^\alpha_j = <^\alpha_i \restriction j$ (in §5 we choose such sequences, generally this choice has to be closed under limits, has no maximal member; the default value is the usual order). A simple case of the $<^\lambda_i$

(the one, which we already use) is when the contractor ζ_α choose a linear order $<_\alpha$ of λ and let $<^\alpha_i = <_\alpha \restriction i$). If not said otherwise we allow to add instances of $\Gamma^{\rm na}$. We demand that all the bigness notions are nice: also we can replace $\langle \Omega^\alpha_j : j < i\rangle$ by one bigness notion Ω^α. We may use games to describe the constructions as in [She83b], [HLS93], see also [She92, AP] or [She94, AP].

The Context Continued 4.3. We have $\bar S = \langle S_i : \ i < \lambda\rangle$, a partition of λ to stationary sets, and $\bar W = \langle W_\alpha : \alpha < \lambda^+\rangle$ a partition of λ^+ such that for every regular $\theta \le \lambda$ the set $\{\delta < \lambda^+ : {\rm cf}(\delta) = \theta$ and $\delta \in W_\alpha\}$ is a stationary subset of λ^+ (actually here we use W_α only for $\alpha < \lambda$). In applications for each $\alpha < \lambda^+$ we assign a "contractor" who can make sure the model $\mathfrak{B}_{\alpha+1}$ of T which we shall construct will have some properties. Let $W^\theta_\alpha = \{\delta \in W_\alpha : {\rm cf}(\delta) = \theta\}$.

Now we start with the formal description.

Preliminaries 4.4. We choose by induction on $\alpha < \lambda^+, \mathbf{c}^\alpha_i$ for $i < \lambda$ such that:

(a) $\bar{\mathbf{c}}_\alpha = \langle \mathbf{c}^\alpha_i : i < \lambda\rangle$ is an increasing continuous sequence of subsets of α

(b) $\alpha = \bigcup\limits_{i<\lambda} \mathbf{c}^\alpha_i$

(c) $|\mathbf{c}^\alpha_i| < \lambda$

(d) $\beta \in \mathbf{c}^\alpha_i \Rightarrow \mathbf{c}^\beta_i = \mathbf{c}^\alpha_i \cap \beta$

(e) $\mathbf{c}^0_i = \emptyset, \alpha \in \mathbf{c}^{\alpha+1}_0$, $0 \in \mathbf{c}^{1+\alpha}_0, [\aleph_0 \le {\rm cf}(\alpha) < \lambda \,\&\, \mathbf{c}^\alpha_i \ne \{0\} \Rightarrow \alpha = \sup(\mathbf{c}^\alpha_i)]$.

The Construction Definition 4.5. We define a game $\eth = \eth_{\bar{\mathbf{c}}}$ between the portagonist and antagonist player, the antagonist choices are divided to the work of various so called contractors and they are actually independent sub-players. All the other choices are of the protagonist; the protagonist wins a play when always there is a legal move. The order of the choices is first by $\alpha < \lambda^+$ and then by $\varepsilon < \lambda$. During a play the following are chosen.

For $\alpha < \lambda^+, \mathfrak{B}_\alpha$ and given α and $\mathfrak{B}_\alpha$ toward with choosing $\mathfrak{B}_{\alpha+1}$ by induction on $\epsilon < \lambda$, ordinal we choose an $i^\alpha_\epsilon = i_{\alpha,\epsilon}$, set $u^\alpha_{i_{\alpha,\epsilon}}$, and for $j \in u^\alpha_{i_{\alpha,\epsilon}}$ a type p^α_j and a set A^α_j, and (for ϵ and α) $E_\alpha \cap i^\alpha_{\epsilon+1}$ and $\langle <^\alpha_j : j \in u^\alpha_{i_{\alpha,\epsilon}}\rangle, \bar u^\alpha_\epsilon = \langle u^\alpha_j : j \in E_\alpha \cap i^\alpha_{\epsilon+1}\rangle, E^+_\alpha \cap i^\alpha_{\epsilon+1}, \langle \Gamma^\alpha_i, \bar c^\alpha_i : i \in E^+_\alpha \cap i^\alpha_{\epsilon+1}\rangle$ and $\bar\Omega^\alpha = \langle \Omega^\alpha_j : j \in E^+_\alpha \cap i^\alpha_{\epsilon+1}\rangle$ (and for some α's also $<^\alpha_0$) and $\langle \bar a_{\alpha,i} : i < \lambda\rangle$ such that:

(A) (a) $\mathfrak{B}_\alpha$ is a model of T with universe $\lambda \times \alpha$ (so we stipulate $\mathfrak{B}_0$ is an empty model and for notational simplicity ignore the case $|\mathfrak{B}_{\alpha+1} \setminus \mathfrak{B}_\alpha| < \lambda$; alternatively you may ask that the universe of $\mathfrak{B}_\alpha$ is $\gamma_\alpha < \lambda \times \alpha$ with no serious changes)

(b) $\beta < \alpha \Rightarrow \mathfrak{B}_\beta \prec \mathfrak{B}_\alpha$ (so $\mathfrak{B}_\alpha$ is $\prec$-increasing continuous)

(c) essentially $\mathfrak{B}_{\alpha+1} \setminus \mathfrak{B}_\alpha = \cup\{\bar a_{\alpha,i} : i < \lambda\}$, more exactly as we would like to allow elements appearing $\bar a_{\alpha,i}$ to be equal to a member of $\mathfrak{B}_\alpha$ and as we may like not to use $\Gamma^{\rm tr}$, we demand just $\mathfrak{B}_{\alpha+1} = {\rm acl}_{\mathfrak{B}_{\alpha+1}}\big(\mathfrak{B}_\alpha \cup \bigcup\limits_{i<\lambda} \bar a_{\alpha,i}\big)$

(d) $\langle A^\alpha_j : j < \lambda\rangle$ is an increasing sequence of subsets of $\mathfrak{B}_\alpha$ each of cardinality $< \lambda$ and $\bigcup\limits_{j<\lambda} A^\alpha_j = \mathfrak{B}_\alpha$

(B) (a) E_α is a club of λ

(b) if $\beta \in \mathbf{c}^\alpha_i$ and $i \in E_\alpha$ then $E_\alpha \setminus (i+1) \subseteq E_\beta$

(c) $E_\alpha = \{i^\alpha_\epsilon : \epsilon < \lambda\}, i^\alpha_\epsilon = i_{\alpha,\epsilon}$ is increasing continuous with ϵ

(d) $0 \in E_\alpha$ (so $i^\alpha_0 = 0$)

(C) (a) $\bar{u}^\alpha = \langle u^\alpha_i : i \in E_\alpha \rangle, u^\alpha_i$ a closed set of ordinals $< \lambda$ with a last element, $\min(u^\alpha_i) = i$, $\max(u^\alpha_i) < \lambda$

(b) $E^+_\alpha = \bigcup\limits_{i \in E_\alpha} u^\alpha_i$

(c) $i \in E_\alpha \Rightarrow \min(E_\alpha \setminus (i+1)) > \max(u^\alpha_i)$

(d) $i \in E_\alpha, \beta \in \mathbf{c}^\alpha_i, j \in u^\alpha_i$ then $A^\alpha_j \cap \mathfrak{B}_\beta = A^\beta_j$

(e)(α) if $i \in E_\alpha, \beta \in \mathbf{c}^\alpha_i$, then $i \in E_\beta$ and u^β_i is an initial segment of u^α_i

(β) if $i \in E_\alpha, j \in u^\alpha_i$, $\beta \in \mathbf{c}^\alpha_j, j \notin u^\beta_i$ then $j \in E_\beta$

(γ) $<^\alpha_j$ is a linear, well ordering of $\{i : i \in j \cap E^+_\alpha\}$ increasing with j for $j \in E^+_\alpha$, i.e. if $j_1 < j_2$ then $\mathrm{dom}(<^\alpha_{j_1})$ is an initial segment of $(\mathrm{dom}(<^\alpha_{j_2}), <^\alpha_{j_2})$

(D) (a) $\bar{p}^\alpha = \langle p^\alpha_j : j \in E^+_\alpha \rangle$ is an increasing continuous sequence of types over $\mathfrak{B}_\alpha$ (for $\alpha = 0$ this means consistent with T)

(b) p^α_j is a type in the variables $\{\bar{x}_{\alpha,j_1} : j_1 \in j \cap E^+_\alpha\}$

(c) p^α_j is a complete type over A^α_j (in $\mathfrak{B}_\alpha$)

(d) $\langle \bar{a}_{\alpha,j_1} : j_1 < j \rangle$ realizes in $\mathfrak{B}_{\alpha+1}$ the type p^α_j

(e) Ω^α_j a bigness notion, an instance of $\mathbf{\Upsilon}_{\mathrm{hor}}$ with parameters from A^α_j or at least $A^\alpha_j \cup \bigcup\{\bar{x}_{\alpha,j_1} : j_1 <^\alpha_{j+1} j\}$

(f) the type p^α_j is big for $\langle \Omega^\alpha_{j_1} : j_1 < j \rangle$ by the order $<^\alpha_j$

(g) for $j_1 < j_2$ in E^+_α we have : $p^\alpha_{j_2} \restriction \bigcup\limits_{j<j_1} \bar{x}_{\alpha,j}$ is a nice extension of $p^\alpha_{j_1}$ (i.e. in $\mathfrak{B}_{\alpha+1}$, $A^\alpha_{j_2} \cap \mathrm{acl}\big(A^\alpha_{j_1} \cup \bigcup\limits_{j<j_1} \bar{a}_{\alpha,j}\big) = A^\alpha_{j_1}$); the niceness require nice orthogonality in 4.1(4)$\otimes$, but in our present proof this is automatic

(E) (a) $\bar{c}^\alpha_j \subseteq \mathfrak{B}_\alpha$ is a sequence of length $\lg(\bar{x}_{\Gamma^\alpha_j})$ and $\mathrm{tp}(\bar{c}^\alpha_j, A^\alpha_j, \mathfrak{B}_\alpha)$ is Γ^α_j-big (for $j \in E^+_\alpha$)

(b) Γ^α_j is an instance of $c\ell_{\mathfrak{B}_\alpha}(\mathbf{\Upsilon}^{\mathrm{ver}})$ with parameters from A^α_j

(c) if $i \in E_\alpha, j \in u^\alpha_i, \beta \in \mathbf{c}^\alpha_i$, $j \in u^\beta_i$ <u>then</u>:

(α) $\bar{c}^\beta_j$ is an initial segment of $\bar{c}^\alpha_j$

(β) if $\mathbf{c}^\alpha_i$ has no last element then $\bar{c}^\alpha_j$ is the limit of $\langle \bar{c}^\gamma_j : \beta \leq \gamma \in \mathbf{c}^\alpha_i \rangle$

(γ) Γ^β_j is a restriction (to initial segment of the variables) of Γ^α_j

(note: if $j \in u^\alpha_i \setminus \bigcup\{u^\beta_i : \beta \in \mathbf{c}^\alpha_i\}$, the only restriction on $\bar{c}^\alpha_j$ is: $\in \mathfrak{B}_\alpha$); the <u>easy</u> case is $\bar{c}^\alpha_i = \bar{c}^\beta_i$, and this is the one used, in the general case we need to ensure that the limit in clause (β) exists.

(F) (a) $\mathscr{P}_\alpha$ is a family of cardinality $\leq \lambda$ of: types over $\mathfrak{B}_\alpha$, subsets, relations on $|\mathfrak{B}_\alpha|$ and partial function from $|\mathfrak{B}_\alpha|$ to $|\mathfrak{B}_\alpha|$; and this family is increasing with α, with reasonable closure conditions. E.g. choose $\mathfrak{A}_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$, increasing with α of cardinality λ with

$\lambda + 1 \subseteq \mathfrak{A}$ such that the construction so far belong to it, including $\langle \mathfrak{A}_\beta : \beta < \alpha \rangle$, $\langle E_\beta, E^+_\beta$: $\beta < \alpha \rangle$, $p^\beta_j, A^\beta_j <^\beta_j, u^\beta_j, \Gamma^\beta_j, \bar{c}^\beta_j, \bar{a}^\beta_\eta) : \beta < \alpha, j \in E^*_\beta)$ $\langle i^\beta_\epsilon, u^\beta_{i_{\beta,\epsilon}} : \beta < \alpha, \epsilon < \lambda \rangle$

(b) in $\mathfrak{B}_{\alpha+1}$ all types over $\mathfrak{B}_\alpha$ from $\mathscr{P}_\alpha$ of cardinality $< \lambda$ are realized in $\mathfrak{B}_{\alpha+1}$

(G) In stage α, the construction is done by induction on $\epsilon < \lambda$. First we decide (or are given) what is i^α_ϵ and then $\Omega^\alpha_j, p^\alpha_j$ for $j \in \cup\{u^\beta_{i_\epsilon} : \beta \in \mathbf{c}^\alpha_{i_\epsilon}\}$ by induction on j, and then we continue adding more elements to what will be u^α_i and the corresponding $\Gamma^\alpha_j, \Omega^\alpha_j$ and p^α_j $(j \in u^\alpha_i)$ and lastly choose $i^\alpha_{\epsilon+1}$, $p^\alpha_{i_{\alpha,\epsilon+1}}$. The decisions are distributed among the various contractors, for $j \in E^+_\alpha$ it will be $\zeta^\alpha_j = \zeta_{\alpha,j}$, but $\zeta^\alpha_{i_{\alpha,\epsilon}}$ will have a say on every $j \in u^\alpha_\epsilon$ (and is called "the (major) contractor for (α, ϵ) or the major ϵ-substage in the stage α"), and ζ^α_0 on all $j \in E^+_\alpha$, in particular ζ^α_0 (which is called "the (major) contractor for α)", and written also as ζ_α decide what will be the family of permissible $\langle \Omega^\alpha_j : j < i \rangle$ for $i < \lambda$ (usually unique, in particular whether we have $<^\alpha_i$'s). Let $E^0_\alpha \subseteq \lambda$ be a thin enough club such that $0 \in E^0_\alpha$. For $\epsilon = 0$, $i_{\alpha,\zeta}$ is zero and but $\zeta^\alpha_{i_{\alpha,\epsilon}} = \zeta^\alpha_0$ is the ζ such that $\alpha \in W_\zeta$. For ϵ a limit ordinal let $i = i^\alpha_\epsilon$ be $\bigcup_{\xi<\epsilon} i^\alpha_\xi$ (necessarily it is in E^0_α and $[\xi < \epsilon \Rightarrow \max(u^\alpha_\xi) < i])$ and let $\zeta^\alpha_{i_{\alpha,\epsilon}}$ be the unique ζ such that $i^\alpha_\epsilon \in S_\zeta$. For ϵ successor it will be a member of $E^0_\alpha \cap \bigcap_{\beta \in \mathbf{c}_{\alpha,\epsilon-1}} E_\beta$ (which is $> \max(u^\alpha_{\epsilon-1})$) as decided in stage $\epsilon - 1$.

(H) Also in stage α, in the induction on ϵ we choose $\mathscr{P}_{\alpha,\epsilon}$ as in (F)(1) increasing with ϵ, $\mathscr{P}_\alpha \subseteq \mathscr{P}_{\alpha,\epsilon}$, $\bigcup_{\epsilon<\lambda} \mathscr{P}_{\alpha,\epsilon} \subseteq \mathscr{P}_{\alpha+1}$, $|\mathscr{P}_\alpha| \le \lambda$ and the construction up to ϵ belongs to it. Also (essentially) all types in $\mathscr{P}_{\alpha,\epsilon}$ of cardinality $< \lambda$ over $\mathfrak{B}_\alpha \cup A^\alpha_j$ will be realized in $\mathfrak{B}_{\alpha+1}$ (as in 4.7(2) below).

(I) the division of the decisions:

(a) for any $\alpha < \lambda^+$ the antagonist chooses an index family called "the horizontal contractor" $\bar{\zeta}^{\mathrm{hor}}_\alpha = \langle \zeta^{\mathrm{hor}}_{\alpha,x} : x \in X^{\mathrm{hor}}_\alpha \rangle$ (or we call $\zeta^{\mathrm{hor}}_{\alpha,x}$ a case of $\bar{\zeta}^{\mathrm{hor}}_\alpha$) where X^{hor}_α has cardinality $\le \lambda^+$

(b) for any $\alpha < \lambda^+, i^\varepsilon < \lambda$ the antagonist chooses a non-empty index family called "the vertical contractor" $\bar{\zeta}^{\mathrm{ver}}_\alpha = \langle \zeta^{\mathrm{ver}}_{\alpha,y} : y \in Y^{\mathrm{ver}}_\alpha \rangle$ where Y^{ver}_α has cardinality $\le \lambda$.

(c) By bookkeeping for each stage $\alpha < \lambda^+$ of the construction, exactly one of the cases of the vertical contractors $\zeta^{\mathrm{hor}}_{\beta,x}$are active as the major contractors where $\beta \le \alpha, x \in X^{\mathrm{hor}}_\alpha$; for each $\zeta^{\mathrm{hor}}_{\gamma,x}$ (so $\gamma < \lambda^+, x \in X^{\mathrm{hor}}_\alpha \cap \mathscr{P}_\gamma$) for some $\beta < \lambda^+$, in all $\alpha \in W_\beta$ we have $\zeta^{\mathrm{hor}}_{\gamma,x}$ is "the contractor": this contractor chooses the Ω's

(d) By bookkeeping for each $\zeta^{\mathrm{ver}}_{\alpha,y}$ (so $\alpha < \lambda^+, y \in Y^{\mathrm{ver}}_\alpha$), for every $y \in Y^{\mathrm{ver}}_{\alpha,}$ and $\beta \in (\alpha, \lambda^+)$ we have: for a club of $\epsilon < \lambda$ for some $j \in u^\beta_\epsilon \setminus \{\max(u^\beta_\epsilon)\}$ we have: this contractor choose the Γ^α_j.

(e) some contractor in $Y^{\mathrm{ver}}_{\alpha,\varepsilon}$ choose Γ^{na} which belong to $\mathbf{\Upsilon}_{\mathrm{ver}}$

(f) the antagonist chooses also the $\mathscr{P}_\alpha$ and the $\mathscr{P}_{\alpha,\varepsilon}$.

Theorem 4.6. *The protagonist wins the game $\partial_{\mathbf{c}}$.*

Proof. Note:

(α) for α's with no $<^\alpha_i$'s we can carry the construction by the orthogonality of $\mathbf{\Upsilon}_{\rm hor}, \mathbf{\Upsilon}^{\rm ver}$

(β) for α's with the $<^\alpha_i$ not well ordered (case not used) we need stronger demand on the bigness notion Ω^α_j: it is simple.

(γ) the contractor ζ^α_ϵ can choose a j_1 to be in u^α_ϵ though j_1 belongs some S_ξ, $\xi \neq \zeta^\alpha_\epsilon$.

$\square_{4.6}$

Observation 4.7. *1) $\mathfrak{B}^* = \bigcup\limits_{\alpha<\lambda^+} \mathfrak{B}_\alpha$ is a model of T.*

2) $\|\mathfrak{B}^*\| = \lambda^+$.

Proof. 1) As $\mathfrak{B}_\alpha$ is a model of T, $\prec$-increasing with α.
2) As $\Gamma^{\rm na} \in \mathbf{\Upsilon}_{\rm hor}$ and some ζ_α contractor allows it. $\square_{4.7}$

Discussion 4.8. 1) We can demand only $\mathfrak{B}_{\alpha+2}$ to be quite saturated model, while for limit ordinal $\delta, \mathfrak{B}_{\delta+1}$ can be any algebraically closed set.
2) We can weaken the demand on $\mathbf{\Upsilon}_{\rm hor}, \mathbf{\Upsilon}^{\rm ver}$ to be sets of λ-weak, λ^+-weak g-bigness notion respectively, see Definition 2.18.

If we assume that $(*)$ stated below, then it is enough to demand also on $\mathbf{\Upsilon}^{\rm ver}$ that it is $(< \lambda)$-weak g-bigness notion

$(*)$ there is $f^* \in {}^\lambda\lambda$ such that for $\alpha < \lambda^+$ if f_α is the α the function in ${}^\lambda\lambda$ (e.g. $f_\alpha(i) = {\rm otp}(\mathbf{c}^\alpha_i)$) then $f_\alpha <_{\mathscr{D}_\lambda} f^*$ (where $\mathscr{D}_\lambda$ is the club filter on λ).

§ 5. Proving the compactness

Here we prove our main theorem. For suitable expansion $\mathfrak{C}_*$ of $\mathscr{H}(\chi^*, \in)$, and $\kappa \geq |\tau(\mathfrak{C})|$, $\lambda = (2^\kappa)^+$, there is a κ^+-saturated model $\mathfrak{B}^*$ of $\mathrm{Th}(\mathfrak{C}_*)$ of cardinality λ^+ (or λ^{++}) in which if $\mathbf{b}_1, \mathbf{b}_2$ are Boolean algebras (or rings) in $\mathfrak{B}^*$'s sense, any complete embedding of $\mathbf{b}_1^{\mathfrak{B}^*}$ into $\mathbf{b}_2^{\mathfrak{B}^*}$ is one from $\mathfrak{B}^*$; this gives compactness of appropriate logics.

Question 5.1. Characterize the first order theories $\mathfrak{t}$ such that for any T (as in 1.1(A) or as in 1.1(B)) there are models of T in which each automorphism of any instance of $\mathfrak{t}$ (interpreted there) is definable (by a first order formula with parameters) hence is represented.

We carry this here for Boolean rings (in 5.2).

§ 5(A). **Explanation of the proof of 5.2:**

We shall prove here, in particular the compactness of the logic $\mathbb{L}(\dot{\mathbf{Q}}^{\rm isb})$-where $\dot{\mathbf{Q}}^{\rm isb}$ is quantification over isomorphism of one atomic Boolean ring onto another atomic Boolean ring (not atomless ones as in [Shee]; [we later in the section deal with any Boolean algebra]. In fact we deal with a more general case which says something for any case of the independence property, but here we try to explain the proof for this specific case. Of course we do it in the framework of §4 showing that for any "positive" set of moves, i.e. a strategy for the antagonist, there is a strategy for them, i.e. for the protagonist guaranteeing all such isomorphisms are definable (with parameters). Below we shall survey the proof so we oversimplify in some points. In particular assume 2^κ is regular so we could let $\lambda = 2^\kappa$. Let T be a first order complete theory satisfying $|T| < \lambda = (2^\kappa)^+$.

We build by induction on $\alpha < \lambda^+$ model $\mathfrak{B}_\alpha$ of T of cardinality λ such that $\mathfrak{B}_\alpha$ is $\prec$-increasing, continuous in α and $\mathfrak{B} = \mathfrak{B}_{\lambda^+} = \cup\{\mathfrak{B}_\alpha : \alpha < \lambda^+\}$ should serve. We consider $\mathbf{b}_1, \mathbf{b}_2$ which are definitions by first order formulas with parameters of atomic Boolean rings in $\mathfrak{B}$. For stationarily many α, we think there will be an "undesirable" isomorphism $\mathbf{f}$ from one atomic Boolean ring $\mathbf{b}_1[\mathfrak{B}]$ onto the other, $\mathbf{b}_2[\mathfrak{B}]$ such that $(\mathfrak{B}_\alpha, \mathbf{f}{\restriction}\mathfrak{B}_\alpha) \prec (\mathfrak{B}, \mathbf{f})$ so $\mathbf{b}_\ell[\mathfrak{B}_\alpha] = \mathbf{b}_\ell^{\mathfrak{B}_\alpha}$ is the Boolean ring $\mathbf{b}_\ell$ as interpreted in $\mathfrak{B}_\alpha$. We cannot list and treat all such possibilities and we do not know to guess then (note that G.C.H. may fail here), so we try to add few elements such that the restriction of $\mathbf{f}$ to them will suffice to reconstruct $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\alpha]$. A first approximation is to add $\langle a_i^\alpha : i < 2^\kappa \rangle$ with each $a_i^\alpha \dot{\in} \mathbf{b}_1$ such that for no two disjoint equivalently distinct atoms b_1, b_2 of $\mathbf{b}_1^{\mathfrak{B}_\alpha}$ is $\{i < 2^\kappa : b_1 \leq_{\mathbf{b}_1} a_i^\alpha\} = \{i < 2^\kappa : b_2 \leq_{\mathbf{b}_1} a_i^\alpha\}$, for this use the bigness notion from Definition 3.23. So from $\mathbf{f} \restriction \{a_i^\alpha : i < 2^\kappa\}$ which has fewer possibilities we will be able to reconstruct $\mathbf{f} \restriction \mathfrak{B}_\alpha$. So consider $\beta > \alpha$ such that $\mathfrak{B}_\beta$ is closed under $\mathbf{f}, \mathbf{f}^{-1}$. Now add new $a_\xi^{\alpha,\beta}$ (for $\xi < \kappa$) such that for every distinct $b_1, b_2 \in \mathbf{b}_1^{\rm at}[\mathfrak{B}_{\beta,j}]$ we have $\{i : b_1 \leq_{\mathbf{b}_1} a_i^{\alpha,\beta}\} \neq \{i : b_2 \leq_{\mathbf{b}_1} a_i^{\alpha,\beta}\}$, such a sequence exists as suitable types appear in the $\boldsymbol{\Upsilon}_{\rm hor}$. So if we list the possible $\mathbf{f} \restriction \{a_i^{\alpha,\beta} : i < \kappa\}$, together with α, β we have essentially listed the possible $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\alpha]$! In limit $\delta < \lambda^+$ such that $\lambda^{\mathrm{cf}(\delta)} = \lambda$, we can list $\leq \lambda$ candidates to $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\delta]$ (those such that for unboundedly many $\alpha < \delta, \mathbf{f} \restriction \mathfrak{B}_\alpha$ was listed before δ) so $\mathbf{f}{\restriction}\mathbf{b}_1[\mathfrak{B}_\delta]$ is listed in δ, for a club of such δ's.

The next stage - assume for simplicity that on $\delta < \lambda^+$ we guess $\mathbf{b}_1, \mathbf{b}_2, \mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_1]$. As in older proofs we add $x_\delta \in \mathbf{b}_1[\mathfrak{B}_1]$ and try to omit the type $\{y \in \mathfrak{B}_\delta \,\&\, [\mathbf{f}(a) \leq_{\mathbf{b}_2} y]^{\text{if}[a \leq_{\mathbf{b}_1} x_\delta]} : a \in \mathbf{b}_1^{\text{at}}[\mathfrak{B}_\delta]\}$. It is hard to omit types without $\diamondsuit_\mu$, so we use the special types (as in [She78b]) by preserving bigness for vertical bigness notions as explained below. For x_δ, we define by induction on $j < \lambda$, $p_j \in \mathbf{S}^1(\mathfrak{B}_{\delta,j})$ increasing continuous in $j < \lambda$ such that "$x\dot{e}\mathbf{b}_1$" $\in p_j$, and p_j is big in the sense that for any pairwise distinct $a_0, a_1, \ldots, a_n \in \mathbf{b}_1^{\text{at}}[\mathfrak{B}_\delta] \setminus \mathfrak{B}_{\delta,j}$, the type $p_j \cup \{[a_\ell \leq x_\delta]^{\text{if}[\ell \text{ even}]} : \ \ell \leq n\}$ is consistent in $\mathfrak{B}_\delta$, this is a case of Γ^{ind}. For each $\zeta < \lambda$ we would like to choose $a^\delta_\zeta \in \mathbf{b}_1^{\text{at}}[\mathfrak{B}_\delta]$ and an infinite $\dot{\mathbf{J}}^\delta_j \subseteq \mathfrak{B}_\delta$ to which $\mathbf{f}(a^\delta_j)$ belongs, indiscernible over $\mathfrak{B}_{\delta,j} \cup \{a^\delta_j\}$, and to make it indiscernible over $\mathfrak{B}_{\delta,j} \cup \{a^\delta_j\} \cup \{x_\delta\}$. More generally, we would like to promise that for $\beta > \delta, \{j < \lambda : \dot{\mathbf{J}}^\delta_j$ is an indiscernible sequence over $\mathfrak{B}_{\beta,j} \cup \{a^\delta_j\}$ (in $\mathfrak{B}_\delta$)$\}$ is stationary. To preserve this in limit we promise that this occurs for $j_\epsilon \in u_{\delta,\epsilon}$ for "almost" all $\epsilon \in S$ where $S \subseteq \lambda$ is stationary, almost means except a non-stationary set (so it is clear that having λ almost disjoint stationary subsets of λ is helpful though not actually used).

However we have outsmarted ourselves: if we add $\langle a^\alpha_i : i < 2^\kappa \rangle$ as above, this does not let us fulfill the obligation we have intended to add in order to omit the type - omitting types by the indiscernibility is a strong commitment. There are various directions to try to solve the dilemma, our choice is to weaken the demand on $\langle a^\alpha_i : \ i < 2^\kappa \rangle$ - we demand just that:

(∗) letting $\mathscr{E}_\alpha$ be the following equivalence relation on $\mathbf{b}_1^{\text{at}}[\mathfrak{B}_\alpha]$

$$b_1 \mathbf{E}_\alpha b_2 =: \bigwedge_i [b_1 \leq a^\alpha_i \equiv b_2 \leq a^\alpha_i]$$

we demand:

(∗) for an unbounded set $\mathscr{U}_\delta \subseteq \lambda$, for every $j \in \mathscr{U}_\delta$ and $b \in \mathbf{b}_1^{\text{at}}\ [\mathfrak{B}_{\alpha,j+1}] \setminus \mathbf{b}_1^{\text{at}}[\mathfrak{B}_{\alpha,j}]$, the set $b/\mathscr{E}_\alpha$ is a singleton.

So from $\mathbf{f} \restriction \{a^\alpha_i : \ i < 2^\kappa\}$ we can reconstruct $\mathbf{f} \restriction \bigcup\limits_{j \in \mathscr{U}_\delta} (\mathbf{b}_1^{\text{at}}[\mathfrak{B}_{\alpha,j+1}] \setminus \mathbf{b}_1^{\text{at}}[\mathfrak{B}_{\alpha,j}])$, more exactly $\mathbf{f} \restriction \{b \in \mathbf{b}_1^{\text{at}}(\mathfrak{B}_\alpha) : b/\mathscr{E}_\alpha$ a singleton$\}$ which includes the mapping above. Now it is natural to demand on $\mathfrak{B}_\alpha$ that every definable (with parameters) infinite set has cardinality λ, and when it is a subset of $\mathbf{b}_1^{\text{at}}[\mathfrak{B}_\alpha]$ then it has members in $\mathbf{b}_1^{\text{at}}[\mathfrak{B}_{\alpha,j+1}] \setminus \mathbf{b}_1^{\text{at}}[\mathfrak{B}_{\alpha,j}]$ for every $j < \lambda$ large enough. So if $c_1, c_2 \in \mathbf{b}_1[\mathfrak{B}_\alpha]$ and the symmetric difference is not (really) finite union of atoms, then we can distinguish between $\mathbf{f}(c_1)$, $\mathbf{f}(c_2)$. From the definition of $\mathbf{f} \restriction \{b \in \mathbf{b}_1^{\text{at}}(\mathfrak{B}_\alpha) : b/\mathscr{E}_\alpha$ a singleton $\}$ we can reconstruct the isomorphism $\mathbf{f}$ induce on $\mathbf{b}_1[\mathfrak{B}_\alpha]/$ (truly finite union of atoms) onto $\mathbf{b}_2[\mathfrak{B}_\alpha]/$ (truly finite union of atoms) though if we like to assume just "$\mathbf{f}$ is a complete embedding" we have to use a larger ideal. For our purpose it is enough to show that $\mathbf{f} \restriction \mathfrak{B}_\alpha$ can be reconstructed up to having $\leq \lambda$ possibilities. So assume that $\mathbf{f}_1, \mathbf{f}_2$ are isomorphisms from $\mathbf{b}_1[\mathfrak{B}_\alpha]$ onto $\mathbf{b}_2[\mathfrak{B}_\alpha]$ inducing the same isomorphism above and let $A =: \{b \in \mathbf{b}_1^{\text{at}}[\mathfrak{B}_\alpha] : \ \mathbf{f}_1(b) \neq \mathbf{f}_2(b)\}$. In the case A is infinite, and $\mathfrak{B}_\alpha$ is $\aleph_1$-saturated we get contradiction, how? there is $A' \subseteq A$ infinite such that for $b_1 \neq b_2$ in $A', \langle \mathbf{f}_1(b_1), \mathbf{f}_2(b_1), \mathbf{f}_1(b_2), \mathbf{f}_2(b_2) \rangle$ is with no repetition, so there is $a \in \mathbf{b}_1[\mathfrak{B}_\alpha]$ such that $b \in A' \Rightarrow b \leq a \,\&\, \mathbf{f}_2^{-1}\mathbf{f}_1(b) \cap a = 0_{\mathbf{b}_1} \,\&\, \mathbf{f}_1\mathbf{f}_2^{-1}(b) \cap a = 0_{\mathbf{b}_1}$, and $\mathbf{f}_1(a) \triangle \mathbf{f}_2(a)$ is not finite union of atoms). We assume $\lambda^{\aleph_0} = \lambda$ and $\text{cf}(\delta) > \aleph_0$ - the latter can be waived. All this is not the end - we have just succeed to have for stationarily many $\delta < \lambda^+$ such that $\lambda^{\mathrm{cf}(\delta)} = \lambda, \mathrm{cf}(\delta) > \aleph_0$ and $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\delta]$ being among our guesses and for some $j < \lambda$ for every $a \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\delta] \setminus \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_{\delta,j}]$ there is no infinite set indiscernible over $\mathfrak{B}_{\delta,j} \cup \{a\}$ to which $h(a)$ belongs. We would like to deduce $h(a) \in \mathrm{acl}_{\mathfrak{B}_\delta}(\mathfrak{B}_{\delta,j} \cup \{a\})$. If this does not occur we cannot immediately add x_δ and promise to omit a type as above (for possible lack of $\dot{\mathbf{J}}^\delta_\zeta$'s) but we can add such indiscernibles and then have x_δ (can do it all in $\mathfrak{B}_{\delta+1}$).

Above we were obscure on which bigness notions we use. Actually these come from "random enough sets" (like $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$).

So we have accomplished two things. First, for every $\alpha < \lambda^+$, $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\alpha]$ appear in $\mathscr{P}_{\lambda^+}$ hence for a club of $\delta < \lambda^+$ of appropriate cofinality θ $(< \lambda)$, $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\delta]$ appear in $\mathscr{P}_\delta$ (i.e. immediately). Second, for a club of $\delta < \lambda^+$ of cofinality λ, in $\mathscr{P}_\delta$ we have $\mathbf{f} \restriction B^{3,\mathbf{f}}_{\alpha,\delta}$, where $B^{3,\mathbf{f}}_{\alpha,\delta}$ is a suitable "large" subset of $\mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\delta]$ (i.e. have a member below $d \in \mathbf{b}_1[\mathfrak{B}_\delta]$ if $[x\dot{e}d]$ is $\Gamma^{\mathrm{ms}}_{\bar{c}}$-big). So then the pre-killer contractor "acts". He tries to promise that for stationary many $\epsilon < \lambda$ for some $j \in u^\delta_{\dot{e}} \setminus \{\max(u^\delta_{\dot{e}})\}$ the sequence $\langle (d, d_n) : n < \omega\rangle$ is indiscernible over A^δ_j and $\Gamma^\delta_j = \Gamma^{\mathrm{ids}}$ (so this will be preserved) where $d \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\delta], d_n \in \mathbf{b}_2[\mathfrak{B}_\delta]$ and $\mathbf{f}(d) = d_0$ (so $d \in B^{3,\mathbf{f}}_{\alpha,\delta}$) but $d_0 \notin \mathrm{acl}_{\mathfrak{B}_\delta}(A^\delta_j + d)$.

In later stage $\beta \in \mathscr{C}^3_{\mathbf{f}}, \mathrm{cf}(\beta) = \theta$, we will know $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\beta]$ (i.e. it belongs to $\mathscr{P}_\beta$), and so we "promise" that for stationary many $\epsilon < \lambda$ for some $j \in u^\beta_\epsilon$ the sequence $\langle (d^{\beta,j}, d^{\beta,j}_n : n < \omega\rangle = \langle (d^{\delta,j}, d^{\delta,j}_n) : n < \omega\rangle$ is indiscernible over A^β_j and $e^\delta_j \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\beta]$ satisfies $\mathbf{f}(e^\delta_j) \cap d^{\delta,j}_1 > 0_{\mathbf{b}_2}$ and $a^\delta_\omega \in \mathbf{b}_1[\mathfrak{B}_{\beta+1}]$ satisfies $e^\beta_j \cap a^\delta_\omega = 0_{\mathbf{b}_1}, d^{\beta,j} \le_{\mathbf{b}_1} a^\delta_\omega$ (this is the old way to kill). So we get that for $b \in B^{3,\mathbf{f}}_{\alpha,\lambda^+}, \mathbf{f}(b) \in \mathrm{acl}_{\mathfrak{B}_{\lambda^+}}(A + b)$ where $A \in \mathscr{P}_{\lambda^+} \cap [\mathfrak{B}_{\lambda^+}]^{<\lambda}$, so $\mathbf{f}(b) \in \{f_i(b) : i < i(*)\}$, where

$$\mathfrak{B}_{\lambda^+} \models \text{“}\breve{f}_i \text{ is a partial function from } \mathbf{b}_1^{\mathrm{at}} \text{ to } \mathbf{b}_2\text{”},$$

so let $\mathbf{f}(b) = f_{i(b)}(b)$.

Now as $B^{3,\mathbf{f}}_{\alpha,\lambda^+}$ is large enough, we can show that without loss of generality $i(b)$ depends just on $\mathrm{tp}(b, A, \mathfrak{B}_{\lambda^+})$. Then we show that $\breve{f}_i$ (without loss of generality) satisfies

$$(\forall x, y \in \mathrm{Dom}(\breve{f}_i))[x \neq y \to \breve{f}_i(x) \cap f_i(y) = 0_{\mathbf{b}_2}].$$

Next, we have one such $\dot{f}$ which will define $\mathbf{f} \restriction (\mathbf{b}_1 \restriction (-d))$ where the formula $[x\dot{e}d]$ is $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-small (note: as $\mathbf{b}_1$ is a "finite", it is a Boolean algebra, so $-d$ is legal).

Now for δ in which the relevant contractor works, let $\bar{c} = \langle a^\delta_n : n < \omega\rangle$, we get f as above.

So this translates into: we have a tree whose levels are non-standard integer $\mathbf{n}^* \in \mathfrak{B}_{\lambda^+}$, with inverse order and we have to show that also such trees have no undefinable branch. This needs: replacing nice by strictly nice (see 5.6). We succeed to deal with this thus at last we finish the proof.

Main Lemma 5.2. *Let $T = T^*$ be as in 1.1(B), $\lambda \ge |T|$ regular, $\kappa > \aleph_0$ regular $\lambda^+ \ge \lambda^{<\kappa}$ and: $\lambda^{\aleph_0} = \lambda, \theta = \aleph_0$. Then in the framework of 4.1 we can get that the model $\mathfrak{B} = \mathfrak{B}^* = \mathfrak{B}_{\lambda^+} = \bigcup_{\alpha<\lambda^+} \mathfrak{B}_\alpha$ satisfies:*

1) $\mathfrak{B}^*$ is a model of T of cardinality λ^+.

2) $\mathfrak{B}$ is κ-compact that is every type over $\mathfrak{B}^*$ of cardinality $< \kappa$ is realized, even by λ^+ elements.
3) There are pseudo finite sets $a_\alpha \in \mathfrak{B}$ (for $\alpha < \lambda^+$) increasing by $\subseteq^{\mathfrak{B}}$ such that for every pseudo finite $b \in \mathfrak{B}$, for every large enough $\alpha, \mathfrak{B} \models$ "$b \subseteq a_\alpha$". Also, if $Y \subseteq \mathfrak{B}$ and each $Y \cap a_\alpha$ is represented in $\mathfrak{B}^*$ (i.e. for some $b \in \mathfrak{B}$ we have, for every $c \in \mathfrak{B}$:

$$\mathfrak{B} \models \text{"}c \dot{\in} b\text{"} \Leftrightarrow c \in Y \,\&\, \mathfrak{B} \models \text{"}c \dot{\in} a_\alpha\text{"})$$

then Y is definable (with parameters, by a first order formula) in $\mathfrak{B}$.
4) If $(*)$ below holds and $\mathbf{b}_1, \mathbf{b}_2$ are atomic Boolean rings in $\mathfrak{B}$ (so their set of members is a "set" of $\mathfrak{B}$ not just a definable subset (with parameters)) then every isomorphism from $\mathbf{b}_1$ onto $\mathbf{b}_2$ is represented in $\mathfrak{B}$; where

- $(*)$ *for some $n(*) < \omega, \kappa_{n(*)} < \kappa_{n(*)-1} < \ldots < \kappa_0 = \lambda$ we have $(2^{\kappa_{\ell+1}}) \geq \kappa_\ell$ and $\mathrm{cf}\big([\kappa_\ell]^{\leq \kappa_{\ell+1}}, \subseteq\big) \leq \lambda^+$ and $\lambda^{\kappa_{n(*)}} \leq \lambda^+$. Let κ^*_ℓ be κ^+_ℓ if $\ell \in \{1, \ldots, n(*)\}$ and λ if $\ell = 0$*
  for example
- $(**)$ *2^κ regular, $\lambda = (2^\kappa)^+, \kappa_0 = \lambda, \kappa_1 = 2^\kappa, \kappa_2 = \kappa$ or $\lambda = (2^\kappa)^{++}, \kappa_0 = \lambda, \kappa_1 = (2^\kappa)^+, \kappa_2 = 2^\kappa, \kappa_3 = \kappa$.*

Remark 5.3. 1) From $(*)$ of (4), the demand $\lambda^{\kappa_{n(*)}} \leq \lambda^+$ is used only in proving $\otimes_4$ during Stage E. We can weaken it to:

- $(*)$ We can find $Y_i \subseteq \kappa_{n(*)}$ for $i < \theta, \dot{\mathbf{I}}$ an ideal on θ such that: $2^\theta \leq \lambda^+, T_{\dot{\mathbf{I}}}(\lambda) \leq \lambda^+$ (see [Shed, 3.7=Lc18]) and $(\forall Z \in \mathscr{P}(\theta) \setminus \dot{\mathbf{I}})(\exists^{<\sigma}\alpha)(\exists\beta)(\alpha \neq \beta < \theta \,\&\, \bigwedge_{i \in Z} [\alpha \in Y_i \equiv \beta \in Y_i])$ and $\lambda^{<\sigma} \leq \lambda^+$ (mostly it suffice $\theta = \mathrm{cf}(\theta), \lambda^{\langle\theta\rangle} = \lambda, \lambda^\theta \leq \lambda^+$, which means every tree with $\leq \lambda$ nodes has $\leq \lambda$ θ-branches (no much harm done if we demand $\lambda = \lambda^{<\kappa} \,\&\, \kappa > |T|$)).

2) Recall from [Shee, 0.12=L2.8A] that $\mathbf{f}$ is a complete embedding of the Boolean ring $\mathbf{B}_1$ into the Boolean ring $\mathbf{B}_2$ if it is an embedding and maps every maximal antichain of $\mathbf{B}_1$ to a maximal antichain of $\mathbf{B}_2$; equivalently if $a \in \mathbf{B}_2 \setminus \{0_{\mathbf{B}_2}\}$ then there is $b_1 \in \mathbf{B}_1, b \neq 0_{\mathbf{B}_1}$ such that $\mathbf{B}_1 \models 0 <_{\mathbf{B}_1} c \leq b^1 \Rightarrow \mathbf{B}_2 \models a \cap \mathbf{f}(c) \neq 0^1$.
3) Recall a Boolean ring is like an ideal of a Boolean algebra.

Proof. Stage A: We use 4.1-4.5 (almost as in 4.6) for T^* and Υ_{hor}, Υ^{ver} as in 4.1 such that:

- $(*)_0$ $\Gamma^{\mathrm{na}}, \Gamma^{\mathrm{mt}}, \Gamma^{\mathrm{wm}}$ (wide cases only!), $\Gamma^{\mathrm{av}}_{<\kappa}, \Gamma^{\mathfrak{p}_{\mathrm{uf}}}$ (see 3.30 or see [Shee, 2.11=L2.6]) are in Υ_{hor} and
- $(*)_1$ $\Gamma^{\mathrm{na}}, \Gamma^{\mathrm{ids}}_{\omega+1}, \Gamma^{\mathrm{wm}}$ are in Υ^{ver}.
  Also
- $(*)_2$ $\mathscr{P}_\alpha, \mathscr{P}_{\alpha,i}$ ($\mathscr{P}_\alpha$ increasing in α), $|A_{\alpha,i}| \leq \lambda$ (see 4.5(F)), $\mathscr{P}_{\alpha+1} \supseteq \bigcup_{i<\lambda} \mathscr{P}_{\alpha,i}, \mathscr{P}_\alpha \subseteq \mathscr{P}_{\alpha,0}$ (of course $\mathscr{P}_{\alpha,i}$ increasing in i); they will be, for some χ large enough, the set of objects definable in $(\mathscr{H}(\chi), \in, <^*_\chi)$ from the construction up to this point and finitely many members of $\alpha \cup |\mathfrak{B}_\alpha| \cup (\lambda + 1)$, (for any α and i this include p^α_i).

For this to make sense we have to check the orthogonality condition which holds: we check each one in $\boldsymbol{\Upsilon}^{\mathrm{ver}}$: for Γ^{na} by 2.3(2), for $\Gamma^{\mathrm{ids}}_{\omega+1}$ by 2.5(2); for Γ^{wm}, orthogonality: to Γ^{na} by 2.3(2), to Γ^{mt} by 3.35, to Γ^{wm} by 3.14(2), to $\Gamma^{\mathrm{av}}_{<\kappa}$ by 3.14(3) and to $\Gamma^{\mathfrak{p}_{\mathrm{uf}}}$ by 3.31; may compare with [Shee, 2.17=L2.8A].

We fix a winning strategy for the protagonist and then we decide various things for the antagonist in the form of decisions for various contractors. Each such commitment implies that $\mathfrak{B}$, (the outcome of a play under the restrictions above) satisfies more.

This essentially fits in 4.6 but in some cases things are more complicated. One contractor, ζ_{sat} (the saturator) acts in every stage $\alpha < \lambda^+$ but he uses cases of Γ^{na} only (actually we can let it act in stage α only for α successor of successor ordinals). Another contractor ζ_{br} (the branch killer) do not need to add Ω^α_i's but acts for every $\alpha < \lambda^+$ guaranteeing amalgamation of certain kind exist.

The third real deviation from 4.6 is the coder (see Stage D). We should be careful and show that the demands can be fulfilled.

The fourth deviation is that in the end we have to use a refinement of "niceness", from 5.6-5.15.

Stage B: We assign a contractor called the saturator ζ_{sat}, for $\alpha \in W_{\zeta_{\mathrm{sat}}} \subseteq \lambda^+$, he chooses $\Omega^\alpha_0 = \Gamma^{\mathrm{na}}$, and a type $p(\bar{x})$ to be $\subseteq p^\alpha_0$ if possible such that every member of $\mathscr{P}_\alpha$ which is a type over $\mathfrak{B}_\alpha$ of cardinality $< \lambda$ is eventually chosen. However here we also need for every α for every $i < \lambda$, every type over $A^{\alpha+1}_i$ from $\mathscr{P}_{\alpha,i}$ of cardinality $< \lambda$ which is consistent with $\mathfrak{B}_{\alpha+1}$, is realized in $\mathfrak{B}_{\alpha+1}$ (note: there are just $\leq \lambda$ many). Moreover, if the type is $p(\bar{x}) \in \mathbf{S}^m(B), B \subseteq A^{\alpha+1}_i$, and $p(\bar{x})$ "say" $\bar{x} \cap \mathrm{acl}(B) = \emptyset$ he can demand such type to be realized by a sequence disjoint to $\mathfrak{B}_\alpha$: using Γ^{na} he can; moreover, for stationarity many $i \in E_\alpha \cap S_{\zeta_{\mathrm{sat}}}$, for some $j_1 < j_2$ successive member of u^α_i there is such a sequence $\subseteq A^{\alpha+1}_{j_1} \setminus A^{\alpha+1}_{j_2}$. This is help for (1)+(2) of 5.2 in the absence of Skolem functions. We have Skolem functions here because we use 1.1(B), but if we like to use 1.1(A) and in some continuations we seem to have to be more careful, for this end we list the cases.

Stage C: We turn to (3) of 5.2 to which we assign two contractors: the end extender contractor ζ_{ex} and the branch killer contractor ζ_{br}. In order to satisfy the first phrase of (3) of 5.2 we can for $\alpha \in W_{\zeta_{\mathrm{ex}}}$, add an element $a_\alpha = \bar{a}_{\alpha,0}$ realizing $\mathfrak{p}^{\mathfrak{B}_\alpha}_{\mathrm{uf}}$, i.e. the end extender contractor decides that Ω^α_{1+i} is a case of Γ^{na} and $\Omega^\alpha_0 = \Gamma^{\mathfrak{p}_{\mathrm{uf}}}$ (see 3.31(1) or see [Shee, 2.11]). But for $\alpha \in W'_{\zeta_{\mathrm{br}}} =: \{\alpha \in W_{\zeta_{\mathrm{br}}} : \mathrm{cf}(\alpha) = \lambda$, and $\alpha = \sup(W_{\zeta_{\mathrm{ex}}} \cap \alpha)\}$ we demand more:

$\otimes^c_{1,\alpha}$ for every $\beta > \alpha$ and $b \in \mathfrak{B}_\beta$ we have (α) or (β) where:

- (α) $(\exists \gamma \in W_{\zeta_{\mathrm{ex}}} \cap \alpha)[b \cap a_\gamma \notin \mathfrak{B}_\alpha]$ (so $a_{\gamma'}$ for every large enough $\gamma' \in W_{\zeta_{\mathrm{ex}}} \cap \alpha$ or even pseudofinite set $a \in \mathfrak{B}_\alpha$ extending a_γ is o.k. instead of a_γ, i.e. $b \cap a \notin \mathfrak{B}_\alpha, a \in \mathfrak{B}_\alpha$)
- (β) for some $\bar{b} \subseteq \mathfrak{B}_\alpha$ and formula $\psi = \psi(x, \bar{b})$, for every $c \in \mathfrak{B}_\alpha$:
$$\mathfrak{B}_\beta \models [c \dot{\in} b \equiv \psi(c, \bar{b})].$$

However to do this, the branch killer, for α as above ($\in W'_{\zeta_{\mathrm{br}}}$) at stage α, guarantee:

$\otimes^c_{2,\alpha}$ for some club $E = E^{\mathrm{br}}_\alpha$ of λ, for every $i \in S_{\zeta_{\mathrm{br}}} \cap E$ there is $j = j_i = j^{\mathrm{br}}(\alpha, i) \in u^\alpha_i, j < \max(u^\alpha_i)$, such that $\Gamma^{\alpha+1}_j = \Gamma^{\mathrm{ids}}_{\omega+1}, \bar{c}^{\alpha+1}_j = \langle c^{\alpha+1,j}_n : n \leq$

$\omega\rangle, c^{\alpha+1,j}_\omega$ is $a_{\gamma(\alpha+1,j)}$ ($\in \mathfrak{B}_\alpha$) for some $\gamma(\alpha+1,j) \in W_{\zeta_{\rm ex}} \cap \alpha$, $c^{\alpha+1,j}_0 \in \mathfrak{B}_{\alpha+1}$ realizes $\mathfrak{p}^{\mathfrak{B}_\alpha}_{\rm uf}$ (we could have used $\Gamma^{\rm ids}_\omega$ instead) and $\langle \gamma(\alpha+1,j_i) : i \in S_{\zeta_{\rm br}} \cap E\rangle$ is strictly increasing with limit α.

In stage α itself (i.e. defining $\mathfrak{B}_{\alpha+1}$), for this the branch killer contractor $\zeta_{\rm br}$ chooses $<^\alpha$ (so $<^\alpha_i = <^\alpha \restriction i$) as follows: $j_1 <^\alpha j_2$ if and only if both j_1 and j_2 are $< \lambda$ and exactly one of the following occurs:

- (a) j_1 even, j_2 odd
- (b) both even, $j_1 < j_2$
- (c) both odd, $j_1 < j_2$

Further he decrees Ω^α_j is: $\Gamma^{\mathfrak{p}_{\rm uf}}$ for j even, instance of $\Gamma^{\rm na}$ for j odd.

Now to make $\otimes^c_{1,\alpha}$ and $\otimes^c_{2,\alpha}$ true in stage α is straightforward, but preserving $\otimes^c_{1,\alpha}$ needs care. In stage α itself we can think we first add $\langle \bar a_{\alpha,j} : j < \lambda \text{ even } \rangle$, then the others. When adding $\bar a_{\alpha,0}$ use the properties of $\mathfrak{p}_{\rm uf}$ (see 3.30(2), compare with [Shee, 2.20(2)]) adding $\bar a_{\alpha,2+2i}$ no subset of $\bar a_{\alpha,0}$ is added ([Shee, 2.20(2)]) so it preserves the old $\otimes^c_{1,\alpha'}$ for $\alpha' \in W_{\zeta_{\rm br}} \cap \alpha$, as for $\bar a_{\alpha,2i+1}$ we use only $\Gamma^{\rm na}$ so it is like the successor case below.

As for the case α is limit the preservation is automatic; we are left with the successor case. So now suppose we are in stage α and we would like to define $\mathfrak{B}_{\alpha+1}$ etc. and to preserve $\otimes^c_{1,\beta}$ for $\beta \in \alpha \cap W'_{\zeta_{\rm br}}$. In step $\epsilon < \lambda$ from $S_{\zeta_{\rm br}}$, after defining $i = i^\alpha_\epsilon, u^\alpha_i = u^\alpha_{i^\alpha_\epsilon}$ and $p^\alpha_{j_1}, j_1 = j(1) := \max(u^\alpha_{i^\alpha_\epsilon})$, by some bookkeeping we choose $\beta \in W'_{\zeta_{\rm br}} \cap \mathbf{c}^\alpha_i$, and $y \in \bigcup\{\bar x^\alpha_\xi : \xi < j_1\}$ or just $y = \sigma(\bar x^\alpha_{\xi_1}, \ldots, \bar x^\alpha_{\xi_n})$ for some $n < \omega$, term σ with parameters in $A^\alpha_{i_{\alpha,\epsilon}}$ and $\xi_1, \ldots, \xi_n < j_1$ (if T has no Skolem function: just in their algebraic closure, no real difference). We can find, by $\otimes^c_{2,\beta}$ and $\mathfrak{B}_\alpha$ satisfying clause (E) of 4.5, ordinals $i(*) \in E^{\rm br}_\alpha \cap S_{\zeta_{\rm br}}$ such that letting $j_2 = j(2) := j^{\rm br}(\beta, i(*))$, the sequence $\bar c^\beta_{j_2} = \langle c^\beta_{nj(2)} : n \le \omega\rangle$ is indiscernible over $A^\alpha_{j_2}$ which include $A^\alpha_{j_1}$. We extend $p^\alpha_{j_1}$ to a complete type q over $\mathrm{acl}_{\mathfrak{B}_\alpha}(A^\alpha_{j_2} \cup \{c^{\beta,j_2}_\omega\})$ satisfying the required bigness conditions (concerning $\Omega^\alpha)\epsilon, \epsilon < j_1$ recall $\Gamma^{\rm ids}$ is orthogonal to every instance of $\boldsymbol{\Upsilon}^{\rm ver}$). Remember: $c^{\beta,j(2)}_\omega \in \mathfrak{B}_\beta$ and $c^{\beta,j(2)}_0 \in \mathfrak{B}_{\beta+1}$ does $\subseteq^{\mathfrak{B}_{\beta+1}}$ – extend every pseudo-finite set of $\mathfrak{B}_\beta$. Choose appropriate $i^\alpha_{\varepsilon+1}$, i.e. $A^\alpha_{i\alpha,\epsilon+1} \supseteq \mathrm{dom}(q)$.

<u>First case</u>: q says that $y \cap c^{\beta,j(2)}_\omega$ is not equal to any member of the domain of q or just $\mathrm{dom}(q) \cap \mathfrak{B}_\beta$. Using niceness of the bigness demand on q (for $\langle \Omega^\alpha_\xi : \xi < j_1\rangle$), we can extend q to an appropriate complete type over $A^\alpha_{j_2+1}$ and by clause (D)(7) of 4.5 we get the desired contradiction, $\bigcup\{p^\alpha_i : i < \lambda\}$ will say $y \cap c^{\beta,j(2)}_\omega$ is $\notin \mathfrak{B}_\alpha$. So clearly we succeed in guarantying $\otimes^c_{1,\beta}$.

<u>Second case</u>: q says $y \cap c^{\beta,j(2)}_\omega = d$ for some $d \in \mathfrak{B}_\beta \cap \mathrm{dom}(q)$. As $c^{\beta,j(2)}_\omega, c^{\beta,j(2)}_0$ realize the same type over $A^\alpha_{j_2}$, there is an elementary mapping $\mathbf{g}$ from $\mathrm{acl}_{\mathfrak{B}_\alpha}(A^\beta_{j_2} \cup \{c^{\beta,j(2)}_\omega\})$ onto $\mathrm{acl}_{\mathfrak{B}_\alpha}(A^\beta_{j_2} \cup \{c^{\beta,j(2)}_0\})$ satisfying $\mathbf{g} \restriction A^\beta_{j_2} =$ the identity, now use $\mathbf{g}(q)$ instead of q above, so we know $y \cap c^{\beta,j(2)}_0 \in \mathrm{acl}_{\mathfrak{B}_\alpha}(A^\beta_j \cup \{c^{\beta,j(2)}_0\}) \subseteq \mathfrak{B}_\alpha$ and as $\otimes^c_{1,\beta}$ holds for α and $c^{\beta,j(2)}_0$ does $\subseteq^{\mathfrak{B}_{\beta+1}}$-extends every pseudo-finite member of $\mathfrak{B}_\beta$ we are done.

This argument works in both cases (as we use nice types), and "moving by $\mathbf{g}$" preserved the relevant properties.

<ins>Stage D</ins>: We now start dealing with part (4), but meanwhile, more generally we deal with complete embedding of a pseudo-finite Boolean Algebra into a Boolean ring (both represented in the model); the more restricted case from part (4) of 5.2 will use this. We assign a contractor, the coder $\zeta_{\rm cd}$, to try to code a complete embedding of one "finite" Boolean ring (hence algebra) $\mathbf{b}_1[\mathfrak{B}]$ to another not necessarily "finite" Boolean ring, $\mathbf{b}_2[\mathfrak{B}]$ so, both $\mathbf{b}_1, \mathbf{b}_2$ are Boolean rings in the sense of $\mathfrak{B}$, note that as $\mathbf{b}_1$ is pseudo-finite it is atomic and call its set of atoms $\mathbf{b}_1^{\rm at}$.

By normal bookkeeping we assign to every such pair $(\mathbf{b}_1, \mathbf{b}_2)$ a stationary subset $W_{(\mathbf{b}_1,\mathbf{b}_2)}$ of $\{\delta \in W_{\zeta_{\rm cd}} : {\rm cf}(\delta) = \lambda\}$. For $\delta \in W_{(\mathbf{b}_1,\mathbf{b}_2)}$, the coder decrees that $\Omega^\delta_n = \Gamma^{\rm av}_{\dot D,\langle n:n<\omega\rangle}$ ($\dot D \in \mathscr{P}_{0,0}$ any non principal ultrafilter on ω), and $\Omega^\delta_{\omega+i} = \Gamma^{\rm ms}[\mathbf{b}_1^{\rm at}, \langle 1/x^\delta_n : n < \omega\rangle]$ for $i < \kappa_1$ (see $(*)_1$ of 5.2(4); this is an instance of $\Gamma^{\rm wm}$) (so $u^\delta_0 = \omega + \kappa_1$). All Ω^δ_i (for $i \geq \omega + \kappa_1$) will be instances of $\Gamma^{\rm na}$. Now for $\epsilon < \lambda$ such that $i = i^\delta_\epsilon \in S_{\zeta_{\rm cd}}$, $j_0 = \sup \bigcup\{u^\beta_i : \beta \in \mathbf{c}^\alpha_i\}$), choose $j_1 = j^{\delta,\epsilon}_1$, $j_2 = j^{\delta,\epsilon}_2$, such that $j_0 < j^{\delta,\epsilon}_1 < j^{\delta,\epsilon}_2 < \lambda$ and:

$(*)_3$ $(\mathfrak{B}_\delta \restriction A^\delta_{j_1}, A^\delta_{j_0}) \prec (\mathfrak{B}_\delta, A^\delta_{j_0})$ and

$(*)_4$ $(\mathfrak{B}_\delta \restriction A^\delta_{j_2}, A^\delta_{j_0}, A^\delta_{j_1}) \prec (\mathfrak{B}_\delta, A^\delta_{j_0}, A^\delta_{j_1})$ define u^δ_i as $\bigcup\{u^\beta_i : \beta \in \mathbf{c}^\alpha_i\} \cup \{j_0, j_1, j_2\}$ and demand:

$\otimes_0$ if $d \in \mathfrak{B}_\delta, \mathfrak{B}_\delta \models$ "$d \dot{e} \mathbf{b}_1^{\rm at}$", $d \in A^\delta_{j_2} \setminus A^\delta_{j_1}$ then there is no $d' \in \mathfrak{B}_\delta, \mathfrak{B}_\delta \models$ "$d' \dot{e} \mathbf{b}_1^{\rm at}$", $d' \neq d$" such that
$\{d \leq x^\delta_{\omega+i} \equiv d' \leq x^\delta_{\omega+i} : i < \kappa_1\} \subseteq \bigcup\limits_{j<\lambda} p^\delta_j$ and
for simplicity there is $i < \kappa_1$ such that $[d \leq x^\delta_{\omega+i}] \in p^\delta_{j_2}$.

Note that <ins>this is not</ins> a part of the general machinery of 4.6, but we shall see that it is compatible with it, i.e. this is part of contractor $\zeta_{\rm cd}$'s work i.e. he overtake more control this "at the expense of" "$\zeta_{\rm cd}$ is the main contractor for δ". Now it is reasonable to demand that when p^δ_j is defined, the condition holds for $d \in A_{j_2^{\delta,\epsilon}} \setminus A_{j_1^{\delta,\epsilon}}, d' \in A^\delta_j$ (for every ϵ such that $j^{\delta,\epsilon}_1 < j^{\delta,\epsilon}_2 \leq j$). So how can we preserve this condition when defining p^δ_j (for $j \in E^+_j$)? For $j = 0$ no problem. For limit j there is no problem. So assume, i^δ_ϵ is defined, $j \in u^\delta_{i^\delta_\epsilon}$, p^δ_j is defined and we have to define $p^\delta_{j(*)}$ where $j(*) > j$ is the successor of j in $u^\delta_{i^\delta_\epsilon}$ or j is the last member of $u^\delta_{i^\delta_\epsilon}$ and $j(*) = i^\delta_{\epsilon+1}$. We have to consider what is the constraint.

Note: as the $1/x^\delta_n$ in p^δ_j satisfies

$(*)_4$ for every $n, m < \omega$, $[\bigwedge\limits_n \mathfrak{B}_\delta \models 0 < c < 1/n \Rightarrow \bigwedge\limits_{n,m} \mathfrak{B}_{\delta+1} \models$ "$c < 1/x^\delta_m < 1/n$"$]$

we really have freedom.

<ins>The First Case</ins>: No constraint.

So we have to extend p^δ_j in a nice way (to preserve clause (D7) and $\otimes_0$). By induction on $i < j$ we choose $\bar a_{\delta,i}$ to realize over $A^\delta_{j(*)}$ the right type (say in some saturated M, $\mathfrak{B}_\delta \prec M$); i.e. we preserve (remember we can look at $\mathbf{b}_1$ as the family of subsets of $\mathbf{b}_1^{\rm at}$, in T^*'s sense):

(i) $A^\delta_{j(*)} \cap {\rm acl}_M(A^\delta_j + \{\bar a_{\delta,\xi} : \xi < i\}) = A^\delta_j$

(*ii*) $\mathrm{tp}(\bar{a}_{\delta,i}, A^\delta_{j(*)} \cup \{\bar{a}_{\delta,\xi} : \xi < i\})$ is Ω^δ_i-big

(*iii*) $a_{\delta,i} = \bar{a}_{\delta,i} \in \mathbf{b}_1[M]$ when $\omega \le i < \omega + \kappa_1$

(*iv*) $\big[d\dot{e}^M \mathbf{b}^{\mathrm{at}}_1 \,\&\, d \in (A^\delta_{j(*)} \setminus A^\delta_j) \Rightarrow \neg d\dot{e}^M a_{\delta,i}\big]$ when $\omega \le i < \omega + \kappa_1$

(*v*) if $i \ge \omega$ and $d \in A^\delta_{j(*)} \setminus A^\delta_j$, $e \in \mathrm{acl}(A^\delta_j \cup \bigcup\limits_{\epsilon<i} \bar{a}_{\delta,\epsilon})$, and $M \models e \subseteq \mathbf{b}^{\mathrm{at}}_1, e \le \log_2(a_{\delta,n})$ (equivalently if $2^{-|e|} \ge 1/a_{\delta,n}$)" for each n then $M \models$ "$\neg d\dot{e}e$".

[Why we need (v)? After defining $a_{\delta,i}$ for $i \le \omega$, applying the relevant claim from §3, in order to have freedom for (iv) we need (v).]

This is possible by 3.19(2) (check conditions). If $i < \omega$ by clause (ii) we have exactly one choice, clause (i) is easy by niceness (and uniqueness), clauses (iii)+(iv)+(v) are irrelevant, and clause (v) is easy. For $i = \omega$ clauses (i)-(iv) are immediate, to assumption of (v) implies e is truly finite hence all $\dot{e}$-members are by (i) not in $A^\delta_{j(*)}, \setminus A^\delta_j$. If $\omega < i \le \omega + \kappa_1, i$ limits we have no problem, if i is a successor ordinal we use 3.19(2).

Lastly,if $i \ge \omega + \kappa_1$ we just use niceness.

<u>The Second Case</u>: We have obligation from Stage C.

I.e. $i^\delta_\epsilon \in S_{\zeta_{\mathrm{br}}}$ and let $j_1 = \max\,(u^\delta_\epsilon)$, choose $\beta, i(*), j_2, y$ as in Stage C, "for the successor case" (with j_1, j_2 here standing for $j(*), j$ there). Choose the type q over $\mathrm{acl}_{\mathfrak{B}_\alpha}(A^\alpha_{j_1} \cup \{c^{\beta,j_2}_\omega\})$ as the first case (in our present stage) with $A^\alpha_{j_1}, \mathrm{acl}_{\mathfrak{B}_\alpha}(A^\alpha_{j_1} \cup \{c^{\beta,j_2}_\omega\})$ here standing for $A^\alpha_{j(*)}, A^\alpha_j$ there.

If in Stage C, first case apply, then we can choose appropriate $i^\delta_{\epsilon+1}$ and extend q to a type as required by the proof of first case (in our present stage). If the second case in Stage C apply, the elementary mapping $\mathbf{g}$ preserve the right things so no problems: just like when the first case applies.

<u>The Third Case</u>: Γ^δ_j is defined and equal to $\Gamma^{\mathrm{ids}}_{\omega+1}$. We just first lengthen the indiscernible set to $\dot{\mathbf{I}}$, $|\dot{\mathbf{I}}| > \beth_{(2^\lambda)^+}$ say in M^* where $\mathfrak{B}_\alpha \prec M^*$ this extend p^δ_j to p^+, complete type over $\mathrm{acl}(A^\delta_j \cup \dot{\mathbf{I}})$ which is a nice extension of p big for $\langle \Omega^\delta_\zeta : \zeta \in u^\delta_{i,\dot{e}} \rangle$ such that for some $u \subseteq \kappa_1$ for all $d \in \mathrm{acl}(A^\delta_j \cup \dot{I}) \setminus A^\delta_j$ (note $A^\delta_j = \mathrm{acl}(A^\delta_j)$) we have: $\{(d \le x^\delta_{\omega+i})^{[\mathrm{if}(i \in u)]} : i < \kappa_1\} \subseteq p^+$, where u is chosen such that for no $d \in A^\delta_j$, $\{(d \le x^\delta_{\omega+i})^{\mathrm{if}(i \in u)]} : i < \kappa_1\} \subseteq p$ (note: p^+ exists by 3.19(1)), then use the proof in 2.6. (i.e. $\Gamma^{\mathrm{ids}} \perp \Gamma^{\mathrm{ms}}$, in fact $u = \emptyset$ is O.K.)

<u>The fourth case</u>: Γ^δ_j is defined and equal to $\Gamma^{\mathrm{wm}}_{a,\dot{w},\bar{c}}$.

Similar to first case; for $n < \omega$, we choose $a_{\delta,n}$ (corresponding to x^δ_n) by 3.10(3), i.e. as $\Gamma^{\mathrm{av}}_\omega$, $\Gamma^{\mathrm{wm}}_{a,\dot{w},\bar{c}}$ are orthogonal.

For $x^\delta_{\omega+i}$ $(i < \kappa_1)$ we can use 3.21.

<u>The fifth case</u>: Γ^δ_j is defined and equal to Γ^{na}.

Easy.

<u>The sixth case</u>: $j, j(*)$ are like j_1, j_2 above in $(*)_2 + \otimes_0$ of this stage.

Again by claim 3.19(2) choosing the function h carefully enough remembering $|A^\delta_{j(*)}| < \lambda \le 2^{\kappa_1}$.

<u>Stage E</u>: Assume

⊠ $\mathbf{b}_1$, $\mathbf{b}_2$ as in stage D, for $\mathfrak{B}_{\lambda^+}$, $\mathbf{f}$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ into $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$.

$\mathbf{f}$ will be fixed for stages E—J.

So

$$\mathscr{C}^1_{\mathbf{f}} = \{\delta < \lambda^+ : (\mathfrak{B}_\delta, \mathbf{f}{\restriction}\mathfrak{B}_\delta) \prec (\mathfrak{B}, \mathbf{f})\}$$

is a club of λ^+; let $W_{\mathbf{f}} = \mathscr{C}^1_{\mathbf{f}} \cap W_{(\mathbf{b}_1,\mathbf{b}_2)} \subseteq \mathscr{C}^1_F \cap W_{\zeta_{\rm cd}}$.

Clearly by 5.3(2)

$\otimes$ for $\delta \in \mathscr{C}^1_{\mathbf{f}}$, $\mathbf{f}{\restriction}\mathbf{b}_1[\mathfrak{B}_\delta]$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}_\delta]$ into $\mathbf{b}_2[\mathfrak{B}_\delta]$.

For $\alpha \in W_{\mathbf{f}}$ let $B^*_\alpha := \{b \in \mathbf{b}^{\rm at}_1[\mathfrak{B}^*] : b \in \mathfrak{B}_\alpha$ and for no $b' \neq b$ do we have

$$b' \dot{e}^{\mathfrak{B}_\alpha} \mathbf{b}^{\rm at}_1 \,\&\, b' \in \mathfrak{B}_\alpha \,\&\, \bigwedge_{i<\kappa_1} [b \dot{e}^{\mathfrak{B}^*} a^\alpha_{\omega+i} \equiv b' \dot{e}^{\mathfrak{B}^*} a^\alpha_{\omega+i}]\}.$$

Note that B^*_α depends on α, $\mathfrak{B}_\alpha$, $\mathbf{b}_1$, $\langle a^\alpha_{\omega+i} : i < \kappa_1\rangle$ but not on $\mathbf{f}$. For the rest of stage E we fix α.

For $\delta \in \mathscr{C}^1_{\mathbf{f}}$ for awhile we shall try to show that $\mathbf{f} \restriction B^*_\delta$ belongs to $\mathscr{P}_{\lambda^+}$, this in the following substages:

$\otimes_1$ from $\mathbf{f}^0_\delta = \mathbf{f} \restriction \{a^\delta_{\omega+i} : i < \kappa_1\}$ we can reconstruct $\mathbf{f}^1_\delta = \mathbf{f} \restriction B^*_\delta$, so if $\mathbf{f}^0_\delta = \mathbf{f} \restriction \{a^\delta_{\omega+i} : i < \kappa_1\} \in \bigcup\limits_\alpha \mathscr{P}_\alpha$ then $\mathbf{f}^1_\delta = \mathbf{f} \restriction B^*_\delta \in \mathscr{P}_{\lambda^+}$.

[How? For $d \in B^*_\delta$, $\mathbf{f}(d)$ is the maximal member of $\mathbf{b}_2[\mathfrak{B}_\delta]$ which is $\le \mathbf{f}(c)$ whenever $c \in \{a_{\delta,\omega+i} : i < \kappa_1,\ d \le a_{\delta,\omega+i}\} \cup \{\mathbf{f}(-a_{\delta,\omega+i}) : i < \kappa_1$ and $d \le -a_{\delta,\omega+1}\}$. Note that $-a_{\delta,\omega+i}$ is well defined as $\mathbf{b}_1[\mathbf{B}]$ is a Boolean Algebra, by the first paragraph of stage D. Also note: $\mathbf{f}(d)$ satisfies this as $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\delta]$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}_\delta]$ into $\mathbf{b}_2[\mathfrak{B}_\delta]$ and in $\mathbf{b}_1[\mathfrak{B}_\delta], d$ is a maximal member of $\mathbf{b}_1[\mathfrak{B}_\delta]$ which is $\le c$ whenever $c \in \{a_{\delta,\omega+i} : i < \kappa_1, d \le a_{\delta,\omega+i}\} \cup \{-a_{\delta,\omega+i} : i < \kappa_1, d \le -a_{\delta,\omega+i}\})$.]

$\otimes_2$ if $\ell < n(*)$, $A \subseteq \mathfrak{B}_{\lambda^+}$, $|A| < \kappa_\ell$, <u>then</u> we can find $A' \subseteq \mathfrak{B}_{\lambda^+}$ satisfying $|A'| < \kappa_\ell, A' \in \bigcup\limits_{\alpha<\lambda^+} \mathscr{P}_\alpha$ and $A \subseteq A'$.

[Why? By $(*)$ of 5.2(4) we prove by induction on ℓ. For $\ell = 0$ immediate by κ_ℓ being regular (as $\kappa_0 = \lambda = {\rm cf}(\lambda)$ and have ${\rm cf}([\lambda]^{<\lambda}, \subseteq) = \lambda$). For $\ell + 1$, by the induction hypothesis we can find $A'' \in \bigcup\limits_{\alpha<\lambda^+} \mathscr{P}_\alpha$ such that $A \subseteq A'', |A''| < \kappa^*_\ell$ and now recall ${\rm cf}([\kappa_\ell]^{\le\kappa_{\ell+1}}, \subseteq) \le \lambda^+$ so a cofinal subset of $[\kappa_\ell]^{\le\kappa_{\ell+1}}$ has cardinality $\le \lambda^+$ and belongs to $\bigcup\limits_{\alpha<\lambda^+} \mathscr{P}_\alpha$ hence is included in $\bigcup\limits_{\alpha<\lambda^+} \mathscr{P}_\alpha$.]

$\otimes_3$ if $\ell < n(*), A \subseteq \mathbf{b}^{\rm at}_1[\mathfrak{B}_{\lambda^+}], |A| < \kappa_\ell, A \in \bigcup\limits_{\alpha<\lambda^+} \mathscr{P}_\alpha$ <u>then</u> we can find A', $A_1,\ A_2 \in \bigcup\limits_{\alpha<\lambda^+} \mathscr{P}_\alpha$, such that $A \subseteq A_1$, ${\rm rang}(\mathbf{f} \restriction A_1) = A_2, |A_1| + |A_2| < \kappa_\ell, A_i \subseteq \mathbf{b}_i[\mathfrak{B}_{\lambda^+}]$ for $i = 1, 2$ and $A' \subseteq \mathbf{b}_1[\mathfrak{B}_\delta], |A'| < \kappa_{\ell+1}$ and from $\mathbf{f} \restriction A'$ and A_1, A_2 we can reconstruct $\mathbf{f} \restriction A_1$; i.e. it belongs to $\bigcup\limits_{\alpha<\lambda^+} \mathscr{P}_\alpha$.

[Why ? By $\otimes_2, {\rm cf}([\lambda^+]^{<\kappa_\ell}, \subseteq) = \lambda^+$ so by 5.2(4)$(*)$ and [Shed, 3.11] there is a stationary $\mathscr{S} \subseteq [\lambda]^{\le\kappa_\ell}$ of cardinality $\le \lambda^+$ hence there is a model $N, |N| \in \bigcup\limits_{\alpha<\lambda^+} \mathfrak{A}_\alpha$, such that $N \prec (\mathfrak{B}_{\lambda^+}, \mathbf{f}), A \subseteq N, \|N\| < \kappa_\ell$ and $\mathbf{b}_1, \mathbf{b}_2 \in N$. Let $A_i = N \cap \mathbf{b}_i[\mathfrak{B}_{\lambda^+}]$ for $i = 1, 2$, so $\mathbf{f} \restriction N$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}] \restriction N$ into $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}] \restriction N$;

hence to reconstruct it it suffice to reconstruct $\mathbf{f} \restriction (\mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_{\lambda^+}] \cap N)$. But the saturator guarantee the existence of $a_i \in \mathfrak{B}_{\lambda^+}$ satisfying $a_i \dot{e} \mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ for $i < \kappa_{\ell+1}$ such that: if $d' \neq d'' \in \mathbf{b}_1[N]$ then $\bigvee_{i<\kappa_{\ell+1}} [d' \leq a_i \,\&\, d'' \cap a_i = 0_{\mathbf{b}_1}]$. Let $A' = \{a_i : i < \kappa_{\ell+1}\}$.]

$\otimes_4$ if $A \subseteq \mathbf{b}_1[\mathfrak{B}_{\lambda^+}], |A| \leq \kappa_{n(*)}$ then $\mathbf{f} \restriction A' \in \bigcup_{\alpha<\lambda^+} \mathscr{P}_\alpha$ for some A' satisfying $A \subseteq A' \subseteq \mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ and $|A'| < \lambda$.

[Why? As $\lambda^{\kappa_{n(*)}} \leq \lambda^+$.]

$\otimes_5$ if $A \in \mathscr{P}_{\lambda^+}, A \subseteq \mathbf{b}_1[\mathfrak{B}_{\lambda^+}], |A| < \lambda$ then $\mathbf{f}{\restriction}A \in \mathscr{P}_{\lambda^+}$ (remember that $[\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]]^{<\lambda} \cap \mathscr{P}_{\lambda^+}$ is cofinal in $[\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]]^{<\lambda}$).

[Why? Put together $\otimes_2, \otimes_3, \otimes_4$; i.e. we can prove by induction on $\ell \leq n(*)$ that if $|A| < \kappa_{n(*)-\ell}$ then the conclusion holds; now for $\ell = 0$ use $\otimes_4$ and for $\ell+1$ use $\otimes_3$.]

$\otimes_6$ $\mathbf{f} \restriction B^*_\alpha \in \bigcup_{\alpha<\lambda^+} \mathscr{P}_\alpha$ for $\alpha \in W_{\mathbf{f}}$.

[Why? Put together $\otimes_1$, $\otimes_5$.]

Let $\mathscr{I}^1_\alpha = \mathscr{I}^{1,\mathbf{f}}_\alpha$ be the ideal of $\mathbf{b}_1[\mathfrak{B}_\alpha]$ generated by $\{c : c \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\alpha]$ but $c \notin B^*_\alpha\}$, clearly $\mathscr{I}^1_\alpha \in \mathscr{P}_{\lambda^+}$ but we do not claim any definability in $\mathfrak{B}_{\lambda^+}$. Also clearly $c \in \mathscr{I}^1_\alpha$ <u>if and only if</u> for some $n < \omega$ and $c_1, \ldots c_n \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\alpha] \setminus B^*_\alpha$ we have $\mathbf{b}_1[\mathfrak{B}_\alpha] \models$ "$c = c_1 \cup \ldots \cup c_n$".a

As B^*_α has members in every infinite subset of $\mathbf{b}_1[\mathfrak{B}_\alpha]$ definable in $\mathfrak{B}_\alpha$ with parameters, clearly

$\otimes_7$ for $b \in \mathbf{b}_1[\mathfrak{B}_\alpha]$ we have $b \in \mathscr{I}^1_\alpha \Leftrightarrow \neg(\exists c \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\alpha])(c \in B^*_\alpha \,\&\, c \leq b)$.

Let

$$\mathscr{I}^2_\alpha = \mathscr{I}^{2,\mathbf{f}}_\alpha = \{c \in \mathbf{b}_2[\mathfrak{B}_\alpha] : \text{ for every } b \in B^*_\alpha \text{ we have } \mathbf{f}(b) \cap c = 0_{\mathbf{b}_1}\}.$$

Clearly $\mathscr{I}^2_\alpha \in \mathscr{P}_{\lambda^+}$ is an ideal of $\mathbf{b}_2[\mathfrak{B}_\alpha]$. We define a function $\mathbf{f}^0_\alpha$ with domain $\mathbf{b}_1[\mathfrak{B}_\alpha]$:

$$\mathbf{f}^0_\alpha(d) = \{c \in \mathbf{b}_2[\mathfrak{B}_\alpha] : \begin{array}{l} \text{for every } b \in B^*_\alpha \text{ we have} \\ b \leq_{\mathbf{b}_1} d \Rightarrow \mathbf{f}(b) \leq_{\mathbf{b}_2} c) \text{ and} \\ b \cap d = 0_{\mathbf{b}_1} \Rightarrow \mathbf{f}(b) \cap c = 0_{\mathbf{b}_2})\}. \end{array}$$

It is easy to see that $\mathbf{f}^0_\alpha \in \mathscr{P}_{\lambda^+}$ and $\mathbf{f}^0_\alpha$ is a homomorphism from the Boolean Algebra $\mathbf{b}_1[\mathfrak{B}_\alpha]$ into the Boolean ring $\mathbf{b}_2[\mathfrak{B}_\alpha]\mathscr{I}^2_\alpha$, and $\mathbf{f}(d) \in \mathbf{f}^0_\alpha(d)$, and $\mathscr{I}^1_\alpha$ is the kernel of $\mathbf{f}^0_\alpha$.

We now show (recall that by assumption $\lambda = \lambda^{\aleph_0}, \theta = \aleph_0$):

$\otimes_8$ Assume $\mathrm{cf}(\alpha) \neq \theta$ and the saturator works for unboundedly many $\beta < \alpha$. If $\mathbf{f}', \mathbf{f}''$ are two functions satisfying the information on $\mathbf{f} \restriction \mathfrak{B}_\alpha$ gathered so far (more exactly: $\mathbf{f}' \restriction B^*_\alpha = \mathbf{f}{\restriction}B^*_\alpha = \mathbf{f}'' \restriction B^*_\alpha$, and $\mathbf{f}', \mathbf{f}''$ are complete embeddings of $\mathbf{b}_1[\mathfrak{B}_\alpha]$ into $\mathbf{b}_2[\mathfrak{B}_\alpha]$ hence looking at the definitions of $\mathscr{I}^{1,\mathbf{f}}_\alpha, \mathscr{I}^{2,\mathbf{f}}_\alpha, \mathbf{f}^0_\alpha$, clearly $d \in \mathbf{b}_1[\mathfrak{B}_\alpha] \Rightarrow \mathbf{f}'(d)/\mathscr{I}^2_\alpha = \mathbf{f}(d)/\mathscr{I}^2_\alpha = \mathbf{f}''(d)/\mathscr{I}^2_\alpha)$ <u>then</u> $\{e : e \in \mathbf{b}_1^{\mathrm{at}}[\mathfrak{B}_\alpha]$ and $\mathbf{f}'(e) \neq \mathbf{f}''(e)\}$ has $< \theta$ members.

[Why? Assume $e_i \in \mathbf{b}_1^{\rm at}[\mathfrak{B}_\alpha]$ for $i < \theta$ are pairwise distinct and $\mathbf{f}'(e_i) \neq \mathbf{f}''(e_i)$. So without loss of generality $\mathbf{f}'(e_i) - \mathbf{f}''(e_i) > 0_{\mathbf{b}_2[\mathfrak{B}_\alpha]}$. As $\mathbf{f}''$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}_\alpha]$ into $\mathbf{b}_2[\mathfrak{B}_\alpha]$ clearly for $i < \theta$ there is $e'_i \in \mathbf{b}_1^{\rm at}[\mathfrak{B}_\alpha]$ such that $\mathbf{f}''(e'_i) \cap (\mathbf{f}'(e_i) - \mathbf{f}''(e_i)) > 0_{\mathbf{b}_2[\mathfrak{B}_\alpha]}$. This implies $\mathbf{f}''(e'_i) \neq \mathbf{f}''(e_i)$ hence $e'_i \neq e_i$, and as $\langle e_i : i < \theta \rangle$ is without repetitions, without loss of generality $e'_i \notin \{e_j : j < \theta\}$ for $i < \theta$. As $\mathrm{cf}(\alpha) \neq \theta$ without loss of generality for some $\beta < \alpha, \{e_i, e'_i : i < \theta\} \subseteq \mathfrak{B}_\beta$, hence (as $\lambda = \lambda^{\langle \theta \rangle}$ and less suffice) for some countable $u \in [\theta]^{\aleph_0}, \{(e_i, e'_i) : i < \omega\} \in \mathscr{P}_\beta$, without loss of generality $u = \omega$; hence by the saturater work there is $d \in \mathbf{b}_1[\mathfrak{B}_\alpha]$ such that $e_n \leq d, e'_n \cap d = 0_{\mathbf{b}_1}$. Hence by $(*)_3$ for some $c \in \mathscr{I}^1_\alpha, \mathbf{f}'(d - c) \leq \mathbf{f}''(d), \mathbf{f}''(d - c) \leq \mathbf{f}'(d)$ and so in $\mathbf{b}_2[\mathfrak{B}_\alpha]$ we have

$$\begin{aligned}(\mathbf{f}'(d) - \mathbf{f}''(d)) \cap \mathbf{f}''(e'_n) \geq \ & \mathbf{f}''(-d) \cap \mathbf{f}'(d) \cap \mathbf{f}''(e'_n) = \\ & \mathbf{f}''((-d) \cap e'_n) \cap \mathbf{f}'(d) \geq \\ & \mathbf{f}''(e'_n) \cap \mathbf{f}'(d) \geq \\ & \mathbf{f}''(e'_n) \cap \mathbf{f}'(e_n) \geq \\ & \mathbf{f}''(e'_n) \cap (\mathbf{f}'(e_n) - \mathbf{f}''(e_n)) > 0_{\mathbf{b}_1}\end{aligned}$$

[Why? As $\mathbf{b}_1[\mathfrak{B}_\alpha]$ is a Boolean algebra ($\mathbf{b}_1^{\rm at}$ being "finite"); as $\mathbf{f}''$ is an embedding as $e'_n \cap d = 0$ so $e'_n \leq -d$; as $e_n \leq d$; by Boolean rules; and by the choice e'_i.]

Choose n such that $\neg(e'_n \leq_{\mathbf{b}_1} c)$, so we got contradiction to $\mathbf{f}'(d)/\mathscr{I}^2_\alpha = \mathbf{f}''(d)/\mathscr{I}^2_\alpha$. So $\otimes_8$ really holds.]

From now on we assume the conclusion of $\otimes_8$ holds which suffice for 5.2.

As $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\alpha]$ satisfies the requirements in $\otimes_8$, (there is a least one such $\mathbf{f}'$, and) by $\otimes_8$ there are $\leq \lambda^{\langle \theta \rangle} \leq \lambda^+$ such functions $\mathbf{f}'$ so we conclude

$\otimes_9$ $(\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\alpha]) \in \mathscr{P}_{\lambda^+}$ hence $\beta < \alpha \Rightarrow \mathbf{b} \restriction \mathbf{b}_1[\mathfrak{B}_\beta] \in \mathscr{P}_{\lambda^+}$.

So

$$\begin{aligned}\mathscr{C}^2_{\mathbf{f}} = \{\gamma < \lambda^+ : \ & \gamma \text{ a limit ordinal such that if} \\ & \alpha < \gamma \text{ then } \mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\alpha] \in \mathscr{P}_\beta \text{ for some } \beta \in (\alpha, \gamma)\}\end{aligned}$$

is a club of λ^+ hence (as $\lambda = \lambda^{\langle \theta \rangle}$)

$\otimes_{10}$ if $\alpha \in \mathscr{C}^2_{\mathbf{f}}$ and $\mathrm{cf}(\alpha) = \theta$ then $f_\alpha := \mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\alpha] \in \mathscr{P}_\alpha$ [remember $\mathscr{P}_\alpha \neq \bigcup\limits_{\gamma < \alpha} \mathscr{P}_\gamma$!]. We shall not $\mathbf{f}_\alpha \in \cup\{\mathscr{P}_\alpha : \alpha < \lambda^+\}$ when not necessary i.e. $\mathbf{q}^0_\alpha$ suffice.

Stage F: We have a contractor, the pre-separator, $\zeta_{\rm ps}$ which acts for any (fixed for this stage):

$\otimes_{11}$ $\mathbf{b}_1, \mathbf{b}_2$ as in Stage D, and $\mathbf{p}, c, \bar{c}^1, \bar{c}^2$ as in 3.29 of length say ω with $\mathbf{b}_1^{\rm at}$ for a

(for example those which the contractor $\zeta_{\rm ps}$ posed, they will be fixed in this stage), for stationarily many $\alpha \in W_{\zeta_{\rm ps}}$ such that $\mathrm{cf}(\alpha) = \lambda$ (and the contractor for this α choose $\mathbf{b}_1, \mathbf{b}_2, \bar{c}^1, \bar{c}^2, \mathbf{p}$ which are from $\mathfrak{B}_\alpha$ and so $\bar{c}^1, \bar{c}^2 \in \mathscr{P}_\alpha$), and is quite closed under the saturator work.

Now

$(*)_5$ the pre-separator takes care that for every $i \in E_\alpha \cap S_{\zeta_{\rm ps}}$, there is $j_i \in u^\alpha_i$, $j_i < \max(u^\alpha_i)$, such that Γ^α_i has the form

$$\Gamma^{\rm wm}[\mathbf{b}_1, \dot{w}_{\mathbf{p}}, \bar{c}^1] = \Gamma^{\rm wmg}[\mathbf{b}^{\rm at}_1, \mathbf{p}, \bar{c}^1]$$

(note: $\mathbf{b}_1$ can be considered the power set of $\mathbf{b}^{\rm at}_1$, recall $\mathfrak{B}_\alpha$ "think" that $\mathbf{b}^{\rm at}_1$ is finite hence $\mathbf{b}_1$ is a Boolean Algebra). The set of such α's will be called $W_{\zeta_{\rm sp},\mathbf{b}_1,\mathbf{b}_2,c,\bar{c}^1,\bar{c}^2,\mathbf{p}}$.

We have another contractor, the separator, $\zeta_{\rm sp}$, such that: stationarily many $\alpha \in W_{\zeta_{\rm sp}}, {\rm cf}(\alpha) = \lambda$, are assigned to $\mathbf{b}_1$, $\mathbf{b}_2$, c, $\bar{c}^2, \bar{c}^1, \mathbf{p}$ and $p \in \mathscr{P}_{\alpha,0}$ which is a $\Gamma^{\rm ms}_{\mathbf{b}^{\rm at}_1,\bar{c}^2}$-big type over $\mathfrak{B}_\alpha$ of cardinality $< \lambda$, and the pre-separator has acted in some $\alpha_1 < \alpha$, for the relevant parameters; the separator chooses $\Omega^\alpha_0 = \Gamma^{\rm ms}_{\mathbf{b}^{\rm at}_1,\bar{c}^2}$ and make $p \subseteq p^\alpha_0$ (so x^α_0 will be in $\mathbf{b}^{\rm at}_1[\mathfrak{B}_{\alpha+1}]$). Now for every $i^\alpha_\epsilon \in S_{\zeta_{\rm sp}} \cap E_{\alpha_1}$ with $\alpha_1 \in \mathbf{c}^\alpha_{i^\alpha_\epsilon}$, he took care to have $j \in u^\alpha_{i^\alpha_\epsilon}$, $j < \max(u^\alpha_{i_\epsilon})$ such that for some $d^\alpha_j \in \mathbf{b}_1[\mathfrak{B}_{\alpha_1}]$, the type ${\rm tp}(d^\alpha_j, A^\alpha_j, \mathfrak{B}_\alpha)$ is $\Gamma^{\rm wmg}_{\mathbf{b}^{at}_1,\mathbf{q},\bar{c}^1}$-big and $[x^\alpha_0 \dot{e} d^\alpha_j] \in p_{j'}$ for $j' > j$, but $\Gamma^{\alpha+1}_j = \Gamma^{\rm wmg}_{\mathbf{b}^{\rm at}_1,\mathbf{p},\bar{c}^1}$ and $c^{\alpha+1}_j = d^\alpha_j - a_{\alpha,0}$ (in $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$'s sense!); this is possible by the pre-separator work for some higher j and 3.29 (and the assumption on $\mathbf{p}, c, \bar{c}^1, \bar{c}^2$ in $\otimes_{11}$).

Let $\alpha < \lambda^+$ be such that $\mathfrak{B}_\alpha$ is closed under $\mathbf{f}$ and we fix α for a while. Let $\beta \le \lambda^+$ be such that $\beta \ge \beta^*_\alpha = \min\{\beta : \beta > \alpha \text{ and } \mathbf{f}_\alpha \in \mathscr{P}_\beta\}$ where below α there is α_1 as above. We define an equivalence relation $\mathscr{E}_{\alpha,\beta}$ on $\mathbf{b}^{\rm at}_1[\mathfrak{B}_\beta]$: $d_1 \mathscr{E}_{\alpha,\beta} d_2$ <u>if and only if</u> for every $b \in \mathbf{b}_1[\mathfrak{B}_\alpha]$, we have $d_1 \le b \Leftrightarrow d_2 \le b$.

Let $B^1_{\alpha,\beta} = \{d \in \mathbf{b}^{\rm at}_1[\mathfrak{B}_\beta] : d/\mathscr{E}_{\alpha,\beta} \text{ is a singleton}\}$.

Note: as the separator do his for stationarily many $\alpha' < \lambda^+$ of cofinality λ, we can use α such that $\mathfrak{B}_\alpha$ is closed under $\mathbf{f}$. Let $B^2_{\alpha,\beta} = B^1_{\alpha,\beta} \setminus \mathfrak{B}_\alpha$,

$$B^3_{\alpha,\beta} = B^{3,\mathbf{f}}_{\alpha,\beta} = \{a_{\gamma,0} : \begin{array}{l}\gamma \in [\alpha,\beta) \text{ and in stage } \gamma \text{ the separator acts} \\ (\text{for the parameters } \mathbf{b}_1, \mathbf{b}_2, c, \mathbf{p}, \bar{c}^1, \bar{c}^2)\}.\end{array}$$

Recall that there is $\alpha_1 < \alpha$ such that the pre-separator act for those parameters. Clearly $B^3_{\alpha,\beta} \subseteq B^2_{\alpha,\beta} \subseteq B^1_{\alpha,\beta}$, moreover

$(*)_6$ for every $\beta_1 \in [\beta, \lambda^+)$ we have $B^3_{\alpha,\beta} \subseteq B^3_{\alpha,\beta_1}$ and $B^3_{\alpha,\beta} \subseteq B^2_{\alpha,\beta_1}$.

[Why? $B^3_{\alpha,\beta} \subseteq B^3_{\alpha,\beta_1}$ holds trivially. As for $B^3_{\alpha,\beta} \subseteq B^2_{\alpha,\beta}$ (this is the whole point of the work of the separators, that is, assume $a_{\gamma,0} \in B^3_{\alpha,\beta}$ and without loss of generality $\beta = \beta_1$ if $a_{\gamma,0} \notin B^2_{\alpha,\beta}$ then there is $d \in a_{\gamma,0}/\epsilon_{\alpha,\beta} \setminus \{a_{\gamma,0}\}$, but for unboundedly many $j \in E^+_{\gamma+1}$ the separator in stage γ, for j choose $c^{\gamma+1}_j = d^\alpha_j - a_{\gamma,0}$ and $A^{\gamma+1}_j \cup \{a_{\gamma,0}, d\} \subseteq A^\beta_j$, and ${\rm tp}(c^{\gamma+1}_j, A^\beta_j, \mathfrak{B}_\beta)$ is $\Gamma^{\rm wmg}_{\mathbf{b}^{\rm at}_1,\mathbf{q},\bar{c}^1}$-big hence $d' \in \mathbf{b}^{\rm at}_1[\mathfrak{B}_\beta] \cap A^\beta_j \Rightarrow \neg(d' \dot{e} c^{\gamma+1}_j)$ hence in particular $\neg(d \dot{e} d^\alpha_j)$ but $(a_{\gamma,0} \dot{e} d^\alpha_j)$ and $d^\alpha_j \in \mathbf{b}^{\rm at}_1[\mathfrak{B}_\alpha]$ so d^α_j exemplify $\neg(d \dot{e}_{\alpha,\beta} a_{\gamma,0})$ so $(*)_5$ holds].

For $\beta \in (\beta^*_\alpha, \lambda^+)$ we define a relation $R_\beta = R_{\alpha,\beta}$:

$\boxtimes$ $cR_\beta b$ iff: $c \in B^3_{\alpha,\beta}, b \in \mathbf{b}_2[\mathfrak{B}_\beta]$, and for every $d \in \mathbf{b}_1[\mathfrak{B}_\alpha]$ and $d' \in \mathbf{b}_2[\mathfrak{B}_\alpha]$ such that $\mathbf{f}(d) = d' \mod \mathscr{I}^2_\alpha$ equivalently, $d' \in \mathbf{f}^0_\alpha(d))$ we have $c \le_{\mathbf{b}_1} d \Rightarrow b \le_{\mathbf{b}_2} d'$ and $\mathbf{b}_1 \models c \cap d = 0_{\mathbf{b}_1} \Rightarrow \mathbf{b}_2 \models b \cap d' = 0_{\mathbf{b}_2}$.

Now

$(*)_7$ if $\beta_1 < \beta_1 \le \alpha$ are as above then $R_{\alpha,\beta_1} = R_{\alpha,\beta_2} \cap (\mathfrak{B}_\beta \times \mathfrak{B}_\beta)$

$(*)_8$ $R_\beta \in \mathscr{P}_\beta$.

[Why? As $\mathbf{f}^0_\alpha$, belongs to $\mathscr{P}_\beta$ etc.]

$(*)_9$ if $cR_\beta b$ then $b \notin 0_{\mathbf{b}_2}$.

[Why? Think].

$(*)_{10}$ If $(\mathbf{f}(c) = b) \in \mathfrak{B}_\beta$, and $c \in B^3_{\alpha,\beta}$ then $cR_\beta b$.

[Why? Let $d \in \mathbf{b}_1[\mathfrak{B}_\alpha]$ and $d' \in \mathbf{b}_2[\mathfrak{B}_\alpha]$ be such that $\mathbf{f}(d) = d' \mod \mathscr{I}^2_\alpha$. Now assume $c \leq d$ (in $\mathbf{b}_1[\mathfrak{B}_\beta]$) then $\mathbf{f}(c) \leq \mathbf{f}(d)$ (in $\mathbf{b}_2[\mathfrak{B}_\beta]$) and let $\mathbf{f}(d) - d' \leq \mathbf{f}(d_0) \cup \mathbf{f}(d_1) \cup \ldots \cup \mathbf{f}(d_{n-1})$, for some true natural number n and $d_\ell \dot{e} \mathbf{b}^{\rm at}_1[\mathfrak{B}_\alpha]$ hence

$$\mathbf{f}(c) - d' \leq (\mathbf{f}(c) - \mathbf{f}(d)) \cup (\mathbf{f}(d) - d') = (\mathbf{f}(d) - d') \leq \mathbf{f}(d_0) \cup \ldots \cup \mathbf{f}(d_{n-1})$$

but $c \in \mathbf{b}^{\rm at}_1[\mathfrak{B}_\beta] \setminus \mathfrak{B}_\alpha$, so $c \cap d_\ell = 0$ in $\mathbf{b}_1[\mathfrak{B}_\beta]$ hence $\mathbf{f}(c) \cap \mathbf{f}(d_\ell) = 0$ in $\mathbf{b}_2[\mathfrak{B}_\beta]$ so at last $\mathbf{f}(c) - d' = 0$ in $\mathbf{b}_2[\mathfrak{B}_\beta]$. So $c \leq_{\mathbf{b}_1} d \Rightarrow b \leq_{\mathbf{b}_2} d'$. Similarly $\mathfrak{b}_2[!!\mathfrak{B}_\beta] \models c \cap d = 0 \Rightarrow \mathbf{b}_2[\mathfrak{B}_\beta] \models b \cap d' = 0$.]

$(*)_{11}$ If $\mathbf{f}(c) \in \mathfrak{B}_\beta$ and $cR_\beta b$ then $b \leq \mathbf{f}(c)$.

[Why? If not, $b - \mathbf{f}(c) > 0$ in $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$, so as the embedding $\mathbf{f}$ is complete and $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ is atomic, for some $e \in \mathbf{b}^{\rm at}_1[\mathfrak{B}_{\lambda^+}], 0 <_{\mathbf{b}_1} e$ and we have (in this formula $\mathbf{f}(e), \mathbf{f}(c)$ are members of $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$

$$\mathbf{b}_2[\mathfrak{B}_{\lambda^+}] \models \text{“}\mathbf{f}(e) \cap (b - \mathbf{f}(c)) > 0\text{”}.$$

Now if $e \cap c > 0$ then $c = e$ so $\mathbf{f}(e)\mathbf{f}(c)$ hence $\mathbf{f}(e) \cap (b - \mathbf{f}(c)) = 0$, contradiction.

So clearly $e \cap c = 0$ in $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$, so as $c \in B^3_{\alpha,\beta} \subseteq B^1_{\alpha,\beta}$ there is $d \in \mathbf{b}_1[\mathfrak{B}_\alpha]$ such that $c \leq d, e \cap d = 0$ (in $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$), hence $\mathbf{f}(c) \leq \mathbf{f}(d), \mathbf{f}(e) \cap \mathbf{f}(d) = 0$ in $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$, and by the definition of R_β, as $cR_\beta b$ holds.

Also $b \leq \mathbf{f}(d)$, hence $b \cap \mathbf{f}(e) = 0$ in $\mathbf{b}_2$ so $b - \mathbf{f}(e) = 0$ in $\mathbf{b}_2[!\mathfrak{B}_{\lambda^+}]$ so we have gotten a contradiction to the choice of e. Hence $b \leq \mathbf{f}(c)$.]

$(*)_{12}$ $b = \mathbf{f}(c)$ if $cR_\beta b$ and $(\mathfrak{B}_\beta, \mathbf{f}) \prec (\mathfrak{B}_{\lambda^+}, \mathbf{f})$.

[Why? If not, by $(*)_{11}$ above we have $b < \mathbf{f}(c)$ so by the “$\mathbf{f}$ is a complete embedding” for some e we have $\mathfrak{B}_{\lambda^+} \models \text{“}e \dot{e} \mathbf{b}^{\rm at}_1 \,\&\, \mathbf{f}(e) \cap (\mathbf{f}(c) - b) > 0_{\mathbf{b}_2}\text{”}$. So as e is an atom of $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ clearly $e \leq_{\mathbf{b}_1} c e \cap c = 0_{\mathbf{b}_1}$. First assume $e \cap c = 0_{\mathbf{b}_1}$ hence $0_{\mathbf{b}_2} = \mathbf{f}(e \cap c) = \mathbf{f}(e) \cap \mathbf{f}(2) \geq \mathbf{f}(e) \cap (\mathbf{f}(c) - b) > 0_{\mathbf{b}_2}$ contradiction. Second assume if $e \leq_{\mathbf{b}_1} c$ recall that $c \in B^3_{\alpha,\beta} \subseteq \mathbf{b}^{\rm at}_1\ [\mathfrak{B}_\beta] \setminus \mathfrak{B}_\alpha$ so $e = c$ contradiction to $cR_\beta b$ as the later implies $b \neq 0_{\mathbf{b}_2}$].

We can conclude

$(*)_{13}$ if $\beta \in (\beta^*_\alpha, \lambda^+)$ and $\mathbf{f}{\restriction}\mathbf{b}_1[\mathfrak{B}_\beta]$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}_\beta]$ into $\mathbf{b}_2[\mathfrak{B}_\beta]$ (which occurs for a club of β's) then $\mathbf{f} \restriction B^3_{\alpha,\beta} \in \mathscr{P}_\beta$.

What have we gained compared to $\otimes_1 0$ in the end of stage E? Here this works for all confinalities (for a club of β-s). Let $\bar{c} = \langle c_i : i < \omega \rangle$ be an increasing sequence for $\mathbf{b}^{\rm at}_1$ and let $\mathscr{C}^3_{\mathbf{f}} = \mathscr{C}_{\mathbf{f},\bar{c}} = \{\delta : \delta < \lambda^+$ is a limit ordinal $> \beta^*_\alpha$ and $\mathbf{f}$ is a complete embedding of $\mathbf{b}_1[\mathfrak{B}_\delta]$ into $\mathbf{b}_2[\mathfrak{B}_\beta]$ and if $\mathfrak{B}_\delta \models \text{“}d \dot{e} \mathbf{b}_1$ and $|d|/|a| \geq c_i\text{”}$ for every $i < \lg(\bar{c})$ then there is $e \in \mathbf{b}^{\rm at}_1[\mathfrak{B}_\delta], e \leq d$ such that $e \in B^3_{\alpha,\beta}\}$.

Clearly it is a club of λ^+. Now we can note more (but shall not use it below).

$(*)_{14}$ for every c, $\bar{c}^1$, $\bar{c}^2$ as in 3.4 for some club $\mathscr{C}^4_{\mathbf{f}} \subseteq \mathscr{C}^3_{\mathbf{f}} \cap \mathscr{C}^1_{\mathbf{f}}$ of λ^+ we have

$(*)_{15}$ for $\delta \in \mathscr{C}_{\mathbf{f}}$ there is $\mathbf{f}^2_\delta \in \mathscr{P}_\delta$ which is a function from $\mathbf{b}_1[\mathfrak{B}_\delta]$ into $\mathbf{b}_2[\mathfrak{B}_\delta]$ and $e \in \mathbf{b}_1[\mathfrak{B}_\delta] \Rightarrow \mathbf{f}(e) \triangle \mathbf{b}^2_\delta(e) \in \mathscr{I}^\delta_{\mathbf{f},\bar{c}}$ where

$$\mathscr{I}^\delta_{\mathbf{f},\bar{c}} = \{\mathbf{f}(b) : \mathbf{b}_1[\mathfrak{B}_\delta] \models \text{“}b \subseteq a \text{ has } \leq c_i|a| \text{ elements” for some } i < \lg(\bar{c})\}.$$

Note that now $\mathbf{p}, \bar{c}^2$ disappear, as by 3.29 such $\mathbf{p}, \bar{c}^2$ exists.

Stage G: Assume

$$\otimes\ \mathbf{b}_1,\ \mathbf{b}_2,\ \alpha,\ \bar{c},\ \mathbf{f} \restriction \mathbf{b}^{\mathrm{at}}_1[\mathfrak{B}_\alpha], \text{ and } \mathbf{f}{\restriction}B^{3,\mathbf{f}}_{\alpha,\beta} \in \mathscr{P}_\beta \text{ for } \beta \in \mathscr{C}^3_{\mathbf{f}} \text{ as above.}$$

We have a pre-killer contractor ζ_{pk} such that: for stationarily many (even for every) $\delta \in \mathscr{C}^3_{\mathbf{f}} \cap W_{\zeta_{\mathrm{pk}}}$ of cofinality λ, (for each candidate $\mathbf{f}$ for $\mathbf{f}{\restriction}B^3_{\alpha,\delta}$) for stationarily many $\epsilon \in S_{\zeta_{\mathrm{pk}}}$, if possible we ensure that for some $j \in u^\delta_{i_{\delta,\dot{e}}} \setminus \{\max(u^\delta_{i_{\delta,\dot{e}}})\}$, we have $\Gamma^{\delta+1}_j = \Gamma^{\mathrm{ids}}_{\langle(d_{\delta,j},d_{\delta,j,n}):n<\omega\rangle}$ so $\bar{c}^\delta_j = \langle(d_{\delta,j}, d_{\delta,j,n}) : n < \omega\rangle$ where $d_{\delta,j} \in B^3_{\alpha,\delta} \subseteq \mathbf{b}^{\mathrm{at}}_1[\mathfrak{B}_\delta], d_{\delta,j,n} \in \mathbf{b}_2[\mathfrak{B}_\delta], \mathbf{f}(d_{\delta,j}) = d_{\delta,j,0}$ and $\langle d_{\delta,j,n} : n < \omega\rangle$ is indiscernible over $A^\delta_j \cup \{d_{\delta,j}\}$ with $d_{\delta,j,0} \neq d_{\delta,j,1}$, of course, and without loss of generality $\neg(d_{\delta,j,0} \leq_{\mathbf{b}_2} d_{\delta,j,1})$ as we can find $d_{\delta,j,-1}, d_{\delta,j,-2}, \ldots$ such that $\langle d_{\delta,j,n} : n \in \mathbb{I}\rangle$ is an indiscernible over $A^\delta_j \cup \{d_{\delta,j}\}$. Later the automorphism killer contractor ζ_{ak} is active for our case for stationarily many $\beta \in \mathscr{C}^3_{\mathbf{f}} \cap W_{\zeta_{\mathrm{ak}}}$ with $\mathrm{cf}(\beta) = \theta$ (so $\mathbf{f}{\restriction}\mathbf{b}_1[\mathfrak{B}_\beta] \in \mathscr{P}_\beta$ by $\otimes_{10}$, of course, the automorphism killer contractor deal there with all such candidates as he does not know which is really necessary). For stationarily many $\epsilon \in S_{\zeta_{\mathrm{ak}}} \subseteq \lambda$ he ensure for $\delta \in \beta \cap \mathscr{C}^3_{\mathbf{f}} \cap W_{\zeta_{\mathrm{pk}}}$ as above, if possible, that for some $j \in u^\beta_{i_{\beta,\dot{e}}} \setminus \{\max(u^\beta_{i_{\beta,\dot{e}}})\}$ and $j_1 < \lambda$, $d_{\delta,j_1} \notin \mathrm{acl}_{\mathfrak{B}_\beta}(A^\beta_j)$, $\langle d_{\delta,j_1,n} : n < \omega\rangle$ is indiscrenible over $A^\beta_j + d_{\delta,j_1}$ and $\Gamma^\beta_j = \Gamma^{\mathrm{ids}}_{\bar{c}^\delta_j}$. He choose $e^\beta_j \in \mathbf{b}^{\mathrm{at}}_1[\mathfrak{B}_\beta]$ such that $\mathbf{b}^{\mathrm{at}}_1[\mathfrak{B}_\beta] \models$ "$e^\beta_j \cap d_{\delta,j} = 0$" and $(\mathbf{f} \restriction \mathbf{b}^{\mathrm{at}}_1[\mathfrak{B}_\beta])(e^\beta_j) \cap d_{\delta,j,1} > 0_{\mathbf{b}_2}$ (if there is no one, his candidate for $\mathbf{f} \restriction \mathbf{b}_1[\mathfrak{B}_\beta]$ is faked, failing coming from a complete embedding, so can be forgotten) and he make $e^\beta_j \in A^\beta_{\min(u^\beta_j \setminus (j+1))}$. Then he let $\Omega^\beta_n = \Gamma^{\mathrm{av}}_{\dot{D},\langle m:m<\omega\rangle}$ (for $n < \omega$), $\Omega^\beta_\omega = \Gamma^{\mathrm{ms}}_{\mathbf{b}^{\mathrm{at}}_1,\langle 1/x_{\beta,n}:n<\omega\rangle}$ be as in stage D (alternatively $a_{\beta,n} > n/|\mathbf{b}^{\mathrm{at}}_1|$ for $n < \omega, a_{\beta,n} < b$ if $b \in \mathrm{acl}_{\mathfrak{B}_{\beta+1}}(\mathfrak{B}_\beta \cup \{a_{\beta,m} : m < \omega\})$ and $n < \omega \Rightarrow n/|\mathbf{b}^{\mathrm{at}}_1| < b$ (check in stage D)), and he demand:

$(*)_{16}$ for a club of $\epsilon < \lambda$, for j as above:

- (α) $d_{\delta,j} \leq_{\mathbf{b}_1} a_{\beta,\omega}\ \&\ e^\beta_j \cap a_{\beta,\omega} = 0_{\mathbf{b}_1}$ in $\mathbf{b}_1[\mathfrak{B}_{\beta_1}]$
- (β) $\langle(d_{\delta,j}, d_{\delta,j,n}) : n < \omega\rangle$ is indiscernible over $A^{\beta+1}_j$.

No problem and $\mathbf{f}(a_{\beta,\omega})$ will give a contradiction.

So

$(*)_{17}$ for every $\delta \in \mathscr{C}^3_{\mathbf{f}} \cap W_{\zeta_{\mathrm{pk}}}$ of cofinality λ, for some $j < \lambda$, the pre-killer contractor cannot choose $d_{\delta,j}, d_{\delta,j,n}$ $(n < \omega)$ as above. Which means: if $d \in B^3_{\alpha,\lambda^+} \setminus \mathrm{acl}_{\mathfrak{B}_\delta}(A^\delta_j)$ then $\mathbf{f}(d) \in \mathrm{acl}_{\mathfrak{B}_\delta}(A^\delta_j + d)$ which is equal to $\mathrm{dcl}_{\mathfrak{B}_\delta}(A^\delta_j + d)$ as T^* has Skolem functions.

Now without loss of generality λ^2 divides δ hence there is $\delta' < \lambda^+$ such that $\mathrm{cf}(\delta') = \lambda$ and $A^\delta_j \subseteq \mathfrak{B}'_\delta$ hence for some $j' < \lambda$, $A^\delta_j \subseteq A^{\delta'}_{j'}$ and of course $A^{\delta'}_{j'} \in \bigcup_{\beta<\delta} \mathscr{P}_\beta$.

So using Fodor lemma

$(*)_{18}$ for some A^* for stationarily many δ as above satisfying $A^* \in \cup\{\mathscr{P}_\gamma : \gamma < \delta\}$, in $(*)_{17}$ we can replace A^δ_j by A^*.

Stage H:

We let the automorphism killer contractor act also for $\breve{f} \in A^*$ $\beta \in \mathscr{C}^3_{\mathbf{f}} \cap W_{\zeta_{\rm ak}}$ of cofinality λ for stationarily many $\epsilon \in S_{\zeta_{\rm pk}}$, where $\mathfrak{B}_\beta \models$ "$\breve{f}$ a partial function from $\mathbf{b}^{\rm at}_1$ to $\mathbf{b}_2$" to ensure that for some $j \in u^\beta_{i_{\beta,\epsilon}}, j < \max(u^p_{i_{\beta,\epsilon}})$, he choose in $\mathfrak{B}_\delta$, if possible $\langle d_{\beta,j,n} : n < \omega\rangle$, indiscernible over $A^\delta_j + \breve{f}$ such that $d_{\beta,j,0} \neq d_{\beta,j,1} \in B^3_{\alpha,\beta}$, $\mathbf{f}(d_{\beta,j,0}) = \breve{f}(d_{\beta,j\geq})$ and $\mathbf{f}(d_{\beta,j,1}) \neq \breve{f}(d_{\beta,j,1})$ and are, of course, well defined. If so without loss of generality add $\breve{f}(d_{\beta,j,1}) \notin \{\dot{f}(d_{\beta,j,n}) : n < \omega\}$, let $\Gamma^\beta_j = \Gamma^{\rm ids}_{\bar{c}^\beta_j}$, $\bar{c}^\beta_j = \langle(\breve{f}, d_{\beta,j,n}) : n < \omega\rangle$ and get contradiction as above.

Now easily (possibly shrinking $\mathscr{C}^3_{\mathbf{f}}$, using the freedom in choosing a type for $\zeta_{\rm ak}$; using that T has Skolem functions)

$(*)_{19}$ $\delta \in \mathscr{C}^3_{\mathbf{f}}$, and $A \in [\mathfrak{B}_\delta]^{<\lambda} \cap \cup\{\mathscr{P}_\alpha : \alpha < \delta\}$ contains the relevant parameters and $d^1, d^2 \in B^3_{\alpha,\delta}$ realizes the same non-algebraic type over $\mathrm{acl}_{\mathfrak{B}_\delta}(A^*)$, then for some $d^* \in B^3_{\alpha,\delta}$ we have that for $\ell = 1, 2$ there is an infinite indiscernible sequence to which d^*, d^ℓ belong.

Hence (together with stage G)

$(*)_{20}$ for some $\alpha < \lambda^+$ and $A^* \in \mathscr{P}_{\lambda^+} \cap [\mathfrak{B}_{\lambda^+}]^{<\lambda}$, (without loss of generality $A^* = \mathrm{acl}_{\mathfrak{B}_{\lambda^+}}(A^*) \prec \mathfrak{B}_{\lambda^+}$ and of course $A^* \in \mathscr{P}_{\lambda^+} = \bigcup\{\mathscr{P}_\gamma : \gamma < \lambda^+\}$), for every $\Gamma^{\rm ms}_{\mathbf{b}^{\rm at}_1,\bar{c}}$–big type $p \in \mathbf{S}(A^*, \mathfrak{B}_{\lambda^+})$ from $\mathscr{P}_{\lambda^+}$ such that $[x\dot{e}\mathbf{b}^{\rm at}_1] \in p$ for some $\breve{f}_p \in A^*$ we have

- (α) $\mathfrak{B}_{\lambda^+} \models$ "$\breve{f}_p$ is a (partial) function from $\mathbf{b}^{\rm at}_1$ to $\mathbf{b}_2$"
- (β) for every $d \in B^3_{\alpha,\lambda^+}$ realizing p, we have $\breve{f}_p(d) = \mathbf{f}(d)$
- (γ) $x \neq y \wedge x\dot{e}\mathrm{Dom}(\breve{f}) \subseteq \mathbf{b}^{\rm at}_1 \wedge y\dot{e}\mathrm{Dom}(\breve{f}) \subseteq \mathbf{b}^{\rm at}_1 \Rightarrow \breve{f}_p(x) \cap \breve{f}_p(y) = 0_{\mathbf{b}_2}$ or at least
- $(\gamma)'$ for some i and $\psi(y, x)$, such that $\forall x[|\{y\dot{e}\mathbf{b}^{\rm at}_1 : \psi(y, x)\}| \leq c_i \times |\mathbf{b}^{\rm at}_1|]$ we have $x \neq y \wedge x\dot{e}\mathrm{Dom}(f) \wedge y\dot{e}\mathrm{Dom}(\dot{f}) \subseteq \mathbf{b}^{\rm at}_1$.

[Why? First note that if $b \in \mathrm{acl}_{\mathfrak{B}_{\lambda^+}}(A^* + c), A^*$ as above then for some $\breve{f} \in A^*, \mathfrak{B}_{\lambda^+} \models$ "$\breve{f}$ is a partial function and $\breve{f}(c) = b$"; also if $c \in \mathbf{b}^{\rm at}_1[\mathfrak{B}_{\lambda^+}], b \in \mathbf{b}_2[\mathfrak{B}_{\lambda^*}]$ without loss of generality $\mathfrak{B}_{\lambda^+} \models$ "$\breve{f}$ is a function from $\mathbf{b}^{\rm at}_1$ into $\mathbf{b}_2$"; this takes care of clause (α). Second by the first paragraph of this stage we get clause (β).

Third, concerning clause (γ) we get $p(x) \cup p(y) \cup \{\neg\psi(y, x, \bar{a}) : \bar{a} \subseteq A$, and for some $i < \lg(\bar{c})$ we have $\mathfrak{B}_{\lambda^+} \vdash |\{v\dot{e}\mathbf{b}^{\rm at}_1 : \psi(b, x, \bar{a})\}| \leq c_1 x|\mathbf{b}^{\rm at}_1|\} \cup \{\breve{f}_p(x) \cap \breve{f}_p(y) \neq 0_{\mathbf{b}_2}$ in $\mathbf{b}_2, x \neq y\}$ belong to $\mathscr{P}_{\lambda^+}$ and is not realized in $\mathfrak{B}_{\lambda^+}$ hence by the saturator we can change $\breve{f}_p$ to make clause (γ) true. Now we can extend $\breve{f}_p$ by giving the value $0_{\mathbf{b}_2}$ for $x\dot{e}\mathbf{b}^{\rm at}_1$ on which it is not defined, so we can make $\mathrm{Dom}(\breve{f}_p) = \mathbf{b}^{\rm at}_1$].

Without loss of generality $c_a \times |\mathbf{b}^{\rm at}_1|$ is an integer. So for some $\bar{a}, \mathfrak{B}_{\lambda^+} \models$ "$\langle b_{\dot{n}} : \dot{n} < c_1 \times |\mathbf{b}^{\rm at}_1|\rangle$ is a partition of $\mathbf{b}^{\rm at}_1$, and for each $\dot{n}, a' \neq a'' \wedge a''\dot{e}b_{\dot{n}} \Rightarrow \breve{f}_p(a') \cap \breve{f}_p(a'') = 0_{\mathbf{b}_2}$".

Now we can find for each $\breve{f}_p$ (for $p \in \mathbf{S}(A^*, \mathfrak{B}_{\lambda^+}) \cap \mathscr{P}_{\lambda^+}$ which is $\Gamma^{\rm ms}_{\mathbf{b}^{\rm at}_1,\bar{c}}$-big), in $\mathfrak{B}_{\lambda^+}$ a "finite" sequence $\langle c^{\ell,p}_k : k < k^*_p, \ell < 2\rangle$ such that (in $\mathfrak{B}_{\lambda^+}$), $2k^*_p \le |\mathbf{b}^{\rm at}_1|$ and $c^{\ell,p}_k \dot{e} \mathbf{b}^{\rm at}_1$ are pairwise distinct and $\breve{f}_p(c^{0,p}_k) \cap \breve{f}_p(c^{1,p}_k) > 0_{\mathbf{b}_2}$ and

$$(\forall x, y \dot{e} \mathbf{b}^{\rm at}_1)(x \ne y \,\&\, x, y \notin \{c^{\ell,p}_k : k < k^*_p, \ell < 2\} \rightarrow \breve{f}_p(x) \cap \breve{f}_p(y) = 0_{\mathbf{b}_2}).$$

Let $d^\ell_p = \bigcup\{c^{\ell,p}_k : k < k^*_p\} \dot{e} \mathbf{b}_1$ for $\ell = 1, 2, d^2_p = 1_{\mathbf{b}_1} - d^0_p - d^1_p$, so d^0_p, d^1_p, d^2_p are pairwise disjoint and let $e^\ell_p = \mathbf{f}(d^\ell_p) \dot{e} \mathbf{b}_2$ and $e^{\ell,p}_k = \breve{f}_p(c^{\ell,p}_k)$ (all in $\mathfrak{B}_{\lambda^+}$'s sense).

Lastly, let

$$d^{\ell,*}_p = \bigcup\{c : c \dot{e} \mathbf{b}^{\rm at}_1 \text{ and } c \le_{\mathbf{b}_1} d^\ell_p \text{ and } \breve{f}_p(c) \le e^\ell_p\}$$

so $d^{\ell,*}_p \dot{e} \mathbf{b}_1$.

Now for $\ell = 0, 1, 2$

$(*)_{21}$ $p(x) \cup [x \le d^\ell_p - d^{\ell,*}_p]$ is $\Gamma^{\rm ms}_{\bar{c}}$-small.

[Why? Clearly this type belong to $\mathscr{P}_{\lambda^+}$ hence if $(*)$ fail this there is $b \in B^3_{\alpha,\lambda^+}$ realizing it so $\mathbf{f}(b) = \breve{f}_p(b)$ and as $\mathbf{f}$ is a homomorphism we get contradiction.]

$(*)_{22}$ for every $k < k^*_p$, we have $(e^0_k \le_{\mathbf{b}_1} (d^0_p - d^{0,*}_p)) \vee (e^1_k \le_{\mathbf{b}_1} (d^1_p - d^{1,*}_p))$.

[Why? As $f_p(e^0_k) \cap f_p(e^1_k) > 0_{\mathbf{b}_1}$ whereas $\mathbf{f}(d^0_p) \cap F(d^1_p) = 0_{\mathbf{b}_2}$.]

Hence [just proving that (in $\mathfrak{B}_{\lambda^+} \vee \kappa^*_p$ is "small"]

$(*)_{23,\ell}$ For some $\vartheta_p(x) \in p(x)$ we have $\vartheta_p(x) \,\&\, [x \le_{\mathbf{b}_1} (d^0_p \cup d^1_p) = \bigcup\{e^\ell_k : \ell < 2, k < k^*_p\}]$ is $\Gamma^{\rm ms}_{\bar{c}}$-small.

Let $a_p = \{x \dot{e} \mathbf{b}^{\rm at}_1 : \vartheta(x)\}$ and $a^-_p = \{x \dot{e} \mathbf{b}^{\rm at}_1 : \dot{e}(x) \,\&\, x \le_{\mathbf{b}_1} d^0_p \cup d^1_p\}$ and let $a^+_p = \{x \dot{e} \mathbf{b}^{\rm at}_1 : \vartheta(x) \,\&\, \neg x \le d^0_p \cup d^1_p\}$ so a_p is the disjoint union of a^-_p and a^+_p. Note that $\langle c^{\ell,p}_\kappa : \ell < 2, \kappa < \kappa^*_p\rangle, \langle e^\ell_k : \ell, k\rangle, d^\ell_p, d^{\ell,*}_p, a^-_p, a^+_p$ depend on $\breve{f}_p$ (and not on p) and $\breve{f}_p \in A^*$ so without loss of generality $\in A^*$ for every $\Gamma^{\rm ms}_{\bar{c}}$-big $p \in \mathbf{S}(A, \mathfrak{B}_{\lambda^+}) \cap \mathscr{P}_{\lambda^+}$ we have $[x \le_{\mathbf{b}_1} a^-_p] \notin p$ and for some $j(p) < \lg(\bar{c})$ we have "$|a^-_p| \le c_{j(p)} \times |\mathbf{b}^{\rm at}_1|$ and $\breve{f}_p \restriction a_p$ maps $\mathbf{b}^{\rm at}_1$ into $\mathbf{b}_2, x \ne y \in a^+_p \Rightarrow \breve{f}_p(x) \cap \breve{f}_p(y) = 0_{\mathbf{b}_2}$ and so $\breve{f}_p$ induce an embedding of $\mathbf{b}_1 \restriction a^+_p$ into $\mathbf{b}_2$ called $\breve{f}_p$". (We are identifying $\mathbf{b}_1$ with $\mathscr{P}(\mathbf{b}^{\rm at}_1)$ where $a^+_p := \cup\{x \dot{e} \mathbf{b}^{\rm at}_1 : \breve{f}_p(x) > 0_{\mathbf{b}_2}$ in $\mathbf{b}_2$-s sense$\}$ satisfy $a^+_p \dot{e} \mathbf{b}_1$ so for some $\breve{f}_p$ we have $\mathfrak{B}_{\lambda^+} \models$ "$\breve{f}_p$ is an embedding of $\mathbf{b}_1 \restriction a^+_p$ into $\mathbf{b}_2, \mathbf{b}_1 \restriction a^+_p$ is the sub-boolean ring of $\mathbf{b}_1$ with set of elements $\{d \in \mathbf{b}_1 : d \le a^+_p\}$".

<u>Stage I</u>: Let $c^* \in \mathfrak{B}_{\lambda^+}$ be such that $c_i < c^* \dot{e} (0,1)_{\mathbb{R}}, c^* < 1/n$ and $c^*|\mathbf{b}^{\rm at}_1|$ integer $> n$ for every base n. Let $\dot{k}^* \in \mathbb{N}[\mathfrak{B}_{\lambda^+}]$ be small enough.

In $\mathfrak{B}_{\lambda^+}$ let $\langle \breve{f}_{\dot{n}} : \dot{n} \le \dot{k}^*\rangle$ be a list of functions from $\mathbf{b}^{\rm at}_1$ into $\mathbf{b}_2$ satisfying $x \in \mathbf{b}^{\rm at}_1 \,\&\, y \in \mathbf{b}^{\rm at}_1 \,\&\, x \ne y \Rightarrow \breve{f}_{\dot{n}}(x) \cap \breve{f}_{\dot{n}}(y) = 0_{\mathbf{b}_2}$ including all such members of A^* (exist by the saturator work). Let $e \in \mathfrak{B}_{\lambda^+}$ be an equivalence relation on $\mathbf{b}^{\rm at}_1$ with $< \dot{k}^*$ equivalence classes such that $\mathfrak{B}_{\lambda^+} \models$ "xey" implies $\operatorname{tp}(x, A^*, \mathfrak{B}_{\lambda^+}) = \operatorname{tp}(y, A^*, \mathfrak{B}_{\lambda^+})$.

Clearly

$(*)_{24}$ if $\dot{n}_1, \dot{n}_2 < \dot{k}^*, \dot{m}_1 e \dot{m}_2$ then $\breve{f}_{\dot{n}_1}(\dot{m}_1) \le \breve{f}_{\dot{n}_2}(\dot{m}_1) \Leftrightarrow \breve{f}_{\dot{n}_2}(\dot{m}_2) \le \breve{f}_{\dot{n}_2}(\dot{m}_2)$

$(*)_{25}$ if $a' \neq a'', a'ea''$ and $\dot{n} < \dot{k}^*$ then $\breve{f}_{\dot{n}}(a') \cap \breve{f}_{\dot{n}}(a'') = 0_{\mathbf{b}_2}$.

In $\mathfrak{B}_{\lambda^+}$ let $b^* = \{x\dot{e}\mathbf{b}_1^{\mathrm{at}} : |x/e| \geq \mathbf{k}_1^*\}$, where e.g. $\mathfrak{B}_\lambda =$ "$\dot{m}k_1^* = 2\dot{m}k^*$" and we can choose $\langle a_{\dot{m}} : \dot{m} < \dot{m}^* \rangle$ "randomly in $\mathfrak{B}_{\lambda^+}$ sense such that:

(α) $\dot{m}_1 < \dot{m}_2 < \dot{k}_1^* \Rightarrow \neg(a_{\dot{m}_1} e a_{\dot{m}_2})$

(β) $a_{\dot{m}}\dot{e}b^*, b^* = \bigcup\limits_{\dot{m}<\dot{k}_1^*} (a_{\dot{m}}/e)$.

So almost surely

(γ) if $\dot{m}_1 < m^*, \dot{n}_1, \dot{n}_2 < \dot{k}^*$ and for every $\dot{n}_2 < \dot{k}^*$, if $\breve{f}$ satisfies $\mathrm{Dom}(\breve{f}) = \{0, \ldots, \dot{m}_1^*\} \backslash \{\dot{m}_1\}, (\forall \dot{m}) \breve{f}(\dot{m}) \subseteq \{0, \ldots \dot{k}^*\}$ and $b = \cup\{\breve{f}_{\dot{n}}(a_{\dot{m}})\dot{m} \in \mathrm{Dom}(\breve{f}), \dot{n} \in \breve{f}(\dot{m})\}\dot{e}\mathbf{b}_2$ and for every a $\dot{e}(a_{\dot{m}}/e)\backslash\{a_{\dot{m}}\}$ we have $\breve{f}_{\dot{n}_1}(a) - \breve{f}_{\dot{n}_2}(a)$ is disjoint to b (in $\mathbf{b}_2$), <u>then</u> $a_{\dot{m}_1}$ satisfies this.

So

$(*)_{27}$ for each $\dot{m} < \dot{m}^*$, for some $\dot{n} < \dot{k}^*$ we have $\breve{f}(a_{\dot{m}}) = \breve{f}_{\dot{n}}(a_{\dot{m}})$.

Hence by $(*)_{27}$ we have $a \in (a_{\dot{m}}/e) \Rightarrow \breve{f}(a) = \breve{f}_{\dot{n}}(a)$. Our next aim is to show that the choice of $\dot{n}$ can be done uniformly: in a way represented in $\mathfrak{B}_{\lambda^+}$. For each $\dot{m}_1 \neq \dot{m}_2 < \dot{m}^*$ and $\dot{n}_1, \dot{n}_2 < \dot{k}^*$ choose if possible a member $b = b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2}\dot{e}(a_{\dot{m}_2}/e)$ such that $\breve{f}_{\dot{n}_1}(a_{\dot{m}_1}) \cap \breve{f}_{\dot{n}_2}(b) \neq 0_{\mathbf{b}_2}$, otherwise we let $b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2}$ be $0_{\mathbf{b}_1}$; so of course, without loss of generality the function $(\dot{n}_1, \dot{n}_2, \dot{m}_1, \dot{m}_2) \mapsto b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2}$ is represented in $\mathfrak{B}_{\lambda^+}$.

Let $a^1 =: \{a_{\dot{m}} : \dot{m} < \dot{m}^*\}$ and $a^2 =: \breve{f}(a^1) \in \mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$. Now by stage H for each $\dot{m}_1 < \dot{m}^*$ for some $\dot{n}_0 < \dot{k}^*$ we have $\breve{f}(a_{\dot{m}_1}) = \breve{f}_{\dot{n}_0}(a_{\dot{m}_1})$ hence $\breve{f}_{\dot{n}_0}(a_{\dot{m}_1}) = \breve{f}(a_{\dot{m}_1}) \leq \breve{f}a^1) = a^2$, also we have

$$(*)^{28}_{\dot{n}_1,\dot{m}_1} \quad a \in a_{\dot{m}_1}/e \Rightarrow \mathrm{tp}(a, A^*, \mathfrak{B}_{\lambda^+}) = \mathrm{tp}(a_{\dot{m}_1}, A^*, \mathfrak{B}_{\lambda^+}) \Rightarrow \breve{f}(a) = \breve{f}_{\dot{n}_0}(a).$$

We like to define $\dot{n}_0$ from a, or just from $a_{\dot{m}_1}$ (inside $\mathfrak{B}_{\lambda^+}$).

Clearly

$(*)_{29}$ $\breve{f}_{\dot{n}_0}(a_{\dot{m}_1}) \leq_{\mathbf{b}_2} a^2, \dot{n}_0 < \dot{k}^*$ and $a \in a_{\dot{m}_1}/e \setminus \{a_{\dot{m}_1}\} \Rightarrow \breve{f}_{\dot{n}_0}(a) \cap a^2 = 0_{\mathbf{b}_2}$

$(*)_{30}$ for no $\dot{n}_1 < \dot{k}^*, \dot{n} \neq \dot{n}_0$ do we have

$$\breve{f}_{\dot{n}_1}(a_{\dot{m}_1}) \leq_{\mathbf{b}_2} a^2 \,\&\, \neg(\breve{f}_{\dot{n}_1}(a_{\dot{m}_1}) \leq_{\mathbf{b}_2} \breve{f}_{\dot{n}_0}(a_{\dot{m}_1})).$$

[Why? Assume $\dot{n}_1$ is a counterexample, necessarily by $\mathbf{f}$ being a complete embedding using the maximal antichain $\mathbf{b}_1^{\mathrm{at}} - \bigcup\limits_{a \in b^*} a\} \cup \{a : a \in b^*\}$ of $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$, for some $\dot{m}_2 < \dot{m}^*$ and $a''_{\dot{m}_2}\dot{e}a_{\dot{m}_2}/e$ we have $\breve{f}_{\dot{n}_1}(a_{\dot{m}_1}) - \breve{f}_{\dot{n}_0}(a_{\dot{m}_1})$ computed in $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$ is not disjoint to $\mathbf{f}(a''_{\dot{m}_2})$, but for some $\dot{n}_2 < \dot{m}^*$ we have $\breve{f}(a_{\dot{m}_2}) = \breve{f}_{\dot{n}_2}(a_{\dot{m}_2})$, so necessarily $b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2} \neq 0_{\mathbf{b}_1}$ and is a member of $\mathbf{b}_1$ disjoint to a^1 so $\mathbf{f}(b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2})$ is disjoint (in $\mathbf{b}_2$) to $a^2 = \mathbf{f}(a^1)$.

Also as $b_{\dot{n}_1,\dot{n}_2,\dot{m}_1\dot{m}_2}\dot{e}a_{\dot{m}_2}/e$ and $\mathbf{f}(a_{\dot{m}_2}) = \breve{f}_{\dot{n}_2}(a_{\dot{m}_2})$ clearly $\mathbf{f}(b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2}) = \breve{f}_{\dot{n}_2}(b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2})$ so by the previous sentence $\breve{f}_{\dot{n}_2}(b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2}) \cap a^2 = 0_{\mathbf{b}_2}$, but by the choice of $b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2} (> 0_{\mathbf{b}_1})$, i.e. the previous sentence we have $f_{\dot{n}_2}(b_{\dot{n}_1,\dot{n}_2,\dot{m}_1,\dot{m}_2}) \cap \breve{f}_{\dot{n}_1}(a_{\dot{m}_1}) > 0_{\mathbf{b}_2}$ hence $\neg(\breve{f}_{\dot{n}_1}(a_{\dot{m}_1}) \leq_{\mathbf{b}_2} a_2)$, contradicting the choice of $\dot{n}_1$].

So

$(*)_{31}$ in $\mathfrak{B}_{\lambda^+}$, if $x \dot{e} \mathbf{b}_1^{\rm at} \,\&\, x \le_{\mathbf{b}_1} \mathbf{b}^*$ then $\mathbf{f}(x)$ is $\breve{f}_{\dot{n}(x)}(x)$ where $\dot{n}(x)$ is the unique $\dot{n} < \dot{k}^*$ such that

$$\begin{aligned}(\exists \dot{m} < \dot{m}^*)(x \dot{e} a_{\dot{m}}/e \,\&\, \breve{f}_{\dot{n}}(a_{\dot{m}}) \quad &\le_{\mathbf{b}_2} a^2 \,\&\, (\forall \dot{n}' < \dot{n})[\breve{f}_{\dot{n}'}, (a_{\dot{m}}) \\ &\le a^2 \to \breve{f}_{\dot{n}}(a_{\dot{m}}) \le \breve{f}_{\dot{n},1}(a_{\dot{m}})]),\end{aligned}$$

so $x \mapsto \dot{n}(x)$ is a function in $\mathfrak{B}_{\lambda^+}$.

Stage J:
Let

$$a_{\rm lev} := \{b : b \text{ a natural number } < \log\log|\mathbf{b}_1^{\rm at}|\} \quad (\text{in } \mathfrak{B}_{\lambda^+})$$

though really we are interested just in

$$a_{\rm lev}^- := \{b \dot{e} a_{\rm lev} : b > n \text{ for every (true) natural number (not in } \mathfrak{B}_{\lambda^+}!)$$

for $b \dot{e} a_{\rm lev}$, in $\mathfrak{B}_{\lambda^+}$ let

$$\begin{aligned} I_b := \{\breve{f} : \quad & \breve{f} \text{ a partial function from } \mathbf{b}_1^{\rm at} \text{ into } \mathbf{b}_2 \setminus \{0_{\mathbf{b}_2}\} \text{ such that} \\ & e_1 \neq e_2 \dot{e} \mathbf{b}_1^{\rm at} \to \breve{f}(e_1) \cap \breve{f}(e_2) = 0_{\mathbf{b}_2} \text{ such that} \\ & |\mathbf{b}_1^{\rm at} - {\rm Dom}(\breve{f})| \le 2^b\}\end{aligned}$$

(b-th level). We define a distance function on $I = \bigcup\{I_b : b \dot{e} a_{\rm lev}\}$:

$${\rm dis}(\breve{f}_1, \breve{f}_2) = |\{x \dot{e} \mathbf{b}_1^{\rm at} : x \notin {\rm Dom}(\breve{f}_1) \text{ or } x \notin {\rm Dom}(\breve{f}_2) \text{ or } \breve{f}_1(x) \neq \breve{f}_2(x)\}|.$$

Next define a branch, it is an (outside) function $\hat{H}$, satisfying ${\rm Dom}(\hat{H}) = a_{\rm lev}^-$, $\hat{H}(b) \in I_b, b_1 < b_2$ (in $a_{\rm lev}^-$) $\Rightarrow {\rm dis}(\hat{H}(b_1), \hat{H}(b_2)) < 10^{b_2}$.

Now $\mathbf{f}$ induce a branch $\hat{H}$ as $\boxtimes$ in the end of stage I holds for $b \in a_{\rm lev}^-$.

By 5.6 below we have "no undefinable branch" so there is an equivalent branch H' (see 5.6 below) which is definable in $\mathfrak{B}_{\lambda^+}$ hence is represented say by $\dot{f}$.

Let $\dot{f}_n = \dot{f}(n)$ (n a true natural number in $\mathfrak{B}_\lambda$'s sense).

For each $b \dot{e} a_{\rm lev}^-$, for some $n_b < \omega$ we have ${\rm dis}(\hat{H}(b), \hat{H}(n)) \le |10^b|$ for every $n \ge n_b$ (otherwise $\{n : n \text{ true natural number}\}$ is definable in $\mathfrak{B}_{\lambda^+}$) so some n^* is n_b for arbitrarily small $b \dot{e} a_{\rm lev}^-$ hence (changing 10^b slightly) without loss of generality this holds for every small enough $b \dot{e} a_{\rm lev}^-$.

[Why? As the cofinality of $(\{\dot{n} \in \mathbb{N} : n <_{\mathfrak{B}_{\lambda^+}} \dot{n} \text{ for } n < \omega\}, \ge)$ is uncountable (in fact is λ^+ as $\Omega_{\alpha,0} = \Gamma^{\rm ver}_{\langle n:n<\omega\rangle}$ for unboundedly many $\alpha < \lambda^+$]. In particular $|\mathbf{b}_1^{\rm at} \setminus {\rm Dom}(\dot{f}_{n^*})|$ is really $< 10^b$ for each $b \in a_{\ell ev}$ hence is finite. Assume $\{d \in \mathbf{b}_1^{\rm at}[\mathfrak{B}_{\lambda^+}] : d \dot{e} {\rm Dom}(\dot{f}_{n^*}),\ \dot{f}_{n^*}(d) \neq \mathbf{f}(d)\}$ is infinite. So we can find $d_n^1, d_n^2 \dot{e} \mathbf{b}_1^{at}$, pairwise distinct hence disjoint, satisfying $\mathbf{f}(d_n^2) \cap \dot{f}_{n^*}(d_n^1) \neq 0_{\mathbf{b}_2}$. We can find $b \dot{e} \mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$, such that $d_n^1 \le_{\mathbf{b}_1} b, d_n^2 \cap b = 0_{\mathbf{b}_1}$, let $b' = \mathbf{f}(b)$, and in $\mathfrak{B}_{\lambda^+}$:

$$b'' := \{d' : d' \dot{e} \mathbf{b}_1^{\rm at}, d' \cap_{\mathbf{b}_1} b = 0_{\mathbf{b}_1} \text{ but } \dot{f}_{n^*}(d') \cap b' \neq 0\},$$

so necessarily $\mathfrak{B}_{\lambda^+} \models$ "$|b''|$ is $> n$" for each n, hence for some $c \in a_{\rm lev}^-$ we have $\mathfrak{B}_{\lambda^+} \models$ "$|b''| > c$" and we get easy contradiction.

Thus modulo 5.6 we have finished proving:

(∗) every complete embedding of $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ into $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$ appear in $\mathfrak{B}_{\lambda^+}$ where $\mathfrak{B}_{\lambda^+} \models$ "$\mathbf{b}_1$ is a finite Boolean ring hence algebra, $\mathbf{b}_2$ is a Boolean ring (e.q. algebra)".

Stage K: I assume $\mathfrak{B}_{\lambda^+} \models$ "$\mathbf{b}_1, \mathbf{b}_2$ are atomic Boolean rings" and $\mathbf{f}$ is a isomorphism from $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ onto $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$, and let $\mathscr{Y} = \{a : \mathfrak{B}_{\lambda^+} \models$ "$a \in \mathbf{b}_1$ is $\neq 0_{\mathbf{b}_1}$ and is a finite union of atoms"$\}$. So for every $a \in \mathscr{Y}, \mathbf{f}$ is an isomorphism from Boolean Algebra $\mathbf{b}^1_a[\mathfrak{B}_{\lambda^+}]$ where $\mathbf{b}^a_1 := \mathbf{b}_1 \restriction \{x : x \leq_{\mathbf{b}_1} a\}$ onto $\mathbf{b}_2[\mathfrak{B}_{\lambda^+}]$ where $\mathbf{b}_2^{\mathbf{f}(a)} = \mathbf{b}_2 \restriction \{y : y \leq_{\mathbf{b}_2} \mathbf{f}(a)\}$, i.e. $\mathbf{f}_a = \mathbf{f} \restriction \mathbf{b}^a_1[\mathfrak{B}_{\lambda^+}]$ is in $\mathfrak{B}_{\lambda^+}$ hence by Stage C we are done. $\square_{5.2}$

Remark 5.4. We may try to replace the proof from $(*)_{27}$ till here by:

(∗) for $\dot{m}_1 < \dot{m}^*, \dot{n}_1 < \dot{k}^*$, we have $\mathbf{f}(a_{\dot{m}_1}) = \breve{f}_{\dot{n}_1}(a_{\dot{m}_1})$ <u>if and only if</u> $(\dot{m}_1, \dot{n}_1)$ satisfies $\breve{f}_{\dot{n}_1}(a_{\dot{m}_1}) \leq_{\mathbf{b}_2} a^2$ and if $\breve{f}_{\dot{n}_2}(a_{\dot{m}_1} \leq_{\mathbf{b}_1} a^2$ and $\breve{f})\dot{m})\{\dot{n} < \dot{k}^* : \breve{f}_{\dot{n}}(a_{\dot{m}}) \leq a^2\}$ when $\dot{m} < \dot{m}^* \,\&\, \dot{m} \neq \dot{m}^*$, then $\breve{f}_{\dot{n}_1}(a) - \breve{f}_{\dot{n}_2}(a)$ is disjoint to a^2 for every $aea_{\dot{m}_1}$ and $a^2 = \cup\{\breve{f}_{\dot{n}}(a_{\dot{m}}) : \dot{n} \in \breve{f}'(\dot{m})\} \cup \breve{f}_{\dot{n}_1}(a_{\dot{m}_1})$.

Hence playing with $\dot{k}^*$ we get:

⊠ for every $\dot{k}^*$ such that $\mathfrak{B}_{\lambda^+} \models$ "$n < \dot{k}^* < |\mathbf{b}_1^{\mathrm{at}}|$ for $n < \omega$, there is $b^*_{\dot{k}^*} \in \mathbf{b}_1$ such that $\mathbf{f} \restriction \{a\dot{e}\mathbf{b}_1^{\mathrm{at}} : a \cap b_{\dot{k}^*} = 0_{\mathbf{b}_1}\}$ is represented in $\mathfrak{B}_{\lambda^+}$ hence $\mathbf{f} \restriction \{a\dot{e}\mathbf{b}_1 : a \cap b_{\dot{k}^*} = 0_{\mathbf{b}_1}\}$ is represented in $\mathfrak{B}_{\lambda^+}$ and $|\{a\dot{e}\mathbf{b}_1^{\mathrm{at}} : a \leq_{\mathbf{b}_1} b_{\dot{k}^*}\}| < \dot{k}^*$.

Discussion 5.5. 1) In 5.2 we can add
5) Assume $N = \mathfrak{B}_*^{[\bar{\varphi}]}$ and

(*i*) N is a model of $\mathfrak{t}_n^{\mathrm{ind}}$ and
(*ii*) $(\forall y_1 \neq y_2 \in \theta^N) \Rightarrow (\exists^{\geq n} x \in P^N)[xRy_1 \equiv \neg xRy_2]$.

<u>Then</u> any auto of N is represented in $\mathfrak{B}^*$. The proof is just easier. Without (ii) we get a weaken result. If above we have dealt with complete embedding rather then just isomorphism onto, we can get more. May like to allow such $\mathbf{f}$'s (and get the same result).
2) We may like in 5.2 to allow $\mathbf{b}$ to be non-atomic. One way is combining our proof with [Shee]. We shall give a complete proof elsewhere.

Suppose $\mathfrak{B}_{\lambda^+} \models$ "$\mathbf{b}_1$, $\mathbf{b}_2$ are Boolean rings", $\mathbf{f}$ a complete embedding of $\mathbf{b}_1[\mathfrak{B}_{\lambda^+}]$ into $\mathbf{b}_2[\mathfrak{B}]$, we would like to show that it is representable.

Let us define, inside $\mathfrak{B}_{\lambda^+}$, $Y = \{a : a$ a "finite" subset of $\mathbf{b}_1$ consisting of pairwise disjoint elements such that $\mathfrak{B}_{\lambda^+} \models$ "for every $c \in a$, either c is an atom of $\mathbf{b}_1$ or below $c, \mathbf{b}_1$ is atomless"$\}$ and for $a\dot{e}Y$ let $b_a = \bigcup\{x : x\dot{e}a\} \in \mathbf{b}_1$.

For every (internally) finite $a\dot{e}Y$ let $\mathbf{b}_1[a] :=$ "the sub-ring of $\mathbf{b}_1$ generated by a", it is, in $\mathfrak{B}_{\lambda^+}$, a finite sub-Boolean ring of $\mathbf{b}_1$ and itself is a Boolean algebra and inside $\mathfrak{B}_\lambda$

⊗ if $a' \subseteq \mathbf{b}_1[a]$ is a maximal antichain of $\mathbf{b}_1[a]$, then in $\mathbf{b}_1, a'$ is a maximal family of pairwise disjoint elements which are $\leq b_a$.

Hence $\mathbf{f} \restriction \mathbf{b}_1[a]$ is a complete embedding of $\mathbf{b}_1[a]$ into $\mathbf{b}_2 \restriction \{x \in \mathbf{b}_2 : x \leq_{\mathbf{b}_2} \mathbf{f}(b_a)\}$ which also is a Boolean algebra (and sub-Boolean ring of $\mathbf{b}_2$), hence is represented in $\mathfrak{B}_{\lambda^+}$. So by Stage C, i.e. as we have proved part (3) of 5.2, we have finished proving part (4) too. For $\mathbf{b}$ a not necessarily atomic Boolean ring. (Note: if we have earlier the result only for isomorphism (onto $\mathbf{b}_2$), here we would have to work harder.)

Claim 5.6. *For $\mathfrak{B}_* = \mathfrak{B}_{\lambda^+}$ as in 5.2 we can add:*

- *every branch is equivalent to a definable branch $\hat{H}$ <u>when</u> we assume (compare with 3.35(2)):*
  - $(*)$ (a) $d^*, a \in \mathbb{N}^{\rm ns}[\mathfrak{B}_{\lambda^+}] = \{x : \mathfrak{B}_{\lambda^+} \models$ "$x\dot{\in}\mathbb{N} \,\&\, n < x$" *for every truly finite* $n\}$
    - (b) $\mathfrak{B}_{\lambda^+} \models$ "$\bar{I} = \langle I_d : d \in \mathbb{N}^{d^*}\rangle$ *is a sequence of sets,* dis *is a symmetric two place function from* $I = \bigcup\{I_d : d \in \mathbb{N}^{d^*}\}$ *into* $\mathbb{N}$, *satisfying* $\mathrm{dis}(x,z) \le \mathrm{dis}(x,y) + \mathrm{dis}(y,z)$" *where* $\mathbb{N}^d := \{x \in \mathbb{N} : x < d\}$
    - (c) $\mathfrak{B}_{\lambda^+}$, c *is a function (monotonic for simplicity) from* $\mathbb{N}^{d^*}$ *to* $\mathbb{N}^a$ *such that the value "converge to finite" when the argument does, i.e.:*

$$b\dot{\in}\mathbb{N}^a \,\&\, \bigwedge_n n < b \Rightarrow \quad (\exists d)[\bigwedge_n n < d\dot{\in}\mathbb{N}^{d^*} \,\&\, \forall x \forall d' \,(x\dot{\in}I_{d'} \wedge \bigwedge_n n < d' \le d \Rightarrow c(d') < b)].$$

  - (d) *We call a function $\hat{H}$ (generally, not necessary in $\mathfrak{B}_*$) a c-branch (but may omit c)* $\mathrm{Dom}(\hat{H}) = \mathbb{N}^{d^*} \setminus \{n : n < \omega\}$
  - (e) $\hat{H}(d) \in I_d$ *if*
    - (α) $\bigwedge_n n < d_1 < d_2 \le d^* \Rightarrow \mathrm{dis}(\hat{H}(d_1), \hat{H}(d_2)) \le 2{\times}\max\{c(d_1), c(d_2)\}$
  - (f) *branches $\hat{H}_1, \hat{H}_2$ are i-equivalent if* $\bigwedge_n n < d \le d^* \Rightarrow \mathrm{dis}(\hat{H}_1(d), \hat{H}_2(d)) \le 4 \times c(d)$
  - (g) *a branch $\hat{H}$ is c-definable if for some* $f \in \mathfrak{B}_{\lambda^+}$, $d \in \mathrm{Dom}(\hat{H}) \Rightarrow \hat{H}(d) = f(d)$.

The proof of 5.6 is broken to some definition and claims.

Fact 5.7. Let Γ be a g-bigness notion, $\mathfrak{k} = (\mathbf{K}_{\mathfrak{k}}, \le_{\mathfrak{k}})$ elementary class. <u>Then</u> $p = \mathrm{tp}(\bar{a}, A, M)$ is Γ-big iff $p^* = \mathrm{tp}(\bar{a}, \mathrm{acl}(A), M)$ is Γ-big.

Proof. The "if" is by monotonicity. For the "only if" assume the left, let $M \le_{\mathfrak{k}} N \in \mathbf{K}, N$ is strongly $|A|^+$-saturated, $\bar{a}' \in {}^\alpha N, p' = \mathrm{tp}(a', \mathrm{acl}A, N)$ extend p and is Γ-big. But there is an automorphism f of N mapping $\bar{a}'$ to $\bar{a}, f \restriction A = \mathrm{id}_A$ so $f^{-1}(\mathrm{acl}_M(A)) = \mathrm{acl}_M(A)$. $\square_{5.7}$

Definition 5.8. 1) We say a bigness notion Γ is strict (or strictly nice) <u>if</u> (where $\alpha = \lg(x_\Gamma)$)

- $(*)$ if $p \in \mathbf{S}^\alpha(A, M)$ is Γ-big, $A = \mathrm{acl}_M(A)$ and β is an ordinal <u>then</u> for some $N, \bar{a}_i$ $(i < \beta)$ we have
  - (α) $M \le_{\mathfrak{k}} N$
  - (β) $\bar{a}_i \in {}^\alpha N$

(γ) $\mathrm{tp}(\bar a_i, A \cup \bigcup_{j<i} \bar a_j, N)$ is Γ-big

(δ) if $\bar b \in {}^m(\mathrm{acl}(A \cup \bar a_i)), \Omega = \Gamma^{\mathrm{mt}}_{a,\mathbf{d},\bar c}$ is an a-bigness notion (see Definition 3.34), $A_\Omega \subseteq A, \mathrm{tp}(\bar b, A, N)$ is Ω-big then $\mathrm{tp}(\bar b, A \cup \bigcup_{j<i} \bar a_j, N)$ is Ω-big; hence $\mathrm{tp}(\bar b, A \cup \bigcup_{j\neq i} a_j, N)$ is Ω-big.

2) We say $p = \mathrm{tp}(\bar a, A_2, M)$ is a strict (or strictly nice) extension of $p \restriction A_1$, where $A_1 \subseteq A_2 \subseteq M$ <u>when</u>: if $m < \omega$, $\bar b \in {}^m(\mathrm{acl}_M(A_2 + \bar a)), \Omega = \Gamma^{\mathrm{mt}}_{a,\mathrm{dis},\bar c}$ is a bigness notion (see Definition 3.34), $A_\Omega \subseteq A_1, A_2 \subseteq A_1, \mathrm{tp}(\bar b, \mathrm{acl}(A_1), M)$ is Ω-big <u>then</u> $\mathrm{tp}(\bar b, \mathrm{acl}(A_2), M)$ is Ω-big.
3) A global bigness notion Γ is strict (or strictly nice) when: if $p = \mathrm{tp}(\bar a, A_2, M)$ is Γ-big and $A_\Gamma \subseteq A_1$ and $A_1 \subseteq A_2 \subseteq M$ <u>then</u> p has a strictly nice extension $\mathrm{tp}(\bar a', A_2, N), M \leq_{\mathfrak{k}} N$, which is Γ-big. We may omit the "nice" one and leave the "strictly".

Definition 5.9. For g-bigness notions Γ_1, Γ_2, we say Γ_1 is strictly orthogonal to Γ_2 or $\Gamma_1 \perp_s \Gamma_2$ when:

($*$) if $A_{\Gamma_1} \cup A_{\Gamma_2} \subseteq A \subseteq M \in \mathbf{K}, \mathrm{tp}(\bar a_\ell, A, M)$ is Γ_ℓ-big for $\ell = 1, 2$ <u>then</u> we can find N, $\bar a_1'$, $\bar a_2'$ such that:
- (α) $M \leq_{\mathfrak{k}} N \in \mathbf{K}$
- (β) $\bar a_1', \bar a_2' \subseteq N$
- (γ) $\mathrm{tp}(\bar a_\ell', A, N)$ extend $\mathrm{tp}(\bar a_\ell, A, N)$ for $\ell = 1, 2$
- (δ) $\mathrm{tp}(\bar a_\ell', A + \bar a_{3-\ell}', N)$ is Γ_ℓ-big for $\ell = 1, 2$
- ($\varepsilon)_1$ $\mathrm{tp}(\bar a_1', A + \bar a_2', N)$ is a strictly nice extension of $\mathrm{tp}(\bar a_1', A)$ that is: if $\Omega = \Gamma^{\mathrm{mt}}_{a,\mathrm{dis},\bar c}$, a bigness notion, $A_\Omega \subseteq \mathrm{acl}(A)$ and $\bar b \subseteq \mathrm{acl}(A + \bar a_1')$ and $\mathrm{tp}(\bar b, A, N)$ is Ω-big then $\mathrm{tp}(\bar b, A + \bar a_2', N)$ is Ω-big.

Claim 5.10. *Assume T has Skolem function (as in 1.1(B)) If $\Gamma_1 \perp_s \Gamma_2$ <u>then</u> $\Gamma_2 \perp_s \Gamma_1$, moreover we have*

$\otimes$ *in ($*$) inside Definition 5.9 above it follows that*

($\varepsilon)_2$ $\mathrm{tp}(\bar a_2', A + \bar a_1', N)$ *is strictly nice extension of* $\mathrm{tp}(\bar a_2', A, N)$, *i.e. if* $\Omega = \Gamma^{\mathrm{mt}}_{a,\mathrm{dis},\bar c}$ *a bigness notion,* $A_\Omega \subseteq \mathrm{acl}(A)$ *and* $\bar b \subseteq \mathrm{acl}(A + \bar a_2'), \mathrm{tp}(\bar b, A, N)$ *is Ω-big <u>then</u>* $\mathrm{tp}(\bar b, A + \bar a_1', N)$ *is Ω-big.*

So we can say "Γ_1, Γ_2 are strictly nicely orthogonal", i.e. this is symmetric relation.

Proof. Use 3.34(2). Let us prove $(\epsilon)_2$ from $\otimes$, so assume that $\Omega = \Gamma^{\mathrm{mt}}_{\mathrm{dis},\bar c}$, $\bar b \subseteq \mathrm{acl}(A + \bar a_2')$ is a counterexample, in particular $A_\Omega \subseteq A$ and

($*$) $\bar b \subseteq \mathrm{acl}(A + \bar a_2')$ is a counter example, so
- (a) $\mathrm{tp}(\bar b, \mathrm{acl}_{N_*}(A), N)$ is Ω-big
- (b) $\mathrm{tp}(\bar b, \mathrm{acl}_N(A + \bar a_1'), N)$ is Ω-small.

Now $\mathrm{tp}(\bar b, A_{\bar a_1})$ is Ω-small, but $\Gamma^{\mathrm{mt}}_{a,\mathrm{dis},\bar c}$ is a local bigness notion hence by 5.7 there is a formula $\vartheta(\bar y, \bar a_1') \in \mathrm{tp}(\bar b, A + \bar a_1')$, possibly with parameter from A, which is Ω-small, so there are $n < \omega$ and $i < \lg(\bar c)$ such that $\{\mathrm{dis}(\bar y_\ell, \bar y_m) \geq c_i : \ell < m < n\} \cup \{\vartheta(y_\ell, \bar a_1') : \ell \leq n\}$ is inconsistent and without loss of generality n is minimal.

Clearly, it follows that (we can add $\bigwedge_{k<\ell<n} \mathrm{dis}(\bar y_k, \bar y_\ell) \geq c_i$ but no need)

$$N \models (\exists \bar y_0 \ldots \bar y_{n-1}[\bigwedge_{\ell<n} \vartheta(\bar y_\ell, \bar a'_1) \wedge (\forall \bar y)(\vartheta_\ell(\bar y, \bar a_1) \rightarrow \bigwedge_{\ell<n} \mathrm{dis}(\bar y, \bar y_\ell) < c_i)].$$

As T has Skolem functions clearly there are $\bar b_0, \ldots, \bar b_{n-1} \subseteq \mathrm{acl}(A + \bar a'_1)$ such that $N \models \bigwedge_{\ell<n} \vartheta(\bar b_\ell, \bar a'_1) \wedge (\forall \bar y)(\vartheta_0, (\bar y, \bar a'_1) \rightarrow \bigwedge_{\ell<n} \mathrm{dis}(\bar y, \bar b_\ell) < c_i)$. We can substitute $\bar b$ for $\bar y$ so for some $\ell < n, \bar b^* \in \{\bar b_0, \ldots \bar b_{n-1}\}$ we have $N \models \mathrm{dis}(\bar b, \bar b^*) < c_i$.

Now first,

Case 1: if $\mathrm{tp}(\bar b^*, \mathrm{acl}_N(A), N)$ is not Ω-big

Then as in the previous sentences, for some $\bar b^{**} \subseteq \mathrm{acl}_N(A)$ and $j < \lg(\bar c)$ we have

$$N \models \mathrm{dis}(\bar b^*, \bar b^{**}) \leq c_j,$$

so together with the previous sentence $N \models \mathrm{dis}(\bar b, \bar b^{**}) \leq \max\{i, j\} + 1$, contradiction to $\mathrm{tp}(\bar b, \mathrm{acl}_N(A), N)$ is Ω-big.

Second,

Case 2: $\mathrm{tp}(\bar b^*, \mathrm{acl}_N(A), N)$ is Ω-big

We can still note that $\mathrm{tp}(\bar b^*, \mathrm{acl}_N(A + \bar a'_1), N)$ is not Ω-big contradicting $(\epsilon)_1$ of $(*)$ of 5.9 by 5.7.

Together we have gotten a contradiction. $\square_{5.10}$

Claim 5.11. *If Γ_1, Γ_2 are orthogonal g-bigness notions and Γ_1 is strict then Γ_1, Γ_2 are strictly orthogonal.*

Proof. Let $p_\ell \in \mathbf{S}^{\alpha(\Gamma_\ell)}(A, \mathfrak{C})$ for $\ell = 1, 2$ and $A_\Gamma \subseteq A$.

Now let $\lambda = |T| + |A| + |\alpha(\Gamma_1)| + |\alpha(\Gamma_2)| + \aleph_0 + |\{\Omega : \Omega$ is as in 5.9$(*)(\varepsilon)_2$ for A and we choose $A_\alpha, \bar a_{1,\alpha}$ by induction on $\alpha \leq \lambda^+$ such that:

$(*)_{1,\alpha}$ (a) $A_\alpha = \cup\{\bar a_{1,\beta} : \beta < \alpha\} \cup A$

(b) $\mathrm{tp}(\bar a_{1,\alpha}, A_\alpha, \mathfrak{C})$ is a Γ_1-big strictly nice extension of p_1.

This is possible by the assumption. We can find $\bar a_2$ realizing p_2 such that $\mathrm{tp}(\bar a_2, A_{\lambda^+}, \mathfrak{C})$ is Γ_2-big. It is enough to prove that for some $\alpha < \lambda^+$

$(*)_{2,\alpha}$ for every finite $\bar b \subseteq \mathrm{acl}(A + \bar a_{1,\alpha}, \mathfrak{C})$ and $\Omega = \Gamma^{\mathrm{mt}}_{a,\mathrm{dis},\bar c}$ with parameters from A, if $\mathrm{tp}(\bar b, \mathrm{acl}(A), \mathfrak{C})$ is Ω-big then $\mathrm{tp}(\bar b, \mathrm{acl}(A + \bar a_2), \mathfrak{C})$ is Γ-big.

Toward contradiction assume this fails for every α and let $(\Omega_\alpha, \bar b_\alpha, i_\alpha)$ witness this so $\Omega_\alpha = \Gamma^{\mathrm{mt}}_{a,\mathrm{dis},\bar c_\alpha}, i_\alpha < \ell g(\bar c_\alpha)$. By the choice of λ for some $\alpha < \beta < \lambda^+$ we have $(\Omega_\alpha, \bar b_\alpha, i_\alpha) = (\Omega_\beta, \bar b_\beta, i_\beta)$.

By transitivity of distance we get contradiction to $(*)_{1,\beta}(b)$. $\square_{5.11}$

Claim 5.12. *1) If T is as in 1.1(2) or at least T has Skolem function then every bigness notion is a strictly nice bigness notion.*
2) If the bigness notion Γ_1, Γ_2 are orthogonal then they are strictly orthogonal.

Proof. 1) By 3.35(2).
2) As in the proof of 1.18(3). $\square_{5.12}$

Now we should check our notion, and revise the construction in §4 and 5.2. For local bigness notions we get better results.

Claim 5.13. *Assume T is as in 1.1(2) ,(or just has Skolem function in a strong enough sense) and $\mathfrak{k} = (\mathbf{K}, \le) = (mod_T, \prec)$. Assume Γ is a local bigness notion, $A_\Gamma \subseteq A \subseteq B \subseteq N$ and $A = \mathrm{acl}_N(A), B = \mathrm{acl}_N(B), p = \mathrm{tp}(d, B, N) \in \mathbf{S}^{\alpha(\Gamma)}(B, N)$ is Γ-big and $\mathrm{acl}_N(A+\bar{d}) \cap B = A$, i.e. niceness, and $\Omega = \Gamma^{\mathrm{mt}}_{a,\mathrm{dis},\bar{c}}$ is a bigness notion with $A_\Omega \subseteq A$.*

Then we get strict niceness, and even

- (α) *if $\bar{b}_1 \in B$ and $\mathrm{tp}(\bar{b}, A, N)$ is Ω-big then $\mathrm{tp}(\bar{b}_1, \mathrm{acl}(A + \bar{d}), N)$ is Ω-big*
- (β) *if $\bar{b}_2 \in \mathrm{acl}_N(A + d)$ and $\mathrm{tp}(\bar{b}_2, A, N)$ is Ω-big then $\mathrm{tp}(\bar{b}_2, B, N)$ is Ω-big.*

Proof. For transparency we use singleton. We can find a two-place function $\breve{f} = \breve{f}_\omega$ definable in N such that:

$(*)_1$ (α) $\mathrm{Dom}(\breve{f}) = \{(d, c) : d\dot{\in}a, c\dot{\in}\mathbb{R}^+\}$

(β) $N \models (\forall y\dot{\in}a)[\mathrm{dis}(y, \breve{f}(y, c)) < c \,\&\, \breve{f}(y, c)\dot{\in}\mathbb{R}^+]$ for every $c \in (\mathbb{R}^+)^N$

(γ) $N \models (\forall y, y')[\breve{f}(y, c) = z \wedge \mathrm{dis}(y', z) < c/2 \rightarrow \breve{f}(y', \mathbf{c}, \zeta)]$ for every $c \in (\mathbb{R}^+)^N$.

[Why? Here we use "T as in 1.1(B) (and a is "a set, not a class"). That is, there is b for some $F_1 \in \tau_N$ such that $N \models$ "for every appropriate $a, \mathrm{dis}, c, F_1(a, c, \mathrm{dis})$ is a maximal subset of a such that $(\forall xy)[x \neq y\dot{\in}a \rightarrow \mathrm{dis}(x, y) \ge c]$". Clearly exist and let $F_2 \in \tau_M$ such that $N \models$ "for every appropriate $a, \mathrm{dis}, c, b\dot{\in}a, x = F_2(b, a, c, \mathrm{dis})$ is a member of $F_1^N(a, c, \mathrm{dis})$ such that $\mathrm{dis}(x, b) < c$", exists by the maximality of $F_1(a, c, \mathrm{dis})$. So $(b, c) \mapsto F_2(b, a, c, \mathrm{dis})$ is a function as required.]

Now if $A_\Omega \subseteq C \subseteq N$ then

- $(*)$ $\mathrm{tp}(d, b, C, N)$ is Ω-big <u>if and only if</u> for every large enough $i < \lg(\bar{c})$ we have $f_\Omega(b, c_i) \notin \mathrm{acl}_N(A)$.

$\square_{5.13}$

Claim 5.14. *1) In claim 3.19(1), (2) we can strengthen clause (γ) to*

- $(\gamma)^+$ $\mathrm{tp}(b', \mathrm{acl}_N(A'), N)$ *is a strict extension of* $\mathrm{tp}(a, A, M^*)$.

2) In claim 3.21 we can strengthen clause (β) to

- $(\beta)^+$ $\mathrm{tp}(b', \mathrm{acl}(A + b_2), \mathfrak{C})$ *is a strict extension of* $\mathrm{tp}(b_1, \mathrm{acl}(A), \mathfrak{C})$.

3) In 3.26(2) we can replace "nicely" (in clause (δ) of the conclusion) by strictly.

Proof. We can just use 5.13, (its assumption is O.K. for our application.
1) In the beginning of the proof of 3.19 we reduce it to the proof of $q^*(x)$ being $\Gamma^{\mathrm{ms}}_{a,\bar{c}}$-big, and there we get (2) by applying proof to $\bigcup_{i<\lambda} A'_i$; to get $(\gamma)^+$ we just need to require there that $\mathrm{tp}(f_i(A'), \mathrm{acl}(\bigcup_{j<i} f_j(A')), M^*)$ is strict.
2) Similarly.
3) Similarly: when we waive "nicely" we just replace "p_ζ nicely extend $\mathrm{tp}(d^*, A)$" by "p_ζ strictly extend $\mathrm{tp}(d^*, A)$". $\square_{5.14}$

Claim 5.15. *We can repeat §4, replace "g-bigness notions" by "strict bigness notions" (automatic if we use T^* and in (D)(7)), "nice extension" by "strict extension".*

Proof. Straightforward (or use 5.13(4)).

Proof of 5.6:

In the proof of 5.2 we replace nicely by strictly nice and as we work assuming 1.1(2), by 5.13(4), this holds automatically. The additional point is similar to Stage C. We add two contractors: the pre-pseudo branch killer $\zeta_{\rm pr}$, and the pseudo branch killer $\zeta_{\rm pk}$.

For $\beta \in W_{\zeta_{\rm pr}}$ of cofinality λ, assigned to our parameters (so the relevant parameters are in $\mathfrak{B}_\alpha$ for some $\alpha < \beta$), we demand

$\otimes_{1,\beta}$ for stationarily many $\epsilon \in S_{\zeta_{\rm pr}}$, for some $j \in u^{\beta+1}_{i_{\beta+1,\epsilon}}$, non-maximal in $u^{\beta+1}_{i_{\beta+1,\epsilon}}$, satisfying $j \geq \max(u^\beta_{i_{\beta,\epsilon}})$, we have:

- (a) $\bar{c}^{\beta+1} = \langle \bar{c}^{\beta+1,j}_n : n < \omega\rangle$ is indiscernible over $A^{\beta+1}_j$
- (b) $\bar{c}^{\beta+1,j}_n = \langle c^{\beta+1,j}_{n,i} : i < i(*)\rangle, i(*) < \lambda$
- (c) $\{c^{\beta+1,j}_{0,i} : i < i(*)\}$ is the universe of an (elementary) submodel of $\mathfrak{B}_\beta$, including $A^{\beta+1}_j \cap \mathfrak{B}_\beta$
- (d) $\mathfrak{B}_\beta \models$ "$c^{\beta+1,j}_{0,\ell} \dot{\in} \mathbb{N}$ and $m < 2c^{\beta+1,j}_{0,\ell} < c^{\beta+1,j}_{0,\ell+1} < \dot{n}$", $\dot{n} \in A^{\beta+1}_j \cap \mathfrak{B}_\beta, \mathfrak{B}_\beta \models$ "$m < \dot{n} \in \mathbb{N}$" for every true natural number m, all this for any $\ell < \omega$
- (e) $\mathfrak{B}_{\beta+1} \models$ "$c^{\beta+1,j}_{0,\ell_1} > c^{\beta+1,j+1}_{0,\ell_2}$ for $\ell_1, \ell_2 < \omega$
- (f) $\mathrm{tp}(\bar{c}^{\beta+1,j}_n, \bigcup_{\ell<n} \bar{c}^{\beta+1,j}_\ell \cup A^{\beta+1}_j)$ is a strict extension of $\mathrm{tp}(\bar{c}^{\beta+1}_n, A^{\beta+1}_j)$
- (g) $A^{\beta+1}_{j+1} \cap \mathfrak{B}_\beta = \mathrm{Rang}(\bar{c}^{\beta+1,j}_0)$.

For every $\alpha < \lambda^+$

$\otimes_{2,\alpha}$ if $\beta < \alpha$ and $\zeta_{\rm pr}$ acted in β for a pseudo tree as above, <u>then</u> for every $\breve{f} \in I[\mathfrak{B}_{\alpha+1}] \setminus I[\mathfrak{B}_\alpha]$, for stationary many $\epsilon \in S_{\zeta_{\rm pr}}$, for some $j < \lambda$ as in $\otimes_{1,\beta}$ we have:

(∗) (a) $\epsilon \in E_\beta \cap E_\alpha, \beta \in \mathbf{c}^\alpha_{i^\alpha_\epsilon}, j \in u^\alpha_{i^\alpha_\epsilon}$

(b) one of the following occurs:

(α) $\mathrm{tp}(\breve{f}, A^\alpha_{j+1}, \mathfrak{B}_{\alpha+1})$ is $\Gamma^{\rm mt}_{{\rm dis},(\bar{c}^{\beta,j}_0 \restriction \omega)}$-big

(β) there are $\ell < \omega$ and $\breve{f}' \in I[\mathfrak{B}_\alpha]$ such that, in $\mathfrak{B}_{\alpha+1}$:

(i) $\mathrm{dis}(\breve{f}', \breve{f}) \leq c^{\beta,j}_{0,\ell}$

(ii) for every $\breve{f}'' \in I[\mathfrak{B}_\beta]$ we have $dis(\breve{f}', \breve{f}'') > c^{\beta,j}_{0,\ell+1}$

(γ) there is $\breve{f}' \in A^\beta_j$ such that $\mathrm{dis}(\breve{f}, \breve{f}') \in \mathbb{N}$ is $< \dot{n}$ for every non-standard $\dot{n} \in \mathbb{N}[\mathfrak{B}_\beta]$.

Why can we do this? Having p^β_j first define p^β_{j+1} ignoring clause (b) of (∗), if it holds, fine; assume not. As (α) fail by observation 3.37 for some $\breve{f}' \in A^\alpha_{j+1}$ and $\ell < \omega$ we have: p^α_{j+1} says that "$\mathrm{dis}(\breve{f}, \breve{f}') \leq c^{\beta,j}_{0,\ell}$".

If $\mathrm{tp}(\breve{f}', A^\beta_{j+1}, \mathfrak{B}_\alpha)$ is $\Gamma^{\mathrm{mt}}_{\mathrm{dis},\bar{c}^{\beta,j}_0 \restriction \omega}$-big, possibly increasing ℓ we get possibility (β) as we are using the strict version in $(D)(7)$ of 4.5. So assume that $\mathrm{tp}(\breve{f}', A^\beta_{j+1}, \mathfrak{B}_\alpha)$ is $\Gamma^{\mathrm{mt}}_{\mathrm{dis},\bar{c}^{\beta,j}_0} \restriction w$-small. So using 3.37 again and transitivity for some $\breve{f}'' \in A^\beta_{j+1}$ (possibly increasing ℓ) we have $\mathrm{dis}(\breve{f}, \breve{f}'') < c^{\beta,j}_{0,\ell}$. We can find i such that $\breve{f}'' = c^{\beta,j}_{0,i}$ where $i < i^*$. So we can find q, another candidate for p^α_{j+1}, which say $\mathrm{dis}(\breve{f}, c^{\beta,j}_{1,i}) < c^{\beta,j}_{1,\ell}$ but recall $c^{\beta,j}_{1,\ell} < c^{\beta,j}_{0,\ell}$. So to get clause (β) it suffices to have in $\mathfrak{B}_\alpha$ that $\mathrm{tp}(c^{\beta,j}_{1,i}, A^\beta_{j+1})$ is $\Gamma^{\mathrm{mt}}_{\mathrm{dis},\bar{c}^{\beta,j}_0 \restriction \omega}$-big. By the choice of $\langle \bar{c}^{\beta,j}_n : n < \omega \rangle$ this holds except when $\breve{f}'' = c^{\beta,j}_{0,i} \in A^\beta_j$.

But then $\mathrm{dis}(\breve{f}, \breve{f}'')$ belongs to $A^{\beta+1}_j$, so it is a "$\mathfrak{B}_{\alpha+1}$-natural number", i.e. a members of $A^{\beta+1}_j \cap \mathfrak{B}_\beta \cap \mathbb{N}[\mathfrak{B}_{\beta+1}]$ smaller than all "non standard" member of $\mathbb{N}[\mathfrak{B}_\beta]$; as this holds for unboundedly many $\beta < \lambda$ clearly clause (γ) of $(*)(b)$ holds.

So having guaranteed the relevant $\otimes_{1,\beta}$, $\otimes_{1,\alpha}$, we apply it to a "branch" $\langle g(\dot{n}) : \dot{n} \in \mathbb{N}^a[\mathfrak{B}_{\lambda^+}] \setminus \{m : m < \omega\} \rangle$, for each $\beta < \lambda^+$ satisfying, $\mathrm{cf}(\beta) = \lambda$ and $\alpha \in (\beta, \lambda^+)$ such that $\mathrm{tp}(a^\alpha_0, \mathfrak{B}_\alpha, \mathfrak{B}_{\alpha+1})$ is $\Gamma^{\mathrm{av}}_{\dot{D},\langle n:n<\omega \rangle}$ and $\dot{n} \in \mathfrak{B}_\beta \Rightarrow (\dot{n}) \in \mathfrak{B}_\beta$.

So we can find $\breve{f}''_\beta \in I[\mathfrak{B}_\beta]$ such that $\mathrm{dis}(\breve{f}''_\beta, g(a^\alpha_0)) \in \mathbb{N}[\mathfrak{B}_{\lambda^+}]$ is smaller than all non standard $\mathbf{n} \in \mathbb{N}[\mathfrak{B}_\beta]$. By Fodor lemma for some $\breve{f}^*$ for stationary many β, $\breve{f}''_\beta = \breve{f}^*$. This finish the proof. $\square_{5.15}$

Remark 5.16. 1) It seems we can use guessing of clubs as in [She03] (more [She97]) and "$\check{\mathscr{D}}_\lambda$ is λ^+-saturated" (when $\lambda > \aleph_1$) to deal with 5.6, but the present look simpler and did not check.
2) We can also in stage C in the proof of 5.2 deal with weaker notions of trees (with distance instead equality).
3) By 5.2 and the previous chapter, we can conclude the compactness of the quantifier on complete embeddings of one boolean ring to another.
4) It is more natural for general T (in order to save claim 5.10) to replace Γ^{mt} in Definition 5.8,5.9 by Γ^{mt} as in the following definition.

Definition 5.17. Assume

(a) T is as in 1.1(1), i.e. a complete first order theory
(b) M^* is a model of T
(c) $\bar{\varphi} = (\varphi_1(x), \varphi_1(x), \varphi_2(x,y), \varphi_3(xy,z), \varphi_4(x,y,z)) = (\varphi_{\mathrm{dom}}(x), \varphi_{\mathrm{rang}}(x), x \le y, \varphi^{\mathrm{dis}}_{\le z}(x,y), \varphi_{\mathrm{subadd}}(x,y,z))$ are first order formulas (possibly with parameters), or φ (the intended meaning of $\varphi_{\mathrm{dom}}(M^*)$ is the "space, and of $\varphi_{\mathrm{rang}}(M)$ the possible distances
(d) the formula $\varphi^{\mathrm{dis}}_{\le z}(x,y)$ is such that:

$$\varphi^{\mathrm{dis}}_{\le z}(x,y) \Rightarrow \varphi_{\mathrm{dom}}(x) \,\&\, \varphi_{\mathrm{dom}}(y) \,\&\, \varphi_{\mathrm{rang}}(z) \text{ and } \mathrm{dis}_{\le z_1}(x,y) \& z_1 \le z_2 \Rightarrow \mathrm{dis}_{\le z_2}(x,y)$$

(the intended meaning of $\varphi^{\mathrm{dis}}_{\le z}(x,y)$ is the distance from x to y is $\le z$)

(e) $\varphi_{\mathrm{subadd}}(x,y,z)$ is a first order formula such that

$$\begin{aligned}\varphi_{\mathrm{subadd}}(x,y,z) \Rightarrow \varphi_{\mathrm{rang}}(x) \quad & \& \,\varphi_{\mathrm{rang}}(y) \,\&\, \varphi_{\mathrm{rang}}(z) \,\& \\ & x \le z \wedge y \le z \forall x,y(\varphi_{\mathrm{rang}}(x) \wedge \varphi_{\mathrm{rang}}(y) \to (\exists z)\varphi_{\mathrm{subadd}}(x,y,z))\end{aligned}$$

the (intended meaning of $\varphi_{\mathrm{subadd}}(x,y,z)$ is $x + y \le z$)

(f) $\varphi^{\rm dis}_{\leq z_1}(x_1 x_2) \wedge \varphi^{\rm dis}_{\leq z_2}(x_2, x_3) \wedge \varphi_{\rm subadd}(z_1, z_2, z_3) \Rightarrow \varphi^{\rm dis}_{\leq z_3}(x_1, x_3)$

(g) $x_1 \leq x_2$, is a partial directed order of $\varphi_{\rm Mod}(M^*)$

(h) $\bar{c} = \langle c_i : i < \delta \rangle, c_1 \in \varphi_{\rm rang}(M^*), i < j \Rightarrow \varphi_{\rm subadd}(x_i, x_i, x_{i+}1)$.

We define $\Gamma = \Gamma^{\rm mt}_{\bar{\varphi}, \bar{c}}$ as the following local bigness notion: $\vartheta(x, a)$ is Γ-big (in $N, M^* \prec N$) if and only if

$$\{\vartheta(x_n) : n < \omega\} \cup \{\neg \varphi^{\rm dis}_{\leq c_i}(x_n, x_m) : n \neq m < \omega\}$$

is consistent.

Remark 5.18. More generally we can use any dependency relation.

§ 6. Constructing models in $\aleph_1$ under CH

This section has little dependence on the earlier parts. In Rubin Shelah [RS80] models in $\aleph_1$ were constructed using: in two cases $\diamondsuit_{\aleph_1}$ and in one CH. Here we prove all of them under CH and get further results (for example the results on ordered fields). The construction here was promised in [She83b]. The omission of types in 6.2 continue [RS80], hence Keisler [Kei70] or [Kei71] which deal with the quantifier $\exists^{\geq \aleph_1}$.

Context 6.1. T^* as in the context 1.1(B) T^* countable. M^* a countable model of T^* (usually well founded), with universe ω for simplicity.

Definition 6.2. 1) For a formula $\varphi(x,\bar y)$ (here first order) and term σ of T^* we shall define the formula $(\dot{\mathbf{Q}}^{\mathrm{id}}_\sigma x)\varphi(x,\bar y)$, assuming σ is a term whose set of free variables does not include x and for notational simplicity is $\subseteq \bar y$, so we can write $\sigma(\bar y)$; now $(\dot{\mathbf{Q}}^{\mathrm{id}}_{\sigma(\bar y)}x)\varphi(x,\bar y)$ means "$\sigma=\sigma(\bar y)$ is an $\aleph_1$-complete ideal[9] and $\{x\dot\in \mathrm{Dom}(\sigma) : \varphi(x;\bar y)\}\dot\in\sigma(\bar y)$ <u>or</u> $\sigma(\bar y)$ is not an $\aleph_1$-complete ideal".
2) If M is a model of T^*, p a type over M (i.e., a set of formulas $\varphi(\bar x,\bar a)$, $\bar a\subseteq M$, $\bar x$ a fixed finite sequence) we say: p is suitably omitted by M <u>if</u>:

(∗) if $\bar b\subseteq M^*$, $n<\omega$, and $M\models(\dot{\mathbf{Q}}_{\sigma_n}y_n)\dots(\dot{\mathbf{Q}}_{\sigma_1}y_1)(\exists\bar x)\psi(\bar x,y_1\dots y_n,\bar b)$, <u>then</u> for some $\varphi(\bar x,\bar a)\in p$, $M\models(\dot{\mathbf{Q}}_{\sigma_n}y_n)\dots(\dot{\mathbf{Q}}_{\sigma_1}y_1)(\exists\bar x)[\psi(\bar x,y_1,\dots,y_n,\bar b)\,\&\,\neg\varphi(\bar x,\bar a)]$.

3) Let M be a model of T^*, and Δ be a finite set of formulas of the form $\varphi(\bar x;\bar y_\varphi)$, p a set formulas of the form $\varphi(\bar x,\bar a)$, where $\bar a\subseteq M$, $\varphi(\bar x;\bar y)\in\Delta$. We say that p is strongly undefined over M <u>if</u> there are no sequences $\langle\psi_\varphi(\bar y_\varphi,\bar c_\varphi):\varphi(\bar x,\bar y_\varphi)\in\Delta\rangle$ where $\bar c_\varphi\subseteq M$ such that:

$$\varphi(\bar x,\bar a)\in p\Rightarrow M\models\psi_\varphi[\bar a,\bar c_\varphi],$$

$$\neg\varphi(\bar x,\bar a)\in p\Rightarrow M\models\neg\psi_\varphi[\bar a,\bar c_\varphi].$$

Observation 6.3. *1) In 6.2(3) if p is strongly undefined type over M, <u>then</u> it is suitably omitted over M; if p is suitable omitted by M <u>then</u> p is omitted by M.*
2) If $a\in M$ and $A\subseteq\{b:M\models b\dot\in a\}$ is not represented in M, <u>then</u>

$$p:=\{x\subseteq a\,\&\,[b\dot\in x]^{\mathrm{if}(b\in A)}:M\models b\dot\in a\}$$

is strongly undefined in M.
3) If $M_n\prec M_{n+1}$ for $n<\omega, p$ a type over M_0 suitably omitted by M_n for each n, <u>then</u> p is suitably omitted by $\bigcup\limits_{n<\omega}M_n$.

Construction 6.4. We describe a construction of an elementary extension M^{**} of M^* of cardinality $\aleph_1$; we leave some points for latter fulfillment.

<u>**Step A**</u>: Let $\chi>\aleph_1$, and $x\in\mathscr{H}(\chi)$ be given. Let $\tau_*\in\mathscr{H}(\chi)$ be a countable vocabulary extending τ_{T^*}, having infinitely many n-place predicates and n-place function symbols in τ_{T^*} for each $n<\omega$.

<u>**Step B**</u>: We choose a list $\bar\eta=\langle\eta_\alpha:\alpha<\omega_1\rangle$ of ${}^{\omega_1>}2$, such that

[9] We can restrict ourselves to a class R (of T^*) or allow non-first order φ, and get other variants. If we would like to have $|\omega^M|=\aleph_1$, we start with a $\aleph_1$-saturated model (by section 5) and apply the theorems below to it.

$$[\eta_\alpha \triangleleft \eta_\beta \Rightarrow \alpha < \beta].$$

So $\langle\rangle = \eta_0$, and for simplicity: $\eta_\alpha{}^\frown\langle\ell\rangle \in \{\eta_{\alpha+k} : 0 < k < \omega\}$ for $\ell = 1, 2$, and

$$\lg(\eta_\alpha) \text{ is a limit ordinal } \equiv \alpha \text{ is limit.}$$

Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a partition of ω_1 to pairwise disjoint stationary sets such that $\min(S_\alpha) > 1 + \alpha$ and each S_α is non-small, see [Shed, 3.1(2)] and history there.

By induction on $\alpha < \omega_1$ we choose N_α and M_{η_α} and $\beta(\alpha), \mathbf{G}_{\eta_\alpha}$ such that:

- (a) $N_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$, $\alpha \subseteq N_\alpha$, $\{x, \bar\eta\} \in N_\alpha$, $\langle N_\gamma : \gamma \leq \beta\rangle \in N_\alpha$ for $\beta < \alpha$ and $\langle M_{\eta_\gamma}, G_{\eta_\gamma}, b_{\eta_\gamma} : \gamma \leq \beta\rangle \in N_\alpha$ for $\beta < \alpha$,
- (b) the sequence $\langle N_\beta : \beta \leq \alpha\rangle$ is increasing and continuous, each N_α is countable,
- (c) M_{η_β} is a model of T^*, with universe $\omega(1 + \lg(\eta_\beta)), M_{\langle\rangle} = M^*$ (but for convenience β as a member of those models is called a_β),
- (d) for $\eta_\alpha \triangleleft \eta_\beta$, then $M_{\eta_\alpha} \prec M_{\eta_\beta}$, and if p belongs to N_β and is a suitably omitted type over M_{η_α}, then M_{η_β} suitably omits it too,
- (e) if $\lg(\eta_\alpha)$ is a limit ordinal, then $M_{\eta_\alpha} = \bigcup\{M_{\eta_\alpha\restriction i} : i < \lg(\eta_\alpha)\}$,
- (f) if $\lg(\eta_\alpha) = \gamma + 1$, $\gamma \in S_{\beta(\gamma)}, M_{\eta_\alpha\restriction\gamma} \models$ "$a_{\beta(\gamma)}$ is an $\aleph_1$–complete ideal on $\mathrm{Dom}(a_{\beta(\gamma)}) = \bigcup\{y : y\dot{e}a_{\beta(\gamma)}\}$", <u>then</u> let $y_{\eta_\alpha} = a_{\beta(\gamma)}$; otherwise let y_{η_α} be the ideal of non-stationary subsets of ω_1 in the sense of $M_{\eta_\alpha\restriction\gamma}$,
- (g) if $\lg(\eta_\alpha) = \gamma + 1$, then
  - (i) $\mathbf{G}_{\eta_\alpha}$ is an N_α-generic subset of $\mathbb{P}_{\eta_\alpha}$, where:
    - $\mathbb{P}_{\eta_\alpha} := \{\varphi : \varphi = \varphi(x_{\omega(1+\alpha)}, x_{\omega(1+\alpha)+1}, \ldots, x_{\omega(1+\alpha)+n}; \bar b)$ for some $n < \omega, \bar b \subseteq M_{\eta_\alpha\restriction\gamma}$ and $M_{\eta_\alpha\restriction\gamma} \models (\dot{\mathbf{Q}}_{y_{\eta_\alpha}} x_{\omega(1+\alpha)})(\exists x_{\omega(1+\alpha)+1}) \ldots (\exists x_{\omega(1+\alpha)+n})\varphi\}$
    - the order on $\mathbb{P}_{\eta_\alpha}$ is naturally defined
  - (ii) $\mathbf{G}_{\eta_\alpha} = \{\varphi(x_{\omega(1+\alpha)}, \ldots, x_{\omega(1+\alpha)+n}, \bar b) : n < \omega,\ \bar b \subseteq M_{\eta_\alpha\restriction\gamma}$ and $M_{\eta_\alpha} \models \varphi[a_{\omega(1+\alpha)}, \ldots, a_{\omega(1+\alpha)+n}; \bar b]\}$.

There is no problem to carry out the construction.

Note: in order to have a good definition of "suitably omitted", we restrict the family $\mathbb{P}_{\eta_\alpha}$ to be quite well defined, loosing some cases.

Lastly,for $\nu \in {}^{\omega_1}2$, for $M_\nu = \bigcup\limits_{i<\omega_1} M_{\nu\restriction i}$.

<u>**Step C**</u>: We choose $N_{\omega_1} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$, $\|N_{\omega_1}\| = \aleph_1$, $\langle N_\alpha, M_{\eta_\alpha} : \alpha < \omega_1\rangle \in N_{\omega_1}$, and $\omega_1 \subseteq N_{\omega_1}$. Let $\langle \bar f_i : i < \omega_1\rangle$ list the sequences from N_{ω_1} of length $\leq \omega$ of functions $f \in N_{\omega_1}$ such that $\mathrm{Rang}(f) \subseteq \{0, 1\}$ and (where τ_* is from Step A)

$$\mathrm{Dom}(f) = \{(\eta_\alpha, M^+_{\eta_\alpha}) : \quad \alpha < \omega_1, M^+_{\eta_\alpha} \text{ is an expanion of } M_{\eta_\alpha} \text{ by at most countably many relations and functions from } \tau_*\}.$$

Let $\bar f_i = \langle f_{i,n} : n < \alpha_i\rangle, \alpha_i \leq \omega$. We shall choose $n(i) < \alpha_i$ and let $f_i = f_{i,n(i)}$.

Let $\langle S'_i : i < \omega_1\rangle$ be a partition of ω_1 to stationary subsets, and if $2^{\aleph_0} < 2^{\aleph_1}$ non-small sets and, if S_α is not small, then even $S'_i \cap S_\alpha$ is not small (as above, [Shed, 3.1(2)].

For each i let $\langle l^i_\alpha : \alpha \in S'_i \rangle$ be such that for $j < \omega_1$, $\langle \ell^i_\alpha : \alpha \in S'_i \cap S_j \rangle$ is a weak diamond for f_i (see [Shed, 3.1(2)]). So for every $\nu \in {}^{\omega_1}2$ and expansion M^+_ν of M_ν by at most countably many relations and functions from τ_*,

$$\{\beta \in S'_i : \nu \restriction \beta = \eta_\alpha \text{ and } f_i(\eta_\alpha, M^+_\nu \restriction (\omega(1+\alpha))) = \ell^i_\alpha\}$$

is stationary.

Choose $\nu^* \in {}^{\omega_1}2$ such that $\nu^*(\alpha) = 1 - \ell^i_\alpha$ if $\alpha \in S'_i$ is not a successor ordinal. Let $M^{**} = M_{\eta_{\nu^*}}$.

Claim 6.5. *1) For $\eta \in {}^{\omega_1 \geq}2$, M_η has the same natural numbers as M^*, but when $\lg(\eta)$ is ω_1 or just is large enough, ${\omega_1}^{M_\eta}$ is not well ordered.*
2) For $\nu \in {}^{\omega_1}2$ and $a \in M_\nu$ we have

- *$M_\nu \models$ " a is countable" iff $\{b : M_\nu \models$ "$b \dot{\in} a$"$\}$ is countable (in particular M^{**} satisfies this).*

3) For $\nu \in {}^{\omega_1}2$, if $M_\nu \models$ "a is an uncountable set", then for stationary many $\alpha, M_{\nu \restriction (\alpha+1)} \models$ "a_α is a countable subset of a and $b \dot{\in} a_\alpha$", whenever $M_{\nu \restriction \alpha} \models$ "$b \dot{\in} a$".
4) If $M_\nu \models$ "a is a stationary subset of ω_1", then the set $\{\alpha < \omega_1 : M_\nu \models$ 'a_α is an ordinal and $a_\alpha \dot{\in} a$" and $\bigwedge_{b \in M_\nu} [b \dot{\in} a_\alpha \Rightarrow b \in M_{\nu \restriction \alpha}]\}$ is a stationary subset of ω_1.
5) Moreover, for stationary many α, a_α satisfies: $M_\nu \models$ "a_α is countable", and $M_\nu \models$ "$b \dot{\in} a_\alpha$" $\Leftrightarrow b \in M_{\nu \restriction \alpha}$.

Proof. Straightforward. $\square_{6.5}$

Claim 6.6 (CH)**.** *Assume $\omega^{M_{\langle\rangle}}$ is well ordered.*
1) If $\nu \in {}^{\omega_1}2, M_\nu \models$ "$|a| = \aleph_0$ and y is a family of subsets of a", then $M_\nu \models$ "y is non-meagre" iff $\{\{x : M_\nu \models x \dot{\in} b\} : b \dot{\in} y\}$ is a non-meagre subset of the power set of $\{x : M_\nu \models x \dot{\in} a\}$.
2) Assume $\nu \in {}^{\omega_1}2$ and $M_\nu \models$ "$\mathbf{b}_1$ is a Boolean ring of subsets of a including the singletons, $|a| = \aleph_0$, $\mathbf{b}_2$ is a Boolean ring, and $\mathbf{b}_1$ is not meagre". Then every complete embedding $\mathbf{f}$ of $\mathbf{b}_1^{M^\nu}$ into $\mathbf{b}_2^{M^\nu}$ is represented in M_ν.
3) Assume $\nu \in {}^{\omega_1}2$ and $M_\nu \models$ "$\mathbf{b}_1$ is a Boolean ring of subsets of a_1 including all the finite ones, $a \subseteq a_1, |a| = \aleph_0, \{b \cap a : b \in \mathbf{b}_1\}$ is a non-meagre family of subsets of a and $\mathbf{b}_2$ is a Boolean ring". Then for every embedding $\mathbf{f}$ of $\mathbf{b}_1^{M_\nu}$ into $\mathbf{b}_2^{M_\nu}$ the following condition is satisfied:

- $(*)$ *for some $\dot{g} \in M_\nu$, we have:*
  - *(i) $\dot{g}$ is a function with domain a (in M_ν),*
  - *(ii) for $b \dot{\in}^{M_\nu} a, \dot{g}(b)$ is an ideal of $\mathbf{b}_2$,*
  - *(iii) $b_1 \dot{\in}^{M_\nu} a, b_2 \dot{\in}^{M_\nu} a, b_1 \neq b_2 \Rightarrow \dot{g}(b_1) \cap \dot{g}(b_2) = \{0_{\mathbf{b}_2}\}$,*
  - *(iv) for $b \dot{\in}^{M_\nu} a, \mathbf{f}(\{b\}) \dot{e}^{M_\nu} \dot{g}(b)$.*

Proof. 1) Check.
2) Follows by 3).
3) By CH (and the choice of the N_α's), for some $\alpha_0, \{a, \mathbf{f} \restriction \{\{b\} : b \dot{\in} a\}\} \in N_{\alpha_0}$, $\eta_{\alpha_0} \lhd \nu$, and all parameters are in $M_{\eta_{\alpha_0}}$. Let $\mathbf{b}'_1 = \{b \cap a : b \in \mathbf{b}_1\}$, and let $\dot{g} \in M_\nu$ be the function from $\mathbf{b}_1$ to $\mathbf{b}'_1$ such that $\dot{g}(b) = a \cap b$. For some $\alpha \in (\alpha_0, \omega_1)$, $y_{\nu \restriction \alpha}$ is (in M^*_ν) the ideal of meagre subsets of $\mathbf{b}'_1 \subseteq \mathscr{P}(a)$ included in $\mathbf{b}'_1$ ($\mathbf{b}'_1 \notin y_{\nu \restriction \alpha}$ as $\mathbf{b}_1$ is not meagre).

We now note that if $\eta_{\beta_\ell+1} = \eta_\alpha\hat{}\langle\ell\rangle$, then $M_{\eta_{\beta_\ell+1}}$ cannot suitably omit

$$p := \{[\mathbf{b}_2 \models \text{"}\mathbf{f}(b) \leq y \ \&\ f(c) \cap y = 0\text{"}] \ \&\ y\dot{e}\mathbf{b}_2 : \mathbf{b}_1[M_{\eta_{\beta_\ell+1}}] \\ \models \text{"}b \leq a_{\beta_\ell} \ \&\ a_{\beta_\ell} \cap c = 0_{\mathbf{b}_1}\text{"}, \text{ and } b, c \in B_1[M_{\eta_{\beta_\ell+1}}] \text{ are atoms } \subseteq a)\}.$$

(as in $M_\nu, \mathbf{f}(a_{\beta_\ell})$ realizes it). Hence, there is a suitable support

$$\dot{\mathbf{Q}}_{\sigma_0} x_0 \ldots \dot{\mathbf{Q}}_{\sigma_{n-1}} x_{n-1} \, \exists x \, \varphi(x, x_0, \ldots, x_{n-1}, a_{\beta_\ell}, \bar{b})$$

of $p, \bar{b} \subseteq M_{\eta_\beta}$. So some $t \in \mathbf{G}_{\eta_{\beta_\ell+1}}$ forces this (for $\not\leq_{\eta_{\beta_\ell+1}}$ see clause (g) of 6.4). Using this t we can define r as required. $\square_{6.6}$

Discussion 6.7. What occurs in 6.6 if we omit the assumptions "$\omega^{M_{\langle\rangle}}$ is well founded"? We should replace "finite" by "finite in the sense of M_ν", in particular (see 6.6(2)) for every complete embedding $\mathbf{f}^*$ of $\mathbf{b}_1[M_\nu]$ into $\mathbf{b}_2[M_\nu]$ for some $a', \breve{f}' \in M_\nu$, $M_\nu \models$ "a' is a finite subset of a, $b\dot{e}\mathbf{b}_2, \mathbf{f}$ a complete embedding of $\mathbf{b}_1 \restriction \{x\dot{e}\mathbf{b}_1 : x \cap a = 0_{\mathbf{b}_1}\}$ into $\mathbf{b}_2 \restriction \{y\dot{e}\mathbf{b}_2 : y \cap b = 0_{\mathbf{b}_2}\}$ and $\breve{f}^{M_\nu} = \mathbf{f} \restriction \mathrm{Dom}(\mathbf{f}^{M_\nu})$".

The next claim says that for $\nu_0 \neq \nu_1 \in {}^{\omega_1}2$, the models M_{ν_0}, M_{ν_1} has "very little in common over $M_{\nu_0 \cap \nu_1}$".

Claim 6.8. *1) Assume $\eta \in {}^{\omega_1>}2, \eta\hat{}\langle l\rangle \lhd \nu_l \in {}^{\omega_1}2$, and for each $n < \omega : M_\eta \models$ "$\dot{T}$ is a tree with set of levels $(W, \leq_W)$ which is an $\aleph_1$-directed partial order, a_n a level of the tree, $a_n <_W a_{n+1}$", and $\{a_n : n < \omega\}$ is cofinal in $(W, \leq_W)^{M_\eta}$. If, for $n < \omega$,*

$$M_\eta \models \text{`}b_n \dot{e} \dot{T} \text{ is in level } a_n\text{"}$$

and, for $\ell < \omega$,

$$M_{\nu_\ell} \models \text{"}b^\ell \dot{e} \dot{T} \ \&\ b_n \leq_{\dot{k}} \ b^\ell \leq_k b^{\ell+1}\text{"}$$

<u>then</u> for some c,

$$M_\eta \models \text{`}c \text{ is a branch of } \dot{T} \text{ with uncountable cofinality and } b_n \in c\text{"}$$

for $n < \omega$.
2) If A_1, $A_2 \subseteq M_\eta$ are disjoint and no (first order) formula (with parameters in M_η) separates them, $\eta \in {}^{\omega_1>}2$, $\eta\hat{}\langle\ell\rangle \lhd \nu_\ell \in {}^{\omega_1\geq}2$, and $A_1 \cup A_2 \in N_{\lg(\eta)+1}$ (for example $A_1 \cup A_2$ is represented in M_η), <u>then</u> for at least one ℓ, in M_{ν_l} no formula separates them.

Remark 6.9. Note: if $A_1 \cup A_2 = \{b : M_\eta \models b\dot{e}a\}$, then: [$A_1$, A_2 not separated in M_ν] means [A_1 not represented in M_ν].

Proof. 1) Let $\eta = \eta_{\alpha(0)}$. Assume that there is no c as required. We prove by induction on $\alpha \in [\alpha(0), \omega_1]$ the statement when we replace M_{ν_l} by $\bigcup\{M_{\eta_\beta} : \beta \leq \alpha$ and $\eta_\beta \lhd \nu_\ell\}$. This is enough – for $\alpha = \omega_1$ we get the result.
For $\alpha = \alpha(0)$ this is trivial.
For α limit - nothing new arises.
The only case we have to prove something is $\eta_\alpha \lhd \nu_\ell$, α a successor. We can consider all the countably many possible $b^{1-\ell} \cup \{M_{\eta_\beta} : \beta < \alpha$ and $\eta_\beta \lhd \nu_{1-\ell}\}$, so $\langle b_n : n < \omega\rangle$ is determined up to $\aleph_0$ possibilities, as really the identity of $\langle b_n : n < \omega\rangle$ is not important just the branch which $\langle b_n : n < \omega\rangle$ determines and all those branches

belong to $N_{\alpha-1}$. So the type $p_0 = \{x \dot{e} \dot{T} \wedge b_n <_{\dot{T}} x : n < \omega\} \in N_\alpha$, and we just have to prove that it is omitted. Let η_β be the ($\triangleleft$–) predecessor of M_{η_α}. By the induction hypothesis, M_{η_β} omits p_0; if we fail, by the construction it is not omitted by M_{η_β}. But omitting p_0 is equivalent to omitting

$$p = \{x \dot{e} \dot{T} \,\&\, [b <_{\dot{T}} x]^{\text{if}[\bigvee_n b < b_n]} : M_{\eta_\beta} \models \text{“} b \dot{e} \dot{T} \text{”}\},$$

so by 6.3(1) the type p is not strongly undefined. But by T^*'s choice this means it is represented in M_{η_β}, a contradiction.
2) Same proof. $\square_{6.8}$

Conclusion 6.10. *1) If $M_{\nu^*} \models$ “$\dot{T}$ is a tree with δ levels, $\mathrm{cf}(\delta)$ is regular uncountable”, <u>then</u> every full branch of $\dot{T}^{M_{\nu^*}}$ (i.e., a linear ordered subset which has members in an unbounded set of levels) is represented in M_{ν^*}.*
2) The set of levels of $\dot{T}$ can be partially ordered as long as it is $\aleph_1$–directed (in the sense of T^*), and we get the same result.

Proof. 1) By 6.8 + 6.4 Step C, i.e., consider expansions of N_ν by a branch B of $\dot{T}^{N_\nu}$ (i.e., a unary relation). Pick i such that $f_i(\eta_1, (M_{\eta_\alpha}, B)) = 0$ iff for some $\nu \in {}^{\omega_1}2$, $\eta_\alpha{}^\frown\langle 0\rangle \triangleleft \nu$ and B', a full branch of $\dot{T}^{M_\nu}$, $(M_{\eta_a}, B) \prec (M_\nu, B')$.
2) Similar. $\square_{6.10}$

Definition 6.11. For an atomic Boolean ring $\mathbf{B}$:
1) $\mathbf{B}$ is non-meagre, if, identifying $b \in \mathbf{B}$ with $\{x : x \in \mathbf{B}^{\mathrm{at}}; x \leq_{\mathbf{B}} b\}$, $\mathbf{B}$ is a non-meagre family of subsets of $\mathbf{B}^{\mathrm{at}}$ ($\mathbf{B}^{\mathrm{at}}$ is the set of “atoms” of $\mathbf{B}$), i.e., $\mathbf{B}$ can be represented as a countable union $\bigcup\limits_{n<\omega} Y_n$, each Y_n nowheredense (i.e., for every finite $a_1 \subseteq a_2 \subseteq \mathbf{B}^{\mathrm{at}}$ there are finite $b_1 \subseteq b_2 \subseteq \mathbf{B}^{\mathrm{at}}$ such that $a_1 \subseteq b_1, a_2 \setminus a_1 \subseteq b_2 \setminus b_1$, and for no $c \in \mathbf{B}$ do we have $\cap b_2 = b_1$).

Observation 6.12. *1) If $Y \subseteq \mathscr{P}(X)$ (i.e., is a family of subsets of X), $X' \subseteq X$, and $\{y \cap X' : y \in Y\}$ is a meagre subset of $\mathscr{P}(X')$, then Y is a meagre subset of $\mathscr{P}(X)$.*
2) If Y is a meagre (or nowheredense) subset of $\mathscr{P}(X)$, <u>then</u> the set $\{a \subseteq X : a$ is countable, $\{y \cap a : y \in Y\}$ is meagre (or nowheredense) subset of a $\mathscr{P}(a)\}$ is a club of $[X]^{\aleph_1} = \{a \subseteq X : |a| = \aleph_0\}$.

Question 6.13. Phrase the statement which suffices for the proof instead CH (it seems the existence of a non-meagre set of cardinality $\aleph_1$ suffices).

Conclusion 6.14. *Assume $\omega^{M_{\langle\rangle}}$ is well ordered, and suppose $M_{\nu^*} \models$ “$\mathbf{b}_1$ is a Boolean ring of subsets of a including the singletons, $|a| > \aleph_0$, $\mathbf{b}_1$ is non-meagre and $\mathbf{b}_2$ is a Boolean algebra”. <u>Then</u> every complete embedding of $\mathbf{b}_1$ into $\mathbf{b}_2$ is represented in M_{ν^*}.*

Proof. In M_{ν^*} let $(W, \leq)$ be the set of countable subsets of $\mathbf{b}_1^{\mathrm{at}} \cup \mathbf{b}_2$ ordered by inclusion. We define a tree $\dot{t}$ with W as a set of levels by:

$$\dot{t} = \{(c, f) : c \dot{e} W \text{ and } f \text{ is a function from } c \cap \mathbf{b}_1^{\mathrm{at}} \text{ into } \mathbf{b}_2\}.$$

We define the order of $\dot{t}$ by $(c_1, f_1) \leq (c_2, f_2) \Leftrightarrow c_1 \subseteq c_2 \,\&\, f_1 \subseteq f_2$ (defined in M_{ν^*}).

Now, if $\mathbf{f}$ is a complete embedding of $\mathbf{b}_1^{M_{\nu^*}}$ into $\mathbf{b}_2^{M_{\nu^*}}$, then for a club E of $\alpha < \omega_1$, the restriction $\mathbf{f} \restriction \mathbf{b}_1[M_{\nu^* \restriction \alpha}]$ is a complete embedding of $\mathbf{b}_1[M_{\nu^* \restriction \alpha}]$ into $\mathbf{b}_2[M_{\nu^* \restriction \alpha}]$ (see [Shee]). Let the ordinal γ be such that

$$M_{\nu^*} \models \text{"}a_\gamma = \text{ the non-stationary ideal on } [\mathbf{b}_1[M_{\nu^*}] \cup \mathbf{b}_2[M_{\nu^*}]^{\aleph_0}\text{"}.$$

So for $\alpha \in E \cap S_\gamma$, $\{b : M_{\nu^*} \models b\dot{e}a_{\omega(1+\alpha)}\}$ is exactly $(\mathbf{b}_1[M_{\nu^*}] \cup \mathbf{b}_2[M_{\nu^*}]) \cap M_{\nu^* \restriction \alpha}$.

Now we apply 6.8(2) with $\mathbf{f}, a_{\omega(1+\alpha)} \cap \mathbf{b}_1^{\rm at}[M_{\nu^* \restriction \alpha}], \mathbf{b}_1^{\rm at}$, $\mathbf{b}_1$, $\mathbf{b}_2$ here standing for $\mathbf{f}$, a, a_1, $\mathbf{b}_1$, $\mathbf{b}_2$ there. We get $\dot{g} = \dot{g}_\alpha \in M_{\nu^*}$ as there. But $\alpha \in E$, so $\dot{g} = \mathbf{f} \restriction \mathbf{b}_1^{\rm at}[M_{\nu^* \restriction \alpha}]$. Clearly $\dot{g}_\alpha \dot{e} \dot{t}_{a_{\omega(1+\alpha)}}$ and for $\alpha < \beta$ from $E \cap S_\gamma$ we have: $(W, \leq) \models$ "$a_{\omega(1+\alpha)} \leq a_{\omega(1+\beta)}$", $\dot{t} \models$ "$\dot{g}_\alpha \leq \dot{g}_\beta$"; and $\{a_{\omega(1+\alpha)} : \alpha \in E \cap S_\gamma\}$ is cofinal in $(W, \leq)$, and $\{\dot{g}_\alpha : \alpha < \omega_1\}$ induce a branch of $\dot{t}$.

By 6.10(2) we finish. $\square_{6.14}$

Conclusion 6.15. *Assume $\omega^{M_{\langle\rangle}}$ is well ordered and*

$M_{\nu^*} \models$ "
(a) $\mathbf{b}_1$ *is a Boolean ring,*
$\boldsymbol{\Xi}$ *is a family of maximal antichains of* $\mathbf{b}_1$,
(b) *for* $\Xi \in \boldsymbol{\Xi}$, *the sub-algebra sub-Boolean ring*
$\mathbf{b}_1^{[\Xi]} = \{x \in \mathbf{b} :$ *for every* $y \in \Xi, x \cap y = 0 \vee x \cap y = y\}$
$\mathbf{b}_1$ *is non meagre, i.e. essentially as a family of subsets of* $b \equiv$
(c) $\boldsymbol{\Xi}$ *is* $\aleph_1$*-directed (order: bigger means finer),*
(d) *for every* $x \in \mathbf{b}_1 \setminus \{0_B\}$, *there are* $\Xi \in \boldsymbol{\Xi}$, *and* $y \in \Xi$, *such that* $0 < y \leq x$ *or at least*
$x = \sup_{\mathbf{b}_1}\{z :$ *for every* $\Xi \in \boldsymbol{\Xi}$ *and* $y \in \mathbf{b}_1^{[\Xi]}$ *we have:*
$x \leq y \Rightarrow z \leq y\}$,
(e) $\mathbf{b}_2$ *is a Boolean ring*".

Then every complete embedding of $\mathbf{b}_1[M_{\nu^*}]$ *into* $\mathbf{b}_2[M_{\nu^*}]$ *is represented in* M_{ν^*}.

Proof. Combine 6.14 and 6.10(2). $\square_{6.15}$

Definition 6.16. For a model $(A, \leq, R)$ of $\mathfrak{t}^{\rm poe}$ (see Definition 3.1) and $X \subseteq A$ we say that:
1) A set $X \subseteq A$ is nwd (nowhere dense) when every cone has a subcone disjoint to it (a cone is $\{x : x_0 \leq x\}$). A set $X \subseteq A$ is meagre if it is a countable union of nowhere dense sets.
2) A set $X \subseteq A$ is non-medium meagre when if the family of countable $a \subseteq A$ satisfying $(*)_a = (*)_a[X]$ is an unbounded subset of $[A]^{\leq \aleph_0}$, where
$(*)_a[X] : (A, \leq, R) \restriction a$ is a model of $\mathfrak{t}^{\rm poe}$, and we cannot find X_n, a nwd subset of a (for $n < \omega$) such that: $a = \bigcup\limits_{n<\omega} X_n$, and for every $c \in A$ there is $n < \omega$ satisfying: $\{b \in a : b \leq c\} \subseteq X_n$.

If $X = A$ we may omit it. We say in this case that X is non-meagre in $(A, \leq, R)$ for a.
3) $X \subseteq A$ is non-weakly meagre when for a stationary set of $a \in [A]^{\leq \alpha_0}$ we have $(*)_a[X]$.

Remark 6.17. 1) For an ordered field or just a dense linear order $(A, <)$ we use $A=$ the set of open intervals of A_1, with $\leq$ being subintervals, R being disjoint.

2) If we can get the parallel of 6.6, 6.11, 6.14 to models of $\mathfrak{t}^{\rm poe}$ (hence to ordered fields), we later get stronger results, the missing point is 6.14(1) – downward monotonicity of non-meagre.

Claim 6.18. *1) If $(A_1, \le, R)$ is a model of $\mathfrak{t}^{\rm poe}$, $X \subseteq A$ is meagre in $(A, \le, R)$, then for some club $S \subseteq [A]^{<\aleph_0}$, for every $a \in S$, we have: X is meagre in $(A, \le, R)$ for a. This in turn means: X is weakly meagre. If X is medium meagre in $(A, \le, R)$, then X is weakly meagre.*

Proof. Should be clear. $\square_{6.18}$

Claim 6.19. [CH]

Assume $\nu \in {}^{\omega_1}2$, for $\ell = 1, 2$

$$M_\nu \models \text{“}(A^\ell, \le^\ell, R^\ell) \text{ is a model of } \mathfrak{t}^{\rm poe}\text{”}$$

and

$$M_\nu \models \text{“}a^\ell \subseteq A^\ell \text{ is countable”}.$$

Also $(A^1, \le^1, R^1) \restriction a^1 \models \mathfrak{t}^{\rm poe}$, and $M_\nu \models$ “for $(A^1, \le^1, R^1)$, $(*)_a$ from 6.16(2) holds”. Then for every embedding f of $(A^1, \le^1, R^1)^{M_\nu} \restriction a$ into $(A^2, \le^2, R^2)^{M_\nu}$ mapping a^1 into a^2 we have:

- $\otimes$ *for every cone C of $(A^1, \le^1, R^1)^{M_\nu} \restriction a$, on some subcone C' of $(A^1, \le^1, R^1)^{M_\nu} \restriction a$, we have:*
  - $(*)$ *there is $r \in M_\nu$ such that:*
    - *(i)* $M_\nu \models$ “$\dot g$ is a function with domain $\{x \dot e a^1 : x \dot e C'\}$,
    - *(ii)* for $x \dot e^{M_\nu} \mathrm{Dom}(\dot g)$ we have: $\dot g(x)$ is a subset of A^2,
    - *(iii)* if $b_1, b_2 \dot e^{M_\nu} \mathrm{Dom}(\dot g)$, and $b_1 R^1 b_2$, then $(\exists x_1 \dot e r(b_1))(\exists x_2 \dot e \dot g(b_2))[x_1; R^2; x_2$ hence x_1, x_2 are $\le^1$-incomparable],
    - *(iv)* for $b \dot e^{M_\nu} \mathrm{Dom}(\dot g)$, we have: $f(b) \in \dot g(b)$.

Proof. Straightforward (like the proof of 6.6(2)). $\square_{6.19}$

Claim 6.20. [CH]

Assume that $M_{\nu^*} \models$ “$(A_\ell, \le^\ell, R^\ell)$ is a model of $\mathfrak{t}^{\rm poe}$ and for $\ell = 1$ non-medium meagre” and $\mathbf{f}$ is a dense embedding (see [Shee], i.e., on branches) of $(A^1, \le^1, R^1)^{M_{\nu^*}}$ into $(A^2, \le^2, R^2)^{M_{\nu^*}}$. Then for a dense set of cones C (of $(A^1, \le^1, R^1)$, $\mathbf{f} \restriction C$ is represented in M_{ν^*}.

Proof. Like the proof of 6.14 (using Fodor lemma). $\square_{6.20}$

Definition 6.21. 1) We say that $\mathbf{B}$ is a partial Boolean algebra if the functions $(x \cap y, x \cup y, x - y, 0, 1)$ are partial (but $0^{\mathbf{B}}$ well defined), so the identities are interpreted as “if at least one side is well defined then so is the other and they are equal”. (So a Boolean ring is a partial Boolean algebra.) Let $a \le b$ mean $a \cap b = a$, so $\neg[b \cap a = 0]$ means $b \cap a$ is an element $\ne 0$ or undefined.
2) Let $\mathbf{B}$ be a partial Boolean algebra. A set $\Xi \subseteq \mathbf{B}$ is called a maximal antichain of $\mathbf{B}$ when:

$(*)$ (a) $a \in \Xi \Rightarrow a \ne 0$

(b) $a \neq b \in \Xi \Rightarrow a \cap b = 0$

(c) $b \in \mathbf{B} \setminus \{0\} \Rightarrow \bigvee\limits_{a \in \Xi} (\exists c)[a \cap b = c \neq 0_{\mathbf{B}}]$.

3) For a partial Boolean algebra $\mathbf{B}$ and a maximal antichain Ξ, let $\mathbf{B}^\Xi$ be a partial Boolean algebra with universe $\mathbf{B}$ and $\mathbf{B}^\Xi \models$ "$b \leq c$" iff for every $a \in \Xi, [b \cap a \neq 0 \Rightarrow c \cap a \neq 0]$.
4) For $\mathbf{B}, \Xi$ as above, and $Y \subseteq \mathbf{B}$ we call Y nowhere dense for $(\mathbf{B}, \Xi)$, if for every partial finite function h from Ξ to $\{0,1\}$ there is a finite function h^+ from Ξ to $\{0,1\}$ extending h and such that for no $c \in Y$ do we have $h^+(a) = 0 \Rightarrow a \cap c = 0, h^+(a) = 1 \Rightarrow a \leq c$.

We say Y is μ-meagre for $(\mathbf{B}, \Xi)$ <u>if</u> it is the union of $< \mu$ nowhere dense for $(\mathbf{B}, \Xi)$ sets; if $\mu = \aleph_1$ we omit it.

We say $\mathbf{B}$ is μ-meagre over Ξ when $\mathbf{B}$ is μ-meagre for $(\mathbf{B}, \Xi)$ as a subset of $\mathbf{B}$.

Claim 6.22. *Assume $\omega^{M_{\langle\rangle}}$ is well ordered. If $\mathbf{b}_1^{M^{**}}$, $\mathbf{b}_2^{M^{**}}$ are Boolean rings in $M^{**} = M_{\nu_*}$, Ξ as in 6.12(a),(c),(d) and*

$(b)^-$ for every $\Xi \in \mathbf{\Xi}$, $\mathbf{b}_1^\Xi$ is not meagre (in the sense of N^{**})

<u>*then*</u> *every complete embedding of $\mathbf{b}_1^{M^{**}}$ into $\mathbf{b}_2^{M^{**}}$ is represented in M^{**}.*

Proof. Straightforward. $\square_{6.22}$

Theorem 6.23. *[CH] 1) The logic $\mathbb{L}$ extended by the following quantifiers is still $\aleph_0$-compact (getting models of cardinality $\aleph_1$:*

(A) complete embedding of one Boolean ring to another,

(B) embedding of one ordered field into another with dense range.

2) In the logic $\mathbb{L}_{\aleph_1,\aleph_0}$ extended by $\exists^{\geq \aleph_1}$ and the following quantifiers we still cannot characterize well ordering of order type $\leq \omega_1$:

(A) non-meagreness of a family of subsets of a countable set,

(B) complete embedding of a non meagre B.A. into a Boolean algebra,

(C) dense embedding of a non meagre ordered field considering the interval under inclusion as a model of $\mathfrak{t}^{\rm poe}$.

3) The logic $\mathbb{L}$ extended by $\exists^{\geq \aleph_1}$ and the quantifier from (A),(B),(C) of part (2), is $\aleph_0$-compact, getting models of cardinality $\aleph_1$.

Proof. 1) Assume we are given such a theory T in this logic. First use §5 to get an $\aleph_1$-compact model N^* of T (e.g. in $\mathbf{L}[A], A \subseteq 2^{\aleph_2}$ code T and $\mathscr{P}(\aleph_2)$, which satisfies (A)+(B), then create a model $\mathfrak{C}^*$ of T^* in which N^* is a member. Let $M^* \prec \mathfrak{C}^*, \|M^*\| = \aleph_0$, $N^* \in M^*$, and apply this section.
2) Should be clear. $\square_{6.23}$

Remark 6.24. You can add in 6.23(2),(3) also the quantifier (aa X), i.e., make "for stationary many countable $x \subseteq y$" be standard. For this in 6.5(5) we should replace "stationary many α" by "club many α", and so restrict somewhat the Ξ-s which we may use.

Claim 6.25. *Assume $\omega^{M_{\langle\rangle}}$ is well ordered. For $\mathbf{b}_1$, $\mathbf{b}_2 \in M_{\nu^*}$ such that $\mathbf{b}_\ell[M_{\nu^*}]$ is a triple $(P^\ell, Q^\ell, R, P^\ell)$ with the strong independence property (this means satisfying $\aleph_0$ sentences).*
1) Claim 6.6, Def.6.11, conclusion 6.14 generalize naturally to dense embedding (see [Shee]*).*
2) In 6.23(2) we can add:

- (D) *dense embedding of one interpretation of a model of the strong independence property into another.*

Proof. No new point. $\square_{6.25}$

Remark 6.26. 1) We can in 6.2 and say that T^* suitably omit Γ <u>when</u> $\Gamma \subseteq \{\varphi(\bar{x}) \in \mathbb{L}(\tau_{T^*})\}$ and

- $(*)$ if $T^* \cup \{(\dot{\mathbf{Q}}_{\sigma_n} y_n) \ldots (\dot{\mathbf{Q}}_{\sigma_1} y_1)(\exists \bar{x})\psi(\bar{x}, y_i, \ldots, y_n)\}$ is consistent then for semi $\varphi(\bar{x}) \in p, T^* \cup \{(\dot{\mathbf{Q}}_{\sigma_n} y_n) \ldots (\dot{\mathbf{Q}}_{\sigma_1} y_1)(\exists \bar{x})(\psi(\bar{x}, y_1 \ldots y_n) \wedge \neg\varphi(\bar{x}))$ and has the "omitting type theorem".

2) We can replace here $(\dot{\mathbf{Q}}_\sigma y)$ by $(\dot{\mathbf{Q}}_\Gamma y)$ where Γ is a bigness notion and $\lg \bar{y} = \lg \bar{x}_\Gamma$.

Theorem 6.27. *[*CH*]*

Let T be countable and complete first order theory. <u>Then</u> T has a model M^* of cardinality $\aleph_1$ such that:

- (A) *If $\mathbf{b}_1, \mathbf{b}_2$ are interpretations of Boolean rings in M^*, every complete embedding of $\mathbf{b}_1$ to $\mathbf{b}_2$ is definable (from parameters) in M^**
- (B) *if $\mathbb{F}_1, \mathbb{F}_2$ are interpretation of dense linear ordered in M^*, every dense embedding of $\mathbb{F}_1$ into $\mathbb{F}_2$ is on a dense set of interval definable (from parameters) in M^* .*
- (C) *The parallel to interpretation of the theorem $\dot{k}^{\mathrm{ind}}$.*

Proof. Use [She83b]+the theorem above. $\square_{6.27}$

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