ON CON($\mathfrak{d}_\lambda >$ COV$_\lambda$(MEAGRE))
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ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal λ the dominating number, i.e., the cofinality of ${}^\lambda\lambda$, is strictly bigger than $\text{cov}_\lambda(\text{meagre})$, i.e. the minimal number of nowhere dense subsets of ${}^\lambda 2$ needed to cover it. This answers a question of Matet.

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§ 0. Introduction

Cardinal characteristics were defined, historically, over the continuum. See the celebrated Van Dowen [vD84], for the general topologist perspective and the excellent survey Blass [Bla], Bartoszyński [Bar10] for the set theoretic perspective. In recent years there are many results concerning generalized cardinal characteristics. The idea is to imitate the definition of a given characteristic over the continuum, by translating it to uncountable cardinals.

It is reasonable to distinguish regular cardinals and singular cardinals. Among the regular cardinals, it makes sense to distinguish limit cardinals from successor cardinals. In this paper we focus on strongly inaccessible cardinals. These cardinals and their characteristics behave, in many cases, much like $\aleph_0$, but certainly not always. See Landver [Lan92], Cummings-Shelah [CS95] and Matet-Shelah [MS]. Our main result is the consistency of $\mathrm{cov}_\lambda(\text{meagre}) < \mathfrak{d}_\lambda$ at a super-compact cardinal λ, and we begin with the following definitions:

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

Definition 0.1. The *bounding* and *dominating numbers.*

Let λ be an inaccessible cardinal. For $f, g \in {}^\lambda\lambda$ let:

(a) $f \leq^* g$ if $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$,
(b) $A \subseteq {}^\lambda\lambda$ is *unbounded* if there is no $h \in {}^\lambda\lambda$ such that $f \in A \Rightarrow f \leq^* h$,
(c) $A \subseteq {}^\lambda\lambda$ is *dominating* when for every $f \in {}^\lambda\lambda$ there exists $g \in A$ such that $f \leq^* g$,
(d) *the bounding number* for λ, denoted by $\mathfrak{b}_\lambda$, is $\min\{|A| : A \subseteq {}^\lambda\lambda$ is unbounded in ${}^\lambda\lambda\}$,
(e) *the dominating number* for λ, denoted by $\mathfrak{d}_\lambda$, is $\min\{|A| : A \subseteq {}^\lambda\lambda$ is dominating in ${}^\lambda\lambda\}$.

Notice that the usual definitions of $\mathfrak{b}$ and $\mathfrak{d}$ are $\mathfrak{b}_{\aleph_0}$ and $\mathfrak{d}_{\aleph_0}$ according to Definition 0.1. The definition of $\mathrm{cov}_\lambda(\text{meagre})$ involves some topology.

Definition 0.2. *The meagre covering number.*

Let λ be a regular cardinal.

(a) ${}^\lambda 2$ is the space of functions from λ into 2,
(b) $({}^\lambda 2)^{[\nu]} := \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$ and for $\nu \in {}^{\lambda>}2 := \bigcup\limits_{\alpha<\lambda} {}^\alpha 2$,
(c) $\mathscr{U} \subseteq {}^\lambda 2$ is *open* in the topology $({}^\lambda 2)_{<\lambda}$, iff for every $\eta \in \mathscr{U}$, there exists $i < \lambda$ such that $({}^\lambda 2)^{[\eta\restriction i]} \subseteq \mathscr{U}$,
(d) $\mathscr{U} \subseteq {}^\lambda 2$ is *meagre* iff it is the union of $\leq \lambda$ no-where dense subsets,
(e) $\mathrm{cov}_\lambda(\text{meagre})$ is the minimal cardinality of a family of meagre subsets of $({}^\lambda 2)_{<\lambda}$ which covers this space.

This paper deals with the relationship between $\mathfrak{d}_\lambda$ and $\mathrm{cov}_\lambda(\text{meagre})$. If λ is a successor cardinal then $\mathrm{cov}_\lambda(\text{meagre}) < \mathfrak{d}_\lambda$ is consistent (see (b) below). Matet asked (a personal communication) whether $\mathfrak{d}_\lambda \leq \mathrm{cov}_\lambda(\text{meagre})$ is provable in ZFC, where λ is strongly inaccessible. We give here a negative answer.

For λ a super-compact cardinal and $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mu^\lambda$, we force large $\mathfrak{d}_\lambda$ i.e., $\mathfrak{d}_\lambda = \mu$ and small covering number (i.e., $\mathrm{cov}_\lambda(\text{meagre}) = \kappa$). A similar result should hold also for a wider class of cardinals and we intend to return elsewhere to this subject.

Let us sketch some known results. These results are related to the unequality number and the covering number for category. Recall:

Definition 0.3. *The unequality number.*

Let κ be an infinite cardinal. The *unequality number of* $\kappa, \mathfrak{e}_\kappa$, is the minimal cardinal λ satisfying that there is a set $\mathscr{F} \subseteq {}^\kappa\kappa$ of cardinality λ, such that there is no $g \in {}^\kappa\kappa$ satisfying $(\forall f \in \mathscr{F})(\exists^\kappa \alpha < \kappa)(f(\alpha) = g(\alpha))$.

For $\kappa = \aleph_0, \mathfrak{e}_\kappa = \text{cov}_{\aleph_0}(\text{meagre})$; see Bartoszyński (in [Bar87]) and Miller (in [Mil82]).

Now,

(a) the statement $\mathfrak{e}_\kappa = \text{cov}_\kappa(\text{meagre})$ is valid for $\kappa > \aleph_0$, in the case that κ is strongly inaccessible, by [Lan92]. But if κ is a successor cardinal, it may fail,

(b) if $\kappa < \kappa^{<\kappa}$ then $\text{cov}_\kappa(\text{meagre}) = \kappa^+$. This is due to Landver (in [Lan92]).

We intend also to address:

Problem 0.4. Can we replace "super-compact" by "strongly inaccessible"?

Problem 0.5.

(1) Can we prove the consistency of $\text{cov}_\lambda(\text{meagre}) < \mathfrak{b}_\lambda$?

(2) For λ strongly inaccessible (or just Laver indestructible super-compact) is there a non-trivial λ^+-c.c. $(< \lambda)$-strategically complete forcing notion $\mathbb{Q}$ which is ${}^\lambda\lambda$-bounding?

We say more in subsequent works [She17], [Shea] and in preparation [Shec].

A point which in a previous version was just a step along the way, the referee asked to justify fully, was analyzed to be serious. This was done but eventually is separated to [Sheb]. A posteriori the point is that in the parallel case for $\lambda = \aleph_0$, for full memory FS iteration such a claim is true. In fact, by Judah-Shelah [JS88], if $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$ is FS iteration of Suslin-c.c.c. forcing notion, $\underset{\sim}{\mathbb{Q}}_\beta$ with the generic $\underset{\sim}{\eta}_\beta \in {}^\omega\omega$ and for notational transparency, its definition is with no parameter and $\zeta : \beta(*) \to \alpha(*)$ is increasing and $\mathbb{P} = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \leq \beta(*), \beta < \beta(*) \rangle$ is FS iteration, but $\underset{\sim}{\mathbb{Q}}'_\beta$ is defined exactly as $\underset{\sim}{\mathbb{Q}}_{\zeta(\beta)}$ is but now in $\mathbf{V}^{\mathbb{P}'_\beta}$ rather then in $\mathbf{V}^{\mathbb{P}_{\zeta(\beta)}}$ then $\Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\langle \underset{\sim}{\eta}_{\zeta(\beta)} : \beta < \beta(*) \rangle$ is generic for $\mathbb{P}'_{\beta(*)}$ over $\mathbf{V}$".

Now this is not clear to us for $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions for $\lambda > \aleph_0$. The solution is essentially to change the iteration: to use a "quite generic" $(< \lambda)$-support iteration which "includes" the one we like and use the complete sub-forcing it generates; see [Sheb].

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We try to use standard notation. We use $\theta, \kappa, \lambda, \mu, \chi$ for cardinals and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ for ordinals. We use also i and j as ordinals. We adopt the Cohen convention that $p \leq q$ means that q gives more information, in forcing notions. The symbol $\triangleleft$ is preserved for "being an initial segment". Also recall ${}^{B}A = \{f : f$ a function from B to $A\}$ and let ${}^{\alpha>}A = \cup\{{}^{\beta}A : \beta < \alpha\}$, some prefer ${}^{<\alpha}A$, but ${}^{\alpha>}A$ is used systematically in the author's papers. Lastly, $J^{\rm bd}_\lambda$ denotes the ideal of the bounded subsets of λ.

For exact references to [Sheb] see the introduction there, just before [Sheb, 0.1= Lz6]. The picture of cardinal invariants related to uncountable λ is related but usually quite different than the one for $\aleph_0$, they are more similar if κ is "large" enough, mainly strongly inaccessible.

§ 1. Preliminaries

Definition 1.1. Let λ be super-compact. We say that $h : \lambda \to \mathscr{H}(\lambda)$ *is a Laver diamond (for* λ*)* <u>when</u> for every $x \in \mathbf{V}$ there are a normal fine ultrafilter D over $I = [\mathscr{H}(\chi)]^{<\lambda}$ for some χ, such that $x \in \mathscr{H}(\chi)$ and the Mostowski collapse $\mathbf{j}$ of $\mathbf{V}^I/D$ maps $\langle h(\sup(u \cap \lambda)) : u \in I\rangle/D$ to x (we can use elementary embeddings instead of an ultrafilter).

Notation 1.2. If $\mathbb{P}$ is a forcing notion in $\mathbf{V}$ then $\mathbf{V}^{\mathbb{P}}$ denotes $\mathbf{V}[\mathbf{G}]$ for $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$; we may write $\mathbf{V}[\mathbb{P}]$ instead.

The most straightforward way to increase $\mathfrak{b}_\lambda$ in the classical case of $\aleph_0$ is Hechler forcing (dominating real forcing). A condition is a function $f_p : \omega \to \omega$ which is separated into a finite trunk η_p and the rest of the function. Formally, $p = (\eta_p, f_p)$ where $\eta_p \trianglelefteq f_p$.

If p, q are conditions then $p \leq q$ <u>iff</u> $\eta_p \trianglelefteq \eta_q$ and $f_q(n) \geq f_p(n)$ for every $n \notin \mathrm{dom}(\eta_p)$ hence for every n. A generic object adds a function $g : \omega \to \omega$ which dominates the functions from the ground model. By iterating Hechler reals one increases the bounding number $\mathfrak{b}$.

If $\lambda = \lambda^{<\lambda}$ then one can define the *generalized Hechler forcing* $\mathbb{D}_\lambda$ by replacing ω by λ. The basic step is $(<\lambda)$-complete and λ^+-c.c. and actually λ-centered. Hence one can iterate and increase $\mathfrak{b}_\lambda$.

In [She92, §1,§2] and then Goldstern-Shelah [GS93], Kellner-Shelah [KS12] other invariants are discussed. Consider two functions $f, g : \omega \to (\omega\backslash\{0\})$ going to infinity such that $f \geq g$ and ask about:

- $\mathfrak{c}^+_{f,g} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \prod\limits_i [f(i)]^{g(i)}$ and $(\forall \eta \in \prod\limits_i f(i))(\exists g \in \mathscr{F})[\bigwedge\limits_i \eta(i) \in g(i)]\}$,
- $\mathfrak{c}^-_{f,g} = \min\{\mathscr{F} : \mathscr{F} \subseteq \prod\limits_i f(i)$ and for no $g \in \prod\limits_i [f(i)]^{g(i)}$ do we have $(\forall \eta \in \mathscr{F})(\forall^\infty i)(\eta(i) \in g(i))\}$.

There are relevant forcing notions; we could have used $[f(i)]^{<g(i)}$, this case is generalized here, so below $f = gu$ replaced by $\langle \theta_\varepsilon : \varepsilon < \lambda\rangle$ we shall use a λ^+-c.c. one as in c.c.c. creature forcing (see more in [RS97],[HS]).

For transparency,

Convention 1.3. Below $\lambda, \bar\theta$ are as in 1.4 below.

Definition 1.4. Let λ be inaccessible, $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda\rangle$ be a sequence of regular cardinals $< \lambda$ satisfying $\theta_\varepsilon > \varepsilon$.

(1) We define the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar\theta}$ by:

- (A) $p \in \mathbb{Q}$ <u>iff</u>:
  - (a) $p = (\eta, f) = (\eta^p, f^p)$,
  - (b) $\eta \in \prod\limits_{\zeta<\varepsilon} \theta_\zeta$ for some $\varepsilon < \lambda$, (η is called the trunk of p),
  - (c) $f \in \prod\limits_{\zeta<\lambda} \theta_\zeta$,
  - (d) $\eta \triangleleft f$.
- (B) $p \leq_{\mathbb{Q}} q$ iff:
  - (a) $\eta^p \trianglelefteq \eta^q$,

(b) $f^p \leq f^q$, i.e. $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$,
(c) if $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$, actually follows.

(2) The generic is $\underset{\sim}{\eta} = \bigcup \{\eta^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\bar\theta}}\}$.

The new forcing defined above is not $(< \lambda)$-complete anymore. By fixing a trunk $\eta \in \Pi_{\varepsilon<\zeta}\theta_\varepsilon$ one can define a short increasing sequence $\bar p = \langle p_\varepsilon : \varepsilon < \zeta\rangle$ of conditions which goes up to θ_ζ at the ζ-th coordinate that is, $\langle p_\varepsilon(\zeta) : \zeta < \theta_\zeta\rangle$ is increasing and hence has no upper bound in θ_ζ, so $\bar p$ has no upper bound in $\mathbb{Q}$. However, this forcing is $(< \lambda)$-strategically complete (see Definition 1.6(2), below) since the COM (= completeness) player can increase the trunk at each move.

Remark 1.5. The forcing is parallel to the creature forcing from [She92, §1,§2], [KS12] but they are ${}^\omega\omega$-bounding.

Recall,

Definition 1.6.

(1) We say that a forcing notion $\mathbb{P}$ is *α-strategically complete* <u>when</u> for each $p \in \mathbb{P}$ in the following game $\Game_\alpha(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy.

A play lasts α moves; in the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$ and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq_{\mathbb{P}} q_\beta$.

The player COM *wins* a play if he has a legal move for every $\beta < \alpha$.

(2) We say that a forcing notion $\mathbb{P}$ is $(< \lambda)$-*strategically complete* <u>when</u> it is α-strategically complete for every $\alpha < \lambda$.

Basic properties of $\mathbb{Q}_{\bar\theta}$ are summarized and proved in [GS12, §2].

The following fact describes some immediate connections between various concepts of completeness:

Fact 1.7.

(a) If $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta\rangle$ is a $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions, <u>then</u> $\mathbb{P}_\delta$ is also $(< \lambda)$-strategically complete (see e.g. [She00]),
(b) If $\mathbb{P}$ is $(< \lambda)$-strategically complete forcing notion <u>then</u> $({}^{\lambda>}\mathrm{Ord})^{\mathbf{V}} = ({}^{\lambda>}\mathrm{Ord})^{\mathbf{V}^{\mathbb{P}}}$, and consequently λ is strongly inaccessible in $\mathbf{V}^{\mathbb{P}}$,
(c) like (a) replacing "$(< \lambda)$-strategically complete" by "$(< \lambda)$-complete"or by "α-strategically complete",
(d) if $\mathbb{P}$ is $(< \lambda)$-complete then $\mathbb{P}$ is $(< \lambda)$-strategically complete.

Definition 1.8. For an ordinal $\alpha_* := \alpha(*)$ let $\mathbf{Q}_{\lambda,\bar\theta,\alpha(*)}$ be the class of quintuple $\mathbf{q} = (\bar u, \bar{\mathscr{P}}, \bar{\mathbb{P}}, \underset{\sim}{\bar{\mathbb{Q}}}, \underset{\sim}{\bar\eta})$ consisting of (omitting α_* means for some α_* and we let $\ell g(\mathbf{q}) = \alpha_{\mathbf{q}} = \alpha_*$):

(a) $\bar u = \langle u_\alpha : \alpha < \alpha_*\rangle$ and $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \alpha_*\rangle$ where $\mathscr{P}_\alpha \subseteq [u_\alpha]^{\leq\lambda}, u_\alpha \subseteq \alpha$, without loss of generality $\mathscr{P}_\alpha$ is closed under subsets (but is not necessarily an ideal),
(b) $\langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \leq \alpha_*, \beta < \alpha_*\rangle$ is a $(< \lambda)$-support iteration and let $\mathbb{P}_{\mathbf{q},0} = \mathbb{P}_{\mathbf{q},0,\alpha(\mathbf{q})}$; but we may write $\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta$,

(c) each $\mathbb{P}_\alpha$ is $(<\lambda)$-strategically complete and λ^+-c.c.,
(d) $\underset{\sim}{\eta}_\beta \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ is the generic of $\underset{\sim}{\mathbb{Q}}_\beta$ where $\underset{\sim}{\eta}_\beta$, the generic of $\underset{\sim}{\mathbb{Q}}_\beta$ (defined in clause (e) below) is $\cup\{\eta_p : p \in \mathbf{G}_{\mathbb{Q}_\beta}\}$,
(e) if $\alpha < \alpha_*$ $\mathbf{G} \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$ then $\underset{\sim}{\eta}_\alpha[\mathbf{G}]$ in $(\prod_{\varepsilon<\lambda} \theta_\varepsilon, <_{J^{\rm bd}_\lambda})$ dominates every $\nu \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ from $\mathbf{V}[\langle \underset{\sim}{\eta}_\gamma : \gamma \in u\rangle]$ when $u \in \mathscr{P}_\alpha$; moreover, in $\mathbf{V}[\mathbf{G}]$:
  $(*)$ $\underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}]$ is the sub-forcing of $\mathbb{Q}_{\bar\theta}$ consisting of the $p \in \mathbb{Q}_{\bar\theta}$ such that: for some $\bar s, \underset{\sim}{f}, \eta_p$ (so $\eta_p = \eta$, etc.) we have:
    (α) $p = (\eta, f) = (\eta_p, f_p)$ so $\eta \in \prod_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda$,
    (β) $\bar s = \langle (u_i, f_i) : i < i_*\rangle$,
    (γ) $i_* < \lambda$,
    (δ) for each $i < i_*$ we have $u_i \in \mathscr{P}_\beta, \eta \triangleleft f_i \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ and $f_i \in \mathbf{V}[\langle \underset{\sim}{\eta}_\gamma[\mathbf{G}] : \gamma \in u_i\rangle]$,
    (ε) $f = \sup\{f_i : i < i_*\}$, i.e. $\varepsilon < \lambda \Rightarrow f(\varepsilon) = \cup\{f_i(\varepsilon) : i < i_*\}$;we may add $i_* < \theta_{\lg(\eta)}$.
(f) <u>notation</u>: so $u_{\mathbf{q},\alpha} = u_\alpha, \mathbb{P}_{\mathbf{q},\alpha} = \mathbb{P}_\alpha$, etc., but when $\mathbf{q}$ is clear from the context we may omit it.

Definition 1.9. For $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar\theta,\alpha(*)}$,

(1) We let $\alpha \le \alpha_*, \mathbb{P}_{1,\alpha} = \mathbb{P}^{\mathbf{q}}_{1,\alpha}$ be essentially the completion of $\mathbb{P}_\alpha$; we express it by:

$(*)_1$ the elements of $\mathbb{P}_{1,\alpha} = \mathbb{P}_{\mathbf{q},1,\alpha}$ are of the form $\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)}$, where:
  (α) $i_* = i(*) \le \lambda$,
  (β) $\gamma_i < \alpha$ for $i < i_*$,
  (γ) $\mathbf{B}$ is a λ-Borel function from ${}^{i(*)}(\prod_{\varepsilon<\lambda} \theta_\varepsilon)$ into $\{0,1\} = \{\text{false}, \text{true}\}$; $\mathbf{B}$ is from $\mathbf{V}$, of course, such that $\nVdash_{\mathbb{P}_{\mathbf{q}}}$ "$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)} = 0$".
$(*)_2$ the order is natural: $\mathbb{P}_{1,\alpha} \models$ "$\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(1)} \le \mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}, \ldots)_{i<i(2)}$" iff $\Vdash_{\mathbb{P}_\alpha}$ "if $\mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}[\underset{\sim}{\mathbf{G}}], \ldots)_{i<i(2)}$ is equal to 1 then so is $\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(1)}$".

(2) For $\mathscr{U} \subseteq \alpha_*$ let $\mathbb{P}_{\mathscr{U}} = \mathbb{P}^{\mathbf{q}}_{\mathscr{U}}$ be the sub-forcing of $\mathbb{P}_{1,\alpha(\mathbf{q})}$ consists of $\{\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma(i)}, \ldots)_{i<i(*)} \in \mathbb{P}_{1,\alpha(\mathbf{q})} : i(*) \le \lambda$ and $\gamma_i \in \mathscr{U}$ for every $i < i(*)\}$.

Claim 1.10.

(1) For any sequence $\langle u_\alpha, \mathscr{P}_\alpha : \alpha < \alpha_*\rangle$ as above, i.e. as in clause (a) of Definition 1.8, there is one and only one $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar\theta,\alpha_*}$ with $u_{\mathbf{q},\alpha} = u_\alpha, \mathscr{P}_{\mathbf{q},\alpha} = \mathscr{P}_\alpha$ for $\alpha < \alpha_*$.

(1A) For $\alpha \le \alpha_*$, the forcing notions $\mathbb{P}_{\mathbf{q},1,\alpha}, \mathbb{P}_{\mathbf{q},1,\mathscr{U}}$'s are well defined and are as demanded in Definition 1.9.

(2) For every $\alpha \le \alpha_*$ $\mathbb{P}^\bullet_{\mathbf{q},\alpha} = \mathbb{P}^\bullet_{\mathbf{q},0,\alpha}$ is $\mathbb{P}_{\mathbf{q},\alpha} = \mathbb{P}_{\mathbf{q},0,\alpha}$ restricted to the set of $p \in \mathbb{P}_{\mathbf{q},\alpha}$ (from Definition 1.8) satisfying the following is dense in $\mathbb{P}_{\mathbf{q},\alpha}$:

$(*)$ *if $\beta \in \mathrm{dom}(p)$, then $q = p(\beta)$ is a $\mathbb{P}_\beta$-name of a member of $\underset{\sim}{\mathbb{Q}}_\beta$ such that:*
  (a) *$\eta_q, i_q, \langle u_{q,i} : i < i_q\rangle$ are objects (not just $\mathbb{P}_\beta$-names),*
  (b) *$\underset{\sim}{f}_q = \sup\{\underset{\sim}{f}_i : i < i_q\}$, each $\underset{\sim}{f}_i$ is a $\mathbb{P}_\beta$-name of a member of $\prod_{\varepsilon<\lambda} \theta_\varepsilon$,*
  (c) *each $\underset{\sim}{f}_i$ has the form $\mathbf{B}_{q,i}(\ldots, \underset{\sim}{\eta}_{\gamma(i,j)}, \ldots)_{j<j(*)\le\lambda}$ where $\{\gamma(i,j) : j < j(*)\} \subseteq u_{q,i}$ and $\mathbf{B}_q$ is a Borel function from ${}^{j(*)}(\prod_{\varepsilon<\lambda} \theta_\varepsilon)$ into $\prod_{\varepsilon<\lambda} \theta_\varepsilon$,*

(d) $p(\beta) = (\underset{\sim}{\eta}_q, \underset{\sim}{f}_q)$.

(2A) Abusing our notation we may identify $\mathbb{P}^\bullet_\alpha$ *with* $\mathbb{P}_\alpha$.

(3) Above for every $v \subseteq \alpha$ *and* $j_* < \lambda$ *the set of* $p \in \mathbb{P}^\bullet_\alpha$ *such that* $v \subseteq \mathrm{dom}(p) \wedge (\forall \beta \in \mathrm{dom}(p))(\ell g(\eta_{p(\beta)}) > j_*)$ *is dense.*

(4) $\mathbb{P}^\bullet_{\mathbf{q},0,\alpha} \lessdot \mathbb{P}_{\mathbf{q},1,\alpha}$ *moreover* $\mathbb{P}^\bullet_{\mathbf{q},0,\alpha}$ *is dense in* $\mathbb{P}_{\mathbf{q},1,\alpha}$ *and in* $\mathbb{P}_{\mathbf{q},0,\alpha}$ *and* $\mathscr{U}_1 \subseteq \mathscr{U}_2 \subseteq \alpha \le \alpha_{\mathbf{q}} \Rightarrow \mathbb{P}_{\mathbf{q},1,\mathscr{U}_q} \lessdot \mathbb{P}_{\mathbf{q},1,\mathscr{U}_2} \lessdot \mathbb{P}_{\mathbf{q},1,\alpha}$ *so* $\mathbb{P}_{\mathbf{q},1,\{\beta:\beta<\alpha\}} = \mathbb{P}_{\mathbf{q},1,\alpha}$ *and* $|\mathbb{P}_{\mathbf{q},1,\mathscr{U}}| \le |\mathscr{U}|^\lambda$.

(5) If $\alpha < \alpha_*$ *and* $u \in \mathscr{P}_\alpha$ *then* $\underset{\sim}{\eta}_\alpha \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ *dominates every* $\nu \in (\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\underset{\sim}{\bar\eta}\restriction u]}$ *and* $\mathbf{V}[\mathbf{G}_{\mathbb{P}_{\mathbf{q},1,\mathscr{U}}}] = \mathbf{V}[\langle \eta_\alpha : \alpha \in \mathscr{U}\rangle]$, *where* $\eta_\alpha = \underset{\sim}{\eta}[\mathbf{G}_{\mathbb{P}_{\mathbf{q},1,\alpha}}]$.

(6) Assume $\mathbf{G} \subseteq \mathbb{P}_\mathbf{q}$ *is generic over* $\mathbf{V}, \eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ *and* $\eta'_\alpha \in (\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbf{G}]}$ *for* $\alpha < \alpha_*$ *and* $\{(\alpha,\varepsilon) : \alpha < \alpha_*, \varepsilon < \alpha$ *and* $\eta_\alpha(\varepsilon) \ne \eta'_\alpha(\varepsilon)\}$ *has cardinality* $< \lambda$. *Then for some (really unique)* $\mathbf{G}'$ *we have* $\mathbf{G}' \subseteq \mathbb{P}_\mathbf{q}$ *is generic over* $\mathbf{V}$ *and* $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$ *and* $\underset{\sim}{\eta}_\alpha[\mathbf{G}] = \eta'_\alpha$ *for* $\alpha < \alpha_*$.

(7) Like (6) for $\mathbb{P}^\mathbf{q}_\mathscr{U}$.

Proof. See [Sheb, 1.13=Lc8, 1.16=Lc11]. $\square_{1.10}$

Theorem 1.11. *For any ordinal* α_* *there is a quadruple* $(\mathbf{q}, \delta_*, \mathscr{U}_*, h)$ *such that:*

(A) (a) $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar\theta}$ *and let* $\delta_* = \ell g(\mathbf{q})$,
(b) $\mathscr{U}_* \subseteq \delta_*$ *has order type* α_*,
(c) h *is the order preserving function from* α_* *onto* $\mathscr{U}_*$,
(d) *if* $\alpha \in \mathscr{U}_*$ *then* $\mathscr{U}_* \cap \alpha \in \mathscr{P}_{\mathbf{q},\alpha}$,
(e) *if* $\mathrm{cf}(\alpha_*) > \lambda$ *then in* $\mathbf{V}^{\mathbb{P}_\mathbf{q}}$ *the set* $\{\underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}_*\}$ *is cofinal in* $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon, \le_{J^{\mathrm{bd}}_\lambda})$.

(B) *if* $\mathscr{U}_1 \subseteq \mathscr{U}_*, \mathscr{U}_2 \subseteq \mathscr{U}_*, \mathrm{otp}(\mathscr{U}_1) = \mathrm{otp}(\mathscr{U}_2)$ *and* g *is the order preserving function from* $\mathscr{U}_1$ *onto* $\mathscr{U}_2$, *<u>then</u>* g *induces an isomorphism* $\hat g$ *from* $\mathbb{P}_{\mathbf{q},\mathscr{U}_1}$ *onto* $\mathbb{P}_{\mathbf{q},\mathscr{U}_2}$ *mapping* $\underset{\sim}{\eta}_\beta$ *to* $\underset{\sim}{\eta}_{g(\beta)}$ *for* $\beta \in \mathscr{U}_1$.

Proof. By [Sheb, 2.14=Le38], in particular clause (A)(e) is justified by clause (E) there. $\square_{1.11}$

Definition 1.12.

(1) Let "$\mathbb{P}$ is a $(< \lambda)$-directed complete " mean:

($*$) if J is a directed partial order of cardinality $< \lambda$ and $p_s \in \mathbb{P}$ for $s \in J$ and $s \le_J t \Rightarrow p_s \le_\mathbb{P} p_t$ <u>then</u> the set $\{p_s : s \in J\}$ has an upper bound in $\mathbb{P}$.

(2) Assume $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda\rangle$ and $\theta_\varepsilon = \mathrm{cf}(\theta_\varepsilon) \in (\varepsilon, \lambda)$. We say "$\mathbb{P}$ is $(< \lambda, \theta_*, \bar\theta)$-directed complete" <u>when</u>:

($*$) if $\mathbb{P} \in N \prec (\mathscr{H}(\chi), \in)$ and $\lambda_N = N \cap \lambda$ is inaccessible $< \lambda$ and $N^{<\lambda_N} \subseteq N$, $\|N \cap \theta_*\| < \theta_{\lambda_N}$ and $\mathbf{G} \subseteq \mathbb{P} \cap N$ is $(N, \mathbb{P})$-generic <u>then</u> $\mathbf{G}$ has a common upper bound

Definition 1.13. We say that h is a $(< \lambda, \theta_*, \bar\theta)$-Laver diamond when:

(a) λ is a super-compact cardinal,
(b) $\theta_* > \lambda$,
(c) $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda\rangle$, where $\varepsilon < \theta_\varepsilon = \mathrm{cf}(\theta_\varepsilon) < \lambda$,

(d) for every x and $\chi > \lambda$ such that $x \in \mathscr{H}(\chi)$ there are $\mathbf{j}, \mathbf{M}$ such that:
- $\bullet_1$ $\mathbf{M}$ is a transitive class,
- $\bullet_2$ $\mathbf{j}$ is an elementary embedding of $\mathbf{V}$ into $\mathbf{M}$,
- $\bullet_3$ $\mathbf{M}^{\le \chi} \subseteq \mathbf{M}$,
- $\bullet_4$ $\mathbf{j}$ is the identity on $\mathscr{H}(\lambda)$,
- $\bullet_5$ $\mathbf{j}(h)(\lambda) = x$ and $\mathbf{j}(\bar{\theta})(\lambda) = \theta_*$,

Observation 1.14.

(1) There are enough cases of h as in 1.13, for example, for every Laver diamond h,

(a) *if $\theta = (\beth_{\lambda^+})^+$ and $\langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ defined by $\theta_\varepsilon = \beth(\|\varepsilon\|^+)^+$ then h is a Laver diamond for $(\lambda, \theta_*, \bar{\theta})$,*

(b) *there is $\bar{\theta}$ such that for every regular cardinal $\theta_* > \lambda$, h is a Laver diamond for $(\lambda, \theta_*, \bar{\theta})$.*

(2) Being a $(< \lambda)$-directed complete forcing notion, is preserved by $(< \lambda)$-support iteration. Similarly for being "$(< \lambda, \theta_*, \bar{\theta})$-directed complete".

(3) If the forcing notion $\mathbb{Q}$ is $(< \lambda)$-directed complete and $\boxdot_{<\lambda, \theta_*, \bar{\theta}}$ from Definition 2.2(2) below then $\mathbb{Q}$ is $(< \lambda, \theta_*, \bar{\theta})$-directed complete .

Proof. Easy, e.g for clause (b) in part (1), let $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ where for $\varepsilon < \lambda$ we let θ_ε be the first regular cardinal θ such that $\ge \varepsilon$, and $h(\varepsilon) = (\theta, \chi)$, for some χ. $\square_{1.14}$

§ 2. The forcing

In this section we prove the main result of the paper, which reads as follows:

Theorem 2.1. *Assume,*

(a) *λ is super-compact*
(b) $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.

Then for some $(< \lambda)$-strategically complete λ^+-cc forcing notion hence $\mathbb{P}$ not collapsing cardinals $\geq \lambda$, λ is still super-compact in $\mathbf{V}^{\mathbb{P}}$ and $\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa, \mathfrak{d}_\lambda = \mu$.

Proof. Let $\theta_* = \mathrm{cf}(\theta) > 2^\mu$ and $\bar{\theta}$ be such that there is a Laver diamond for $(< \lambda, \theta_*, \bar{\theta})$, justified by 1.14. By Lemma 2.3(1) below we can (by 2.3(2) below) force $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$ while maintaining the super-compactness of λ. By Lemma 2.7 we force $\mathfrak{d}_\lambda = \mu \wedge \mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$ using a forcing notion $\mathbb{P}$ which is λ^+-c.c. and $(< \lambda)$-strategically complete. Notice that λ is still super-compact in the generic extension, so we are done. $\square_{2.1}$

Definition 2.2.

(1) For λ super-compact we define $\boxdot_\lambda$ by:

$\boxdot_\lambda$ for any regular cardinal $\chi > \lambda$ and forcing notion $\mathbb{P} \in \mathscr{H}(\chi)$ which is $(< \lambda)$-strategically complete (see Definition 1.6(2)) the following set $\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi,\mathbb{P}}$ is a stationary subset of $[\mathscr{H}(\chi)]^{<\lambda}$: $\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi,\mathbb{P}}$ is the set of N's such that for some $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, \mathbb{A} = \mathbb{A}_N, M = M_N, \mathbf{G} = \mathbf{G}_N$ we have (and we may say $(\lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ is a $\boxdot_\lambda$-witness for $N \in \mathscr{S}_{\chi,\mathbb{P}}$ or for $(N, \mathbb{P}, \chi)$):
  (a) $N \prec (\mathscr{H}(\chi)^{\mathbf{V}}, \in)$ and $\mathbb{P} \in N$,
  (b) the Mostowski collapse of N is $\mathbb{A}$ and let $\mathbf{j}_N : N \to \mathbb{A}$ be the unique isomorphism,
  (c) $N \cap \lambda$ is a strongly inaccessible cardinal called $\lambda_N = \lambda(N)$, such that ${}^{\lambda(N)>}N \subseteq N$,
  (d) $\mathbb{A} \subseteq M := (\mathscr{H}(\chi_N), \in), M$ is transitive as well as $\mathbb{A}$,
  (e) $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$ is generic over $\mathbb{A}$ for the forcing notion $\mathbf{j}_N(\mathbb{P})$,
  (f) $M = \mathbb{A}[\mathbf{G}]$.

(2) Assume $\lambda, \theta_*, \bar{\theta}$ satisfies clauses (b), (c) of Def 1.13 and λ is super-compact. We define $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$ by:

$\boxdot_{<\lambda,\theta_*,\bar{\theta}}$ As in part (1) adding $\|N \cap \theta_*\| < \theta_{\lambda_N}$ and even $2^{\|N\cap\theta_*\|} < \theta_{\lambda_N}$.

Our first lemma is closed to Laver's indestructibility. It consists of two parts. In the first part we prove that one can force $\boxdot_\lambda$ at a super-compact cardinal λ while preserving its super-compactness. In the second part, we deal with a more informative version $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$. See 1.14, this can be done in an indestructible manner. Namely, any further extension of the universe by a $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$-directed complete forcing notion will preserve the $(< \lambda, \theta_*, \bar{\theta})$-Laver diamond and the principle $\boxdot_{\lambda,\theta_*,\bar{\theta}}$.

Lemma 2.3.

(1) If λ is super-compact (in the universe $\mathbf{V} = \mathbf{V}_0$) then after some preliminary forcing $\mathbb{R}$ of cardinality λ, getting a universe $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{R}}$, in $\mathbf{V}_1$ the cardinal λ is still super-compact and $\boxdot_\lambda$ from 2.2 holds.

(2) Moreover (in part (1)), if (in $\mathbf{V}_0$) h is a $(< \lambda, \theta_*, \bar{\theta})$-Laver diamond and $\mathbf{V}_1 \models$ "$\mathbb{P}$ is $(< \lambda, \theta_*, \bar{\theta})$-directed complete" (see 1.12(3)) then in $\mathbf{V}_1^{\mathbb{P}}$ (λ is still supercompact and) the statement $\boxdot_{<\lambda,\bar{\theta}_*,\bar{\theta}}$ holds also in $\mathbf{V}_1^{\mathbb{P}}$.

Remark 2.4.

(1) The following is a major point in 2.3 and has caused some confusion. In the definition of $\boxdot_\lambda$ we demand only that the forcing notion is $(< \lambda)$-strategically complete.

To clarify, note:

(A) In the proof of $\boxplus_\lambda$ holding (i.e. 2.3(1)) we use Easton support iteration $\langle \mathbb{P}_\varepsilon, \underset{\sim}{\mathbb{Q}}_\varepsilon : \varepsilon < \lambda \rangle$ where $\underset{\sim}{\mathbb{Q}}_\varepsilon$ is a $\mathbb{P}_\varepsilon$-name of a $(< \lambda_\varepsilon)$-strategically complete forcing notion from $\mathscr{H}(\lambda)$, where $\lambda_\varepsilon = \min\{\theta : \theta \text{ regular} > \varepsilon \text{ and } \leq \|\mathbb{P}_\varepsilon\|\}$,
(B) For without loss of generality in the the proof of 2.6 we use the relevant forcing being $(< \lambda)$-directed complete but not so in 2.7).

(2) We may e.g. restrict χ to be strong limit.

Proof. (1) This is similar to the proof in Laver [Lav78] using Laver's diamond, see Definition 1.1, but we elaborate.

By Laver [Lav78] without loss of generality there is a Laver diamond $h : \lambda \to \mathscr{H}(\lambda)$. Let $E = \{\theta : \theta < \lambda \text{ is a strong limit cardinal and } \alpha < \theta \Rightarrow h(\alpha) \in \mathscr{H}(\theta)\}$, clearly a club of λ and let $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$ list $\{\theta \in E : \theta \text{ is strongly inaccessible}\}$ in increasing order.

We now define $\mathbf{q}_\varepsilon$ and $\bar{\chi}^\varepsilon$ by induction on $\varepsilon \leq \lambda$ such that:

$(*)_0$
(a) $\mathbf{q}_\varepsilon = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \leq \varepsilon, \xi < \varepsilon \rangle$ is an Easton support iteration (so $\mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi$ do not depend on ε),
(b) $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$,
(c) $\bar{\chi}^\varepsilon = \langle \chi_\zeta : \zeta < \varepsilon \rangle$ where each χ_ξ is a regular cardinal $\in [\kappa_\xi, \kappa_{\xi+1})$,
(d) $\underset{\sim}{\mathbb{Q}}_\xi \in \mathscr{H}(\chi_{\xi+1})$ is a $\mathbb{P}_\xi$-name of a $(< \kappa_\xi)$-strategically complete forcing notion,
(e) if $h(\xi) = (\underset{\sim}{\mathbb{Q}}, \chi)$ and the pair $(\underset{\sim}{\mathbb{Q}}, \chi)$ satisfies the requirements on $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\xi)$ in clauses (c),(d) then $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\xi) = h(\xi)$.

Concerning clause (b) which says "$\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$", note that for ζ a limit ordinal letting $\kappa_{<\zeta} = \cup\{\kappa_\xi : \xi < \zeta\}$ we have $\kappa_{<\zeta}$ is strong limit and:

- if $\kappa_{<\zeta}$ is regular, equivalently strongly inaccessible then $\kappa_{<\zeta} = \kappa_\zeta$ and $\mathbb{P}_\zeta = \cup\{\mathbb{P}_\xi : \xi < \zeta\}$ and so $\mathbb{P}_\zeta \subseteq \cup\{\mathscr{H}(\kappa_\xi) : \xi < \zeta\} = \mathscr{H}(\kappa_{<\zeta}) = \mathscr{H}(\kappa_\zeta)$,
- if $\kappa_{<\zeta}$ is singular, then $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_{<\zeta}^+) \subseteq \mathscr{H}(\kappa_\zeta)$ as κ_ζ is inaccessible $> \kappa_{<\zeta}$.

Easily we can carry the induction so $\mathbf{q}_\lambda$ is well defined, $\mathbb{P}_\lambda = \cup\{\mathbb{P}_\varepsilon : \varepsilon < \lambda\} \subseteq \cup\{\mathscr{H}(\kappa_\varepsilon) : \varepsilon < \lambda\} = \mathscr{H}(\lambda)$ and "$\xi < \lambda \Rightarrow \mathbb{P}_\lambda/\mathbb{P}_\xi$ is $(< \kappa_\xi)$-strategically complete" hence $\mathbb{P}_\lambda/\mathbb{P}_\xi$ adds no new sequence of length $< \kappa_\xi$ of ordinals. Clearly it is enough to prove that in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\boxdot_\lambda$.

Toward contradiction assume $\chi, \mathbb{P}, \mathscr{S} = \mathscr{S}_{\chi,\mathbb{P}}$ form a counter-example in $\mathbf{V}^{\mathbb{P}_\lambda}$, hence there are $p_* \in \mathbb{P}_\lambda$ and $\mathbb{P}_\lambda$-names $\underset{\sim}{\chi}, \underset{\sim}{\mathbb{P}}, \underset{\sim}{\mathscr{S}}, \underset{\sim}{E}$ such that $p_* \Vdash_{\mathbb{P}_\lambda}$ "$\underset{\sim}{\chi} > \lambda$ is regular, $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\underset{\sim}{\chi})$ is $(< \lambda)$-strategically complete and $\underset{\sim}{\mathscr{S}}_{\underset{\sim}{\chi},\mathbb{P}}$ is defined as in $\boxdot_\lambda$ and $\underset{\sim}{E} \subseteq [\mathscr{H}(\underset{\sim}{\chi})^{\mathbf{V}[\mathbb{P}_\lambda]}]^{<\lambda}$ is a club disjoint to $\underset{\sim}{\mathscr{S}}$".

As we can increase p_*, without loss of generality $\underset{\sim}{\chi} = \chi$ and let $x = (\chi, \underset{\sim}{\mathbb{P}})$; and as $\mathbf{V} \models$ "λ is super-compact and h is a Laver diamond" for some $(I, D, \mathbf{M}, \mathbf{j}, \mathbf{j}_0, \mathbf{j}_1)$ we have:

$(*)_1$ (a) $\mathbf{M}$ is a transitive class,
(b) $\mathbf{M}$ is a model of ZFC,
(c) ${}^{\chi}\mathbf{M} \subseteq \mathbf{M}$,
(d) $\mathbf{j}$ is an elementary embedding from $\mathbf{V}$ into $\mathbf{M}$,
(e) $\mathrm{crit}(\mathbf{j}) = \lambda$,
(f) $\mathbf{j}(h)(\lambda) = (\chi, \underset{\sim}{\mathbb{P}})$,
(g) $I = [\mathscr{H}(\chi_1)]^{<\lambda}$ and $\chi_1 > \chi$,
(h) D is a fine normal ultrafilter on I,
(i) $\mathbf{j}_0$ is the canonical elementary embedding of $\mathbf{V}$ into $\mathbf{V}^I/D$,
(j) $\mathbf{M}$ is the Mostowski Collapse of $\mathbf{V}^I/D$,
(k) $\mathbf{j}_1$ is the canonical isomorphism from $\mathbf{V}^I/D$ onto $\mathbf{M}$,
(l) $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$.

Moreover, by Definition 1.1,

$(*)_2$ $x = \mathbf{j}_1(\langle (\sup(u \cap \lambda) : u \in I\rangle/D)$.

Let $\mathbf{q} = \mathbf{j}(\mathbf{q}_\lambda)$ so $\mathbf{q} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \leq \mathbf{j}(\lambda), \xi < \mathbf{j}(\lambda)\rangle$ and $\zeta < \lambda \Rightarrow \mathbb{P}^{\mathbf{q}}_\zeta = \mathbb{P}_\zeta$, etc.
So,

$(*)_3$ in $\mathbf{M}$ the pair $x = (\chi, \underset{\sim}{\mathbb{P}})$ satisfies:
(a) $\chi \in (\lambda, \mathbf{j}(\lambda))$,
(b) $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\chi)$,
(c) $\underset{\sim}{\mathbb{P}}$ is a $\mathbb{P}_\lambda$-name of a $(< \lambda)$-strategically complete forcing notion.

[Why? Because $[\mathbf{M}]^\chi \subseteq \mathbf{M}$ hence $\mathscr{H}(\chi^+)^{\mathbf{V}} \subseteq \mathbf{M}$].
Now,

$(*)_4$ the following sets belong to D:
(a) $\mathscr{S}_1 = \{u \in I : x \in u$ and $(\mathscr{H}(\chi_1), \in){\restriction}u \prec (\mathscr{H}(\chi_1), \in)\}$
(b) $\mathscr{S}_2 = \{u \in \mathscr{S}_1 : u \cap \lambda$ is an inaccessible cardinal we call $\lambda_u\}$,
(c) $\mathscr{S}_3 = \{u \in \mathscr{S}_2$: the Mostowski Collapse of $(\mathscr{H}(\chi_1), \in){\restriction}u$ is isomorphic, for some $\chi^\dagger$ to $(\mathscr{H}(\chi^\dagger), \in)\}$.

[Why? As D is a fine and normal ultrafilter on I.]

$(*)_5$ if $u \in \mathscr{S}_3 \subseteq I$ then let χ_u be the χ guaranteed to exist by $u \in \mathscr{S}_3$ and let $\mathbf{j}_u$ be the Mostowski collapse of $(\mathscr{H}(\chi_1), \in){\restriction}u$ onto $(\mathscr{H}(\chi_u), \in)$

$(*)_6$ for every formula $\varphi = \varphi(-) \in \mathbb{L}(\{\in\})$ the following are equivalent:
(a) $(\mathscr{H}(\chi_1), \in) \models \varphi[x]$,
(b) $(\mathscr{H}(\chi_1), \in)^I/D \models \varphi[\langle h(u \cap \lambda) : u \in I\rangle/D]$,
(c) $\mathscr{X}^1_\varphi \in D$ where $\mathscr{X}^1_\varphi = \{u \in I : x \in u$ and $(\mathscr{H}(\chi_1), \in){\restriction}u \models \varphi[x]\}$,
(d) $\mathscr{X}^2_\varphi \in D$ where $\mathscr{X}^2_\varphi = \{u \in I : p_*, x \in u$ and $(\mathscr{H}(\chi_u), \in) \models \varphi[\mathbf{j}_u(x)]\}$ on,
(e) $\mathscr{X}^3_\varphi \in D$ where $\mathscr{X}^3_\varphi = \{u \in I : x \in u, \chi_u = \mathrm{otp}(\chi{\restriction}u)$ and $(\mathscr{H}(\chi_u), \in) \models \varphi[\mathbf{j}_u(x)]\}$.

[Why? We have (a) $\Leftrightarrow$ (c) as D is a fine normal ultrafilter on $I = \mathscr{H}(\chi_1)$; we have (c) $\Leftrightarrow$ (d) as $\mathbf{j}_u$ is an isomorphism from $(\mathscr{H}(\chi_1), \in){\restriction} u$ onto $\mathscr{H}(\chi_u)$; we have (d) $\Leftrightarrow$ (e) by the choice of D; lastly, (b) $\Leftrightarrow$ (c) by Łos theorem.]

Hence,

$(*)_7$ there is N as required in $\mathbf{V}^{\mathbb{P}}$.

[Why? Choose $u \in I$ which belongs to all the sets from D mentioned in $(*)_4 + (*)_6$. Let $\zeta = u \cap \lambda$, so it is inaccessible, even measurable, and $\mathbf{j}_u(x) = \mathbf{j}_u(\chi, \underset{\sim}{\mathbb{P}}) = h(\zeta)$ so (by the choice of $\mathbf{q}$) $h(\zeta) = (\chi, \underset{\sim}{\mathbb{Q}}_\zeta)$ and $\underset{\sim}{\mathbb{Q}}_\zeta$ is a $\mathbb{P}_{\mathbf{q},\zeta}$-name.

Let $\mathbf{G}$ be a generic subset of $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_\lambda$ to which p_* belongs, $\mathbf{G}_\zeta = \mathbf{G} \cap \mathbb{P}_{\mathbf{q},\zeta}$, hence it is a generic subset of $\mathbb{P}_{\mathbf{q},\zeta}$ over $\mathbf{V}$ hence a generic subset of $\mathbf{j}_u(\mathbb{P}_{\mathbf{q}}) \in \mathscr{H}(\chi_\zeta)$ and let $N = ((\mathscr{H}(\chi_1), \in){\restriction} u)[\mathbf{G}], \mathbb{A} = (\mathscr{H}(\chi_\zeta)^{\mathbf{V}[\mathbf{G}_\zeta]}, \in), M = \mathbb{A}^{\underset{\sim}{\mathbb{Q}}_\zeta[\mathbf{G}_\zeta]}$. Easily N is as promised, contradiction to the choice of p_*.]

So we are done proving part (1).

(2) Let $\mathbb{Q}$ be a forcing notion in $\mathbf{V}$ which is $(< \lambda, \theta_*, \bar{\theta})$-directed complete, $\underset{\sim}{\mathbb{P}}$ is a $\mathbb{Q}$-name of a $(< \lambda)$-strategically complete forcing notion. Let χ_1 be large enough so that $\lambda, \mathbb{Q}, \underset{\sim}{\mathbb{P}} \in \mathscr{H}(\chi_1)$ and it suffices to prove that in $\mathbf{V}^{\mathbb{Q}}$, the set $\mathscr{S}_{\chi_1,\mathbb{Q}}$ is stationary. So let $\underset{\sim}{E}$ be $\mathbb{Q}$-name and let $p \in \mathbb{P}$ be such that $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{E}$ a club of $[\mathscr{H}(\chi_1)]^{<\lambda}$ disjoint to $\underset{\sim}{\mathscr{S}}_{\chi_1,\underset{\sim}{\mathbb{P}}}$", no need to use a name for χ_1 as we can increase p.

Let $\chi \gg \chi_1$; now $\mathbb{Q} * \underset{\sim}{\mathbb{P}} \in \mathscr{H}(\chi)$ is a $(< \lambda)$-strategically complete forcing notion and without loss of generality codes (χ_1, E).

As $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$ holds in $\mathbf{V}$ we can apply it to the forcing $\mathbb{P}_{\geq p} * \underset{\sim}{\mathbb{Q}}$ so we can find a tuple $(N, \lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ witnessing it, in particular, $(p, \emptyset) \in \mathbf{G}_N, \mathbb{P} * \underset{\sim}{\mathbb{Q}} \in N$ so $\chi_1, \underset{\sim}{E} \in N$. Let $\mathbf{G}_{\mathbb{P}}$ be a subset of $\mathbb{P}$ generic over $\mathbf{V}$ which extends $\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N$ for some $\underset{\sim}{q}'\}$, possible because $\mathbf{G}_N$ is in $\mathbf{V}$, a subset of $\mathbb{P}$ which has an upper bound, this is the only place we use "$\mathbb{P}$ is $(< \lambda, \theta_*, \bar{\theta})$-directed complete ".

Next, let $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_{\mathbb{P}}], N_1 = N[\mathbf{G}_{\mathbb{P}}], E_1 = \underset{\sim}{E}[\mathbf{G}_{\mathbb{P}}], \mathbb{A}_1 = \mathbb{A}[\mathbf{j}''_N(\mathbf{G}_{\mathbb{P}} \cap N)] = \mathbb{A}[\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N\}], \mathbf{G}_1 = \{\underset{\sim}{q}[\mathbf{j}(\underset{\sim}{q}) : (p, \underset{\sim}{q}) \in \mathbf{G}_N\}$.

Now recalling p forces $\underset{\sim}{E}$ is disjoint to $\underset{\sim}{\mathscr{S}}$ clearly,

$(*)$ $N_1 \in E_1$.

Hence,

$(*)$ $N_1 \notin \mathscr{S}$.

But easily in $\mathbf{V}_1$ we have: $(\lambda_N, \chi_N, \mathbf{j}_1, \mathbb{A}_1, M_1 = M, \mathbf{G}_1)$ witnesses $N_1 \in \mathscr{S} \cap E_1$, a contradiction to the choice of $\underset{\sim}{E}$. $\square_{2.3}$

Discussion 2.5. Suppose that one wishes to force an inequality between two cardinal characteristics. There are two general approaches, which can be labeled as Top-down and Bottom-up. In the Bottom-up strategy one begins with a universe in which many characteristics are small, e.g. by assuming $2^\lambda = \lambda^+$, and then increases some of them while trying to keep the smallness of the rest. In the Top-down strategy one begins with a universe in which many characteristics are large. The forcing aims to decrease some of them while keeping the large value of the rest.

We shall use the Top-down approach, so we begin by increasing $\mathfrak{b}_\lambda$ (and $\mathfrak{d}_\lambda$) to some $\mu = \mathrm{cf}(\mu) > \lambda$. Notice that $\mathfrak{b}_\lambda$ is a relatively small characteristic and, in particular, always $\mathfrak{b}_\lambda \leq \mathfrak{d}_\lambda$. The next step will be to decrease $\mathrm{cov}_\lambda(\mathrm{meagre})$ in such a way that maintains the fact that $\mathfrak{d}_\lambda = \mu$. We shall increase $\mathfrak{b}_\lambda$ by using the generalization to λ of Hechler forcing. This is a standard way to achieve this goal, but we spell out the proof since it demonstrates the way that we employ Lemma 2.3.

Claim 2.6. *Assume that:*

(a) λ *is super-compact,*
(b) $\lambda < \mu = \mathrm{cf}(\mu) = \mu^\lambda$ *and,*
(c) $\boxdot_{<\lambda,\theta_*,\bar\theta}$ *and* $\theta_* > (2^\mu)^+$.

Then one can force $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$ *while keeping the super-compactness of* λ *and the principle* $\boxdot_{<\lambda,\theta_*,\bar\theta}$ *and 2.3 still holds by 1.14(2).*

Proof. Begin with the preparatory forcing of Lemma 2.3 to make λ indestructible and to force $\boxdot_{<\lambda,\theta_*,\bar\theta}$, hence it will be preserved by any further $(<\lambda)$-directed complete forcing. By 2.3 as in the applications of Laver-indestructibility we can assume that GCH holds above λ after the preparatory forcing. In particular, if $\mu = \mathrm{cf}(\mu) > \lambda$ then $\mu^\lambda = \mu$ follows.

Let $\mathbb{D}_\lambda$ be the generalized Hechler forcing. So a condition $p \in \mathbb{D}_\lambda$ is a pair (η_p, f_p) such that $\eta_p \in {}^{<\lambda}\lambda$, $f_p \in {}^\lambda\lambda$ and $\eta_p \trianglelefteq f_p$. If $p, q \in \mathbb{D}_\lambda$ then $p \le q$ iff $\eta_p \trianglelefteq \eta_q$ and $f_p(\alpha) \le f_q(\alpha)$ for every $\alpha \in \lambda$.

Let $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \mu, \beta < \mu\rangle$ be a $(<\lambda)$-support iteration of the generalized Hechler forcing notions for λ. Explicitly, $\underset{\sim}{\mathbb{Q}}_\alpha$ is the $\mathbb{P}_\alpha$-name of $\mathbb{D}_\lambda$ in $\mathbf{V}^{\mathbb{P}_\alpha}$ for every $\alpha < \mu$. Denote the generic λ-Hechler for $\underset{\sim}{\mathbb{Q}}_\alpha$ by $\underset{\sim}{f}^*_\alpha$. So $\mathbb{P}_\mu$ is the limit and choose a generic $\mathbf{G} \subseteq \mathbb{P}_\mu$. We claim that $\mathbf{V}[\mathbf{G}] \models$ "$\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$" as witnessed by $\langle f^*_\alpha : \alpha < \mu\rangle$. Notice that $2^\lambda = \mu$ in $\mathbf{V}[\mathbf{G}]$, so it is sufficient to prove that $\mathfrak{b}_\lambda = \mu$ in $\mathbf{V}[\mathbf{G}]$.

Since λ is regular, each $\underset{\sim}{\mathbb{Q}}_\alpha$ is $(<\lambda)$-complete and even $(<\lambda)$-directed complete. By Fact 1.14(2) $\mathbb{P}_\alpha$ is $(<\lambda)$-directed complete as well, for every $\alpha \le \mu$. Likewise, each $\underset{\sim}{\mathbb{Q}}_\alpha$ satisfies $*^\omega_\lambda$ so $\mathbb{P}_\mu$ is λ^+-c.c. (see [She78] or [She22]). It follows that $\mathbf{V}[\mathbf{G}]$ preserves cardinals and cofinalities. Moreover, no new $(<\lambda)$-sequences of ordinals are introduced. Notice also that $\mathbb{P}_\mu$ is $(<\lambda)$-directed complete and so (by 1.14(3)) it is $(<\lambda, \theta_*, \bar\theta)$-directed complete .

By 2.3(2) this implies $\mathbf{V}[\mathbf{G}] \models$ "λ is super-compact and $\boxdot_{<\lambda,\theta_*,\bar\theta}$ holds".

The main point is that $\{f^*_\alpha : \alpha < \mu\}$ is a cofinal family in $({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$. For this, assume that $\Vdash_{\mathbb{P}_\mu}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$". For every $\alpha < \lambda$ fix a maximal antichain $\langle p_{\alpha,i} : i < i_\alpha \le \lambda\rangle$ of conditions which force a value to $\underset{\sim}{f}(\alpha)$. Let $\delta = \sup(\cup\{\mathrm{dom}(p_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\})$.

Since $\lambda < \mu = \mathrm{cf}(\mu)$ we see that $\delta < \mu$, and clearly $\underset{\sim}{f}$ is a $\mathbb{P}_\delta$-name. We conclude, therefore, that $\underset{\sim}{f}$ is dominated by $\underset{\sim}{f}^*_{\delta+1}$ and hence $\{\underset{\sim}{f}^*_\alpha : \alpha < \mu\}$ exemplifies $\mathfrak{b}_\lambda = \mu$. This fact completes the proof. $\square_{2.6}$

Our second lemma is the main burden of the proof. The statement of the theorem requires λ to be super-compact, in order to obtain the indestructibility properties given by Lemma 2.3. The combinatorial part given in Lemma 2.7 below requires only strong inaccessibility. However, we assume super-compactness in order to keep $\boxdot_{<\lambda,\theta_*,\bar\theta}$,

Lemma 2.7. *Assume that:*

(a) λ *is super-compact,*

(b) $\lambda < \mu = \mathrm{cf}(\mu) = \mu^\lambda$,
(c) $\boxdot_{<\lambda,\theta_*,\bar\theta}$ *and* $\theta_* \geq (2^\mu)^+$.

Then there exists a λ^+-c.c. $(<\lambda)$-strategically forcing notion $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathfrak{d}_\lambda = \mu \wedge \mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$"; also $\Vdash_{\mathbb{P}}$ "λ is supercompact".

Proof. By claim 2.6 without loss of generality $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$. In particular, λ is super-compact and $\boxdot_{<\lambda,\theta_*,\bar\theta}$ holds in the generic extension. Let $\langle f^*_\alpha : \alpha < \mu\rangle$ witness $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$ and without loss of generality $\alpha < \beta < \mu \Rightarrow f^*_\alpha <_{J^{\mathrm{bd}}_\lambda} f^*_\beta$.

Recalling Definitions 1.8, 1.9, Claim 1.10, Theorem 1.11, in $\mathbf{V}$ there are $\beta(*), \mathbf{q}, \bar u, \mathscr{U}_*, \ldots$ such that:

$(*)_1(A)$ $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar\theta,\beta(*)}$ so in particular we have (in $\mathbf{q}$):
- (a) $\langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \leq \beta(*), \beta < \beta(*)\rangle$ is a $(<\lambda)$-support iteration,
- (b) $\bar u = \langle u_\beta : \beta < \beta(*)\rangle$, $\bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*)\rangle$,
- (c) $u_\beta \subseteq \beta$, $\mathscr{P}_\beta \subseteq [u_\beta]^{\leq\lambda}$ is closed under subsets,
- (d) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ has generic $\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$, and is $(<\lambda)$-strategically complete,
- (e) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ is as in 1.8(e) so is $\subseteq \mathbb{Q}_{\bar\theta}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha:\alpha\in u_\beta\rangle]}$ and $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda}\theta_\varepsilon$" and $\underset{\sim}{\bar\eta} = \langle \eta_\beta : \beta < \beta(*)\rangle$,
- (f) $\mathscr{U}_* \subseteq \beta(*)$ has order type κ and $\langle \beta^*_i : i \leq \kappa\rangle$ lists $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order, and $\beta(*) = \sup(\mathscr{U}_*)$,
- (g) if $\beta \in \mathscr{U}_*$ then $\mathscr{U}_* \cap \beta \subseteq u_\beta$ and $[\mathscr{U}_* \cap \beta]^{\leq\lambda} \subseteq \mathscr{P}_\beta$ and $\Vdash_{\mathbb{P}_{0,\beta+1}}$ "if $\nu \in \mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}_* \cap \beta\rangle] \cap \prod\limits_{\varepsilon<\lambda}\theta_\varepsilon$ then $\nu <_{J^{\mathrm{bd}}_\lambda} \underset{\sim}{\eta}_\beta$",
- (h) if $\alpha \leq \beta(*)$ then $\mathbb{P}_{0,\alpha}$ is $(<\lambda)$-strategically complete, $(<\lambda,\theta_*,\bar\theta)$-directed complete and λ^+-c.c.,
- (i) $\mathbb{P}_{1,\alpha}, \mathbb{P}_{1,\mathscr{U}}$ are as in 1.9,
- (j) $\theta_* \geq (2^\mu)^+$.

(B) letting $\mathbb{P}'_i = \mathbb{P}_{\mathbf{q},1,\{\beta^*_j:j<i\}}$ for $i \leq \kappa$ we have:
- (a) The sequence $\bar{\mathbb{P}}' = \langle \mathbb{P}'_i : i \leq \kappa\rangle$ of forcing notions is $\lessdot$-increasing, and is continuous for ordinals $i \leq \kappa$ of cofinality $> \lambda$ see [Sheb, 2.5(8)=Lb14(8)], but the continuity will not be used,
- (b) $\mathbb{P}'_i$ is $(<\lambda)$-strategically complete for $i \leq \kappa$,
- (c) $(\prod\limits_{\varepsilon<\lambda}\theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup\{(\prod\limits_{\varepsilon<\lambda}\theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa\}$,
- (d) The sequence $\langle \mathbb{P}_{1,\beta} : \beta \leq \beta(*)\rangle$ is a sequence of forcing notions, $\lessdot$-increasing and if $\beta \leq \beta(*)$ then $\mathbb{P}_{0,\beta} \lessdot \mathbb{P}_{1,\beta}$, in fact is dense in it and if $i \leq \kappa$ then $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$.

We shall mention more properties later.

[Why are there such objects? We apply 1.11 and 1.8 and 1.10, that is [Sheb]].

Also,

$(*)_2$
- (a) recall $\langle \beta^*_i : i \leq \kappa\rangle$ lists $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order,
- (b) for $i < \kappa$ let $\underset{\sim}{g}'_i$ be $\underset{\sim}{\eta}_{\beta^*_i}$ (to avoid excessive subscripts),
- (c) let $\underset{\sim}{\bar g}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$,
- (d) $\underset{\sim}{\mathscr{P}}_\alpha = \mathscr{P}_{\mathbf{q},\alpha}$ and without loss of generality $u_\alpha = \cup\{u : u \in \mathscr{P}_\alpha\}$ for $\alpha < \beta(*)$.

$(*)_3$ if $u \in \mathscr{P}_\alpha, \alpha < \beta(*)$ then $\Vdash_{\mathbb{P}_{0,\alpha+1}}$ "$\underset{\sim}{\eta}_\alpha \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\langle \underset{\sim}{g}_\beta:\beta\in u\rangle]}$", the order being modulo $J^{\rm bd}_\lambda$.

[Why? By the choice of the forcing, see 1.4 or $(*)_1(A)(g)$ above].

$(*)_4$ we have

(a) $\Vdash_{\mathbb{P}'_\kappa}$ "$\underset{\sim}{\bar g}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and cofinal in $(\prod_{\varepsilon<\lambda} \theta_\varepsilon, <_{J^{\rm bd}_\lambda}$)".

(b) for every $p \in \mathbb{P}_{\beta(*)}$ and $\alpha \in \mathrm{dom}(p) \cap \mathscr{U}_*$, for every large enough $i < \kappa$, we have $p \Vdash_{\mathbb{P}_{0,\beta(*)}}$ "$\underset{\sim}{f}_{p(\alpha)} \le \underset{\sim}{g}'_i = \underset{\sim}{\eta}_{\beta^*_i} \mod J^{\rm bd}_\lambda$",

(c) $\Vdash_{\mathbb{P}_{0,\beta(*)}}$ "$\underset{\sim}{\bar g}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and cofinal in $\left(\Pi_{\varepsilon<\lambda\theta_\varepsilon}, <_{J^{\rm bd}_\lambda}\right)$".

[Why? Clause (a) holds by $(*)_3$ noting that $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup\{(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa\}$ which holds by 1.11(A)(d). Clause (b) holds by 1.11(A)(e) alternatively, in [Sheb] this notation means:

- if $\alpha \in \mathscr{U}_*$ and $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}, \alpha(i) \in u_\alpha \cap (t/E'_{\mathbf{n}})$ for $i < \lambda$ and $\mathbf{B}$ a λ-Borel function from ${}^\lambda(\Pi_{\varepsilon<\lambda}\theta_\varepsilon)$ into $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$ <u>then</u> for every $i < \kappa$ large enough $\Vdash$"$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\alpha(\varepsilon)}, \ldots)_{i<\lambda} \le_{J^{\rm bd}_\lambda} g'$".

.

Clause (c) holds by $(*)_1$(A)(g)].

Now,

$(*)_5$ $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathrm{cov}_\lambda(\mathrm{meagre}) \le \kappa$".

[Why? First, notice that we can look at $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ instead of ${}^\lambda 2$.

Second, for each $\varepsilon < \lambda, i < \kappa$ the set $B_{\varepsilon,i} = \{\eta \in \prod\limits_{\xi<\lambda} \theta_\xi$: for every $\zeta \in [\varepsilon, \lambda)$ we have $\eta(\zeta) \le \underset{\sim}{g}'_i(\zeta) < \theta_\zeta\}$ is closed nowhere dense, and by $(*)_4$ we have $\mathbf{V}^{\mathbb{P}'_\kappa} \models$ "$\prod_{\zeta<\lambda} \theta_\zeta = \cup\{B_{\varepsilon,i} : \varepsilon < \lambda, i < \kappa\}$". In fact, $\langle B_{0,i} : i < \kappa\rangle$ suffice.

Alternatively we have $\langle \underset{\sim}{g}'_i : i < \kappa\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing cofinal in $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$ and let $\mathscr{W}_{i,\zeta} := \{\eta : \eta \in {}^\lambda 2$ and for every $\varepsilon \in [\zeta, \lambda)$ we have either $\eta \restriction [\Sigma_{\xi<\varepsilon}\theta_\xi, \Sigma_{\xi\le\varepsilon}\theta_\xi)$ is constantly zero or $\min\{\alpha : \Sigma_{\xi<\varepsilon}\theta_\xi + \alpha \in \eta^{-1}(\{1\})\} < \underset{\sim}{g}'_i(\varepsilon)\}$. So $\mathscr{W}_{i,\zeta}$ is a closed nowhere dense subset of ${}^\lambda 2$ and $\cup\{\mathscr{W}_{i,\zeta} : i < \kappa, \zeta < \lambda\} = {}^\lambda 2$ and $\kappa\times\lambda$ has cardinality $\lambda + \kappa = \kappa$ because if $f \in {}^\lambda 2$ then we define $\nu_f \in \Pi_{\varepsilon<\lambda}\theta_\varepsilon$ as follows: for $\varepsilon < \lambda$:

(a) if $f \restriction [\Sigma_{\xi<\varepsilon}\theta_\xi, \Sigma_{\xi\le\varepsilon}\theta_\varepsilon)$ is not constantly zero then we let $\nu_f(\varepsilon) = \min\{\alpha : f(\Sigma_{\xi<\varepsilon}\theta_\varepsilon + \alpha) = 1\}$;

(b) if otherwise then let $\nu_f(\varepsilon) = 0$.

So there are $i < \kappa$ and $\varepsilon < \lambda$ such that: $\zeta \in [\varepsilon, \lambda) \Rightarrow \nu_f(\zeta) < \underset{\sim}{g}'_i(\zeta)$. Now it is easy to check that $f \in \mathscr{W}_{i,\varepsilon}$.]

Lastly,

$(*)_7$ $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathrm{cov}_\lambda(\mathrm{meagre}) \ge \kappa$".

[Why? For $i < \kappa$ let us define the $\mathbb{P}'_{i+1}$-name $\underset{\sim}{\nu}'_i$ of a member of ${}^\lambda 2$ by $\underset{\sim}{\nu}'_i(\varepsilon) = 0$ iff $\underset{\sim}{g}'_i(\varepsilon)$ is even. Now clearly $\Vdash_{\mathbb{P}'_{i+1}}$ "$\underset{\sim}{\nu}'_i$ is a λ-Cohen sequence over $\mathbf{V}^{\mathbb{P}'_i}$". (But let us elaborate; $\underset{\sim}{\nu}'_i$ is also a $\mathbb{P}_{\beta^*_i+1}$-name and $\Vdash_{\mathbb{P}_{\beta^*_i+1}}$ "$\underset{\sim}{\nu}'_i$ is λ-Cohen over $\mathbf{V}^{\mathbb{P}_{\beta^*_i}}$ hence over $\mathbf{V}^{\mathbb{P}'_i}$"; the last hence because $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$. As $\mathbb{P}_{\beta^*_i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ and $\mathbb{P}'_{i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ we are done).

Also every closed nowhere dense subset of ${}^\lambda 2$ from $\mathbf{V}^{\mathbb{P}'_\kappa}$ is from $\mathbf{V}^{\mathbb{P}'_i}$ for some $i < \kappa$. So if $p \Vdash$ "cov$_\lambda$(meagre) $< \kappa$" then for some $\zeta < \kappa$ and $\underset{\sim}{A}_\varepsilon (\varepsilon < \zeta)$ we have $p \Vdash$ "$\underset{\sim}{A}_\varepsilon$ is a closed no-where dense subset of ${}^\lambda 2$ for $\varepsilon < \zeta$" and $p \Vdash$ "$\bigcup\limits_{\varepsilon<\zeta} \underset{\sim}{A}_\varepsilon$ is equal to the set of ${}^\lambda 2$". Without loss of generality each $\underset{\sim}{A}_\varepsilon$ is a $\mathbb{P}_{i(\varepsilon)}$-name, $i(\varepsilon) < \kappa$ and recall that κ is regular. Hence $i = \sup\{i(\varepsilon) : \varepsilon < \zeta\} < \kappa$ and $\underset{\sim}{g}'_i$ gives a contradiction to the choice of $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \zeta\rangle$; so $(*)_6$ holds indeed.]

The reader may look at some explanation in 2.9.

Now we come to the main and last point recalling $\langle f^*_\alpha : \alpha < \mu\rangle$ from Claim 2.6

$(*)_7$ $\Vdash_{\mathbb{P}'_\kappa}$ "no $\underset{\sim}{f} \in ({}^\lambda\lambda)$ dominates $\{f^*_\alpha : \alpha < \mu\}$".

We shall show that it suffices to prove $(*)_7$ for proving Lemma 2.3(2), and then that $(*)_7$ holds, thus finishing.

Why it suffices? As $\langle f^*_\alpha : \alpha < \mu\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and cf$(\mu) = \mu > \lambda$, this implies $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda \ge \mu$". Also in $\mathbf{V}, \mu^\lambda = \mu > \kappa > \lambda$ and $|\mathbb{P}'_\kappa| = \kappa^\lambda$ by (A)(g) of 1.10(4) which is $\le \mu$ and $\mathbb{P}'_\kappa$ satisfies the λ^+-c.c. hence $\Vdash_{\mathbb{P}'_\kappa}$ "$2^\lambda = \mu$", hence together $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda = \mu$". Also by $(*)_1(B)(b)$, "$\mathbb{P}'_\kappa$ is $(< \lambda)$-strategically complete and λ^+-c.c." and by $(*)_5 + (*)_6$ we know that "cov$_\lambda$(meagre) $= \kappa$" so we are done; hence $(*)_7$ is really the last piece missing. The rest of the proof is dedicated to proving that $(*)_7$ holds.

We shall use further nice properties of $\mathbb{P}'_j, \underset{\sim}{g}'_i (j \le \kappa, i < \kappa)$ which hold by $(*)_1 + (*)_2$ (and $(*)_3, (*)_4$) and their proof, i.e. 1.10, 1.11 and see [Sheb, 2.12=Lb35, 2.13=Lb38].

$\boxplus_1$ (a) (α) $\langle \underset{\sim}{g}'_\gamma : \gamma < \kappa\rangle$ is generic for $\mathbb{P}'_\kappa$, i.e., if $\mathbf{G}$ is a subset of $\mathbb{P}'_\kappa$ generic over $\mathbf{V}$ and $g'_i = \underset{\sim}{g}'_i[\mathbf{G}]$ <u>then</u> $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_i : i < \kappa\rangle]$,

(β) if in addition $\nu \in ({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$ <u>then</u> for some $\rho \in ({}^\lambda\kappa)^{\mathbf{V}}$ and λ-Borel function $\mathbf{B} \in \mathbf{V}$ we have $\nu = \mathbf{B}(\langle g'_{\rho(\varepsilon)} : \varepsilon < \lambda\rangle)$

(b) if in $\mathbf{V}[\mathbf{G}],''_\gamma \in \prod\limits_{\zeta<\lambda} \theta_\zeta$ for $\gamma < \kappa$ and the set $\{(\gamma, \zeta) : \gamma < \kappa$ and $\zeta < \lambda$ and $g''_\gamma(\zeta) \ne g'_\gamma(\zeta)\}$ has cardinality $< \lambda$ then $\bar{g}'' = \langle g''_\gamma : \gamma < \kappa\rangle$ is generic for $\mathbb{P}'_\kappa$ and $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$; similarly for $\mathbb{P}_{\beta(*)}$,

(b)$^+$ similarly for any $\gamma_\bullet \le \kappa$ and $\langle g''_\gamma : \gamma < \gamma_\bullet\rangle$, really follows,

(c) $\Vdash_{\mathbb{P}'_\kappa}$ "$\underset{\sim}{g}'_\gamma$ dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\gamma]}$" for $\gamma < \kappa$,

(d) if $\langle \zeta(\gamma) : \gamma < \kappa\rangle$ is an increasing sequence of ordinals $< \kappa$ (from $\mathbf{V}$!), then $\langle g'_{\zeta(\gamma)} : \gamma < \kappa\rangle$ is generic for $\mathbb{P}'_\kappa$ (over $\mathbf{V}$);

(e) if $\gamma \le \kappa$ then $\mathbb{P}'_\gamma$ is $(< \lambda)$-strategically complete and satisfies the λ^+-c.c..

We shall use $\boxplus_1$ freely, this (mainly $\boxplus_1(d)$) had been the motivation for [Sheb].

To prove $(*)_7$ assume toward contradiction that this fails, and hence for some condition $p^* \in \mathbb{P}'_\kappa$ and $\mathbb{P}'_\kappa$-name $\underset{\sim}{f}$ and λ-Borel function $\mathbf{B}$ (from $\mathbf{V}$) and $\rho_\bullet \in {}^\lambda\kappa$ we have:

$\circledast_0$ $p^* \Vdash_{\mathbb{P}'_\kappa}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$ and it does dominate $\{f^*_\alpha : \alpha < \mu\}$, equivalently $({}^\lambda\lambda)^{\mathbf{V}}$" and $\underset{\sim}{f} = \tilde{\mathbf{B}}(\langle \underset{\sim}{g}'_{\rho_\bullet(i)} : i < \lambda\rangle)$.

Now let χ be regular large enough and we choose $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda\rangle$ such that:

$\circledast_1$ (a) N_ε is as in $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$ for the forcing notion $\mathbb{P}_{0,\beta(*)}$ (equivalently $\mathbb{P}_{1,\beta(*)}$, <u>not</u> $\mathbb{P}'_\kappa$), that is $N_\varepsilon \in \mathscr{S}_{\chi,\mathbb{P}_{1,\beta(*)}}$ see $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$ of 2.2(2),

(b)
- $\lambda_\varepsilon = \lambda(\varepsilon) := \mathrm{otp}(N_\varepsilon \cap \lambda) > \lambda^-_\varepsilon := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\} \geq \Sigma\{\lambda_\zeta : \zeta < \varepsilon\} \geq \varepsilon$,
- $\bar{N} \restriction \varepsilon \in N_\varepsilon$ andotp$(N_\varepsilon \cap \kappa) < \theta_{\lambda(\varepsilon)}$, moreover $\mathrm{otp}(\theta_* \cap N_\varepsilon) < \theta_{\lambda(\varepsilon)}$, (really $\mathbb{P}_{1,\beta(*)} \in \mathscr{H}(\theta_*)$) hence $\bigcup\limits_{\zeta<\varepsilon} N_\zeta \subseteq N_\varepsilon$,
- λ_ε is an inaccesible cardinal.

(c) $\lambda, \kappa, \mu, \theta_*, \bar{\theta}, \mathbf{q}, \bar{\mathbb{P}}', \mathscr{U}_*, p^*, \underset{\sim}{f}, \mathbf{B}, \rho_\bullet, \bar{g}'$ belong to N_ε.

Next choose $f^* \in {}^\lambda\lambda$, i.e. $\in ({}^\lambda\lambda)^{\mathbf{V}}$, such that:

$\circledast_2$ for arbitrarily large $\varepsilon < \lambda$ for some $\zeta \in [\lambda^-_\varepsilon, \lambda_\varepsilon)$ we have $f^*(\zeta) > \lambda_\varepsilon$, (we can demand more: for every large enough $\varepsilon < \lambda$).

For $\varepsilon < \lambda$ let $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, \mathbb{A}_\varepsilon, \mathbf{G}^+_\varepsilon)$ be a witness for $(N_\varepsilon, \mathbb{P}_{1,\beta(*)}, \chi)$ recalling χ was chosen after $\circledast_1$(a) we have $\boxdot_{<\lambda,\theta_*,\bar{\theta}}$ from Definition 2.2(2) so $\lambda_\varepsilon \in (\varepsilon, \lambda)$ is strongly inaccessible and $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda^-_\zeta < \lambda_\zeta$, recalling $\circledast_1$ and noting $\langle \lambda^-_\varepsilon : \varepsilon < \lambda\rangle$ is an increasing continuous sequence of cardinals below λ. Let $\mathbf{G}^\dagger_\varepsilon = \mathbf{G}^+_\varepsilon \cap \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$ recalling $\mathbf{G}^+_\varepsilon \subseteq \mathbf{j}_\varepsilon(\mathbb{P}_{1,\beta(*)})$ and noting $\mathbf{G}^\dagger_\varepsilon \subseteq \mathbf{j}_\varepsilon(\mathbb{P}'_\kappa) \subseteq \mathbb{A}_\varepsilon$.
Let (for $\varepsilon < \lambda$):

$\circledast_3$ (a) $v_\varepsilon = N_\varepsilon \cap \kappa$,

(b) $\kappa_\varepsilon = \kappa(\varepsilon) = \mathrm{otp}(v_\varepsilon)$ and so $\kappa_\varepsilon = \mathbf{j}_\varepsilon(\kappa)$ etc,

(c) $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < \kappa_\varepsilon\rangle$ list v_ε in increasing order,

(d) for $i < \mathrm{otp}(v_\varepsilon)$, equivalently $i < \mathbf{j}_\varepsilon(\kappa) = \kappa_\varepsilon$ let $\eta^\varepsilon_i = (\mathbf{j}_\varepsilon(\underset{\sim}{g}'_{\gamma_i(\varepsilon)}))[\mathbf{G}^\dagger_\varepsilon] \in \prod\limits_{\zeta<\lambda_\varepsilon} \theta_\zeta \cap \mathbb{A}[\mathbf{G}^\dagger_\varepsilon]$,

(e) let $\bar{\eta}^\varepsilon = \langle \eta^\varepsilon_i : i < \kappa_\varepsilon\rangle$.

Note that clearly,

$\circledast_4$ for each $\varepsilon < \lambda$ we have:

(a) $\bar{\eta}^\varepsilon$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_\kappa))$, moreover

(b) if we change $\eta^\varepsilon_i(\zeta)$ (legally, i.e. to an ordinal $< \theta_\zeta$) for $< \lambda_\varepsilon$ pairs $(i,\zeta) \in \mathrm{otp}(v_\varepsilon) \times \lambda_\varepsilon$ and get $\bar{\eta}'$, <u>then</u> also $\bar{\eta}'$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_\kappa))$, clearly $N_\varepsilon[\bar{\eta}^\varepsilon] = N_\varepsilon[\bar{\eta}']$,

(c) there is a unique $\mathbf{G}'_\varepsilon$ a subset of $\mathbb{P}_{1,\beta(*)} \cap N_\varepsilon$ generic over N_ε such that $\mathbf{j}''_\varepsilon(\mathbf{G}'_\varepsilon) = \mathbf{G}^+_\varepsilon$ so $\mathscr{H}(\chi_\varepsilon) = \mathbb{A}_\varepsilon[\mathbf{j}''_\varepsilon(\mathbf{G}'_\varepsilon)]$
[Why this equality holds? By $\circledast_1$(a) and the first line after $\circledast_2$ i.e. 2.2(2); recall $\mathbf{G}^\dagger_\varepsilon = \mathbf{G}^+_\varepsilon \cap \mathbf{j}_\varepsilon(\mathbb{P}'_\kappa)$ and we have $(\mathbf{j}_\varepsilon(\underset{\sim}{\eta}_{\gamma_i(\varepsilon)}[\mathbf{G}'_\varepsilon]) = \eta^\varepsilon_i$,]

(d) like $\boxplus_1$ with $\mathbf{V}, \mathbb{P}_{1,\beta(*)}, \mathbf{G}, \lambda$ there standing for $\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}_{1,\beta(*)})\mathbf{G}^+_\varepsilon, \lambda_\varepsilon$ here.

Hence we have,

$\circledast'_4$ for $\varepsilon < \lambda$,

(a) (α) let $\Xi^\dagger_\varepsilon := \{\bar\nu : \bar\nu = \langle \nu_i : i < \kappa_\varepsilon\rangle$ and for some $\mathbf{G}^\dagger \subseteq \mathbb{P}' \cap N_\varepsilon$ generic over N_ε we have $\nu_i \in \Pi_{\xi<\lambda_\varepsilon}\theta_\xi$ satisfies $\xi < \lambda_\varepsilon \Rightarrow$ some $\psi \in \mathbf{G}^\dagger$ - forces that for each $i < \kappa_\varepsilon$, $\underset{\sim}{g}'_{\gamma_i(\varepsilon)} \restriction \xi = \nu_i \restriction \xi\}$,

(β) for $\bar\nu \in \Xi^\dagger_\varepsilon$ let $\mathbf{G}^\dagger_{\bar\nu}$ be like $\mathbf{G}^\dagger$ above, (it is uniquely determined by $\bar\nu$),

(γ) let $\Xi^+_\varepsilon = \{\bar\nu \in \Xi^\dagger_\varepsilon$: there is a subset $\mathbf{G}^+$ of $\mathbf{j}_\varepsilon(\mathbb{P}_{1,\beta(*)})$ extending $\mathbf{G}^\dagger_{\bar\nu}$ and is generic over $\mathbb{A}_\varepsilon = \mathbf{j}''_\varepsilon(N_\varepsilon)$ such that $\mathbb{A}_\varepsilon[\mathbf{G}^+] = \mathscr{H}(\chi_\varepsilon)\}$ (so $\mathbf{j}^{-1}_\varepsilon(\mathbf{G}^+)$ is a subset of $\mathbb{P}_{1,\beta(*)} \cap N_\varepsilon$ generic over N_ε and the Mostowski collapse of $N_\varepsilon[\mathbf{j}^{-1}_\varepsilon(\mathbf{G}^+)]$ is $\mathscr{H}(\chi_\varepsilon)$).

(b) (α) Let $\Xi^\bullet_\varepsilon$ be the set of pairs $(\bar\nu, \mathbf{G}^+) = (\bar\nu, \mathbf{G}(+))$ such that: $\bar\nu \in \Xi^\dagger_\varepsilon$ and $\mathbf{G}^+ \subseteq \mathbf{j}_\varepsilon(\mathbb{P}_{1,\beta(*)})$ extending $\mathbf{G}^\dagger_{\bar\nu}$ and is generic over $\mathbb{A}_\varepsilon$.

(β) We may write $(\bar\nu, \mathbf{G}^+_{\bar\nu})$ or just $\bar\nu$ though actually $\bar\nu$ does not determine $\mathbf{G}^+_{\bar\nu}$; but of course $\bar\nu$ determines $\mathbf{G}^\dagger_{\bar\nu} = \mathbf{G}^+_{\bar\nu} \cap \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$.

(γ) Note that $\bar\eta^\varepsilon$ belongs to $\Xi^\dagger_\varepsilon$ and even $(\bar\eta^\varepsilon, \mathbf{G}) \in \Xi^+_\varepsilon$ when $\mathbf{G} = \mathbf{G}^+_\varepsilon$.

[Why? See [Sheb, 3.28-3.32=Le53-Le67] and [Sheb, 4.27 = Le70]].

By the assumption toward contradiction, $\circledast_0$, and $\mathbb{P}'_\kappa$ being $(< \lambda)$-strategically complete, recalling $\boxplus_1$, there are $\zeta(*), p^{**}$ and p^+ (recall $p^* \in \mathbb{P}'_\kappa \lessdot \mathbb{P}_{1,\beta(*)}$ is from $\circledast_0$ and $\mathbb{P}_{0,\beta(*)}$ being dense in $\mathbb{P}_{1,\beta(*)}$) such that:

$\circledast_5$ (a) $p^* \le p^{**} \in \mathbb{P}'_\kappa$ and $p^+ \in \mathbb{P}_{0,\beta(*)}$ satisfies $\mathbb{P}_{1,\beta(*)} \models$ "$p^{**} \le p^+$"; (we may add that $\mathbb{P}'_\kappa \models$ "$p^{**} \le \phi$" $\Rightarrow \phi, p^+$ are compatible in $\mathbb{P}_{1,\beta(*)}$),

(b) $\zeta(*) < \lambda$,

(c) $p^{**} \Vdash_{\mathbb{P}'_\kappa}$ "$f^*(\zeta) < \underset{\sim}{f}(\zeta)$ whenever $\zeta(*) \le \zeta < \lambda$" where f^* is from $\circledast_2$,

(d) if $\gamma \in$ Dom(p^+) then $\eta^{p^+(\gamma)}$ is an object (not just a $\mathbb{P}_{0,\gamma}$-name) and has length $\ge \zeta(*)$ (recall that $\eta^{p^+(\gamma)}$ is the trunk of the condition $p^+(\gamma)$, see clause $(\alpha)(b)$ of Definition 1.4(1)).

Note that possibly Dom$(p^+) \nsubseteq \cup\{v_\varepsilon : \varepsilon < \lambda\}$. Choose $\varepsilon(*)$ such that:

$\circledast'_5$ $\varepsilon(*) < \lambda$ satisfies $\lambda_{\varepsilon(*)} > \zeta(*) + |\mathrm{dom}(p^+)|$ and $\mathrm{dom}(p^+) \cap \cup\{v_\varepsilon : \varepsilon < \lambda\} \subseteq v_{\varepsilon(*)}$ and $\gamma \in \mathrm{dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$;

Recall clause (d) of $\circledast_5$ and $|\mathrm{dom}(p^+)| < \lambda$ as $p^+ \in \mathbb{P}_{0,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is the limit of a $(< \lambda)$-support iteration.

By $\circledast_2$ we can add $(\exists\zeta)[\lambda^-_{\varepsilon(*)} \le \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$. Our intention is to find $q \in \mathbb{P}_{0,\beta(*)}$ above p^+ which (in $\mathbb{P}_{1,\beta(*)}$) is above some $q' \in \mathbb{P}'_\kappa$ which is $(N_{\varepsilon(*)}, \mathbb{P}'_\kappa)$-generic, that is it forces $\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_\kappa}$ to include a generic subset of $(\mathbb{P}'_\kappa)^{N_{\varepsilon(*)}}$ hence is induced by some $\bar\nu$ as in $\circledast'_4$, recalling $\circledast_4(b)$. Toward this in $\circledast_6$ below the intention is that $p^+_{\kappa(\varepsilon(*))}$ will serve as q.

Let $\kappa(*) = \kappa_{\varepsilon(*)}$ and γ_i for $i < \kappa(*)$ be such that[1] $\langle \gamma_i : i < \kappa(*)\rangle$ list $\{\beta^*_i : i \in v_{\varepsilon(*)}\} = \mathscr{U}_{**} = N_{\varepsilon(*)} \cap \mathscr{U}_*$ in increasing order; recall $\mathscr{U}_* = \{\beta^*_i : i < \kappa\}$ and $i < j < \kappa \Rightarrow \beta^*_i < \beta^*_j$ and $v_{\varepsilon(*)} \subseteq \kappa$ has order type $\kappa(\varepsilon(*))$ so $\gamma_i = \gamma_i(\varepsilon(*))$ from $\circledast_3$. Next let $\gamma_{\kappa(*)} = \kappa(*)$ so $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in v_{\varepsilon(*)}\} = \kappa(*) = \mathbf{j}_{\varepsilon(*)}(\kappa)$. Recall that $\kappa = \mathrm{cf}(\kappa) > \lambda, \mathrm{otp}(v_{\varepsilon(*)}) = \mathrm{otp}(N_{\varepsilon(*)} \cap \kappa)$ hence $N_{\varepsilon(*)} \models$ "$\kappa(*) = \kappa_{\varepsilon(*)}$ is a regular cardinal $> \lambda_{\varepsilon(*)}$"; hence:

[1]This is used in $\boxplus_3$ and the proof of $(*)_6$. Not to be confused with $\bar\gamma^\varepsilon$ of $\circledast_3(c)$.

(*) $\kappa(*)$ is really a regular cardinal so call it σ.

Now we define a game $\eth$ as follows[2]:

$\boxplus_2$ (A) each play lasts $\kappa(*)+1=\sigma+1$ moves and in the i-th move:

(a) if $i=j+1$ the antagonist player chooses $\xi_j=\xi(j)<\sigma$ such that $j_1<j \Rightarrow \zeta(j_1)<\xi(j)$,

(b) then, if $i=j+1$ the protagonist chooses $\zeta_j=\zeta(j)\in(\xi(j),\sigma)$, but there are more restrictions implicit in $\boxplus_3$ below,

(c) in any case (that is, also in the cases $i\le\sigma$ is a limit ordinal or zero) the protagonist also chooses $p_i^+,\bar\nu^i$ such that $\boxplus_3$ below holds.

(B) in the end of the play the protagonist wins the play iff he always has a legal move and in the end:

(a) p_σ^+ is $(\mathbb{P}'_{\gamma(*)},N_{\varepsilon(*)})$-generic, note the condition is not a member of the same forcing, so we mean that p_σ^+ forces (for $\mathbb{P}_{1,\beta(*)}$) that the intersection of the generic with $\mathbb{P}'\cap N_{\varepsilon(*)}$ is generic over $N_{\varepsilon(*)}$,

(b) $\{\zeta(i):i<\sigma\}\in\mathbb{A}_{\varepsilon(*)}$; note that trivially it belongs to $M_{\varepsilon(*)}=\mathbb{A}_{\varepsilon(*)}[\mathbf{G}_\varepsilon^+]\mathscr{H}(\chi_\varepsilon)$, see $\circledast_4(c)$.

(c) note that we do not demand that $\bar\nu'=\langle\nu_{\gamma_i}:i<\sigma\rangle$ belongs to Ξ_ε^+, we demand only that it belongs to $\Xi_\varepsilon^\dagger$ and even Ξ'_ε.

where,

$\boxplus_3$ (a) $p_i^+\in\mathbb{P}_{0,\gamma_i}$,

(b) if $j<i$ then $\mathbb{P}_{0,\gamma_i}\models$ "$p_j^+\le p_i^+$",

(c) if $\gamma\in\cup\{\mathrm{Dom}(p_j^+):j<i\}$ then $p_i^+\restriction\gamma\Vdash_{\mathbb{P}_{0,\gamma_i}}$ "$\underset{\sim}{\eta}^{p_i^+(\gamma)}$ has length $\ge i$ and $\ge\lambda_{\varepsilon(*)}$" moreover $\underset{\sim}{\eta}^{p_i^+(\gamma)}$ is an object, $\eta^{p_i^+(\gamma)}$,

(d) $\mathbb{P}_{0,\gamma_i}\models$ "$p^+\restriction\gamma_i\le p_i^+$", ($p^+$ is from $\circledast_5$(a)),

(e) (α) $\bar\nu^i=\langle\nu_{\gamma_j}:j<i\rangle$ and $\nu_{\gamma_j}\in\prod\limits_{\iota<\lambda_{\varepsilon(*)}}\theta_\iota$,

(β) $\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),i}$ is a subset of $\mathbb{P}_{\gamma_i}\cap N_{\varepsilon(*)}$ generic over $N_{\varepsilon(*)}$,

(γ) $\mathbf{G}^{\dagger}_{\varepsilon(*),i}=\mathbf{j}''_{\varepsilon(*)}(\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),i})$ so is a subset of $\mathbf{j}''_\varepsilon(\mathbb{P}'_{\gamma_2})$ generic over $\mathbb{A}_{\varepsilon(*)}$,

(δ) $\nu_j=\underset{\sim}{\eta}_{\gamma_j}[\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),i}]$ for $j<i$.

(f) for $j<i$ we have $\nu_{\gamma_j}\trianglelefteq\eta^{p_i^+(\gamma_j)}$ so $p_i^+\restriction\gamma_j\Vdash$ "$\nu_{\gamma_j}\vartriangleleft\underset{\sim}{g}'_{\gamma_j}$" recalling $\boxplus_1$,

(g) for $j<i$ (recall $\bar\eta^\varepsilon$ is from $\circledast_3(d)$) we have (α) or (β), where:

(α) $\nu_{\gamma_j}=\eta^{\varepsilon(*)}_{\gamma_{\zeta(j)}}$ recalling $\eta^{\varepsilon(*)}_{\gamma_j}$ is from $\circledast_3(d)$,

(β) $\gamma_j\in\mathrm{Dom}(p^+)$ and $\{\iota<\lambda_{\varepsilon(*)}:\eta^{\varepsilon(*)}_{\zeta(j)}(\iota)\ne\nu_{\gamma_j}(\iota)\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$.

(h) p_i^+ is an upper bound of $\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),i}$ hence is a $(N_{\varepsilon(*)},\mathbb{P}'_{\gamma_i})$-generic in the natural sense, (actually follows from clause (g), see later in $\odot$ in the beginning of the proof of 2.8).

We shall prove,

[2]The idea is to scatter the $\eta^{\varepsilon(*)}_{\gamma_i}$'s. Why not use the original places? as then we shall have a problem in $\circledast_6$; the scattering is helpful because we are relying on 1.10 and 1.11.

$\circledast_6$ in the game $\Game$:
  (a) the antagonist has no winning strategy,
  (b) at stage i, if $\langle \zeta(j) : j < i \rangle \in \mathbb{A}_\varepsilon$ then the protagonist has a legal move, moreover for any $\zeta(i) \in (\xi(i), \sigma)$ large enough the protagonist can choose it.

Why $\circledast_6$ suffice?

By clause (a) of $\circledast_6$ we can choose a play $\langle (\xi(i), \zeta(i), p_i^+, \bar{\nu}^i, \mathbf{G}^{\dagger\dagger}_{\varepsilon(*),\sigma}, \mathbf{G}^{\dagger}_{\varepsilon(*),\nu}) : i \leq \sigma \rangle$ in which the protagonist wins. Recalling $\mathbb{P}'_\kappa \lessdot \mathbb{P}_{1,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is a dense subforcing of $\mathbb{P}_{1,\beta(*)}$, clearly,

$\circledast_7$ there is p such that:
  (a) $p \in \mathbb{P}'_\kappa$,
  (b) if $\mathbb{P}'_\kappa \models$ "$p \leq p'$" hence $p' \in \mathbb{P}'_{\gamma(*)}$ <u>then</u> p', p^+_σ are compatible in $\mathbb{P}_{1,\beta(*)}$,
  (c) p is above p^{**} and it forces that $\underset{\sim}{g}'_{\gamma_i} \restriction \lambda_{\varepsilon(*)} = \nu_{\gamma_{\zeta(i)}}$ for $i < \sigma$ and $\mathbf{j}_{\varepsilon(*)}(\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\beta(*)}} \cap N_{\varepsilon(*)}) = \mathbf{G}^{\dagger}_{\langle \nu_{\gamma_i} : i < \sigma \rangle} \in \Xi^{\dagger}_{\varepsilon(*)}$.

Then on the one hand,

$\circledast_{7.1}$ $p \in \mathbb{P}'_\kappa$ being above p^{**} forces $f^* \restriction [\zeta(*), \lambda) < \underset{\sim}{f} \restriction [\zeta(*), \lambda)$ hence $f^* \restriction [\zeta(*), \lambda_{\varepsilon(*)}) < \underset{\sim}{f} \restriction [\zeta(*), \lambda_{\varepsilon(*)})$ recalling that $\zeta(*) < \lambda_{\varepsilon(*)}$, see $\circledast_5$ and the choice of $\varepsilon(*)$ immediately after $\circledast_5$.

On the other hand,

$\circledast_{7.2}$ $\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),\sigma}$ is a subset of $\mathbb{P}' \cap N_\varepsilon$ generic over N_ε.

[Why? By $\circledast_2$(e)(β) and the choice of the play.]

$\circledast_{7.3}$ p^+_σ is an upper bound of $\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),\nu}$.

[Why? By $\circledast_2$(h) and the choice of the play.]

$\circledast_{7.4}$ p is an upper bound of $\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),\nu}$ in $\mathbb{P}$.

[Why? By $\circledast_7$(b), $\circledast_{7.1}$ and $\circledast_{7.2}$].

$\circledast_{7.5}$ p is $(N_{\varepsilon(*),\sigma}, \mathbb{P}')$-generic.

[Why? If p^+_i is not an upper bound of $\mathbf{G}^{\dagger\dagger}_{\varepsilon(*),i}$ (in $\mathbb{P}_{1,\gamma_i}$) <u>then</u> there are p''_i and $r \in G^{\dagger\dagger}_{\varepsilon(*),i}$ such that $\mathbb{P}_{1,\gamma_i} \models$"$p^+_i \leq p'$" and p', r are compatible in $\mathbb{P}_{1,\gamma_i}$. As $\mathbb{P}_{0,\gamma_i}$ is a dense subset of $\mathbb{P}_{1,\gamma_i}$ and $\mathbb{P}'_{\gamma_i} \cap N_{\varepsilon(*)}$ is a subset of $\mathbb{P}_{1,\gamma_i}$ of cardinality $\leq \|N_{\varepsilon(*)}\| < \lambda_\varepsilon$, there is $p'' \in \mathbb{P}_{0,\gamma_i}$ above p'_i which decide the value of $\mathbb{P}'_{\gamma_i} \cap N_{\varepsilon(*)}$].
As $\underset{\sim}{f} \in N_{\varepsilon(*)}$ it follows from $\circledast_{7.5}$ that:

$\circledast_{7.6}$ $p \Vdash$ "$\underset{\sim}{f} \restriction \lambda_{\varepsilon(*)}$ is a function from $\lambda_{\varepsilon(*)}$ to $\lambda_{\varepsilon(*)}$".

Together $\circledast_{7.1}$ and $\circledast_{7.6}$ give a contradiction by the choice of f^* in $\circledast_2$ and of $\varepsilon(*)$ above which implies that $\underset{\sim}{f}(\zeta) > f^*(\zeta) > \lambda_{\varepsilon(*)}$ for some $\zeta < \lambda_{\varepsilon(*)}$ hence $\circledast_6$ is enough. In Lemma 2.8 below we show that $\circledast_6$ is true; so we are done. $\square_{2.7}$

Lemma 2.8. *The statement $\circledast_6$ is true.*

Proof. Note that:

$\odot$ in $\boxplus_3$, clause (h) follows.

[Why? By [Sheb, 3.93 = Le70], particularly part (5), we have $\circledast_4$(c) and the choice of the $\mathbf{G}^{\dagger\dagger}_\varepsilon$ after $\circledast_2$. In particular recall that [Sheb, 3.43 = Le70] says:

$\boxplus$ If $\mathbf{m}$ is reasonable (see [Sheb, 2.13 = Le36(3)]) then for every $p \in \mathbb{P}_\mathbf{m}$ and $s \in \mathrm{dom}(p) \cap M_\mathbf{m}$, for every large enough $t \in M_\mathbf{m}$ we have $p \Vdash_{\mathbb{P}_\mathbf{m}}$ "$\underset{\sim}{f}_{p(s)} \le \underset{\sim}{\eta}_t$ mod J^{bd}_λ"].

Let us prove $\circledast_6$; first, assuming clause (b) which is proved below, for clause (a) choose any strategy $\mathbf{st}$ for the antagonist and fix a partial strategy $\mathbf{st}'$ for the protagonist choosing $(p^+_i, \bar\nu^i)$ depending on the previous choices and $\xi(i) < \kappa_{\varepsilon(*)}$ such that it is a legal move if relevant and possible. So the only freedom left for the protagonist is to choose the $\zeta(i)$. So (recalling $\boxplus_2(A)(a)$) we have in $\mathbf{V}$ a function $F : {}^{\sigma >}\sigma \to \sigma$ (so F depends on $\mathbf{st}$ and $\mathbf{st}'$) such that:

$(*)_F$ playing the game such that the antagonist uses $\mathbf{st}$ and the protagonist uses $\mathbf{st}'$, arriving at the i-th move, $\bar\zeta = \langle \zeta(j) : j < i \rangle$ is well defined and if $\bar\zeta \in N_{\varepsilon(*)}$ then for the protagonist any choice $\zeta_i \in (F(\bar\zeta), \sigma) \cap \mathscr{U}_{**}$ is legal.

Note that F belongs to $\mathscr{H}(\chi_\varepsilon)$ unlike $p^+_\varepsilon, \bar\nu^\varepsilon$. Now we have to find an increasing sequence $\bar\zeta = \langle \zeta(i) : i < \sigma \rangle$ from $\mathbb{A}_{\varepsilon(*)}$ not just from $M_{\varepsilon(*)} = \mathscr{H}(\chi_{\varepsilon(*)})^\mathbf{V}$ such that $F(\bar\zeta{\restriction}i) < \zeta(i) < \sigma$ and $\bar\zeta \in \mathbb{A}_{\varepsilon(*)}$. Why possible? As $F \in \mathscr{H}(\chi_{\varepsilon(*)})$ and $\mathscr{H}(\chi_{\varepsilon(*)}) = M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_{\varepsilon(*)}]$ where $\mathbf{G}^+_{\varepsilon(*)}$ is a subset of $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{1,\kappa}) \in \mathbb{A}_{\varepsilon(*)}$ generic over $\mathbb{A}_{\varepsilon(*)}$ and $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c. and $\sigma = \mathrm{cf}(\sigma) > \lambda_{\varepsilon(*)}$ this[3] is possible. That is, there is a $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$-name $\underset{\sim}{F}_* \in \mathbb{A}_{\varepsilon(*)}$ such that $F = \underset{\sim}{F}_*[\mathbf{G}^+_{\varepsilon(*)}]$ and we define in $\mathbb{A}_{\varepsilon(*)}$ the function $F' : {}^{\sigma >}\sigma \to \sigma$ by $F'(\langle \zeta(j) : j < i \rangle) = \sup\{\xi + 1 : \xi \in \{\zeta(j) + 1 : j < i\}$ or $\xi < \sigma$ and $\nVdash_{\mathbf{j}(\mathbb{P}_{0,\beta(*)})}$ "$\underset{\sim}{F}(\langle \zeta(j) : j < i \rangle) \neq \xi$"$\}$; clearly this is O.K.

We are left with proving $\circledast_6(b)$.

Case 1: $i = 0$.

Let $p^+_0 = p^+ \restriction \gamma_0$.

Case 2: i limit.

By clauses (b) and (c) of $\boxplus_3$, there is $p^+_i \in \mathbb{P}_{0,\gamma_i}$ which is an upper bound (even l.u.b.) of $\{p^+_j : j < i\}$. Note that $\bar\nu_i = \langle \nu_j : j < i \rangle$ and it satisfying $\boxplus_2(B)(b)$ and $\boxplus_3$(e)(α)(f)(g), so by $\boxplus_1(b)^+$ there is $\mathbf{G}^\dagger_{\varepsilon(*),i}$ as required in $\boxplus_3(\beta), (\gamma)$.

So we are done with Case 2.

Case 3: $i = j + 1$ and $\gamma_j \notin \mathrm{dom}(p^+)$.

Clearly γ_i is in $\mathscr{U}_*$ the successor of γ_j and $(\exists \iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$. As in case 4 below but easier by the properties of the iteration and [Sheb, §3C].

Case 4: $i = j + 1$ and $\gamma_j \in \mathrm{dom}(p^+)$. Again γ_i is in $\mathscr{U}_*$ the successor of γ_j and $(\exists \iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$.

First we find p'_j such that:

$\circledast_8$ (a) $p^+_j \le p'_j \in \mathbb{P}_{0,\gamma_j}$,
(b) if $\gamma \in \mathrm{dom}(p^+_j)$ then $p'_j \restriction \gamma \Vdash$ "$\ell g(\underset{\sim}{\eta}^{p'_j(\gamma)}) > i(*) = \sigma$" (see $\boxplus_3(c)$),

[3] In fact $\mathbf{V} \models$ "$\mathbb{P}'_\kappa$ satisfies the κ-c.c." suffices.

(c) p'_j forces [4] a value to the pair $(\eta^{p^+(\gamma_j)}, \underset{\sim}{f}^{p^+(\gamma_j)} \restriction \lambda_{\varepsilon(*)})$; we call this pair $q_j = (\eta^{q_j}, f^{q_j})$.

[Why? This should be clear.]

Second,

$\circledast_9$ p^+_j hence p'_j is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$-generic and $\langle \nu_{\gamma_{j(1)}} : j(1) < j\rangle$ induces the generic.

[Why? By clause (h) of $\boxplus_2$, see $\odot$ above. Alternatively As in the proof of $\circledast''_7$ of Lemma 2.7 when we assume that we have carried the induction, by $\boxplus_2$, clause (g) and $\circledast_4$].

Now,

$\circledast_{10}$ (a) $f^{q_j} \in (\prod_{\zeta<\lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_{\varepsilon(*)}]}$; recalling that f^{q_j} is from clause (c) of $\circledast_8$.

(b) for every large enough $\zeta \in (\xi(i), \sigma)$ we have:
- $f^{q_j} \le \eta^{\varepsilon(*)}_\zeta \mod J^{\rm bd}_{\lambda_\varepsilon}$.

[Why? Clause $\circledast_{10}$(a) holds because $f^{q_j} \in (\prod\limits_{\zeta<\lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbf{V}}$, hence belongs to $\mathscr{H}(\chi_{\varepsilon(*)})$ which is the universe of $M_{\varepsilon(*)}$ so $f^{q_j} \in M_{\varepsilon(*)}$. But $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_{\varepsilon(*)}] = \mathscr{H}(\chi_{\varepsilon(*)}, \in)$ and $\bar{\eta}^{\varepsilon(*)} = \langle \mathbf{j}_{\varepsilon(*)}(\underset{\sim}{\eta}_\gamma) : \gamma \in \mathscr{U}_* \cap N_{\varepsilon(*)}\rangle$; recalling $\bar{\eta}^{\varepsilon(*)}$ is a generic for $\mathbf{j}_\varepsilon(\mathbb{P}'_\kappa)$.

For clause $\circledast_{10}$(b) recall $(*)_4(b)$. Hence $N_{\varepsilon(*)}$ satisfies the parallel statement, so $N_{\varepsilon(*)}$ satisfies: if we force by $\mathbb{P}$ then $\{\underset{\sim}{\eta}_\gamma : \gamma \in \mathscr{U}_* \cap N_{\varepsilon(*)}\}$ is cofinal in $(\Pi_{\varepsilon<\lambda_{\varepsilon(*)}} \theta_\varepsilon, \le_{J^{\rm bd}_{\lambda_{\varepsilon(*)}}})$. Note that for every ε, otp$(\mathscr{U}_* \cap N_\varepsilon)$ has cofinality $> \lambda_\varepsilon$ by the choice of N_ε.

This is a crucial point: this is justified by clause (A)(e) of 1.11.

Applying $\mathbf{j}_{\varepsilon(*)}$ and recalling $\mathbb{A}_{\varepsilon(*)}[\mathbf{G}^+_{\varepsilon(*)}] = \mathscr{H}(\chi_{\varepsilon(*)})$ we are done proving $(*)_{10}$].

Now we choose $\zeta(j) > \sup\{\zeta(j_1) : j_1 < j\}$ as in clause (b) of $\circledast_{10}$ and $\nu_j = \eta_{\zeta(j)}$; so here we obey the promise "for every large enough $\zeta(i)$". Next choose $p^+_i \in \mathbb{P}'_\kappa$ such that $p^+_i \restriction \gamma_j = p'_j, \eta^{p^+_i(\gamma_i)} = \nu_j$ and $f^{p^+_i(\gamma_j)} \restriction [\lambda_\varepsilon, \lambda) = f^{p^+(\gamma_j)} \restriction [\lambda_\varepsilon, \lambda)$ and $\nu_{\zeta(j)} \triangleleft f^{p^+_i(\gamma_j)}$".

Lastly, we choose p^+_i above p'_i as in the proof of Case 2, so we have finished Case 4.

We have carried the induction hence proved $\circledast_6(b)$ so we are done proving 2.8. $\square_{2.8}$

Discussion 2.9.

(1) The reader may justly wonder why we use $\mathbb{A}' = A[\underset{\sim}{\bar{g}}'] = \mathbb{A}[\underset{\sim}{\bar{g}} \restriction \mathscr{U}_*]$ rather than simply $\mathbb{A}[\underset{\sim}{\bar{g}}]$. Of course, nothing is lost by it, but why the extra complication?

(2) The answer is that we are committed to p^+ so $\mathbb{P}_{0,\beta(*)}$, and it is not clear why it can be increased to a condition q which is $(N_\varepsilon(*))$-generic and forces the desired statement (i.e. contradicting "f dominates $\langle f^*_\alpha : \sigma < \mu\rangle$"). We succeed to do this using $\bar{\nu}'$ which is almost equal to a suitable sub-sequence of $\eta^{\varepsilon(*)}$. So during the proof we used: if $\zeta(i) \in \mathscr{U}_*$ is increasing with $i < \kappa$ then also $\langle \underset{\sim}{\eta}''_{\zeta(i)} : i < \kappa\rangle$ is

[4] recall that $\eta^{p^+(\gamma_j)}$ is an object, not a name and p^+_j is $(N_{\varepsilon(*)}, \mathbb{P}_{\gamma_j})$-generic

generic over $\mathbf{V}$ for the sub-forcing of $\mathbb{P}_{1,\beta(*)}$ generated by $\underset{\sim}{\bar{g}}\restriction\mathscr{U}_*$; see $\circledast''_7$ inside the proof of $\circledast_6$ inside 2.8. But using $\mathscr{U}_* = \beta(*)$, we do not know this.

(3) Now in the parallel case for $\lambda = \aleph_0$ with FS-iteration with full memory, such claim is true, see §0.

(4) But we do not know the parallel of (3) for λ, so we use a substitute using $\mathscr{U}_*$, i.e. $\mathbb{P}'_\kappa$.

Claim 2.10. *In 2.7 we can add $\mathbb{P}_\kappa$ "λ is supercompact".*

Proof. Recall our forcing actually is $\mathbb{R} \times \underset{\sim}{\mathbb{P}}^1 * \underset{\sim}{\mathbb{P}}^2$, where $\mathbb{R}$ is the preparatory forcing build from Laver's diamond, so $\mathbb{R} = \mathbb{R}_\lambda$, limit of the Easton support iteration $\langle \mathbb{R}^1_\alpha, \underset{\sim}{\mathbb{R}}^0_\beta : \alpha \le \lambda, \beta < \lambda\rangle, \mathbb{P}_1$ forces $\mathfrak{d}_\lambda = \mathfrak{b}_\lambda = \mu$, and $\underset{\sim}{\mathbb{P}}^2$ is $\mathbb{P}'$ use in 2.7 (all over $\mathbf{V}_0$).

For every μ we can find (in $\mathbf{V}_0$) a transitive class $\mathbf{M}, \mathbf{M}^{<\chi} \subseteq M$ and elementary embedding $\mathbf{j} : \mathbf{V} \to \mathbf{M}$, with critical cardinal λ.

Repeating the proof of preservation of supercompactness it is enough find an upper bound for $\mathbf{j}''(\mathbf{G}^2_{\mathbb{P}})$, what is done as in the proof 2.8, with $\gamma_2 = i$. We elaborate in [Sheb, 4.28 = Le23] $\square_{2.10}$

References

[Bar87] Tomek Bartoszyński, *Combinatorial aspects of measure and category*, Fundamenta Mathematicae **127** (1987), 225–239.

[Bar10] ______, *Invariants of measure and category*, Handbook of Set Theory, Vols. 1, 2, 3 (Matthew Foreman and Akihiro Kanamori, eds.), Springer, Dordrect, 2010, pp. 491–555.

[Bla] Andreas Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), vol. 1, Springer, pp. 395–490.

[CS95] James Cummings and Saharon Shelah, *Cardinal invariants above the continuum*, Ann. Pure Appl. Logic **75** (1995), no. 3, 251–268, arXiv: math/9509228. MR 1355135

[GS93] Martin Goldstern and Saharon Shelah, *Many simple cardinal invariants*, Arch. Math. Logic **32** (1993), no. 3, 203–221, arXiv: math/9205208. MR 1201650

[GS12] Shimon Garti and Saharon Shelah, *A strong polarized relation*, J. Symbolic Logic **77** (2012), no. 3, 766–776, arXiv: 1103.0350. MR 2987137

[HS] Haim Horowitz and Saharon Shelah, *Saccharinity with ccc*, arXiv: 1610.02706.

[JS88] Haim I. Judah and Saharon Shelah, *Souslin forcing*, J. Symbolic Logic **53** (1988), no. 4, 1188–1207. MR 973109

[KS12] Jakob Kellner and Saharon Shelah, *Creature forcing and large continuum: the joy of halving*, Arch. Math. Logic **51** (2012), no. 1-2, 49–70, arXiv: 1003.3425. MR 2864397

[Lan92] A. Landver, *Baire numbers, uncountable cohen sets and perfect-set forcing*, Journal of Symbolic Logic **57** (1992), 1086–1107.

[Lav78] Richard Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, Israel J. of Math. **29** (1978), 385–388.

[Mil82] Arnold W. Miller, *A characterization of the least cardinal for which the baire category theorem fails*, Proceedings of the American Mathematical Society **86** (1982), 498–502.

[MS] Pierre Matet and Saharon Shelah, *Positive partition relations for $P_\kappa(\lambda)$*, arXiv: math/0407440.

[RS97] Andrzej Rosłanowski and Saharon Shelah, *Norms on possibilities. II. More ccc ideals on 2^ω*, J. Appl. Anal. **3** (1997), no. 1, 103–127, arXiv: math/9703222. MR 1618851

[Shea] Saharon Shelah, *Bounding forcing with chain conditions for uncountable cardinals.*

[Sheb] ______, *Corrected Iteration*, arXiv: 2108.03672.

[Shec] ______, *Creature iteration for inaccessibles.*

[She78] ______, *A weak generalization of MA to higher cardinals*, Israel J. Math. **30** (1978), no. 4, 297–306. MR 0505492

[She92] ______, *Vive la différence. I. Nonisomorphism of ultrapowers of countable models*, Set theory of the continuum (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 26, Springer, New York, 1992, arXiv: math/9201245, pp. 357–405. MR 1233826

[She00] ______, *Was Sierpiński right? IV*, J. Symbolic Logic **65** (2000), no. 3, 1031–1054, arXiv: math/9712282. MR 1791363

[She17] ______, *A parallel to the null ideal for inaccessible λ: Part I*, Arch. Math. Logic **56** (2017), no. 3-4, 319–383, arXiv: 1202.5799. MR 3633799

[She20] ______, *On* con($\mathfrak{d}_\lambda > \mathrm{cov}_\lambda$(meagre)), Trans. Amer. Math. Soc. **373** (2020), no. 8, 5351–5369, arXiv: 0904.0817. MR 4127879

[She22] ______, *Forcing axioms for λ-complete μ^+-c.c*, MLQ Math. Log. Q. **68** (2022), no. 1, 6–26, arXiv: 1310.4042. MR 4413641

[vD84] Eric K. van Douwen, *The integers and topology*, Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), Elsevier Science Publishers, 1984, pp. 111–167.

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