THE TURING DEGREES AND KEISLER'S ORDER

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ABSTRACT. There is a Turing functional Φ taking A' to a theory T_A whose complexity is exactly that of the jump of A, and which has the property that $A \leq_T B$ if and only if $T_A \trianglelefteq T_B$ in Keisler's order. In fact, by more elaborate means and related theories, we may keep the complexity at the level of A without using the jump.

Keisler's order (Definition 1.1 below) is a pre-order on complete countable first-order theories introduced by Keisler in 1967 [6], often thought of as a partial order on the equivalence classes. Informally, this order puts $T_1 \trianglelefteq T_2$ if it is "harder" for the regular ultrapowers of T_2 to be saturated than those of T_1. Sorting out the structure of this order has long been an important test problem for model theory. For orientation, we note the following. A minimum and maximum class exist [6]. The union of the smallest two classes is precisely the stable theories [13]. The maximum class includes clearly complicated theories like Peano arithmetic [6], but also any theory with linear order (with the strict order property) [13], and indeed with SOP_2 [9]. In between, no classes have yet been characterized, but we know that the random graph is in the minimum unstable class [8], [10].

A recent breakthrough in [11] has shown that Keisler's order has the maximum number of classes, continuum many, and that this is already witnessed by theories which look like "filtered random graphs" – indeed, so-called simple unstable rank 1 theories. Recall that by [13], all NIP theories (informally, those without any randomness) fall into three classes. The recent work shows that near the random graph, things are very different, due in part to interactions of model theory and finite combinatorics (see [11], §3). Indeed, [11, §12] shows that Keisler's order embeds $\mathcal{P}(\omega)/\operatorname{fin}$, in this region. At this point it was natural to ask (as was recorded in [11, 13.7], and noticed by readers of that paper, who encouraged us) whether Keisler's order embeds the "gold standard" for complexity, the Turing degrees.

The aim of this paper is to answer this question positively, hopefully as groundwork for future theorems. First we give an embedding, with no information about complexity, of an arbitrary partial order with the countable predecessor property into Keisler's order (Theorem 1.2). Then we show that in the case of the Turing degrees the complexity can be meaningfully calibrated: Theorem 3.11 shows that there is a Turing machine Φ, which on input A' produces a theory T_A, which is uniform in the jump and degree invariant, and which has the property that $A \leq_T B$ if and only if $T_A \trianglelefteq T_B$ in Keisler's order. As will be discussed below, this can be seen as a best possible effective version of Theorem 1.2 on the Turing degrees. In the last section, we show (surprisingly) that for related theories we may stay at the level of the complexity of A, avoiding the jump.

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Are there substantive connections between the structure of Turing degrees and classes of simple unstable theories? We do not assert this, but a priori, it may seem no less likely than the connections to cardinal invariants of the continuum [9].

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1. A baseline proof

Definition 1.1 (Keisler's order, [6]). *Let T_1, T_2 be complete countable first-order theories. We say $T_1 \trianglelefteq T_2$ if for every infinite λ, every regular ultrafilter $\mathcal{D}$ on λ, every model $M_1 \models T_1$, and every model $M_2 \models T_2$, if $(M_2)^\lambda/\mathcal{D}$ is λ^+-saturated, then $(M_1)^\lambda/\mathcal{D}$ is λ^+-saturated.*

Regular ultrafilters are easy to find (all nonprincipal ultrafilters on ω are regular; consistently all ultrafilters are regular), see [3] Chapter 4.3. Their significance here is that they make Keisler's order about *theories*, not models, since by a lemma of Keisler, if $M \equiv N$ in a countable language and $\mathcal{D}$ is a regular ultrafilter on λ, then either both $M^\lambda/\mathcal{D}$ and $N^\lambda/\mathcal{D}$ are λ^+-saturated, or neither is. See [6] 2.1a. For more on Keisler's order, see e.g. [6], [13, Ch. VI], [7, Ch. 1], [10, §1], or [1, §§2-3].

For our first proof, we will need the following theorem of [11], which establishes the surprising fact that Keisler's order has continuum many classes. (For orientation, note that up to to renaming of symbols in the language, there are really only continuum many complete countable theories, and of course each class must contain at least one theory; so there could not be *more* than continuum many classes.)

Theorem A (see Theorem 11.3 of [11]). *There exist continuum many complete countable simple theories $\langle T_\alpha : \alpha < 2^{\aleph_0} \rangle$ such that for any countable $u, v \subseteq 2^{\aleph_0}$, $T_u \trianglelefteq T_v$ if and only if $u \subseteq v$, where T_u denotes the disjoint union of the theories $\{T_\alpha : \alpha \in u\}$, and similarly for T_v.*

From Theorem A we may now easily derive:

Theorem 1.2. *Let $(\mathcal{T}, \leq)$ be a partial order which satisfies:*

(1) $|\mathcal{T}| \leq 2^{\aleph_0}$

(2) for every $\mathfrak{b} \in \mathcal{T}$, the set $\{\mathfrak{a} \in \mathcal{T} : \mathfrak{a} \leq \mathfrak{b}\}$ is at most countable.

Then $(\mathcal{T}, \leq)$ embeds in Keisler's order. That is, there is a map f from $\mathcal{T}$ to the set of complete countable first order theories such that for any two $\mathfrak{a}, \mathfrak{b} \in \mathcal{T}$, $\mathfrak{a} \leq \mathfrak{b}$ in $\mathcal{T}$ if and only if $f(\mathfrak{a}) \trianglelefteq f(\mathfrak{b})$.

Proof. Start with the family of theories from Theorem A above. Fix an injection g from $\mathcal{T}$ to this family, notation $\mathfrak{a} \mapsto T_{\alpha(\mathfrak{a})}$, possible by condition (1). Note that any two elements in the range of g are $\trianglelefteq$-incomparable. Now define $T_\mathfrak{a}$ to be the disjoint union of $\{T_{\alpha(\mathfrak{b})} : \mathfrak{b} \leq \mathfrak{a}\}$. This remains a countable theory by condition (2). By construction, $\mathfrak{b} \leq \mathfrak{a}$ if and only if $T_\mathfrak{b} \trianglelefteq T_\mathfrak{a}$. □

Conclusion 1.3. *There is an embedding of the partial order of Turing degrees into Keisler's order.*

Proof. The Turing degrees as a partial order satisfy the hypotheses of 1.2. □

Observe that Theorem 1.2 (or 1.3) tells us nothing a priori about the complexity of the embedding f. It is natural to hope that restricting the embedding to the

Turing degrees, which come equipped with natural notions of complexity, it may be possible to determine the complexity of the map in a meaningful way.

2. Discussion

One reason to hope for more than 1.3 on the Turing degrees is that the theories T_α in Theorem A are themselves parametrized by subsets of ω. Briefly, each theory involves two key ingredients: a fast-growing sequence of sparse finite graphs $\bar{E} = \langle E_n : n < \omega \rangle$, and a subset $A \subseteq \omega$. The set A represents the "active levels" $n \in \omega$ where constraints coming from E_n apply; at "lazy levels" $n \in \omega \setminus A$, E_n is replaced by a complete graph on the same set of vertices, which corresponds to no constraints. (In [11], instead of a set A we often write its characteristic function ξ, and there we list separately the sequence $\bar{m}$ of integers giving the sizes of the vertex sets of the E's.) The data of $\bar{E}$ and A (or ξ), and implicitly $\bar{m}$, is part of a "parameter," denoted $\mathfrak{m}$, which then determines a theory $T_{\mathfrak{m}}$. In [11], we used a fixed sequence $\bar{E}$ and continuum many sets A which were "suitably independent" (in the sense of Engelking-Karlowicz [4], see [11] 6.20-6.22) to produce the many different theories. This is a compact way of simulating many independent growth rates of graph sequences. So we can consider the parameters (thus, the theories) as given by certain subsets of ω.

Convention 2.4. *For the rest of the paper, we fix a sequence of graphs $\bar{E}$ as in* [11] §6*, so a parameter in the sense of* [11] *is specified by the additional data of the set of active levels, and it makes sense to write "$\mathfrak{m}[A]$" and "$T_{\mathfrak{m}[A]}$".*

However, consider the following theorem.[1]

Theorem B (translation of [11] Conclusion 10.25)**.** *Let $\lambda \geq \aleph_1$. Let $\mathcal{I}$ be an ideal on ω and A a set which is almost disjoint from every $B \in \mathcal{I}$. Then there exists a regular ultrafilter $\mathcal{D}$ on λ which handles every $T_{\mathfrak{m}[B]}$ for $B \in \mathcal{I}$ and does not handle $T_{\mathfrak{m}[A]}$. (In the notation of Keisler's order, $T_{\mathfrak{m}[A]} \ntrianglelefteq T_{\mathfrak{m}[B]}$ for every such $T_{\mathfrak{m}[B]}$.)*

Note that in the language of Theorem B, if $\mathcal{I}$ is countable and T is the theory corresponding to the "disjoint union" of the theories $T_{\mathfrak{m}[B]}$ for $B \in \mathcal{I}$, then a regular ultrafilter $\mathcal{D}$ handles every $T_{\mathfrak{m}[B]}$ if and only if it handles T. Theorem B suggests that in order for Keisler's order to separate theories, their sets of active levels should be quite different in the sense of the ideals they generate. Here is a theorem which says, in a certain case, this is a characterization: condition (1) says, informally, "some ultrafilter picks out precisely these theories from our family to saturate."

Theorem C (Theorem 11.10 from [11])**.** *There is a family of parameters $\{\mathfrak{m}[A] : A \subseteq \omega\}$ such that each $T_{\mathfrak{m}[A]}$ is countable, complete, and simple*[2] *and the following are equivalent for any $\lambda \geq 2^{\aleph_0}$ and any set $\mathbf{X} \subseteq \mathcal{P}(\omega)$:*

(1) There exists a regular ultrafilter $\mathcal{D}$ on λ such that $\mathbf{X} = \{A \subseteq \omega : \mathcal{D}$ is $(\lambda^+, T_{\mathfrak{m}[A]})$-good $\}$.
(2) $\mathbf{X} \supseteq [\omega]^{<\aleph_0}$ is an ideal.

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[1] "Almost" means "mod finite," and "handles" means "produces λ^+-saturated ultrapowers of" in the sense of Keisler's order.

[2] indeed with the only dividing coming from equality. "Simple" in the model theoretic sense means: there is $\kappa = \kappa(T)$ so that every type does not fork over a set of size $< \kappa$. See e.g. [5].

So, before connecting the Keisler complexity of (the theory arising from) a set of active levels to the Turing complexity of a set of natural numbers, we should deal with the fact that Turing degrees are rarely ideals. This motivates the opening move of the next section.

3. Second proof

Observation 3.5. *There is an injection $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ which is computable and has computable inverse, so that writing $\mathcal{H}$ for the range of h we have:*

(1) *the elements of $\mathcal{H}$ are almost disjoint, i.e. if $A, B \subseteq \omega$ are disjoint then $h(A) \cap h(B)$ is finite.*
(2) *for any two disjoint nonempty $\mathbf{X}, \mathbf{Y} \subseteq \mathcal{H}$ and for $\mathcal{I}, \mathcal{J}$ the ideals of subsets of $\mathbb{N}$ generated by $\mathbf{X}, \mathbf{Y}$ respectively, and for every $A \in \mathcal{I}$, A is almost disjoint from every element of $\mathcal{J}$.*
(3) *if in (2) we replace disjoint by "$\mathbf{X} \setminus \mathbf{Y} \neq \emptyset$" then there exists $A \in \mathcal{I}$ such that A is almost disjoint from every element of $\mathcal{J}$.*

Proof. Fix a computable bijection between $\mathbb{N}$ and the internal nodes of a binary tree of countable height. Identify each $A \subseteq \omega$ with its characteristic function, which uniquely determines a branch, and let h send A to the set of integers assigned to nodes on that branch. $\square$

For the rest of the paper, fix $h : \mathcal{P}(\omega) \to \mathcal{H}$ as in 3.5. Recall that $X \leq_1 Y$ means there is a total computable $1 : 1$ function such that $x \in X$ iff $f(x) \in Y$.

Remark 3.6. *A and $h(A)$ are clearly Turing-equivalent, even $\equiv_1$. Thus any Turing degree $\mathbf{a} \subseteq \mathcal{P}(\omega)$ naturally corresponds computably to a countably infinite $\mathbf{a}_{\mathcal{H}} = \{h(A) : A \in \mathbf{a}\} \subseteq \mathcal{H}$.*

Convention 3.7. *For the rest of the paper, we fix an enumeration $\{\psi_e : e < \omega\}$ of Turing machines, so that the program ψ_e is computable from the index e and vice versa, and as usual let ψ_e^A denote ψ_e with A on the input tape.*

Notation 3.8. *Let $X_{h,e,A} = h(\ \{n < \omega : \psi_e^A(n) \text{ halts and outputs } 1\}\ )$, and let $H^A = \langle X_{h,e,A} : e < \omega \rangle$.*

Observation 3.9. *Suppose $A, B \subseteq \omega$.*

(1) *The complexity of H^A is exactly that of A', the jump of A.*
(2) *If A is not Turing-reducible to B, then $H^A \setminus H^B \neq \emptyset$.*

Proof. (1) On one hand, $A' = \{n : \varphi_n^A(n) \text{ halts}\}$ can be read off from H^A, since there is a computable function f which takes an index n to an index $f(n)$ such that for any m, $\psi_{f(n)}(m)$ halts and outputs 1 if and only if $\psi_n(m)$ halts. On the other, each $X_{h,e,A}$ is c.e. in A so is $\leq_1 A'$ by the Jump Theorem [14, 3.4.3].

(2) There is a set which is c.e. relative to A and not c.e. relative to B. (By the Jump Theorem, X is c.e. in Y if and only if $X \leq_1 Y'$, and $X \leq_T Y$ if and only if $X' \leq_1 Y'$. Now A' is c.e. in A, but $\neg(A \leq_T B)$ thus $\neg(A' \leq_1 B')$.) $\square$

Corollary 3.10. *If $A, B \subseteq \omega$ and A is not Turing-reducible to B then in the notation of* 3.5*, letting $\mathbf{X} = H^A$ and $\mathbf{Y} = H^B$, and letting $\mathcal{I}, \mathcal{J}$ be the ideals of subsets of $\mathbb{N}$ generated by $\mathbf{X}$, $\mathbf{Y}$ respectively, there is $X \in \mathcal{I}$ which is almost disjoint from every element of $\mathcal{J}$.*

Theorem 3.11. *There is a Turing functional Φ taking A' to a theory T_A which satisfies:*

(a) *each T_A is a set of axioms for a complete, countable, simple unstable theory.*
(b) *Φ is uniform in the jump: the complexity of T_A is exactly A'.*
(c) *if A, B are Turing-equivalent then T_A and T_B are model-theoretically the same (i.e., up to renaming of symbols) and computable from each other.*
(d) *thus Φ is degree-invariant.*
(e) *$A \leq_T B$ if and only if $T_A \trianglelefteq T_B$ in Keisler's order.*

Discussion 3.12. *Theorem* 3.11 *can be seen as a best possible effective version of Theorem* 1.2 *on the Turing degrees, in the sense that we are getting degree invariance with no more power beyond what already accrues from the downward closure in* 1.2.

Proof sketch. The work of Φ is described explicitly in §4 below, but we give the punchline here. Fix in advance the computable signature given in §4, which observe is the union of the partial signatures τ^{ψ_e} for $e < \omega$. When Φ receives a set $A \subseteq \omega$, it divides its computation among $\{\psi^A_e\}_e$ and proceeds to list the axioms as in the §4, with $X_{h,e,A}$ (Notation 3.8) determining the rules for predicates in the partial signature τ^{ψ_e}. There is no nontrivial interaction of predicates across the partial signatures. For each ψ_e, items (0), (1), (3), (4) (5) from §4 are computable and for (2), (6), (7) it suffices to know $X_{h,e,A}$. Also from e.g. (2), one can read off the characteristic function of $X_{h,e,A}$ from the axioms restricted to τ^{ψ_e}.

As for the claims of the Theorem:

(a) This is the work of the earlier paper, see [11] 2.20 and 2.21.

(b) The uniformity follows from the description in §4. The set of axioms T_A has exactly the complexity of the jump of A, since to determine T^A it is both necessary and sufficient to know the characteristic functions of all sets computed by all the ψ^A's.

(c) Suppose $B \leq_T A$ and we are given T_A. In the notation of §4, in order to generate $T_B \restriction \tau^{\psi_i}$ it is sufficient to know what ψ_i computes with B on its input tape. Fixing a given means of computing B from A we can computably produce an index $e = e(i)$ so that ψ_e simulates ψ^B_i, and then we just copy the axioms of $T_A \restriction \tau^{\psi_e}$ replacing each predicate superscripted ψ_e with the corresponding one superscripted ψ_i.

(d) Follows from (c).

(e) If A is Turing-reducible to B, then T_A is interpretable in T_B so $T_A \trianglelefteq T_B$. If A is not Turing-reducible to B, apply 3.10 with $\mathbf{X} = \{X_{h,e,A} : e < \omega\}$ and $\mathbf{Y} = \{X_{h,e,B} : e < \omega\}$, followed by Theorem B. $\square$

Discussion 3.13. *We thank Hirschfeldt for pointing out that this proof also gives an embedding of the* enumeration *degrees,* [2] *Definition 1.1 (and thus, the Turing degrees) into Keisler's order, replacing the ψ_e's by enumeration operators.*

4. The operation of Φ

To justify Theorem 3.11 we look more carefully at the theories constructed in [11] §2, clarifying the computable content. (A motivated model theoretic exposition of these theories is in [11] §3.) Here is a high-level summary of what to look for below. The theories have three kinds of ingredients: a fast-growing sequence of natural numbers, an associated sequence of graphs, and at least one and no more than countably many subsets of ω. The signature involves unary and binary predicates. We will fix in advance the sequence of natural numbers and the sequence of graphs. We then observe that everything about these two sequences, and about the unary predicates, may be computably axiomatized. The only remaining ingredient is to list the axioms relating to the subsets of ω, which control the binary predicates. Formally, our operator will take in $A \subseteq \omega$ and produce the remaining axioms based on the countably many subsets $X_{h,e,A}$. To determine T_A it will be necessary and sufficient to know the characteristic functions of each of these subsets.

Background step and fixing notation. We choose and fix (a) a rapidly growing sequence $\langle m_n : n < \omega \rangle$ of natural numbers[3], and (b) a sequence $\langle E_n : n < \omega \rangle$ of graphs where E_n has m_n vertices which we identify with $[0, \dots, m_n - 1]$. A technical point: in E_n, each vertex is connected to itself.

(a) About $\bar{m}$: the required lower bounds on its growth rate are given in [11] Definition 6.1. We can easily choose this sequence to be computable, for instance by using the formula in [11] 6.1, with equality.
(b) About $\bar{E}$: the requirements on $\bar{E}$ are given in [11] Definition 6.2, expressing that in each E_n any small set of vertices has a common neighbor, and no large set of vertices does. For our purposes here, [11] 6.7 (which proves such sequences exist, via finite random graphs) should be understood as an existence result. Knowing that for each n, *some* graph E_n on m_n vertices exists which satisfies the requirements, we may generate the sequence computably, for instance at stage n lexicographically ordering the graphs on m_n vertices and checking in order until we find the first one which works.

In sum, the data of $\bar{m}$ and $\bar{E}$ can be chosen to be computable, even if perhaps not very efficiently.

Notation: For each n, let $\mathcal{T}_{1,n} = \mathcal{T}_{2,n}$ be the set of sequences η of length n such that $\eta(i) < m_i$ for $i \leq n$. (There are two such: a "left tree" and a "right tree," distinct but symmetric.) Let $\mathcal{T}_{1,\leq n}$, or $\mathcal{T}_{2,\leq n}$ mean all such sequences of length $\leq n$. (So these are the nodes of a finite tree of height n with m_i-branching at level i.) Let $\mathcal{T}_1 = \bigcup_n \mathcal{T}_{1,\leq n} = \mathcal{T}_2 = \bigcup_n \mathcal{T}_{2,\leq n}$ be the sets of all such finite sequences.

We now describe a set of axioms for a complete, countable theory T_A, for any $A \subseteq \omega$. The computable signature is

$$\tau = \{\mathcal{Q}^{\psi_e}, \mathcal{P}^{\psi_e}, Q^{\psi_e}_\eta, P^{\psi_e}_\nu : \eta \in \mathcal{T}_1, \nu \in \mathcal{T}_2, e < \omega\} \cup \{R^{\psi_e} : e < \omega\}$$

where each R^ψ is a binary predicate and the rest are unary. As a reminder, $X_{h,e,A}$ denotes the image under h of the set computed by ψ with A on the input tape.

The axioms we will need to enumerate are the following. Observe in these conditions that writing τ^ψ for the restriction of τ to predicates superscripted by a specific Turing machine ψ, the only nontrivial interactions between predicates are among those in the same τ^ψ, and indeed and one can think of the resulting theory as the disjoint union of its restriction to each τ^ψ.

(0) For each ψ_i, ψ_j a universal axiom stating that $(\mathcal{Q}^{\psi_i} \cup \mathcal{P}^{\psi_i}) \cap (\mathcal{Q}^{\psi_j} \cup \mathcal{P}^{\psi_j}) = \emptyset$ whenever $i \neq j$.

[3]Caution to the reader: the idea is *not* that the m_n's give a subset of $\mathbb{N}$ or otherwise relate to B. It just records that our trees at level $n \in \mathbb{N}$ have branching m_n.

(1) For every $n < \omega$ and each ψ, universal axioms stating that:[4]
  - $\mathcal{Q}^\psi$ and $\mathcal{P}^\psi$ are disjoint. Identify $\mathcal{Q}^\psi$ and $Q^\psi_{\langle\rangle}$, $\mathcal{P}^\psi$ and $P^\psi_{\langle\rangle}$.
  - $\langle Q^\psi_\eta : \eta \in \mathcal{T}_{1,m}\rangle$ partitions $\mathcal{Q}^\psi$ for each $m \leq n$ and this partition satisfies $\eta' \trianglelefteq \eta \in \mathcal{T}_{1,\leq k}$ implies $Q^\psi_{\eta'} \supseteq Q^\psi_\eta$. (Concentric predicates represent advancing along a branch.)
  - $\langle P^\psi_\nu : \nu \in \mathcal{T}_{2,m}\rangle$ partitions $\mathcal{P}^\psi$ for each $m \leq n$ and this partition satisfies $\nu' \trianglelefteq \nu \in \mathcal{T}_{2,\leq k}$ implies $P^\psi_{\nu'} \supseteq P^\psi_\nu$. (Same on the other side.)
  - $R^\psi \subseteq \mathcal{Q}^\psi \times \mathcal{P}^\psi$. ($\mathcal{R}^\psi$ holds between elements of $\mathcal{Q}^\psi$ and of $\mathcal{P}^\psi$.)

(2) If $n \in X_{h,e,A}$, add a universal axiom saying "n is an active level," meaning:
  - $R^{\psi_e}(x, y)$ only if for some $\eta_1 \in \mathcal{T}_{1,n}$ and $\eta_2 \in \mathcal{T}_{2,n}$, we have $Q^{\psi_e}_{\eta_1}(x)$ and $P^{\psi_e}_{\eta_2}(y)$ and in the graph E_n there is an edge between $\eta_1(n-1)$ and $\eta_2(n-1)$. [5]

  Informally, at "active levels" we put new constraints on the behavior of R^{ψ_e}, and at non-active ("lazy") levels there are no new constraints.

  The axioms so far enumerate a universal theory; we would like to axiomatize its model completion. [11] *Corollary 2.20 and Conclusion 2.21 show this model completion exists and is quite simple, for instance it eliminates quantifiers. The remaining axioms give the necessary information.*

(3) For each k, and each $\eta \in \mathcal{T}_{1,\leq k}$ an axiom saying: whether there exists x in $Q^{\psi_e}_\eta$ which is R^{ψ_e}-connected to $y_1, \dots, y_k$ and not to $z_1, \dots, z_k$, all in $\mathcal{P}^{\psi_e}$, depends on the quantifier-free type of $y_1, \dots, y_k, z_1, \dots, z_k$ restricted to the finite signature $\{Q^{\psi_e}_\eta : \eta \in \mathcal{T}_{1,\leq k}\} \cup \{P^{\psi_e}_\eta : \eta \in \mathcal{T}_{2,\leq k}\}$. [This can be expressed in terms of complete formulas in the variables y_i, z_j which specify the unary predicates for each variable, along with equalities and inequalities.]

(4) Parallel to (3), swapping Q/P, $T_{1,k}/T_{2,k}$, and changing the direction of R.

(5) For each k, an axiom saying: there exists x in $\mathcal{Q}^{\psi_e}$ which is R^{ψ_e}-connected to $y_1, \dots, y_k$ and not to $z_1, \dots, z_k$, all in $\mathcal{P}^{\psi_e}$, if and only if ($\{y_1, \dots, y_k\} \cap \{z_1, \dots, z_k\} = \emptyset$ and there exists x R^{ψ_e}-connected to $y_1, \dots, y_k$). [If the formula is not inconsistent, it reduces to the positive part. Since R^{ψ_e} is not symmetric, we have two copies of any such axiom, swapping $\mathcal{P}^{\psi_e}(x)$ for $\mathcal{Q}^{\psi_e}(x)$.]

(6) For each choice of
  - k,
  - $\rho \in \mathcal{T}_{1,k}$ and $\nu_1, \dots, \nu_k \in \mathcal{T}_{2,k}$, such that for all $t \leq k$, if $t \in X_{h,e,A}$ then $(\rho(t-1), \nu_i(t-1)) \in E_t$,

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[4] Informally, the signature has finitely many unary predicates which hard-code the structure of two finite height, finitely branching trees, and a binary predicate which may hold between elements of the left tree and elements of the right tree. This translates [11], Definition 2.15.

[5] Recall η_1, η_2 have domain $\{0, \dots, n-1\}$, so $\eta_i(n-1) \in [0, \dots, m_n - 1]$ for $i = 1, 2$. The condition amounts to writing down a formula which does not refer to E_n or $\mathcal{T}_1$ or $\mathcal{T}_2$ directly but simply disjuncts over the pairs $Q^{\psi_e}_{\eta_1}$, $P^{\psi_e}_{\eta_2}$ (of which there are finitely many) whose indices satisfy the condition. Informally, the pattern of "allowed connections" between the successors of a given node in the left tree and a given node in the right tree, both at level i, is given by E_i at active levels, and by a complete graph at non-active levels.

- complete quantifier-free formula $\theta(y_1, \ldots, y_k)$ in the free variables $y_1, \ldots, y_k$ such that $\theta(y_1, \ldots, y_k)$ implies $P^{\psi_e}_{\nu_1}(y_1) \wedge \cdots \wedge P^{\psi_e}_{\nu_k}(y_k)$,

an axiom saying:

$$(\exists y_1, \ldots, y_n)(\exists x) \left(\theta(y_1, \ldots, y_n) \wedge Q^{\psi_e}_{\rho}(x) \wedge \bigwedge_{1 \leq i \leq k} R^{\psi_e}(x, y_i) \right).$$

(The fact that these are the *only* cases where such x's will exist follows from the list in (2), and notice that it's enough to say this works for some such $y_1, \ldots, y_n$ because of (3).)

(7) Parallel to (6), swapping Q^{ψ_e}'s and P^{ψ_e}'s, $\mathcal{T}_1$ and $\mathcal{T}_2$, and changing the direction of R^{ψ_e}.

From the above axioms, it should be clear that for each $A \subseteq \omega$ and for each ψ_e, the theory T_A restricted to the signature τ^{ψ_e} is determined by the subset $X_{h,e,A}$, and in turn determines it.

Remark 4.14. As it is now clear how the theories depend on subsets of ω, we note that the "$\mathfrak{m}$" notation of Section 2 records this also. That is, each theory $T_A \upharpoonright \tau^{\psi_e}$ here is model-theoretically the same as $T_{\mathfrak{m}[X_{h,e,A}]}$ there, under the interpretation corresponding to erasing the superscript ψ_e from each of the predicates. Moreover, in model theoretic language, our theory T_A and the theory which is the disjoint union of the theories $\{T_{\mathfrak{m}[X_{h,e,A}]} : e < \omega\}$ can each be interpreted in the other.

5. On avoiding the jump

As noted above, Theorem 3.11 is very natural because of its relation to Theorem 1.2. However, we thank the referee for encouraging us to consider whether some other operator may be found which is uniform in A rather than A'. A priori, this may seem unlikely, and indeed, a priori, the above theories do not seem adapted to answer this question; they can even be seen as orthogonal to it, since they were built to interact with each other in some sense as freely as possible. Already in [11] some kind of disjoint union was needed whenever a dependence was called for. This is related to the fact that Keisler's order quantifies over all regular ultrafilters.

In this concluding section we prove, perhaps quite suprisingly, that the answer is yes. The construction will build on what was done above, essentially by modifying the previous section in ways which are important for computability and unimportant for model theory. It is best read with an understanding of the proof of 3.11.

We fix as before an enumeration $\{\psi_e : e < \omega\}$ of Turing machines, and working towards Theorem 5.16 below, we shall now describe the operation of Ψ which takes in $A \subseteq \omega$ and outputs a set of axioms T^*_A. Similarly to the earlier case, T^*_A will be the disjoint union of axioms $T^*_{e,A}$ for $e < \omega$. We shall fix A and e and describe $T^*_{e,A}$.

Let $X_{h,e,A}$ be as in 3.8 and let $\chi_{h,e,A}$ denote the partial characteristic function of the h-image of the set computed by ψ^A_e, that is, for $\mathbf{t} \in \{0,1\}$, $\chi_{h,e,A}(n) = \mathbf{t}$ if and only if ψ^A_e halts on input $h^{-1}(n)$ and outputs $\mathbf{t}$. Let $\chi_{h,e,A}(n,k) = \mathbf{t}$ mean that ψ^A_e halts after exactly k steps on input $h^{-1}(n)$ and outputs $\mathbf{t}$.

The computable signature for $T^*_{e,A}$ will be (note here e is fixed):

$$\tau_{e,A} = \{\mathcal{Q}^{\psi_e}_{\rho}, \mathcal{P}^{\psi_e}_{\rho}\} \cup \{Q^{\psi_e}_{\eta,\rho}, P^{\psi_e}_{\nu,\rho} : \eta \in \mathcal{T}_{1,n}, \nu \in \mathcal{T}_{2,n}, \rho \in {}^{n+1}\omega, n < \omega\} \cup \{R^{\psi_e}\}$$

where R^{ψ_e} is a binary predicate and the rest are unary.

Discussion 5.15 (Informal explanation/intention)**.** We start at level 0 with countably many copies of the predicate $Q_{\langle\rangle}$, indexed as $Q_{\langle\rangle,\langle k\rangle}$ for $k < \omega$. We start running ψ_e^A on input $h^{-1}(0)$, where observe $0 = \lg(\langle\rangle)$. If after exactly k steps the computation halts, specify that $Q_{\langle\rangle,\langle k\rangle}$ is nonempty, else specify it is empty. Do the same on the other side for the $P_{\langle\rangle,\langle k\rangle}$'s. In this case ρ is a sequence of integers of length one. So for all but at most one ρ on each side (and if one exists, it is the same ρ), these predicates will be empty, and all these decisions are clearly computable. If this computation does halt after exactly k_0 steps, then: if the output is 1, add axioms saying that 0 is an active level (i.e., for $Q_{\langle\rangle,\langle k_0\rangle}$ and $P_{\langle\rangle,\langle k_0\rangle}$), if not, just add an axiom saying $\mathcal{R} \subseteq Q_{\langle\rangle,\langle k_0\rangle} \times P_{\langle\rangle,\langle k_0\rangle}$. Now we deal with level 1. For each predicate of the form $Q_{\langle a\rangle,\rho}$ where $\rho = \langle i_0, i_1\rangle$, we have an axiom saying that $Q_{\langle a\rangle,\langle i_0,i_1\rangle} \subseteq Q_{\langle\rangle,\langle i_0\rangle}$ so most of these are immediately empty, unless k_0 from level 0 exists and $i_0 = k_0$. In this case, we run ψ_e^A on input $h^{-1}(1)$ and if after exactly k steps the computation halts, specify that $Q_{\langle a\rangle,\langle k_0,k\rangle} \neq \emptyset$, else specify it is empty. Do the same on the other side for the $P_{\langle a\rangle,\langle k_0,k\rangle}$'s. If the computation does halt after exactly k_1 steps, then if the output is 1, add axioms saying that 1 is an active level, if not, just specify that $\mathcal{R} \subseteq \bigcup_a Q_{\langle a\rangle,\langle k_0,k_1\rangle} \times \bigcup_b P_{\langle b\rangle,\langle k_0,k_1\rangle}$. And so forth. In some sense, we are choosing an "isomorphic copy" of the theory from the previous section which exists as a choice of a computable branch in a tree of computations.

The axioms are as follows. We will temporarily drop the superscript ψ_e, and writing "$Q_{\eta,\rho}$" assumes the two subscripts have appropriate and compatible lengths.

(1) $Q_{\langle\rangle,\rho} \cap P_{\langle\rangle,\rho} = \emptyset$.
(2) $Q_{\eta_1,\rho_1} \supseteq Q_{\eta_2,\rho_2}$ if $\eta_1 \trianglelefteq \eta_2$, $\rho_1 \trianglelefteq \rho_2$, and likewise for P.
(3) The predicates $Q_{\eta^\frown\langle\ell\rangle,\rho}$ partition $Q_{\eta,\rho\restriction_{\lg(\eta)+1}}$, and likewise for P.
(4) Consider $Q_{\eta,\rho}$ where $\lg(\eta) = n$. If ψ_e^A on input $h^{-1}(n)$ halts after exactly $\rho(n)$ steps, after having completed the previous computation, then add an axiom saying $Q_{\eta,\rho}$ is nonempty, otherwise add an axiom saying $Q_{\eta,\rho}$ is empty. Furthermore, if ψ_e^A on input $h^{-1}(n)$ halts after exactly $\rho(n)$ steps and outputs 1, add an axiom saying "n is an active level," i.e. $R(x,y)$ if and only if for some $\eta_1 \in \mathcal{T}_{1,n}$ and $\eta_2 \in \mathcal{T}_{2,n}$ we have $Q_{\eta_1,\rho}(x)$ and $P_{\eta_2,\rho}(y)$ and in E_n there is an edge between $\eta_1(n-1)$ and $\eta_2(n-1)$.
(5) Add the analogues of the computable axioms (3), (4), (5) above, quantifying over ρ of the appropriate length.
(6) For each choice of
   - m,
   - $\eta \in \mathcal{T}_{1,m}$ and $\nu_1, \ldots, \nu_m \in \mathcal{T}_{2,m}$, such that for all $\ell \leq m$, ψ_e^A on input $h^{-1}(\ell)$ halts after exactly $\rho(\ell)$ steps, <u>and</u> if it halts and outputs 1 then $(\eta(\ell-1), \nu_i(\ell-1)) \in E_\ell$ for $1 \leq i \leq m$,
   - complete quantifier-free formula $\theta(y_1, \ldots, y_m)$ in the free variables $y_1, \ldots, y_m$ such that $\theta(y_1, \ldots, y_m)$ implies $P_{\nu_1,\rho}(y_1) \wedge \cdots \wedge P_{\nu_m,\rho}(y_m)$,

   an axiom saying:

$$(\exists y_1, \ldots, y_n)(\exists x) \left(\theta(y_1, \ldots, y_n) \wedge Q_{\eta,\rho}(x) \wedge \bigwedge_{1 \leq i \leq m} R(x, y_i) \right).$$

(7) Parallel to (6), swapping Q's and P's, $\mathcal{T}_1$ and $\mathcal{T}_2$, and changing the direction of R.

Observe the following:

(a) The axioms are computable in A.

(b) There is at most one $\rho_* \in {}^\omega\omega$ such that for all n, for some (in fact, all) $\eta \in \mathcal{T}_{1,n}$ and $\nu \in \mathcal{T}_{2,n}$, we have that $Q_{\eta,\rho_*\restriction_{n+1}}$ is nonempty and $P_{\nu,\rho_*\restriction_{n+1}}$ is nonempty.

(c) If ψ^A_e is not a total function then all but finitely many of the predicates are empty, and thus by the earlier paper [11], T^*_A is Keisler-equivalent to the random graph, which is the minimum simple unstable theory in Keisler's order.

(d) Not only is T^*_A is computable from A, but also we can read off A from T^*_A (for instance, if we choose in advance one of the ψ's which we know computes the identity). So its complexity is exactly that of A.

(e) Using the notation of the previous section, T^*_A is model-theoretically equivalent to the disjoint union of theories whose active levels are those $X_{h,e,A}$ for which ψ^A_e is a total function. Thus, the analysis of the previous section goes through and $T^*_A \trianglelefteq T^*_B$ in Keisler's order if and only if $A \leq_T B$. However, T^*_A is *not* computability-theoretically more complicated than A, because we have distributed the information of the jump across infinitely many copies of the predicates.

So we may conclude:

Theorem 5.16. *There is a Turing machine Ψ sending sets $A \subseteq \omega$ to theories T^*_A which satisfies:*

(a) each T^*_A is a set of axioms for a complete, countable, simple unstable theory.
(b) Ψ is uniform in A: the complexity of T^*_A is exactly that of A.
(c) if A, B are Turing-equivalent then T^*_A and T^*_B are model-theoretically the same (i.e., up to renaming of symbols).
(d) thus Ψ is degree-invariant.
(e) $A \leq_T B$ if and only if $T^*_A \trianglelefteq T^*_B$ in Keisler's order.

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