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κ-Madness and definability

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Assuming the existence of a supercompact cardinal, we construct a model where, for some uncountable regular cardinal κ, there are no $\Sigma^1_1(\kappa)$ κ-mad families.

1 Introduction

The study of higher analogs of descriptive set-theoretic results has gained considerable attention during the past few years. Recent work includes new results on regularity properties, definable equivalence relations, and the connections with classification theory (cf. [4] for a survey and a list of relevant open problems).

In this paper we consider the definability of mad families from the point of view of generalised descriptive set theory. Our basic objects of study are the following:

Definition 1.1 (a) A family $\mathcal{F} \subseteq [\kappa]^\kappa$ is called κ-mad if $|A \cap B| < \kappa$ for every distinct $A, B \in \mathcal{F}$, and $\mathcal{F}$ is $\subseteq$-maximal with respect to this property.

(b) We say that $X \subseteq 2^\kappa$ is $\Sigma^1_1(\kappa)$ if there is a tree $T \subseteq \bigcup_{\alpha<\kappa} \kappa^\alpha \times 2^\alpha$ such that $X = \{\eta \in 2^\kappa :$ there is $\nu \in \kappa^\kappa$ such that $(\nu \restriction \alpha, \eta \restriction \alpha) \in T$ for every $\alpha < \kappa\}$.

Following Mathias's classical result that there are no analytic mad families (cf. [6]), it is natural to investigate the higher analogs of Mathias's result for a regular uncountable cardinal κ. It turns out that under suitable large cardinal assumptions, it is possible to construct a model where no $\Sigma^1_1(\kappa)$ κ-mad families exist, thus consistently obtaining a higher version of the result of Mathias.

The main result of the paper is Theorem 2.8, which will also be stated here:

Main result *The existence of a regular uncountable cardinal κ such that there are no $\Sigma^1_1(\kappa)$ κ-mad families is consistent relative to a supercompact cardinal.*

An important ingredient of the proof is the forcing $\mathbb{Q}_D$ in Definition 2.2. $\mathbb{Q}_D$ is a $(<\kappa)$-complete forcing adding a generic subset of κ that is almost contained in every set from the normal ultrafilter D on κ. We shall prove that such forcing notions destroy $\Sigma^1_1(\kappa)$ κ-mad families. Using a Laver-indestructible supercompact cardinal, we shall iterate those forcings to obtain the desired model

The rest of the paper will be devoted to the proof of the above result.

2 Proof of the main result

Hypothesis 2.1 *We fix a measurable cardinal κ and a normal ultrafilter D on κ.*

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We shall now define a variant of Mathias forcing (we shall use the Jerusalem notation, so $p \le q$ means that q is stronger than p):

Definition 2.2 (A) Let $\mathbb{Q} = \mathbb{Q}_D$ be the forcing notion defined as follows:

(a) $p \in \mathbb{Q}_D$ iff $p = (u, A) = (u_p, A_p)$, where $u \in [\kappa]^{<\kappa}$ and $A \in D$.

(b) $\le\ =\ \le_{\mathbb{Q}_D}$ is defined as follows: $p \le q$ iff

(1) $u_p \subseteq u_q$.

(2) $A_q \subseteq A_p$.

(3) $u_q \setminus u_p \subseteq A_p$.

(4) $\alpha < \beta$ for every $\alpha \in u_p$ and $\beta \in u_q \setminus u_p$.

(B) Let $\underset{\sim}{u}$ be the $\mathbb{Q}_D$-name for $\cup\{u_p : p \in \underset{\sim}{G}\}$.

(C) $p \le^{pr} q$ iff $p \le q$ and $u_p = u_q$.

Observation 2.3 *(a) $\mathbb{Q}_D$ is $(<\kappa)$-complete.*

(b) The sequence $(p_i : i < \kappa)$ has an upper bound if the following conditions hold:

(1) $(p_i : i < \kappa)$ is $\le^{pr}$-increasing.

(2) If $i \in \bigcap_{j<i} A_j$ and $i > \sup(u_{p_0})$, then $j \in [i, \kappa) \to i \in A_{p_j}$.

Proof. (a) By the κ-completeness of D.

(b) By the normality of D, $(u_{p_0}, \underset{i<\kappa}{\Delta} A_{p_i} \setminus u_{p_0})$ is a condition in $\mathbb{Q}_D$. It is easy to see that it is the desired upper bound. □

Claim 2.4 *Let $\mathbb{Q}$ be a forcing notion.*

(α) (A) implies (B) where:

(A) (a) $\mathbf{B}$ is a $\Sigma^1_1(\kappa)$ subset of $[\kappa]^\kappa$ and $\Vdash$ "$\underset{\sim}{X} \in \mathbf{B}$".

(b) $\chi > 2^\kappa$, $N \prec (H(\chi), \in)$, $\{\mathbf{B}, D, \underset{\sim}{X}\} \subseteq N$, $|N| = \kappa$, and $[N]^{<\kappa} \subseteq N$.

(c) $\mathbb{Q}$ is a $(<\kappa)$-complete forcing notion.

(d) $\mathbb{Q} \in N$.

(e) $G \subseteq \mathbb{Q}{\restriction}N$ is generic over N.

(B) $\underset{\sim}{X}[G]$ is $\subseteq \kappa$ and belongs to $\mathbf{B}$.

(β) (A) implies (B) where:

(A) (a) $\mathbf{B}$ is a $\Sigma^1_1(\kappa)$ subset of $[\kappa]^\kappa$ defined by the tree $T \in V$.

(b) $\mathbb{Q}$ is a $(<\kappa)$-complete forcing notion.

(c) $\mathbf{B}^{V^{\mathbb{Q}}}$ is κ-mad in $V^{\mathbb{Q}}$, where $\mathbf{B}^{V^{\mathbb{Q}}}$ is the set of all $\eta \in (2^\kappa)^{V^{\mathbb{Q}}}$ for which there is $\nu \in (\kappa^\kappa)^{V^{\mathbb{Q}}}$ such that $(\nu{\restriction}\alpha, \eta{\restriction}\alpha) \in T$ for all $\alpha < \kappa$, where T is as in Definition 1.1(b) for $\mathbf{B}$.

(B) $\mathbf{B}^V$ is κ-mad in V, where $\mathbf{B}^V$ is defined as above with V replacing $V^{\mathbb{Q}}$.

Proof. (α) For $\alpha \le \kappa$, let $T_\alpha = 2^\alpha \times \kappa^\alpha$, and for $\alpha < \beta < \kappa$ and $(\eta, \nu) \in T_\beta$, let $(\eta, \nu){\restriction}\alpha = (\eta{\restriction}\alpha, \nu{\restriction}\alpha) \in T_\alpha$. Let $T_* = \bigcup_{\alpha<\kappa} T_\alpha$, then T_κ is the set of κ-branches through T_*. There is a subtree $T \subseteq T_*$ such that $\{\eta : \exists\nu((\eta, \nu) \in \lim(T))\} = \mathbf{B}$ (where η is interpreted as $\{\alpha : \eta(\alpha) = 1\}$ and $\lim(T) = \{(\eta, \nu) : (\eta{\restriction}\alpha, \nu{\restriction}\alpha) \in T$ for all $\alpha < \kappa\}$), hence there are $(\underset{\sim}{\eta}, \underset{\sim}{\nu})$ such that $\Vdash$ "$(\underset{\sim}{\eta}, \underset{\sim}{\nu}) \in \lim(T)$ and $\underset{\sim}{X} = \{\alpha : \underset{\sim}{\eta}(\alpha) = 1\}$". Without loss of generality, $T, \underset{\sim}{\eta}, \underset{\sim}{\nu} \in N$. For each $\alpha < \kappa$, let $I_\alpha \in N$ be a dense open subset of $\mathbb{Q}$ where $I_\alpha = \{p \in \mathbb{Q} : p$ forces a value to $(\underset{\sim}{\eta}, \underset{\sim}{\nu}){\restriction}\alpha\}$. For each $\alpha < \kappa$, choose $p_\alpha \in G \cap I_\alpha$ and let $(\eta_\alpha, \nu_\alpha) \in T_\alpha$ be the value forced by p_α for $(\underset{\sim}{\eta}, \underset{\sim}{\nu}){\restriction}\alpha$. For every $\alpha < \beta < \kappa$, p_α and p_β are compatible, and hence $\eta_\alpha \le \eta_\beta$ and $\nu_\alpha \le \nu_\beta$. Let $(\eta, \nu) := (\bigcup_{\alpha<\kappa} \eta_\alpha, \bigcup_{\alpha<\kappa} \nu_\alpha) \in \lim(T)$, then $N[G] \models$ "$\underset{\sim}{X}[G] = \{\alpha : \eta(\alpha) = 1\}$", hence $\underset{\sim}{X}[G] \in \mathbf{B}$. This completes the proof of (α).

(β) Obviously, each element of $\mathbf{B}^V$ has cardinality κ and $\mathbf{B}^V$ is a κ-almost disjoint family. Let $C \in [\kappa]^\kappa$, by assumption (A)(c), $\Vdash_{\mathbb{Q}}$ "there is $D \in \mathbf{B}$ such that $|C \cap D| = \kappa$". Therefore, for some $\mathbb{Q}$-name $\underset{\sim}{\tau}$, $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in \mathbf{B}$ and $|C \cap \underset{\sim}{\tau}| = \kappa$". Fix a large enough χ and $N \prec (H(\chi), \in)$ such that $|N| = \kappa$, $[N]^{<\kappa}$ and $\{\underset{\sim}{\tau}, \mathbf{B}, C\} \subseteq N$. By the

$({<}\kappa)$-completeness of $\mathbb{Q}$, there is $G \subseteq \mathbb{Q}{\restriction}N$ which is generic over N. By part (α) of the claim, $\underset{\sim}{\tau}[G] \in \mathbf{B}^V$ and $|C \cap \underset{\sim}{\tau}[G]| = \kappa$, hence $\mathbf{B}^V$ is κ-mad in V. $\square$

Claim 2.5 *Letting $\mathbb{Q}$ denote a general forcing notion, there are no $(\mathbb{Q}, \underset{\sim}{u}, D, \mathbf{B})$ such that:*

(a) $\mathbb{Q}$ is a $({<}\kappa)$-complete forcing notion.
(b) D is a normal ultrafilter on κ.
(c) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{u} \in [\kappa]^\kappa$ and $\underset{\sim}{u} \subseteq^* A$ for every $A \in D$".
(d) $\mathbf{B} \in V$ is a $\Sigma^1_1(\kappa)$ subset of $[\kappa]^\kappa$.
(e) $\mathbf{B}^V$ is κ-mad in V.
(f) $\mathbf{B}^{V^{\mathbb{Q}}}$ is κ-mad in $V^{\mathbb{Q}}$.

Proof. Suppose towards contradiction that there are $(\mathbb{Q}, \underset{\sim}{u}, D, \mathbf{B})$ as above. Hence $\mathbf{B}$ is a $\Sigma^1_1(\kappa)$ κ-mad family in V. Fix a sequence $(A^*_i : i < \kappa) \in V$ of pairwise distinct members of $\mathbf{B}$. Let $F : \kappa \times \kappa \to \kappa$ be the function defined as

$$F(i, \alpha) := \text{the } \alpha\text{th member of } A^*_i \setminus \bigcup_{j<i} A^*_j \in [\kappa]^\kappa$$

(recalling that κ is regular and $\mathbf{B}$ is κ-almost disjoint).

Now define the following $\mathbb{Q}$-names:

(1) $\underset{\sim}{\alpha}_i$ is $\min\{\underset{\sim}{u} \setminus (i+1)\}$.

(2) $\underset{\sim}{\beta}_i$ is $F(i, \underset{\sim}{\alpha}_i)$.

(3) $\underset{\sim}{v} = \{\underset{\sim}{\beta}_i : i \in \underset{\sim}{u}$ satisfies that $\mathrm{otp}(i \cap \underset{\sim}{u})$ is even$\}$.

Let E be the filter on κ generated by the sets $\{\{F(i, \alpha) : i < \alpha$ are from $A\} : A \in D\}$. By Rowbottom's theorem, for every $A \in D$ and $X \subseteq \kappa$, if $f_X : [A]^2 \to \{0, 1\}$ is defined by $f_X(i, \alpha) = 0$ iff $F(i, \alpha) \in X$, then there exists a monochromatic $B \subseteq A$ such that $B \in D$. It follows that E is an ultrafilter. As F is injective, each set in E has cardinality κ. By the κ-completeness of D, E is also κ-complete.

Subclaim 1 $E \cap \mathbf{B} = \varnothing$.

Proof. Let $C \in \mathbf{B}$.

Case I: $C = A^*_j$ for some $j < \kappa$. Let $A \in D$ such that $\min(A) > j$, then by the definition of F, $\{F(i, \alpha) : i < \alpha$ are from $A\} \cap A^*_j = \varnothing$. It follows that $C \notin E$.

Case II: $C \in \mathbf{B} \setminus \{A^*_i : i < \kappa\}$. In this case, define $f : \kappa \to \kappa$ by $f(i) = \sup(A^*_i \cap C) + i + 1$ and let $H = \{\delta < \kappa : \delta$ is a limit ordinal such that $f(i) < \delta$ for all $i < \delta\}$. So $H \subseteq \kappa$ is a club, hence $H \in D$ and $H^* := \{F(i, \alpha) : i < \alpha$ are from $H\} \in E$. Suppose that $F(i, \alpha) \in H^*$; if $F(i, \alpha) \in C$, then $\alpha \leq F(i, \alpha) < f(i) < \alpha$, a contradiction. It follows that $C \notin E$.

This proves Subclaim 1. $\square$

We shall now return to the proof of the main claim. Suppose towards contradiction that $\mathbf{B}^{V^{\mathbb{Q}}}$ is κ-mad in $V^{\mathbb{Q}}$. As $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{v} \in [\kappa]^\kappa$", there is a $\mathbb{Q}$-name $\underset{\sim}{\tau}$ of a member of $\mathbf{B}^{V^{\mathbb{Q}}}$ such that $\Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{v} \cap \underset{\sim}{\tau}| = \kappa$". For every $p \in \mathbb{Q}$, let $B^+_p = \{\alpha < \kappa : p \nVdash$ "$\alpha \notin \underset{\sim}{\tau}$"$\}$.

Subclaim 2 $B^+_p \in E$.

Proof. Suppose towards contradiction that $B^+_p \notin E$, then there is some $C_p \in D$ such that $B^+_p \cap \{F(i, \alpha) : i < \alpha$ are from $C_p\} = \varnothing$. Therefore, if $i < \alpha$ are from C_p then $p \Vdash$ "$F(i, \alpha) \notin \underset{\sim}{\tau}$". Recalling that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{u} \subseteq^* C_p$", it follows that $p \Vdash$ "$\underset{\sim}{\alpha}_i \in C_p$ for i large enough", and also $p \Vdash$ "for i large enough, $i \in \underset{\sim}{u} \to i \in C_p$". Therefore, $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\beta}_i = F(i, \underset{\sim}{\alpha}_i) \notin \underset{\sim}{\tau}$ for every large enough $i \in \underset{\sim}{u}$". Recalling the definition of $\underset{\sim}{v}$, it follows that $p \Vdash$ "$|\underset{\sim}{v} \cap \underset{\sim}{\tau}| < \kappa$", contradicting the choice of $\underset{\sim}{\tau}$. It follows that $B^+_p \in E$, which completes the proof of Subclaim 2. $\square$

For every $p \in \mathbb{Q}$, let $B_p^- = \{\alpha < \kappa : p \nVdash \text{“}\alpha \in \underset{\sim}{\tau}\text{”}\}$.

Subclaim 3 $B_p^- \in E$.

Proof. Suppose not, then $B_* := \kappa \setminus B_p^- \in E$ (hence $B_* \in [\kappa]^\kappa$) and $p \Vdash$ “$B_* \subseteq \underset{\sim}{\tau}$”. By the κ-madness of $\mathbf{B}$, there is $C \in \mathbf{B}$ (in V) such that $|C \cap B_*| = \kappa$. As $p \Vdash$ “$B_* \cap C \subseteq \underset{\sim}{\tau}, \underset{\sim}{\tau} \in \mathbf{B}$ and $\mathbf{B}$ is κ-mad”, it follows that $p \Vdash$ “$\underset{\sim}{\tau} = C$”. We shall derive a contradiction by showing that $\Vdash_{\mathbb{Q}}$ “$|\underset{\sim}{v} \cap C| < \kappa$”: Choose i_* such that $C \neq A_i^*$ for every $i \in [i_*, \kappa)$. It follows that $|C \cap A_i^*| < \kappa$ for every $i \in [i_*, \kappa)$. Now repeat the argument of Case II in the proof of Subclaim 1 and choose f, H and H^* as there. As $H \in D$, $\Vdash_{\mathbb{Q}}$ “for large enough i, $i \in \underset{\sim}{u} \to i, \underset{\sim}{\alpha_i} \in H$”. Repeating the same argument as in Subclaim 1, $\Vdash_{\mathbb{Q}}$ “for large enough $i \in \underset{\sim}{u}$, $\underset{\sim}{\beta_i} = F(i, \underset{\sim}{\alpha_i}) \in H^*$, hence $\underset{\sim}{\beta_i} \notin C$”. It follows that $\Vdash_{\mathbb{Q}}$ “$|\underset{\sim}{v} \cap C| < \kappa$”, leading to a contradiction. This completes the proof of Subclaim 3. □

Observation 2.6 *(A) Given $p_1, p_2 \in \mathbb{Q}$ and $\alpha < \kappa$, there exist (q_1, q_2, β) such that:*

(a) $p_l \le_{\mathbb{Q}} q_l$ $(l = 1, 2)$.

(b) $\beta \in [\alpha, \kappa)$.

(c) $q_1 \Vdash$ “$\beta \in \underset{\sim}{\tau}$”.

(d) $q_2 \Vdash$ “$\beta \notin \underset{\sim}{\tau}$”.

(B) As in (A), with (d) replaced by the following:

(d′) $q_2 \Vdash$ “$\beta \in \underset{\sim}{\tau}$”.

Proof. By the previous subclaims, $B_{p_1}^+ \cap B_{p_2}^-, B_{p_1}^+ \cap B_{p_2}^+ \in E$, hence there exist $\beta \in (B_{p_1}^+ \cap B_{p_2}^-) \setminus \alpha$ and $\delta \in (B_{p_1}^+ \cap B_{p_2}^+) \setminus \alpha$. By the definitions of $B_p^{+/-}$, there exist $q_1 \ge p_1$ and $q_2 \ge p_1$ such that (q_1, q_2, β) are as required, and similarly for δ and (B). This proves the observation. □

Let $\chi = (2^\kappa)^+$ and $N \prec (H(\chi), \in)$ such that $|N| = \kappa$, $N^{<\kappa} \subseteq N$, $\kappa \subseteq N$ and $\underset{\sim}{\tau}, D, \mathbf{B} \in N$. Let $(I_i : i < \kappa)$ list the dense open subsets of $\mathbb{Q}$ from N. We shall now choose (p_i^1, p_i^2, γ_i) by induction on $i < \kappa$ such that:

(a) $p_i^1, p_i^2 \in \mathbb{Q} \cap N$ and $\gamma_i \in N$.

(b) $i < j \to p_i^l \le_{\mathbb{Q}} p_j^l$ $(l = 1, 2)$.

(c) If $i = 4j + 1$, then $p_i^1, p_i^2 \in I_j$.

(d) $\gamma_i \in \kappa \setminus \bigcup\limits_{j<i} (\gamma_j + 1)$.

(e) If $i = 4j + 2$, then $p_i^1 \Vdash$ “$\gamma_{4j+2} \in \underset{\sim}{\tau}$” and $p_i^2 \Vdash$ “$\gamma_{4j+2} \in \underset{\sim}{\tau}$”.

(f) If $i = 4j + 3$, then $p_i^1 \Vdash$ “$\gamma_{4j+3} \in \underset{\sim}{\tau}$” and $p_i^2 \Vdash$ “$\gamma_{4j+3} \notin \underset{\sim}{\tau}$”.

(g) If $i = 4j + 4$, then $p_i^1 \Vdash$ “$\gamma_{4j+4} \notin \underset{\sim}{\tau}$” and $p_i^2 \Vdash$ “$\gamma_{4j+4} \in \underset{\sim}{\tau}$”.

Observation 2.7 *It is possible to choose (p_i^1, p_i^2, γ_i) as above for each $i < \kappa$.*

Proof. Case I: $i = 0$. This is trivial.

Case II: i is a limit ordinal: As $N^{<\kappa} \subseteq N$ and $(p_j^l : j < i), (\gamma_j : j < i) \in N$, we can find p_i^1 and p_i^2 using the $(<\kappa)$-completeness of $\mathbb{Q}$ and elementarity. As κ is regular, there is no problem to choose γ_i.

Case III: $i = 4j + 1$: As $p_j^1, p_j^2, I_j \in N$, by elementarity there exist p_i^1 and p_i^2 as required.

Case IV: $i = 4j + 2$: Use Observation 2.3(B).

Case V: $i = 4j + 3$: Use Observation 2.3(A).

Case VI: $i = 4j + 4$: Use Observation 2.3(A), with (p_i^2, p_i^1) here standing for (p_1, p_2) there.

Finally, let $G_l = \{q \in \mathbb{Q} \cap N : q \le_{\mathbb{Q}} p_i^l$ for some $i < \kappa\}$ $(l = 1, 2)$, then $G_l \subseteq \mathbb{Q} \cap N$ is generic over N. By Claim 1(α), $C_l := \underset{\sim}{\tau}[G_l] \in \mathbf{B}$. By the choice of (p_i^1, p_i^2, γ_i), $\{\gamma_{4i+2} : i < \kappa\} \subseteq C_1 \cap C_2$, hence $C_1 \cap C_2 \in [\kappa]^\kappa$. Similarly, $|\{\gamma_{4i+3} : i < \kappa\}| = \kappa$ and $\{\gamma_{4i+3} : i < \kappa\} \subseteq C_1 \setminus C_2$, hence $C_1 \neq C_2$. This contradicts the κ-madness of $\mathbf{B}$ in V. □

This completes the proof of Claim 2. □

Before formulating and proving our main result, we remind the reader of the notion of a Laver-indestructible supercompact cardinal. It was shown by Laver in [5] that if κ is supercompact, then there is a κ-c.c. forcing $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, κ is supercompact and its supercompactness is preserved under forcing with $(<\kappa)$-directed closed posets. This will be used in the proof below to construct our iteration, where we have to guarantee that for unboundedly many $\alpha < \delta$ we can find a normal ultrafilter on κ in $V^{\mathbb{P}_\alpha}$.

Theorem 2.8 *If κ is a Laver-indestructible supercompact cardinal, then there is a generic extension where κ is supercompact, and there are no $\Sigma^1_1(\kappa)$ κ-mad families.*

Proof. We recall the following strong version of κ^+-c.c. (cf., e.g., [7, 8]). A forcing $\mathbb{Q}$ satisfies $*^1_{\kappa,\mathbb{Q}}$ if:

(a) $\mathbb{Q}$ is $(<\kappa)$-complete.
(b) If $\{p_\alpha : \alpha < \kappa^+\} \subseteq \mathbb{Q}$, then for some club $E \subseteq \kappa^+$ and regressive function f on E we have $(\delta_1, \delta_2 \in E \wedge f(\delta_1) = f(\delta_2)) \to p_{\delta_1}, p_{\delta_2}$ are compatible.
(c) Every two compatible conditions in $\mathbb{Q}$ have a least upper bound.

Obviously, $*^1_{\kappa,\mathbb{Q}}$ implies κ^+-c.c. By [7], $*^1_{\kappa,\mathbb{Q}}$ is preserved under $(<\kappa)$-support iterations.

It is easy to verify that $\mathbb{Q} = \mathbb{Q}_D$ satisfies $*^1_{\kappa,\mathbb{Q}}$ when D is a normal ultrafilter on κ (e.g., fix a bijection $g : [\kappa]^{<\kappa} \to \kappa$, and for every $\{p_\alpha : \alpha < \kappa^+\}$, let $E = (\kappa, \kappa^+)$ and let $f : E \to \kappa^+$ be defined by $f(\alpha) = g(u_\alpha)$ where $p_\alpha = (u_\alpha, A_\alpha)$)

Let $(\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta)$ be a $(<\kappa)$-support iteration such that:

(a) $cf(\delta) > \kappa$.
(b) Each $\underset{\sim}{\mathbb{Q}}_\beta$ is $*^1_{\kappa,\underset{\sim}{\mathbb{Q}}_\beta}$.
(c) $\delta = \sup\{\alpha < \delta : \text{in } V^{\mathbb{P}_\alpha}, \underset{\sim}{\mathbb{Q}}_\alpha = \mathbb{Q}_{\underset{\sim}{D}_\alpha}$, where $\underset{\sim}{D}_\alpha$ is a $\mathbb{P}_\alpha$-name of a normal ultrafilter on $\kappa\}$.

As κ is a Laver indestructible supercompact cardinal, there is an iteration as above. Suppose towards contradiction that there is a $\Sigma^1_1(\kappa)$ κ-mad family $\mathbf{B}$ in $V^{\mathbb{P}_\delta}$. $\mathbf{B} = \{\eta : \exists \nu((\eta, \nu) \in \lim(T))\}$ for a suitable tree T. By the fact that $cf(\delta) > \kappa$ and $\mathbb{P}_\delta$ is κ^+-c.c., it follows that $T \in V^{\mathbb{P}_\beta}$ for some $\beta < \delta$. Let $\gamma \in [\beta, \delta)$ such that $\underset{\sim}{\mathbb{Q}}_\gamma = \mathbb{Q}_{\underset{\sim}{D}_\gamma}$ where $\underset{\sim}{D}_\gamma$ is a $\mathbb{P}_\gamma$-name of a normal ultrafilter on κ. By Claim 1(β), $\mathbf{B}^{V^{\mathbb{P}_\gamma}}$ is κ-mad in $V^{\mathbb{P}_\gamma}$.

Applying Claim 2 to $V_1 = V^{\mathbb{P}_\gamma}$, $\mathbb{Q} = \mathbb{P}_\delta/\mathbb{P}_\gamma$, and $D = \underset{\sim}{D}_\gamma$, it follows that $\mathbf{B}$ is not κ-mad in $V^{\mathbb{P}_\delta}$, a contradiction. It follows that there are no $\Sigma^1_1(\kappa)$ κ-mad families in $V^{\mathbb{P}_\delta}$. □

3 Open problems

We conclude by listing some of the open problems following from our work. Following the main result of the paper, one may ask whether it is possible to get an implication instead of just consistency:

Question 3.1 Suppose that κ is supercompact, is there a $\Sigma^1_1(\kappa)$ κ-mad family?

Question 3.2 What is the consistency strength of ZFC + "for some uncountable regular cardinal κ, there are no $\Sigma^1_1(\kappa)$ κ-mad families"?

It is known by [2, 6, 9] that ZF + DC + "there are no mad families" is consistent ([9] shows that it holds in Solovay's model while in [2] we obtain a consistency result relative to ZFC).

Question 3.3 (a) What is the consistency strength of ZF + DC + "there exists a regular uncountable cardinal κ such that there are no κ-mad families"?
(b) Suppose that $\kappa > \aleph_0$ is regular, does DC_κ imply the existence of a κ-mad family?

It is known by [1, 3] that Borel maximal eventually different families and maximal cofinitary groups exist, therefore it is natural to investigate the κ-version of those results:

Question 3.4 (a) Does ZFC imply that there are κ-Borel κ-maximal eventually different families for every (or at least for some) regular uncountable cardinal κ?

(b) Similarly, replacing regular uncountable cardinals by successor cardinals, inaccessible non-Mahlo cardinals, etc.

Question 3.5 (a) Does ZFC imply that there are κ-Borel κ-maximal cofinitary groups for every (or at least for some) regular uncountable cardinal κ?

(b) Similarly, replacing regular uncountable cardinals by successor cardinals, inaccessible non-Mahlo cardinals, etc.

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