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BETWEEN REDUCED POWERS AND ULTRAPOWERS, II.

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ABSTRACT. We prove that, consistently with ZFC, no ultraproduct of countably infinite (or separable metric, non-compact) structures is isomorphic to a reduced product of countable (or separable metric) structures associated to the Fréchet filter. Since such structures are countably saturated, the Continuum Hypothesis implies that they are isomorphic when elementarily equivalent.

The trivializing effect of the Continuum Hypothesis (CH) to the structure of the continuum has been known at least since the times of Sierpiński and Gödel ([24]). The particular instance of this phenomenon that we are concerned with in the present paper is the existence of highly non-canonical isomorphisms between massive quotient structures of cardinality $\mathfrak{c} = 2^{\aleph_0}$. The operation of taking a reduced product[1] $\prod_{\mathcal{F}} A_n$ of a sequence (A_n) of first-order structures often results in a countably saturated structure.[2] This is the case with the two most commonly used reduced products: ultraproducts associated with nonprincipal ultrafilters on $\mathbb{N}$ and reduced products associated with the Fréchet filter. If each A_n has the cardinality of at most $\mathfrak{c}$ (in particular, if it is countable or separable[3]), then so does $\prod_{\mathcal{F}} A_n$, and the CH implies that the latter structure is saturated. By a classical theorem of Keisler, elementarily equivalent and saturated first-order structures of the same cardinality are isomorphic (see [5, Theorem 5.1.13]). Therefore CH implies that the isomorphism of such reduced products reduces (no pun intended) to elementary equivalence.

In [13], this observation was combined with computation of the theory of the structure (K denotes the Cantor space)

$$C(K, A) = \{f : K \to A \mid f \text{ is continuous}\} \tag{0.1}$$

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[1]Reduced products can be defined with respect to a filter or with respect to the dual ideal, the two definitions resulting in the same object; see Remark 1.1.

[2]Saturation is a model-theoretic property that enables transfinite constructions of isomorphisms and automorphisms; see e.g., [5, Chapter 5] or [12, Chapter 16]. It will not be used in this paper.

[3]Our results apply both to classical, discrete, structures and metric structures, as in [2].

for a separable (or countable discrete) structure A to prove that CH implies (Fin denotes the ideal of finite subsets of $\mathbb{N}$)

$$\prod_{\mathcal{U}} C(K, A) \cong \prod_{\text{Fin}} A \tag{0.2}$$

for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ ([13, Theorem E]).[4] This result is the basis for [13, Theorem B], asserting that under CH there exists an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that the quotient map from $\prod_{\text{Fin}} A$ to $\prod_{\mathcal{U}} A$ has a right inverse for every countable (or separable metric) structure A. In the case when A is a C^*-algebra, this simplifies some intricate arguments in Elliott's classification program for nuclear, separable C^*-algebras (see the upcoming [4], also [53] and [38] for related applications of ultrapowers). Although the assumption of CH can be removed from the applications of (0.2) to the Elliott classification programme ([13, Theorem A]), the question whether (0.2) can be proved in ZFC remained.

A well-known instance of the trivializing effect of CH is Parovičenko's theorem from general topology. Stated in the dual, Boolean-algebraic, form, it asserts that CH implies that all atomless, countably saturated, Boolean algebras of cardinality $\mathfrak{c}$ are isomorphic. In [51] it was proved that the conclusion of Parovičenko's theorem is equivalent to CH. An alternative proof of this fact is given by the main result of [19] (or by [30]), asserting that if CH fails then there are $2^{\mathfrak{c}}$ nonisomorphic ultrapowers of the countable atomless Boolean algebra associated with nonprincipal ultrafilters on $\mathbb{N}$. By Łoś's Theorem all of these Boolean algebras are elementarily equivalent (which in this case reduces to being atomless, see [5, Exercise 1.5.3]), and they are countably saturated and of cardinality $\mathfrak{c}$, being ultrapowers associated with countably incomplete ultrafilters. Clearly, at most one of these ultrapowers can be isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Theorem A and Theorem B show that in two of the most popular models of ZFC in which CH fails (models of forcing axioms and Cohen's original model of ZFC), none of these ultrapowers is isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

Theorem A. *The Proper Forcing Axiom, PFA, implies that* $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *is not isomorphic to an ultraproduct of Boolean algebras associated with a nonprincipal ultrafilter on* $\mathbb{N}$.

Each of our other main results applies to wider class of structures. The first one is primarily concerned with a larger class of quotient algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ associated with Borel ideals on $\mathbb{N}$ in place of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. The study of quotient Boolean algebras of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for an ideal $\mathcal{I}$ on $\mathbb{N}$, dates back at least to Erdös and Ulam (see [8]). The space $\mathcal{P}(\mathbb{N})$ is identified with the Cantor space, and thus equipped with a canonical compact metrizable topology. If $\mathcal{I}$ includes the ideal Fin of finite subsets of $\mathbb{N}$, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless and therefore elementarily equivalent to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. We will consider only ideals $\mathcal{I}$ that include Fin.

The question of countable saturation of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is a bit subtler. In [26] it was proved that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is countably saturated for every F_σ ideal $\mathcal{I}$ that includes Fin. The essence of the Just–Krawczyk construction is encapsulated in the concept of a *layered ideal* in [11], where it was proved that if $\mathcal{I}$ is a layered ideal that includes Fin then $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is countably saturated. (This class of ideals properly includes that of the F_σ ideals; for example, for any additively indecomposable countable ordinal

[4] In [13], $\prod_{\text{Fin}} A$ was denoted A^∞ and $\prod_{\mathcal{U}} A$ was denoted $A^{\mathcal{U}}$, following the notation favoured by operator-algebraists. In the present paper we adopt the notation favoured by logicians and apologize to any stray operator algebraists; see however Corollary D.

α the ideal $\{X \subseteq \alpha |$ the order type of X is strictly smaller than $\alpha\}$ is layered.) Since the quotient over an analytic ideal that includes Fin necessarily has cardinality $\mathfrak{c}$, these results show that all quotients over layered analytic ideals that include Fin are isomorphic under CH. This conclusion extends to reduced products of countable Boolean algebras, $\prod_{\mathcal{I}} A_n$, associated with layered ideals ([21, Theorem 2.7]).

In [27, Corollary 1.4] it was proved that if an ideal $\mathcal{I}$ includes Fin then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is countably saturated if and only if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has no $(\aleph_0, \aleph_0)$-gaps. In [36] it was proved that $\mathcal{F}$ on $\mathbb{N}$ has the property that every reduced product of the form $\prod_n A_n/\mathcal{F}$ is countably saturated if and only if the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathcal{F}$ is countably saturated. Results analogous to the latter one have been established in [35] for filters on $\aleph_1$ and in [39] for filters on an arbitrary cardinal.

Note that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is canonically isomorphic to the reduced product, $\prod_{\mathcal{I}} A$, where A is taken to be the 2-element Boolean algebra. Also note that all of these reduced products are *projectively definable* in the sense that there is an $n \in \mathbb{N}$ and $\mathbf{\Sigma}^1_n$-formulas φ, $\varphi_\wedge$, $\varphi_\vee$, and $\varphi_\setminus$, such that the set $\{x \in \mathbb{R} | \varphi(x)\}$ equipped with the operations $\wedge$, $\vee$, and $\setminus$ defined by $\varphi_\wedge$, $\varphi_\vee$, and $\varphi_\setminus$ is a Boolean algebra.[5]

Theorem B. *In a model obtained by adding at least $\mathfrak{c}^+$ Cohen reals to a model of ZFC the following holds. If $\mathfrak{B}$ is a projectively definable Boolean algebra then $\mathfrak{B}$ is not isomorphic to an ultraproduct of countable Boolean algebras associated with a nonprincipal ultrafilter on $\mathbb{N}$.*

In particular, for any analytic ideal $\mathcal{I}$ on $\mathbb{N}$, the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not isomorphic to an ultraproduct of countable Boolean algebras associated with a nonprincipal ultrafilter on $\mathbb{N}$.

The proof of Theorem B applies to a wider class of models, but not to the Sacks models; see §8.

The next result applies to a yet wider range of structures. For the order property (OP) and the robust order property see Definition 1.2. Any theory in which the order property is witnessed by an atomic formula has the robust order property. In particular, the classes of atomless Boolean algebras and dense linear orderings have the robust OP. Another relevant property specific to the theory of atomless Boolean algebras is that it is preserved by taking arbitrary reduced products. (Theories with this property are axiomatized by Horn sentences, see e.g., [5, Theorem 6.2.5'].) This is not shared by many other theories (e.g., linear orders or fields), and it is because of this fact that in the following result the theories of A_n and B_n are a priori unrelated.

Theorem C. *There exists a forcing extension in which for every countable theory T that has the robust order property the following holds.*

For every sequence (A_n) of countable structures in the language of T, every sequence (B_n) of countable models of T, and every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the following are true.

(1) *The ultraproduct $\prod_{\mathcal{U}} B_n$ is not isomorphic to $\prod_{\mathrm{Fin}} A_n$ or even to an elementary submodel thereof.*
(2) *If the order property of T is witnessed by a quantifier-free formula then $\prod_{\mathcal{U}} B_n$ does not embed into $\prod_{\mathrm{Fin}} A_n$.*

[5] We avoid using the simpler term projective Boolean algebras in order to avoid confusion with Boolean algebras that are projective objects in the category of Boolean algebras.

Since the original impetus for these results drew from the Elliott classification program of C^*-algebras, we'll explicitly state the relevant corollary. If A is a C^*-algebra, then the structure $C(K, A)$ as in (0.1) is isomorphic to the tensor product $A \otimes C(K)$, where $C(K)$ is the algebra of continuous complex-valued functions on K. By [13, Theorem E], for a separable C^*-algebra A and a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, CH implies that the ultrapower[6] $(A \otimes C(K))^{\mathcal{U}}$ is isomorphic to $A^\infty := \ell_\infty(A)/c_0(A)$ (the latter algebra, known as the asymptotic sequence algebra, is the analog of the reduced power $\prod_{\mathrm{Fin}} A$) and the isomorphism extends the identity on A (A is routinely identified with its diagonal copies in $A^{\mathcal{U}}$ and A^∞).

Corollary D. *There exists a forcing extension in which the following holds for every separable C^*-algebra A and every ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

1. *$(A \otimes C(K))^{\mathcal{U}}$ is not isomorphic to A^∞.*
2. *$(A \otimes C(K))^{\mathcal{U}}$ is not isomorphic to a C^*-subalgebra of B^∞ for any separable C^*-algebra B.*

The related conclusion, that $C(K)^{\mathcal{U}}$ is not isomorphic to a C^*-subalgebra of ℓ_∞/c_0, is known to be relatively consistent with ZFC and its variant (known as *Woodin's condition*) plays an important role in Woodin's proof of automatic continuity for homomorphisms of Banach algebras ([6]).

The question of the existence of a universal structure among the ultrapowers of a fixed countable (or separable metric) structure is closely related to the questions answered in theorems stated above. Partial answers to this question (which are easy consequences of the earlier work of one of the authors) are given in the brief §7.

Notation. We write $X \Subset Y$ as a shorthand for '$X \subseteq Y$ and X is finite' (this is sometimes denoted by the formula $X \in [Y]^{<\aleph_0}$). The ideal of finite subsets of a set X is denoted Fin_X (some authors prefer $[X]^{<\aleph_0}$). For the Fréchet ideal $\mathrm{Fin}_{\aleph_0}$ we write Fin.

Rough outline. Our proofs use model theory (§1, §7) and set theory (§2, §4). In §1 we discuss the order property (OP) of first-order theories, discrete and continuous. Standard facts about partially ordered sets (posets) are recalled in §2, and depletions of posets are introduced and studied in §3. In §4 we define a functor $E \mapsto \mathbb{H}_E$ from the category of partial orderings into the category of forcing notions. The material from §3 is used to prove that the $\mathbb{H}_E$ forces that E embeds into the reduced product $\prod_{n\in\mathbb{N}}(n, <)$ (n is identified with $\{0, \dots, n-1\}$) in a particularly gentle way. Theorem C and Corollary D are proved in §5, while Theorem A and Theorem B are proved in §6. In §7 we record results about the existence of a universal model among the ultrapowers of countable models of T associated with ultrafilters on $\mathbb{N}$. Some concluding remarks and questions can be found in §8.

1. Reduced products, the order property, continuous logic

In this section we recall the pertinent definitions. It should be emphasized that the first-order theory T is not assumed to be complete.

[6]By $A^{\mathcal{U}}$ we denote the *norm* ultrapower of A, and not the tracial ultrapower. The norm ultrapower is commonly denoted $A_{\mathcal{U}}$.

1.1. **Reduced products.** We will use the following convention. Suppose that A_n are structures of the same countable language, $\bar{a}_n$ is a tuple in A_n for all $n \in \mathbb{N}$, and all of these tuples are of the same sort. (If the language is single-sorted, then the sort of a tuple is its arity. Note that the natural languages associated with the unbounded metric structures, such as C^*-algebras, are multisorted, see [15].) Then $\bar{a}$ denotes the tuple $(\bar{a}_n)$ in $\prod_n A_n$ of the same sort.

If $\mathcal{F}$ is a filter on $\mathbb{N}$ and A_n, for $n \in \mathbb{N}$, are structures of the same language $\mathcal{L}$, then the reduced product $\prod_{\mathcal{F}} A_n$ is defined as follows. Its domain is the quotient of $\prod_n A_n$ over the relation $\bar{a} \sim_{\mathcal{F}} \bar{b}$ if $\{n | a_n = b_n\} \in \mathcal{F}$. The function symbols in $\mathcal{L}$ are interpreted in the natural way (note that $\sim_{\mathcal{F}}$ is a congruence). If $k \geq 1$ and $R(x(0), \dots, x(k-1))$ is a k-ary relation symbol and $\bar{a}(0), \dots, \bar{a}(k-1)$ is a k-tuple, then we let $\prod_{\mathcal{F}} A_n \models R(\bar{a}(0), \dots, \bar{a}(k-1))$ if and only if the set

$$\{n | A_n \models R(a_n(0), \dots, a_n(k-1))\}$$

belongs to $\mathcal{F}$. The image of $\bar{a}$ in the reduced product $\prod_{\mathcal{F}} A_n$ under the quotient map is also denoted $\bar{a}$, by a standard and innocuous abuse of notation.

If $\mathcal{F}$ is the Fréchet filter (i.e., the filter of cofinite subsets of $\mathbb{N}$), then $\prod_{\mathcal{F}} A_n$ is denoted $\prod_{\mathrm{Fin}} A_n$. (This is yet another standard and innocuous abuse of notation; Fin denotes the ideal dual to the Fréchet filter, and the reduced products are sometimes defined with respect to the dual ideals.) If $\mathcal{U}$ is an ultrafilter (i.e., a proper filter maximal with respect to the inclusion), then $\prod_{\mathcal{U}} A_n$ is called the ultraproduct.

When all structures A_n are equal to some A, the corresponding reduced products (ultraproducts) are called reduced powers (ultrapowers).

Remark 1.1. Reduced products are sometimes defined with respect to a filter $\mathcal{F}$, and sometimes with respect to the dual ideal $\mathcal{F}^*$, as convenient. The only difference is in the notation. In this paper we will use the two notions—reduced products with respect to filters and reduced products with respect to ideals—interchangeably.

1.2. **The order properties.** This combinatorial property of a first-order theory marks the watershed between well-behaved and wild (see [40]).

Definition 1.2. Suppose that T is a first-order theory, not necessarily complete.

(1) If $\varphi(\bar{x}, \bar{y})$ is an asymmetric formula (with $\bar{x}$ and $\bar{y}$ of the same sort) in the language of T consider the asymmetric binary relation $\lhd_\varphi$ on a model A of T, defined by

$$\bar{a} \lhd_\varphi \bar{b} \text{ if } A \models \varphi(\bar{a}, \bar{b}). \tag{1.1}$$

Some $\bar{a}_j$, for $j < n$, in A form a $\lhd_\varphi$-*chain* if for all $i \neq j$ we have $\bar{a}_i \lhd_\varphi \bar{a}_j$ if and only if $i < j$.

(2) If every model of T has an arbitrarily long finite $\lhd_\varphi$-chain, we say that the pair (T, φ) *has the order property, OP* ([40]).[7]

(3) The pair (T, φ) has the *robust order property* if it has the order property and in addition for models A_n, for $n \in \mathbb{N}$, of T and $\bar{a}$ and $\bar{b}$ in $\prod_{\mathrm{Fin}} A_n$ we have $\prod_{\mathrm{Fin}} A_n \models \varphi(\bar{a}, \bar{b})$ if and only if the set

$$\{n \mid A_n \not\models \varphi(\bar{a}_n, \bar{b}_n)\}$$

is finite. (Note that it is not required that $\prod_{\mathrm{Fin}} A_n$ models T.)

[7] One says that φ has the order property when T is clear from the context.

(4) The pair (T, φ) is said to have the *strict order property* (SOP) if the relation $\lhd_\varphi$ is a partial ordering on every model of T.

The relation between the order property and the robust order property depends on the analysis of the relation between the theories of A_n and the theory of $\prod_{\mathrm{Fin}} A_n$, as given by the Feferman–Vaught theorem ([22] and [23] for continuous logic, also see [12, §16.3]). We will need only the following easy case.

Lemma 1.3. *If a pair (T, φ) has the order property and $\varphi(\bar{x}, \bar{y})$ is atomic, or a negation of an atomic formula, then the pair (T, φ) has the robust order property.*

Proof. Fix models $A_n \models T$ for $n \in \mathbb{N}$ and suppose φ is an atomic formula. If $\bar{a}_n$ and $\bar{b}_n$ are tuples of the appropriate sort in A_n such that $A_n \models \varphi(\bar{a}_n, \bar{b}_n)$, then (writing $\bar{a}$ for the element of the product that has the representing sequence $(\bar{a}_n)$), we have $\prod_n A_n \models \varphi(\bar{a}, \bar{b})$ and moreover for any filter $\mathcal{F}$ on $\mathbb{N}$ we have $\prod_{\mathcal{F}} A_n \models \varphi(\bar{a}, \bar{b})$. This also applies to the negation of φ, and the assertion follows immediately. $\square$

The proof of Lemma 1.3 uses only the fact that φ satisfies the variant of Łoś's Theorem for reduced products asserting that $\prod_{\mathcal{F}} A_n \models \varphi(\bar{a}, \bar{b})$ if and only if $\{n \mid A_n \models \bar{a}_n, \bar{b}_n)\} \in \mathcal{F}$. A larger class of formulas with this preservation property, called h-formulas, has been isolated in [37].

1.3. **Continuous logic.** For more details on continuous logic see [2] (see [15] or [12, §16] for operator algebras). That said, this subsection is targeted at the readers already familiar with continuous logic, and its aim is to convince these readers that the proofs of the continuous versions of our main results are analogous to the proofs in the discrete case.

The value of a formula $\varphi(\bar{x})$ evaluated in a model M, at a tuple $\bar{a}$ of the appropriate sort, is denoted $\varphi(\bar{a})^M$ and defined by recursion on the complexity of φ.

1.4. **Reduced products in continuous logic.** If $\mathcal{F}$ is a filter on $\mathbb{N}$, then the reduced product $\prod_{\mathcal{F}} A_n$ of metric structures of the same language is defined as follows.[8] With $\mathcal{F}^* := \{A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A \notin \mathcal{F}\}$ (the coideal of all sets positive with respect to $\mathcal{F}$), on $\prod_n A_n$ define pseudometric

$$d_{\mathcal{F}}(\bar{a}, \bar{b}) := \inf_{X \in \mathcal{F}^*} \sup_{m \notin X} d_m(a_m, b_m).$$

The universe A of $\prod_{\mathcal{F}} A_n$ is the completion of the quotient space with respect to $d_{\mathcal{F}}$.

Every predicate symbol $R(\bar{x})$ in the language is interpreted as a function into $\mathbb{R}$, and the syntax rules of continuous logic require that its interpretation in each A_n respects the same modulus of uniform continuity. For $\bar{a} \in \prod_n A_n$ let

$$R(\bar{a}) := \inf_{X \in \mathcal{F}^*} \sup_{m \notin X} R^{A_m}(\bar{a}_m).$$

Then $d_{\mathcal{F}}(\bar{a}, \bar{b}) = 0$ implies $R(\bar{a}) = R(\bar{b})$, and one defines the interpretation of R in A by $R^A([\bar{a}]) := R(\bar{a})$ (where $[\bar{a}]$ is the equivalence class of $\bar{a}$ in $\prod_{\mathcal{F}} A_n$).

Function symbols are interpreted in the obvious manner (using the fact that they are also required to respect the same modulus of uniform continuity in all A_n).

For more details see e.g., [2, §5] (for ultraproducts) and [12, §16.2 and Definition D.2.13] for the general case.

[8] Needless to say, the definition of a reduced product with a filter on some other set is analogous.

1.5. **Order property in continuous logic.** The following is the continuous analog of Definition 1.2.

Definition 1.4. Suppose that T is a theory in a continuous language, not necessarily complete.

(1) If $\varphi(\bar{x},\bar{y})$ is an asymmetric formula (with $\bar{x}$ and $\bar{y}$ of the same sort) in the language of T consider the asymmetric binary relation $\triangleleft_\varphi$ on a model A of T, defined by
$$\bar{a} \triangleleft_\varphi \bar{b} \text{ if } \varphi(\bar{a},\bar{b})^A = 0 \text{ and } \varphi(\bar{b},\bar{a})^A = 1.$$
Some $\bar{a}_j$, for $j<n$, in A form a $\triangleleft_\varphi$-*chain* if for all $i \neq j$ we have $\bar{a}_i \triangleleft_\varphi \bar{a}_j$ if and only if $i<j$.

(2) If every model of T has an arbitrarily long finite $\triangleleft_\varphi$-chain, we say that the pair (T,φ) *has the order property, OP* ([17, Definition 5.2]).

(3) The pair (T,φ) has the *robust order property* if for models A_n, for $n \in \mathbb{N}$, of T, and all $\bar{a}$ and $\bar{b}$ in $\prod_{\mathrm{Fin}} A_n$ we have $\prod_{\mathrm{Fin}} A_n \models \varphi(\bar{a},\bar{b})$ if and only if for all sufficiently small $\varepsilon>0$ the set
$$\{n \mid \varphi^{A_n}(\bar{a}_n,\bar{b}_n) < \varepsilon \text{ and } \varphi^{A_n}(\bar{b}_n,\bar{a}_n) > 1-\varepsilon\}$$
is finite. (As before, it is not required that $\prod_{\mathrm{Fin}} A_n$ models T.)

(4) The pair (T,φ) is said to have the *strict order property* (SOP) if the relation $\triangleleft_\varphi$ is a partial ordering on every model of T.

Therefore by replacing φ with $f(\varphi)$ for a suitable piecewise continuous function f, the order property of a continuous theory as well as its robustness are witnessed by a discrete (i.e., 0-1 valued) formula. Because of this, we will provide proofs of our results only in the case of discrete theories, with understanding that they carry on virtually unchanged to the continuous context. A proof of the following is analogous to the proof of Lemma 1.3 and therefore omitted.

Lemma 1.5. *If T is a continuous theory, a pair (T,φ) has the order property, and $\varphi(\bar{x},\bar{y})$ is an atomic formulas then the pair (T,φ) has the robust order property.* □

2. Background on posets

In this section we warm up by stating and proving some well-known results. Consider the following two partial quasi-orderings on $\mathbb{N}^{\mathbb{N}}$ (by $\forall^{\infty j}$ we denote the quantifier 'for all but finitely many $j \in \mathbb{N}$'):
$$f \leq^* g \Leftrightarrow (\forall^\infty j) f(j) \leq g(j)$$
$$f <^* g \Leftrightarrow (\forall^\infty j) f(j) < g(j).$$
Any proper initial segment of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ is included in one of the form $(\{f \in \mathbb{N}^{\mathbb{N}} \mid f \leq \eta\}, \leq^*)$ for some $\eta \in \mathbb{N}^{\mathbb{N}}$. Such an initial segment is isomorphic to $(\prod_k \eta(k), \leq^*)$ (if $f \leq^* \eta$, then the pointwise minimum of f and η is an element of $\prod_k \eta(k)$ equal to f modulo finite) and these structures will be our main focus. The following is essentially a bounded variant of [9, Proposition 0.1].

Lemma 2.1. *There are $\eta \in \mathbb{N}^{\mathbb{N}}$ and $\Phi\colon (\prod_k k, \leq^*) \to (\prod_k \eta(k), <^*)$ such that for all f and g, if $f \leq^* g$ and $g \not\leq^* f$ then $\Phi(f) <^* \Phi(g)$.*

A morphism Φ as guaranteed by Lemma 2.1 is called *strictly increasing.*

Proof. Recursively define η by $\eta(0) := 1$ and $\eta(n+1) := \sum_{j\leq n} j\eta(j) + 1$ for $n \geq 0$. By rewriting the recursive definition of η, one sees that for every $n \geq 1$ and every $m \geq 0$ we have

$$(m+1)\eta(n) > \sum_{j<n} j\eta(j) + m\eta(n). \tag{2.1}$$

Fix $f \in \prod_k k$. Let $\Phi(f)(0) := 0$ and for $n \geq 0$ let

$$\Phi(f)(n+1) := \sum_{j\leq n} f(j)\eta(j).$$

Then, since $f(j) < j$ for all j, we have $\Phi(f)(n+1) < \sum_{j\leq n} j\eta(j) < \eta(n+1)$ for all $n \geq 1$, and therefore $\Phi(f)$ belongs to $\prod_k \eta(k)$.

Suppose that f and g are in $\prod_k k$. If $n \geq 1$ is such that $g(n) > f(n)$, then (2.1) with $m := f(n)$ implies

$$g(n)\eta(n) \geq (f(n)+1)\eta(n) > \sum_{j<n} j\eta(j) + f(n)\eta(n) \geq \Phi(f)(n+1)$$

and therefore $\Phi(g)(n+1) > \Phi(f)(n+1)$. This implies that if f and g are in $\prod_k k$, then every $n \geq 1$ such that $f(n) < g(n)$ in addition satisfies $\Phi(f)(n+1) < \Phi(g)(n+1)$.

It remains to prove that $f \leq^* g$ and $g \not\leq^* f$ together imply $f(k) < g(k)$ for all sufficiently large k. Fix such f and g, and let $n \geq 1$ be such that $f(n) < g(n)$ and $f(k) \leq g(k)$ for all $k \geq n$. Then, as we have just seen, $\Phi(f)(n+1) < \Phi(g)(n+1)$. Since $\Phi(f)(k+1) = \Phi(f)(k) + f(k)\eta(k)$, for every $k \in \mathbb{N}$ the conditions $\Phi(f)(k) < \Phi(g)(k)$ and $f(k+1) \leq g(k+1)$ together imply that $\Phi(f)(k+1) < \Phi(g)(k+1)$. By induction on $k \geq n+1$ one proves that $\Phi(f)(k) < \Phi(g)(k)$ for all $k \geq n+1$, as required. □

The universal structure obtained in Lemma 2.2 is very similar to the Rado graph, also known as the (countably infinite) random graph, and it ought to be well-known. It was however easier to include a proof than to look for it in the literature.

Lemma 2.2. *There exists an injectively universal countable structure $(C, \lhd)$ with an asymmetric binary relation $\lhd$. This universality property is absolute between transitive models of a sufficiently large fragment of* ZFC.

Proof. Let $C := \mathbb{N}$ and define the relation $\lhd$ as follows. If $m < n$ are in $\mathbb{N}$ and $n = \sum_j d_j(n)3^j$ is the ternary expansion of n (so that $d_j(n) \in \{0,1,2\}$ for all j) then let $m \lhd n$ if $d_m(n) = 1$, $n \lhd m$ if $d_m(n) = 2$, and let m and n unrelated if $d_m(n) = 0$. The structure $(C, \lhd)$ has the following property resembling the random graph:

(*) If F and G, are disjoint finite subsets of C, then there exists $n \in C$ such that $m \in F$ implies $m \lhd n$, $m \in G$ implies $n \lhd m$, and $m \notin F \cup G$ implies that m and n are unrelated.

To see this, let $n := \sum_{m\in F} 3^m + \sum_{m\in G} 2\cdot 3^m$.

Given the property (*) of $(C, \lhd)$, every countable $(A, \lhd')$ can be isomorphically embedded into $(C, \lhd)$ by recursion. Since (*) is a first-order property, it is absolute between transitive models of a sufficiently large fragment of ZFC (see e.g., [32, Lemma II.4.3]). □

3. The depletion of a poset

The notion of the *depletion* of a linear ordering given in Definition 3.1 appears implicitly in [9].

Definition 3.1. Suppose that I is a finite linear ordering whose elements are listed in the increasing order as $\xi(i)$, for $i < m$ and $m = |I| \geq 2$.[9] Also suppose A and $F(\xi)$, for $\xi \in I$, are disjoint sets, and $\leq$ is a partial ordering on a set that includes $E := A \cup \bigcup_{\xi \in I} F(\xi)$. Define a binary relation $\ll_I$ on E as follows.

If x and y belong to E, we let $x \ll_I y$ if and only $x \leq y$ and in addition one of the following applies.

(1) Both x and y belong to $A \cup F(\xi)$ for some $\xi \in I$.
(2) There are $i < j$ such that $x \in F(\xi(i))$ and $y \in F(\xi(j))$ and one of the following holds.
   (a) There exists $a \in A$ such that $x \leq a$ and $a \leq y$
   (b) With $k = j - i$, there are $x_l \in F(\xi(i + l))$ for $0 \leq l \leq k$ such that $x_0 = x$, $x_k = y$, and $x_l \leq x_{l+1}$ for all $l < k$.
(3) There are $i > j$ such that $x \in F(\xi(i))$ and $y \in F(\xi(j))$ and one of the following holds.
   (a) There exists $a \in A$ such that $x \leq a$ and $a \leq y$
   (b) With $k = i - j$, there are $x_l \in F(\xi(j + l))$ for $0 \leq l \leq k$ such that $x_0 = y$, $x_k = x$, and $x_l \geq x_{l+1}$ for all $l < k$.

The elements x_i, for $i \leq k$ as in (2b) or (3b) comprise an *I-walk between x and y*, or an *I-walk with endpoints x and y*. The relation $\ll_I$ is called the *depletion of $\leq$ given by I, A, and $(F(\xi)|\xi \in I)$*. When I is clear from the context, we write $\ll$ for $\ll_I$.

Take note of the fact that a depletion depends on the order on I in an essential way. In addition, it clearly depends on A, the choice of $F(\xi)$ for $\xi \in I$, and the order on $A \cup \bigcup_{\xi \in I} F(\xi)$. Hence writing $\ll_{(A,(F(\xi)|\xi \in I),\leq)}$ in place of $\ll_I$ may appear to be a more reasonable choice. Fortunately, A and $F(\xi)$, for ξ in some set including I, as well as the ordering $\leq$ on some set that includes $A \cup \bigcup_\xi F(\xi)$, will (unlike I) be fixed and clear from the context in any given instance in which a depletion is used.

It should be emphasized that in both (2b) and (3b) it is required that the 'walk' between x and y hits $F(\xi(k))$ for all $i \leq k \leq j$.

In order to help the reader internalize these definitions, we state two lemmas whose easy proofs are omitted.

Lemma 3.2. *With the notation as in Definition 3.1, if $|I| \leq 2$, then $\ll_I$ agrees with the restriction of $\leq$ to $A \cup \bigcup_{\xi \in I} F(\xi)$.* □

Lemma 3.3. *With the notation as in Definition 3.1, if $s \subseteq t \subseteq I$, if $\{\min(s), \max(s)\} = \{\min(t), \max(t)\} = \{\xi, \eta\}$, and $x(\zeta)$, for $\zeta \in t$, is a t-walk, then $x(\zeta)$, for $\zeta \in s$, is an s-walk with the same endpoints.*

On the other hand, it is possible that $\xi(\zeta)$, for $\zeta \in s$, is an s-walk but there is no t-walk with the same endpoints that extends it. □

[9] If this appears excessively pedantic, note that the conditions (2b) and (3b) are sensitive to leaving gaps in I. We note that I is not necessarily a set of ordinals.

The (admittedly rather dull) Lemma 3.4 will be instrumental in a critical place in the proof of Theorem 4.13.

Lemma 3.4. *Suppose that I, A, $F(\xi)$, for $\xi \in I$, and an ordering $\leq$ on $A \cup \bigcup_{\xi\in I} F(\xi)$ are as in Definition* 3.1*. The depletion $\ll$ of $\leq$ given by these parameters is a partial ordering whose graph is included in the graph of $\leq$.*

Proof. Fix A, I enumerated in the increasing order as $\xi(i)$, for $i < m$, $F(\xi)$, for $i < m$, and an ordering $\leq$ on $E := A \cup \bigcup_{i<m} F(\xi(i))$.

We will write $\ll$ for $\ll_s$. It is clear from the definition that $x \ll y$ implies $x \leq y$ and that $\ll$ and $\leq$ agree on $A \cup F(i)$ for every i. Therefore $\ll$ is antisymmetric and reflexive, and it will suffice to prove that it is transitive.

Towards this end, fix x, y, and z such that $x \ll y$ and $y \ll z$. Then $x \leq y$ and $y \leq z$, and therefore $x \leq z$. If x and z belong to $A \cup F(\xi(i))$ for some i, then $x \ll z$ by (1). Therefore if at least one of $x \in A$ or $z \in A$ holds then $x \ll z$, and we may assume

(xz) $x \in F(\xi(i))$ and $z \in F(\xi(j))$ for some distinct i and j.

If $y \in A$ then $x \ll z$ by (2a). Similarly, if there exists $a \in A$ such that $x \leq a$ and $a \leq y$, then $x \ll z$. Also, if there exists $a \in A$ such that $y \leq a$ and $a \leq z$, then $x \ll z$. We can therefore assume (in addition to (xz) above) that

(y) $y \in F(\xi(n))$ for some n

and that both $x \ll y$ and $y \ll z$ are witnessed by instances of (2b).

The proof now reduces to the analysis of the ordering of the set $\{i, j, n\}$. Claim 3.5 will help when discussing the possible cases.

Claim 3.5. Suppose that $i < m$, $0 < k \leq m - i$, $x \in F(\xi(i))$ and $y \in F(\xi(i + k))$.

(1) Assume there is no $a \in A$ such that $x \leq a$ and $a \leq y$. Then $x \ll y$ if and only if there are $x_l \in F(\xi(i+l))$ for all $0 \leq l \leq k$ such that $x_0 = x$, $x_k = y$, and $x_l \leq x_{l+1}$ for all $0 \leq l < k$.

(2) Assume there is no $a \in A$ such that $y \leq a$ and $a \leq x$. Then $y \ll x$ if and only if there are $x_l \in F(\xi(i+l))$ for all $0 \leq l \leq k$ such that $x_0 = x$, $x_k = y$, and $x_{l+1} \leq x_l$ for all $0 \leq l < k$.

Proof. (1) For the direct implication, note that the assumptions imply that (2b) of Definition 3.1 applies. Let $x_0 := x$, $x_k := y$, and for $0 < l < k$ let x_l be a witness for (2b) of Definition 3.1. These objects are clearly as required.

For the converse implication, assume that x_l for $0 \leq l \leq k$ are as in the statement of the claim. Then clearly (2b) of Definition 3.1 applies.

The proof of (2) is analogous and therefore omitted. □

Back to our proof. With i, j, and n as in (xz) and (y), if $i \leq n \leq j$, then part (1) of Claim implies that $x \ll z$. If $i < j < n$, then the witnessing sequence for $x \ll y$ contains $t \in F(\xi(j))$, such that $x \ll t$ and $t \ll y$. But then (since $\ll$ implies $\leq$) $t \leq z$, and $t \ll z$ since both t and z belong to $F(\xi(j))$. A proof in the case when $n < i < j$ is similar and uses part (2) of the Claim. This proves our claim in the case when $i < j$.

The proof in the case when $i > j$ is analogous. □

Lemma 3.6 extends Lemma 3.3 and stands in contrast to the situation described in Remark 3.7.

Lemma 3.6. *With the notation as in Definition* 3.1 *and I not necessarily finite, if $s \subseteq t \Subset I$, then for any two x and y in $A \cup \bigcup_{\xi \in s} F(\xi)$ we have that $x \ll_t y$ implies $x \ll_s y$.*

Proof. Fix $x \ll_t y$ in $A \cup \bigcup_{\xi \in s} F(\xi)$. A glance at Definition 3.1 shows that we may assume that $x \in F(\xi)$ and $y \in F(\eta)$ for some distinct ξ and η in s, since in any other situation $x \ll_s y$ follows immediately. If there is $z \in A$ such that $x \le z \le y$, then clearly $x \ll_w y$ whenever $\{\xi, \eta\} \subseteq w \Subset I$. We may therefore assume that there is a t-walk with endpoints x and y. We denote the 'steps' of s by $x(\zeta)$, for $\zeta \in t$ and $\xi \le \zeta \le \eta$. Lemma 3.3 implies that $x(\zeta)$, for $\zeta \in s$ and $\xi \le \zeta \le \eta$ is an s-walk with endpoints x and y, hence $x \ll_s y$. $\square$

Remark 3.7. Suppose that we are in the situation of Lemma 3.4 and s is a proper subset of t. The fact that in (2) and (3) the 'walk' between x and y is required to hit every $F(\xi(k))$ implies that the graph of $\ll_s$ may be a proper superset of the graph of the restriction of $\ll_t$ to the domain of $\ll_s$.

Here is another easy lemma with an omitted proof that complements Remark 3.7.

Lemma 3.8. *Suppose that we are in the situation of Lemma* 3.4 *and $s \subseteq t$ is convex (i.e., if $\xi < \eta < \zeta$ are in t and $\{\xi, \zeta\} \subseteq s$ then $\eta \in s$). Then $\ll_s$ agrees with the restriction of $\ll_t$ to $A \cup \bigcup_{\xi \in I} F(\xi)$.* $\square$

Proposition 3.9 is a relative to a result of Kurepa ([33]) and to [9, Theorem 7.1]. The role that this proposition plays in the proof of our Theorem 4.13 is analogous to the role that [9, Theorem 7.1] had played in the proof of [9, Theorem 9.1]. For reader's convenience, we include a proof. If I is a linear ordering, then I^* denotes the converse ordering.

Proposition 3.9. *Suppose that κ is an uncountable cardinal, A and $F(\xi)$, for $\xi < \kappa$, are disjoint, and $\le$ is a partial ordering of $E := A \cup \bigcup_{\xi < \kappa} F(\xi)$. In addition suppose that A is countable, all $F(\xi)$ are finite, and E has neither κ nor κ^*-chains.*

Then there exists a cofinal $X \subseteq \kappa$ such that for any two distinct elements ξ and η of X the following condition holds.

$*_{(\xi,\eta)}$ *there is $s \Subset \kappa$ such that $\{\xi, \eta\} = \{\min(s), \max(s)\}$ and there is no s-walk with endpoints in $F(\xi)$ and $F(\eta)$.*[10]

Proof. Let $\mathbb{P}$ be the poset of all $X \subseteq \kappa$ such that $0 \in X$ which satisfy the condition $*_{(\xi,\eta)}$ for all distinct ξ and η in X, ordered by the inclusion. This poset is clearly closed under unions of chains, and therefore Zorn's lemma implies that it has a maximal element, X. (Readers not fond of the Axiom of Choice will notice that since κ is well-ordered, the existence of X can be proved by recursion in ZF.) We claim that X satisfies the requirements. Towards obtaining a contradiction, suppose that X is not cofinal in κ.

If X has no maximal element, then let $\alpha := \sup(X)$ and $Y := X \cup \{\alpha\}$. In order to verify that Y belongs to $\mathbb{P}$, it suffices to verify $*_{(\xi,\alpha)}$ for all $\xi \in X$. Fix $\xi \in X$. We need to prove that there exists t such that there is no t-walk with endpoints in $F(\xi)$ and $F(\alpha)$. Let $\eta := \min(X \setminus (\xi + 1))$ and (using $*_{(\xi,\eta)}$) let $s \Subset \kappa$ be such that $\{\xi, \eta\} = \{\min(s), \max(s)\}$ and there is no s-walk with endpoints $F(\xi)$ and $F(\eta)$. Let $t := s \cup \{\alpha\}$. Then $\{\xi, \alpha\} = \{\min(t), \max(t)\}$ and Lemma 3.3 implies there is no t-walk with endpoints in $F(\xi)$ and $F(\alpha)$.

[10] We emphasize that s is not necessarily a subset of X.

Now suppose that X has a maximal element, ξ. Assume for a moment that there are $\eta > \xi$ and $s \Subset \kappa$ which witnesses that $*_{(\xi,\eta)}$ holds. We claim that $X \cup \{\eta\}$ belongs to $\mathbb{P}$. Fix $\zeta \in X$ such that $\zeta < \xi$. Since $X \in \mathbb{P}$, there is $t \Subset \kappa$ such that $\{\zeta, \xi\} = \{\min(t), \max(t)\}$ and there is no t-walk with endpoints in $F(\zeta)$ and $F(\xi)$. With $w := s \cup t$ we have $\{\zeta, \eta\} = \{\min(w), \max(w)\}$ and Lemma 3.3 implies that there is no w-walk with endpoints in $F(\zeta)$ and $F(\eta)$. Therefore $*_{(\zeta,\eta)}$ holds. Since ζ was an arbitrary element of X below ξ (and since $*_{(\xi,\eta)}$ holds), this implies $X \cup \{\eta\}$ is an element of $\mathbb{P}$ strictly larger than X; contradiction.

It will therefore suffice to find $\eta > \xi$ such that $*_{(\xi,\eta)}$ holds. Assume that such η does not exist. Then for every $\eta > \xi$ and every $s \Subset \kappa$ which satisfies $\{\xi, \eta\} = \{\min(s), \max(s)\}$ there is an s-walk with endpoints in $F(\xi)$ and $F(\eta)$. For each s fix a walk, $x(s, \zeta)$, for $\zeta \in s$, with endpoints in $F(\xi) = \min(s)$ and $F(\eta) = \max(s)$.

Let $\mathcal{U}$ be an ultrafilter on Fin_κ which for every $s \Subset \kappa$ includes the set $\{t \Subset \kappa | s \subset t\}$ (such $\mathcal{U}$ exists, since the family of sets of this form has the finite intersection property). Fix $\xi \leq \zeta < \kappa$. Since $F(\zeta)$ is finite, there exists a unique $y(\zeta) \in F(\zeta)$ such that $\{s | x(s, \zeta) = y(\zeta)\} \in \mathcal{U}$. Clearly $y(\zeta)$, for $\xi \leq \zeta < \kappa$, is a κ-chain or a κ^*-chain (as decided by $\mathcal{U}$); contradiction.

Therefore there exist $\eta > \xi$ and $s \Subset \kappa$ such that $\{\xi, \eta\} = \{\min(s), \max(s)\}$ such that there is no s-walk with endpoints in $F(\xi)$ and $F(\eta)$. As already pointed out, this implies $X \cup \{\eta\} \in \mathbb{P}$; contradiction.

Therefore any maximal element of $\mathbb{P}$ is cofinal in κ, as required. $\square$

4. Gently embedding posets into reduced powers

In the present section we assume that the reader is familiar with the basics of forcing as presented in e.g., [32] or [44]. The present section is largely based on [9], and Theorem 4.13 is a close relative to [9, Theorem 9.1].

The category of partially ordered sets is considered with respect to the order-embeddings, i.e., injections $f\colon E \to E'$ such that $a \leq_E b$ if and only if $f(a) \leq_{E'} f(b)$. The category of forcing notions is considered with respect to regular embeddings (also known as complete embeddings, [32, Definition III.3.65]). If a forcing notion $\mathbb{H}_0$ is a regular subordering of a forcing notion $\mathbb{H}_1$, we then write $\mathbb{H}_0 \lessdot \mathbb{H}_1$. Notably, $\mathbb{H}_0 \lessdot \mathbb{H}_1$ is equivalent to the assertion that for every generic filter $G \subseteq \mathbb{H}_1$, $G \cap \mathbb{H}_0$ is also generic. In other words, $\mathbb{H}_1$ can be considered as a two-step iteration of $\mathbb{H}_0$ followed by the naturally defined quotient forcing notion.

If κ is an uncountable cardinal, a forcing notion $\mathbb{P}$ is said to have *precaliber* κ if every set of κ conditions in $\mathbb{P}$ has a subset of cardinality κ such that each of its finite subsets has a common lower bound. Precaliber $\aleph_1$ is a strong form of the countable chain condition. For example, if $\mathbb{P}$ has precaliber $\aleph_1$ then it is *productively ccc*, in the sense that the product of $\mathbb{P}$ with any ccc poset is ccc. (We will not need this fact.)

Theorem 4.1. *There is a functor from the category of partially ordered sets into the category of forcing notions $E \mapsto \mathbb{H}_E$ with the following properties.*

(1) *$\mathbb{H}_E$ has precaliber κ for every uncountable regular cardinal κ.*[11]
(2) *$\mathbb{H}_E$ forces that there is an embedding $\Upsilon\colon E \to (\prod_k k, \leq^*)$ (thus for all a and b in E we have $a \leq_E b$ if and only if $\Upsilon(a) \leq^* \Upsilon(b)$).*

[11] Here k is identified with the linearly ordered set $\{0, 1, \ldots, k-1\}$ of cardinality k.

(3) *If $\kappa > \mathfrak{c}$ is a regular cardinal and neither κ nor its reverse κ^* embed into E, then $\mathbb{H}_E$ forces that κ does not embed into $\prod_{\mathrm{Fin}}(A_n, \lhd)$ for every sequence $(A_n, \lhd_n)$ of countable structures equipped with an asymmetric binary relation.*

Proof. The proof of this theorem will occupy most of the present section. For $\mathbb{H}_E$ see Definition 4.2, (1) is Lemma 4.4, (2) is Lemma 4.6, and (3) is Theorem 4.13. □

In the Definition 4.2 and elsewhere, if $\mathrm{dom}(f) \subseteq \mathbb{N}$ then $f \restriction m$ denotes the restriction of f to $m = \{0, \dots, m-1\}$.

Definition 4.2. For a partially ordered set E, $\mathbb{H}_E$ is the forcing notion defined as follows. The conditions of $\mathbb{H}_E$ are triples $p = (D_p, n_p, f_p)$, where $D_p \Subset E$, $n_p \in \mathbb{N}$, and $f_p \colon D_p \to \prod_{m<n_p} m$.

The ordering is defined by letting $p \leq q$ (p *extends* q) if the following conditions hold.

(1) $D_p \supseteq D_q$, $n_p \geq n_q$, $f_p(a) \restriction n_q = f_q(a)$ for all $a \in D_q$, and
(2) for all a and b in D_q, if $a \leq_E b$ then $f_p(a)(j) \leq f_p(b)(j)$ for all $j \in [n_q, n_p)$.

In order to relax the notation, if (p_ξ) is an indexed family of conditions in $\mathbb{H}_E$ we write $p_\xi = (D_\xi, n_\xi, f_\xi)$.

Lemma 4.3. *Suppose that E is a poset, $R \Subset E$, $m \geq 2$ and p_i, for $i < m$, are conditions in $\mathbb{H}_E$ such that the following holds whenever $i \neq j$.*

(1) *We have $D_i \cap D_j = R$.*
(2) *All $a \in R$ satisfy $f_i(a) \restriction \min(n_i, n_j) = f_j(a) \restriction \min(n_i, n_j)$.*

Then some $q \in \mathbb{H}_E$ extends all p_i.[12]

Proof. Let $D_q := \bigcup_{i<m} D_i$ and $n_q := \max_{i<m} n_i$. If $i < m$ is such that $n_i = n_q$, then for $a \in D_i$ let $f_q(a) = f_i(a)$. Then $f_q(a)$ is well-defined for $a \in R$ by (2). For $i < m$ such that $n_i < n_q$ and for $a \in D_i \setminus R$, let (with $\max \emptyset = 0$)

$$f_q(a)(j) := \max\{f_q(b)(j) \mid b \in R, b \leq_E a\}$$

for $n_i \leq_E j < n_q$. This defines $q \in \mathbb{H}_E$. We will prove that $q \leq p_i$ for all $i < m$.

Clearly, q and p_i satisfy (1) of Definition 4.2 for all $i < m$. Fix $i < m$.

If $n_i = n_q$ then (2) of Definition 4.2 is vacuous, hence $q \leq_E p_i$.

Suppose $n_i < n_q$. To check that $q \leq p_i$, we need to verify (2) of Definition 4.2. Fix a and b in D_i such that $a \leq_E b$. If there is no $c \in R$ such that $c \leq_E b$, then for all $j \in [n_i, n_q)$ we have $f_q(a)(j) = f_q(b)(j) = 0$. If there is $c \in R$ such that $c \leq_E b$, then $\{c \mid c \leq_E a\} \subseteq \{c \mid c \leq_E b\}$ and by the definition of f_q we have $f_q(a)(j) \leq f_q(b)(j)$.

Thus (2) of Definition 4.2 holds, and $q \leq p_i$. □

Lemma 4.4. *The poset $\mathbb{H}_E$ has precaliber κ for every uncountable regular cardinal κ.*

Proof. Fix a family p_ξ, for $\xi < \kappa$, in $\mathbb{H}_E$. By the Δ-system lemma and passing to a subfamily of the same cardinality, we may assume that there exists $R \Subset E$ such that $D_\xi \cap D_\eta = R$ for all distinct ξ and η below κ. By the pigeonhole principle (using the assumption that κ has uncountable cofinality), we may also assume that there exists n such that $n_\xi = n$ for all ξ. Also, since there are only finitely many

[12]We write $q \leq p$ if q extends p.

possibilities for $f_\xi(a)$, for $a \in R$, we may assume that the functions f_ξ agree on R and therefore we are in the situation of Lemma 4.3. Therefore, after this refining argument, Lemma 4.3 implies that every finite subset of $\{p_\xi \mid \xi < \kappa\}$ has a common lower bound. $\square$

A proof of Lemma 4.5 is straightforward and therefore omitted.

Lemma 4.5. *For any poset E the following holds.*

(1) *For every n and every $a \in E$, the set*
$$\mathcal{D}(\mathbb{H}_E, n, a) := \{p \in \mathbb{H}_E \mid n_p \geq n, a \in F_p\}$$
is dense in $\mathbb{H}_E$.

(2) *If $b \not\leq_E a$ in E, then for every $n \in \mathbb{N}$ the set*
$$\mathcal{E}(\mathbb{H}_E, n, a, b) := \{p \in \mathbb{H}_E \mid a \in F_p, b \in F_p, \text{ and } (\exists k \geq n) f_p(a)(k) < f_p(b)(k)\}$$
is dense in $\mathbb{H}_E$. $\square$

Lemma 4.6. *If E is a poset and $G \subseteq \mathbb{H}_E$ is a generic filter, then*
$$\Upsilon_G(a)(j) := f_p(a)(j)$$
for $p \in G$ defines a strictly increasing function $\Upsilon_G \colon E \to (\prod_k k, \leq^*)$.

Proof. By genericity, G intersects all dense sets defined in Lemma 4.5 and therefore Υ is a strictly increasing map from E into $(\prod_k k, \leq^*)$. $\square$

If E is a subordering of E' then every $p \in \mathbb{H}_E$ is (literally) a condition in $\mathbb{H}_{E'}$. We will therefore identify $\mathbb{H}_E$ with a subordering of $\mathbb{H}_{E'}$.

Lemma 4.7. *If E' is a poset and E is a subposet of E', then $\mathbb{H}_E$ is a regular subordering of $\mathbb{H}_{E'}$.*

Proof. The identity map from $\mathbb{H}_E$ into $\mathbb{H}_{E'}$ is clearly an order-embedding. It suffices to prove that there exists a *reduction* (or *projection*) $\pi \colon \mathbb{H}_{E'} \to \mathbb{H}_E$: A map such that for every $p \in \mathbb{H}_{E'}$ we have $p \leq \pi(p)$ and every $q \in \mathbb{H}_E$ such that $q \leq \pi(p)$ is compatible with p ([32, Lemma III.3.72]). Let
$$\pi_E(p) := (D_p \cap E, n_p, f_p \restriction (D_p \cap E)).$$
Clearly, $p \leq \pi_E(p)$. If $q \leq \pi(p)$, then $D_q \cap D_p = D_p \cap E$ and $f_p(a)(j) = f_q(a)(j)$ for all $a \in D_p \cap D_q$ and all $j < n_p$. By Lemma 4.3, p and q are compatible. $\square$

In the situation when E is a subordering of E', as in Lemma 4.7, we will need a description of the quotient forcing $\mathbb{H}_{E'}/\dot{G}$, for a generic $G \subseteq \mathbb{H}_E$. If for some $k \in \mathbb{N}$ we have $s \in \prod_{n<k} n$ and $f \in \prod_k k$, then
$$s \sqsubset f \text{ stands for } s = f \restriction k.$$

Definition 4.8. If $E \subseteq E'$ are partial orderings and $\Upsilon \colon E \to (\prod_k k, \leq^*)$ is a strictly increasing function, a forcing notion $\mathbb{H}_{E'}(E, \Upsilon)$ is defined as follows. The conditions in $\mathbb{H}_{E'}(E, \Upsilon)$ are the triples $p = (D_p, n_p, f_p)$, where $D_p \Subset E'$, $n_p \in \mathbb{N}$, $f_p \colon D_p \to \prod_{j<n} n$, and for $a \in E$ we have $f_p(a) \sqsubset \Upsilon(a)$.

The ordering is inherited from $\mathbb{H}_{E'}$. Therefore $p \leq q$ (*p extends q*) if the following conditions hold.

(1) $D_p \supseteq D_q$, $n_p \geq n_q$, $f_p(a) \restriction n_q = f_q(a)$ for all $a \in D_q$, and
(2) for all a and b in D_q, if $a \leq_{E'} b$ then $f_p(a)(j) \leq f_p(b)(j)$ for all $j \in [n_q, n_p)$.

Thus $\mathbb{H}_{E'}(E,\Upsilon)$ is a subordering of $\mathbb{H}_{E'}$ consisting of those conditions that 'agree' with Υ on E and $\mathbb{H}_{E'}(E,\Upsilon)$ generically adds an embedding from $E'\setminus E$ into $\prod_k k$. Note that $\mathbb{H}_{E'}(E,\Upsilon)$ is not necessarily separative; this will not cause any issues.

The proofs of the two parts of Lemma 4.10 are virtually identical to the proofs of [9, Theorem 4.2] and [9, Lemma 4.3], respectively.

Definition 4.9. For a poset E' and $a\in E'$ let

$$L(a):=\{b\in E' \mid b\leq_{E'} a\}$$

and

$$R(a):=\{b\in E' \mid b\geq_{E'} a\}.$$

Lemma 4.10. *Suppose E' is a poset, E is a subposet of E', and $\dot{G}$ is the canonical name for the $\mathbb{H}_E$-generic filter.*

(1) *With the projection $\pi_E\colon \mathbb{H}_{E'}\to\mathbb{H}_E$ as in the proof of Lemma 4.7, the map $p\mapsto(\pi_E(p),p)$ from $\mathbb{H}_E$ into $\mathbb{H}_{E'}*\mathbb{H}_{E'}/\dot{G}$ is a dense embedding.*
(2) *$\mathbb{H}_E$ forces that $\mathbb{H}_{E'}/\dot{G}$ is forcing-equivalent to $\mathbb{H}_{E'}(E,\Upsilon_{\dot{G}})$ ($\Upsilon_{\dot{G}}$ is the generic embedding, see Lemma 4.6).*
(3) *If $X\subseteq E$ is such that for every $a\in E'\setminus E$ the set $X\cap L(a)$ is cofinal in $L(a)$ and the set $X\cap R(a)$ is coinitial in $R(a)$, then $\mathbb{H}_E$ forces that $\mathbb{H}_{E'}(E,\Upsilon_{\dot{G}})$ and $\mathbb{H}_{E'\setminus(E\setminus X)}(E\cap X,\Upsilon_{\dot{G}}\restriction X)$ are forcing-equivalent.* □

The following is [9, Lemma 5.1] (see also [3, Lemma 2.5]).

Lemma 4.11. *Suppose $\mathbb{P}_0$ and $\mathbb{P}_1$ are forcing notions and $\dot{f}_j$ is a $\mathbb{P}_j$-name for an element of $\prod_k k$ for $j<2$. If $\mathbb{P}_0\times\mathbb{P}_1\Vdash \dot{f}_0\leq^*\dot{f}_1$ then the set of all $p\in\mathbb{P}_0\times\mathbb{P}_1$ such that there exist $m\in\mathbb{N}$ and $h\in\prod_k k$ which satisfy $p\Vdash\dot{f}_0\leq^m\check{h}$ and $p\Vdash\check{h}\leq^m\dot{f}_1$ is dense in $\mathbb{P}_0\times\mathbb{P}_1$.* □

In combination with Lemma 4.11, Lemma 4.12 will be used in a crucial place in the proof of Theorem 4.13.

Lemma 4.12. *Suppose $(E,\leq)$ is a poset and A,B, and D are subsets of E such that $E=A\cup B$, $D=A\cap B$, and for every $a\in A$ and every $b\in B$ the following conditions hold.*

(1) *$a\leq b$ if and only if $a\leq d$ and $d\leq b$ for some $d\in D$, and*
(2) *$a\geq b$ if and only if $a\geq d$ and $d\geq b$ for some $d\in D$.*

Then $\mathbb{H}_D$ forces that ($\dot{G}$ is the canonical name for the generic filter in $\mathbb{H}_D$) $\mathbb{H}_E(D,\Upsilon_{\dot{G}})$ and $\mathbb{H}_A(D,\Upsilon_{\dot{G}})\times\mathbb{H}_B(D,\Upsilon_{\dot{G}})$ are forcing equivalent.

With the assumptions of Lemma 4.12 it can be proved that the function

$$\Xi\colon \mathbb{H}_E(D,\Upsilon_{\dot{G}})\to\mathbb{H}_A(D,\Upsilon_{\dot{G}})\times\mathbb{H}_B(D,\Upsilon_{\dot{G}})$$

defined by $\Xi(p):=(\pi_A(p),\pi_B(p))$ is a dense embedding, but we will not need this fact.

Proof. We use the notation from Lemma 4.10 and write $\dot{G}(X)$ for the canonical name for the generic filter for $\mathbb{H}_X$ (or $\mathbb{H}_X(Y,\Upsilon)$ for some Y and Υ) where X is A,B,D, or E.

By Lemma 4.10 (1) with E and A in place of E' and E, $\mathbb{H}_E$ is forcing equivalent to $\mathbb{H}_A*\mathbb{H}_E(A,\Upsilon_{\dot{G}(A)})$. By the same lemma with A and D in place of E' and E,

$\mathbb{H}_A$ is forcing equivalent to $\mathbb{H}_D * \mathbb{H}_A(D, \Upsilon_{\dot{G}(D)})$. Therefore $\mathbb{H}_E$ is forcing equivalent to the iteration

$$\mathbb{H}_D * \mathbb{H}_A(D, \Upsilon_{\dot{G}(D)}) * \mathbb{H}_E(A, \Upsilon_{\dot{G}(A)}). \tag{4.1}$$

The assumptions imply that $L(b) \cap D$ is cofinal in $L(b)$ and $R(b) \cap D$ is coinitial in $R(b)$, for every $b \in B$. Therefore by Lemma 4.10 (3) applied with $\mathbb{H}_E(A, \Upsilon_{\dot{G}(D)})$, E, A, and D in place of $\mathbb{H}_{E'}(E, \Upsilon_{\dot{G}})$, E', E, and X, we conclude that $\mathbb{H}_A$ forces that $\mathbb{H}_E(A, \Upsilon_{\dot{G}(A)})$ is forcing equivalent to $\mathbb{H}_B(D, \Upsilon_{\dot{G}(D)})$ (also recall that $\mathbb{H}_D$ is a regular subordering of $\mathbb{H}_A$, and that $\Upsilon_{G(A)}$ extends $\Upsilon_{G(D)}$). Since $\mathbb{H}_B(D, \Upsilon_{\dot{G}(D)})$ does not depend on $\dot{G}(A)$, but only on its intersection with $\mathbb{H}_D$, the iteration in (4.1) is forcing equivalent to

$$\mathbb{H}_D * (\mathbb{H}_B(D, \Upsilon_{\dot{G}(D)}) \times \mathbb{H}_A(A, \Upsilon_{\dot{G}(D)}))$$

and $\mathbb{H}_D$ forces that $\mathbb{H}_E(A, \Upsilon_{\dot{G}(A)})$ is forcing equivalent to the product of $\mathbb{H}_B(D, \Upsilon_{\dot{G}(D)})$ and $\mathbb{H}_A(A, \Upsilon_{\dot{G}(D)})$, as claimed. $\square$

In the proof of Theorem 4.13, for f and g in $C^{\mathbb{N}}$ (with $(C, \triangleleft)$ as guaranteed by Lemma 2.2) we will write $f \triangleleft^n g$ if $f(j) \triangleleft g(j)$ for all $j \geq n$. A proof of Theorem 4.13 is analogous to, but shorter than, the proof of [9, Theorem 9.1] (a baroque writeup of this proof with an ample supply of limiting examples and all sorts of digressions (many of which were warranted) can be found in [9]).

Theorem 4.13. *Suppose κ is a regular cardinal such that $\lambda^{\aleph_0} < \kappa$ for every cardinal $\lambda < \kappa$ and E is a partial ordering such that neither κ nor κ^* embeds into E. Then $\mathbb{H}_E$ forces that $\prod_{\mathrm{Fin}}(A_n, \triangleleft_n)$ has no κ-chains for any sequence $(A_n, \triangleleft_n)$, for $n \in \mathbb{N}$, of countable sets with asymmetric binary relations.*

Proof. Since C as in Lemma 2.2 is universal, $\mathbb{H}_E$ forces that $\prod_{\mathrm{Fin}}(A_n, \triangleleft_n)$ has a κ-chain for some sequence $(A_n, \triangleleft_n)$ (not necessarily in the ground model) if and only if $\mathbb{H}_E$ forces that $(C^{\mathbb{N}}, \triangleleft^*) := \prod_{\mathrm{Fin}}(C, \triangleleft)$ has a κ chain. It will therefore suffice to prove the theorem with the additional assumption that $(A_n, \triangleleft_n) = (C, \triangleleft)$ for all n.

Assume that $\dot{f}_\xi$, for $\xi < \kappa$, is a name for a κ-chain in $(C^{\mathbb{N}}, \triangleleft^*)$. (We emphasize that this means that for all $\xi < \eta$, $\mathbb{H}_E$ forces both $\dot{f}_\xi \triangleleft^* \dot{f}_\eta$ and $\dot{f}_\eta \not\triangleleft^* \dot{f}_\xi$.) The ccc-ness of $\mathbb{H}_E$ implies that for every limit ordinal ξ there exists a countable $E(\xi) \subseteq E$ such that $\dot{f}_\xi$, $\dot{f}_{\xi+1}$, and $\dot{f}_{\xi+2}$ are $\mathbb{H}_{E(\xi)}$-names. Since κ is regular and $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$, the Δ-system lemma for countable sets implies that any family of κ countable sets includes a Δ-system of cardinality κ. By passing to a subfamily, we may assume that the sets $E(\xi)$ form a Δ-system with a countable root A.

For every limit ordinal ξ fix $q_\xi \in \mathbb{H}_E$ and $n \in \mathbb{N}$ such that

$$q_\xi \Vdash_{\mathbb{H}_E} \dot{f}_\xi \triangleleft^n \dot{f}_{\xi+1} \triangleleft^n \dot{f}_{\xi+2}. \tag{4.2}$$

By the pigeonhole principle and passing to a subfamily if necessary, we may assume that n as in (4.2) is the same for all ξ. Lemma 4.7 implies that $\mathbb{H}_{E(\xi)}$ is a regular subordering of $\mathbb{H}_E$. We may therefore assume that $q_\xi \in \mathbb{H}_{E(\xi)}$ and we have

$$q_\xi \Vdash_{\mathbb{H}_{E(\xi)}} \dot{f}_\xi \triangleleft^n \dot{f}_{\xi+1} \triangleleft^n \dot{f}_{\xi+2}.$$

Writing $q_\xi = (D_\xi, n_\xi, f_\xi)$, let $F(\xi) := D_\xi \setminus A$. Note that each $F(\xi)$ is finite.

Fix $\xi < \kappa$ for a moment and fix a generic filter $G \subseteq \mathbb{H}_{A \cup F(\xi)}$ such that $q_\xi \in G$. Recall that the domain of C is $\mathbb{N}$, and is therefore equipped with a well-ordering (as a matter of fact, the first well-ordering known to man). For $j \in \mathbb{N}$ we can let $h_\xi(j)$ be the least element of C (in this well-ordering) such that

$$r \Vdash \dot{f}_{\xi+1}(j) = \check{c}$$

for some r in the quotient $\mathbb{H}_E(A \cup F(\xi), \Upsilon_G)/G$ (see Lemma 4.10).

This defines $h_\xi \in C^{\mathbb{N}}$ in $V[G]$. Use the Maximality Principle ([32, Theorem IV.7.1]) to choose an $\mathbb{H}_{A\cup F(\xi)}$-name $\dot{h}_\xi$ for this function.

Claim 4.14. The condition q_ξ forces that $\dot{f}_\xi \lhd^n \dot{h}_\xi \lhd^n f_{\xi+2}$.

Proof. If there are $r \leq q_\xi$ in $\mathbb{H}_E$ and $j \geq n$ such that $r \Vdash \dot{f}_\xi(j) \ntrianglelefteq \dot{h}_\xi(j)$, fix a generic filter G in $\mathbb{H}_E$ containing r. Then in $V[G]$ we have $\dot{f}_\xi \ntrianglelefteq^n \dot{f}_{\xi+1}$, although $q_\xi \in G$; contradiction. An analogous argument gives that there is no $j \geq n$ such that some $r \leq_E q_\xi$ forces that $\dot{h}_\xi(j) \ntrianglelefteq \dot{f}_{\xi+2}(j)$. □

The pairs $(q_\xi, \dot{h}_\xi)$ are indexed by limit ordinals below κ. We re-enumerate them preserving the order and obtain conditions q_ξ and names $\dot{h}_\xi$ for $\xi < \kappa$. Since $\mathbb{H}_E$ has the ccc, some condition $q \in \mathbb{H}_E$ forces that κ of the q_ξ's belong to the generic filter. Therefore q forces that the family of all h_ξ such that q_ξ belongs to the generic filter is a κ-chain in $(C^{\mathbb{N}}, \lhd^*)$.

By Proposition 3.9, there exists a cofinal $X \subseteq \kappa$ such that for any two distinct elements $\xi < \eta$ of X there is $s \Subset \kappa$ such that $\{\xi, \eta\} = \{\min(s), \max(s)\}$ and there is no s-walk with endpoints in $F(\xi)$ and $F(\eta)$.

Fix $\xi \in X$, let $\xi' := \min(X \setminus (\xi+1))$, and fix $s \Subset \kappa$ such that $\{\xi, \xi'\} = \{\min(s), \max(s)\}$ and there is no s-walk with endpoints in $F(\xi)$ and $F(\xi')$. We will analyze the relation between the names $\dot{h}_\xi$ and $\dot{h}_{\xi'}$.

Let $\xi(j)$, for $j < n$,[13] be an increasing enumeration of s, so that in particular $\xi(0) = \xi$ and $\xi(n-1) = \xi'$. Consider the depletion $\ll_s$ of $\leq_E$ given by A and $F(\eta)$, for $\eta \in s$. By Lemma 3.4, $\ll_s$ is a partial ordering on

$$A' := A \cup \bigcup_{j<n} F(\xi(j)).$$

For $i < j < n$ let

$$A(i,j) := A \cup F(\xi(i)) \cup F(\xi(j)),$$

ordered by $\ll_s$. Note that if $|i-j| > 1$ then $A(i,j)$ is ordered by a restriction of the depletion associated with s which can differ from the natural depletion $\ll_{(\xi(i),\xi(j))}$ (Remark 3.7). However, for $i < n-1$, Lemma 3.8 implies that the restriction of $\ll_s$ to $A(i, i+1)$ agrees with the depletion $\ll_{\{\xi(i),\xi(i+1)\}}$, which by Lemma 3.2 agrees with the ordering induced from E. By Lemma 4.7, we have $\mathbb{H}_{A(i,i+1)} \lessdot \mathbb{H}_E$.

Since A' is ordered by $\ll_s$, which possibly disagrees with the ordering of A' inherited from E, the posets $\mathbb{H}_{A'}$ and $\mathbb{H}_E$ are possibly unrelated. On the other hand, each $A(i,j)$ is a subordering of A' and therefore Lemma 4.7 implies $\mathbb{H}_{A(i,j)} \lessdot \mathbb{H}_{A'}$.

Since $\dot{h}_{\xi(i)}$ and $\dot{h}_{\xi(j)}$ are $\mathbb{H}_{A(i,j)}$-names and $\mathbb{H}_E$ forces $\dot{h}_{\xi(i)} \lhd^* \dot{h}_{\xi(i+1)}$ for all $i < n-1$, we have that $\mathbb{H}_{A'}$ forces $\dot{h}_{\xi(i)} \lhd^* \dot{h}_{\xi(i+1)}$ for all $i < n-1$. By transitivity, $\mathbb{H}_{A'}$ forces $\dot{h}_{\xi(0)} \lhd^* \dot{h}_{\xi(n-1)}$. Since $\dot{h}_{\xi(0)}$ and $\dot{h}_{\xi(n-1)}$ are $\mathbb{H}_{A(0,n-1)}$-names and $\mathbb{H}_{A(0,n-1)} \lessdot \mathbb{H}_{A'}$, we have that $\mathbb{H}_{A(0,n-1)}$ forces $\dot{h}_{\xi(0)} \lhd^* \dot{h}_{\xi(n-1)}$.

[13]This n is unrelated to the n appearing in 4.2. No danger of confusion here.

Since there is no s-walk whose endpoints are some $x \in F(\xi(0))$ and some $y \in F(\xi(n-1))$, we have that $x \ll_s y$ implies there is $a \in A$ such that $x \leq a \leq y$, and that $y \ll_s x$ implies there is $a \in A$ such that $y \leq a \leq x$. This means that the assumptions of Lemma 4.12 with E, A, B, and D replaced with $A \cup F(\xi(0)) \cup F(\xi(n-1))$, $A \cup F(\xi(0))$, $A \cup F(\xi(n-1))$, and A, respectively, (sorry!) are satisfied. Therefore if $G \subseteq \mathbb{H}_A$ is a generic filter then the quotient $\mathbb{H}_{A\cup A(0,n-1)}/G$ is forcing-equivalent to the product

$$\mathbb{H}_{A\cup F(\xi(0)}(A, \Upsilon_G) \times \mathbb{H}_{A\cup F(\xi(n-1))}(A, \Upsilon_G).$$

Most importantly, the names $\dot{h}_{\xi(0)}$ and $\dot{h}_{\xi(n-1)}$ are added by the two factors of this product. By Lemma 4.11, there exist a condition $p_\xi \in \mathbb{H}_A$ and an $\mathbb{H}_A$-name $\dot{g}_\xi$ (recall that $\xi = \xi(0)$) such that

$$p_\xi \Vdash \dot{h}_{\xi(0)} \leq^* \dot{g}_\xi \leq^* \dot{h}_{\xi(n-1)}.$$

Since $\mathbb{H}_A$ is countable, there is $q \in \mathbb{H}_A$ such that $Y = \{\xi \in X | p_\xi = q\}$ is a cofinal subset of X (and of κ). Therefore q forces that $\mathbb{H}_A$ adds a strictly increasing κ-chain $\dot{g}_\xi$, for $\xi < \kappa$, to $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Since A is countable, $\mathbb{H}_A$ cannot add more than $\mathfrak{c}$ reals; contradiction. □

The robustness of the robust order property (Definition 1.2) is used in Proposition 4.15.

Proposition 4.15. *Suppose that the pair (T, φ) has the robust order property and E is any poset. Then $\mathbb{H}_E$ forces the following.*

1. *The poset E embeds into $\prod_{\mathrm{Fin}}(A_n, \triangleleft_\varphi)$ for every sequence (A_n) of models of T.*
2. *For any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there is strictly increasing map from E into $\prod_{\mathcal{U}}(A_n, \triangleleft_\varphi)$ whose range is linearly ordered by $\triangleleft_\varphi$.*

Proof. The first part is almost obvious, but proving it in some detail will also provide a proof of the second part.

By Lemma 2.1, there are $\eta \in \mathbb{N}^{\mathbb{N}}$ and $\Phi\colon \prod_{\mathrm{Fin}}(n, \leq^*) \to \prod_{\mathrm{Fin}}(\eta(n), <^*)$ such that $f \leq^* g$ and $g \not\leq^* f$ implies $(\forall^\infty n)\Phi(f)(n) < \Phi(g)(n)$. Since A_n is a model of T, there exists a $\triangleleft_\varphi$-chain C_n of length $\eta(n)$ in A_n. By identifying this chain with $(\eta(n), \leq)$, we obtain $\Phi\colon \prod_{\mathrm{Fin}}(n, \leq^*) \to \prod_{\mathrm{Fin}}(A_n, \triangleleft_\varphi)$ such that $f \leq^* g$ and $g \not\leq^* f$ implies $\Phi(f)(n) \triangleleft_\varphi \Phi(g)(n)$ and $\Phi(g)(n) \not\triangleleft_\varphi \Phi(f)(n)$ for all but finitely many n. By composing the embedding of E into $\prod_{\mathrm{Fin}}(n, \leq^*)$ provided by Theorem 4.1 with Φ, we obtain an $\mathbb{H}_E$-name for an embedding of $\Xi\colon E \to \prod_{\mathrm{Fin}}(A_n, \triangleleft_\varphi)$ (this proves the first part; read on for the proof of the second part) that in addition has the property that $a <_E b$ implies

$$(\forall^\infty n)(\Xi(f)(n) \triangleleft_\varphi \Xi(g)(n) \text{ and } \Xi(g)(n) \not\triangleleft_\varphi \Xi(f)(n)).$$

Let $\mathcal{U}$ be a nonprincipal ultrafiter on $\mathbb{N}$ and let $\pi_{\mathcal{U}}$ denote the quotient map from $\prod_{\mathrm{Fin}} A_n$ to $\prod_{\mathcal{U}} A_n$. Then the displayed formula implies that the restriction of $\pi_{\mathcal{U}}$ to $\Xi[E]$ is strictly increasing. The range of this map is the ultraproduct of the $\triangleleft_\varphi$-chains C_n, and therefore linearly ordered by Łoś's Theorem. This proves the second part. □

5. Proofs of Theorem C and Corollary D

The proof of Theorem C will use the following result (see [9, Theorem 3.2] for a proof) .

Theorem (Galvin). *For every uncountable cardinal κ there exists a partial ordering E_κ such that E_κ has no infinite chains but for every linear ordering $\mathcal{L}$ such that there are neither κ-chains nor κ^*-chains in $\mathcal{L}$ there is no strictly increasing map $\Phi\colon E \to \mathcal{L}$.* □

Proof of Theorem C. Fix a theory T with the robust order property. We will prove that the Levy collapse of the continuum to $\aleph_1$ followed by $\mathbb{H}_E$ for E provided by Galvin's theorem forces both conclusions of Theorem C.

These proofs have a common initial segment that we now present.

In the extension by the Levy collapse of the continuum to $\aleph_1$, let $\kappa > \mathfrak{c}$ be a regular cardinal ($\kappa = \aleph_2$ will do). Let E be the poset as guaranteed by Galvin's theorem stated at the beginning of this section. We will prove that the Levy collapse followed by $\mathbb{H}_E$ is the forcing notion as promised in the statement of Theorem C.

Fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, a sequence (A_n) of countable structures in the language of T, and a sequence (B_n) of countable models of T. Proposition 4.15 implies that $\mathbb{H}_E$ adds a strictly increasing map from E into $\prod_{\mathcal{U}}(B_n, \triangleleft_\varphi)$ whose range is linearly ordered by $\triangleleft_\varphi$. By the choice of E, there exists a κ-$\triangleleft_\varphi$-chain or a κ^*-$\triangleleft_\varphi$-chain in $\prod_{\mathcal{U}}(B_n, \triangleleft_\varphi)$.

On the other hand, by Theorem 4.1 there are neither κ-chains nor κ^*-chains in $\prod_{\mathrm{Fin}}(A_n, \triangleleft_\varphi)$.

From this point on the proofs of (1) and (2) differ.

(1) We need to prove that $\prod_{\mathcal{U}} B_n$ is not isomorphic to an elementary submodel of $\prod_{\mathrm{Fin}} A_n$. Since elementary embeddings preserve $\triangleleft_\varphi$, this is immediate from the fact that the former contains a κ- or κ^*-$\triangleleft_\varphi$-chain and the latter does not.

(2) Suppose in addition that φ is quantifier-free. Then all embeddings preserve $\triangleleft_\varphi$, and $\prod_{\mathcal{U}} B_n$ cannot be isomorphic to a submodel of $\prod_{\mathrm{Fin}} A_n$. □

Proof of Corollary D. Suppose that A is a separable C^*-algebra and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. If $\mathcal{U}$ is principal, then $(A \otimes C(K))^{\mathcal{U}}$ is isomorphic to $A \otimes C(K)$ while A^∞ is nonseparable. We may therefore assume that A is infinite-dimensional and that $\mathcal{U}$ is nonprincipal.

The theory of infinite-dimensional C^*-algebras has the order property witnessed by an atomic formula ([16, Lemma 5.3]). Therefore the theory of $A \otimes C(K)$ has the robust order property, and Theorem C (2) implies that $(A \otimes C(K))^{\mathcal{U}}$ does not embed into B^∞ for any C^*-algebra B. □

6. Proofs of Theorem A and Theorem B: Tie points

The contents of this section is rather accurately described by its title.

Definition 6.1. Suppose X is a compact Hausdorff space. A point $x \in X$ is a *tie point* if there are closed subsets A and B of X such that $A \cup B = X$ and $A \cap B = \{x\}$ (in symbols, $A \bowtie_x B$).

Two subsets $\mathcal{I}$ and $\mathcal{J}$ of a Boolean algebra $\mathfrak{B}$ are *orthogonal* if $a \wedge b = 0_{\mathcal{B}}$ for all $a \in \mathcal{I}$ and all $b \in \mathcal{J}$. The following is proved by parsing the definitions.

Proposition 6.2. *Suppose $\mathfrak{B}$ is a Boolean algebra. The following are equivalent for an ultrafilter $\mathcal{U}$ on $\mathfrak{B}$.*

(1) *The complement of $\mathcal{U}$ is equal to the union of two orthogonal ideals.*
(2) *$\mathcal{U}$ is a tie-point in the Stone space of $\mathfrak{B}$.* □

Definition 6.3. By analogy with true P-points, an ultrafilter $\mathcal{U}$ in a Boolean algebra is called a *true tie point* if the ideals as in Proposition 6.2 (2) can be chosen so that each one of them is generated by a linearly ordered subset.

The salient point of the proof of the following Lemma is the observation that true tie points are Σ_1-definable, but the reader may choose to ignore this remark and read the proof instead.

Lemma 6.4. *Every ultraproduct of countable atomless Boolean algebras has a true tie point.*

Proof. Every ultrafilter in a countable atomless Boolean algebra is a true tie point, since the generating sets of order type ω can be chosen by recursion. Suppose $\prod_{\mathcal{U}} C_n$ is an ultraproduct of countable atomless Boolean algebras. If $\mathcal{U}$ is principal, then $\prod_n C_n$ is isomorphic to one of the C_n's and the assertion follows from the first sentence of this proof.

Now assume $\mathcal{U}$ is nonprincipal. For every C_n fix a true tie point p_n and linearly ordered generating sets $\mathcal{A}_n$ and $\mathcal{B}_n$ for the ideal $C_n \setminus p_n$. Then $(C_n, \mathcal{A}_n, \mathcal{B}_n)$ is an expansion of C_n to the language with two additional unary predicates. Each one of these structures satisfies the following: Both $\mathcal{A}_n$ and $\mathcal{B}_n$ are linearly ordered, $A \wedge B = \emptyset$ for all $A \in \mathcal{A}_n$ and $B \in \mathcal{B}_n$, and for every $X \in C_n$ either X or its complement belongs to $\mathcal{A}_n \cup \mathcal{B}_n$. These are all first-order statements, and they imply that the complement of $\mathcal{A}_n \cup \mathcal{B}_n$ is an ultrafilter.

The ultraproduct $\prod_{\mathcal{U}}(C_n, \mathcal{A}_n, \mathcal{B}_n)$ is an expansion of $\prod_n C_n$ and by Łoś's Theorem the sets $\mathcal{A} := \prod_{\mathcal{U}} \mathcal{A}_n$ and $\mathcal{B} := \prod_{\mathcal{U}} \mathcal{B}_n$ generate ideals of $\prod_{\mathcal{U}} C_n$ whose complement is a true tie point. □

Proof of Theorem A. We need to prove that PFA implies $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is not isomorphic to an ultraproduct of Boolean algebras associated with a nonprincipal ultrafilter on $\mathbb{N}$. By [50] (see [7, Corollary 1.9]), PFA implies that there are no tie points in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, while there are tie points in an ultraproduct of countable atomless Boolean algebras by Lemma 6.4. □

The following will be used in the proof of Theorem B.

Lemma 6.5. *The poset for adding at least $\mathfrak{c}^+$ Cohen reals forces that every projectively definable atomless Boolean algebra $\mathfrak{B}$ has no true tie points.*

Proof. Suppose that $\kappa \geq \mathfrak{c}^+$ and let $\mathbb{C}_\kappa$ denote the poset for adding κ Cohen reals. Fix an $n \in \mathbb{N}$ and $\mathbf{\Sigma}^1_n$-formulas φ, $\varphi_\wedge$, $\varphi_\vee$, and $\varphi_\setminus$, which define $\mathfrak{B}$. We will only need the $\mathbf{\Delta}^1_{n+1}$ formula $\varphi_<$, that defines the relation $a < b$ in $\mathfrak{B}$. By passing to an intermediate forcing extension, without a loss of generality we may assume that the reals coding these formulas belong to the ground model.

By genericity, no nonprincipal ultrafilter on $\mathbb{N}$ in the forcing extension is generated by fewer than κ subsets of $\mathbb{N}$. (This is well-known, but here is a sketch of the proof: After adding κ Cohen reals, for every $\mathcal{X} \subseteq \mathcal{U}$ of cardinality less than κ there is a Cohen real Y generic over $V[\mathcal{X}]$. For every infinite $X \subseteq \mathbb{N}$, the set of all

$Y \subseteq \mathbb{N}$ such that $X \cap U$ and $X \setminus Y$ are both infinite is comeager. Therefore $\mathcal{X}$ does not 'decide' whether $Y \in \mathcal{U}$ or $\mathbb{N} \setminus Y \in \mathcal{U}$.) Assume p is a true tie point in $\mathfrak{B}$ and let $\mathcal{A}$ and $\mathcal{B}$ be the linearly ordered (modulo $\mathcal{I}$) sets whose complements generate $\mathfrak{B} \setminus p$. By genericity, at least one of $\mathcal{A}$ and $\mathcal{B}$ has cofinality greater than $\mathfrak{c}$. By interchanging $\mathcal{A}$ and $\mathcal{B}$, we may assume that the cofinality of $\mathcal{A}$ is $\kappa > \mathfrak{c}$.

The proof is completed by Kunen's isomorphism of names argument ([31]; see §8 for disambiguation) that we now sketch.

Suppose that $\dot{f}_\xi$, for $\xi < \kappa$, is a $\mathbb{C}_\kappa$-name for a strictly increasing chain cofinal in $\mathcal{A}$. In particular, f_ξ, for $\xi < \mathfrak{c}^+$, is a name for a strictly increasing chain in $\mathcal{A}/\mathcal{I}$. This will suffice to obtain a contradiction. Since each $\dot{f}_\xi$ is a name for a real, it is coded by a sequence of antichains and the union, denoted D_ξ, of the supports of all conditions in these antichains is a countable subset of κ. Since $\mathfrak{c}^+$ is regular and $\lambda < \mathfrak{c}^+$ implies $\lambda^{\aleph_0} < \mathfrak{c}^+$, by passing to a cofinal subset we may assume that the sets D_ξ form a Δ-system with root R. By a counting argument and passing to a cofinal subset again we may assume that the restrictions of $\dot{f}_\xi$ to R agree, and that $\dot{f}_\xi$ and $\dot{f}_\eta$ are isomorphic for all ξ and η. This means that for $\xi < \eta$ there is an automorphism $\Phi_{\xi\eta}$ of $\mathbb{C}_\kappa$ that sends $\dot{f}_\xi$ to $\dot{f}_\eta$, for any two $\xi < \eta < \mathfrak{c}^+$. However, since $\mathbb{C}_\kappa$ forces that $\varphi_<(\dot{f}_\xi, \dot{f}_\xi)$ is true and the real coding the asymmetric formula $\varphi_<$ is in the ground model, this is a contradiction that completes the proof. □

Proof of Theorem B. If $\mathcal{I}$ does not include the Frèchet filter then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has atoms, and therefore cannot be isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. We may therefore assume that $\mathcal{I}$ incudes the Frèchet filter.

In the model obtained by adding at least $\mathfrak{c}^+$ Cohen reals to a model of ZFC, suppose that $\mathfrak{B}$ is a projectively definable Boolean algebra. By Lemma 6.5, there are no true tie points in $\mathfrak{B}$. By Lemma 6.4, in every model of ZFC there is a true tie point in any ultraproduct of countable atomless Boolean algebras. □

7. The existence of universal ultrapowers

Hitherto unbeknownst to the junior (!) author, some questions closely related to those resolved in our main results have easy answers, collected in this section.

Suppose that T is a theory in a countable (or separable) language and let

$$\operatorname{Ult}_T = \{A^{\mathcal{U}} | A \models T, A \text{ is countable (separable), and } \mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}\},$$
$$\operatorname{Mod}_{\mathfrak{c}}(T) = \{A | A \models T, |A| = \mathfrak{c}\}.$$

This conclusion, assumed for all T, is by [17, Theorem 5.6]. CH implies that all $M \in \operatorname{Ult}_T$ are saturated, and therefore isomorphic. This conclusion is by [17, Theorem 5.6] (also [19]) equivalent to CH. In some applications it suffices to know that among the ultrapowers of a model A of T there exists one which is (injectively) universal. We'll say that a set of models has a $\preceq$-*universal element* if it has an element universal under elementary embeddings, and that it has a $\hookrightarrow$-*universal element* if it has an element universal under not necessarily elementary embeddings.

As pointed out in [47, p. 181], this sort of nitpicking (distinguishing between $\hookrightarrow$ and $\preceq$) is in general unnecessary, since one can make the difference disappear by expanding the language by Skolem functions. We nevertheless nitpick because the existence of Skolem functions in continuous logic is a delicate problem.

Proposition 7.1. *Suppose that T is a first-order theory. Then* Ult_T *has a $\preceq$-universal ($\hookrightarrow$-universal) element if and only if* $\operatorname{Mod}_{\mathfrak{c}}(T)$ *does.*

Proof. It is well-known that every model of T of cardinality $\mathfrak{c}$ is isomorphic to an elementary submodel of an ultrapower of a countable model of T. This follows from the results of [40, Chapter VI.5] or [19]. Therefore any $M \in \mathrm{Ult}_T$ which is $\preceq$-universal is also $\preceq$-universal for $\mathrm{Mod}_{\mathfrak{c}}(T)$, and similarly for $\hookrightarrow$-universality. $\square$

Proposition 7.2. *If T is a stable theory in a countable language, then Ult_T has a $\preceq$-universal, and therefore a $\hookrightarrow$-universal, model (in ZFC).*

Proof. In [17, Theorem 5.6] it was proved that if T is stable then all ultrapowers $A^{\mathcal{U}}$ for a countable (or separable) A and $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ are saturated, and therefore isomorphic. $\square$

We can therefore assume that T is not stable, or equivalently, that it has the order property (for the continuous case, this equivalence is in [17, Theorem 5.5], generalizing classical result of the second author, [40]).

Question 7.3. *Suppose that T is a theory in a countable language with the order property and CH fails. Can $\mathrm{Mod}_{\mathfrak{c}}(T)$ (equivalently, Ult_T) have a $\preceq$-universal, or a $\hookrightarrow$-universal, model?*

We give some partial answers to this question. The strict order property (SOP) of T is the strengthening of the order property in which the witnessing formula φ is required to define a (partial) ordering on every model of T.

Proposition 7.4. *Suppose that T is a theory in a countable language with the SOP. If there exists a cardinal κ such that $\kappa^+ < \mathfrak{c} = \mathrm{cf}(\mathfrak{c}) < 2^\kappa$ and T is complete, then Ult_T does not have a $\preceq$-universal model.*

If the SOP is witnessed by a quantifier-free formula, then even if T is not complete, Ult_T does not even have a $\hookrightarrow$-universal model.

Proof. The existence of κ as stated implies there is no $\hookrightarrow$-universal linear order of cardinality $\mathfrak{c}$ by [28, Theorem 3.10]. By [28, Theorem 5.5], this implies that $\mathrm{Mod}_{\mathfrak{c}}(T)$ has no $\preceq$-universal element. As explained in [28], if the SOP is witnessed by a quantifier-free formula, then $\mathrm{Mod}_{\mathfrak{c}}(T)$ has no $\hookrightarrow$-universal element. $\square$

The SOP_4 ([43, Definition 2.5]) is a technical weakening of the strict order property, hence the following is a strengthening of Proposition 7.4.

Proposition 7.5. *Suppose that T is a theory in a countable language with the SOP_4. If there exists a cardinal κ such that $\kappa^+ < \mathfrak{c} = \mathrm{cf}(\mathfrak{c}) < 2^\kappa$ and T is complete, then Ult_T does not have a $\preceq$-universal model.*

If the SOP_4 is witnessed by a quantifier-free formula, then even if T is not complete, Ult_T does not even have a $\hookrightarrow$-universal model.

Proof. This is [43, Theorem 2.12]. $\square$

The *olive property* is a collection of properties of a first-order theory introduced in [46, Definition 1.8 and Definition 2.1]. One talks about the (Δ, η, k, m)-olive property, but for our purposes, Δ is the set of all formulas in the language of T if T is complete or the set of all quantifier-free formulas otherwise. The parameter m is the arity of the tuples witnessing the order property and can be suppressed. The roles of η and k are laid out in [46, Definition 2.1], and we will say that 'T has the olive property' if it has the (η, k)-olive property for some $\eta \in \{0,1\}^n$ and k. We only remark that by [46, Theorem 3.1] the theory of groups has the olive property (by [49] the theory of groups fails SOP_4), hence (since the olive property for groups is witnessed by quantifier-free formulas), the following applies to it.

Proposition 7.6. *Suppose that T is a theory in a countable language with the olive property. If there exists a cardinal κ such that $\kappa^+ < \mathfrak{c} = \mathrm{cf}(\mathfrak{c}) < 2^\kappa$ and T is complete, then Ult_T does not have a $\preceq$-universal model.*

If the olive property is witnessed by a quantifier-free formulas, then even if T is not complete, Ult_T does not even have a $\hookrightarrow$-universal model.

Proof. This is [46, Theorem 2.9 (i) and (ii)]. For the quantifier-free case, use part (ii) and the $\lambda - (\eta, k, m)$-olive property from [46, Definition 2.3 (ii)]. □

With a strengthened cardinal arithmetic assumption one can say more. For a cardinal μ, a set of models $\mathcal{A}$ is said to have a *$\preceq$-basis of cardinality μ* if there is $\mathbb{B} \subseteq \mathcal{A}$ of cardinality μ such that every $A \in \mathcal{A}$ elementarily embeds into some element of $\mathbb{B}$. It is said to have a *$\hookrightarrow$-basis of cardinality μ* if there is $\mathbb{B} \subseteq \mathcal{A}$ of cardinality μ such that every $A \in \mathcal{A}$ embeds (not necessarily elementarily) into some element of $\mathbb{B}$.

Proposition 7.7. *Suppose that T is a theory in a countable language with the SOP, SOP_4, or the olive property. If there exists a cardinal κ such that $\kappa^+ < \mathfrak{c} = \mathrm{cf}(\mathfrak{c})$ and $\mathfrak{c}^+ < 2^\kappa$ and T is complete, then $\mathrm{Mod}_\mathfrak{c}$ does not have a $\preceq$-basis of cardinality less than 2^κ.*

If the SOP, the SOP_4, or the olive property, is witnessed by quantifier-free formulas, then even if T is not complete, $\mathrm{Mod}_\mathfrak{c}(T)$ does not have a $\hookrightarrow$-basis of cardinality less than 2^κ.

Proof. We will prove the $\preceq$-case, starting with the following.

Claim 7.8. For every $\mathbb{A} \subseteq \mathrm{Mod}_\mathfrak{c}(T)$ with $|\mathbb{A}| = 2^\kappa$ there are $M_A \in \mathrm{Mod}_\mathfrak{c}(T)$ such that $A \prec M_A$ for $A \in \mathbb{A}$ and for every choice of $N_A \in \mathrm{Mod}_\mathfrak{c}(T)$ with $M_A \preceq N_A$ for $A \in \mathbb{A}$ there exists $X \subseteq \mathbb{A}$ of cardinality 2^κ such that N_A does not embed into N_B for all distinct A and B in X.

Proof. With κ as in the assumptions, a function $\mathrm{inv}\colon \mathrm{Mod}_\mathfrak{c}(T) \to [\mathcal{P}(\kappa)]^\mathfrak{c}$ with the following properties exists.

(1) If M_0 and M_1 are in $\mathrm{Mod}_\mathfrak{c}(T)$ and M_0 is elementarily embeddable into M_1 then $\mathrm{inv}(M_0) \subseteq \mathrm{inv}(M_1)$.
(2) If the property in question is witnessed by quantifier-free formulas, M_0 and M_1 are in $\mathrm{Mod}_\mathfrak{c}(T)$, and M_0 is embeddable (not necessarily elementarily) into M_1, then $\mathrm{inv}(M_0) \subseteq \mathrm{inv}(M_1)$.
(3) If $M_0 \in \mathrm{Mod}_\mathfrak{c}(T)$ and $S \subseteq \kappa$ then there exists $M_1 \in \mathrm{Mod}_\mathfrak{c}(T)$ such that $M_0 \prec M_1$ and $S \in \mathrm{inv}(M_1)$.

For SOP, inv is $\mathrm{INV}(M, \bar{C})$ for a fixed κ-scale $\bar{C}$ (see [28, §3(4)] and [28, Lemma 3.7]). For SOP_4, this is $\mathrm{INV}_\varphi(M, \bar{C})$, where φ witnesses SOP_4 and $\bar{C}$ is a club system (see [43, Definition 2.13 (b)]). For the olive property, see [46, Remark 1.9].

Fix $\mathbb{A} \subseteq \mathrm{Mod}_\mathfrak{c}(T)$ with $|\mathbb{A}| = 2^\kappa$. Let S_A, for $A \in \mathbb{A}$, be distinct subsets of κ. By a realizing types argument and (3), there are $M_A \in \mathrm{Mod}_\mathfrak{c}(T)$ such that $A \prec M_A$ and $S_\xi \in \mathrm{inv}(M_A)$. Fix N_A such that $M_A \preceq N_A$. Since $|\mathrm{inv}(N_A)| = \mathfrak{c}$ when $|A| = \mathfrak{c}$, by Hajnal's free subset theorem ([25, Theorem 1] applied with $m = 2^\kappa$ and $n = \mathfrak{c}$ to the function $A \mapsto \mathrm{inv}(N_A)$) there exists $X \subseteq \mathbb{A}$ of cardinality 2^κ such that $S_A \notin \mathrm{inv}(N_B)$ for all distinct A and B in X, and therefore N_A, for $A \in X$, are as required. □

Suppose towards contradiction that $\mathrm{Mod}_{\mathfrak{c}}$ has a $\preceq$-basis $\mathbb{B}$ of cardinality less than 2^κ. Since Ult_T has $2^{\mathfrak{c}}$ elements ([19]), we can fix $\mathbb{A} \subseteq \mathrm{Ult}_T$ of cardinality 2^κ. With M_A, for $A \in \mathbb{A}$, as provided by Claim 7.8, for every A there is $N_A \in \mathbb{B}$ such that $M_A \preceq N_A$. Since $|\mathbb{B}| < 2^\kappa$, the conclusion of Claim 7.8 fails; contradiction.

A proof of the $\hookrightarrow$-case of Proposition is analogous, using the modification of the Claim in which it is allowed that $N_\xi \hookrightarrow N'_\xi$. □

8. Concluding remarks and questions

The question that initiated the research reported here remains open:

Question 8.1. *Suppose that there is a countable (or separable) structure A whose theory has the order property and $\prod_{\mathcal{U}} A$ is isomorphic to $\prod_{\mathrm{Fin}} A$ for some $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$. Does it follow that the CH holds?*

Our main results show that in some models of ZFC in which CH fails the premise of Question 8.1 fails as well. The methods of [45], [42], and [41] may be relevant to the possibility of giving negative answer to this question.

Our proof of Theorem B uses the well-known technique introduced by Kunen in the proof of [31, Theorem 12.7]. It may be worth pointing out that, although the proof is well-known, the actual statement of the theorem isn't quite as well-known as it should be. This theorem asserts that in the standard model for adding $\kappa > \mathfrak{c}$ Cohen reals no well-ordering of $\mathbb{R}$ belongs to the σ-algebra generated by arbitrary rectangles on $\mathbb{R}$. The conclusion of this result is equivalent to the assertion that there are no κ-chains in any Borel ordering on a Polish space, and it is often misstated as the weaker assertion that there are no κ-chains in $\mathbb{N}^{\mathbb{N}}/\mathrm{Fin}$.

The proof of Theorem C uses a forcing notion related to the forcing $\mathbb{C}_\kappa$ for adding κ side-by-side Cohen reals and an analysis of names which is to some extent similar to Kunen's. (This forcing belongs to the class of *semicohen* forcing notions, see [29].) The two results are however different, since the forcing $\mathbb{H}_E$ used in the proof of Theorem C can add an ω_2-chain to some Borel poset $(\mathbb{N}^{\mathbb{N}}, \rho)$ without adding an ω_2-chain to $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ (this has been proved for a close relative of $\mathbb{H}_E$ in [9, Theorem 2.1]).

The argument of the proof of Theorem B works for many other forcings that add more than $\mathfrak{c}$ reals, as long as one can uniformize the names and there are no ultrafilters on $\mathbb{N}$ with small generating sets in the extension. The latter does not apply to the Sacks forcing. As a matter of fact, after adding $\mathfrak{c}^+$ Sacks reals to a model of CH with countable supports (by either countable support product or countable support iteration), there exists a selective $\aleph_1$-generated ultrafilter on $\mathbb{N}$, and it is a true tie point ([1]). It is therefore not clear whether in some of the Sacks models $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is isomorphic to an ultraproduct of countable atomless Boolean algebras. If so, then this would have to be an $\aleph_1$-generated ultrafilter. The most obvious choice would be an ultrafilter generated by a ground-model selective ultrafilter (there are $2^{\aleph_1}$ such ultrafilters by [1]). As all of these ultrafilters 'look the same' (see [54] for an interpretation of this assertion) this suggests the following test question.

Question 8.2. *Suppose that in either one of the Sacks models $\mathcal{U}$ and $\mathcal{V}$ are $\aleph_1$-generated selective ultrafilters. Is it true that $(\mathbb{N}, \leq)^{\mathcal{U}} \cong (\mathbb{N}, \leq)^{\mathcal{V}}$?*

One could ask an analogous question for countable models of other countable first-order theories with the order property; $(\mathbb{N}, \leq)$ just appears to provide the simplest interesting instance of this question. The ideas from [48, §2 and §4] may be relevant to this problem in the case of Boolean algebras.

Question 7.3 on the existence of universal ultrapowers in the absence of CH tackled in §7 also remains open. See [47] for the bigger picture.

We conclude with a few words on 'definable' reduced products $\prod_{\mathcal{F}} A_n$. If $\mathcal{F}$ is an analytic filter on $\mathbb{N}$ (i.e., one that is analytic as a subset of $\mathcal{P}(\mathbb{N})$, given its Cantor-set topology) that extends the Fréchet filter, then the restriction of $\mathcal{F}$ to any $\mathcal{F}$-positive set is not an ultrafilter (because all analytic sets, unlike the nonprincipal ultrafilters, have the universal property of Baire.) Therefore the Feferman–Vaught theorem ([22], and for the metric case [23] or [12, §16.3]) implies that if all A_n are elementarily equivalent, and if $\mathcal{F}$ is analytic and extends the Fréchet filter then $\prod_{\mathcal{F}} A_n$ is elementarily equivalent to $\prod_{\mathrm{Fin}} A_n$. Many (but not all) of the reduced products $\prod_{\mathcal{F}} A_n$ are countably saturated[14] and therefore isomorphic to $\prod_{\mathrm{Fin}} A_n$ if CH holds. The following question is somewhat vague—the answer is clearly a function of the theories of the A_ns, among other things— but see the discussion in the paragraphs following it for clarification.

Question 8.3. *Given a sequence A_n, for $n \in \mathbb{N}$, of structures of the same language, for what Borel filters $\mathcal{F}$ on $\mathbb{N}$ is $\prod_{\mathcal{F}} A_n \cong \prod_{\mathrm{Fin}} A_n$ provable in ZFC for every sequence of countable (separable) models A_n?*

In the case when each A_n is the two-element Boolean algebra—i.e., the case of the quotients of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$—the isomorphism is provable if and only if there is a continuous $f\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ that lifts such an isomorphism ([20]) and in many (conjecturally, all) cases this is equivalent to the Rudin–Keisler isomorphism of the underlying ideals ([10, Corollary 3.4.2] and [20, Corollary 3]). For current state of the art in this subject see [14].

In the case when all A_n are Boolean algebras, Question 8.3 is really a question about abelian C^*-algebras. This is because the category of Boolean algebras is, via the Stone duality, equivalent to the category of compact, zero-dimensional, Hausdorff spaces and the latter category is, by the Gelfand–Naimark duality, equivalent to the category of unital, abelian, C^*-algebras with real rank 0 (see [12, §1.3]). By this observation and the main result of [18], PFA implies that two such reduced products are isomorphic if and only if there is an (appropriately defined) 'trivial' isomorphism between them. For example, PFA implies that $\prod_{\mathrm{Fin}} B \ncong \mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ if B is the atomless countable Boolean algebra. The ultimate extension of the result of [18] to the coronas of arbitrary separable C^*-algebras was proved in [34] and [52]; see also the survey [14].

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