GENERAL NON-STRUCTURE THEORY
E59

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This is a revised version of [She87b, Ch.III,§1-§3], has existed (and somewhat revised) for many years. Was mostly ready in the early nineties, and public to some extent. For the sake of [LS03] we add a paragraph in section 1(D). For the sake of [SU] we add in the end of §2. Recently this work was used and continued in Farah-Shelah [FS10]. This was written as Chapter III of the book [Shear], which hopefully will materialize some day, but in meanwhile it is [Shej]. The intentions were: [Shei] (revising [She86]) for Ch.I, [AGS] for Ch.II, [Shej] for Ch.III, [Shec] for Ch.IV, [Shel] for Ch.V, [Shea] for Ch.VI, [Shed] for Ch.VII, [Sheh], a revision of [She85] for Ch.VIII, [Shee] for the appendix, and probably [She04], [Shef], [Sheg], and [Shek]. References like [Shee, q17=Lc2] means that c2 is the label of 3.19 in [Shee], will only help the author if changes in the paper [Shee] will change the number. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.

Abstract. The theme of the first two sections, is to prepare the framework of how from a "complicated" family of so called index models $I \in K_1$ we build many and/or complicated structures in a class K_2. The index models are characteristically linear orders, trees with $\kappa+1$ levels (possibly with linear order on the set of successors of a member) and linearly ordered graph, for this we phrase relevant complicatedness properties (called bigness).

We say when $M \in K_2$ is represented in $I \in K_1$. We give sufficient conditions when $\{M_I : I \in K^1_\lambda\}$ is complicated where for each $I \in K^1_\lambda$ we build $M_I \in K^2$ (usually $\in K^2_\lambda$) represented in it and reflecting to some degree its structure (e.g. for I a linear order we can build a model of an unstable first order class reflecting the order). If we understand the structures in K_2 well enough we can even build, e.g. rigid members of K^2_λ.

Note that we mention "stable", "superstable", but in a self contained way, not relying in any way on stability theory, just using an equivalent definition which is useful here and explicitly given. We also frame the use of generalizations of Ramsey and Erdös-Rado theorems to get models in which any I from the relevant K_1 is reflected. We give in some detail how this may apply to specific cases: Boolean Algebras, the class of separable reduced Abelian $\dot{p}$-group and how we get relevant models for ordered graphs (in some cases via forcing).

In the third section we show stronger results concerning linear orders. If for each linear order I of cardinality $\lambda > \aleph_0$ we can attach a model $M_I \in K_\lambda$ in which the linear order can be embedded such that for enough cuts of I, their being omitted is reflected in M_I, then there are 2^λ non-isomorphic cases.

But in the end of the second section we show how the results on trees with $\omega+1$ levels (on which concentrate [Shea] gives results on linear ordered (not covered by §3), on trees with $\omega+1$ levels see [Shea]. To get more we prove explicitly more on such trees. Those will be enough for results in model theory of Banach space of Shelah-Usvyatsov [SU], see more in [Sheb].

Annotated Content

§ 0. Introduction

A major result presented in this paper is (in earlier proofs we have it only in "most" cases):

Theorem 0.1. *If $\psi \in \mathbb{L}_{\chi^+,\omega}, \varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\chi^+,\omega}, \ell g(\bar{x}) = \ell g(\bar{y}) = \partial$ and ψ has the $\varphi(\bar{x},\bar{y})$–order property (see Definition 1.2(5)) then $\dot{\mathbb{I}}(\lambda,\psi) = 2^\lambda$ provided that for example:*

- $\lambda \geq \chi + \aleph_1, \partial < \aleph_0$ *or* $\lambda = \lambda^\partial + \chi + \partial^{++} + \aleph_1$ *or* $\lambda^{\partial^+} < 2^\lambda, \lambda \geq \chi$.

Proof. When $\lambda \geq \chi + \aleph_1, \partial < \aleph_0$, by Theorem 3.25(3), $\dot{\mathbb{I}}(\lambda,\psi) = 2^\lambda$.

So we can assume that $\lambda \geq \chi$ and $\partial \geq \aleph_0$. When $\lambda^\partial = \lambda$ or $\lambda^{(\partial^+)} < 2^\lambda$ the conclusion holds by 3.28(a), 3.28(b), respectively, using $\kappa = \partial^+$ and the existence of such models follows from 1.18(3), see (b) there, they are as required by 3.11(4). So we are done. $\square_{0.1}$

Note that although some notions connected to stability appear, they are not used in any way which require knowing them: we define what we use and at most quote some results. In fact, the proof covered problems with no (previous) connection to stability. For understanding and/or checking, the reader does not need to know the works quoted below: they only help to see the background. Also the citation to [Shec], [Shea], [Shed] are just to point additional information, and are not needed for understanding.

Generally the strategy here is the construction of many models (up to isomorphism in this paper) in K_λ ($:= \{M \in K : \|M\| = \lambda\}$) goes as follows. We are given a class K of models (with fixed vocabulary), and we are trying to prove that K has many complicated members. To help us, we have a class K^1 of "index models" (this just indicates their role; supposedly they are well understood; they usually are a class of linear orders or a class of trees). By the "non-structure property of K", for some formulas φ_ℓ (see below), for every $I \in K^1_\lambda$ there is $M_I \in K_\lambda$ and $\bar{a}_t \in M_I$ for $t \in I$, which satisfies (in M_I) some instances of $\pm\varphi_\ell$.

We may demand on M_I:

(0) nothing more (except the restriction on the cardinality), or
(1) $\langle \bar{a}_t : t \in I \rangle$ behaves nicely: like a skeleton (see 3.1(1)), or even
(2) M_I is "embedded" in a model built from I in a simple way (Δ–represented; see Definition 2.1(c)), or
(3) M_I is built from I in a simple way, an the extreme case being $\mathrm{EM}_\tau(I,\Phi)$; see Definition 1.8 where $\tau = \tau(M_I)$ of course.

Now even for (0) we can have meaningful theorems (see [Shec, a2] and [Shec, 1.3]); but we cannot have all we would naturally like to have — see [Shec, b17] (i.e., we cannot prove much better results in this direction, as shown by a consistency proof).

Though it looks obvious by our formulation, experience shows that we must stress that the formulas φ_ℓ need not be first order, they just have to have the right vocabulary (but in results on "no M_i embeddable in M_j" this usually means embedding preserving $\pm\varphi_\ell$ (but see 1.30). So they are just properties of sequences in the structures we are considering preserved by the morphism we have in mind.

Another point is that though it would be nice to prove

$$[I \ncong J \quad \Rightarrow \quad M_I \ncong M_J];$$

this does not seem realistic. What we do is to construct a family

$$\{I_\alpha : \alpha < 2^\lambda\} \subseteq K^1_\lambda$$

such that for $\alpha \neq \beta$, in a strong sense I_α is not isomorphic to (or not embeddable into) I_β (see 2.2, 3.11, more in [Shea, a2], [Shea, 1.4]), such that now we have $M_{I_\alpha}, M_{I_\beta}$ not isomorphic for $\alpha \neq \beta$. We are thus led to the task of constructing such I_α's, which, probably unfortunately, splits to cases according to properties of the cardinals involved. Sometimes we just prove $\{\alpha : M_\alpha \cong M_\beta\}$ is small for each β.

A point central to [Shei], [AGS], [Shed],[Shef] and [Sheg] but incidental here, is the construction of a model which is for example rigid or has few endomorphisms, etc. This is done in details in §(2D) for Boolean Algebras (and for many relatives of "rigid" and classes of Boolean Algebras, in [She83] and better and more in [Shed]).

The methods here can be combined with [She87a] or [She84] to get non-isomorphic $\mathbb{L}_{\infty,\lambda}$–equivalent models of cardinality λ; Instead of "$\mathbb{L}_{\infty,\lambda}$-equivalent non-isomorphic model of T" we can consider equivalence by stronger games, e.g. $\mathrm{EF}_{\alpha,\lambda}$-equivalence started in Hyttinen-Tuuri [HT91], and then Hyttinen-Shelah [HS94], [HS95], [HS99]; See Väänänen [Vaa95] on such games and for more (in the 2010-th), see [She08].

In the next few paragraphs we survey the results of this paper. In this survey we omit some parameters for various defined notions. These parameters are essential for an accurate statement of the theorems. We suppress them here trying to make the reading easier while still communicating essential points.

Classically Ehrenfeucht Mostowski model of a theory T, are ones generated by an indiscernible sequence $\langle a_t : t \in I\rangle$ for I a linear order which are models of some $T_1 \supseteq T$ with Skolem functions. In §1 we deal with a generalization, I not necessary a linear order so write $\mathrm{GEM}(I, \Phi)$. This is how in a natural way we construct a model from an "index model". The proof of existence many times rely on partition theorems. We give definitions, deal with the framework, quote important cases, and present general theorems for getting the GEM models, i.e., getting templates; we then, as an example, deal with random graphs for theories in $\mathbb{L}_{\kappa^+,\omega}$.

In §2 we discuss a more general method of so called "representability" (from [She83]). This is a natural way to get for "a model gotten from an index model I" that "I is complicated" implies "M is complicated". We discuss applications (to separable reduced Abelian $\dot{p}$-groups and to Boolean algebras), but the aim is to explain; full proofs of fuller results will appear later (see [Shea, §3], [Shed] respectively). We introduce two strongly contradictory notions, the Δ–representability of a structure M in the "free algebra" (i.e., "polynomial algebra") of an index model (Definition 2.1) and the $\varphi(\bar{x}, \bar{y})$–unembeddability of one index model in another. Now, to show that a class K has many models it suffices if for some formula φ, one first shows that:

(a) an index class K_1 has many pairwise φ–unembeddable structures,

second, that

(b) for each $I \in K_1$, there is a suitable model M_I which is Δ–representable in the free algebra on I, which in some sense reflects the structure I

and finally that

(c) if $M_I \cong M_J$ or just M_I is embeddable into M_J and M_J is Δ–representable in the free algebras on J then I is φ–embeddable in J.

However, for building for example a rigid model of cardinality λ, it is advisable to use $\langle I_\alpha : \alpha < \lambda \rangle$ such that I_α is φ–unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta$. (See the beginning of section 2(D) and 2.15, more in [Shed]). Generally having a suitable sequence of $I \in K_1$ is expressed by "K_1 has a suitable bigness property". Note that from having "many complicated $I \in K^\kappa_{\rm tr}$ (tree with say κ levels)" we can deduce such existence for the class of linear order, see 2.25.

Now, §3 does not depend on §2. The point is that in this section our non-isomorphisms proofs are so strong that they do not need even "representability", we use a much weaker property. In §3 we extend and simplify the argument showing that an unstable first order theory T has 2^λ models of cardinality λ if $\lambda \geq |T| + \aleph_1$. Rather than constructing Ehrenfeucht–Mostowski models we consider a weaker notion — that of a linear order J indexing a weak (κ, φ)-skeleton like sequence in a model M. In this section, K_1 is the class of linear orders. The formula $\varphi(\bar{x}, \bar{y})$ need not be first order and after 3.25 may have infinitely many arguments. Most significantly we make no requirement on the means of definition of the class K of models (for example first order, $\mathbb{L}_{\infty,\infty}$, etc). We require only that for each linear order J there is an $M_J \in K$ and a sequence $\langle \bar{a}_s : s \in J \rangle$ which is (κ, φ)–skeleton like in M_J.

Ehrenfeucht and Mostowski [EM56] built what are here $\mathrm{GEM}_\tau(I, \Phi)$ for I a linear order and first order T where $\tau = \tau_T$. Ehrenfeucht [Ehr57], [Ehr58] (and Hodges in [Hod73] improved the set theoretic assumption) proved that if T has the property (E) then it has at least two non-isomorphic models (this property is a precursor of being unstable).

Recall that the property (E) says that: some formula $R(x_1, \dots x_n)$ is asymmetric on some infinite subset of some model of T; note that (E) is not equivalent to being unstable as the theory of random graphs fails it. Morley [Mor65] proves that for well ordered I, the model generated by I is stable in appropriate cardinalities, to deduce that non-totally transcendental countable theories are not categorical in any $\lambda > \aleph_0$. See more in [She90, VII,VIII]; by it if $T \subseteq T_1$ are unstable, complete first order and $\lambda \geq |T_1| + \aleph_1$ then T_1 has 2^λ models of cardinality λ with pairwise non-isomorphic reducts to τ_T. On the cases for $\mathbb{L}_{\chi^+,\omega}, \lambda > \chi$, see Grossberg-Shelah [GS86b], [GS] which continue [She71a].

This paper is a revised version of sections §1,§2,§3 of chapter III of [She87b]. Meanwhile see recent works of Will Boney and Malliaris-Shelah [MS22].

In the intended book on non-structure, this was suppose to be Ch.III. For later chapters §2 is essential to some of the later parts of non-structure (see [Shec], [Shea] [Shed]) them but not §1 or §3 still but better read 1.1-1.9. This work is continued in [Sheb].

We thank the referee for many suggestions to improve the presentation.

§ 0(A). **Preliminaries.**

Notation 0.2. 1) We use $\mathscr{L}$ to denote a logic, τ to denote a vocabulary (i.e. a set of predicates and function symbols).
2) A language L is a set of sentences (and formulas, e.g. $\mathscr{L}(\tau)$, see 0.3 below).
3) For a model $M, \tau(M) = \tau_M$ is the vocabulary of M.
4) $\mathbb{L}$ is first order logic, $\mathbb{L}_{\lambda,\kappa}$ is like first order logic allowing $\bigwedge\limits_{i<\alpha} \psi_i, (\exists \bar{x}_{[u]})\psi$, where $\bar{x}_{[u]} = \langle x_i : i \in u\rangle, |u| < \kappa, \alpha < \lambda$.

Definition 0.3. 1) A logic $\mathscr{L}$ consists of:

- (a) a class function, giving for every vocabulary τ a language $L = \mathscr{L}(\tau) = \mathscr{L}_\tau$, i.e. a set of sentences and formulas L and naturally defined formula L will denote such language or just a set of formulas (usually with some closure properties)
- (b) a satisfaction relation $\models_{\mathscr{L}}$ such that $M \models_L \psi$ implies M is a model, $\psi \in \mathscr{L}(\tau_M)$
- (c) if M_1, M_2 are isomorphic τ-models and $\psi \in \mathscr{L}(\tau)$ then $M_1 \models \psi \Leftrightarrow M_2 \models \psi$.

§ 1. Models from Indiscernibles

§ 1(A). **Background.**

We survey here [She90, Ch.VIII,§3] (already in [She78]), which was the starting point for the other works appearing or surveyed in this paper and [Shec], [Shel]. So we concentrate on building many models for first order theories, using G.E.M. models, i.e., in all respects taking the easy pass. Our aim there was

Theorem 1.1. *If T is a complete first order theory, unstable and $\lambda \geq |T| + \aleph_1$, then $\dot{\mathbb{I}}(\lambda, T) = 2^\lambda$.*

(This is reproved here in 3.25) where

Definition 1.2. A first order theory T is unstable when for some first order formula $\varphi(\bar{x}, \bar{y})$ $(n = \ell g(\bar{x}) = \ell g(\bar{y}))$ in the vocabulary τ_T of T of course, for every λ there is a model M of T and $\bar{a}_i \in {}^n M$ for $i < \lambda$ such that

$$M \models \varphi[\bar{a}_i, \bar{a}_j] \text{ if } i < j (< \lambda).$$

Definition 1.3. For a theory T and vocabulary $\tau \subseteq \tau_T$, let
$\dot{\mathbb{I}}(\lambda, T) =$ the number of models of T of cardinality λ, up to isomorphism,
$\dot{\mathbb{I}}_\tau(\lambda, T) =$ the number of τ-reducts of models of T of cardinality λ, up to isomorphism.

Definition 1.4. 1) For a class K of models and a set Δ of formulas:

$\dot{\mathbb{I}}(\lambda, K) =$ the number of models in K of cardinality λ up to isomorphism,
$\dot{\mathbb{I}}(K) =$ the number of models in K up to isomorphism,
$\dot{I}\dot{E}_\Delta(\lambda, K) = \sup\{\mu$: there are $M_i \in K_\lambda$, for $i < \mu$, such that for $i \neq j$ there is no Δ-embedding of M_i to $M_j\}$.

see part (2); and we may write τ instead $\Delta = \mathbb{L}(\tau_K)$, may omit τ and Δ when it is $\mathbb{L}(\tau_M)$.
2) $f : M \longrightarrow N$ is a Δ-embedding (of M into N) when (f is a function from $|M|$ into $|N|$ and) for every $\varphi(\bar{x}) \in \Delta$ and $\bar{a} \in {}^{\ell g(\bar{a})}|M|$, we have:

$$M \models \varphi[\bar{a}] \Rightarrow N \models \varphi[f(\bar{a})].$$

(so if $(x \neq y) \in \Delta$ then f is one to one).

Definition 1.5. 1) A sentence $\psi \in \mathbb{L}_{\chi^+,\aleph_0}$ is ∂-unstable when there are $\alpha < \partial$ and a formula $\varphi(\bar{x}, \bar{y})$ from $\mathbb{L}_{\chi^+,\aleph_0}$ with $\ell g(\bar{x}) = \ell g(\bar{y}) = \alpha$ such that ψ has the φ-order property, i.e., for every λ there is a model M_λ of ψ and a sequence $\bar{a}_\zeta$ of length α from (i.e. of elements of) M_λ such that for $\zeta, \xi < \lambda$ we have:

$$M_\lambda \models \varphi[\bar{a}_\zeta, \bar{a}_\xi] \Leftrightarrow \zeta < \xi.$$

If $\partial = \aleph_0$ we may omit it.
2) For κ regular and T first order, we say $\kappa < \kappa(T)$ when there are first order formulas $\varphi_i(\bar{x}, \bar{y}_i) \in \mathbb{L}(\tau_T)$ for $i < \kappa$ and for every λ there is a model M_λ of T and for $i \leq \kappa, \eta \in {}^i\lambda$ a sequence $\bar{a}_\eta$ from M_λ, with

$$i < \kappa \Rightarrow \ell g(\bar{a}_\eta) = \ell g(\bar{y}_i)$$

$$i = \kappa \Rightarrow \ell g(\bar{a}_\eta) = \ell g(\bar{x})$$

such that: if $\nu \in {}^i\lambda,\ \eta \in {}^\kappa\lambda,\ \nu \triangleleft \eta$ then $M_\lambda \models \varphi_{i+1}[\bar{a}_\eta, \bar{a}_{\nu\hat{}\langle\alpha\rangle}] \Leftrightarrow \eta(i) = \alpha$. [We shall not use this except in 1.11 below.]
3) T, a first order theory, is unsuperstable if $\aleph_0 < \kappa(T)$ [but we shall use it only in 1.11].

§ 1(B). GEM **models.**

Definition 1.6. 1) $\langle \bar{a}_t : t \in I\rangle$ is Δ-indiscernible (in M) when

- (a) I is an index model (usually linear order or tree), i.e., it can be any model but its role will be as an index set,
- (b) Δ is a set of formulas in the vocabulary of M (i.e. in $\mathscr{L}_{\tau(M)}$ for some logic $\mathscr{L}$)
- (c) the Δ-type in M of $\bar{a}_{t_0}\hat{}\ldots\hat{}\bar{a}_{t_{n-1}}$ for any $n < \omega$ and $t_0, \ldots t_{n-1} \in I$ depends only on the quantifier free type of $\langle t_0, \ldots, t_{n-1}\rangle$ in I.
  - Recall that the Δ-type of $\bar{a}$ in M is $\{\varphi(\bar{x}) \in \Delta : M \models \varphi(\bar{a})\}$, where $\bar{a}, \bar{x}$ are indexed by the same set. So the length of $\bar{a}_t$ depend just on the quantifier free type which $\ell g(\bar{a}_t)$ realizes in I.
  - When Δ is closed under negations for any $\varphi(\bar{x})$ we have $\varphi(\bar{x})$ belongs or $\neg\varphi(\bar{x})$.
  - If we allow $\varphi(\bar{x}) \in \Delta, \kappa > \alpha = \ell g(\bar{x}) \geq \omega$ and we allow $\langle t_i : i < \alpha\rangle$ above, then we say (Δ, κ)-indiscernible.

2) For a logic $\mathscr{L}$, "$\mathscr{L}$-indiscernible" will mean Δ-indiscernible for $\mathscr{L}_{\tau(M)}$, the set of $\mathscr{L}$-formulas in the vocabulary of M. If $\Delta, \mathscr{L}$ are not mentioned we mean first order logic.
3) Notation: Remember that if $\bar{t} = \langle t_i : i < \alpha\rangle$ then $\bar{a}_{\bar{t}} = \bar{a}_{t_0}\hat{}\bar{a}_{t_1}\hat{}\ldots$.

Many of the following definitions are appropriate for counting the number of models in a pseudo elementary class. Thus, we work with a pair of vocabularies, $\tau \subseteq \tau_1$. Often τ_1 will contain Skolem functions for a theory T which is $\subseteq \mathscr{L}(\tau)$.

Convention 1.7. For the rest of this section all predicates and function symbols have finite number of places (and similarly $\varphi(\bar{x})$ means $\ell g(\bar{x}) < \omega$).

Definition 1.8. 1) $M = \mathrm{GEM}(I, \Phi)$ when for some vocabulary $\tau = \tau_\Phi = \tau(\Phi)$ (called L_1^Φ in [She90, Ch.VII]) and sequences $\bar{a}_t (t \in I)$ we have:

- (i) M is a τ_Φ-structure and is generated by $\{\bar{a}_t : t \in I\}$,
- (ii) $\langle \bar{a}_t : t \in I\rangle$ is quantifier free indiscernible in M,
- (iii) Φ is a function, taking (for $n < \omega$) the quantifier free type of $\bar{t} = \langle t_0, \ldots, t_{n-1}\rangle$ in I to the quantifier free type of $\bar{a}_{\bar{t}} = \bar{a}_{t_0}\hat{}\ldots\hat{}\bar{a}_{t_{n-1}}$ in M (so Φ determines τ_Φ uniquely).

1A) Pedantically we should say $\mathbf{\bar a} = \langle \bar a_t : t \in I\rangle$ is a witness for $M = \mathrm{GEM}(I,\Phi)$ or $(M,\mathbf{\bar a})$ is a $\mathrm{GEM}(I,\Phi)$ pair when the above holds, but abusing notation we usually write M instead of $(M,\mathbf{\bar a})$.
1B) If $\tau \subseteq \tau_\Phi$ let $\mathrm{GEM}_\tau(I,\Phi)$ be the τ-reduct of $\mathrm{GEM}(I,\emptyset)$.
2) A function Φ as above is called a template and we say it is proper for I <u>when</u> there is M such that $M = \mathrm{GEM}(I,\Phi)$. We say Φ is proper for K if Φ is proper for every $I \in K$, and lastly Φ is proper for (K_1,K_2) if it is proper for $K_1, \tau(K_2) \subseteq \tau_\Phi$ and $\mathrm{GEM}_{\tau(K_2)}(I,\Phi) \in K_2$ for $I \in K_1$.
3) For a logic $\mathscr{L}$, or even a set $\mathscr{L}$ of formulas in the vocabulary of M, we say that Φ is almost $\mathscr{L}$-nice (for K) <u>when</u> it is proper for K and:

(∗) for every $I \in K, \langle \bar a_t : t \in I\rangle$ is $\mathscr{L}$-indiscernible in $\mathrm{GEM}(I,\Phi)$.

4) In part (3), Φ is $\mathscr{L}$-nice <u>when</u> it is almost $\mathscr{L}$-nice and

(∗∗) for $J \subseteq I$ from K we have $\mathrm{GEM}(J,\Phi) \prec_{\mathscr{L}} \mathrm{GEM}(I,\Phi)$.

5) In part (3) we say that Φ is $(\mathscr{L},\tau)$-nice <u>when</u> $\tau \subseteq \tau_\Phi$, it is almost $\mathscr{L}$-nice and (see 1.8(1B)).

(∗∗∗) for $I \subseteq J$ from K we have $\mathrm{GEM}_\tau(J,\Phi) \prec_{\mathscr{L}} \mathrm{GEM}_\tau(I,\Phi)$.

In the book [She90], always $\mathbb{L}(\tau_\Phi)$-nice Φ were used and $\mathrm{GEM}(I,\Phi), \mathrm{GEM}_\tau(I,\Phi)$ here are $\mathrm{EM}^1(I,\Phi), \mathrm{EM}(I,\Phi)$ there.

Definition 1.9. 1) Saying "a GEM-model" will mean "a model of the form $\mathrm{GEM}_\tau(I,\Phi)$" where Φ, I, τ are understood from the context.
2) We identify $I \subseteq {}^{\kappa\geq}\lambda$ which is closed under initial segments, with the model $(I, P_\alpha, \cap, <_{\mathrm{lx}}, \triangleleft)_{\alpha\leq\kappa}$, where:
   $P_\alpha = I \cap {}^\alpha\lambda$,
   $\rho = \eta \cap \nu$ if $\rho = \eta{\restriction}\alpha$ for the maximal α such that $\eta{\restriction}\alpha = \nu{\restriction}\alpha$,
   $\triangleleft =$ being initial segment of (including equality),
   $<_{\mathrm{lx}} =$ the lexicographic order.
3) Similarly to (2), for any linear order J, every $I \subseteq {}^{\kappa\geq}J$ which is closed under initial segments is identified with $(I, P_\alpha, \cap, <_{\mathrm{lx}}, \triangleleft)_{\alpha\leq\kappa}$ ($\leq_{\mathrm{lx}}$ is still well defined).
4) K^κ_{tr} is the class of such models, i.e., models isomorphic to such I, i.e., to $(I, P_\alpha, \cap, <_{\mathrm{lx}}, \triangleleft)_{\alpha\leq\kappa}$ for some $I \subseteq {}^{\kappa\geq}J$ which is closed under initial segments, J a linear order (tr stands for tree). We call I standard if J is an ordinal or at least well ordered.
5) K_{or} is the class of linear orders.

Remark 1.10. The main case here is $\kappa = \aleph_0$. We need such trees for $\kappa > \aleph_0$, for example if we would like to build many κ-saturated models of T, $\kappa(T) > \kappa$, κ regular. If $\kappa(T) \leq \kappa$ there may be few κ–saturated models of T.

In [She90, Ch.VIII] we have also proved:

Lemma 1.11. *1) If $T \subseteq T_1$ are complete first order theories, T is unstable as exemplified by $\varphi = \varphi(\bar x, \bar y)$, say $n = \ell g(\bar x) = \ell g(\bar y)$, <u>then</u> for some template Φ proper for the class of linear orders and nice for first order logic, $|\tau_\Phi| = |T_1| + \aleph_0$ and for any linear order I and $s,t \in I$ we have*

$$\mathrm{EM}(I,\Phi) \models \varphi[\bar a_s, \bar a_t] \text{ iff } I \models s < t.$$

2) If $T \subseteq T_1$ are complete first order theories and T is unsuperstable, <u>then</u> there are first order $\varphi_n(\bar{x}, \bar{y}_n) \in \mathbb{L}(\tau_T)$ and a template Φ proper for every $I \subseteq {}^{\omega\geq}\lambda$ such that for any such I we have:

- (a) *$\eta \in {}^{\omega}\lambda, \nu \in {}^{n}\lambda$ implies $\mathrm{EM}(I, \Phi) \models \varphi_n[\bar{a}_\eta, \bar{a}_\nu]$ <u>iff</u> $\eta \restriction n = \nu$*
- (b) *$\mathrm{EM}(I, \Phi) \models T_1$ and Φ is $\mathbb{L}(\tau_\Phi)$-nice, $|\tau_\Phi| = |T_1| + \aleph_0$ (note that for $\eta_1, \eta_2 \in I$ we have $\eta_1 \neq \eta_2 \Rightarrow \bar{a}_{\eta_1} \neq \bar{a}_{\eta_2}$)*[1].

3) If $T \subseteq T_1$ are complete first order theories and $\kappa = \mathrm{cf}(\kappa) < \kappa(T)$ <u>then</u>

- (a) *there is a sequence of first order formulas $\varphi_i(\bar{x}, \bar{y}_i)$ (for $i < \kappa$) witnessing $\kappa < \kappa(T)$ i.e. there are a model M of T and sequences $\bar{a}_\eta$ for $\eta \in {}^{\kappa\leq}\lambda$ such that for $\eta \in {}^{\kappa}\lambda, \nu \in {}^{i}\lambda, i < \kappa, \alpha < \lambda$ we have $M \models \varphi_i[\bar{a}_\eta, \bar{a}_{\nu\hat{}\langle\alpha\rangle}]$ <u>iff</u> $\alpha = \eta(i)$,*
- (b) *for any $\langle \varphi_i(\bar{x}, \bar{y}) : i < \kappa \rangle$ as in (a) there is a nice template Φ proper for K^κ_{tr} such that for any λ:*
  - (α) *if $\eta \in {}^{\kappa}\lambda$, $\nu \in {}^{i}\lambda, i < \kappa, \alpha < \lambda$ then*
  $$\mathrm{GEM}({}^{\kappa\geq}\lambda, \Phi) \models \varphi_i[\bar{a}_\eta, \bar{a}_{\nu\hat{}\langle\alpha\rangle}] \text{ \underline{iff} } \alpha = \eta(i);$$
  - (β) $\mathrm{GEM}(I, \Phi) \models T_1$,
  - (γ) *Φ is $\mathbb{L}(\tau_\Phi)$-nice,*
  - (δ) $|\tau_\Phi| = |T_1| + \aleph_0$.

Proof. See [She90, Ch.VII,§3], but here we can consider the conclusion as the definition of unstable or unsuperstable and of $\kappa < \kappa(T)$, respectively (note that there we use a partition theorem and possibly replace $\varphi_i(\bar{x}, \bar{y}_i)$ by $\varphi_i(x, \bar{y}'_i)\hat{}\neg\varphi_i(x, \bar{y}''_i)$. $\square_{1.11}$

Remark 1.12. On K^ω_{tr} for $\mathbb{L}_{\lambda^+, \aleph_0}$ we need the Ramsey property defined below, see 1.19 (and 1.20+ 1.21).

In [She90, Ch.VIII,§2] we actually proved:

Theorem 1.13. *1) If $\lambda > |\tau_\Phi|$, and $\Phi, \tau_\Phi, \langle \varphi_n : n < \omega \rangle$ are as in Lemma 1.11(2) (and Φ is almost $\mathbb{L}$-nice) <u>then</u>: we can find $I_\alpha \subseteq {}^{\omega\geq}\lambda$ (for $\alpha < 2^\lambda$), $|I_\alpha| = \lambda$ such that for $\alpha \neq \beta$ there is no one-to-one function from $\mathrm{GEM}(I_\alpha, \Phi)$ onto $\mathrm{GEM}(I_\beta, \Phi)$ preserving the $\pm\varphi_n$ for $n < \omega$.*
2) If λ is regular, also for $\alpha \neq \beta$ there is no one-to-one function from $\mathrm{GEM}(I_\alpha, \Phi)$ into $\mathrm{GEM}(I_\beta, \Phi)$ preserving the $\pm\varphi_n$ for $n < \omega$.
3) The φ_n's do not need to be first order, just their vocabularies should be $\subseteq \tau_\Phi$. But instead of "Φ is almost $\mathbb{L}(\tau_\Phi)$-nice" we need just "Φ is almost $\{\pm\varphi_n(\ldots, \sigma_\ell(\bar{x}_\ell), \ldots)_{\ell < \ell(n)} : n < \omega, \sigma_\ell$ terms of $\tau_\Phi\}$-nice" and we should still demand (as in all this section)

- $(*)$ *the $\bar{a}_\eta$ are finite (and we are assuming that the functions are finitary).*

4) So if as in Lemma 1.11, $\varphi_n \in \mathscr{L}(\tau)$ <u>then</u> $\{M_\alpha \restriction \tau : \alpha < 2^\lambda\}$ are 2^λ non-isomorphic models of T of cardinality λ.

[1] In fact $\mathrm{EM}(I, \Phi)$ is well defined for $I \in K^\omega_{\mathrm{tr}}$.

Proof. Part (2) is proved in 2.7. Fully this was proved in [She90, §2 of Ch.VIII] (though it is not explicitly claimed, it was used elsewhere and there is no need to change the proofs). Also we shall later (in [Shea, p2]) prove better theorems, mainly getting 1.13(2) also for singular λ. $\square_{1.13}$

Remark 1.14. 1) Applying 1.13, we usually look at the τ-reducts of the models $\mathrm{GEM}(I,\Phi)$ as the objects we are interested in, where the φ_n's are in the vocabulary τ. E.g., for $T \subseteq T_1$ first order, T unsuperstable, we use $\varphi_n \in \mathbb{L}(\tau_T)$.
2) The case $\lambda = |\tau_\Phi|$ is harder. In [She90, Ch.VIII,§2,§3], the existence of many models in λ is proved for T unstable, $\lambda = |\tau_\Phi| + \aleph_1$ and there (in some cases) "T_1, T first order" is used.

§ 1(C). **Finding Templates.**

How do we find templates Φ as required in 1.11 and parallel situations?

Quite often in model theory, partition theorems (from finite or infinite combinatorics) together with a compactness argument (or a substitute) are used to build models. Here we phrase this generally. Note that the size of the vocabulary (μ in the "(μ,λ)-large")) is a variant of the number of colours, whereas λ is usually μ; it becomes larger if our logic is complicated.

Definition 1.15. Fix a class K (of index models) and a logic (or logic fragment) $\mathscr{L}$.
1) An index model $I \in K$ is called (μ,λ,χ)-Ramsey for the logic $\mathscr{L}$ <u>when</u>:

- (a) the cardinality of I is $\le \chi$ and every qf (= quantifier free) type p (in $\tau(K)$) which is realized in some $J \in K$ is realized in I,
- (b) for every vocabulary τ_1 of cardinality $\le \mu$, a τ_1-model M_1 and an indexed set $\langle \bar b_t : t \in I\rangle$ of finite sequences from $|M_1|$ with $\ell g(\bar b_t)$ determined by the quantifier free type which t realizes in I <u>there is</u> a template Φ, which is proper for K, with $|\tau_\Phi| \le \lambda$ such that ($\tau_1 \subseteq \tau_\Phi$ and):
  - $(*)$ for any $\tau(K)$-quantifier free type $p, I_1 \in K$ and $s_0,\dots,s_{n-1} \in I_1$ for which $\langle s_0,\dots,s_{n-1}\rangle$ realizes p in I_1 and for any formula
  $$\varphi = \varphi(x_0,\dots,x_{m-1}) \in \mathscr{L}(\tau_1)$$
  and τ_1-terms $\sigma_\ell(\bar y_0,\dots,\bar y_{n-1})$ for $\ell = 0,\dots,m-1$ we have
  - $(**)$ if for every $t_0,\dots,t_{n-1} \in I$ such that $\langle t_0,\dots,t_{n-1}\rangle$ realizes p in I we have $M_1 \models \varphi[\sigma_0(\bar b_{t_0},\dots,\bar b_{t_{n-1}}),\sigma_1(\bar b_{t_0},\dots,\bar b_{t_{n-1}}),\dots,\sigma_{m-1}(\bar b_{t_0},\dots,\bar b_{t_{n-1}})]$ <u>then</u> $\mathrm{GEM}(I_1,\Phi) \models \varphi[\sigma_0(\bar a_{s_0},\dots,\bar a_{s_{n-1}}),\sigma_1(\bar a_{s_0},\dots,\bar a_{s_{n-1}}),\dots,\sigma_{m-1}(\bar a_{s_0},\dots,\bar a_{s_{n-1}})]$.

2) The class K of index models is called explicitly (μ,λ,χ)-Ramsey for the logic $\mathscr{L}$ <u>iff</u> some $I \in K$ of cardinality $\le \chi$ is (μ,λ)-Ramsey for $\mathscr{L}$. A class $K' \subseteq K$ of index models is called (μ,λ,i,χ)-Ramsey (inside K, which is usually understood from context), <u>when</u>:

- (a) every member of K' has cardinality $\le \chi$ and every quantifier free type p in $\tau(K')$ realized in some $J \in K$ is realized in some $I \in K'$,

(b) for every vocabulary τ_1 of cardinality $\le \mu$ and τ_1-models M_I for $I \in K'$, and $\bar{b}_{I,t} \in {}^{k(I,t)}(M_I)$, where $k(I,t) < \omega$ depends just on $\mathrm{tp}_{\mathrm{qf}}(\langle t\rangle, \emptyset, I)$ there is a template Φ proper for K with $|\tau_\Phi| \le \lambda$ such that $\tau^1 \subseteq \tau_\Phi$ we have $(*)$ only in $(**)$ we should also say "every $I \in K'$". Let "(μ,χ)-Ramsey" mean "(μ,μ,χ)-Ramsey". Let "μ-Ramsey" mean "(μ,χ)-Ramsey for some χ".

3) In all parts of 1.15, 1.16, 1.17, if $\mathscr{L}$ is first order logic, we may omit it. If $\chi = \mu$ we may omit χ. Similarly in other parts of 1.16, 1.17.
4) For $f : \mathrm{Card} \longrightarrow \mathrm{Card}$, we say K is f-Ramsey for $\mathscr{L}$ when it is $(\mu, f(\mu))$-Ramsey for $\mathscr{L}$ for every (infinite) cardinal μ. We say K is Ramsey for $\mathscr{L}$ if it is (μ,μ)-Ramsey for $\mathscr{L}$ for every μ.
5) We say K is $*$-Ramsey for $\mathscr{L}$ if it is f-Ramsey for $\mathscr{L}$ for some $f : \mathrm{Card} \longrightarrow \mathrm{Card}$.

Definition 1.16. Let K be a class of (index) models and $\mathscr{L}$ a logic.
1) We say $I \in K$ is (almost) $\mathscr{L}$-nicely (μ,λ,χ)-Ramsey for K when 1.15(1) holds, but in addition Φ is (almost) $\mathscr{L}$-nice. Similarly replacing I by a set $K' \subseteq K$.
2) The class K is called explicitly (almost) $\mathscr{L}$-nicely (μ,λ,χ)-Ramsey when some $I \in K$ is (almost) $\mathscr{L}$-nicely (μ,λ,χ)-Ramsey.
3) For $f : \mathrm{Card} \longrightarrow \mathrm{Card}$, we say K is (almost) $\mathscr{L}$-nicely f-Ramsey when for every μ we have: K is (almost) $\mathscr{L}$-nicely $(\mu, f(\mu))$-Ramsey for every (infinite) cardinal μ. We omit f for the identity function.
4) We say K is (almost) $\mathscr{L}$-nicely $*$-Ramsey when for some f, it is (almost) $\mathscr{L}$-nicely f-Ramsey.

Definition 1.17. In 1.15, 1.16 we add "strongly" when we strengthen 1.15(1) by asking in $(*)$ in addition that for any $\tau(K)$-quantifier free type p and $s_0,\dots,s_{n-1} \in I_1$ such that $\langle s_0,\dots,s_{n-1}\rangle$ realizes p in I_1 we can find some $t_0,\dots,t_{n-1}$ suitable for all $\varphi,\sigma_0,\dots$ simultaneously (this helps for omitting types).

Theorem 1.18. *1) For $\mathbb{L}$, the class of linear orders is nicely Ramsey, moreover every infinite order is (μ,λ)-Ramsey for any $\mu \le \lambda$.*
2) For $\mathbb{L}_{\aleph_1,\aleph_0}$ the class of linear orders is nicely $*$-Ramsey. In fact nicely f-Ramsey for the functions $f(\mu) = \beth_{(2^\mu)^+}$.
3) For any fragment of $\mathbb{L}_{\lambda^+,\aleph_0}$ or of $\Delta(\mathbb{L}_{\lambda^+,\aleph_0})$ (see, e.g. [Mak85]*) of cardinality λ, the class of linear orders is nicely f-Ramsey when $f(\mu) = \beth_{(2^\mu)^+}$, even strongly; moreover is strongly nicely f-Ramsey.*
4) K^ω_{tr} (and even K^κ_{tr}) is Ramsey for $\mathbb{L}$. For the definition of K^ω_{tr} see 1.9 above.
5) The class K_{org} of linear ordered graphs is explicitly nicely Ramsey. The class $K_{\mathrm{or},n}$ of linearly orders expanded by an n-place relation is explicitly nicely Ramsey.

Proof. 1) This is the content of the Ehrenfeucht-Mostowski proof that E.M. models exist and it use the finitary Ramsey theorem as used in the proof of 1.11(1). see [She90, Ch.VII].
2) By repeating the proof of Morley's omitting type theorem which use the Erdös-Rado theorem, see [She90, Ch.VII,§5]; the generalization to uncountably vocabulary (and many types) was noted by C.C.Chang.Overfull
3) Like 1.18(2); see [She72, Theorem 2.5], and more in [GS86b], [GS].
4) By [She90, Ch.VII,§3] (we use the compactness of $\mathbb{L}$ and partition properties of trees).
5) By the Nessetril-Rodl theorem (see e.g. [GRS90]). $\square_{1.18}$

By Grossberg-Shelah [GS86a] (improving [She90, Ch.VII], where compactness of the logic $\mathscr{L}$ was used, but no large cardinals):

Theorem 1.19. *$K^\omega_{\rm tr}$ is the nicely $*$-Ramsey for $\mathbb{L}_{\lambda^+,\aleph_0}$ if for example there are arbitrarily large measurable cardinals (in fact, large enough cardinals consistent with the axiom $\mathbf{V} = \mathbf{L}$ suffice).*

We shall not repeat the proof.

Lemma 1.20. *Suppose K_1, K_2, K_3 are classes of models, Φ is a proper template for $(K_1, K_2), \Psi$ proper template for (K_2, K_3) then there is a unique template Θ that is proper for (K_1, K_3) and for $I \in K_1$*

$$\mathrm{GEM}(I, \Theta) = \mathrm{GEM}(\mathrm{GEM}_{\tau(K_2)}(I, \Phi), \Psi)).$$

In this case we may write Θ as $\Psi \circ \Phi$.

Proof. Straightforward. $\square_{1.20}$

Lemma 1.21. *Suppose K is a class of index models, $\tau = \tau(K)$ and*

- $(*)$ *there is a template Ψ proper for K such that $\tau_K \subseteq \tau_\Psi, |\tau_\Psi| = |\tau_K| + \aleph_0$ and for $I \in K$: if $\mathrm{GEM}_{\tau(K)}(I, \Psi) \in K$ and $J := \mathrm{GEM}_{\tau(K)}(I, \Psi)$ is strongly $(\aleph_0, \mathrm{qf})$-homogeneous over I, i.e., if $\bar t = \langle t_1, \dots, t_n\rangle, \bar s = \langle s_1, \dots, s_n\rangle$ realize the same quantifier free type in I, then some automorphism of J takes $\bar a_{\bar t}$ to $\bar a_{\bar s}$.*

We conclude that: if K is (μ, λ, χ)-Ramsey for $\mathscr{L}$ and $|\tau_\Psi| \le \mu$ then K is almost $\mathscr{L}$-nicely (μ, λ, χ)-Ramsey for $\mathscr{L}$.

Proof. Just chase the definitions. $\square_{1.21}$

Remark 1.22. 1) E.g. for $\mathscr{L} \subseteq \mathbb{L}_{\aleph_1,\aleph_0}$ we get in 1.21 even $\mathscr{L}$-nice.
2) The assumption $(*)$ of 1.21(1) holds for $K_{\rm or}, K^\omega_{\rm tr}, K^\kappa_{\rm tr}$ (as well as the other K's from [Shea]).

Conclusion 1.23. *The parallel of 0.1 for $K^\omega_{\rm tr}$ instead $K_{\rm or}$ holds if $\lambda > \mu$.*

Proof. By 1.13 (or use [Shea]). $\square_{1.23}$

§ 1(D). **How Forcing Helps.**

We return to the general Ramsey properties for other classes (not just linear orders and trees). For compact logics, finitary generalization of Ramsey theorem suffices. More generally, certainly it is nice to have them for $\mathscr{L} = \mathbb{L}_{\lambda^+,\aleph_0}$, and even $\Delta(\mathbb{L}_{\lambda^+,\aleph_0})$, so we need a partition theorem generalizing Erdös-Rado theorem, i.e., the case with infinitely many colours. We may for example look at ordered graphs as index models, a quite natural one. It consistently holds ([She89]) though unfortunately it does not necessarily hold (Hajnal-Komjath [HK97]). However, our main point is that this is enough when the consistency is by forcing with e.g. complete enough forcing notion. So the consistency result in [She89] yields a "real", ZFC theorem here. The following is an abstract version of the omitting type theorem for getting models for arbitrarily large cardinality.

Claim 1.24. *Assume that*

(a) *K is a definition of a class of models with vocabulary τ (the "index models"); where τ and the parameters in the definition belongs to $\mathscr{H}(\chi^+)$,*

(b) *$\mathscr{L}$ is a definition of a logic or logic fragment, the parameters of the definition belong to $\mathscr{H}(\chi^+)$ and $\lambda \geq \chi$,*

(c) *in the definition of "Φ is (almost) $\mathscr{L}$-nice" for Φ proper for K with $|\tau_\Phi| < \chi$ (see 1.8(3), (4); so without loss of generality $\Phi \in \mathscr{H}(\chi)$) it suffices to restrict ourselves to I of cardinality $< \chi$,*

(d) *$\mathbb{P}$ is a forcing notion not adding subsets to λ, and preserving clauses (a), (b) and (c) (i.e., the definitions of K and $\mathscr{L}$ have these properties) and no new quantifier free complete n-types are realized in $I \in K$,*

(e) *in $\mathbf{V}^{\mathbb{P}}$, there is a member I^* of K, which is (χ, λ)-Ramsey for $\mathscr{L}$ (or an almost $\mathscr{L}$-nicely (χ, λ)-large) [or an $\mathscr{L}$-nicely (χ, λ)-Ramsey] or such a subset K' of K. For $I \in K$ let $\mathbf{P}^n_I = \{p : p$ is complete quantifier-free τ_K-type realized by some $\bar{t} \in {}^n I\}$. Let $\mathbf{P}_n$ be $\mathbf{P}^n_{I^*}$ or $\cup\{\mathbf{P}^n_I : I \in K\}$ according to the case above; if $q \in \mathbf{P}^n_I$ as exemplified by $\bar{t} \in {}^n I$ let $\mathrm{proj}_\ell(q)$ be the quantifier-free type which t_ℓ realizes in I*

(f) *$\tau_0 \in \mathscr{H}(\chi^+)$ is a vocabulary, $q_* \in \mathbf{P}_1$ and $\langle \Omega_q : q \in \mathbf{P}_n$ for some $n < \omega\rangle$ are such that for every $q \in \mathbf{P}_n$ we have: $\Omega_q \subseteq \{p(\bar{x}_0, \ldots, \bar{x}_{n-1}) : p$ an $\mathscr{L}(\tau_0)$-type in the variables $\bar{x}_0, \ldots, \bar{x}_{n-1}$ where $\bar{x}^\ell = \langle x_{\ell,i} : i < \alpha_{\mathrm{proj}_\ell(q)}\rangle \in \mathscr{H}(\chi^+)$ for some $n < \omega\}$, and in $\mathbf{V}^{\mathbb{P}}$, for every $I \in K$ (in the $\mathbf{V}^{\mathbb{P}}$'s sense) or just $I = I^*$ [or just $I \in K'$, according to the case in clause (e)], there is a τ_0-model M_I and $\bar{b}^I_t \in {}^{\alpha_t}(M_I)$ for $t \in I$ such that:*

  (α) *$\alpha_t = \alpha_q$ if q is the quantifier free-τ_0-1-type which t realizes in I,*

  (β) *for no $t_0, \ldots, t_{n-1} \in I$, does $\langle t_0, \ldots, t_{n-1}\rangle$ realize in I the complete quantifier free $\tau_\kappa - n$-type q and $p = p(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \Omega_q$, does $\bar{b}^I_{t_0}{}^\frown \bar{b}^I_{t_1}{}^\frown \ldots {}^\frown \bar{b}^I_{t_{n-1}}$ realizes p and $\alpha_{t_\ell} = \ell g(\bar{x}_\ell)$.*

Then we can conclude that there is a Φ such that:

(ℵ) *Φ is an (almost) $\mathscr{L}$-nice template Φ, proper for K,*

(ℶ) *$\Phi \in \mathscr{H}(\lambda^+)$ hence also $\tau_\Phi \in \mathscr{H}(\lambda^+)$*

(ℷ) *if $M = \mathrm{GEM}(I, \Phi)$, and $t_0, \ldots, t_{n-1} \in I$, and $\bar{t} = \langle t_0, \ldots, t_{n-1}\rangle$ realizes the complete quantifier free $\tau_\kappa - n$-type q then $\bar{a}_{\bar{t}}$ does not realize in M any $p \in \Omega_q$.*

Proof. Straightforward. $\square_{1.24}$

Claim 1.25. *Assume that*

(a) *K is a class of (index) models,*

(b) *κ is a cardinal, for $\alpha < (2^\kappa)^+$ the structure $I_\alpha \in K$ realizes all quantifier free τ_K-types (in $< \omega$ variables) realized in some $I \in K$, and their number is $\leq \kappa$,*

(c) *if $n < \omega, \alpha < \beta < (2^\kappa)^+, N$ is a model, $\tau(N) \leq \kappa, \alpha^*_r < \kappa^+$ for a complete quantifier free $\tau_K - 1$-type r realized in $I_\beta, \bar{b}_r \in {}^{\alpha^*_r} N$, then we can find $I'_\alpha \subseteq I_\beta$ isomorphic to I_α such that*

(∗) *if* $\bar t, \bar s \in {}^m(I'_\alpha), m \le n$ *and they realize the same quantifier free type in* I'_α *then* $\bar b_{\bar t} = \langle \bar b_{t_\ell} : \ell < m\rangle$ *and* $\bar b_{\bar s} = \langle \bar b_{\bar s_\ell} : \ell < m\rangle$ *realizes the same quantifiers free type in* N*,*

(d) τ *is a vocabulary,* $|\tau| \le \kappa, \psi \in \mathbb{L}_{\kappa^+,\aleph_0}(\tau)$ *and* $\alpha^*_p < \kappa^+$ *for* p *a complete quantifier free* $\tau_K - 1$*-type realized in every* $I_\alpha, \mathscr{L} \subseteq \mathbb{L}_{\kappa^+,\aleph_0}(\tau)$ *is a fragment of cardinality* κ *to which* ψ *belongs,*

(e) *for every* $\alpha < (2^\kappa)^+$*, there is a model* N_α *of* ψ *with* $\bar b^\alpha_t \in {}^{\alpha^*_t}(N_\alpha)$ *for* $t \in I_\alpha$*, where* $\alpha^*_t = \alpha^*_{\mathrm{tp}_{\mathrm{qf}}(t,\emptyset,I_\alpha)}$.

<u>Then</u> *there is a* $\mathscr{L}$*-nice template* Φ*, such that:*

⊗ *for* $I \in K, m < \omega$ *and* $\bar t \in {}^m I$ *we have: the* $\mathscr{L}$*-type which is* $\bar a_{\bar t}$*-realized in* $\mathrm{GEM}(I,\Phi)$ *is realized in some* N_α *by some* $\bar b_{\bar s}$*, where* $\mathrm{tp}_{\mathrm{qf}}(\bar s,\emptyset,I_\alpha) = \mathrm{tp}_{\mathrm{qf}}(\bar t,\emptyset,I)$*.*

In other words, $\{I_\alpha : \alpha < (2^\kappa)^+\}$ *is* κ*-Ramsey for* $\mathscr{L}$*.*

Proof. We can expand N_α by giving names to all formulas in $\mathscr{L}$ and adding Skolem functions (to all first order formulas in the new vocabulary), so we have a τ^+-model $N^+_\alpha, \tau^+ \supseteq \tau = \tau(\psi), |\tau^+| \le \kappa$, correspondingly we extend $\mathscr{L}$ to a fragment $\mathscr{L}^+$ of $\mathbb{L}_{\kappa^+,\aleph_0}(\tau^+)$ of cardinality κ.

By induction on $n < \omega$ we choose A_n, f_n, $\langle I^n_\alpha : \alpha \in A_n\rangle$ such that:

(i) A_n is an unbounded subset of $(2^\kappa)^+$,

(ii) f_n is an increasing function from $(2^\kappa)^+$ onto A_n such that $\alpha < f_n(\alpha)$,

(iii) I^n_α is a submodel of I_α isomorphic to $I_{f_n^{-1}(\alpha)}$,

(iv) if $n > m > 0, \alpha_1, \alpha_2 < (2^\kappa)^+, \bar t^1 \in {}^m(I^n_{f_n(\alpha_1)}), \bar t^2 \in {}^m(I^m_{f(\alpha_2)}), \mathrm{tp}_{\mathrm{qf}}(\bar t^1,\emptyset,I_{f_n(\alpha_1)}) = \mathrm{tp}_{\mathrm{qf}}(\bar t^2,\emptyset,I_{f_n(\alpha_2)})$, then the quantifier free type of $\bar b_{\bar t^1}$ in $N_{f_n(\alpha_1)}$ is equal to the quantifier free type of $\bar b_{\bar t^2}$ in $N_{f_n(\alpha_2)}$,

(v) $A_{n+1} \subseteq A_n$ and $\alpha \in A_{n+1} \Rightarrow I^{n+1}_\alpha \subseteq I^{n+1}_\alpha$.

For $n = 0$ let $A_0 = (2^\kappa)^+$ and $I^0_\alpha = I_\alpha$.

For $n+1$, for each α we apply assumption (c) to $N_{f_n(\alpha+n+1)}$, $I^n_{f_n(\alpha+n+1)}$, $\langle \bar b^\alpha_t : t \in I^n_{f_n(\alpha+n+1)}\rangle$, getting $J^n_{f_n(\alpha+n+1)}$. We define an equivalence relation E_n on $(2^\kappa)^+$ as follows: $\alpha\ E_n\ \beta$ if and only if $\mathrm{tp}(\bar b^{f_n(\alpha+n+1)}_{\bar s},\emptyset,N_{f_n(\alpha+n+1)}) = \mathrm{tp}(\bar b^{f_n(\alpha+n+1)}_{\bar t},\emptyset,N_{f_n(\beta+n+1)})$, whenever $m < \omega, \bar s \in {}^m(J^n_{f_n(\alpha+n+1)}), \bar t \in {}^m(J^n_{f_n(\beta+n+1)})$ and $\mathrm{tp}_{\mathrm{qf}}(\bar s,\emptyset,I_{f_n(\alpha+n+1)}) = \mathrm{tp}_{\mathrm{qf}}(\bar t,\emptyset,I_{f_n(\beta+n+1)})$.

Clearly E_n has $\le 2^\kappa$ equivalence classes, so some equivalence class B is unbounded in $(2^\kappa)^+$. Let

$$A_{n+1} = \{f_n(\alpha+n+1) : \alpha \in B\}, \quad f_{n+1}(\alpha) = f_n(\min(B \setminus \alpha)+n+1),$$

and $I^{n+1}_{f_n(\alpha+n+1)} = J^n_{f_n(\alpha+n+1)}$ for $\alpha \in B$.

Having completed the induction, clearly we have gotten Φ, as the limit. $\square_{1.25}$

Recall that for the class of ordered graphs (and other expansions of linear orders by predicates) we do not have in ZFC strong partition relations, so the following conclusion is what we know.

Conclusion 1.26. *Assume that*

(a) *$\mathscr{L}$ a fragment of $\mathbb{L}_{\kappa^+,\aleph_0}$, T is theory in $\mathscr{L}(\tau)$, and $\theta \geq \kappa + |T| + |\tau| + |\mathscr{L}|$,*

(b) *$\varphi_\alpha = \varphi_\alpha(x_0, \ldots, x_{k_\alpha - 1}) \in \mathscr{L}(\tau)$ for $\alpha < \alpha(*)$ (where $\alpha(*) < \kappa^+$ may be finite),*

(c) *for some $\mu > \theta$, in any forcing extension of $\mathbf{V}$ by a μ-complete forcing notion the following holds for any λ:*

> *if R_α is a subset of $[\lambda]^{k_\alpha}$ for $\alpha < \alpha(*)$ then for some model M of T and $a_\alpha \in M$ for $\alpha < \lambda$ we have: if $\alpha < \alpha(*), \gamma_0 < \ldots < \gamma_{k_\alpha - 1} < \lambda$, then $M \models \varphi_\alpha[a_{\gamma_0}, \ldots, a_{\gamma_{k_\alpha - 1}}] \Leftrightarrow \{\gamma_0, \ldots, \gamma_{k_\alpha - 1}\} \in R_\alpha$*

(d) *Let K be the class of $(I, <, R_0, \ldots, R_\alpha, \ldots)_{\alpha < \alpha(*)}, (I, <)$ linear order, R_α a symmetric irreflexive k_α-place relation on I.*

<u>Then</u> we can find a complete $T_1 \supseteq T$ with Skolem functions, and a template Ψ proper for K and nice, such that:

(α) *$\tau \subseteq \tau_\Psi$ (even τ_Ψ extends τ), and $|\tau_\Psi| \leq \theta$ and $|T_1| \leq \theta$,*

(β) *Ψ is nice for $\mathscr{L}$ and $\mathrm{GEM}(I, \Psi) \vDash T_1$ for $I \in K$,*

(γ) *if $\alpha < \alpha(*)$, and $I \models t_0 < \ldots < t_{k_\alpha - 1}$ then:*

$$\mathrm{GEM}(I, \Psi) \models \varphi_\alpha[a_{t_0}, \ldots, a_{t_{k_\alpha - 1}}] \text{ iff } I \models R_\alpha(t_0, \ldots, t_{k_\alpha - 1}).$$

Proof. We would like to apply 1.25, e.g., with $I_\alpha \in K$ being of cardinality $\beth_{\omega\alpha+1}(\theta)$, and being $\beth_{\omega\alpha}(\theta)^+$-saturated for quantifier free types in the natural sense (such N_α exists by the compactness theorem). However why does assumption (c) of 1.25 hold? By [She89] there is a θ^+-complete forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ this will hold; it would not make a real difference if we replace $\beth_{\omega\alpha+1}(\theta)$ by other suitable cardinal. But by 1.24 this suffices (as our assumptions are absolute enough). $\square_{1.26}$

Remark 1.27. For first order T, this help in Laskowski-Shelah [LS03].

The next conclusion fulfills our promise that for T with the OTOP (omitting type order property) we can in ZFC prove that existence of suitable templates, inspite of the formula exemplifying the order property not being first order.

Conclusion 1.28. *If T is first order countable with the OTOP (see [She90, Ch.XII], the omitting type order property) <u>then</u> for some sequence $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}, \bar{z}) : i < i(*) \rangle$ of first order formulas in $\mathbb{L}(\tau_T)$ and template Φ proper for linear orders we have:*

(α) *$\tau_T \subseteq \tau_\Phi$, $|\tau_\Phi| = |\tau_T| + \aleph_0$,*

(β) *$\mathrm{GEM}_{\tau(T)}(I, \Phi) \models T$ for $I \in K_{\mathrm{org}}$,*

(γ) *if $I \in K_{\mathrm{org}}$ and $s, t \in I$ then*

$$\mathrm{GEM}_{\tau(T)}(I, \Phi) \models (\exists \bar{x}) \bigwedge_{i < i(*)} \varphi_i(\bar{x}, \bar{a}_s, \bar{a}_t) \text{ iff } I \models sRt.$$

Proof. Similarly to the previous conclusion: OTOP is defined in [She90, Ch.XII,4.1,p.608], in a way giving clause (e) of 1.25 above directly, but we need to know that it is absolute (or just preserved by λ-complete forcing), which holds by [She90, Ch.XII,4.3,p.609]. $\square_{1.28}$

For a T is a stable first order T with DOP interesting as the main case is for κ-saturated models of T", not for pseudo elementary classes. In this case, we can prove the result in ZFC directly, see more in the beginning of §(2A).

Now Claim 1.25 apply to the class of linear orders, so a natural question is to find a parallel also for the class $K^\omega_{\rm tr}$, which is the aim of the next claim, see more in Grossberg-Shelah [GS86a] and more lately in [S$^+$a].

It is still conceivable that there are suitable partition relations which are enough, see discussion in [S$^+$b]. Anyhow what we have is:

Conclusion 1.29. *Claim 1.25 applies to the class of trees with ω levels.*

Proof. By the proof in [She90, Ch.VII,§3], i.e., looking at what we use and applying the Erdös-Rado theorem. $\square_{1.29}$

Discussion 1.30. We may consider and get similar results for the following:

- $(*)_1$ we say f is a $(\bar\varphi_1, \bar\varphi_2)$-homomorphism from M_1 into M_2 <u>when</u>:
  - (a) M_1, M_2 are models, not necesarily of the same vocabulary
  - (b) f is a function from M_1 into M_2
  - (c) $\bar\varphi_\ell = \langle \varphi^\ell_i(\bar x_{n_i]}) : i < i_* \rangle$ for $\ell = 1,2$ where φ^ℓ_i is a formula in the vocabulary $\tau(M_\ell)$
  - (d) if $i < i_*$ and $a_n \in M_2$ for $n < n_*$ and $M_1 \models \varphi^1_i[\ldots, a_n, \ldots]_{n<n_*}$ <u>then</u> $M_2 \models \varphi^2_i[\ldots, f(a_n), \ldots]_{n<n_*}$
- $(*)_2$ above we replace "homomorphism" by "embedding" when f is one to one; and in clause (d) we replace implication by equivalence.

Note that $(*)_2$ is a special case of $(*)_1$, i.e. restricting ourselves in $(*)_1$ to the case:

- (a) for every $i < i_*$ for some $j < \ell_*, \varphi^\ell_j$ is equivalent to $\neg\varphi^\ell_i$
- (b) $(\exists i < i_*)(n_i = 2$ and $\varphi^\ell_i = (x_0 \neq x_1))$.

§ 2. Models Represented in Free Algebras and Applications

This section presents a framework, which tries to separate the model theory and combinatorics of [She90, Ch.VIII] and improve it. We shall prove the combinatorics in 2.7(1), §(2E) and more in [Shec] and [Shea]; here we do basic combinatorics and we try to show how to apply it. More applications and more combinatorics are in [Shed].

We sometimes need τ_Φ with function symbols with infinitely many places and deal with logics $\mathscr{L}$ with formulas with infinitely many variables. The example in §(2D). illustrates why.

§ 2(A). **Representation, non-embeddability and bigness.**

We also sometimes would like to rely on a well ordered construction, i.e., on the universe of $\mathscr{M}_{\mu,\kappa}$ (see Definition 2.1 below) there is a well ordering which is involved in the definition of indiscernibility (see 2.1). This means that we have in addition an arbitrary well-order relation. E.g., we would like to build many non-isomorphic $\aleph_1$–saturated models for a stable not superstable first order theory, with the DOP (dimensional order property, see [She90, Ch.X]) so for some $\varphi(\bar{x}, \bar{y})$ (not first order), for any cardinal λ for some model M of T, we have a family $\{\bar{a}_\alpha : \alpha < \lambda\}$ of sequences of length $\leq |T|$ in M with $M \models \varphi[\bar{a}_\alpha, \bar{a}_\beta]$ iff $\alpha < \beta$. The formula φ says: there are z_α ($\alpha < |T|^+$) such that $\bar{x}\char`^\bar{y}\char`^\langle z_\alpha : \alpha < |T|^+\rangle$ realizes a type p. So there is a template Φ proper for $K_{\rm or}$ such that for $I \in K_{\rm or}$ and $s, t \in I$ we have

$$\mathrm{EM}_{\tau(T)}(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t] \text{ iff } I \models s < t$$

($<$ a relevant order), but we need to make them $\aleph_1$-saturated. Ultrapowers may well destroy the order. The natural thing is to make M_I $\aleph_1$-constructible over $\mathrm{GEM}_{\tau(T)}(I, \Phi)$, that is its set of elements is $\{b_\alpha : \alpha < \alpha\}, b_\alpha$ realizing over $\mathrm{EM}_\tau(I, \Phi) \cup \{b_\beta : \beta < \alpha\}$ in M_I a complete type which is $\aleph_1$-isolated. So not only are the $\bar{a}_t$ infinite and the construction involves infinitary functions, but *a priori* the quite arbitrary order of the constructions may play a role.

With some work we can eliminate the well order of the construction for this example (using symmetry, the non-forking calculus, see [CS16]) but there is no guarantee generally and certainly it is not convenient, for example see the constructions in [She83, §3]. Moreover, generally it is better to delete the requirement that the universe of the model is so well defined.

This motivates the following definition.

Definition 2.1.

(a) $\tau(\mu, \kappa) = \tau_{\mu,\kappa}$ is the vocabulary with function symbols
$$\{F_{i,j} : i < \mu,\ j < \kappa\},$$
where $F_{i,j}$ is a j-place function symbol and κ is a regular cardinal

(b) $\mathscr{M}_{\mu,\kappa}(I)$ is the free τ-algebra generated by I for $\tau = \tau_{\mu,\kappa}$

(c) we may write $\mathscr{M}_\mu(I)$ when $\kappa = \aleph_0$ and $\mathscr{M}(I)$ when $\mu = \aleph_0 = \kappa$.

We use the following notation in the remainder of this definition:

- Let $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$. For $\bar{a} = \langle a_i : i < \alpha\rangle \in {}^\alpha M$ let for $i < \alpha$, $f(a_i) = \sigma_i(\bar{t}_i)$, where $\bar{t}_i$ is a sequence of length $< \kappa$ from I and σ_i is a term from $\tau_{\mu,\kappa}$.
- Now if $\alpha < \kappa$ then there is one sequence $\bar{t}$ of members of I of length $< \kappa$ such that
$$\bigwedge_i \mathrm{Rang}(\bar{t}_i) \subseteq \mathrm{Rang}(\bar{t});$$
so we can find terms σ'_i satisfying $f(a_i) = \sigma'_i(\bar{t})$, so without loss of generality $\bar{t}_i = \bar{t}$, we let $\bar{\sigma} = \langle \sigma_i : i < \alpha\rangle$ and $\bar{\sigma}(\bar{t})$ be $\langle \sigma_i(\bar{t}) : i < \alpha\rangle$, so $f(\bar{a}) = \bar{\sigma}(\bar{t})$.

Now

(c) we say that M is Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$ <u>when</u> there is a function $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$ which is a Δ-representation of M where this means: the Δ-type of $\bar{a} \in {}^{\kappa>}M$ (i.e., $\mathrm{tp}_\Delta(\bar{a}, \emptyset, M)$) can be calculated from the sequence of terms $\langle \sigma_i : i < \alpha\rangle$ and $\mathrm{tp}_{\mathrm{qf}}(\langle \bar{t}_i : i < \alpha\rangle, \emptyset, I)$ where $f(\bar{a}) = \langle \sigma_i(\bar{t}_i) : i < \alpha\rangle$ (from (b), so if $f(\bar{a}) = \bar{\sigma}(\bar{t})$ from then can be calculated $\bar{\sigma}$ and $\mathrm{tp}_{\mathrm{qf}}(\bar{t}, \emptyset, I)$). We may say "$M$ is Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$ by f"; similarly below.

(d) We say that M is weakly Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$ <u>when</u> some function $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$ is a weak Δ-representation of M in $\mathscr{M}_{\mu,\kappa}(I)$ which means:
there is a well-ordering $<$ of the universe of $\mathscr{M}_{\mu,\kappa}(I)$ such that for $\bar{a} \in {}^\alpha M$ the Δ-type of $\bar{a}$ can be computed from the information described in (c) and the order $<$ restricted to the family of subterms of the terms $\langle \sigma_i(\bar{t}_i) : i < \alpha\rangle$.

[We introduce weak representability to deal with the dependence on the order of a construction, (cf. the discussion in section 2(D))].

(e) (α) We say $\bar{a}^1 \sim \bar{a}^2 \mod \mathscr{M}_{\mu,\kappa}(I)$ and may say $\bar{a}^1, \bar{a}^2$ are similar in $\mathscr{M}_{\mu,\kappa}(J)$ <u>when</u> for $i = 1, 2$ we have $\bar{a}_i = \langle \sigma^i_j(\bar{t}^i_j) : j < \alpha\rangle$, $\sigma^1_j = \sigma^2_j$ and
$$\mathrm{tp}_{\mathrm{qf}}(\langle \bar{t}^1_j : j < \alpha\rangle, \emptyset, I) = \mathrm{tp}_{\mathrm{qf}}(\langle \bar{t}^2_j : j < \alpha\rangle, \emptyset, I)$$

(β) for the case of weak representability we write $\bar{a}^1 \sim \bar{a}^2 \mod (\mathscr{M}_{\mu,\kappa}(I), <)$ and may say $\bar{a}^1, \bar{a}^2$ are similar in $(\mathscr{M}_{\mu,\kappa}(J), <)$ <u>when</u> in addition the mapping
$$\{\langle \sigma(\bar{t}^1_i), \sigma(\bar{t}^2_i)\rangle : i < \alpha,\ \sigma \text{ is a subterm of } \sigma^1_i = \sigma^2_i\}$$
is a $<$-isomorphism (and both sides are linear orders). We write $\bar{a}^1 \sim_A \bar{a}^2 \mod \ldots$ if $\bar{a}^1{}^\frown\bar{b} \sim \bar{a}^2{}^\frown\bar{b} \mod \ldots$ whenever $\bar{b} \in {}^{\kappa>}A$ where $A \subseteq \mathscr{M}_{\mu,\kappa}(I)$. (This latter is especially important when we work over a set of parameters). We might, for instance, insist that $\bar{t}^1_i$ and $\bar{t}^1_j$ realize the same Dedekind cut over $I_0 \subseteq I$. (So "M is Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$" means: $f(\bar{a}^1)$ similar to $f(\bar{a}^2) \mod \mathscr{M}_{\mu,\kappa}$ implies $\bar{a}^1$ and $\bar{a}^2$ realize the same Δ-type in M.)

(f) (α) We say the representation is full <u>when</u>:
$$c_1 \sim c_2 \mod \mathscr{M}_{\mu,\kappa}(I)) \text{ implies } [c_1 \in \mathrm{Rang}(f) \Leftrightarrow c_2 \in \mathrm{Rang}(f)].$$

(β) We say the weak representation is full <u>when</u> we replace $\mathscr{M}_{\mu,\kappa}(I)$ by $(\mathscr{M}_{\mu,\kappa}(I), <)$, where $<$ is a given well ordering from clause (d).

(g) If Δ is the family of quantifier free formulas it may be omitted.

(h) For $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$, let $\bar{a} \sim \bar{b} \mod (f, \mathscr{M}_{\mu,\kappa}(I))$ means

$$f(\bar{a}) \sim f(\bar{b}) \mod \mathscr{M}_{\mu,\kappa}(I).$$

Similarly, $\bar{a} \sim \bar{b} \mod (f, \mathscr{M}_{\mu,\kappa}(I), <)$ means

$$f(\bar{a}) \sim f(\bar{b}) \mod (\mathscr{M}_{\mu,\kappa}(I), <).$$

2) We may restrict ourselves to well orderings $<$ of $\mathscr{M}_{\mu,\kappa}(I)$ which respect subterms; this means that if $\sigma_1(\bar{t}_1)$ is a subterm of $\sigma_2(\bar{t}_2)$ then $\sigma_1(\bar{t}_1) \leq \sigma_2(\bar{t}_2)$.

Now we define a very strong negation (when φ is "right") to even weak representability.

Definition 2.2. 1) For index models I, J we say I is strongly $\varphi(\bar{x}, \bar{y})$-unembeddable for $\tau(\mu, \kappa)$ into J <u>when</u> for every $f : I \longrightarrow \mathscr{M}_{\mu,\kappa}(J)$ and well ordering $<$ (of $\mathscr{M}_{\mu,\kappa}(J)$) there are sequences $\bar{s}, \bar{t}$ of members of I such that $I \models \varphi[\bar{s}, \bar{t}]$ and $\bar{s}, \bar{t}$ have "similar" (2.1(e)) images in $(\mathscr{M}_{\mu,\kappa}(J), <)$. If we delete the well ordering, we get only "I is $\varphi(\bar{x}, \bar{y})$-unembeddable". If φ clear from the context we may omit it. Note that the formula $\varphi(\bar{x}, \bar{y})$ should be in the vocabulary τ_I; here almost always we have $\tau_J = \tau_I$ but this is not really necessary.
2) K has the [strong] $(\chi, \lambda, \mu, \kappa)$-bigness property for $\varphi(\bar{x}, \bar{y})$ <u>when</u> there are $I_\alpha \in K_\lambda$ for $\alpha < \chi$ such that for $\alpha \neq \beta$ we have I_α is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable for $\tau(\mu, \kappa)$ into I_β.
3) K has the full [strong] $(\chi, \lambda, \mu, \kappa)$-bigness property for $\varphi(\bar{x}, \bar{y})$ <u>when</u> there are $I_\alpha \in K_\lambda$ for $\alpha < \chi$ such that, for $\alpha < \chi$, I_α is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable for $\tau(\mu, \kappa)$ into $\sum\limits_{\beta<\chi, \beta\neq\alpha} I_\beta$ (where $\sum\limits_{\beta\in u} I_\beta$, when all the I_β are τ-models for some fixed vocabulary τ, is a τ-model I with universe the disjoint union $\bigcup\limits_{\beta\in u} |I_\beta|$; if those universes are not pairwise disjoint we use $\bigcup\limits_{\beta\in u} (\{\beta\} \times (I_\beta))$; for a predicate $P \in \tau$, $P^I = \bigcup\limits_{\beta\in u} P^{I_B}$, for every function symbol $F \in \tau$, F^I is the (partial) function $\bigcup\limits_{\beta\in u} F^{I_\beta}$).
4) Saying "I is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable into J for function f satisfying Pr" means we restrict ourselves (in 2.2(1)) to function f from I to $\mathscr{M}_{\mu,\kappa}(J)$ satisfying Pr.
5) The most popular restriction is "f finitary on some P" which means that for every $\eta \in P^I$ for some $n < \omega$, $\tau_{\mu,\kappa}$-term σ and $\eta_0, \ldots, \eta_{n-1} \in J$ we have $f(\eta) = \sigma(\eta_0, \ldots, \eta_{n-1})$. We say f is strongly finitary if in addition σ has only finitely many subterms.
6) Clearly (4) induces parallel variants of 2.2(2), 2.2(3).

Remark 2.3. 1) This definition is used in proving that the model constructed from I is not isomorphic to (or not embeddable into) the model constructed from J.
2) We may in 2.2(1) and the other variants, add: moreover, given $A \subseteq J$ of cardinality $< \kappa$ we demand that $\bar{x}, \bar{y}$ are similar over A. This does not make a real difference so far.

3) About the connection to $\dot{I}\dot{E}(\lambda, T_1, T)$ see [Shea]. Clearly "representable in $\mathscr{M}(I)$" is intended to be weaker relative of being a $\mathrm{GEM}_\tau(I,\Phi)$-model; the next claim shows that this is the case indeed.

Claim 2.4. *1) If Φ is proper for the index model I and $\mu = |\tau_\Phi|$ then $\mathrm{GEM}(I,\Phi)$ can be* qf*-represented in $\mathscr{M}_{\mu,\aleph_0}(I)$.*
*2) If Φ is weakly $\mathscr{L}$-nice, then we can replace "*qf*" by $\mathscr{L}$.*
3) We can add in parts (1) and (2) that the representation is full when I is a linear order which has neither first element nor last element.

Proof. Easy, but we elaborate.
1) Let $(M,\bar{\mathbf{a}})$ be $\mathrm{GEM}(I,\Phi), \bar{\mathbf{a}} = \langle \bar{a}_s : s \in I\rangle$ for transparency assume $\ell g(\bar{a}_s) = 1$ for $s \in I$; otherwise use suitable unary functions.

For each $n < \omega$ let $\langle \sigma_{n,i}(\bar{x}_{[n]}) : i < i_n\rangle$ list the τ_Φ-term with sets of free variables included in $\bar{x}_{[n]}$; as $\mu \geq |\tau_\Phi|$ and μ is infinite, without loss of generality $i_n \leq \mu$. Now we define a function f with doman M as follows:

- $(*)$ if $a \in M$ and $M \models$ "$a = \sigma_{n,i}(\ldots, s_\ell, \ldots)_{\ell<n}$" and $s_0 <_I \ldots <_I s_{n-1}$, then $f(a) = F_{n,i}(\ldots, s_\ell, \ldots)_{\ell<n}$ for some $n < \omega, i < i_n$.

The choice is not unique but each $f(a)$ is defined in at least one way; so choose one of the values; it is easy to check that f is as required, but this is not so nice way to define f, so we give a better proof.

For transparency assume we choose $\langle \sigma_{n,i}(\bar{x}_{[n]}) : i < i_n, n < \omega\rangle$ as above. For each such pair (n,i) let $u = u(n,i)$ be a subset of n of minimal cardinality such that:

- $(*)^1_{u,n,i}$ if $s_0 <_I \ldots <_I s_{n-1}$ and $t_0 <_I \ldots <_I t_{n-1}$ then $(\forall i \in u)(s_i = t_i) \Rightarrow \sigma^\mu_{n,i}(s_0,\ldots,s_{n-1}) = \sigma^M_{n,i}(t_0,\ldots,t_{n-1})$.

Easily:

- $(*)^2_{n,i}$ $u = u(n,i)$ is unique.

[Why? As I is infinite (really less is necessary).]

Next

- $(*)_3$ for $a \in M$
  - (a) $n = n_a$ is the minimal n such that $a \in \{\sigma^M_{n,i}(\ldots, a_\ell, \ldots)_{\ell<n} : i < i_n$ and $s_0 <_I s_1 <_I \ldots <_I s_{n-1}$ are from $I\}$
  - (b) fixing $n = n_a, i = i_a$ is the minimal i such that $a \in \{\sigma^M_{n,i}(\ldots, a_{s_\ell}, \ldots)_{\ell<n} : s_0 <_I \ldots <_I s_{n-2}\}$
  - (c) choose $\langle s_{a,\ell} : \ell < n\rangle$ such that $a = \sigma^M_{n,i}(s_{a,0},\ldots,s_{a,n-1})$ and $s_{a,0} <_I \ldots s_{a,n-1}$
  - (d) let $u_a = u(n_a, i_a)$ and $m_a = m_{n,i} = |u_a|$
  - (e) for $n < \omega, i < i_n$ let $h_{n,i}$ be the unique increasing function from $m_{n,i}$ onto $u_{n,i}$.

Lastly, we define a function f with domain M

- $(*)_4$ for $a \in M$ we let $f(a) = F_{m,j}(\ldots, s_{h(\ell)}, \ldots)_{\ell<m}$ <u>when</u>:
  - $\bullet_1$ $n = n_a, i = i_a, s_0 <_I \ldots <_I s_{n-1}, h = h_a$

$\bullet_2$ $a = \sigma^M_{n,i}(a_{s_0}, \ldots, a_{s_{n-1}})$.

[Why? We have to prove that f is well defined. Now above we have proved that there are $n, i, m, h, \bar{s} = \langle s_\ell : \ell < n\rangle$ as proved above but $\bar{s}$ is not necessarily unique; however, as "I has neither first nor last element".]

$(*)_5$ f is a qf-representation of M in $\mathscr{M}_{\mu,\aleph_0}(I)$.

[Why? Reread the definitions.]
2),3) Easy, too. $\square_{2.4}$

Remark 2.5. We can omit the extra assumption on I in 2.4, if we add the following reasonable assumption:

- if $\sigma(x_0, \ldots, x_{n-1})$ is a τ_Φ-term, and $i < n$ then $(a) \Rightarrow (b)$ where:
  - (a) if J is a linear order and $(M, \bar{\mathbf{a}}) = \mathrm{GEM}(J, \Phi)$ and $s_0 < \ldots < s_{n-1}$ and $t \in J$ and $j < i \Rightarrow s_j <_I t$ and $j \in (i,n) \Rightarrow t <_I s_j$ then $\sigma^M(\ldots, a_{s_j}, \ldots, a_{s_i}, \ldots, a_{s_\ell}, \ldots)_{j<i, i\in(i,n)} = \sigma^M(\ldots, a_{s_j}, \ldots, a_t, \ldots, a_{s_\ell})_{j<i,\ell\in(i,n)}$
  - (b) there is a function symbol F (or just a term) such that $F \in \tau_\Phi$ has arity $n-1$ and if $s_0 <_I \ldots <_I \ldots s_{n-1}, J, (M, \bar{\mathbf{a}})$ are as above then

$$F^M(\ldots, a_{s_j}, \ldots; \ldots a_{s_\ell}, \ldots)_{j<i,\ell\in(i,n)} = \sigma^M(a_{s_0}, \ldots, a_{s_{n-1}}).$$

§ 2(B). **Example: Unsuperstability.**

The following example illustrates the application of this method. We first fix K^ω_{tr} (see 1.9) as the class of index models and fix a formula φ_{tr} (see 2.6 below); note that we shall prove (for regular uncountable cardinals, see 2.7(1), §(2E) here; more is said in 2.7(2) which is proved in [Shea]) that for many pairs $I, J \in K^\omega_{\mathrm{tr}}, I$ is $\varphi_{\mathrm{tr}}(\bar{x}, \bar{y})$-unembeddable in J. In 2.9 we apply this to unsuperstable T. Lastly, in 2.11 below we choose for each $I \in K^\omega_{\mathrm{tr}}$ a reduced separable Abelian $\dot{p}$-group $\mathbb{G}_I$ which is representable in $\mathscr{M}_{\omega,\omega}(I)$. In 2.13 below we show that: [I is φ_{tr}-unembeddable in J implies $\mathbb{G}_I \not\cong \mathbb{G}_J$]; thus the number of reduced separable Abelian $\dot{p}$-groups of cardinality λ is at least as great as the number of trees in K^ω_{tr} with cardinality λ which are pairwise φ_{tr}-unembeddable. Here we prove in 2.7(1) that for any regular uncountable λ that 2^λ is the number. We showed in [She83] that this number is 2^λ for regular λ and many singulars. But as said in 1.13 for every uncountable λ we get 2^λ pairwise non-isomorphic such groups in λ, using $\mathbb{G}_I$ as below.

We may like to strengthen "$\mathbb{G}_I \not\cong \mathbb{G}_J$" to "$\mathbb{G}_I$ not embeddable in $\mathbb{G}_J$". Doing this depends on two points. One concerns singular cardinals, for them the needed family of $I \in K_\lambda$ exists by 2.7(2) which is proved only in [Shea]. The second point depends on the exact notion of embeddability we use; here we use so called "pure embeddings", see 2.11(4) (we shall return to this in [Shea, 3.22]).

Example 2.6. For the class of $I \in K^\omega_{\mathrm{tr}}$ we let:

$$\begin{aligned}\varphi_{\mathrm{tr}}(x_0, x_1 : y_0, y_1) := \quad & [x_0 = y_0] \text{ and } P_\omega(x_0) \text{ and} \\ & \bigvee_{n<\omega} [P_n(x_1) \text{ and } P_n(y_1) \text{ and } P_{n-1}(x_1 \cap y_1)] \text{ and} \\ & [x_1 \lhd x_0 \wedge y_1 \ntriangleleft y_0] \text{ and } y_1 <_{\mathrm{lx}} x_1]\end{aligned}$$

in other words, when for transparency we restrict ourselves to standard $I \subseteq {}^{\omega\geq}\lambda$: $x_0 = y_0 \in {}^{\omega}\lambda$, and for some $n < \omega$ and $\alpha < \beta < \lambda$ we have

$$x_1 = (x_0 \restriction n)\char`^\langle\alpha\rangle \triangleleft x_0$$

and

$$y_1 = (x_0 \restriction n)\char`^\langle\beta\rangle$$

We quote

Theorem 2.7. *Let $K = K^{\omega}_{\rm tr}$, trees with $\omega + 1$ level.*
1) If $\lambda > \mu$ is regular, then K has the strong $(2^\lambda, \lambda, \mu, \aleph_0)$-bigness property for $\varphi = \varphi_{\rm tr}(\bar{x}_{[2]}; \bar{y}_{[2]})$.
1A) We can add "full (strong)".
2) If $\lambda > \mu$ then K has the full strong $(\lambda, \lambda, \mu, \aleph_0)$-bigness property for $\varphi = \varphi_{\rm tr}(\bar{x}_{[2]}, \bar{y}_{[2]})$.

Proof. 1) Let $S_* = \{\delta < \lambda : {\rm cf}(\delta) = \aleph_0\}$ and for each $\delta \in S_*$ let $\eta_\delta \in {}^{\omega}\delta$ be increasing with limit δ. Now for $S \subseteq S_*$ let $I_\delta = \{\eta_\delta : \delta \in S\} \cup {}^{\omega>}\lambda$. Now we consider I_S as a member of $K^{\omega}_{\rm tr}$ as usual.

The main point is:

$\boxplus$ if $S_1, S_2 \subseteq S_*$ and $S_1 \backslash S_2$ is a stationary subset of λ, then I_{S_1} is $\varphi_{\rm tr}$-unembeddable into I_{S_2}.

Why is $\boxplus$ true? Let $f : I_{S_1} \to \mathscr{M}_\mu(I_{S_2})$ and $<_2$ a well ordering of $\mathscr{M}_\mu(S_2)$, let χ be such that $x = \{S_1, S_2, I_{S_1}, \mathscr{M}_\mu(S_2); f, <_2\}$ belongs to $\mathscr{H}(\chi)$. Choose a $\prec$-increasing continuous sequence $\langle N_\alpha : \alpha < \lambda\rangle$ such that $N_\alpha \prec (\mathscr{H}(\chi), \in)$ is of cardinality $< \lambda, \alpha \subseteq N_\alpha, \beta < \alpha \Rightarrow \langle N_\gamma : \gamma \leq \beta\rangle \in N_\alpha$ and $\lambda \in N_\alpha$. Let $E = \{\delta < \lambda : \delta$ is a limit ordinal such that $N_\delta \cap \lambda = \delta\}$, clearly a club of λ. But by the assumption of $\boxplus$, $S_1 \backslash S_2$ is stationary, hence we can find $\delta \in S_1 \cap E \backslash S_2$. Now as $N_\delta \cap \lambda = \delta$, there is $\varepsilon < \delta$ and $n < \omega$ such that $\eta_\delta \restriction n \in N_\varepsilon$ and $N_{\varepsilon+2} \cap \lambda \subseteq \eta_\delta(n)$.

Having chosen η_δ and n we choose $\beta = \eta_\delta(n)$ and let $\alpha \in (N_{\varepsilon+1} \backslash N_\varepsilon) \cap \lambda$ be similar enough to β.

So $\boxplus$ holds. We know that there is a sequence $\langle S_\varepsilon : \varepsilon < \lambda\rangle$ of pairwise disjoint stationary subsets of S_* and for every $u \subseteq \lambda$ let

$$S^\bullet_u = \cup\{S_\varepsilon : \text{ for some } \zeta < \lambda, \varepsilon \in \{2\zeta, 2\zeta + 1\} \text{ and } \varepsilon = 2\zeta + 2 \Leftrightarrow \zeta \in u\}.$$

Clearly $u \neq v \subseteq \lambda \Rightarrow S^\bullet_u \backslash S^\bullet_2$ is a stationary subset of λ so we are done by $\boxplus$.
1A) By §(1A) and the proof of part (1) we can prove this.
2) On this, the existence, see [Shea, j2(2)] (which we deduce from [Shea, b11(2)]). $\square_{2.7}$

Remark 2.8. In 2.7 we choose the last cardinal κ as $\aleph_0$. There is interest in choosing $\kappa > \aleph_0$, but we shall not deal with it here, see [Sheb].

The connection of the bigness properties from 2.2 to the results on $\dot{I}\dot{E}(\lambda, T_1, T)$ is done by:

Claim 2.9. *Assume that*

(a) *Φ, φ_n are as in the conclusion of 1.11(1), $\mu \geq |\tau_\Phi| + \aleph_0$,*
(b) *$I, J \in K^\omega_{\rm tr}$, I is strongly $\varphi_{\rm tr}$-unembeddable into J for $\tau_{\mu,\aleph_0}$,*
(c) *$\tau_0 \subseteq \tau_\Phi$ is a vocabulary including that of the φ_n's.*

Then $\mathrm{GEM}_{\tau_0}(I, \Phi)$ cannot be elementarily embedded into $\mathrm{GEM}_{\tau_0}(J, \Phi)$. Moreover, no function from $\mathrm{GEM}(I, \Phi)$ into $\mathrm{GEM}(J, \Phi)$ preserves the formulas $\pm\varphi_n$ (for $n < \omega$).

Proof. Straightforward but we elaborate. Let f be a function from the model $M_I = \mathrm{GEM}(I, \Phi)$ into $M_J = \mathrm{GEM}(J, \Phi)$ which preserve $\pm\varphi_n$; and let $\langle a_s : s \in I\rangle$ witness this; by 2.4(2) there is a function g from M_2 into $\mathscr{M}_\mu(J)$ which is a $\{\pm\varphi_n : n < \omega\}$-representation.

Define a function $h : I \to {}^{\ell g(\bar{x})}(\mathscr{M}_\mu(J))$ by $h(s) = g(f(\bar{a}_s))$ for $s \in I$.

Recalling "I is φ_k-unembeddable into $\mathscr{M}_\mu(J)$" there are $\eta = x_1 = x_2 \in P^I_\omega, n < \omega, \nu \in P^I_n$ and $y_1 <_{\rm lex} y_2$ in $\mathrm{suc}_I(\nu)$ such that $y_2 \triangleleft_I \eta$ and $(h(\eta), h(y_2), (h(\eta), h(y_1))$ are similar.

But by the choice of $\Phi, M_2 \models \varphi_n[a_\eta, a_{y_2}] \wedge \neg\varphi_n[a_\eta, a_{y_2}]$ hence by the choice fo $f, M_2 \models \varphi_n[f(\bar{a}_\eta), f(\bar{a}_{y_2}))\wedge\neg\varphi_n[f(\bar{a}_\eta), f(\bar{a}_{y_1})]$ and by the choice of $g, (g\circ f(\bar{a}_\eta), g\circ f(\bar{a}_{y_2}))$ and $(g \circ f(\bar{a}_\eta), g \circ f(\bar{a}_{y_2}))$ cannot be similar, contradiction. $\square_{2.9}$

§ 2(C). Example: Separable reduced Abelian $\dot{p}$-groups.

Discussion 2.10. We present the definition of this class of groups in 2.11(1),(2); see on it in [Fuc73], [Fuc74], [EM02] and [GT12]; but no need to read any of them.

From out point of view, this class is closely related to $K^\omega_{\rm tr}$. One way to express it is to derive a tree with $(\omega + 1)$ levels from such a group $\mathbb{G}$: the n-th level consists of $\mathbb{G}/E_n$ where E_n is the following equivalence relation on $\mathbb{G} : xE_ny$ if $\mathbb{G} \models (\exists z)(p^n z = x - y)$ and the ω-th level consists of $\{x\}$ for $x \in \mathbb{G}$; the order is the inverse of inclusions. Of course, this is not a good representation; there is much redundancy. For example, $\mathbb{G}/E_2$ is just a vector space over the field with $\bar{p}$ elements, so we better replace it by a basis. This motivates 2.11(3),(4).

In more detail, we can explicate y^n_η/E_m:

$\boxplus_1$
$$\begin{aligned} y^n_\eta &= x_{\eta\restriction n} + \dot{p}y^{n+1}_\eta \\ &= x_{\eta\restriction n} + \dot{p}(x_{\eta\restriction(n+1)} + \dot{p}y^{n+2}_\eta) \\ &= x_{\eta\restriction n} + \dot{p}x_{\eta\restriction(n+1)} + \dot{p}^2y^{n+2}_\eta) \\ &= x_{\eta\restriction n} + \dot{p}x_{\eta\restriction(n+1)} + \dot{p}^2x_{\eta\restriction(n+2)} + \dot{p}^3y^{n+3}_\eta. \end{aligned}$$

So for $m \geq n$

$\boxplus_2$ $y^n_\eta = \sum\limits_{\ell=n}^{m} \dot{p}^{\ell-n}x_{\eta(n+1)} + \dot{p}^{m-n}y^m_{\eta\restriction m}$

$\boxplus_3$ y^n_η belongs to $(\sum\limits_{\ell=n}^{m-n-1} \dot{p}^\ell x_{\eta(n+1)})/E_{m-n}$.

In the limit we get 2.11(4); also we shall use:

$\boxplus_4$ $\langle \dot{p}^n x_\eta : \eta \in P^I_n\rangle$ are independent over $\mathbb{G}_{3n}$ = the subgroup of $\mathbb{G}$ generated by $\{\dot{p}^{n+1}x_\nu : \nu \in P^I_n$ for some $\ell > n\}$, in the sense that, e.g.

- if $\langle \eta_\ell : \ell \leq \ell_* \rangle$ are pairwise distinct members of P^I_n and $k_\ell \in \{1, \ldots, \dot{p}-1\}$ for $\ell \leq k$ then $\Sigma\{k_\ell \dot{p}^n x_{\eta_\ell} : \ell \leq \ell_*\}$ is not E_n-equivalent to any member of $\mathbb{G}_n$.

(See more in [Shea, §3]; as p denote types we use $\dot{p}$ for prime numbers.)

Definition 2.11. 1) A separable reduced Abelian $\dot{p}$-group $\mathbb{G}$ is a group $\mathbb{G}$ which satisfies (we use additive notation):

(a) $\mathbb{G}$ is commutative (that is "Abelian"),
(b) for every $x \in \mathbb{G}$ for some n, x has order $\dot{p}^n$ (i.e., $\dot{p}^n x$ is zero and n is minimal),
(c) $\mathbb{G}$ has no divisible non-trivial subgroup (= is reduced),
(d) every $x \in \mathbb{G}$ belongs to some 1-generated subgroup which is a direct summand of $\mathbb{G}$ (= is separable).

2) Any such group is a normed space:

$$\|x\| = \inf\{2^{-n} : (\exists y \in \mathbb{G})\dot{p}^n y = x\}.$$

3) For a tree $I \in K^\omega_{\mathrm{tr}}$ we define the $\dot{p}$-group $\mathbb{G}_I$ as follows, $\mathbb{G}_I$ is generated (as an Abelian group) by

$$\{x_\eta : \eta \in \bigcup_{n<\omega} P^I_n\} \cup \{y^n_\eta : \eta \in P^I_\omega \text{ and } n < \omega\},$$

freely except for the relations:

(a) $\dot{p}^{n+1} x_\eta = 0$ for $\eta \in P^I_n$
(b) $\dot{p}^{n+1} y^n_\eta = 0$ for $\eta \in P^I_\omega$
(c) $y^n_\eta - \dot{p} y^{n+1}_\eta = x_{\eta \restriction n}$.

4) For Abelian groups $\mathbb{G}_1, \mathbb{G}_2$, an embedding f of $\mathbb{G}_1$ into $\mathbb{G}_2$ is pure <u>when</u> for every $\lambda \in \mathbb{G}_i$ and $n \geq 2, x$ is divisible by n in $\mathbb{G}_1$ <u>iff</u> $f(x)$ is divisible by n in $\mathbb{G}_2$.

Discussion 2.12. It is well known that $\mathbb{G}_I$ is a reduced separable Abelian $\dot{p}$-group. Also note that we have essentially said

$$y^n_\eta = \sum\{\dot{p}^{\ell-n} x_{\nu_\ell} : \ell \text{ satisfies } n \leq \ell < \omega, \nu_\ell \in P^I_\ell \text{ and } \nu_\ell \lhd \eta\}$$

(the infinitary sum is well defined as $\mathbb{G}_I$ is a normed space). What do we need to apply our framework to the class of Separable reduced Abelian Groups? We need to prove $\mathbb{G}_I$ is represented in I for $I \in K^\omega_{\mathrm{tr}}$, done in 2.13 and to derive "no isomorphism of $\mathbb{G}_I, \mathbb{G}_J$" from "$I$ is φ_{tr}-unembeddable into J", done in 2.14.

Of course, we can replace "h is an isomorphism from $\mathbb{G}_I$ onto $\mathbb{G}_J$", e.g. by "h embeds $\mathbb{G}_I$ into $\mathbb{G}_J$ which preserves "x is not divisible by $\bar{p}^n$" for every n.

It is easy to see that

Fact 2.13. $\mathbb{G}_I$ is a reduced separable Abelian $\dot{p}$-group which is represented in $\mathscr{M}(I)$.

We shall prove now

Claim 2.14. *1) If I is $\varphi_{\rm tr}$-unembeddable into J <u>then</u> $\mathbb{G}_I \not\cong \mathbb{G}_J$.*
2) Moreover there is no pure embedding of $\mathbb{G}_I$ into $\mathbb{G}_J$.

Proof. Let g be an isomorphism from $\mathbb{G}_I$ onto $\mathbb{G}_J$ and $h : \mathbb{G}_J \longrightarrow \mathscr{M}(J)$, where h witnesses that $\mathbb{G}_J$ is representable in $\mathscr{M}(J)$.

Let $f : I \longrightarrow \mathbb{G}_I$ be:

$$f(\eta) = \begin{cases} \sum\limits_{1 \le \ell \le \ell g(\eta)} \dot{p}^{\ell-1} x_{\eta\restriction\ell} & \text{if} \quad \eta \in \bigcup\limits_{n<\omega} P^I_n, \\ y^1_\eta & \text{if} \quad \eta \in P^I_\omega. \end{cases}$$

So $(h \circ g \circ f) : I \longrightarrow \mathscr{M}_{\omega,\omega}(J)$. Now we use the fact that I is $\varphi_{\rm tr}$-unembeddable into J.

So suppose

$$I \models \varphi_{\rm tr}[\eta_0, \nu_0; \eta_1, \nu_1] \text{ and } h \circ g \circ f(\eta_0, \nu_0) \sim h \circ g \circ f(\eta_1, \nu_1).$$

Invoking the definition of $\varphi_{\rm tr}$: for some $\eta := \eta_0 = \eta_1 \in P^I_\omega$ and for some n,

$(*)$ (a) $\nu_1 \lhd \eta_1$
(b) $\nu_1 \in P^I_n$
(c) $\nu_0 \in P^I_n$
(d) $\nu_1\restriction(n-1) = \nu_0\restriction(n-1)$
(e) $\nu_0(n-1) < \nu_1(n-1)$.

For $i = 0, 1$ let

$$z_{\nu_i} = \sum\{\dot{p}^{\ell-1} x_\nu : \nu \lhd \nu_i, \nu \in P^I_\ell \text{ and } 1 \le \ell \le n\}.$$

Now $\mathbb{G}_I \models$ "$\dot{p}^n$ divides $(y^1_\eta - z_{\nu_0})$", hence, as g is an isomorphism, $\mathbb{G}_J \models$ "$\dot{p}^n$ divides $(g(y^1_\eta) - g(z_{\nu_0}))$", which means $\mathbb{G}_J \models$ "$\dot{p}^n$ divides $(g \circ f(\eta) - g \circ f(\nu_0))$".

Similarly, $\mathbb{G}_J \models$ "$\dot{p}^n$ does not divide $(g \circ f(\eta) - g \circ f(\nu_1))$", but

$$h \circ g \circ f(\langle \eta_0, \nu_0 \rangle) \sim h \circ g \circ f(\langle \eta_1, \nu_1 \rangle) \mod \mathscr{M}(J),$$

a contradiction, proving 2.14. $\square_{2.14}$

§ 2(D). **An Example: Rigid Boolean Algebras.**

We would like to build complete Boolean algebras without non-trivial one-to-one endomorphisms. How do we get completeness? We build a Boolean algebra, $\mathbf{B}_0$ and take its completion. Even when $\mathbf{B}_0$ satisfies the c.c.c. we need the term $\bigcup\limits_{n<\omega} x_n$ to represent elements of the Boolean algebra from the "generators" $\{\bar{a}_t : t \in I\}$.

On rigidity we still can get considerable amounts of information by the general theory. When we try to construct many models of K (no one embeddable into the others) we need

$(*)$ there are 2^λ index models I of cardinality λ each $\varphi_K(\bar{x}, \bar{y})$-unembeddable into any other.

But when you intend to construct rigid, indecomposable, etc., you need:

(∗∗) there are $\{I_\alpha \in K : \alpha < \lambda\}, I_\alpha, \varphi_K$-unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta$ (and I_α has cardinality λ).

Why?

Example 2.15. Constructing Rigid Boolean Algebras. (See more, and for more details, in [Shed, §2].) For $I \in K^\omega_{\rm tr}$ let $\mathrm{BA}_{\rm tr}(I)$ be the Boolean Algebra freely generated by $\{a_\eta : \eta \in I\}$ except the relations

$$a_\eta \leq a_\nu \text{ when } \nu \in P^I_\omega, n < \omega, \eta = \nu{\restriction}n.$$

First, choose a sequence $\langle I_\alpha : \alpha < \lambda\rangle$ of members of $K^\omega_{\rm tr}$ each of cardinaltiy λ. Naturally, we choose I_α: for $\alpha < \lambda$ such that I_α is $\varphi_{\rm tr}$-unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta, |I_\alpha| = \lambda$.

We shall choose a sequence $\langle \mathbf{B}_i, a_j : i \leq \lambda, j < \lambda\rangle$ such that $\mathbf{B}_i$ is a Boolean algebra, $\subseteq$-increasingly continuous with $i, a_i \in \mathbf{B}_i$ and if $i < \lambda$ and $a \in \mathbf{B}_i \backslash \{0,1\}$ then $a = a_j$ for some $j \in [i, \lambda)$. Start with $\mathbf{B}_0 = \mathrm{BA}_{\rm tr}(I_0)$, then successively for some $a_i \in \mathbf{B}_i, 0 < a_i < 1$, take

$$\mathbf{B}_{i+1} = (\mathbf{B}_i{\restriction}(1-a_i)) + ((\mathbf{B}_i{\restriction}a_i) * \mathrm{BA}_{\rm tr}(I_i)),$$

$$\mathbf{B}_\lambda = \bigcup_{i<\lambda} \mathbf{B}_i = \{a_i : i < \lambda\}.$$

(In such situations we say that $\mathbf{B}_{i+1}$ is a result of the $\mathrm{BA}_{\rm tr}(I_i)$-surgery of $\mathbf{B}_i$ at a_i. That is, below $1-a_i$ we add nothing and below a_i we use the free product of $\mathbf{B}_i{\restriction}a_i$ and $\mathrm{BA}_{\rm tr}(I_i)$.)

The point is that each $a \in \mathbf{B}_\lambda \setminus \{0,1\}$ was "marked" by some I_α, (the α such that $a_\alpha = a$). Now $\mathrm{BA}_{\rm tr}(I_\alpha)$ is embeddable into $\mathbf{B}_\lambda{\restriction}a_\alpha$; but $\mathbf{B}_\lambda{\restriction}(1-a_\alpha)$ is weakly represented in $\mathscr{M}(\sum_{\beta\neq\alpha} I_\beta)$. So for no automorphism f of $\mathbf{B}_\lambda$ do we have, $f(a_\alpha) \leq 1 - a_\alpha$, which suffices to get "$\mathbf{B}_\lambda$ is rigid"; in fact, it has no one-to-one endomorphism. If we are trying to get stronger rigidity and/or $\mathbf{B}_\lambda \models$ c.c.c., and/or $\mathbf{B}_\lambda$ is complete, we may have to change $K^\omega_{\rm tr}$ and/or $\varphi_{\rm tr}$.

This illustrates the need for some of the complications in definition 2.1, 2.2. E.g., the weak representation and the uncountable κ (for complete Boolean Algebras). That is, if we like to get a complete Boolean Algebra, we may find a regular uncountable κ, build a κ-c.c. Boolean Algebra $\mathbf{B}_1$ satisfying the κ-c.c. and then use the completion $\mathbf{B}_2$ of $\mathbf{B}_1$. Now even if $\mathbf{B}_1$ is represented in $\mathscr{M}_{\mu,\aleph_0}(I), \mu = \mu^{<\kappa}$ then $\mathbf{B}_2$ is naturally represented in $\mathscr{M}_{\mu,\kappa}(I)$.

§ 2(E). **Closure sums.**

As exemplified in §(2D), we like to have cases of the "K has full (strong) $(\chi, \lambda, \mu, \kappa)$-bigness for φ", which means having sequences $\langle I_\alpha : \alpha < \chi\rangle$ of member of K of cardinality λ such that I_α is φ-unembeddable into $\Sigma\{I_\beta : \beta \in \lambda\backslash\{\alpha\}\}$. For this, it is helpful to have classes K clsoed under sums, which is defined and investigated in this subsection.

The definition below (variants of closure under sums) are satisfied by the cases we shall deal with and enable us to translate results e.g. from the full (strong) $(\lambda,\lambda,\mu,\kappa)$-bigness to the (strong) $(2^\lambda,\lambda,\mu,\kappa)$-bigness.
Of course:

Definition 2.16. 1) We say that the class K of τ-structures; with τ a relational vocabulary for transparency, is closed under sums when for every sequence $\langle I_s : s \in S\rangle$ of members of K, pairwise disjoint for simplicity, also I belongs to K where I is the τ- structure which is the union of $\langle I_s : s \in S\rangle$; that is the set of elements of I is the union of the sets of elements of I_s for $s \in S$ and $P^I = \cup\{P^{I_s} : s \in S\}$ for every predicate P from τ.

To deal with more general cases

Definition 2.17. 1) Let τ be a vocabulary with no individual constant and no function symbols or with function symbols being interpreted as partial functions (so $(\exists y)(F(\bar{x}) = y)$ is really a predicate).

For τ-models M_s for $s \in S$ not necessarily pairwise disjoint, $M = \sum\limits_{s\in S} M_s$ is defined by:

(a) M is a τ-model
(b) the universe of M is $\{(s,a) : s \in S \text{ and } a \in M\}$
(c) for a predicate $P \in \tau, P^M = \cup\{P^{M_s} : s \in S\}$
(d) similarly for function symbols.

2) We define sums for a class K of τ_K-models with τ_K with individual constants but only when $M{\restriction}c\ell(\{\emptyset\})$ for $M \in K$ are pairwise isomorphic. That is, defining $M = \sum\limits_{s\in S} M_s$ we identify $(s,a),(t,b)$ where $a = \sigma^{M_s}, b = \sigma^{M_t}$ and σ is a term of τ with no free variables.

But in many cases which interest us, this is only almost true, hence we define:

Definition 2.18. 1) We say that K is almost (μ,κ)-closed under sums for λ and ψ where $\psi = \psi(\bar{x},\bar{y}), \ell g(\bar{x}) = \ell g(\bar{y})$, when for every $I_\alpha \in K$ (for $\alpha < \alpha_0 \le \lambda$), I_α of cardinality $\le \lambda$, there are $J, g, h_\alpha (\alpha < \alpha_0)$ such that:

(a) $J \in K$ and $|J| \le \lambda$,
(b) $h_\alpha : I_\alpha \longrightarrow J$, and for any $x_0,\dots,y_0,\dots \in I_\alpha, I_\alpha \models \psi[\langle x_0,\dots\rangle,\langle y_0,\dots\rangle]$ implies $J \models \psi[\langle h_\alpha(x_0),\dots\rangle,\langle h_\alpha(y_0),\dots\rangle]$,
(c) $g : J \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum\limits_{\alpha<\alpha_0} I_\alpha)$ satisfies, for any $\gamma < \kappa, \bar{x},\bar{y} \in {}^\gamma J$,

$\boxdot_0$ if $g(\bar{x}) \approx g(\bar{y}) \mod \mathscr{M}_{\mu,\kappa}(\sum\limits_{\alpha<\alpha_0} I_\alpha)$ then $\bar{x} \approx \bar{y} \mod \mathscr{M}_{\mu,\kappa}(J)$.

2) We replace "almost" by "semi", when in clause (c) above we weaken $\boxdot_0$ to:

$\boxdot_1$ if $g(\bar{x}) \approx g(\bar{y}) \mod (\mathscr{M}_{\mu,\kappa}(\sum\limits_{\alpha<\alpha_0} I_\alpha), R)$ then $\bar{x} \approx \bar{y} \mod \mathscr{M}_{\mu,\kappa}(J)$, where we define

$$R = \{\langle\langle i,\eta\rangle,\langle j,\nu\rangle\rangle : \eta \in I_i, \nu \in I_j \text{ and } i < j\} \subseteq (\sum_{\alpha<\alpha_0} I_\alpha) \times (\sum_{\alpha<\alpha_0} I_\alpha).$$

3) We add "strongly" to "close" in part (1) when we strengthen clause (c) to:

(c)$^+$ $g : J \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha)$ such that for any well ordering $<_0$ of $\mathscr{M}_{\mu,\kappa}(J)$ (as in 2.1(d)), there is a well ordering $<_1$ of $\mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha)$ such that: for any $\gamma < \kappa$ and $\bar{x}, \bar{y} \in {}^\gamma J$ and $A \subseteq J$ of cardinality $< \kappa$,

$\boxdot_2$ if $g(\bar{x}) \approx g(\bar{y}) \mod (\mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha), <_1)$,

<u>then</u> $\bar{x} \approx \bar{y} \mod (\mathscr{M}_{\mu,\kappa}(J), <_0)$.

4) We add strongly in part (2) <u>iff</u> we strengthen (c) to (c)$^+$, only using $(\mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha), <_1, R)$.
5) We may omit "(μ, κ)" above if $\mathrm{Rang}(g) \subseteq J$.
6) We say that K is essentially closed under sums for λ <u>iff</u> in part (1) in addition, $\mathrm{Rang}(h_\alpha)$, $\mathrm{Rang}(g)$ are unions of equivalence classes of (R is from part (2))

$$\approx \mod J, \qquad \approx \mod (\sum_{\alpha<\alpha_0} I_\alpha, R), \qquad \text{respectively.}$$

Remark 2.19. We could have made, for example $h_\alpha : I_\alpha \longrightarrow \mathscr{M}_{\mu,\kappa}(J)$, or in the definition of sum expand by R, without serious changes in the paper.

The following claim gives the obvious properites of "closure under sums" its holding for the classes we are mainly interested in and the use of closure similar for implications among bigness properties.

Claim 2.20. *0) "K is closed under sums" implies "K is essentially closed under sums", which implies "K is almost closed under sums", which implies "K is almost (μ, κ)-closed under sums". If $\mu_1 \leq \mu_2, \kappa_1 \leq \kappa_2$ then "K is almost (μ_1, κ_1)-closed under sums" implies "K is (μ_2, κ_2)-closed under sums".*

In all above implications we can add "strongly" to both sides (when relevant, related).
1) If K is closed under sums, <u>then</u> the full (strong) $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property implies the (strong) $(\chi_1, \lambda, \mu, \kappa) - \psi$-bigness property, where $\chi_1 = \min\{2^\chi, 2^\lambda\}$.
2) In (1), instead of "K closed under sums" it is enough to assume that K is (strongly) almost closed under sums for λ, ψ.
3) The classes defined in 1.9 above $K^\kappa_{\mathrm{tr}}, K_{\mathrm{or}}$ are almost closed under sums and almost strongly closed under sums.
4) The relations defined in 2.2(2), (3), (6) have obvious monotonicity properties in χ, μ, κ; and for all our K, for λ too. For example

$$\chi \leq \chi' \Rightarrow [(\chi', \lambda, \mu, \kappa)\text{-bigness} \Rightarrow (\chi, \lambda, \mu, \kappa)\text{-bigness}]$$

$$\mu \leq \mu' \& \kappa \leq \kappa' \Rightarrow [(\chi, \lambda, \mu', \kappa')\text{-bigness} \Rightarrow (\chi, \lambda, \mu, \kappa)\text{-bigness}].$$

Proof. 0) Obvious.
1) So we assume K has the full $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property. Without loss of generality $\langle I_\alpha : \alpha < \chi \rangle$ are pairwise disjoint.

As K has the [strong] full $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property, there are $I_\alpha \in K$ (for $\alpha < \chi$), each of cardinality λ, such that I_α is ψ-unembeddable intao $\sum_{\beta \neq \alpha} I_\beta$.

<u>Case 1</u>: $\chi \leq \lambda$.

For $U \subseteq \chi$ let $J_U = \sum_{\alpha \in U} I_\alpha$. Let $\mathscr{P}$ be a collection of subsets of χ such that $|\mathscr{P}| = 2^\chi$ and $U \neq V \in \mathscr{P} \Rightarrow U \nsubseteq V$. Suppose $U, V \in \mathscr{P}, f : J_U \longrightarrow M(J_V)$. Choose $\alpha \in U \setminus V$. Thus $f{\restriction}I_\alpha : I_\alpha \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum_{\beta \neq \alpha} I_\beta)$ and the desired conclusion follows.

<u>Case 2</u>: $\lambda < \chi$.

Take a family $\mathscr{W}$ of subsets of λ, each of cardinality λ, such that

$$U \neq V \in H \Rightarrow U \nsubseteq V$$

and proceed as in Case 1.

2) As K has the [strong] full $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property, there are $I_\alpha \in K$ (for $\alpha < \chi$), each of cardinality λ, such that I_α is ψ-unembeddable into $\sum_{\beta \neq \alpha} I_\beta$. By the assumption of (2) (that K is almost (strongly) closed under sums) for every $U \subseteq \chi, |U| \leq \lambda$ let J_U, g_U, h^U_α $(\alpha \in U)$ satisfy clauses (a), (b), (c) of Definition 2.18(1) for $\sum_{\alpha \in U} I_\alpha$. As in the proof of (1), it suffices to show:

- $(*)$ if $U, V \subseteq \chi, |U| \leq \lambda, |V| \leq \lambda, U \setminus V \neq \emptyset$ and $f : J_U \longrightarrow \mathscr{M}_{\mu,\kappa}(J_V)$, <u>then</u> for some $\bar{a}, \bar{b} \in {}^{\ell g(\bar{x})}(J_U), J_U \models \psi[\bar{a}, \bar{b}]$ and $f(\bar{a}) \approx_A f(\bar{b}) \mod \mathscr{M}_{\mu,\kappa}(J_V)$; or mod $(\mathscr{M}_{\mu,\kappa}(J_V), <)$ for the strong version.

Choose $\alpha \in U \setminus V$.

In the strong case let $<_0$ be a well ordering of $\mathscr{M}_{\mu,\kappa}(J_V)$ (as in 2.1(d),2.18(3)); choose a well ordering $<_1$ of $\mathscr{M}_{\mu,\kappa}(\sum_{\alpha < \alpha_0} I_\alpha)$ as guaranteed by Definition 2.18(3); in the non-strong case let $<_0$, $<_1$ be the empty relations.

Now define

$$g^*_V : \mathscr{M}_{\mu,\kappa}(J_V) \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum_{i \in V} I_i)$$

by

$$g^*_V(\tau(x_0, \ldots)) = \tau(g_V(x_0), \ldots).$$

Consider the sequence of mappings:

$$I_\alpha \underset{h^U_\alpha}{\longrightarrow} J_U \underset{f}{\longrightarrow} \mathscr{M}_{\mu,\kappa}(J_V) \underset{g^*_V}{\longrightarrow} \mathscr{M}_{\mu,\kappa}(\sum_{i \in V} I_i).$$

So $g^*_V \circ f \circ h^U_\alpha : I_\alpha \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum_{i \in V} I_i)$. As $\sum_{i \in V} I_i$ is a submodel of $\sum_{i \neq \alpha} I_i$, also without loss of generality $\mathscr{M}_{\mu,\kappa}(\sum_{i \in V} I_i)$ is a submodel of $\mathscr{M}_{\mu,\kappa}(\sum_{i \neq \alpha} I_i)$. But we know that I_α is ψ-unembeddable into $\sum_{i \neq \alpha} I_i$. Hence there are $\bar{x}, \bar{y} \in I_\alpha$ such that:

- (i) $I_\alpha \models \psi[\bar{x}, \bar{y}]$,
- (ii) $g^*_V \circ f \circ h^U_\alpha(\bar{x}) \approx g^*_V \circ f \circ h^U_\alpha(\bar{y}) \mod (\mathscr{M}_{\mu,\kappa}(\sum_{i \in V} I_i), <_1)$.

By (i) and clause (b) from 2.18(1),

(iii) $J_U \models \psi[\bar{x}', \bar{y}']$, where $\bar{x}' = h^U_\alpha(\bar{x}), \bar{y}' = h^U_\alpha(\bar{y})$.

By (ii) and the definition of $\bar{x}', \bar{y}'$,

(iv) $g^*_V(f(\bar{x}')) \approx g^*_V(f(\bar{y}')) \mod (\mathscr{M}_{\mu,\kappa}(\sum_{i\in V} I_i), <_1)$.

By (iv), clause (c) of 2.18(1) or clause $(c)^+$ of 2.18(3), the definition of $\mathscr{M}_{\mu,\kappa}(\sum_{i\in V} I_i)$, and of g^*_V,

(v) $f(\bar{x}') \approx f(\bar{y}') \mod (\mathscr{M}_{\mu,\kappa}(J_V), <_0)$.

So we have proved $(*)$ (by (iii) and (v)), which suffices.
3)-6) Left to the reader. $\square_{2.20}$

Claim 2.21. *The following classes are almost (and also semi) (μ,κ)-closed under sums for λ*

- (a) $K_{\rm or}$ *(the class linear orders)*
- (b) $K^\omega_{\rm tr}$ *(trees with $\omega+1$ levels)*
- (c) $K^\kappa_{\rm tr}$ *(trees with $\kappa+1$ levels)*
- (d) $K_{\rm org}$ *(ordered graphs).*

Proof. Case (a)
If $\langle I_\alpha : \alpha < \alpha_0\rangle$ is a sequence of linear orders then we let:

- (i) $J = \cup\{\{\alpha\} \times I_\alpha : \alpha < \alpha_0\}$
- (ii) $(\alpha_1, t_1) <_J (\alpha_2, t_2)$ <u>if and only if</u> $\alpha_1 < \alpha_2 \vee (\alpha_1 = \alpha_2 \,\&\, t_1 <_{I_{\alpha_1}} t_2)$
- (iii) $h_\alpha : I_\alpha \to J$ is $h_\alpha(t) = (\alpha, t)$
- (iv) $g : J \to \sum_{\alpha<\alpha_0} I_\alpha$ is the identity.

Now check

Case (b):
Given $\langle I_\alpha : \alpha < \alpha_0\rangle$ the unique we identify the member of $P^{J_\alpha}_0$ for $\alpha < \alpha_0$ but make then otherwise disjoint and take the union.

Case (c):
Similar to case (b).

Case (d):
Similar to case (a). $\square_{2.21}$

Another way to present those matters is to do it around the following definition and claim. That is, we note (in 2.22, 2.23 another sufficient condition for implications of the form "if I is unembeddable into J_2 then it is unembeddable into J_1").

Definition 2.22. 1) We say that J_2 does (μ,κ)-dominate J_1 <u>when</u> there is a function g from $\mathscr{M}_{\mu,\kappa}(J_1)$ into $\mathscr{M}_{\mu,\kappa}(J_2)$ such that: if $\xi < \kappa$ and $\bar{a}, \bar{b} \in {}^\xi(\mathscr{M}_{\mu,\kappa}(J_1))$ and $g(\bar{a}) \cong g(\bar{b}) \mod \mathscr{M}_{\mu,\kappa}(J_2)$ then $\bar{a} \cong \bar{b} \mod \mathscr{M}_{\mu,\kappa}(J_1)$.
2) We say that J_2 strongly (μ,κ)-dominate J_1 <u>when</u>:

- if $<_2$ is a well ordering of $\mathscr{M}_{\mu,\kappa}(J_2)$ then there is a well ordering $<_1$ of $\mathscr{M}_{\mu,\kappa}(J_1)$ satisfying:uch that there is a function g from $\mathscr{M}_{\mu,\kappa}(J_1)$ into $\mathscr{M}_{\mu,\kappa}(J_2)$ satisfying: if $\xi < \kappa$ and $\bar{a}, \bar{b} \in {}^{\xi}(\mathscr{M}_{\mu,\kappa}(J_1))$ and $g(\bar{a}) \cong g(\bar{b}) \mod (\mathscr{M}_{\mu,\kappa}(J_2), <_2)$ then $\bar{a} \cong \bar{b} \mod (\mathscr{M}_{\mu,\kappa}(J_1, <_1))$.

3) We say J_1, J_2 are [strongly] (μ, κ)-equivalent <u>when</u> J_2 [strongly] dominate J_1 and vice versa.

Claim 2.23. *If I is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable into J_2 and J_2 [strongly] (μ, κ)-dominate J_1 <u>then</u> I is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable into J_1.*

Proof. First, apply Definition 2.22 without the "strong"; we have to prove that I is $\varphi(\bar{x}, \bar{y})$-unembeddable into J_2 for $\tau_{\mu,\kappa}$. So assume that $f_1 : I \to \mathscr{M}_{\mu,\kappa}(J_1)$ and we should find $\bar{a}, \bar{b} \in {}^{\ell g(\bar{x})}I$ as in Definition 2.2(1). By the assumption of 2.23 there is a function $g : \mathscr{M}_{\mu,\kappa}(J_1) \to \mathscr{M}_{\mu,\kappa}(J_2)$ as in Definition 2.22(1). So $f_2 := g \circ f_1$ is a well defined function from I into $\mathscr{M}_{\mu,\kappa}(J_2)$, so recalling that we know "I is $\varphi(\bar{x}, \bar{y})$-unembeddable into J_2 for $\tau_{\mu,\kappa}$", it follows that there are sequences $\bar{s}, \bar{t}$ from ${}^{\ell g(\bar{x})}I$ such that:

- $\bullet_1$ $f_2(\bar{s}), f_2(\bar{t})$ are similar in $\mathscr{M}_{\mu,\kappa}(J_2)$
- $\bullet_2$ $I \models \varphi[\bar{s}, \bar{t}]$.

We now apply our assumption "$J_2, (\mu, \kappa)$-dominate J_1 as exemplified by the function g" to the sequences $f_1(\bar{s}), f_2(t) \in {}^{\ell g(\bar{x})}(\mathscr{M}_{\mu,\kappa}(J_1))$. The assumption "$g(f_1(\bar{s})) \cong g(f_2(\bar{t})) \mod \mathscr{M}_{\mu,\kappa}(J_2)$" holds by the choice of $\bar{s}, \bar{t}$. Hence the conclusion in 2.22 holds which means that $f_1(\bar{s}) \cong f_1(\bar{t}) \mod \mathscr{M}_{\mu,\kappa}(J_1)$ so $\bar{s}, \bar{t}$ are as required.

The proof of the strong version is similar. $\square_{2.23}$

§ 2(F). Back to linear orders.

As we have remarked in the introduction to this paper, results on trees can be translated to results on linear orders; this is done seriously in [Shel]. Originally this was neglected as the results on unsuperstable T (and trees with $\omega + 1$ levels) give the results on unstable theories (and linear orders). Anyhow, now we deal with the simplest case parallel to [She90, Ch.VIII,2.1], see more in [Sheb].

Definition 2.24. 1) For any $I \in K^{\kappa}_{\mathrm{tr}}$ we define $\mathbf{or}(I)$ as the following linear order (See Def 1.11(4)).

<u>set of elements</u> is chosen as $\{(t, \ell) : \ell \in \{1, -1\}, t \in I\}$

<u>the order</u> is defined by $(t_1, \ell_1) < (t_2, \ell_2)$ if and only if $t_1 \triangleleft t_2 \wedge \ell_1 = 1$ or $t_2 \triangleleft t_1 \wedge \ell_2 = -1$ or $t_1 = t_2 \wedge \ell_1 = -1 \wedge \ell_2 = 1$ or $t_1 <_{\mathrm{lx}} t_2 \wedge (t_1, t_2$ are $\triangleleft$-incomparable).

2) Let $\varphi_{\mathrm{or}} = \varphi_{\mathrm{or}}(x_0, x_1; y_0, y_1)$ be the formula $x_0 < x_1 \wedge y_1 < y_0$.

3) Let $\varphi^{\kappa}_{\mathrm{tr}} = \varphi^{\kappa}_{\mathrm{tr}}(x_0, x_1; y_0, y_1)$ be (this is for K^{κ}_{tr}, for $\kappa = \aleph_0$ see example 2.6)

$$\begin{aligned}\varphi_{\mathrm{tr}}(x_0, x_1 : y_0, y_1) := \quad & [x_0 = y_0] \text{ and } P_\kappa(x_0) \wedge \bigvee_{\epsilon<\kappa} [P_{\epsilon+1}(x_1) \wedge P_{\epsilon+1}(y_1) \wedge \\ & P_\epsilon(x_1 \cap y_1)] \wedge [x_1 \triangleleft x_0 \wedge \neg(y_1 \triangleleft y_0)] \text{ and } y_1 <_{\mathrm{lx}} x_1].\end{aligned}$$

Claim 2.25. *1) Assume that $I, J \in K^{\kappa}_{\mathrm{tr}}$*

(a) *If I is strongly $\varphi^{\kappa}_{\mathrm{tr}}$-unembeddable for $\tau_{\mu,\kappa}$ into J <u>then</u> $\mathbf{or}(I)$ is strongly $\varphi_{\mathbf{or}}$-unembeddable for $\tau_{\mu,\kappa}$ into $\mathbf{or}(J)$*

(b) *similarly without "strongly".*

2) If $K^\kappa_{\rm tr}$ has the strong $(\chi,\lambda,\mu,\kappa)$-bigness property <u>then</u> $K_{\rm or}$ has the strong $(\chi,\lambda,\mu,\kappa)$-bigness property.
3) In part (2) we may add "full" and/or omit "strong" in the assumption and the conclusion.

Proof. The main point is that:

(∗) if $I \models \varphi^\kappa_{\rm tr}(x_0,x_1;y_0,y_1)$ then **or** $(I) \models \varphi((x_0,1),(x_1,1);(y_0,1),(y_1,1))$.

But (∗) is easy to verify. $\square_{2.25}$

Remark 2.26. 1) We deal mainly with $K^\omega_{\rm tr}$, recall 2.7, i.e. see more in [Shea, p2], so by it we know that $K^\omega_{\rm or}$ has the full strong $(\lambda,\lambda,\mu,\aleph_0)$-bigness property when $\mu<\lambda$.
2) For κ regular uncountable, there are parallel results, noting that obviously $K^\kappa_{\rm or}$ has the full strong $(\chi,\lambda,\mu,\kappa)$ when λ is regular $>|\alpha|^{<\kappa}+\mu$ for every $\alpha<\lambda$ and $\lambda\le\chi$.

It seems reasonable to conjecture that the parallel of [Shea, p2(2)] holds, but we have not tried to work on it, see part (3) of the remark.
3) The results below (on $\varphi_{{\rm or},\alpha,\beta,\pi}$) seem to me a natural step but have actually set down to phrase and prove them for Usvyatsov-Shelah [SU].
4) Even for $\kappa=\aleph_0$ we do not deal with λ singular below, it seems reasonable that this, i.e., the parallel of [Shea, §1] holds, but the results below are more than sufficient for its purpose, as for $\chi>\mu$ singular we can use the result here for $(\chi,\lambda,\mu,\kappa)$ for any regular $\lambda\in(\mu,\chi)$.
5) In 2.16 we use α,β well orders.

It seems reasonable that we can say more for a more general case but again this was not required.
6) We use freely the obvious observation 2.27 below (see also 2.21(a)). Note that the "essentially" version dealt with in 2.27 was not covered by 2.21.

Observation 2.27. *1) $K_{\rm or}$ is essentially closed under sums for λ and $\varphi_{\rm or}$, recalling Definitions 2.18(6), 2.21.*
2) Similar for $\varphi_{{\rm or},\alpha,\beta,\pi}$ defined below.

We have seen above how from a "complicated" sequence of members of $K^\omega_{\rm tr}$ we can derive one of the members of $K_{\bf or}$. But this does not indicate that linear order can be reduced to this case. We know that we can derive linear orders from members of $K^\kappa_{\rm tr}$ for each $\kappa>\aleph_0$, but clearly the class $K^\kappa_{\rm tr}$ is more complicated than the class $K^\omega_{\rm tr}$. Anyhow we have suggested a way to express in our framework that $K_{\rm Pr}$ is complicated, instead of $\varphi_{\bf or}(\bar x,\bar y)$ saying $x_0<x_1\wedge y_2<y_0$, below we use a possibly infinite $\bar x$ and φ says $\bar x=\langle x_i : i<\alpha\rangle$ is increasing, $\langle y_{\bar y(i)} : i<\alpha\rangle$ is increasing, where π is a permutation of α.

Note that 2.30 implies the results of $K^\omega_{\rm tr}$ for regular $\lambda\ge\mu$.

Definition 2.28. We define the following (quantifier free infinitary) formulas for the vocabulary $\{<\}$. For any ordinal α,β and a one-to-one function π from α onto β, and we let $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar x,\bar y)$ where $\bar x=\bar x^\alpha=\langle x_i : i<\alpha\rangle$ and $\bar y=\bar y^\alpha=\langle y_i : i<\alpha\rangle$, be

$$\bigwedge\{x_i<x_j : i<j<\alpha\} \text{ and } \bigwedge\{y_i<y_j : i,j<\alpha \text{ and } \pi(i)<\pi(j)\}.$$

Claim 2.29. *Assume $\chi \ge \lambda = \mathrm{cf}(\lambda) > \mu^{<\kappa}, \kappa = \mathrm{cf}(\kappa)$ and $\gamma < \lambda \Rightarrow |\gamma|^{<\kappa} < \lambda$.*
1) For (α, β, π) as in Definition 2.28, such that $\alpha, \beta \le \lambda$, the class K_{or} has the full strong $(\lambda, \chi, \mu, \kappa)$-bigness property for $\varphi_{\mathrm{or},\alpha,\beta,\pi}(\bar{x}, \bar{y})$.
2) For (α, β, π) as in Definition 2.28 such that $\alpha, \beta \le \lambda$, the class K_{or} has the strong $(2^\lambda, \chi, \mu, \kappa)$ bigness property for $\varphi_{\mathrm{or},\alpha,\beta,\pi}$.
3) In fact in both part (1) and (2) we can find examples which satisfies the conclusion for all triples (α, β, π) as there simultaneously.

Proof. 1) By 2.30 below because there are λ pairwise disjoint stationary sets $S \subseteq S^\lambda_{\aleph_0}$.
2) By part (1) and 2.27(1) and 2.20(1).
3) Check the proof. $\square_{2.29}$

Claim 2.30. *Assume $\kappa = \mathrm{cf}(\kappa) \le \mu, \mu^{<\kappa} < \lambda = \mathrm{cf}(\lambda) \le \lambda_1, \kappa \le \partial = \mathrm{cf}(\partial) < \lambda$ and $\gamma < \lambda \Rightarrow |\gamma|^{<\kappa} < \lambda$.*

If $I, J \in K^\kappa_{\mathrm{or}}$ satisfies $\circledast$ below and $\alpha_*, \beta_* \le \lambda$ and π is a one-to-one function from α_* onto α_* <u>then</u> (recalling Definition 2.24) $\mathbf{or}(I)$ is strongly $\varphi_{\mathrm{or},\alpha_*,\beta_*,\pi}(\bar{x}^{\alpha_*}, \bar{y}^{\alpha_*})$-unembeddable for (μ, κ) into $\mathbf{or}(J)$ where:

$\circledast$ (a) *$S_1, S_2 \subseteq S^\lambda_\partial$ such that $S_1 \setminus S_2$ is a stationary subset of λ*
(b) *$\bar{\eta} = \langle \eta_\delta : \delta \in S_1 \cup S_2 \rangle$ where η_δ is an increasing sequence of ordinals $< \delta$ with limit δ of length ∂*
(c) *for every $\alpha < \lambda$ the set $\{\eta_\delta \restriction i : \delta \in S, i < \partial$ and $\sup \mathrm{Rang}(\eta_\delta \restriction i) \le \alpha\}$ has cardinality $< \lambda$; actually follows*
(d) *$I \in K^\kappa_{\mathrm{tr}}$ is $\{\eta_\delta \restriction i : i \le \partial, \delta \in S_1\} \cup \{\langle \alpha \rangle : \alpha < \lambda_1\}$*
(e) *$J \in K^\kappa_{\mathrm{tr}}$ is $\{\eta_\delta \restriction i : i \le \partial, \delta \in S_1\} \cup \{\langle \alpha \rangle : \alpha < \lambda_1\}$.*

Proof. By 2.23 it is enough to prove that $\mathrm{or}(I)$ is strongly $\varphi_{\mathrm{or},\alpha^*,\pi}(\bar{x}^{\alpha_*}, \bar{y}^{\alpha_*})$-unembeddable into $\mathscr{M}_{\mu,\kappa}(J)$.

So let f be a function from $\mathbf{or}(I)$ into $\mathscr{M}_{\mu,\kappa}(\mathbf{or}(J))$ so actually a function from $I \times \{1, -1\}$ into $\mathscr{M}_{\mu,\kappa}(J)$, and $<_*$ a well ordering of $\mathscr{M}_{\mu,\kappa}(J)$ but we "forget" to deal with it, as there are no problems, and let χ be large enough. Let $\bar{N} = \langle N_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence of elementary submodels of $(\mathscr{H}(\chi), \in)$ such that $I, J, \lambda, \bar{\eta}, \mathscr{M}_{\mu,\kappa}(J \times \{-1, 1\}), f, <_*$ belong to N_0 and $N_\alpha \cap \lambda \in \lambda, \bar{N} \restriction (\alpha + 1) \in N_{\alpha+1}$ for every $\alpha < \lambda$; as it happens "$\alpha_*, \beta_*, \pi \in N_0$" is not needed. So $E := \{\delta < \lambda : N_\delta \cap \lambda = \delta\}$ is club of λ hence we can choose $\delta \in E \cap S_1 \setminus S_2$.

For any $\eta \in I$, clearly $f((\eta, 1))$ is well defined and $\in \mathscr{M}_{\mu,\kappa}(J)$ so let $f((\eta, 1)) = \sigma_\eta(\bar{\nu}_\eta)$ where σ_η is a $\tau_{\mu,\kappa}$-term, $\bar{\nu}_\eta = \langle (\nu_{\eta,\epsilon}, \iota_{\eta,\epsilon}) : \epsilon < \epsilon_\eta \rangle, \nu_{\eta,i} \in J$ and $\iota_{\eta,\epsilon} \in \{1, -1\}, \epsilon_\eta < \kappa$.

Let $\epsilon_* = \epsilon_{\eta_\delta}, \iota_\epsilon = \iota_{\eta_\delta,\epsilon}, i^*_\epsilon = \lg(\nu_{\eta_\delta,\epsilon})$, so $i^*_\varepsilon \le \partial$ for $\epsilon < \epsilon_*$ and let $j^*_\epsilon = \sup\{j \le i^*_\epsilon : \sup \mathrm{Rang}(\nu_{\eta_\delta,\epsilon} \restriction j) < \delta\}$. By our assumption $j^*_\epsilon = \partial$ implies that $i_\epsilon = \partial$ hence as $\delta \notin S_2$ it follows that $\sup \mathrm{Rang}(\nu_{\eta_\delta,\epsilon} \restriction j^*_\varepsilon) < \delta$ hence by clause (c) of the assumption $\nu_{\eta_\delta,\epsilon} \restriction j^*_\varepsilon \in N_\delta$. Also $\alpha < \delta \Rightarrow J \cap {}^{\kappa>}\alpha \subseteq N_{\alpha+1}$ because $N_\delta \cap \lambda \in \lambda$, it has cardinality $< \lambda$ and it belongs to $N_{\alpha+1}$; also let $\nu^*_\epsilon = \nu_{\eta_\delta,\epsilon} \restriction j^*_\epsilon$, it too belongs to N_δ.

So $\{\nu^*_\epsilon : \epsilon < \epsilon_*\} \subseteq N_\delta$, and it has cardinality $< \kappa$ as $\alpha < \lambda \to |\alpha|^{<\kappa} < \lambda$ and $\mathrm{cf}(\delta) = \partial \ge \kappa$ it follows that $\bar{\nu}^* = \langle \nu^*_\epsilon : \epsilon < \epsilon_* \rangle \in N_\delta$.

Let $u_* = \{\epsilon < \epsilon_* : j^*_\epsilon < i^*_\epsilon\}$. For $\epsilon \in u_*$ let $\alpha^*_\epsilon = \min(N_\delta \cap (\lambda + 1) \setminus \nu_{\eta_\delta,\epsilon}(j^*_\epsilon))$, so as above also $\bar{\alpha}^* := \langle \alpha_\epsilon : \epsilon \in u_* \rangle$ belongs to N_δ.

Now for $\eta \in {}^{\partial>}\lambda$ we define $\mathscr{U}_\eta$ as the set of $\beta \in S_1$ such that:

$(*)_{\eta,\beta}$ (a) $\eta \triangleleft \eta_\beta$
(b) $\sigma_{\eta_\beta} = \sigma_*$ so $\epsilon_{\eta_\beta} = \epsilon_*$
(c) $\lg(\nu_{\eta_\beta,\epsilon}) = i^*_\epsilon$ for $\epsilon < \epsilon_*$
(d) $\nu_{\eta_\beta,\epsilon} \restriction j^*_\epsilon = \nu^*_\epsilon$ for $\epsilon < \epsilon_*$
(e) $\iota_{\eta_\beta,\epsilon} = \iota_\epsilon$ for $\epsilon < \epsilon_*$

Note

⊛ if $\eta \triangleleft \eta_\delta$ then
(a) $\delta \in \mathscr{U}_\eta$ and $\mathscr{U}_\eta \in N_\delta$
(b) $\mathrm{cf}(\alpha^*_\epsilon) = \lambda$ for $\epsilon \in u_*$
(c) if $\bar\alpha \in \prod\limits_{\epsilon \in u_*} \alpha^*_\epsilon$ <u>then</u> for arbitrarily large $\beta \in \mathscr{U}_\eta$ we have $\epsilon \in u_* \Rightarrow \nu_{\eta_\beta,\epsilon}(j^*_\epsilon) \in (\alpha_\epsilon, \alpha^*_\epsilon)$
(d) $\mathscr{U}_\eta$ is an unbounded subset of S_1.

[Why? Clause (a) directly. Why clause (b)? Clearly $\nu_{\delta,\varepsilon}(j^*_\varepsilon) \ge \delta$ hence $\nu_{\delta,\varepsilon}(j^*_\varepsilon) \in \lambda \backslash N_\delta$ but $N_\delta \cap \lambda \in \lambda$ hence $\mathrm{cf}(\alpha^*_\varepsilon) = \lambda$ follows. Why clause (d)? Otherwise $\sup(\mathscr{U}_\eta)$ is $< \lambda$ and it belongs to N_δ because $\mathscr{U}_\eta \in N_\delta$, hence $\sup(\mathscr{U}_\eta) \in N_\delta \cap \delta$ so $\sup(\mathscr{U}_\eta) < \delta$ contradicting clause (a). Clause (c) is proved similarly.]

Next let Λ be the set of $\eta \in {}^{\partial >}\lambda$ such that

$\odot_\eta$ for every $\bar\alpha \in \prod\limits_{\epsilon \in u_*} \alpha^*_\epsilon$ there is $\beta \in \mathscr{U}_\eta$ such that $\epsilon \in u_* \Rightarrow \nu_{\eta_\beta,\epsilon}(j^*_\epsilon) \in (\alpha_\epsilon, \alpha^*_\epsilon)$.

So

$(*)_1$ $\eta_1 \triangleleft \eta_2 \wedge \eta_2 \in \Lambda \Rightarrow \eta_1 \in \Lambda$
$(*)_2$ $\epsilon < \kappa \Rightarrow \eta_\delta \restriction \epsilon \in \Lambda$.

Hence

$(*)_3$ for some $\eta_* \in \Lambda$ the set $\mathscr{W} = \{\gamma < \lambda : \eta_*\hat{\ }\langle\gamma\rangle \in \Lambda\}$ is an unbounded subset of λ.

Let $\langle \gamma_\zeta : \zeta < \lambda\rangle$ list $\mathscr{W}$ in increasing order, and let $\alpha_*, \beta_* \le \lambda$ and π be a one-to-one function from α_* onto β_*.

Now first we choose $\delta(1,\zeta) \in S_1$ by induction on $\zeta < \alpha_*$ such that:

$(*)_4$ (a) $\delta(1,\zeta) \in \mathscr{U}_{\eta_*\hat{\ }\langle\gamma_\zeta\rangle}$ i.e. $\gamma_\zeta \in \mathscr{W}$
(b) if $\epsilon \in u_*$ then $\nu_{\eta_{\delta(1,\zeta)},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \mathrm{sub}\{\nu_{\eta_{\delta(1,\xi)},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

This is easy.

Second we choose $\delta(2,\zeta) \in S_1$ by induction on $\zeta < \beta_*$ such that:

$(*)_5$ (a) $\delta(2,\zeta) \in \mathscr{U}_{\eta_*\hat{\ }\langle\gamma_\xi\rangle}$ when $\pi(\xi) = \zeta$
(b) if $\epsilon \in u_*$ then $\nu_{\eta_{\delta(2,\zeta)},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \sup\{\nu_{\eta_{\delta(2,\xi)},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

Let $\bar a = \langle a_\zeta : \zeta < \alpha\rangle$, $\bar b = \langle b_\zeta : \zeta < \alpha\rangle$ from ${}^\alpha I$ be chosen as follows: $a_\zeta = (\eta_{\delta(1,\zeta)}, 1), b_\zeta = (\eta_{\delta(1,\pi(\zeta))}, 1)$ for $\zeta < \alpha$.

Now check, e.g.:

$(*)_6$ $a_{\zeta(1)} <_{\mathrm{or}(I)} a_{\zeta(2)}$ <u>iff</u> $\gamma_{\zeta(1)} < \gamma_{\zeta(2)}$ <u>iff</u> $\zeta(1) < \zeta(2)$

$(*)_7$ $b_{\zeta(1)} <_{\mathrm{or}(I)} b_{\zeta(2)}$ iff $\gamma_{\pi(\zeta)(1)} < \gamma_{\pi(\zeta)(2)}$ iff $\pi(\zeta)(1) < \pi(\zeta)(2)$.

$\square_{2.30}$

§ 3. Order Implies Many Non-Isomorphic Models

In this section (in a self contained way) we prove that not only the old result that any unstable (first order) T has in any $\lambda \geq |T| + \aleph_1$, the maximal number ($2^\lambda$) of pairwise non-isomorphic models holds, but for example that for any template Φ proper for linear orders, if the formula $\varphi(\bar{x}, \bar{y})$ with vocabulary τ, linearly orders $\{\bar{a}_s : s \in I\}$ in $\mathrm{EM}_\tau(I, \Phi)$ (Ehrenfeucht-Mostowski model, see §1) for every I, then the number of non-isomorphic models of the form $\mathrm{EM}_\tau(I, \Phi)$ of cardinality λ up to isomorphism is 2^λ when $\lambda \geq |\tau_\Phi| + \aleph_1$.

Dealing with this problem previously, the author (in the first attempt [She71b]) excluded some of the cardinals λ which satisfy $\lambda = |\tau_\Phi| + \aleph_1$ and in the second [She90, Ch.VIII,§3], replaced the $\mathrm{EM}_\tau(I, \Phi)$ with some kind of restricted ultrapower (of itself). Subsequently (in [She80]) we proved that for some unsuperstable first order complete theory T, and a first order theory T_1 extending T, $|T_1| = \aleph_1$, $|T| = \aleph_0$ the class

$$\mathrm{PC}(T_1, T) = \{M{\restriction}\tau(T) : M \models T_1\}$$

may be categorical in $\aleph_1$, "may be categorical" mean that some forcing extension this holds for some T, T_1; in fact if the original universe $\mathbf{V}$ satisfies CH, we may choose T, T_1 in $\mathbf{V}$.

We also prove there for $T =$ the theory of dense linear order, that we may, i.e. in some forcing extension, have a universal model in $\aleph_1$ even though CH fails. We then thought that the use of ultrapower in [She90, Ch.VIII,§3] was necessary. This is not true. (We thank Rami Grossberg for a stimulating discussion which directed me to this problem again).

By the present theorem we can get the theorem also for the number of models of $\psi \in \mathbb{L}_{\lambda^+,\aleph_0}$ in λ ($> \aleph_0$) when ψ is unstable. Incidentally the proof is considerably easier (than in [She90, Ch.VIII,§3].

Note that we do not need to demand $\varphi(\bar{x}, \bar{y})$ to be first-order; a formula in any logic is O.K.; it is enough to demand $\varphi(\bar{x}, \bar{y})$ to have a suitable vocabulary. This is because an isomorphism from N onto M preserves satisfaction of such φ and its negation. However, the length of $\bar{x}$ (and $\bar{y}$) is crucial. Naturally we first concentrate on the finite case (in 3.1–3.25). But when we are not assuming this, we can, "almost always" save the result. In first reading, it may be advisable to concentrate on the case "λ is regular", $\varphi = \varphi(x, y)$ an asymmetric formula, I a linear order.

§ 3(A). **Skeleton like sequence and invariants.**

For this section, the notion "$\langle \bar{a}_t : t \in I \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y})$-skeleton like inside M" from Definition 3.1(3),(4) is central and in Definition 3.1 the reader can concentrate on it but it relies on 3.1(1) for the case $\Lambda = \{\varphi(\bar{x}, \bar{y}), \psi(\bar{y}, \bar{x})\}, \dot{\mathbf{J}} = {}^{\ell g(\bar{x})}M = {}^{\ell g(\bar{y})}M$.

Definition 3.1. Let M be a model, I an index model; for $s \in I$, $\bar{a}_s$ is a sequence from M, the length of $\bar{a}_s$ depends on the quantifier-free type of s over $\emptyset$ in I only; Λ is a set of formulas of the form $\varphi(\bar{x}, \bar{a})$, $\bar{a}$ from M, $\varphi = \varphi(\bar{x}, \bar{y})$ a formula which has a vocabulary contained in $\tau(M)$ and $\dot{\mathbf{J}}$ a set of sequences from M.

1) We say that $\langle \bar{a}_s : s \in I \rangle$ is κ-skeleton like inside M for[2] Λ when: for every $\varphi(\bar{x}, \bar{a}) \in \Lambda$, there is $J \subseteq I$, $|J| < \kappa$ such that:

(∗) $\varphi(\bar{x}, \bar{y}) \in \Lambda$, if $s, t \in I$ and $\mathrm{tp}_{\mathrm{qf}}(t, J, I) = \mathrm{tp}_{\mathrm{qf}}(s, J, I)$ and $\ell g(\bar{a}_s) = \ell g(\bar{x})$ then
$$M \models \text{“}\varphi[\bar{a}_s, \bar{a}] \equiv \varphi[\bar{a}_t, \bar{a}]\text{”}.$$

2) Variants:

(a) If $\Lambda = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}_\varphi) \in \Delta \text{ and } \bar{a} \in \dot{\mathbf{J}}\}$ then we may write $(\Delta, \dot{\mathbf{J}})$ instead of Λ

(b) if $\Delta = \{\varphi(\bar{x}, \bar{y})\}$ we may write $\varphi(\bar{x}, \bar{y})$ instead of Δ

(c) if $A \subseteq M$ and
$$\dot{\mathbf{J}} = \{\bar{a} : \bar{a} \text{ is from } A, \text{ and for some } \varphi(\bar{x}, \bar{y}) \in \Delta,\ \ell g(\bar{a}) = \ell g(\bar{y})\}$$
we may write A instead of $\dot{\mathbf{J}}$

(d) if $|M| = A$ we may write M instead A, and we may omit it if clear from the context.

3) Supposing $\psi(\bar{x}, \bar{y}) := \varphi(\bar{y}, \bar{x})$, I a linear order, we say $\langle \bar{a}_s : s \in I \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M for $\dot{\mathbf{J}}$ when: $\varphi(\bar{x}, \bar{y})$ is asymmetric (at least in M) with vocabulary contained in $\tau(M)$, $\ell g(\bar{a}_s) = \ell g(\bar{x}) = \ell g(\bar{y}), \langle \bar{a}_s : s \in I \rangle$ is κ-skeleton like inside M for $(\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}, \dot{\mathbf{J}})$ and for every $s, t \in I$ we have:
$$M \models \varphi[\bar{a}_s, \bar{a}_t] \text{ iff } I \models s < t.$$

4) In (1), (3), if M is clear from the context then we may omit “inside M”. In part (3), if $\dot{\mathbf{J}} = {}^\alpha|M|, \alpha = \ell g(\bar{x}) = \ell g(\bar{y})$ then we may omit it.

Discussion 3.2. Note that Definition 3.1 requires considerably more than “the $\bar{a}_s$ are ordered by φ” and even than “the $\bar{a}_s$ are order indiscernibles, ordered by φ”, but much less than “$M = \mathrm{EM}_\tau(I, \Phi)$”.

We may view Definition 3.1 as follows. An EM models $M_1 = M^1_I$ for a theory T is a model of some theory $T_1 \supseteq T$ with Skolem functions, $|T_1| = |T| + \aleph_0$ such that for some linear order I and elements $a_s \in M_1$ we have:

(∗) (a) M_1 is the Skolem hull of $\{a_s : s \in I\}$
(b) the sequence $\langle a_s : s \in I \rangle$ is an indiscernible sequence in M_1; we may call it “the skeleton”.

So the model of T we are interested in is $M_I = M_1 {\restriction} \tau(T)$; it is natural to assume properties of I are reflected in properties of M_I and so of $M = M_I {\restriction} \tau(M)$. The motivation in the original work [EM56] was understanding the automorphism group of M; the automorphism group of I is naturally embeddded into the automorphism group of M_I hence of M. Anyhow here we are interested in getting 2^λ pairwise non-isomorphic models of cardinality λ.

So naturally we consider:

(c) fixing a formula $\varphi(x, y)$ and pairs $(M, \langle a_s : s \in I \rangle)$ as above such that $M \models \varphi[a_s, a_t]$ iff $s <_I t$

[2]The simplest example is: Λ the set of first order formulas with parameters from M.

(d) a family $\langle I_\alpha : \alpha < 2^\lambda \rangle$ of I's of cardinality λ, pairwise very different.

Furthermore, we would like not to restrict our models to such specific ones. So we look for a definition of "the sequence $\langle a_s : s \in I \rangle$ of elements of M is in some sense like the situation is (b) above". This is the motivation behind Definition 3.1 above and, in fact, (a) + (b) gives an example of it as proved in 3.3 below. Ideally in M_I we can reconstruct I in some sense, i.e. $I/\cong$. While very nice it seemed too much to hope for. So we may try to use "the I_α's are pairwise very different" but, even better, we shall define an invariant $\mathrm{inv}(I)$ of I and reconstruct it to a large extent.

So we may hope to replace "from $M_I/\cong$ we can define $I/\cong$" by "from $M_I/\cong$ we can define $\mathrm{inv}(I)$". We shall actually arrive to something very close to it: if we are given a model M of cardinality λ then there are at most λ invariant $\mathbf{i}$ such that

- for some linear order I and $a_s \in M$ for $s \in I$ such that the pair $(M, \langle a_s : s \in I \rangle)$ is as above (for our fix $\varphi(x,y)$) satisfying $\mathrm{inv}(I) = \mathbf{i}$.

Now if the set $\{\mathrm{inv}(I) : I$ a linear order of cardinality $\lambda\}$ has cardinality $\le \lambda$ all this gives nothing, but if this set has cardinality 2^λ we are done. Note that actually we use $\varphi(\bar{x}_n, \bar{y}_n), \bar{a}_s \in {}^n M$ for some n.

Claim 3.3. *1) Assume Φ is an almost $\mathscr{L}$-nice template proper for linear orders (see Definition 1.8). <u>Then</u> for any linear order I, the sequence $\langle \bar{a}_t : t \in I \rangle$ is $\aleph_0$-skeleton like for $\mathscr{L}$ inside $\mathrm{EM}(I,\Phi)$; in fact, $\mathscr{L}(\tau_\Phi)$ may be any set of formulas in the vocabulary τ_Φ, e.g. $\mathscr{L}(\tau_\Phi)$, first order logic for τ_Φ.*
2) In part (1), if I is $\aleph_0$-homogeneous (i.e., for any $n < \omega$ and $t_0 <_I \ldots <_I t_{n-1}, s_0 <_I \ldots <_I s_{n-1}$, there is an automorphism of I mapping t_ℓ to s_ℓ for $\ell < n$), <u>then</u> we can omit "almost $\mathscr{L}$-nice".

Proof. 1) Let $\varphi = \varphi(\bar{x},\bar{y}) \in \mathscr{L}(\tau_\Phi), \bar{b} \in {}^{\ell g(\bar{y})}M$, so for some finite sequence $\bar{t}$ from I and a sequence $\bar{\sigma}$ of τ_Φ-terms we have $\bar{b} = \bar{\sigma}(\bar{t})$. So if s_1, s_2 realize the same quantifier free type over $\bar{t}$ in I, by indiscernibility (i.e., almost $\mathscr{L}$-niceness) <u>then</u> $\mathrm{EM}(I,\Psi) \models$ "$\varphi[\bar{a}_{s_1},\bar{b}] = \varphi[\bar{a}_{s_2},\bar{b}]$". So $\mathrm{rang}(\bar{t})$ is as required.
2) Should be clear. $\square_{3.3}$

Remark 3.4. 1) Note that part 3.3(1) says that being skeleton-like really is a property of the skeleton of EM-models.
2) Note that 3.3(1) apply to $\mathrm{EM}_\tau(I,\Phi)$ whenever $\tau \subseteq \tau_\Phi$.

We now will proceed to assign invariants to linear orders. We prove that there are enough linear orders with well defined pairwise distinct invariants. This is related to (but does not rely on) proofs from the Appendix to [She78]=[She90], where different terminology was employed. Speaking very roughly, we discussed there only $\mathrm{inv}^\alpha_\kappa$ where $\kappa = \aleph_0$. The assertion in the appendix of [She90] that two linear orders are contradictory corresponds to the assertion here that the invariants are defined and different.

Notation 3.5. In the following, for any regular cardinal $\mu > \aleph_0, \mathscr{D}_\mu$ denotes the filter on μ generated by the closed unbounded sets.
2) If D is a filter on μ and $X \subseteq \mu$ intersects each member of D, then $D + X$ denotes the filter generated by $D \cup \{X\}$.
2A) Similarly $D + \mathscr{A}$ for $\mathscr{A} \subseteq \mathscr{P}(\mu)$.

3) For a linear order $I = (I, <_I)$ the cofinality $\mathrm{cf}(I)$ of I is

$$\mathrm{Min}\{|J| : J \subseteq I \text{ and } (\forall s \in I)(\exists t \in J) I \models s < t\}.$$

4) I^* is the inverse linear order and $\mathrm{cf}^*(I)$ is the cofinality of I^* sometimes called the coinitiality of I.
5) For a linear order I and a cardinal κ, we define a filter on the regular cardinal $\mathrm{cf}(I)$

$$\mathscr{D}(\kappa, I) := \mathscr{D}_{\mathrm{cf}(I)} + \{\delta < \mathrm{cf}(I) : \kappa \leq \mathrm{cf}(\delta)\}.$$

6) For a filter D on λ (here mainly $\lambda = \mathrm{cf}(I)$), two functions f and g from λ to some set X, are equivalent $\mod D$ <u>when</u> $\{\alpha : f(\alpha) = g(\alpha)\} \in D$.
7) We write f/D for the equivalence class of f for this equivalence relations <u>when</u> $f : \lambda \to X$ but we allow $f(\alpha)$ to be undefined for some α's as long as $\{\alpha < \lambda : f(\alpha)$ well defined$\} \in D$.

Definition 3.6. 1) For a regular cardinal κ (for example $\aleph_0$) and an ordinal α we define the invariant $\mathrm{inv}^\alpha_\kappa(I)$ for linear orders I (sometimes undefined), by induction on α, by cases:

<u>Case 1</u>: $\alpha = 0, \mathrm{inv}^\alpha_\kappa(I)$ is the cofinality of I if $\mathrm{cf}(I)$ is $\geq \kappa$, and is undefined otherwise.

<u>Case 2</u>: $\alpha = \beta + 1$.
Let $I = \bigcup\limits_{i<\mathrm{cf}(I)} I_i$, where I_i is increasing and continuous with i and I_i is a proper initial segment of I. For $\delta < \mathrm{cf}(I)$ let $J_\delta = (I \setminus I_\delta)^*$ (recalling X^* denotes the inverse order of X, recalling 3.5(4)).
If $\mathrm{cf}(I) > \kappa$ and for some club $\mathscr{C}$ of $\mathrm{cf}(I)$:

$(*)_\mathscr{C}$ $[\delta \in \mathscr{C}$ and $\mathrm{cf}(\delta) \geq \kappa] \Rightarrow \mathrm{inv}^\beta_\kappa(J_\delta)$ is well defined,

<u>then</u> we let

$$\mathrm{inv}^\alpha_\kappa(I) = \langle \mathrm{inv}^\beta_\kappa(J_\delta) : \mathrm{cf}(\delta) \geq \kappa, \delta < \mathrm{cf}(I)\rangle / \mathscr{D}(\kappa, I).$$

Otherwise (i.e., there is no such $\mathscr{C}$ or $\mathrm{cf}(I) \leq \kappa$) $\mathrm{inv}^\alpha_\kappa(I)$ is not defined.

<u>Case 3</u>: α is limit
$\mathrm{inv}^\alpha_\kappa(I) = \langle \mathrm{inv}^\beta_\kappa(I) : \beta < \alpha\rangle$.
2) If $\mathbf{d} = \mathrm{inv}^\alpha_\kappa(I)$ <u>then</u> "the cofinality of $\mathbf{d}$" means $\mathrm{cf}(I)$, clearly well defined.

Remark 3.7. 1) Really just $\alpha = 0, 1, 2$ are used. For regular λ, $\alpha = 1$ suffices, but for singular λ, $\alpha = 2$ is used (see 3.11). In [She71b] all the α's were used as Solovay's theorem was not used.
2) The following lemma will be helpful as we will try to deal with cases of inv inside models and try to prove that it is quite independent of a (relevant) choice of representatives.

Observation 3.8. *1) If $\beta \leq \alpha$ and $\mathrm{inv}^\alpha_\kappa(I) = \mathrm{inv}^\beta_\kappa(J)$, and both are well defined <u>then</u> $\mathrm{inv}^\beta_\kappa(I), \mathrm{inv}^\beta_\kappa(J)$ are well defined and equal.*

2) If I, J are linear orders, $\mathrm{inv}^\alpha_\kappa(I)$ is well defined, $\mathbf{E}$ is a convex equivalence relation on J, $f : J \xrightarrow{\text{onto}} I$ preserves $\le$, and $(f(x) = f(y)) \equiv (x\mathbf{E}y)$, <u>then</u> $\mathrm{inv}^\alpha_\kappa(J)$ is well defined and $\mathrm{inv}^\alpha_\kappa(J) = \mathrm{inv}^\alpha_\kappa(J)$.
3) Assume that $\psi(\bar x, \bar y) = \varphi(\bar y, \bar x)$ and $\varphi_\ell(\bar x, \bar y) \in \{\varphi(\bar x, \bar y), \neg\varphi(\bar x, \bar y), \psi(\bar x, \bar y), \neg\psi(\bar x, \bar y)\}$ for $\ell = 1, 2$. <u>Then</u> $\langle \bar a_s : s \in I\rangle$ is $(\kappa, \varphi_1(\bar x, \bar y))$-skeleton like in M if and only if $\langle \bar a_s : s \in I^*\rangle$ is $(\kappa, \varphi_2(\bar x, \bar y))$-skeleton like in M; also (by the asymmetry assumption) in M we have $\varphi(\bar x, \bar y) \vdash \neg\psi(\bar x, \bar y)$ and $\psi(\bar x, \bar y) \vdash \neg\varphi(\bar x, \bar y)$.

Remark 3.9. 1) To understand the aim of 3.10 below, think we may be considering $\langle a_s : s \in I\rangle, \langle b_t : t \in J\rangle$ such that for some linear order U, and $\langle \bar c_t : t \in U\rangle$ we have $\bar c_t \in {}^{\ell g(\bar x)}M$ and $\langle \bar a_s : s \in I\rangle\hat{}\langle \bar c_u : u \in U\rangle$ and $\langle \bar b_t : t \in J\rangle\hat{}\langle \bar c_u : u \in U\rangle$ are both $(\kappa, \varphi(x, y))$-skeleton like in M and $\mathrm{cf}(U^*) \ge \kappa$.
2) We can omit assumption (c) in 3.10, so the conclusion will tell us that if one of $\mathrm{inv}^\alpha_\kappa(I)$, $\mathrm{inv}^\alpha_\kappa(J)$ is well defined then both are, but presently there is no real gain.
3) In 3.8(2), we cannot replace the assumption "$\mathrm{inv}^\alpha_\kappa(I)$ is well defined" by "$\mathrm{inv}^\alpha_\kappa(J)$ is well defined".

Lemma 3.10. *Suppose that κ is a regular cardinal, I, J are linear orders, and $\bar a_s$ (for $s \in I$), $\bar b_t$ (for $t \in J$) are from M, and $\varphi(\bar x, \bar y)$ is a $\tau(M)$-formula ($\kappa > \ell g(\bar x) = \ell g(\bar y) = \ell g(\bar a_s) = \ell g(\bar b_t)$), and $\psi(\bar x, \bar y) := \varphi(\bar y, \bar x)$.*
Assume:

- *(a) (α) for every $s \in I$ for every large enough $t \in J, M \models \varphi[\bar a_s, \bar b_t]$,*
  - *(β) for every $t \in J$ for every large enough $s \in I, M \models \neg\varphi[\bar a_s, \bar b_t]$,*
- *(b) (α) $\langle \bar a_s : s \in I\rangle$ is $(\kappa, \varphi(\bar x, \bar y))$-skeleton like inside M,*
  - *(β) $\langle \bar b_t : t \in J\rangle$ is $(\kappa, \varphi(\bar x, \bar y))$-skeleton like inside M,*
- *(c) $\mathrm{inv}^\alpha_\kappa(I)$, $\mathrm{inv}^\alpha_\kappa(J)$ are defined.*

<u>Then</u> $\mathrm{inv}^\alpha_\kappa(I) = \mathrm{inv}^\alpha_\kappa(J)$.

Proof. By induction on α.

<u>First Case</u>: $\alpha = 0$
Assume not, so $\mathrm{inv}^0_\kappa(I) \ne \mathrm{inv}^0_\kappa(J)$. Then by Definition 3.6 we have $\mathrm{cf}(I), \mathrm{cf}(J)$ are distinct (and $\ge \kappa$). By symmetry, without loss of generality $\mathrm{cf}(I) > \mathrm{cf}(J)$, so $\mathrm{cf}(I) > \kappa$.
Let $\langle t_\zeta : \zeta < \mathrm{cf}(J)\rangle$ be increasing unbounded in J. For each $\zeta < \mathrm{cf}(J)$ (by clause (a)(β) of 3.10 and 3.8) there is $s_\zeta \in I$ such that:

$$s_\zeta \le s \in I \Rightarrow M \models \neg\varphi[\bar a_s, b_{t_\zeta}].$$

As $\mathrm{cf}(I) > \mathrm{cf}(J)$ there is $s \in I$ such that $\bigwedge\limits_{\zeta < \mathrm{cf}(J)} s_\zeta < s$. Now, the set

$$\{t \in J : M \models \neg\varphi[\bar a_s, \bar b_t]\}$$

includes each t_ζ (as $s_\zeta < s \in I$), and hence it is unbounded in J, contradicting clause (a)(α) of 3.10.

<u>Second Case</u>: $\alpha = \beta + 1$
By the first case and Definition 3.6 we have $\mathrm{cf}(I) = \mathrm{cf}(J) \ge \kappa$. Let $\lambda = \mathrm{cf}(I) = \mathrm{cf}(J)$; let

$$I = \bigcup_{i<\lambda} I_i,$$

where I_i is increasing continuous in i, I_i a proper initial segment of I and $[i \neq j \Rightarrow I_i \neq I_j]$.

Similarly let

$$J = \bigcup_{i<\lambda} J_i.$$

Choose $s_i \in I_{i+1} \setminus I_i$ and $t_i \in J_{j+1} \setminus J_j$. By assumption (a), for every $i < \lambda$ there is $j_i < \lambda$ such that:

$(\alpha)'$ if $t \in J \setminus J_{j_i}$ then $M \models \varphi[\bar{a}_{s_i}, \bar{b}_t]$,
$(\beta)'$ if $s \in I \setminus I_{j_i}$ then $M \models \neg\varphi[\bar{a}_s, \bar{b}_{t_i}]$.

Let

$$\mathscr{C} = \{\delta < \lambda : \delta \text{ is a limit ordinal and } i < \delta \Rightarrow j_i < \delta\};$$

it is a club of λ. For $\delta \in \mathscr{C}$ let $I^\delta = (I \setminus I_\delta)^*$ and let $J^\delta = (J \setminus J_\delta)^*$. By Definition 3.6 above it suffices to prove, for $\delta \in \mathscr{C}$ satisfying $\mathrm{cf}(\delta) \geq \kappa$ and $\mathrm{inv}^\beta_\kappa(I^\delta), \mathrm{inv}^\beta_\kappa(J^\delta)$ are defined, that:

$(*)_\delta$ $\mathrm{inv}^\beta_\kappa(I^\delta) = \mathrm{inv}^\beta_\kappa(J^\delta)$.

For this we use the induction hypothesis, but we have to check that the assumptions (a), (b), (c) hold for this case.

Now clause (c) is part of the assumption of $(*)_\delta$, and clause (b) is inherited from the same property of $\langle \bar{a}_s : s \in I\rangle$, $\langle \bar{b}_t : t \in J\rangle$; lastly clause (a) follows from $(\alpha)' + (\beta)'$ above as $\delta \in \mathscr{C}$. In detail, if $t \in J^\delta$ then $J \models$ "$t_j < t$" for $j < \delta$. Hence, for $i < \delta, M \models \varphi[\bar{a}_{s_i}, \bar{b}_t]$ (by clause $(\alpha)'$ above). So by clause (b)(β) from the assumptions, for every large enough $s \in I^\delta$ we have $M \models \varphi[\bar{a}_s, \bar{b}_t]$, which means that $\langle \bar{a}_s : s \in I^\delta\rangle$, $\langle \bar{a}_t : t \in J^\delta\rangle$ satisfy clause (a)(α). Similarly clause (a)(β) holds.

Third Case: α is limit

Immediate by Definition 3.6. $\square_{3.10}$

Lemma 3.11. *1) If λ, κ are regular, $\lambda > \kappa$, then there are 2^λ linear orders I_α (for $\alpha < 2^\lambda$), each of cardinality λ, with pairwise distinct $\mathrm{inv}^1_\kappa(I_\alpha)$ (for $\alpha < 2^\lambda$), each well defined.*
2) If $\lambda > \kappa$, κ is regular, then there are linear orders I_α (for $\alpha < 2^\lambda$), each of cardinality λ with pairwise distinct $\mathrm{inv}^2_\kappa(I_\alpha)$ (for $\alpha < 2^\lambda$), each well defined.
3) If in (2) we have $\lambda \geq \theta = \mathrm{cf}(\theta) > \kappa$, then we can have $\mathrm{cf}(I_\alpha) = \theta$ if we use inv^3_α. Similarly, if in part (1) we have $\lambda \geq \theta = \mathrm{cf}(\theta) > \kappa$, then we can have $\mathrm{cf}(I_\alpha) = \theta$ if we use inv^2_κ; of course can use $\mathrm{inv}^\alpha_\kappa$ for $\alpha \geq 2$ (similarly elsewhere).

Remark 3.12. The construction of the linear orders is "hinted at" by the proof 3.10, and by the properties of stationary sets. Alternatively see the inductive construction in [She90, Appendix 3.7,3.8]. or see [She71b] where $\mathrm{inv}^\alpha_\kappa(1), \alpha < \lambda^+, \lambda = |I|$ are used.

Proof. 1) So $\lambda > \kappa$ are regular. The set $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary and hence we can find a partition $\langle S_\epsilon : \epsilon < \lambda \rangle$ of S into pairwise disjoint stationary subsets (well known, see Solovay's theorem). For $u \subseteq \lambda$ we define I_u as the set

$$\{(\alpha, \beta) : \alpha < \lambda \text{ and } \alpha \in \bigcup_{\epsilon \in u} S_\epsilon \Rightarrow \beta < \kappa^+ \text{ and } \alpha \in \lambda \setminus \bigcup_{\epsilon \in u} S_\epsilon \Rightarrow \beta < \kappa\}$$

linearly ordered by:

(∗) $(\alpha_1, \beta_1) <_I (\alpha_2, \beta_2)$ iff $\alpha_1 < \alpha_2 \vee (\alpha_1 = \alpha_2 \text{ and } \beta_1 > \beta_2)$.

(Yes! we mean $\beta_1 > \beta_2$ not $\beta_1 < \beta_2$). By the proof of 3.10 above clearly $\langle I_u : u \subseteq \lambda \rangle$ is as required.
2) So we have $\lambda > \kappa$, $\kappa = \mathrm{cf}(\kappa)$.
Let $\lambda = \sum_{i < \mathrm{cf}(\lambda)} \lambda_i$, λ_i increasing continuous $> \kappa$, let $\theta = \mathrm{cf}(\lambda) + \kappa^+$, or just $\kappa^+ + \mathrm{cf}(\lambda) \le \theta = \mathrm{cf}(\theta) \le \lambda$. Recall $2^\lambda = \lambda^{\Sigma\{\lambda_i^+ : i < \mathrm{cf}(\lambda)\}} = \prod_{i < \mathrm{cf}(\lambda)} 2^{\lambda_i^+}$, this motivates the following.

Let $h : \theta \longrightarrow \mathrm{cf}(\lambda)$ be such that for any $i < \mathrm{cf}(\lambda)$ the set $\{\delta < \theta : \mathrm{cf}(\delta) = \kappa$ and $h(\delta) = i\}$ is stationary.

For each i, let $\langle I_{i,\epsilon} : \epsilon < 2^{\lambda_i^+} \rangle$ be as in the proof of (1) for λ_i^+. For any $\nu \in \prod_{i < \mathrm{cf}(\lambda)} 2^{\lambda_i^+}$ let $J_\nu = \sum_{\alpha < \theta} J^*_{\nu,\alpha}$ with $J_{\nu,\alpha} \cong I_{h(\alpha),\nu(h(\alpha))}$.
3) Let $\langle I_\epsilon : \epsilon < 2^\lambda \rangle$ be as guaranteed in part (2) (or part (1) if λ is regular). For each $\epsilon < 2^\lambda$, let $J_\epsilon = \sum_{i < \theta} J^*_{\epsilon,i}$ where $J_{\epsilon,i} \cong I_\epsilon$; now the sequence $\langle I_\epsilon : \epsilon < 2^\lambda \rangle$ is as required. $\square_{3.11}$

Discussion 3.13. Instead considering $\bar{c} \in {}^\partial M$ we may add a filter D on ∂ and consider only $\bar{c}/D$, or even $\bar{c}/E_\mu, E_M$, a definable equivalence relation on ${}^\partial M$.

§ 3(B). **Representing Invariants.**

Now we would like essentially to attach the invariants of a linear order I to a model M which has a skeleton-like sequence indexed by I. In (α) (in Definition 3.14 below) we define what it means for a sequence indexed by I to (κ, θ)-represent the (φ, ψ)-type of $\bar{c}$ over A.

Definition 3.14. Let $A \subseteq M, \bar{c} \in M$ and $\varphi(\bar{x}, \bar{y})$ be an asymmetric formula with vocabulary contained in $\tau(M)$ and $\psi(\bar{x}, \bar{y}) =: \varphi(\bar{y}, \bar{x})$.

- (α) We say that $\langle \bar{a}_s : s \in I \rangle$ does (κ, θ)-represents the tuple $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ <u>when</u>: I is a linear order, $\mathrm{cf}(I) \ge \kappa$ and for some linear order J of cofinality $\theta \ge \kappa$ disjoint to I, there are $\bar{a}_t \in {}^{\ell g(\bar{x})}A$ for $t \in J$, such that:
  - (i) for every large enough $t \in I$, $\bar{a}_t$ realizes $\mathrm{tp}_{\{\varphi(\bar{x},\bar{y}),\psi(\bar{x},\bar{y})\}}(\bar{c}, A, M)$, and
  - (ii) $\langle \bar{a}_s : s \in J + (I)^* \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M (recalling I^* denotes the inverse of I).
- (β) We say that $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ has a (κ, θ, α)-invariant <u>when</u>:

(i) if for $\ell = 1, 2$, $\langle \bar{a}^\ell_s : s \in I_\ell \rangle$ does (κ, θ)-represents $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ and $\mathrm{inv}^\alpha_\kappa(I_\ell)$ are well defined[3] for $\ell = 1, 2$ then $\mathrm{inv}^\alpha_\kappa(I_1) = \mathrm{inv}^\alpha_\kappa(I_2)$,

(ii) some $\langle \bar{a}_s : s \in I \rangle$ does (κ, θ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ and $\mathrm{inv}^\alpha_\kappa(I)$ is well defined.

(γ) Let $\mathrm{INV}^\alpha_{\kappa,\theta}(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ be $\mathrm{inv}^\alpha_\kappa(I)$ when $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ has (κ, θ, α)-invariant and $\langle \bar{a}_s : s \in I \rangle$ does (κ, θ)-represent it

(δ) Let "(κ, α)-invariant" means "(κ, θ, α)-invariant for some regular $\theta \geq \kappa$". Similarly for "κ-represents" and $\mathrm{INV}^\alpha_\kappa(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ (justified by Fact 3.16 below).

Discussion 3.15. Below each of Definition 3.17, Lemmas 3.19 and 3.22, and the proof of Theorem 3.24 have 3 cases. In the easiest case $\lambda = \|M\|$ is regular. When λ is singular the computation of $\mathrm{inv}^\alpha_\kappa(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ is easier when $\mathrm{cf}(\lambda) > \kappa$ (second case). The third case arises when $\lambda > \kappa > \mathrm{cf}(\lambda)$.

The relative easiness of the regular case is caused by the fact that any two increasing representations of a model with cardinality λ must "agree" on a club. In the second case we are able to restrict the first argument to a cofinal, increasing cont sequence of subsets of M of smaller cardinalities. For the third case we must construct a "dual argument", noticing that a a long sequence of member of such cofinal sequence, must concentrate on one member of the representation.

Fact 3.16. Suppose that for $\ell = 1, 2$, the sequence $\langle \bar{a}^\ell_s : s \in I_\ell \rangle$ does (κ, θ_ℓ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$. Then $\theta_1 = \theta_2$.

Proof. So let for $\ell = 1, 2$ the sequence $\langle \bar{a}^\ell_s : s \in J_\ell \rangle$ witness that $\langle \bar{a}^\ell_s : s \in I_\ell \rangle$ does (κ, θ_ℓ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$, i.e., they are as in (α) of Definition 3.14. Assume toward contradiction that $\theta_1 \neq \theta_2$ and by symmetry without loss of generality $\theta_1 < \theta_2$. Let $\langle s_\ell(\alpha) : \alpha < \theta_\ell \rangle$ be an increasing unbounded sequence of members of J_ℓ for $\ell = 1, 2$. So for each $\alpha < \theta_1$ we have

$$t \in I_1 \quad \Rightarrow \quad M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^1_t]$$

and hence by clause (i) of (α) of Definition 3.14 we have $M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{c}]$ recalling $\bar{a}^1_{s_1(\alpha)} \subseteq A$, so for every large enough $t \in I_2, M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^2_t]$. But $\langle \bar{a}^2_t : t \in J_2 + (I_2)^* \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M, hence for some $\beta_\alpha < \theta_2$ we have

$$s_2(\beta_\alpha) \leq t \in J_2 \Rightarrow M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^2_t]$$

and so $\beta(*) = \sup\{\beta_\alpha + 1 : \alpha < \theta_1\} < \theta_2$ (as $\theta_1 < \theta_2 = \mathrm{cf}(\theta_2)$). So $M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^2_{s_2(\beta(*))}]$ for $\alpha < \theta_1$.

But $t \in I_2 \ \Rightarrow \ M \models \neg\varphi[\bar{a}^2_t, \bar{a}^2_{s_2(\beta)}]$ and hence $M \models \neg\varphi[\bar{c}, \bar{a}^2_{s_2(\beta)}]$. Therefore, for every large enough $t \in I_1, M \models \neg\varphi[\bar{a}^1_t, \bar{a}^2_{s_2(\beta)}]$ and hence for every large enough $t \in J_1$, $M \models \neg\varphi[\bar{a}^1_t, \bar{a}^2_{s_2(\beta)}]$. Hence this holds for $t = s_1(\alpha)$, α large enough, a contradiction to the previous paragraph. See more in 3.21(2). $\square_{3.14}$

Definition 3.17. Let $\varphi(\bar{x}, \bar{y})$ be an asymmetric formula with vocabulary $\subseteq \tau(M)$ (where $\ell g(\bar{x}) = \ell g(\bar{y})$ is finite), and let M be a model of cardinality $\lambda, \lambda > \kappa, \kappa$ regular, α be an ordinal.

[3]but see Fact 3.21(2) and Remark 3.9(2)

0) A representation of the model M is an increasing continuous sequence $\bar{M} = \langle M_i : i < \mathrm{cf}(\lambda)\rangle$ such that $\|M_i\| < \lambda$, and $M = \bigcup_{i<\mathrm{cf}(\lambda)} M_i$.
0A) Similarly for sets.
1) For a regular cardinal λ:

$$\mathbf{INV}^\alpha_\kappa(M, \varphi(\bar{x}, \bar{y})) = \{\mathbf{d} : \begin{array}{l} \text{for every representation } \langle A_i : i < \lambda\rangle \text{ of } |M|, \\ \text{there are } \delta < \lambda \text{ and } \bar{c} \in M \text{ (of course, } \ell g(\bar{c}) = \ell g(\bar{x})) \\ \text{such that } \mathrm{cf}(\delta) \geq \kappa \text{ and } \mathbf{d} = \mathrm{INV}^\alpha_\kappa(\bar{c}, A_\delta, M, \varphi(\bar{x}, \bar{y})), \\ \text{(so in particular the latter is well defined) } \}. \end{array}$$

2) For regular cardinals $\theta > \kappa$ such that $\lambda > \mathrm{cf}(\lambda) = \theta$ we let

$$\mathscr{D}_{\theta,\kappa} = \mathscr{D}_\theta + \{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\}$$

and

$$\mathbf{INI}^\alpha_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y})) = \{\langle \mathbf{d}_i : i < \theta\rangle / \mathscr{D}_{\theta,\kappa} : \begin{array}{l} \text{for every representation } \langle A_i : i < \theta\rangle \text{ of } |M|, \\ \text{there is } S \in \mathscr{D}_{\theta,\kappa} \text{ satisfying:} \\ \text{for every } \delta \in S \text{ there is } \bar{c}_\delta \in M \text{ such that} \\ \mathbf{d}_\delta = \mathrm{INV}^\alpha_\kappa(\bar{c}_\delta, A_\delta, M, \varphi(\bar{x}, \bar{y})) \\ \text{so is well defined and the cofinality of } \mathbf{d}_\delta \text{ is } > |A_\delta|\}. \end{array}$$

3) For regular cardinals $\kappa > \theta$, $\lambda > \theta > \kappa + \mathrm{cf}(\lambda)$ and a function h with domain a stationary subset of $\{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\}$ and range a set of regular cardinals $< \lambda$, we let

$$\mathscr{D}_{\theta,h} = \mathscr{D}_\theta + \{\{\delta < \theta : h(\delta) \geq \mu \quad (\text{hence } \delta \in \mathrm{Dom}(h))\} : \mu < \lambda\},$$

and assuming that $\mathscr{D}_{h,\lambda}$ is a proper filter we let:

$$\mathbf{INV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(x, y)) = \{\langle \mathbf{d}_i : i < \theta\rangle / \mathscr{D}_{\theta,h} : \begin{array}{l} \text{for every representation } \langle A_i : i < \mathrm{cf}(\lambda)\rangle \text{ of } |M|, \\ \text{there are } \gamma < \mathrm{cf}(\lambda) \text{ and } S \in \mathscr{D}_{h,\lambda}, S \subseteq \mathrm{Dom}(h), \text{ satisfying} \\ \text{the following for each } \delta \in S, \text{ if } h(\delta) > |A_\gamma| \text{ \underline{then} for some} \\ \bar{c}_\delta \in M, \text{ we have } \mathbf{d}_\delta = \mathrm{INV}^\alpha_\kappa(\bar{c}_\delta, A_\gamma, M, \varphi(\bar{x}, \bar{y})) \\ \text{so is well defined and the cofinality of } \mathbf{d}_\delta \text{ is } > |A_\gamma|\}. \end{array}$$

Remark 3.18. Of course, also in 3.17(1) we could have used $\langle \mathbf{d}_i : i < \lambda\rangle / \mathscr{D}_\lambda$ as the invariant.

Lemma 3.19. *Suppose $\varphi(\bar{x}, \bar{y})$ is a formula in the vocabulary of M, $\ell g(\bar{x}) = \ell g(\bar{y}) < \omega$.*
1) If $\lambda > \aleph_0$ is regular, M a model of cardinality λ, κ regular $< \lambda$, <u>then</u> $\mathbf{INV}^\alpha_\kappa(M, \varphi(\bar{x}, \bar{y}))$ has cardinality $\leq \lambda$.
2) If λ is singular, $\theta = \mathrm{cf}(\lambda) > \kappa$, <u>then</u> $\mathbf{INV}^\alpha_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$ almost has cardinality $\leq \lambda$, which means: there are no $\mathbf{d}^\zeta_i$ (for $i < \theta$, $\zeta < \lambda^+$) such that:

(*i*) *for $\zeta < \lambda^+$, $\langle \mathbf{d}^\zeta_i : i < \theta\rangle / \mathscr{D}_{\theta,\kappa} \in \mathbf{INV}^\alpha_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$,*
(*ii*) *for $i < \theta, \zeta < \xi < \lambda^+$, we have $\mathbf{d}^\zeta_i \neq \mathbf{d}^\xi_i$.*

3) If λ is singular, θ, κ are regular, $\kappa + \mathrm{cf}(\lambda) < \theta < \lambda$, h is a function from some stationary subset of $\{i < \theta : \mathrm{cf}(i) \geq \kappa\}$ into

$$\{\mu < \lambda : \mu \text{ is a regular cardinal }\}$$

such that $\mathscr{D}_{\theta,h}$ is a proper filter, <u>then</u> $\mathbf{INV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$ almost has cardinality $\leq \lambda$, which means: there are no $\mathbf{d}^\zeta_i (i < \theta, \zeta < \lambda^+)$ such that:

(i) *for $\zeta < \lambda^+$, $\langle \mathbf{d}^\zeta_i : i < \theta \rangle / \mathscr{D}_{\theta,h} \in \mathbf{INV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$,*

(ii) *for $i < \theta, \zeta < \xi < \lambda^+$, we have $\mathbf{d}^\zeta_i \neq \mathbf{d}^\xi_i$.*

Proof. Straightforward. $\square_{3.19}$

§ 3(C). Harder Results.

We now show that (for example for the case λ regular) if $|I| \leq \lambda$ and $\mathrm{inv}^\alpha_\kappa(I)$ is well defined then there is a linear order J such that: if a model M has a (κ, φ)-skeleton like sequence inside M of order-type J then $\mathrm{inv}^\alpha_\kappa(I) \in \mathbf{INV}^\alpha_\kappa(M, \varphi)$.

Again, the proof splits into three cases depending on the cofinality of λ. The following result provides a detail needed for the proof.

Claim 3.20. *Suppose that κ is a regular cardinal and $\langle \bar{a}_t : t \in J \rangle$ is a (κ, φ)-skeleton like inside M and $I \subseteq J$. If for each $s \in J \setminus I$ either $\{t \in I : t < s\}$ or the inverse order on $\{t \in I : t > s\}$ has cofinality less than κ (for example 1) <u>then</u> $\langle \bar{a}_t : t \in I \rangle$ is (κ, φ)-skeleton like for M.*

Proof. As usual let $\psi(\bar{x}, \bar{y}) := \varphi(\bar{y}, \bar{x})$. We must show that for every $\bar{a} \in {}^{\ell g(\bar{x})}M$ there is an $I_{\bar{a}} \subseteq I$ with $|I_{\bar{a}}| < \kappa$ such that: if $s, t \in I$ and $\mathrm{tp}_{\mathrm{qf}}(s, I_{\bar{a}}, I) = \mathrm{tp}_{\mathrm{qf}}(t, I_{\bar{a}}, I)$ then

$$M \models \text{“}\varphi(\bar{a}_s, \bar{a}) \equiv \varphi(\bar{a}_t, \bar{a})\text{” and } M \models \text{“}\psi(\bar{a}_s, \bar{a}) \equiv \psi(\bar{a}_t, \bar{a})\text{”}.$$

We know that there is such a set $J_{\bar{a}}$ for J and $\bar{a}$ and for each $s \in J_{\bar{a}}$ choose a set X_s of $< \kappa$ elements of I such that X_s tends to s, i.e., to the cut that s induces in I (either from above or below). (So if $s \in I$, $X_s = \{s\}$; otherwise use the assumption). Let $I_{\bar{a}} = \bigcup\limits_{s \in J_{\bar{a}}} X_s$; as κ is regular, $|X_s| < \kappa$ for $s \in J_{\bar{a}}$ and $|J_{\bar{a}}| < \kappa$ clearly $I_{\bar{a}}$ has cardinality $< \kappa$; also trivially $J_{\bar{a}} \subseteq I$.

Now it is easy to see that if t_1 and $t_2 \in I$ have the same quantifier free type over $I_{\bar{a}}$, then they have the same quantifier free type over $J_{\bar{a}}$, and the claim follows. $\square_{3.20}$

Also the following will be used in proving 3.22:

Fact 3.21. 1) Suppose $\langle \bar{a}_s : s \in J + I^* \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M and both J and I have cofinality $\geq \kappa$. <u>Then</u> for every $\bar{b} \in M$ there exist $s_0 \in J$ and $s_1 \in I^*$ such that:

- if $s_0 < t_\ell < s_1$ (in $J + I^*$) for $\ell = 0, 1$, <u>then</u> $M \models$ “$\varphi(\bar{a}_{t_0}, \bar{b}) \equiv \varphi(\bar{a}_{t_1}, \bar{b})$” and $M \models$ “$\psi(\bar{a}_{t_0}, \bar{b}) \equiv \psi(\bar{a}_{t_1}, \bar{b})$”.

2) Suppose that, for $\ell = 1, 2$, $\langle \bar{a}^\ell_s : s \in I^\ell \rangle$ does (κ, θ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ and $\langle \bar{a}^\ell_s : s \in J^\ell \rangle$ witnesses this. Then $\mathrm{inv}^\alpha_\kappa(I^1) = \mathrm{inv}^\alpha_\kappa(I^2)$ for every ordinal α such that they are both well defined.

Proof. 1) Easy.
2) As we can replace I^ℓ by any end segment, without loss of generality

(∗) for $\ell = 1, 2$ for every $t \in I^\ell$, $\bar{a}_t$ realizes $\mathrm{tp}_{\{\varphi(\bar{x},\bar{y}), \psi(\bar{x},\bar{y})\}}(\bar{c}, A, M)$.

We shall use Lemma 3.10 (with I^1, I^2 here standing for I, J there and ψ for φ). Conditions (b),(c) from 3.10 are met trivially, for (b) using 3.8. By similar arguments in condition (a) it is enough to prove clause (α).

Let us prove (a)(α) from 3.10. So suppose it fails, i.e., $s \in I^1$ but for arbitrarily large $t \in I^2$, $M \models \neg\varphi[\bar{a}^1_s, \bar{a}^2_t]$.

Since $\langle \bar{a}^2_t : t \in J^2 + (I^2)^* \rangle$ is (κ, φ)-skeleton like inside M, the preceding Fact 3.21(1) yields that for arbitrarily large $t \in J^2$, $M \models \neg\varphi[\bar{a}^1_s, \bar{a}^2_t]$. Since $\bar{a}^1_s$ and $\bar{c}$ realize the same $\{\varphi, \psi\}$-type over A_δ (see definition 3.14(α) and (*) above), and as $\bar{a}^2_t \subseteq A_\delta$ for $t \in J^2$, this implies $M \models \neg\varphi[\bar{c}, \bar{a}^2_t]$, so this holds for arbitrarily large $t \in J^2$. Choose such $t_0 \in J^2$, this quickly contradicts the choice of J^2 and I^2. For, it implies that for every $t \in I^2$ (as $\bar{c}, \bar{a}^2_t$ realize the same $\{\varphi, \psi\}$-type over A_δ) we have

$$M \models \neg\varphi[\bar{a}^2_t, \bar{a}^2_{t_0}],$$

which is impossible as $\langle \bar{a}_s : s \in J^2 + (I^2)^* \rangle$ is (κ, φ)-skeleton like (see Definition 3.1(3) the last phrase). $\square_{3.21}$

Lemma 3.22. *Assume $\ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0$ and $\varphi = \varphi(\bar{x}, \bar{y})$.*
1) Let $\lambda > \aleph_0$ be regular. If I is a linear order of cardinality $\leq \lambda$, and $\mathrm{inv}^\alpha_\kappa(I)$ is well defined, then for some linear order J of cardinality λ the following holds:

(∗) *if M is a model of cardinality λ, $\bar{a}_s \in {}^{\ell g(\bar{x})}M$, $\langle \bar{a}_s : s \in J \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M (hence $\varphi(\bar{x}, \bar{y})$ is asymmetric), then $\mathrm{inv}^\alpha_\kappa(I) \in \mathbf{INV}^\alpha_\kappa(M, \varphi(\bar{x}, \bar{y}))$.*

2) Let λ be singular, $\theta = \mathrm{cf}(\lambda) > \kappa$, $\lambda = \sum_{i<\theta} \lambda_i$, where the sequence $\langle \lambda_i : i < \theta \rangle$ is increasing continuous. Suppose that for $i < \theta$, I_i is a linear order of cofinality $> \lambda_i$ and cardinality $\leq \lambda$ such that $\mathrm{inv}^\alpha_\kappa(I_i)$ is well defined. Then for some linear order J of cardinality λ the following holds:

(∗∗) *if M is a model of cardinality λ, $\bar{a}_s \in {}^{\ell g(x)}M$ for $s \in J$, $\langle \bar{a}_s : s \in J \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton inside M, (so $\varphi(\bar{x}, \bar{y})$ asymmetric), then $\langle \mathrm{inv}^\alpha_\kappa(I_i) : i < \theta \rangle / \mathscr{D}_{\theta,\kappa}$ belongs to $\mathbf{INV}^\alpha_\kappa(M, \varphi(\bar{x}, \bar{y}))$.*

3) Let λ be singular, θ, κ be regular, $\lambda > \theta > (\mathrm{cf}(\lambda) + \kappa)$, $\lambda = \sum_{i<\mathrm{cf}(\lambda)} \lambda_i$, λ_i increasing continuous. If, for $i < \theta$, I_i is a linear order such that $\mathrm{inv}^\alpha_\kappa(I_i)$ is well defined, then for some linear order J of cardinality λ the following holds:

(∗∗∗) *if M is a model of cardinality λ, $\bar{a}_s \in {}^{\ell g(x)}M$ for $s \in J$, $\langle \bar{a}_s : s \in J \rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M, (so $\varphi(\bar{x}, \bar{y})$) asymmetric), h is a function from a stationary subset of $\{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\}$ with range a set of regular cardinals $< \lambda$ but $> \theta$ such that $\mathrm{cf}(I_i) \geq h(i)$ and $\mathscr{D}_{\theta,h}$ is a proper filter then $\langle \mathrm{inv}^\alpha_\kappa(I_i) : i < \theta \rangle / \mathscr{D}_{\theta,h}$ belongs to $\mathbf{INV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$.*

Proof. 1) We must choose a linear order J of cardinality λ such that: if J indexes a $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like sequence inside M, a model of cardinality λ, then

$$\mathrm{inv}^{\alpha}_{\kappa}(I) \in \mathbf{INV}^{\alpha}_{\kappa}(M, \varphi(\bar{x}, \bar{y})).$$

For this, for any continuous increasing decomposition $\bar{A}$ of $|M|$, we must find a sequence $\bar{c} \in M$ and an ordinal $\delta < \lambda$ of cofinality κ with

$$\mathrm{INV}^{\alpha}_{\kappa}(\bar{c}, A_\delta, M, \varphi(\bar{x}, \bar{y})) = \mathrm{inv}^{\alpha}_{\kappa}(I).$$

To obtain $\bar{c}$, we shall use a function from λ to J. Let I_α for $\alpha < \lambda$ be pairwise disjoint linear orders isomorphic to I.

Let $J = \sum_{\alpha<\lambda} I^*_\alpha$ (where I^* means we use the inverse of I as an ordered set). Suppose $\langle \bar{a}_s : s \in J\rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M, (hence $\varphi(\bar{x}, \bar{y})$) is asymmetric) and M has cardinality λ. For $\alpha < \lambda$ let $s(\alpha) \in I_\alpha$ and let $\langle A_\alpha : \alpha < \lambda\rangle$ be an increasing continuous sequence such that $M = \{A_\alpha : \alpha < \lambda\}, |A_\alpha| < \lambda$. By the definition of $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like (Definition 3.1(1)), for every $\bar{a} \in {}^{\ell g(\bar{x})}M$, here is a subset $J_{\bar{a}}$ of J of cardinaltiy $< \kappa$ such that: if $s, t \in J \setminus J_{\bar{a}}$ induces the same Dedekind cut on $J_{\bar{a}}$, then $M \models$ "$\varphi[\bar{a}_s, \bar{a}] \equiv \varphi[\bar{a}_t, \bar{a}]$" and $M \models$ "$\varphi[\bar{a}, \bar{a}_s] \equiv \varphi[\bar{a}, \bar{a}_t]$". Since λ is regular, for some closed unbounded subset $\mathscr{C}^*$ of λ, for every $\delta \in \mathscr{C}^*$ we have:

$(*)$ (i) $\bar{a}_{s(\alpha)} \in {}^{\ell g(\bar{x}}(A_\delta)$ for $\alpha < \delta$,

(ii) $J_{\bar{a}} \subseteq \sum_{\beta<\delta} I^*_\beta$ for $\bar{a} \in {}^{\ell g(\bar{x})}(A_\delta)$.

So it is enough to prove that for any $\delta \in \mathscr{C}^*$ of cofinality κ we have

$$\mathrm{inv}^{\alpha}_{\kappa}(I) = \mathrm{INV}^{\alpha}_{\kappa}(\bar{a}_{s(\delta)}, A_\delta, M, \varphi(\bar{x}, \bar{y})).$$

Let $\mathscr{C} \subseteq \delta$ be closed unbound of order types $\mathrm{cf}(\delta)$. Now we shall show that $\langle \bar{a}_s : s \in I_\delta\rangle$ does κ-represents $(\bar{a}_{s(\delta)}, A_\delta, M, \varphi(\bar{x}, \bar{y}))$: the required θ and J in Definition 3.14(α) are $\mathrm{cf}(\delta)$ and $\langle \bar{a}_{s(\beta)} : \beta \in \mathscr{C}\rangle$, and clause (i) of 3.14(α) holds by $(*)(ii)$ above and clause (ii) of 3.14(α) holds by claim 3.20 with $J, \{s(\beta) : \beta \in \mathscr{C}\} \cup I^*_\delta$ here standing for J, I there.

So (see Definition 3.14(γ)) it is enough to show that $(\bar{a}_{s(\delta)}, A_\delta, M, \varphi(\bar{x}, \bar{y}))$ has a (κ, α)-invariant. Now in Definition 3.14(β), part (ii) is obvious by the above; so it remains to prove (i).

Let $\theta =: \mathrm{cf}(\delta)$. So assume that for $\ell = 1, 2$,

$$\langle \bar{a}^\ell_s : s \in I^\ell\rangle \text{ weakly } (\kappa, \theta)\text{-represents } (\bar{a}_{s(\delta)}, A_\delta, M, \varphi(\bar{x}, \bar{y})).$$

Let J^ℓ, $\langle a^\ell_t : t \in J^\ell\rangle$ exemplify this (so each $\bar{a}^\ell_t$ belongs to A_δ) and let $J^*_\ell = J^\ell + (I^\ell)^*$ and assume $\mathrm{inv}^{\alpha}_{\kappa}(I^\ell)$ are well defined. We have to prove that $\mathrm{inv}^{\alpha}_{\kappa}(I^1) = \mathrm{inv}^{\alpha}_{\kappa}(I^2)$. This follows by 3.21(2) above.

2),3) Similar to the proof of part (1), using $J = \sum_{i<\theta}(I_i)^*$ where $I_i \cong I$ are pairwise disjoint. So left to the reader (or see the proof of case (d) and formulation of case (e) in Theorem 3.28). $\square_{3.22}$

Remark 3.23. 1) In the proof of 3.22, instead "$\theta = \mathrm{cf}(\delta) = \kappa$" we can use $\theta = \mathrm{cf}(\delta) \geq \kappa$. For this we should relax the requirements "$\langle \bar{a}_s : s \in \{s(\beta) : \beta \in \mathscr{C}$ or $s \in I_\delta\}\rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like" to

- $(*)$ for every $\bar{a} \in {}^{\ell g(\bar{x})}M$ there is $J_{\bar{a}}$ such that:
  - (i) $J_{\bar{a}} \subseteq \{s(\beta) : \beta \in \mathscr{C}\} \cup I_\delta$
  - (ii) $J_{\bar{a}} \cap \{s(\beta) \in \mathscr{C}\}$ is a bounded subset of $\{s(\beta) : \beta \in cC\}$
  - (iii) $J_{\bar{a}} \cap I_\delta$ has cardinality $< \kappa$
  - (iv) if $s, t \in (\{s(\beta) : \beta \in \mathscr{C}\} \cup I_\delta) \backslash J_{\bar{a}}$ realizes the same Dedekind cut of $(\{s(\beta) : \beta \in \mathscr{C}\} \cup I_\delta$ then $M \models \varphi[a_s, \bar{a}] \equiv \varphi[\bar{a}_t, \bar{a}]$ and $M \models \varphi[\bar{a}, \bar{a}_s] \equiv \varphi[\bar{a}, \bar{a}_t]$.

2) As mentioned about clause (c) in 3.10 is redundant, but using this has some consequences here.

Theorem 3.24. *Suppose that $\lambda > \kappa$, K_λ is a family of τ-models, each of cardinality $\lambda, \varphi(\bar{x}, \bar{y})$ is an asymmetric formula with vocabulary $\subseteq \tau$, and $\ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0$. Further, suppose that for every linear order J of cardinality λ there are $M \in K_\lambda$ and $\bar{a}_s \in M$ for $s \in J$ such that $\langle \bar{a}_s : s \in J\rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like in M.*

Then, in K_λ, there are 2^λ pairwise non-isomorphic models.

Proof. First let $\lambda > \aleph_0$ be regular.

By 3.11(1) there are linear order I_ζ (for $\zeta < 2^\lambda$) each of cardinality λ, such that $\mathrm{inv}^1_\kappa(I_\zeta)$ are well defined and pairwise distinct. Let J_ζ relate to I_ζ as guarantee by 3.22(1). Let $M_\zeta \in K_\lambda$ be such that there are $\bar{a}^\zeta_s \in M_\zeta$ for $s \in J_\zeta$ such that $\langle \bar{a}^\zeta_s : s \in J_\zeta\rangle$ is $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M_ζ (exists by an assumption). By 3.22(1), that is by our choice of J_ζ, we have

$$\mathrm{inv}^1_\kappa(I_\zeta) \in \mathbf{INV}^1_\kappa(M_\zeta, \varphi(\bar{x}, \bar{y})).$$

Clearly,

$$M_\zeta \cong M_\xi \Rightarrow \mathbf{INV}^1_\kappa(M_\zeta, \varphi(\bar{x}, \bar{y})) = \mathbf{INV}^1_\kappa(M_\xi, \varphi(\bar{x}, \bar{y})),$$

and hence

$$M_\zeta \cong M_\xi \quad \Rightarrow \quad \mathrm{inv}^1_\kappa(I_\zeta) \in \mathbf{INV}^1_\kappa(M_\xi, \varphi(\bar{x}, \bar{y})).$$

So if for some $\xi < 2^\lambda$, the number of $\zeta < 2^\lambda$ for which $M_\zeta \cong M_\xi$ is $> \lambda$, then $\mathbf{INV}^1_\kappa(M_\xi, \varphi(\bar{x}, \bar{y}))$ has cardinality $> \lambda$ (remember $\mathrm{inv}^1_\kappa(I_\zeta)$ were pairwise distinct for $\zeta < 2^\lambda$). But this contradicts 3.19(1).

So

$$\{(\zeta, \xi) : \zeta, \xi < 2^\lambda \text{ and } M_\zeta \cong M_\xi\},$$

which is an equivalence relation on 2^λ, satisfies: each equivalence class has cardinality $\leq \lambda$. Hence there are 2^λ equivalence classes and we finish.

For λ singular the proof is similar. If $\mathrm{cf}(\lambda) > \kappa$, we can choose $\theta = \mathrm{cf}(\lambda)$ and use $\mathrm{INV}^2_{\kappa,\theta}$, 3.11(2), 3.22(2), 3.19(2) instead of $\mathbf{INV}^1_{\kappa,\theta}$, 3.11(1), 3.22(1), 3.19(1) respectively.

If $\mathrm{cf}(\lambda) \leq \kappa$, let $\theta = \kappa^+$ so $\lambda > \theta > \kappa + \mathrm{cf}(\lambda)$. Hence we can find a mapping

$$h : \{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\} \longrightarrow \{\mu : \mu = \mathrm{cf}(\mu) < \lambda\}$$

such that for each $\mu = \mathrm{cf}(\mu) < \lambda$ the set

$$\{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa \text{ and } h(\delta) \geq \mu\}$$

is stationary. Now we can use $\mathbf{INV}^{2,h}_{\kappa,\theta}$, 3.11(2), 3.22(3), 3.19(3) instead $\mathbf{INV}^1_\kappa$, 3.11(1), 3.22(1), 3.19(1) respectively.

$\square_{3.24}$

Conclusion 3.25. *1) If T_1 is a first order, $T \subseteq T_1$, T is unstable and complete, $\lambda \geq |T_1| + \aleph_1$, then there are 2^λ pairwise non-isomorphic models of T of cardinality λ which are reducts of models of T_1.*
2) If $T \subseteq T_1$ are as above, $\lambda \geq |T_1| + \kappa^+$, $\lambda = \lambda^{<\kappa}$, κ is regular, then there are 2^λ pairwise non-isomorphic models of T of cardinalty λ which are reducts of models M^1_i of T_1 such that M_i, M^1_i are κ-compact and κ–homogeneous. [Really we can get strongly homogeneous; see [Shel, §1]*].*
3) Assume that $\psi \in \mathbb{L}_{\kappa^+,\aleph_0}(\tau_1)$, $\tau \subseteq \tau_1$, ψ has the order property for $\mathbb{L}_{\kappa^+,\aleph_0}(\tau)$, i.e., for some formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\kappa^+,\aleph_0}(\tau)$ satisfying $\ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0$, for arbitrarily large μ, there is a model M of ψ and $\bar{a}_i \in M$ for $i < \mu$ such that

$$M \models \varphi[\bar{a}_i, \bar{a}_j] \text{ iff } i < j.$$

Then for $\lambda \geq \kappa + \aleph_1$, ψ has 2^λ models of cardinality λ, with pairwise non-isomorphic τ-reducts.

Proof. 1) Let $\varphi = \varphi(\bar{x}, \bar{y})$ be a first order formula exemplifying "T is unstable" (see Definition 1.2). By 1.11(1) there is a template Φ proper for linear orders such that $|\tau_\Phi| = |\tau_1|$ and for any linear order I, $\mathrm{EM}(I, \Phi)$ is a model of T_1 satisfying $\varphi[\bar{a}_s, \bar{a}_t]$ if and only if $I \models s < t$. Clearly $\mathrm{EM}_{\tau(T_1)}(I, \Phi)$ has cardinality $\geq |I|$ but $\leq |\tau_\Phi| + |I| + \aleph_0$. So for every $\lambda \geq |T_1| + \aleph_0 = |\tau_\Phi| + \aleph_0$ and linear order I of cardinality λ the model $M = \mathrm{EM}_{\tau(T)}(I, \Phi)$ is a $\tau(T)$–model, a reduct of a model of T_1, hence M is a model of T of cardinality exactly λ, and by 3.11(4) the sequence $\langle \bar{a}_t : t \in I \rangle$ is κ-skeleton like in M. So we have the assumption of 3.24, hence its conclusion as required.
2) By [She90, Ch.VII,p2], or case II of the proof of Theorem 3.6 (there) we have the assumption of 3.24; but [Shel, §1] supersedes upon this.
3) See 1.18(3) and Definition 1.15 why the assumption of 3.24 holds. $\square_{3.25}$

§ 3(D). **Using Infinitary Sequences.**

Now we turn our attention to the case in which the sequences on which $\varphi(\bar{x}, \bar{y})$ speaks are infinite. We shall need at some point in Theorem 3.28 the following:

Fact 3.26. If $\tau_2 = \tau_1 \cup \{c_i : i \in I\}$, c_i are individual constants, K_ℓ is a class of τ_ℓ-models (for $\ell = 1, 2$), $M \in K_2 \Rightarrow M {\restriction} \tau_1 \in K_1$, and $\mu = \dot{\mathbb{I}}(\lambda, K_2) > \lambda^{|I|}$, then $\dot{\mathbb{I}}(\lambda, K_1) \geq \mu$ (so if $\mu = 2^{\lambda + |\tau_1|}$, equality holds).

Proof. Straightforward (or see [She90, Ch.VIII,1.3]). $\square_{3.26}$

Definition 3.27. We say $\langle \bar a_s : s \in I\rangle$ is $(\kappa, \mu, < \lambda, \varphi(\bar x, \bar y))$-skeleton like inside M, (if $\mu = \lambda$ we may omit μ) <u>iff</u>:

(*i*) for $s, t \in I$ we have

$$M \models \varphi[\bar a_s, \bar a_t] \text{ \underline{if and only if} } I \models s < t,$$

(*ii*) for every $\bar c \in {}^{\ell g(\bar a_s)}M$ for some $J \subseteq I$, $|J| < \kappa$ and $(*)$ of 3.1(1) holds, and

(*iii*) moreover, for each $A \subseteq M$, $|A| < \mu$, there is $J \subseteq I$, $|J| < \lambda$ such that for every $\bar c \in {}^{\ell g(\bar x)}A$, the statement $(*)$ of 3.1 holds for J.

Theorem 3.28. *Suppose $\partial < \kappa < \lambda$ are cardinals, κ regular. Assume K is a class of τ-models, $\varphi = \varphi(\bar x, \bar y)$ is a formula with vocabulary $\subseteq \tau$, and $\partial = \ell g(\bar x) = \ell g(\bar y)$, and*

$(*)$ *$K = K_\lambda$ and for every linear order I of cardinality λ there are $M_I \in K_\lambda$ and a sequence $\langle \bar a_t : t \in I\rangle$ which is $(\kappa, \varphi(\bar x, \bar y))$-skeleton like inside M_I.*

We can conclude that $\dot{\mathbb{I}}(K) = 2^\lambda$ <u>if</u> at least one of the following conditions holds:

(*a*) $\lambda = \lambda^\partial$

(*b*) $\lambda^\kappa < 2^\lambda$

(*c*) *We replace the assumption $(*)$ by:*

$(*)_0$ $K = K_\lambda$,

$(*)_1$ $\lambda^\partial < 2^\lambda$, $\mathrm{cf}(\lambda) > \partial$,

$(*)_2$ *for every linear order J of cardinality λ there are $M_J \in K_\lambda$ and a $(\kappa, < \lambda, \varphi(\bar x, \bar y))$-skeleton like inside M_J sequence $\langle \bar a_s : s \in J\rangle$ (where $\bar a_s \in {}^\partial|M_J|$), see Definition 3.27 below.*

(*d*) *We replace the assumption $(*)$ by: for some $\lambda(0) \le \lambda(1) \le \lambda \le \lambda(3) < 2^\lambda$, $\mu(0) \le \mu(1) \le 2^\lambda$ with $\lambda(1)$ and $\mu(1)$ being regular, we have:*

$(*)_0$ $K = K_{\lambda(3)}$,

$(*)_1$ $\lambda^\partial < 2^\lambda$,

$(*)_2$ *for every linear order J of cardinality λ there is $M_J \in K_{\lambda(3)}$ (of cardinality $\lambda(3)$) and $\langle \bar a_s : s \in J\rangle$ (where $\bar a_s \in {}^\partial|M_J|$) which is $(\kappa, \lambda(0), < \lambda(1), \varphi(\bar x, \bar y))$-skeleton like inside M_J (see Definition 3.27 below),*

$(*)_{3,\mu(0),\lambda(0)}$ *for $J \in K^{\mathrm{or}}_\lambda (= (K_{\mathrm{or}})_\lambda)$ and a set $A \subseteq M_J$ (M_J is from $(*)_2$) if $|A| < \lambda(0)$ then:*

(*i*) *$\mu(0) > |\mathbb{S}^\partial_{\{\varphi,\psi\}}(A, M_J)|$, or at least*

(*ii*) *$\mu(0) > \big|\big\{\mathrm{Av}_{\{\varphi,\psi\}}(\langle \bar b_i : i < \kappa\rangle, A, M_J : \bar b_i \in A$ for $i < \kappa$, the average is well defined and is realized in $M\big\}\big|$, where $\mathrm{Av}_\Delta(\langle b_i : i < \kappa\rangle, A, M_J) := \{\varphi(\bar x, \bar a)^{\mathbf{t}} : \varphi(\bar x, \bar y) \in \Delta, \mathbf{t}$ a truth value, $\bar a \in A$ and and for all but a bounded set of $i < \kappa$, $M_J \models \varphi[\bar b_i, \bar a]^{\mathbf{t}}\}$,*

$(*)_{4,\lambda,\mu(1),\mu(0),\lambda(0)}$ *if $\dot{\mathbf{I}}_i \subseteq {}^\partial\lambda(3)$ and $|\dot{\mathbf{I}}_i| = \lambda$ for $i < \mu(1)$, <u>then</u> for some $B \subseteq \lambda(3)$ we have:*

$$|B| < \lambda(0) \text{ and } |\{i : |\dot{\mathbf{I}}_i \cap {}^\partial B| \ge \kappa\}| \ge \mu(0).$$

(e) *We replace assumption* $(*)$ *by: for some* $\lambda_{0,\epsilon} \le \lambda_{1,\epsilon} \le \lambda \le \lambda_3, \mu_{0,\epsilon} \le \mu_1 \le 2^\lambda$, *for* $\epsilon < \epsilon(*)$, μ_1 *is regular and:*

$(*)_0$ $K = K_{\lambda_3}$,

$(*)_1$ $\lambda^\partial < 2^\lambda$,

$(*)_2$ *for every linear order* J *of cardinality* λ *there is* $M_J \in K_{\lambda(3)}$ *and* $\langle \bar a_s : s \in J\rangle$ *(where* $\bar a_s \in {}^\partial|M_J|$*) which for each* $\epsilon < \epsilon(*)$ *is* $(\kappa_1, < \lambda_{0,i}, < \lambda_{1,i}, \varphi(\bar x, \bar y))$*-skeleton like inside* M_J,

$(*)_{3,\mu_{0,\epsilon},\lambda_{0,\epsilon}}$ *if* $\epsilon < \epsilon(*)$ *and* $J \in K^{\rm or}_\lambda (= (K_{\rm or})_\lambda)$ *and a set* $A \subseteq M_J$ *(*M_J *is from* $(*)_2$*) if* $|A| < \lambda_{0,\epsilon}$ *then:*

(i) $\mu_{0,\epsilon} > |\mathbf{S}^\partial_{\{\varphi,\psi\}}(A, M_J)|$ *or at least*

(ii) $\mu_{0,\epsilon} > |\big\{\mathrm{Av}_{\{\varphi,\psi\}}(\langle \bar b_i : i < \kappa\rangle, A, M_J) : \bar b_i \in A$ *for* $i < \kappa$, *the average is well defined and is realized in* $M\big\}|$, *where*

$$\begin{aligned}\mathrm{Av}_\Delta(\langle b_i : i<\kappa\rangle, A, M_J) := \quad & \{\varphi(\bar x, \bar a)^{\mathbf t} : \varphi(\bar x, \bar y) \in \Delta, \mathbf{t} \text{ a truth value,} \\ & \bar a \in A \text{ and for all but a bounded set of } i < \kappa, \\ & M_J \models \varphi[\bar b_i, \bar a]^{\mathbf t}\},\end{aligned}$$

$(*)_4$ *there are* $h_\alpha : \lambda \longrightarrow \{\theta : \theta$ *regular,* $\kappa \le$ *theta* $\le \lambda\}$ *for* $\alpha < 2^\lambda$ *such that: if* $S \subseteq 2^\lambda$, $|S| \ge \mu(1)$ *and* $f_\alpha : \lambda \longrightarrow {}^\partial(\lambda_3)$ *for* $\alpha \in S$, *<u>then</u> we can find* $\epsilon < \epsilon(*), B \subseteq \lambda_3$ *satisfying:* $|B| < \lambda_{0,\epsilon}$ *and the set* $\{\alpha :$ *the closure of* $\{\zeta < \lambda : f_\alpha(\zeta) \subseteq B\}$ *has a member* δ *of cofinality* κ *such that* $h_\alpha(\delta) \ge \lambda_{1,\epsilon}\}$ *has* $\ge \mu_{0,\epsilon}$ *members. [Note:* $\mathrm{cf}(\delta) = \kappa' \ge \kappa$ *can be allowed if* $(*)_{3,\mu_{0,\epsilon},\lambda_{0,\epsilon}}$ *is changed accordingly].*

(f) *For some* $\mu < \lambda$, *there is a linear order of cardinality* μ *with* $\ge \lambda$ *Dedekind cuts each with upper and lower cofinality* $\ge \kappa$ *and* $2^{\mu+\partial} < 2^\lambda$.

(g) *there is* $\mathscr{P} \subseteq [\lambda^\partial]^\kappa$ *of cardinality* $< 2^\lambda$ *such that every* $X \subseteq \lambda^\partial$ *of cardinality* λ *contains at least one of them (and* $(*)$*); (can use similar considerations in other places).*

Proof. <u>Case (a)</u>:

In Definition 3.14 we can replace A by $\dot{\mathbf J}$, a set of sequences of length ∂ from M, which means that clause (i) in (α) of 3.14 now becomes:

$(i)'$ for every large enough $t \in I$, for every $\bar b \in \dot{\mathbf J}$ we have $M \models \varphi[\bar c, \bar b] = \varphi[\bar a_t, \bar b]$ and $M \models \psi[\bar c, \bar b] \equiv \psi[\bar a_t, \bar b]$.

Thus in Definition 3.17, replace $\langle A_i : i < \lambda\rangle$ by $\langle \dot{\mathbf J}_i : i < \mathrm{cf}(\lambda))\rangle, {}^\partial|M| = \bigcup\limits_i \dot{\mathbf J}_i, |\dot{\mathbf J}_i| < \lambda$, $\dot{\mathbf J}_i$ increasing continuous with i. No further changes in 3.1-3.24 is needed.

Alternatively, we can define $N = F_\partial(M)$ as the model with universe $|M| \cup {}^\partial|M|$, assuming of course $|M|$ is disjoint to ${}^\partial|M|$ such that

$$\tau(N) = \tau(M) \cup \{F_i : i < \partial\},$$

$$R^N = R^M \text{ for } R \in \tau(M),$$

$$G^N(x_1,\dots,x_n) = \begin{cases} G^M(x_1,\dots,x_n) & \text{if } x_1,\dots,x_n \in |M|, \\ x_1 & \text{otherwise} \end{cases}.$$

for function symbol $G \in \tau(M)$ which has n-places and

$$F_i^N(x) = \begin{cases} x(i) & \text{if } x \in {}^{\partial}M, \\ x & \text{if } x \in M \end{cases}$$

for $i < \partial$, so F_i is a new, unary function symbol for $i < \partial$.

Note that $[M_1 \cong M_2$ if and only if $F\partial(M_1) \cong F\partial(M_2)]$, and $\|F\partial(M)\| = \|M\|^{\partial}$, etc. So we can apply 3.24 to the class $\{F\partial(M) : M \in K_\lambda\}$ and the formula $\varphi'(x,y) = \varphi(\langle F_i(x) : i < \partial\rangle, \langle F_i(y) : i < \partial\rangle)$ and we can get the desired conclusion.

Case (b): We use $(\kappa, \varphi(\bar{x},\bar{y}))$-skeleton like sequences $\langle \bar{a}_s : s \in \kappa + (I_\zeta)^*\rangle$ in $M_\zeta \in K_\lambda$ for $\zeta < 2^\lambda$, with $\langle \mathrm{inv}^2_\kappa(I_\zeta) : \zeta < 2^\lambda\rangle$ pairwise distinct, and count the number of models $(M_\zeta, \langle \bar{a}_s : s \in \kappa\rangle)$ up to isomorphism. Then "forget the $\bar{a}_s$, $s \in \kappa$", i.e., use 3.26 below.

Case (c): We revise 3.14–3.25; we use this opportunity to present another reasonable choice in clause (α) of 3.14.

Change 1: In 3.14(α) we replace (i), (ii) by

- $(i)'$ for every formula $\vartheta(\bar{x},\bar{d}) \in \mathrm{tp}_{\{\varphi(\bar{x},\bar{y}),\psi(\bar{x},\bar{y})\}}(\bar{c},A,M)$, for every large enough $t \in I$ we have $M \models \vartheta[\bar{c},\bar{d}] \equiv \vartheta[\bar{a}_t,\bar{d}]$,
- $(ii)'$ $\langle \bar{a}_s : s \in J + (J)^*\rangle$ is $(\kappa, \varphi(\bar{x},\bar{y}))$-skeleton like inside M,
- $(iii)'$ $\theta > \mathrm{cf}(J)$, (actually $\theta \neq \mathrm{cf}(J)$ would suffice, but no real need).

Of course, the meaning of Definition 3.14(β)-(δ) changes, and the reader can check that, e.g., the proof of the Fact 3.16 is still valid.

Change 2: In Definition 3.17(1), inside the definition of $\mathbf{INV}^\alpha_\kappa$, we demand $\mathrm{cf}(\mathbf{d}) = \lambda$ recalling λ is regular.

Change 3: In Definition 3.17(2), inside the definition of $\mathbf{INV}^\alpha_{\kappa,\theta}$ add $\mathrm{cf}(\mathbf{d}_\delta) > \mathrm{cf}(\delta)$ (necessitate by change 1, actually $\mathrm{cf}(\mathbf{d}_\delta) \neq \mathrm{cf}(\delta)$ suffices).

Change 4: In Definition 3.17(3) demand $\mathrm{cf}(\lambda) > \partial$.

Change 5: In 3.19, in all cases the "cardinality $\leq \lambda$" is replaced by "cardinality $\leq \lambda^\partial$" and part (2) becomes like part (3).

Change 6: We replace "$(\kappa, \varphi(\bar{x},\bar{y}))$-skeleton like" by $(\kappa, <\lambda, \varphi(\bar{x},\bar{y}))$-skeleton like. In 3.22(3) add the demand $\mathrm{cf}(\lambda) > \partial$, $h(i) > \mathrm{cf}(i)$.

Change 7: Inside the proof of 3.22(1), now not for every $\bar{a} \in {}^{\ell g(\bar{x})}M$ we define $J_{\bar{a}}$, but for every $A \subseteq M$ of cardinality $< \lambda$ we choose $J_A \subseteq J$, $|J_A| < \lambda$ by Definition 3.27, and in $(*)(ii)$ in the proof there we demand

$$(\forall\alpha<\delta)(\exists\beta<\delta)[\bigcup_{s\in J_{A_\alpha}}\bar a_s\subseteq A\beta].$$

Change 8: In the proof of 3.22(2) let $\langle I_i : i<\theta\rangle$ be as in the statement of 3.22(2), and let $J=\sum_{i<\theta} I^*_i$, and assume $\langle \bar a_s : s\in J\rangle$ is $(\kappa,<\lambda,\varphi(\bar x,\bar y))$-skeleton like inside $M\in K_\lambda$. So let $\langle A_i : i<\theta\rangle$ be a representation of M, and for each $i<\theta$ let $J_{A_i}\subseteq J$, $|J_{A_i}|<\lambda$ be as in Definition 3.27.

Define

$$\begin{aligned}\mathscr{C}=\{\delta<\theta:\ &\delta \text{ is a limit ordinal such that for every } \alpha<\delta\\ &\text{the cardinality of } J_{A_i} \text{ is } <\lambda_\delta\}.\end{aligned}$$

So let $\delta\in C, \mathrm{cf}(\delta)\geq\kappa$. Recall that $\mathrm{cf}(I_\delta)>\lambda_\delta$ so clearly we can find $s(\delta)\in I_\delta$ such that

$$I_\delta\models s(\delta)\leq s\Rightarrow s\notin\bigcup_{i<\delta}J_{A_i}.$$

Now $(\bar c_{s(\delta)},A_\delta,M,\varphi(\bar x,\bar y))$ is as required.

Change 9: In the proof of 3.27(3) let $J=\sum_{\alpha<\theta}I^*_\alpha$ and M, $\langle\bar a_s : s\in J\rangle$, $\langle A_i : i<\mathrm{cf}(\lambda)\rangle$, $J_{A_i}\subseteq J$ be as above, and let $s(\alpha)\in I_\alpha$. As $\mathrm{cf}(\lambda)>\partial$ by $(*)_1$ of the assumption, for each $s\in J$ for some $i(s)<\mathrm{cf}(\lambda)$ we have $\bar c\subseteq A_{i(s)}$, but $\theta=\mathrm{cf}(\theta)>\mathrm{cf}(\lambda)$ hence for some $i(*)<\mathrm{cf}(\lambda)$ the set $W=\{\alpha<\theta : i(\alpha)\leq i(*)\}$ is unbounded in θ. Let $\mathscr{C}=\{\delta<\theta:\delta=\sup(\delta\cap W)\}$. We can choose $\delta\in\mathscr{C}$ of cofinality $\geq\kappa$ such that $h(\delta)>|J_{A_{i(*)}}|$, and continue as in the previous case.

Change 10: Proof of 3.21(2) (necessitated by change 1)

We shall use Lemma 3.10 (with I^1,I^2 here standing for I,J there and ψ for φ). Conditions (b), (c) from 3.10 are met trivially and by similar arguments in condition (a) it is enough to prove clause (α).

Let us prove (a)(α) from 3.10. Let $I^\ell_*\subseteq I^\ell$ be unbounded of order type $\mathrm{cf}(I^\ell)=\theta$ and let $J^\ell_*\subseteq J^\ell$ be unbounded of order type $\mathrm{cf}(J^\ell)$, which is $\neq\theta$. Possibly shrinking those sets the truth values of $\varphi[\bar a^1_s,\bar a^2_t]$ when $s\in I^1_*, y\in J^2\wedge(\exists t')(t'\in J^2_*$ and $t'<_{J^2}t)$ is constant. We can continue as before.

Note that if $\mathrm{cf}(\lambda)>\kappa$ this follows from case (d). If λ is regular, choose $\lambda(0)=\lambda(1)=\lambda(3)=\lambda$ and $\mu(0)=\mu(1)=(\lambda^\partial)^+$ and now the assumptions hold. If λ is singular, let $\epsilon(*)=\mathrm{cf}(\lambda),\chi=(\mathrm{cf}(\lambda)+\kappa)^+\leq\lambda$, $\mu_0=\mu_{1,\epsilon}=(\lambda^\partial)^+$ and let $\{(\lambda_{0,\epsilon},\lambda_{1,\epsilon}):\epsilon<\epsilon(*)\}$ list $\{(\lambda^+_i,\lambda^+_j):i<j<\mathrm{cf}(\lambda)\}$ and choose $h_\lambda=h:\lambda\longrightarrow\{\theta:\theta \text{ regular}, \kappa\leq\theta\leq\lambda\}$ such that $\epsilon<\epsilon(*)=\mathrm{cf}(\lambda)$ implies $\{\delta<\chi:\mathrm{cf}(\delta)=\kappa$ and $h(\delta)=\epsilon\}$ is stationary. Now we can apply case (e).

Case (d): Let $\langle I_\alpha:\alpha<2^\lambda\rangle$ be a sequence of linear orders of cofinality $\mathrm{cf}(\lambda(1))=\lambda(1)$, each of cardinality λ, with pairwise distinct $\mathrm{inv}^2_\kappa(I_\alpha)$ if λ is regular, $\mathrm{inv}^3_\kappa(I_\alpha)$ if λ is singular exists by 3.11. Let $J_\alpha=\sum_{\zeta\leq\lambda}I^*_{\alpha,\zeta}$, where $I_{\alpha,\zeta}$ are pairwise disjoint, $I_{\alpha,\zeta}\cong I_\alpha$. Let M_{J_α} be a model as guaranteed in $(*)_2$ with $\langle\bar a_s:s\in J_\alpha\rangle$ as there. Suppose $\{M_{J_\alpha}/\cong\ :\alpha<2^\lambda\}$ has cardinality $<2^\lambda$, then without loss of generality

$M_{J_\alpha} = M_{J_0}$ for $\alpha < \mu(1)$ and without loss of generality M_{J_0} has universe $\lambda(3)$. Let $s(\alpha,\zeta) \in I_{\alpha,\zeta}$, so

$$\dot{\mathbf{I}}_\alpha := \{\bar{a}_{s(\alpha,\zeta)} : \zeta < \lambda\}$$

is a subset of ${}^{\partial}(\lambda(3))$ of cardinality λ. By $(*)_{4,\lambda,\mu(1),\mu(0),\lambda(0)}$ there is $B \subseteq \lambda(3)$, $|B| < \lambda(0)$ such that

$$S =: \{\alpha < \mu(1) : |\dot{\mathbf{I}}_\alpha \cap {}^{\partial}B| \geq \kappa\}$$

has cardinality $\geq \mu(0)$. Choose for each $\alpha \in S$ a set

$$S_\alpha \subseteq \{\zeta : \bar{a}_{s(\alpha,\zeta)} \subseteq B\},$$

which has order type κ, and let

$$\delta_\alpha =: \sup(S_\alpha).$$

Clearly $\delta_\alpha \leq \lambda$, hence I_{α,δ_α} is well defined. For each $\alpha \in S$, as $\langle \bar{a}_s : s \in J_\alpha\rangle$ is $(\kappa, \lambda(0), < \lambda(1), \varphi(\bar{x},\bar{y}))$-skeleton like and $|B| < \lambda(0)$, there is a subset $J_{\alpha,B}$ of J_α as in Definition 3.27. But I_{α,δ_α} has cofinality $\lambda(1) > |B|$, hence for all large enough $t \in I_{\alpha,\delta_\alpha}$, the type $\mathrm{tp}_{\{\varphi,\psi\}}(\bar{a}_t, B, M_{J_0})$ is the same; choose such t_α. Clearly (for $\alpha \in S$)

$$\mathrm{tp}_{\{\varphi,\psi\}}(\bar{a}_{t_\alpha}, B, M_{J_0}) = \mathrm{Av}_{\{\varphi,\psi\}}(\langle \bar{a}_{s(\alpha,\zeta)} : \zeta \in S_\alpha\rangle, B, M_{J_0}),$$

so by $(*)_{3,\mu(0),\lambda(0)}$ from the assumption of case (d) without loss of generality for some $\alpha \neq \beta$ we get the same type. But I_α, I_β have different (and well defined) inv^2_κ (or inv^3_κ), contradicting 3.21(2).

Case (e):

Similar proof (to (d)).

Case (f):

By 3.26 below.

Case (g):

Similar to case (b). $\square_{3.27}$

In 3.25-3.28 above we do not get anything when $\lambda^\partial = 2^\lambda$, however if we assume that M_J has a clearer structure , e.g., is an EM-model, we can get better results; this will hopefully appear in [Sheb]; also subsection §(3E) hopefully will appear there.

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