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LARGE TURING INDEPENDENT SETS

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Abstract. For a set of reals X and $1 \leq n < \omega$, define X to be n-Turing independent iff the Turing join of any n reals in X does not compute another real in X. X is Turing independent iff it is n-Turing independent for every n. We show the following: (1) There is a non-meager Turing independent set. (2) The statement "Every set of reals of size continuum has a Turing independent subset of size continuum" is independent of ZFC plus the negation of CH. (3) The statement "Every non-meager set of reals has a non-meager n-Turing independent subset" holds in ZFC for $n = 1$ and is independent of ZFC for $n \geq 2$ (assuming the consistency of a measurable cardinal). We also show the measure analogue of (3).

1. Introduction

Let $X \subseteq 2^\omega$ and $1 \leq n < \omega$. We say that X is n-Turing independent iff for every $F \in [X]^{\leq n}$ and $y \in X \setminus F$, the Turing join of F does not compute y. X is Turing independent iff it is n-Turing independent for every $n \geq 1$. In [8], Sacks constructed a Turing independent set of reals of size continuum. One can also construct a Turing independent perfect set $X \subseteq 2^\omega$ by forcing with finite trees (see Lemma 7.1 in [6]). These constructions do not make use of the axiom of choice and therefore cannot produce a non-meager/non-null Turing independent set of reals. This follows from the following.

Fact 1.1. *Suppose $X \subseteq 2^\omega$.*

(a) *If X is non-null and is Lebesgue measurable, then there are $x \neq y$ in X such that $\{k < \omega : x(k) \neq y(k)\}$ is finite.*
(b) *If X is non-meager and has the Baire property, then there are $x \neq y$ in X such that $\{k < \omega : x(k) \neq y(k)\}$ is finite.*

In Section 2, we construct a non-meager Turing independent set. The construction works in ZF + "There exists a non-principal ultrafilter on ω".

Theorem 1.2. *There exists a non-meager Turing independent set of reals.*

The next two sections deal with questions of the following type: Given a "large" $X \subseteq 2^\omega$, must there exist a "large" Turing independent $Y \subseteq X$? In Section 3, we show the following.

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Theorem 1.3. *The following is independent of ZFC plus the negation of CH. Every set of reals of size continuum has a Turing independent subset of size continuum.*

In Section 4, using some facts from [2, 7] about effective randomness/genericity, we prove the following.

Theorem 1.4. *For every non-meager (resp. non-null) $X \subseteq 2^\omega$, there exists a non-meager (resp. non-null) $Y \subseteq X$ such that Y is 1-Turing independent.*

Finally, we show that getting large 2-Turing independent subsets may not be possible.

Theorem 1.5. *Let $n \geq 2$. Each of the following statements is consistent relative to ZFC and its negation is consistent relative to ZFC plus there is a measurable cardinal.*

(a) *For every non-meager $X \subseteq 2^\omega$, there exists a non-meager $Y \subseteq X$ such that Y is n-Turing independent.*
(b) *For every non-null $X \subseteq 2^\omega$, there exists a non-null $Y \subseteq X$ such that Y is n-Turing independent.*

Notation. For $F = \{x_0, x_2, \dots, x_{n-1}\} \subseteq 2^\omega$, the join of F, denoted $\bigoplus_{k<n} x_k$, is the real $y \in 2^\omega$ satisfying $y(nj + k) = x_k(j)$ for every $k < n$ and $n, j < \omega$. $\langle \Phi_e : e < \omega \rangle$ is an effective listing of all Turing functionals. Given $y \in 2^\omega$ and $k < \omega$, we write $\Phi_e^y(k) = n$ iff the eth Turing functional with oracle y converges on input k and outputs n. We write $\Phi_e^y(k) \neq n$ iff either $\Phi_e^y(k)$ diverges or it converges to a value different from n. If the oracle use of the computation "$\Phi_e^y(k) = n$" is included in an initial segment $\sigma \preceq y$, then we also write $\Phi_e^\sigma(k) = n$. For $x, y \in 2^\omega$, define $\Phi_e^y = x$ iff $(\forall k < \omega)(\Phi_e^y(k) = n)$. So $x \leq_T y$ iff for some $e < \omega$, $\Phi_e^y = x$. For $\sigma \in 2^{<\omega}$, define $[\sigma] = \{x \in 2^\omega : \sigma \subseteq x\}$. μ denotes the standard product measure on 2^ω. For $Y \subseteq X \subseteq 2^\omega$, we say that Y is everywhere non-meager (resp. has full outer measure) in X iff for every Borel $B \subseteq 2^\omega$, if $B \cap X$ is non-meager (resp. non-null), then $B \cap Y \neq \emptyset$. Cohen_X is the poset consisting of all finite partial functions from X to 2 ordered by reverse inclusion.

2. A non-meager Turing independent set

Throughout this section, unless stated otherwise, we work in ZF. Although one cannot show in ZF that the meager ideal is a σ-ideal, this doesn't affect the argument below.

Definition 2.1. Let $\bar{\eta} = \langle \eta_k : k \leq N \rangle$ be a finite sequence of members of $2^{<\omega}$. Define $\mathsf{Split}_e(\bar{\eta})$ to be the statement: For every $\langle x_k : k \leq N \rangle$ where each $\eta_k \subseteq x_k \in 2^\omega$, there exists $j \in \mathsf{dom}(\eta_N)$ such that $\Phi_e^X(j) \neq \eta_N(j)$ where $X = \bigoplus_{k<N} x_k$.

Observe that if $\bar{\sigma} = \langle \sigma_k : k \leq N \rangle$, $\bar{\tau} = \langle \tau_k : k \leq N \rangle$, for each $k \leq N$, $\sigma_k \subseteq \tau_k$ and $\mathsf{Split}_e(\bar{\sigma})$ holds, then $\mathsf{Split}_e(\bar{\tau})$ also holds.

Lemma 2.2 (ZF). *Suppose $e < \omega$, $N \geq 1$ and $\bar{\rho} = \langle \rho_k : k \leq N \rangle$ is a finite sequence of members of $2^{<\omega}$. Then there exists $\bar{\eta} = \langle \eta_k : k \leq N \rangle$ such that for every $k \leq N$, $\rho_k \subseteq \eta_k$ and $\mathsf{Split}_e(\bar{\eta})$ holds.*

Proof. Let $j_\star = \min(\omega \setminus \mathsf{dom}(\rho_N))$. First suppose there exists $\langle y_k : k < N \rangle$ such that the following hold.

(a) For every $k < N$, $\rho_k \subseteq y_k \in 2^\omega$.
(b) $\Phi_e^Y(j_\star)$ converges and outputs $i < 2$ where $Y = \bigoplus_{k<N} y_k$.

In this case, fix such $\langle y_k : k < N\rangle$ and i, define $\eta_N = \rho_N \cup \{(j_\star, 1-i)\}$ and choose $\eta_k \subseteq y_k$ for $k < N$ such that $\bigoplus_{k<N} \eta_k$ contains the use of the computation $\Phi_e^Y(j_\star)$.

If there is no such $\langle y_k : k < N\rangle$, then define $\eta_k = \rho_k$ for each $k < N$ and $\eta_N = \rho_N \cup \{(j_\star, 0)\}$. It is clear that $\bar{\eta} = \langle \eta_k : k \leq N\rangle$ satisfies $\mathsf{Split}_e(\bar{\eta})$. □

Lemma 2.3 (ZF). *For each $n < \omega$ there exist k and f satisfying $\dagger(n,k,f)$ where $\dagger(n,k,f)$ says the following: $n < k < \omega$, $f : {}^n2 \to {}^{[n,k)}2$ and for every sequence $\langle \rho_k : k \leq N\rangle$ of pairwise distinct members of ${}^n2$ (where $N \geq 1$) and for every $e < n$, $\mathsf{Split}_e(\bar{\eta})$ holds where $\bar{\eta} = \langle \rho_k^\frown f(\rho_k) : k \leq N\rangle$.*

Proof. Easily follows by repeatedly applying Lemma 2.2. □

Fix a recursive well-ordering $\prec$ of

$$\mathcal{F} = \{(k,f) : k < \omega \text{ and } (\exists n < k)(f : {}^n2 \to {}^{[n,k)}2)\}.$$

Definition 2.4. Using Lemma 2.3, define $\langle k_n : n < \omega\rangle$ and $\langle F_n : n < \omega\rangle$ as follows. For each $n < \omega$, (k_n, F_n) is the $\prec$-least member of $\mathcal{F}$ such that $\dagger(n, k_n, F_n)$ holds. Define the function F by $\mathsf{dom}(F) = 2^{<\omega}$ and for every $\sigma \in 2^{<\omega}$, $F(\sigma) = F_{|\sigma|}(\sigma)$. Define $K : \omega \to \omega$ by $K(0) = 0$ and $K(n+1) = k_{K(n)}$.

Note that $\langle k_n : n < \omega\rangle$, $\langle F_n : n < \omega\rangle$, K and F are all definable without parameters.

Lemma 2.5 (ZF). *Let $\mathcal{U}$ be a non-principal ultrafiter on ω. Let $\mathcal{C}$ be the set of all pairs $(\mathbf{m}, x)$ where $\mathbf{m} = \langle m_k : k < \omega\rangle$ is a strictly increasing sequence in ω with $m_0 = 0$ and $x \in 2^\omega$. Then there exists a function $H : \mathcal{C} \to 2^\omega$ such that the following hold.*

(1) *H is definable from $\mathcal{U}$.*
(2) *For every $(\mathbf{m}, x) \in \mathcal{C}$, if $H(\mathbf{m}, x) = y$, then there are infinitely many $k < \omega$ such that $y \restriction [m_k, m_{k+1}) = x \restriction [m_k, m_{k+1})$.*
(3) *For every $y \in \mathsf{range}(H)$, $\{n < \omega : F(y \restriction K(n)) \subseteq y\} \in \mathcal{U}$. Here K, F are as in Definition 2.4.*

Proof. Fix $(\mathbf{m}, x) \in \mathcal{C}$. Define $\langle n(j) : j < \omega\rangle$ as follows.

(i) $n(0) = 0$.
(ii) $n(j+1) = K(n(j)) + m_{n(j)+1} + 1$.

Note that $\langle n(j) : j < \omega\rangle$ is a strictly increasing sequence in ω such that for each $j < \omega$, both $K(n(j))$ and $m_{n(j)+1}$ are strictly less than $n(j+1)$.

Fix $r_\star < 3$, such that

$$\bigcup\{[n(j), n(j+1)) : j = r_\star \pmod 3\} \in \mathcal{U}.$$

Inductively construct $y \in 2^\omega$ such that for every $j < \omega$, if $j = r_\star \pmod 3$, then the following hold.

(a) $n(j) \leq n < n(j+1) \implies F(y \restriction K(n)) = y \restriction [K(n), K(n+1))$.
(b) $x \restriction [m_{n(j+2)}, m_{n(j+2)+1}) = y \restriction [m_{n(j+2)}, m_{n(j+2)+1})$.

Since $K(n(j+1)) < n(j+2) \leq m_{n(j+2)} < m_{n(j+2)+1} < n(j+3)$, there is no conflict among the two clauses. Define $H(\mathbf{m}, x) = y$. Observe that clause (a) guarantees that $\{n < \omega : F(y \restriction K(n)) \subseteq y\} \in \mathcal{U}$ while clause (b) ensures that there are infinitely many $k < \omega$ such that $y \restriction [m_k, m_{k+1}) = x \restriction [m_k, m_{k+1})$. It is also clear that H is definable from $\mathcal{U}$. □

The following is well-known (for example, see Theorem 2.2.4 in [1]). The proof given there works in ZF.

Lemma 2.6 (ZF). *For every meager $W \subseteq 2^\omega$, there exist $\langle m_k : k < \omega\rangle$ and $x \in 2^\omega$ such that the following hold.*

(i) *$m_0 = 0$, m_k's are strictly increasing in ω.*
(ii) *For every $y \in W$, for all but finitely many $k < \omega$, there exists $n \in [m_k, m_{k+1})$ such that $x(n) \neq y(n)$.*

Proof of Theorem 1.2. We work in ZF + "There exists a non-principal ultrafilter on ω". Fix a non-principal ultrafilter $\mathcal{U}$ on ω. Let $H : \mathcal{C} \to 2^\omega$ be as in Lemma 2.5. Put $Y = \mathsf{range}(H)$. By Lemma 2.6, Y is non-meager so it suffices to show that Y is Turing independent. Suppose not and fix $N \geq 1$ and pairwise distinct members y_0, y_1, …, y_N of Y such that the join of $\{y_0, y_1, \dots y_{N-1}\}$ computes y_N. Put $X = \bigoplus_{k<N} y_k$ and choose $e < \omega$ such that for every $j < \omega$, $\Phi_e^X(j) = y_N(j)$. Define

$$T = \{n < \omega : (\forall k \leq N)(F(y_k \restriction K(n)) \subseteq y_k)\}.$$

Then $T \in \mathcal{U}$. Since y_k's are pairwise distinct, we can find $n \in T$ such that $e < n$ and $\langle y_k \restriction K(n) : k \leq N\rangle$ has pairwise distinct members in $2^{K(n)}$. Define $\bar\eta = \langle y_k \restriction K(n+1) : k \leq N\rangle$. Since $n \in T$, for each $k \leq N$, we must have

$$y_k \restriction K(n+1) = (y_k \restriction K(n))^\frown F_{K(n)}(y_k \restriction K(n)).$$

By Lemma 2.3, it follows that $\mathsf{Split}_e(\bar\eta)$ holds. But this contradicts $\Phi_e^X = y_N$. □

It is unclear how to adapt this argument for the case of measure.

Question 2.7. Must there exist a Turing independent non-null set of reals?

3. Large Turing independent subsets: Cardinality

Given $X \subseteq 2^\omega$, can we find a Turing independent subset of X which has the same cardinality as X? Since X could be a $\leq_T$-chain of size ω_1, we should assume $\omega_2 \leq |X| \leq \mathfrak{c}$. Theorem 3.1 implies that a positive answer is consistent with arbitrarily large continuum.

Theorem 3.1. *Assume $V \models GCH$. Let $\mathbb{P}$ be the forcing for adding κ Cohen reals where $\omega_2 \leq \kappa = \kappa^{\aleph_0}$. Then the following hold in $V^{\mathbb{P}}$.*

(1) *$\mathfrak{c} = \kappa$.*
(2) *For every $\omega_2 \leq \lambda \leq \mathfrak{c}$ and $X \in [2^\omega]^\lambda$ there exists $Y \in [X]^\lambda$ such that for every $n \geq 1$ and $B : (2^\omega)^n \to 2^\omega$ where B is a Borel function coded in V, Y is B-independent which means the following: For every x_0, …, x_{n-1} in Y, $B(x_0, \dots, x_{n-1}) \notin Y \setminus \{x_0, \dots, x_{n-1}\}$.*
(3) *For every $\omega_2 \leq \lambda \leq \mathfrak{c}$ and $X \in [2^\omega]^\lambda$ there exists $Y \in [X]^\lambda$ such that Y is Turing independent.*

A similar result holds in the random real model. The proof is similar to the one we give below for the Cohen case. Note that, in Theorem 3.1, Clause (3) follows from Clause (2).

Proof. Let $\bar c : \kappa \to 2$ be the Cohen_κ-generic sequence added by $\mathbb{P}$. A standard name counting argument shows that $V[\bar c] \models \mathfrak{c} = \kappa$. Fix $\omega_2 \leq \lambda \leq \kappa$ and assume $V[\bar c] \models X = \{x_\alpha : \alpha < \lambda\}$ consists of pairwise distinct members of 2^ω. Since $V \models \mathfrak{c} = \omega_1 < \lambda$, by thinning out X, we can assume that for every $n \geq 1$ and a

Borel function $B : (2^\omega)^n \to 2^\omega$ coded in V, whenever $\beta < \lambda$ and $\alpha_0, \dots, \alpha_{n-1} < \beta$, we have $B(x_{\alpha_0}, \dots, x_{\alpha_{n-1}}) \neq x_\beta$. WLOG, let us assume that the empty condition forces this.

For each $\alpha < \lambda$ and $i < \omega$, choose a maximal antichain $A_{\alpha,i}$ of conditions in $\mathbb{P}$ deciding $\mathring{x}_\alpha(i)$. WLOG, each $A_{\alpha,i} \in [\mathbb{P}]^{\aleph_0}$. Let $\langle p_{\alpha,i,n} : n < \omega \rangle$ be a one-one listing of $A_{\alpha,i}$. Let $\varepsilon_{\alpha,i,n} < 2$ be such that $p_{\alpha,i,n} \Vdash \mathring{x}_\alpha(i) = \varepsilon_{\alpha,i,n}$. Define $W_\alpha = \bigcup\{\mathsf{dom}(p_{\alpha,i,n}) : i, n < \omega\}$. So $W_\alpha \in [\kappa]^{\aleph_0}$.

Case 1. λ is singular. Let $\mu = \mathsf{cf}(\lambda) < \lambda$. Fix a strictly increasing sequence $\langle \lambda_j : j < \mu \rangle$ cofinal in λ such that $\mu < \lambda_0$ and each $\lambda_j = \theta^{++}$ for some $\theta < \lambda$. For each $j < \mu$, using GCH plus the Δ-system lemma (Theorem 1.6, Chapter II in [5]), choose $T_j \subseteq [\lambda_j, \lambda_{j+1})$ such that $|T_j| = \lambda_{j+1}$ and $\langle W_\alpha : \alpha \in T_j \rangle$ forms a Δ-system with root R_j. Put $R = \bigcup\{R_j : j < \mu\}$ and note that $|R| \le \mu$. For each $j < \mu$, the set of $\alpha \in T_j$ for which $(W_\alpha \setminus R_j) \cap R \neq \emptyset$ has size $\le \mu$. By throwing these away, we can assume that for every $\alpha \in T_j$, $(W_\alpha \setminus R_j) \cap R = \emptyset$. By inductively thinning out T_j's once more, we can also assume that for every $i < j$ in μ, $\alpha \in T_i$ and $\beta \in T_j$, $(W_\alpha \setminus R_i) \cap (W_\beta \setminus R_j) = \emptyset$.

For each $j < \mu$, choose $S_j \in [T_j]^{\lambda_{j+1}}$ such that the names $\{\mathring{x}_\alpha : \alpha \in S_j\}$ are pairwise isomorphic in the following sense.

- (i) $\mathsf{otp}(W_\alpha) = \gamma_j$ does not depend on $\alpha \in S_j$. Let $h_\alpha : \gamma_j \to W_\alpha$ be the unique order isomorphism.
- (ii) $h_\alpha^{-1}[R_j] = \Delta_j$ does not depend on $\alpha \in S_j$.
- (iii) For every $i, n < \omega$, $\varepsilon_{i,n} = \varepsilon_{\alpha,i,n}$ and $p_{i,n} = p_{\alpha,i,n} \circ h_\alpha$ do not depend on $\alpha \in S_j$.

Such S_j's exist as $V \models \mathfrak{c} = \omega_1 < \lambda_j = \mathsf{cf}(\lambda_j)$. Put $Y = \{x_\alpha : (\exists j < \mu)(\alpha \in S_j)\}$ and note that $Y \in [X]^\lambda$.

We claim that Y is as required. Suppose not and towards a contradiction, fix $n \ge 1$, $B : (2^\omega)^n \to 2^\omega$ where B is a Borel function coded in V and pairwise distinct $y_0, y_1, \dots, y_n$ in Y such that $B(y_0, \dots, y_{n-1}) = y_n$. For each $0 \le m \le n$, fix $j(m) < \mu$ and $\alpha(m) \in S_{j(m)}$ such that $y_m = x_{\alpha(m)}$. Fix $p \in \mathbb{P}$ such that $p \Vdash B(\mathring{x}_{\alpha(0)}, \dots, \mathring{x}_{\alpha(n-1)}) = \mathring{x}_{\alpha(n)}$. Choose $\beta \in S_{j(n)}$ such that $\beta \notin \{\alpha(m) : 0 \le m \le n\}$ and $(W_\beta \setminus R_{j(n)}) \cap \mathsf{dom}(p) = \emptyset$. Let $\pi : \kappa \to \kappa$ be such that $\pi \restriction W_{\alpha(n)} : W_{\alpha(n)} \to W_\beta$ and $\pi \restriction W_\beta : W_\beta \to W_{\alpha(n)}$ are order preserving bijections and $\pi \restriction (\kappa \setminus (W_{\alpha(n)} \cup W_\beta))$ is the identity. Recall that $W_{\alpha(n)} \cap W_\beta = R_{j(n)}$ and note that $\pi \restriction R_{j(n)}$ is also the identity by Clause (ii) above. Define $\hat{\pi} : \mathbb{P} \to \mathbb{P}$ by $\hat{\pi}(q) = r$ iff $\mathsf{dom}(r) = \pi[\mathsf{dom}(q)]$ and $q(\alpha) = r(\pi(\alpha))$. Then $\hat{\pi}$ is an automorphism of $\mathbb{P}$. Note that for each $0 \le m < n$, $\pi \restriction W_{\alpha(m)}$ is the identity and so $\hat{\pi}(\mathring{x}_{\alpha(m)}) = \mathring{x}_{\alpha(m)}$. Furthermore, by Clauses (i)–(iii) above, $\hat{\pi}(\mathring{x}_{\alpha(n)}) = \mathring{x}_\beta$. Therefore $\hat{\pi}(p) \Vdash B(\mathring{x}_{\alpha(0)}, \dots, \mathring{x}_{\alpha(n-1)}) = \mathring{x}_\beta$. Now by the choice of β, it is easy to see that p and $\pi(p)$ are compatible. Letting $p_\star = p \cup \pi(p)$, we get $p_\star \Vdash B(\mathring{x}_{\alpha(0)}, \dots, \mathring{x}_{\alpha(n-1)}) = \mathring{x}_{\alpha(n)} = \mathring{x}_\beta$. But the empty condition forces that $\mathring{x}_{\alpha(n)} \neq \mathring{x}_\beta$ which is a contradiction.

Case 2. λ is regular. If λ is not the successor of a limit cardinal of countable cofinality, then we can apply the Δ-system lemma and proceed as in Case 1 – In fact, we can find $S \in [\lambda]^\lambda$ such that $\langle W_\alpha : \alpha \in S \rangle$ is a Δ-system and the names $\langle \mathring{x}_\alpha : \alpha \in S \rangle$ are pairwise isomorphic. To deal with the other case, we will use the following.

Lemma 3.2. *Suppose λ is regular uncountable and γ is an infinite ordinal such that $\beth_2(|\gamma|) < \lambda$. Let $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be a sequence of pairwise distinct injective* *functions from γ to ordinals. Then there exists $S \subseteq \lambda$ stationary in λ such that the following holds. For every $k \leq n < \omega$ and a strictly increasing sequence $\bar{\alpha} = \langle \alpha_j : j \leq n \rangle$ of members of S, there exists $\bar{\beta} = \langle \beta_j : j \leq n \rangle$ such that each the following hold.*

(1) *For every $j \leq k$, $\beta_j = \alpha_j$.*
(2) $\beta_n < \beta_{n-1} < \cdots < \beta_{k+1} < \min(S) \leq \alpha_0$.
(3) *$\bar{\alpha}$ and $\bar{\beta}$ are $\bar{f}$-similar which means the following: For every $j, m \leq n$ and $\xi_1, \xi_2 < \gamma$,*

$$f_{\alpha_j}(\xi_1) = f_{\alpha_m}(\xi_2) \iff f_{\beta_j}(\xi_1) = f_{\beta_m}(\xi_2).$$

Proof. WLOG, we can assume that each $f_\alpha : \gamma \to \lambda$. Put $\mu = (2^{|\gamma|})^+$. Then $\mu \leq \beth_2(|\gamma|) < \lambda$. Set $\chi = (\beth_5(\lambda))^+$ and fix a continuously increasing chain $\bar{\mathcal{N}} = \langle \mathcal{N}_\alpha : \alpha < \lambda \rangle$ of elementary submodels of $(\mathcal{H}_\chi, \in, <_\chi)$ such that $\bar{f} \in \mathcal{N}_0$, $\mathcal{H}_\mu \subseteq \mathcal{N}_0$ and for every $\alpha < \lambda$, $|\mathcal{N}_\alpha| < \lambda$, $\mathcal{N}_\alpha \cap \lambda \in \lambda$ and $\bar{\mathcal{N}} \restriction (\alpha + 1) \in \mathcal{N}_{\alpha+1}$. Put $S_0 = \{\delta < \lambda : \mathsf{cf}(\delta) = \mu\}$ and $S_1 = \{\delta \in S_0 : (\forall \alpha < \delta)(\mathsf{range}(f_\alpha) \subseteq \delta)$ and $N_\delta \cap \lambda = \delta\}$. Note that S_1 is stationary in λ. For each $\delta \in S_1$, define

$$J_\delta = \{u \subseteq \gamma : f_\delta \restriction u \in \mathcal{N}_\delta\}.$$

Observe that each J_δ is an ideal on γ. For each $\delta \in S_1$, define $g(\delta)$ to be the least $\alpha < \delta$ such that for every $u \in J_\delta$, $f_\delta \restriction u \in \mathcal{N}_\alpha$. Since $\mathsf{cf}(\delta) = \mu > 2^{|\gamma|}$, $g(\delta)$ is well-defined. Using Fodor's lemma, choose $S \subseteq S_1$ stationary in λ such that $g \restriction S$ is constant. Since $\beth_2(|\gamma|) < \lambda$, we can also assume that $J_\delta = J_\star$ does not depend on $\delta \in S$. Put $\alpha_\star = \min(S)$. We will show that S is as required.

Fix $n \geq 1$. By induction on $n - k$, we'll show that for every strictly increasing sequence $\bar{\alpha} = \langle \alpha_j : j \leq n \rangle$ of members of S, there exists $\bar{\beta} = \langle \beta_j : j \leq n \rangle$ such that Clauses (1)–(3) above hold. If $k = n$, then this is trivial so assume $0 \leq k < n$. By inductive hypothesis, we can fix $\bar{\eta}$ such that the following hold.

(A) For every $j \leq k + 1$, $\eta_j = \alpha_j$.
(B) $\eta_n < \eta_{n-1} < \cdots < \eta_{k+2} < \alpha_\star$.
(C) $\bar{\alpha}$ and $\bar{\eta}$ are $\bar{f}$-similar.

Define $\beta_m = \eta_m$ for $m \neq k + 1$. It suffices to find $\beta_{k+1} < \alpha_\star$ strictly above β_{k+2} such that $\bar{\alpha}$ and $\bar{\beta}$ are $\bar{f}$-similar.

For each $m \neq k + 1$, define

$$u_m = \{\xi < \gamma : f_{\alpha_{k+1}}(\xi) \in \mathsf{range}(f_{\beta_m})\}.$$

We claim that each $u_m \in J_\star$ and $f_{\alpha_{k+1}} \restriction u_m \in \mathcal{N}_{\alpha_\star}$. To see this, using the fact that each f_α is injective, define $h_m : u_m \to \gamma$ by $h_m(\xi) = \xi'$ iff $f_{\alpha_{k+1}}(\xi) = f_{\beta_m}(\xi')$. Since $\mathcal{H}_{\gamma^+} \subseteq \mathcal{N}_{\alpha_{k+1}}$, we get $h_m \in \mathcal{N}_{\alpha_{k+1}}$. Now $f_{\beta_m} \in \mathcal{N}_{\alpha_{k+1}}$ (as $\beta_m < \alpha_{k+1}$), so $f_{\alpha_{k+1}} \restriction u_m = f_{\beta_m} \circ h_m \in \mathcal{N}_{\alpha_{k+1}}$. It follows that $u_m \in J_{\alpha_{k+1}} = J_\star$. That $f_{\alpha_{k+1}} \restriction u_m \in \mathcal{N}_{\alpha_\star}$ follows from the fact that $g \restriction S$ takes a constant value below $\alpha_\star$. Let $w_m = \mathsf{range}(f_{\alpha_{k+1}} \restriction u_m)$.

Define $U = \bigcup\{u_m : m \neq k+1\}$ and $W = \bigcup\{\mathsf{range}(f_{\alpha_{k+1}} \restriction u_m) : m \neq k+1\}$ and note that u_m, w_m, U and W are all in $\mathcal{N}_{\alpha_\star}$. Let X be the set of $\delta \in S_0$ such that $\delta > \beta_{k+2}$ and (a) + (b) + (c) below hold.

(a) $(\forall m \neq k + 1)(f_\delta \restriction u_m = f_{\alpha_{k+1}} \restriction u_m)$.
(b) $(\forall m \neq k + 1)(\forall \xi \in \gamma \setminus u_m)(f_\delta(\xi) \notin w_m)$.
(c) $(\forall m > k + 1)(\forall \xi \in \gamma \setminus u_m)(f_\delta(\xi) \notin \mathsf{range}(f_{\beta_m}))$.

Then X is definable in $\mathcal{H}_\chi$ with parameter from $\mathcal{N}_{\alpha_\star}$. So $X \in \mathcal{N}_{\alpha_\star}$. Furthermore, since $\delta = \alpha_{k+1} \in X \setminus \mathcal{N}_{\alpha_\star}$, it follows that X is unbounded in $\alpha_\star$.

Let $\delta_\star \in X \cap \alpha_\star$. Suppose $m \neq k+1$ and $\xi_1, \xi_2 < \gamma$ are such that $f_{\alpha_{k+1}}(\xi_1) = f_{\alpha_m}(\xi_2)$. Since $\bar\eta$ and $\bar\alpha$ are $\bar f$-similar, we get $f_{\alpha_{k+1}}(\xi_1) = f_{\eta_{k+1}}(\xi_1) = f_{\beta_m}(\xi_2)$. It also follows that $\xi_1 \in u_m$. Since $\delta_\star \in X$, $f_{\delta_\star}(\xi_1) = f_{\alpha_{k+1}}(\xi_1)$. Therefore $f_{\delta_\star}(\xi_1) = f_{\beta_m}(\xi_2)$.

Now assume that $f_{\alpha_{k+1}}(\xi_1) \neq f_{\alpha_m}(\xi_2)$. Put $f_{\alpha_m}(\xi_2) = \eta$. Furthermore, suppose $\eta \in \mathsf{range}(f_{\alpha_{k+1}})$. Choose ξ_3 such that $f_{\alpha_{k+1}}(\xi_3) = f_{\alpha_m}(\xi_2) = \eta$. Repeating the above argument, we get $f_{\delta_\star}(\xi_3) = f_{\beta_m}(\xi_2)$. Since $f_{\delta_\star}$ is injective, it follows that $f_{\delta_\star}(\xi_1) \neq f_{\beta_m}(\xi_2)$. Next, suppose $\eta \notin \mathsf{range}(f_{\alpha_{k+1}})$. If $m > k+1$ and $\xi_1 \in u_m$, then by Clause (a), $f_{\delta_\star}(\xi_1) = f_{\alpha_{k+1}}(\xi_1)$. As $\bar\alpha$ and $\bar\eta$ are $\bar f$-similar, we also have $f_{\alpha_{k+1}}(\xi_1) \neq f_{\beta_m}(\xi_2)$ and therefore $f_{\delta_\star}(\xi_1) \neq f_{\beta_m}(\xi_2)$. If $m > k+1$ and $\xi_1 \notin u_m$, then Clause (c) implies that $f_{\delta_\star}(\xi_1) \neq f_{\beta_m}(\xi_2)$. Finally, if $m < k+1$, then showing $f_{\delta_\star}(\xi_1) \neq f_{\beta_m}(\xi_2) = f_{\alpha_m}(\xi_2)$ boils down to showing the following: For every $m < k+1$, we have $\mathsf{range}(f_{\delta_\star}) \cap \mathsf{range}(f_{\alpha_m}) \subseteq \mathsf{range}(f_{\alpha_{k+1}}) \cap \mathsf{range}(f_{\alpha_m})$.

Construct $\langle (Y_i, W_i) : i < \gamma^+ \rangle$ as follows.

(i) $Y_0 = \{\beta_m : m > k+1\}$, Y_i's are continuously increasing and $Y_i \setminus Y_0 \in [X]^{\leq 2^{|\gamma|}}$.

(ii) $W_i = W \cup \bigcup\{\mathsf{range}(f_\delta) : \delta \in Y_i\}$. Recall that $W = \bigcup\{\mathsf{range}(f_{\alpha_{k+1}} \restriction u_m) : m \neq k+1\}$.

(iii) For each $\delta_1 \in X \setminus Y_i$, there exists $\delta_2 \in Y_{i+1} \setminus Y_i$ such that for every $\xi < \gamma$
   (a) $f_{\delta_1}(\xi) \in W_i \iff f_{\delta_2}(\xi) \in W_i$ and
   (b) $f_{\delta_1}(\xi) \in W_i \implies f_{\delta_1}(\xi) = f_{\delta_2}(\xi)$.

Note that Clause (iii) requires us to add at most $2^{|\gamma|}$ ordinals to $Y_{i+1} \setminus Y_i$. Furthermore, the construction is definable in $(\mathcal{H}_\chi, \in, <_\chi)$ since we can use the well-ordering $<_\chi$ to choose least witnesses for Clause (iii). So $\langle (Y_i, W_i) : i < \gamma^+ \rangle \in \mathcal{N}_{\alpha_\star}$.

We claim that for each $i < \gamma^+$, Y_i and therefore W_i are subsets of $\mathcal{N}_{\alpha_\star}$. As $\langle (Y_i, W_i) : i < \gamma^+ \rangle \in \mathcal{N}_{\alpha_\star}$ and $\gamma^+ + 1 \subseteq \mathcal{H}_\mu \subseteq \mathcal{N}_{\alpha_\star}$, each $Y_i \in \mathcal{N}_{\alpha_\star}$. Since $|Y_i| \leq 2^{|\gamma|} < \mu$ and $\mathcal{H}_\mu \subseteq \mathcal{N}_{\alpha_\star}$, it also follows that $Y_i \subseteq \mathcal{N}_{\alpha_\star}$.

Choose $i_\star < \gamma^+$ such that for every $m \leq k+1$,

$$\mathsf{range}(f_{\alpha_m}) \cap \bigcup_{i<\gamma^+} W_i \subseteq \mathsf{range}(f_{\alpha_m}) \cap W_{i_\star}$$

Using Clause (iii) above with $\delta_1 = \alpha_{k+1}$, get $\delta_{\star\star} = \delta_2 \in Y_{i_\star+1} \setminus Y_{i_\star}$ satisfying (a)+(b) there. Suppose $m < k+1$ and $\eta \in \mathsf{range}(f_{\delta_{\star\star}}) \cap \mathsf{range}(f_{\alpha_m})$. Fix $\xi < \gamma$ such that $f_{\delta_{\star\star}}(\xi) = \eta$. Note that $\eta \in W_{i_\star}$. So by Clause (iii)(a)+(b), we must have $\eta = f_{\delta_{\star\star}}(\xi) = f_{\alpha_{k+1}}(\xi)$. Hence $\eta \in \mathsf{range}(f_{\alpha_{k+1}}) \cap \mathsf{range}(f_{\alpha_m})$. It follows that for every $m < k+1$, $\mathsf{range}(f_{\delta_{\star\star}}) \cap \mathsf{range}(f_{\alpha_m}) \subseteq \mathsf{range}(f_{\alpha_{k+1}}) \cap \mathsf{range}(f_{\alpha_m})$. So we can take $\beta_{k+1} = \delta_{\star\star}$. This concludes the proof of Lemma 3.2. □

Let us return to Case 2 and assume that λ is the successor of a singular cardinality of countable cofinality. Since $V \models \lambda = \mathsf{cf}(\lambda) > \mathfrak{c} = \omega_1$, we can assume that $\{\mathring{x}_\alpha : \alpha < \lambda\}$ consists of pairwise isomorphic names. Fix $\gamma_\star < \omega_1$ such that $\mathsf{otp}(W_\alpha) = \gamma_\star$. For $\alpha < \lambda$, let $f_\alpha : \gamma_\star \to W_\alpha$ be the unique order preserving bijection. Using GCH we can apply Lemma 3.2 with $\gamma = \gamma_\star$ and $\bar f = \langle f_\alpha : \alpha < \lambda \rangle$ to get $S \subseteq \lambda$ satisfying the conclusion there. Let us check that $\{x_\alpha : \alpha \in S\}$ is as required. Towards a contradiction, fix $n \geq 1$, a Borel function $B : (2^\omega)^n \to 2^\omega$ coded in V, $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ in S and $k < n$ ($k = n$ is not possible) such that $B(y_0, y_1, \dots, y_{n-1}) = x_{\alpha_k}$ where $\{y_j : j \le n-1\} = \{x_{\alpha_j} : j \ne k\}$. Since permuting the inputs of B gives rise to another Borel function, we can assume that $B(x_{\alpha_0}, \dots, x_{\alpha_{k-1}}, x_{\alpha_{k+1}}, \dots, x_{\alpha_{n-1}}) = x_{\alpha_k}$. Choose $\bar\beta$ such that Clauses (1)–(3) of Lemma 3.2 hold. Since $\bar\alpha$ and $\bar\beta$ are $\bar f$-similar, we can choose a bijection $\pi : \lambda \to \lambda$ satisfying $f_{\alpha_j} = \pi \circ f_{\beta_j}$ for every $j \le n$. Now we repeat the automorphism argument. Put $\bar d = \bar c \circ \pi$ and let x'_α be the evaluation of $\mathring{x}_\alpha$ via $\bar d$. Then $x'_{\alpha_m} = x_{\alpha_m}$ for $m \le k$ and $x'_{\alpha_m} = x_{\beta_m}$ for $k < m \le n$. Hence $B(x_{\alpha_0}, \dots, x_{\alpha_{k-1}}, x'_{\beta_{k+1}}, \dots, x'_{\beta_n}) = x_{\alpha_k}$. So for some $p \in \mathbb{P}$ with $p \subseteq \bar d$, $p \Vdash_{\mathbb{P}} B(\mathring{x}_{\alpha_0}, \dots, \mathring{x}_{\alpha_{k-1}}, \mathring{x}_{\beta_{k+1}}, \dots, \mathring{x}_{\beta_n}) = \mathring{x}_{\alpha_k}$ which is impossible since $\alpha_k > \max\{\beta_j : j \ne k\}$. This completes the proof of Theorem 3.1. □

Next, we would like to show that it is consistent that CH fails and there exists $X \subseteq 2^\omega$ such that $|X| = \mathfrak{c}$ and X does not even have an infinite Turing independent subset. For this, we will make use of certain locally countable upper semi-lattices described below.

Definition 3.3. Let $(\mathbb{P}, \preceq)$ be a poset.

(1) $\mathbb{P}$ is locally countable iff for every $x \in \mathbb{P}$, $\{y \in \mathbb{P} : y \preceq x\}$ is countable.
(2) $\mathbb{P}$ is an upper semi-lattice iff every finite $F \subseteq \mathbb{P}$ has a $\preceq$-least upper bound (called the join of F).
(3) Suppose $\mathbb{P}$ is an upper semi-lattice. We say that $X \subseteq \mathbb{P}$ is independent in $\mathbb{P}$ iff for every finite $F \subseteq X$ and $y \in X \setminus F$, the join of F is not $\preceq$-above y.

Note that the Turing degrees form a locally countable upper semi-lattice with respect to Turing reduction $\le_T$.

Definition 3.4. Suppose $1 \le n < \omega$, θ and κ are uncountable cardinals, θ is regular and $\kappa \ge \theta$. Let $f : [\kappa]^n \to [\kappa]^{<\theta}$ be such that $a \subseteq f(a)$ for every $a \in [\kappa]^n$.

(i) Define $W_f = \{a \subseteq \kappa : n \le |a| < \aleph_0\}$. For each $a \in W_f$, let $cl_f(a)$ be the $\subseteq$-least subset of κ that contains a and is closed under f. Since θ is regular uncountable, $|cl_f(a)| < \theta$.
(ii) Define the preorder $\le_f$ on W_f by: $a \le_f b$ iff $a \subseteq cl_f(b)$.
(iii) Define the equivalence relation E_f on W_f by aE_fb iff $cl_f(a) = cl_f(b)$. Let $W_f^\star$ be the set of E_f-equivalence classes in W_f. Clearly, $|W_f^\star| = \kappa$ as each E_f-equivalence class has size $< \theta$. For $a \in W_f$, let $[a] \in W_f^\star$ denote the E_f-equivalence class of a.
(iv) For $[a], [b] \in W_f^\star$, define $[a] \preceq_f [b]$ iff $a \le_f b$. Then $(W_f^\star, \preceq_f)$ is a poset in which each element has $< \theta$ predecessors.

We say that $(W_f^\star, \preceq_f)$ is the upper semi-lattice associated with (n, θ, κ, f). That $(W_f^\star, \preceq_f)$ is an upper semi-lattice is justified by the following.

Claim 3.5. For every $[a], [b] \in W_f^\star$, $[a \cup b]$ is the $\preceq_f$-least upper bound of $[a], [b]$ in $W^\star$.

Proof. It is clear that $[a \cup b]$ is an upper bound. Suppose $c \in W_f$ and $a \le_f c$ and $b \le_f c$. Then $a \subseteq cl_f(c)$ and $b \subseteq cl_f(c)$ so $a \cup b \subseteq cl_f(c)$. Hence $[a \cup b] \preceq_f [c]$. So $[a \cup b]$ is the least upper bound. □

Lemma 3.6 (Kuratowski). *Suppose θ is an infinite cardinal, $k < \omega$ and $\kappa = \theta^{+k}$. Then, there exists $F : [\kappa]^{k+1} \to [\kappa]^{<\theta}$ such that for every $a \in [\kappa]^{k+1}$, $a \subseteq F(a)$, and whenever $a \in [\kappa]^{<\aleph_0}$ such that $|a| \ge k+1$, there exists $b \in [a]^{k+1}$ such that $a \subseteq F(b)$.*

Proof. By induction on k. If $k = 0$, then $F : [\theta]^1 \to [\theta]^{<\theta}$ defined by $F(\{\alpha\}) = \alpha+1$ works. Next assume that the result holds for k. Put $\kappa = \theta^{+k}$ and fix a witnessing function $F : [\kappa]^{k+1} \to [\kappa]^{<\theta}$. For each $\alpha < \kappa^+$, fix an injection $h_\alpha : \alpha \to \kappa$. For $a \in [\kappa^+]^{k+1}$ and $\max(a) < \alpha < \kappa^+$, define

$$H(a \cup \{\alpha\}) = \{\xi < \alpha : h_\alpha(\xi) \in F(h_\alpha[a])\} \cup \{\alpha\}.$$

It is easy to check that $H : [\kappa^+]^{k+2} \to [\kappa^+]^{<\theta}$ is as required. □

Lemma 3.7. *Suppose θ is regular uncountable and $k < \omega$. Then, there exists an upper semi-lattice $(\mathbb{P}, \preceq)$ such that for each $p \in \mathbb{P}$, $|\{q \in \mathbb{P} : q \preceq p\}| < \theta$, $|\mathbb{P}| = \theta^{+k}$ and there is no $S \in [\mathbb{P}]^{k+2}$ such that S is independent in $\mathbb{P}$.*

Proof. Put $\kappa = \theta^{+k}$. Using Lemma 3.6, fix $F : [\kappa]^{k+1} \to [\kappa]^{<\theta}$ such that for every $a \in [\kappa]^{k+1}$, $a \subseteq F(a)$, and whenever $a \in [\kappa]^{<\aleph_0}$ such that $|a| \geq k+1$, there exists $b \in [a]^{k+1}$ such that $a \subseteq F(b)$. Let $(\mathbb{P}, \preceq) = (W_F^\star, \preceq_F)$ be the upper semi-lattice associated with $(k+1, \theta, \kappa, F)$ as defined in Definition 3.4. Towards a contradiction, suppose $S = \{[a_n] : 1 \leq n \leq k+2\} \subseteq W_F^\star$ is independent in $(W_F^\star, \preceq_F)$. Let $a = \bigcup\{a_n : 1 \leq n \leq k+2\}$. Then $|a| \geq k+1$ as $|a_n| \geq k+1$ for every n. Choose $b \in [a]^{k+1}$ such that $a \subseteq F(b)$. Since $|b| = k+1$, we can find $1 \leq j \leq k+2$ such that $b \subseteq \bigcup\{a_n : 1 \leq n \leq k+2, n \neq j\}$. It follows that $[a_j]$ is $\preceq_F$-below the join of $\{[a_n] : 1 \leq n \leq k+2, n \neq j\}$: Contradiction. □

In a private communication with the first author, A. Andretta and R. Carroy asked if every locally countable upper semi-lattice of size $\mathfrak{c} > \omega_1$ must have an independent subset of size continuum. Corollary 3.8 shows that the answer is negative.

Corollary 3.8. *Suppose $2 \leq n < \omega$. There exists a locally countable upper semi-lattice $(\mathbb{P}, \preceq)$ of size ω_n such that there is no independent subset of $\mathbb{P}$ of size $n+1$.*

Proof. Apply Lemma 3.7 with $\theta = \omega_1$. □

Proof of Theorem 1.3. The consistency of the statement follows from Theorem 3.1. For the other direction, it suffices to show, for example, that under Martin's axiom (MA) plus $\mathfrak{c} = \omega_5$, there exists $X \in [2^\omega]^{\mathfrak{c}}$ such that X has no Turing independent subset of size 6. Assume MA plus $\mathfrak{c} = \omega_5$. In [10], it was shown that under MA, every locally countable upper semi-lattice of size continuum embeds into the Turing degrees. Using Corollary 3.8, fix a locally countable upper semi-lattice $(\mathbb{P}, \preceq)$ of size ω_5 which has no independent subset of size 6. Let $X \subseteq 2^\omega$ be the range of an upper semi-lattice embedding of $\mathbb{P}$ into the Turing degrees. Then $|X| = \mathfrak{c} = \omega_5$ and since the embedding preserves joins, X has no Turing independent subset of size 6. □

4. Large Turing independent subsets: Measure and category

We first show that under Martin's axiom, every non-meager (resp. non-null) set of reals has a non-meager (resp. non-null) Turing independent subset.

Lemma 4.1 (Sacks). *Suppose $x, y \in 2^\omega$ and x is not computable from y. Then*

$$\{z \in 2^\omega : x \leq_T y \oplus z\}$$

is both meager and null.

Proof. Suppose not and fix a Turing functional Φ and a non-meager (resp. non-null) Borel $B \subseteq 2^\omega$ such that

$$z \in B \implies \Phi^{y \oplus z} = x.$$

Choose $\sigma \in 2^{<\omega}$ such that B is comeager in $[\sigma]$ (resp. has relative measure > 0.9 in $[\sigma]$). We'll show that x is computable from y which is a contradiction.

If B is comeager in $[\sigma]$, then on input k, search for some $\tau \in 2^{<\omega}$ such that $\sigma \preceq \tau$ and $\Phi^{(y \restriction |\tau|) \oplus \tau}(k)$ converges to say s. Then $x(k) = s$ since $B \cap [\tau] \neq \emptyset$.

Next suppose $\mu(B \cap [\sigma]) > 0.9\mu([\sigma])$. On input k, search for $s < 2$ and a finite list $\tau_0, \tau_1, \dots, \tau_n \in 2^{<\omega}$ such that each τ_i extends σ, $\Phi^{(y\restriction|\tau_i|)\oplus\tau_i}(k) = s$ and the measure of $\bigcup\{[\tau_i] : i \le n\}$ is $\ge 0.9\mu([\sigma])$. Then $x(k) = s$ since $B \cap [\tau_i] \neq \emptyset$ for some $i \le n$. To see that this search succeeds, choose a compact $K \subseteq B \cap [\sigma]$ with $\mu(K) \ge 0.9\mu([\sigma])$. For each $v \in K$, fix $\rho = \rho(v) \in 2^{<\omega}$ such that $\sigma \preceq \rho \preceq v$ and $\Phi^{(y\restriction|\rho|)\oplus\rho}(k)$ converges to $x(k)$. As K is compact, there is a finite $\{v_i : i \le n\} \subseteq K$ such that $\{[\rho(v_i)] : i \le n\}$ covers K. So we can take $\tau_i = \rho(v_i)$. $\square$

Note that it also follows that if x is not computable, then $\{z \in 2^\omega : x \le_T z\}$ is both meager and null.

Lemma 4.2. *Assume Martin's axiom. Then every non-meager (resp. non-null) set of reals has an everywhere non-meager (resp. full outer measure) Turing independent subset.*

Proof. First assume that $X \subseteq 2^\omega$ is non-meager. By throwing away a countable subset of X, we can assume that no real in X in computable. It suffices to construct a Turing independent $Y \subseteq X$ such that for every Borel $A \subseteq 2^\omega$, if $A \cap X$ is non-meager, then $A \cap Y \neq \emptyset$. Let $\langle A_\alpha : \alpha < \mathfrak{c}\rangle$ list every Borel subset of 2^ω whose intersection with X is non-meager. Inductively choose $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ such that for each $\alpha < \mathfrak{c}$,

(a) $x_\alpha \in A_\alpha \cap X$ and
(b) for every finite $F \subseteq \{x_\beta : \beta < \alpha\}$, the set $\{x_\alpha\} \cup F$ is Turing independent.

Note that for every nonempty finite $F \subseteq \{x_\beta : \beta < \alpha\}$ and $x \in 2^\omega$ if $\{x\} \cup F$ is not Turing independent then either x is computable from the join of F or for some $y \in F$, y is computable from the join of $\{x\} \cup (F \setminus \{y\})$. By Lemma 4.1 the set of such x's is meager. As there are fewer than continuum many finite subsets of α, under Martin's axiom, the union of all of these meager sets cannot cover $A_\alpha \cap X$. So we can choose x_α's satisfying (a) and (b). Hence $Y = \{x_\alpha : \alpha < \mathfrak{c}\}$ is as required. The proof for the case when $X \subseteq 2^\omega$ is non-null is identical. We just replace meager by null everywhere. $\square$

Recall that $x \in 2^\omega$ is n-generic iff for every Σ^0_n-set $S \subseteq 2^{<\omega}$, there exists $k < \omega$ such that either $x \restriction k \in S$ or no extension of $x \restriction k$ is in S. $x \in 2^\omega$ is n-random iff for every uniformly Σ^0_n-sequence $\langle U_k : k < \omega\rangle$ of open sets in 2^ω with $\mu(U_n) \le 2^{-n}$, x is not in the null set $\bigcap_{n<\omega} U_n$. For $z \in 2^\omega$, the relativized notions "x is n-generic over z" and "x is n-random over z" are obtained by replacing "Σ^0_n" by "Σ^0_n in z". For the proof of Theorem 1.5, we'll need the following facts about effective randomness and genericity.

Fact 4.3 ([2])**.** *Suppose $x, y, z \in 2^\omega$, x is 1-generic over z and $y \le_T x$. If y is 2-generic, then y is also 1-generic over z.*

Fact 4.4 ([7]). *Suppose $x, y, z \in 2^\omega$, x is 1-random over z and $y \leq_T x$. If y is 1-random, then y is also 1-random over z.*

Facts 4.3 and 4.4 imply the following – See Lemma 3.11 in [11].

Lemma 4.5 ([11]). *Suppose Y is a meager (resp. null) set of 2-generic (resp. 1-random) reals. Then the set of reals that compute some member of Y is meager (resp. null).*

Proof. Since Y is meager (resp. null), we can fix $z \in 2^\omega$ such that no real in Y is 1-generic (resp. 1-random) over z. Let W be the set of reals that compute some member of Y. Towards a contradiction, suppose that W is non-meager (resp. non-null). Choose $x \in W$ such that x is 1-generic (resp. 1-random) over z. Choose $y \in Y$ such that $y \leq_T x$. By Fact 4.3 (resp. 4.4), it follows that y is 1-generic (1-random) over z which is impossible. □

Proof of Theorem 1.4. First suppose that $X \subseteq 2^\omega$ is non-meager. By throwing away a meager subset of X, we can assume that each real in X is 2-generic. Towards a contradiction, assume that every 1-Turing independent subset of X is meager. Call $S \subseteq X$ good iff no two distinct reals in S compute the same real in X. Let Y be a maximal good subset of X. For each $e < \omega$, let $W_e = \{x \in X : (\exists y \in Y)(\Phi_e^y = x)\}$. Observe that each W_e is 1-Turing independent and hence meager. It follows that $W = \bigcup\{W_e : e < \omega\}$ is meager. Let T be the set of all reals that compute some member of W. By Lemma 4.5, it follows that T is also meager. We claim that $X \subseteq T$ and therefore we get a contradiction. To see this, suppose $x \in X \setminus T$. Since $Y \subseteq W \subseteq T$, we must have $x \notin Y$. Since Y is a maximal good subset of X, there exist $y \in Y$ and $w \in X$ such that both x and y compute w. But $w \in W$ and hence $x \in T$ which is false. A similar argument works for measure. □

Definition 4.6. Let $\star_M$ be the statement: There exists a non-meager $X \subseteq 2^\omega$ such that the graph of every function from X to X is meager in $2^\omega \times 2^\omega$.

Definition 4.7. Let $\star_N$ be the statement: There exists a non-null $X \subseteq 2^\omega$ such that the graph of every function from X to X is null in $2^\omega \times 2^\omega$.

In [3], starting with a measurable cardinal, Komjáth constructed a ccc forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models \star_M$. In [9], starting with a measurable cardinal, Shelah constructed a ccc forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models \star_N$.

Lemma 4.8 ([3]). *Suppose $X \subseteq 2^\omega$ is non-meager (resp. non-null) and the graph of every function from X to X is meager (null) in $2^\omega \times 2^\omega$. Put $A = X^2$. Then A is non-meager (resp. non-null) in $2^\omega \times 2^\omega$ and for every non-meager (resp. non-null) $B \subseteq A$, there are $x_0 \neq x_1$ and $y_0 \neq y_1$ in X such that $(x_0, y_0), (x_0, y_1), (x_1, y_0)$ are all in B.*

Proof. It is clear that A is non-meager (resp. non-null) in $2^\omega \times 2^\omega$. Suppose $B \subseteq A$ satisfies: There do not exist $x_0 \neq x_1$ and $y_0 \neq y_1$ in X such that $(x_0, y_0), (x_0, y_1), (x_1, y_0)$ are all in B. Let B_0 be the set of those $(x, y) \in B$ for which there does not exist $y' \neq y$ such that $(x, y') \in B$. Let B_1 be the set of those $(x, y) \in B$ for which there does not exist $x' \neq x$ such that $(x', y) \in B$. It is clear that $B = B_0 \cup B_1$. Now observe that $\star_M$ (resp. $\star_N$) implies that each one of B_0, B_1 is meager (resp. null). Hence B is also meager (resp. null). □

Proof of Theorem 1.5. The consistency of the two statements follows from Lemma 4.2. For the consistency of the negations, first note that, instead of 2^ω, we can work in $2^\omega \times 2^\omega$ since the function $(x, y) \mapsto x \oplus y$ preserves all the relevant notions between $2^\omega \times 2^\omega$ and 2^ω. It suffices to show that $\star_M$ (resp. $\star_N$) implies that there is a non-meager (resp. non-null) $A \subseteq 2^\omega \times 2^\omega$ such that for every non-meager (resp. non-null) $B \subseteq X$, there are pairwise distinct a, b, c in B such that $a \leq_T b \oplus c$. But this is obvious by Lemma 4.8. $\square$

In [4], it was shown that it is consistent that there is a non-meager set $X \subseteq \mathbb{R}$ such that for every non-meager $Y \subseteq X$, there are $a < b < c < d$ in Y such that $a - b = c - d$. It follows that one does not need a measurable cardinal in the proof of the independence of the statement in Theorem 1.5(a) for $n \geq 3$.

Question 4.9. Can we prove the consistency of "There exists a non-meager set of reals which has no 2-Turing independent non-meager subset" without assuming the consistency of large cardinals? What about the consistency of "There exists a non-null set of reals which has no n-Turing independent non-null subset" for $n \geq 2$?

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