THE HALPERN–LÄUCHLI THEOREM AT SINGULAR CARDINALS AND FAILURES OF WEAK VERSIONS

NATASHA DOBRINEN AND SAHARON SHELAH

Abstract. This paper continues a line of investigation of the Halpern–Läuchli Theorem at uncountable cardinals. We prove in ZFC that the Halpern–Läuchli Theorem for one tree of height κ holds whenever κ is strongly inaccessible and the coloring takes less than κ colors. We prove consistency of the Halpern–Läuchli Theorem for finitely many trees of height κ, where κ is a strong limit cardinal of countable cofinality. On the other hand, we prove failure of weak forms of Halpern–Läuchli for trees of height κ, whenever κ is a strongly inaccessible, non-Mahlo cardinal or a singular strong limit cardinal with cofinality the successor of a regular cardinal. We also prove failure in L of a weak version for all strongly inaccessible, non-weakly compact cardinals.

1. Introduction

Investigations of the Halpern–Läuchli Theorem on trees of uncountable height commenced with work of the second author in [7]. In that paper, Shelah built on a forcing proof due to Harrington for trees of height ω to show the consistency of a strong version of the Halpern–Läuchli Theorem for trees of height κ, where κ is measurable in certain forcing extensions. A slightly modified version of this theorem was applied by Džamonja, Larson, and Mitchell to characterize the big Ramsey degrees for the κ-rationals in [3] and the κ-Rado graph in [4], for such κ. More recently, consistency strengths of various versions of the Halpern–Läuchli Theorem at uncountable cardinals were investigated in [1], [2], and [9]. This line of investigation is continued in this article.

Let κ be an ordinal. For nodes $\eta, \nu \in {}^{\kappa>}2$, we write $\eta \trianglelefteq \nu$ when η is an initial segment of ν, and write $\eta \triangleleft \nu$ when η is a proper initial segment of ν. The *length* of η, denoted by $\lg(\eta)$, is the ordinal α such that $\eta \in {}^{\alpha}2$. A subset $T \subseteq {}^{\kappa>}2$ is a *subtree* if T is non-empty and closed under initial segments. Similarly to [1], we call a subtree $T \subseteq {}^{\kappa>}2$ *regular* if the following hold:

(1) For all $\eta \in T$ and $\alpha < \kappa$, there is a $\nu \in T$ such that $\eta \trianglelefteq \nu$ and $\lg(\nu) = \max\{\lg(\eta), \alpha\}$;
(2) If $\delta < \kappa$ is a limit ordinal and $\eta \in {}^{\delta}2$ has the property that $\eta \restriction \alpha \in T$ for all $\alpha < \delta$, then $\eta \in T$.

The following is the strong-tree version of the Halpern–Läuchli Theorem for finitely many trees of height κ.

2020 *Mathematics Subject Classification.* 03E02, 03E05, 03E10, 03E35, 05D10.
Key words and phrases. tree, Ramsey theory, singular cardinal.
This work was partially supported by National Science Foundation Grant DMS 1833363. This is paper 1230 on Shelah's list.

Definition 1.1. (1) ($\mathrm{HL}_{n,\theta}(\kappa)$) For finite $1 \leq n < \omega$ and $2 \leq \theta$, we write $\mathrm{HL}_{n,\theta}(\kappa)$ to denote that for any $c : \bigcup\{{}^n({}^\alpha 2) : \alpha < \kappa\} \to \theta$, there are $A, T_0, \dots, T_{n-1}$ satisfying the following:

   (a) $A \in [\kappa]^\kappa$.
   (b) T_ℓ is a regular subtree of ${}^{\kappa>}2$, for each $\ell < n$.
   (c) If $\eta \in T_\ell$, then $\{\eta^\frown\langle 0\rangle, \eta^\frown\langle 1\rangle\} \subseteq T_\ell$ iff $\lg(\eta) \in A$.
   (d) $c \restriction (\bigcup\{\prod_{\ell<n} T_{\ell|\varepsilon} : \varepsilon \in A\})$ is constant, where $T_{\ell|\varepsilon} = \{\eta \in T_\ell : \lg(\eta) = \varepsilon\}$.

(2) A tree $T \subseteq {}^{\kappa>}2$ is called a *strong tree* if there is an $A \in [\kappa]^\kappa$ such that (a)–(c) hold for T.

Variations of the Halpern–Läuchli Theorem will also be investigated in this paper. For their statements, the following notion of suitable triple will be useful.

Definition 1.2. (1) A triple (κ, T, A) is called *suitable* if the following hold:

   (a) $A \in [\kappa]^\kappa$;
   (b) T is a regular subtree of ${}^{\kappa>}2$;
   (c) If $\eta \in T \cap {}^\alpha 2$, then $\{\eta^\frown 0, \eta^\frown 1\} \subseteq T$ iff $\alpha \in A$.

(2) Given $\ell < \omega$, let (0_ℓ) denote the sequence of 0's of length ℓ, and let $\eta^*_\ell = (0_\ell)^\frown 1$, the sequence of length $\ell + 1$ where the first ℓ coordinates are 0 and the last coordinate is 1.

Note that (κ, T, A) is a suitable triple if and only if T is a strong tree with A being the set of lengths of nodes in T which branch.

Definition 1.3 (Halpern–Läuchli Variations)**.** Let $1 \leq n < \omega$ and $2 \leq \theta < \kappa$ be given, with κ strongly inaccessible.

(1) $\mathrm{HL}_{n,\theta}[\kappa]$ abbreviates the following statement: Given a coloring $c : \bigcup\{{}^n({}^\alpha 2) : \alpha < \kappa\} \to \theta$, then there is a suitable triple (κ, T, A) and a color $\theta_* < \theta$ such that for all $\alpha \in A$ and $\overline{\nu} = (\nu_0, \dots \nu_{n-1})$ with $\eta^*_\ell \trianglelefteq \nu_\ell \in T \cap {}^\alpha 2$ for each $\ell < n$, then $c(\overline{\nu}) \neq \theta_*$.

(2) $\mathrm{HL}^+_{n,\theta}(\kappa)$ abbreviates the following statement: Given $\overline{<} = \langle <_\alpha : \alpha < \kappa\rangle$, where $<_\alpha$ is a well-ordering of ${}^\alpha 2$, and given a coloring $c : \bigcup\{[{}^\alpha 2]^n : \alpha < \kappa\} \to \theta$, there is a suitable triple (κ, T, A) so that whenever $\alpha < \kappa$ and $\eta_0, \dots, \eta_{n-1} \in T \cap {}^\alpha 2$ are pairwise distinct, then c is constant on the set

$$\{\overline{\nu} = (\nu_0, \dots \nu_{n-1}) : \bigwedge_{\ell<n} \eta_\ell \triangleleft \nu_\ell \in T,\ \lg(\nu_0) = \dots = \lg(\nu_{n-1}) \in A,$$
$$\text{and } \bar{\nu} \text{ is } <_{\lg(\nu_0)} -\text{increasing}\}.$$

(3) $\mathrm{HL}^+_{n,\theta}[\kappa]$ abbreviates the following statement: Given $\overline{<} = \langle <_\alpha : \alpha < \kappa\rangle$, where $<_\alpha$ is a well-ordering of ${}^\alpha 2$, and given a coloring $c : \bigcup\{[{}^\alpha 2]^n : \alpha < \kappa\} \to \theta$, there is a suitable triple (κ, T, A) so that whenever $\alpha < \kappa$ and $\eta_0, \dots, \eta_{n-1} \in T \cap {}^\alpha 2$ are pairwise distinct, then c misses at least one color on the set

$$\{\overline{\nu} = (\nu_0, \dots \nu_{n-1}) : \bigwedge_{\ell<n} \eta_\ell \triangleleft \nu_\ell \in T,\ \lg(\nu_0) = \dots = \lg(\nu_{n-1}) \in A,$$
$$\text{and } \bar{\nu} \text{ is } <_{\lg(\nu_0)} -\text{increasing}\}.$$

We point out the following straightforward facts.

Fact 1.4. (1) *For any given triple n, θ, κ, the following implications hold:*

$$\begin{array}{ccc} \mathrm{HL}^+_{n,\theta}(\kappa) & \longrightarrow & \mathrm{HL}_{n,\theta}(\kappa) \\ \big\downarrow & & \big\downarrow \\ \mathrm{HL}^+_{n,\theta}[\kappa] & \longrightarrow & \mathrm{HL}_{n,\theta}[\kappa] \end{array}$$

(2) *For $m \leq n$ and $2 \leq \theta \leq \theta'$, the following implications hold:*

$$\begin{aligned} \mathrm{HL}_{n,\theta'}(\kappa) &\longrightarrow \mathrm{HL}_{m,\theta}(\kappa) \\ \mathrm{HL}^+_{n,\theta'}(\kappa) &\longrightarrow \mathrm{HL}^+_{m,\theta}(\kappa) \\ \mathrm{HL}^+_{n,\theta}[\kappa] &\longrightarrow \mathrm{HL}^+_{m,\theta'}[\kappa] \\ \mathrm{HL}_{n,\theta}[\kappa] &\longrightarrow \mathrm{HL}^+_{m,\theta'}[\kappa] \end{aligned}$$

(3) *The following versions are equal:*

$$\begin{aligned} \mathrm{HL}_{n,2}(\kappa) &= \mathrm{HL}_{n,2}[\kappa] \\ \mathrm{HL}^+_{n,2}(\kappa) &= \mathrm{HL}^+_{n,2}[\kappa] \end{aligned}$$

We briefly review some highlights from previous work. Shelah proved in [7] that $\mathrm{HL}^+_{n,\theta}(\kappa)$ holds for all $1 \leq n < \omega$ and $2 \leq \theta < \kappa$ whenever κ is a cardinal with the following property $(*)$: κ is measurable after forcing with $\mathrm{Cohen}(\kappa, \lambda)$, where $\lambda \to (\kappa^+)^{2n}_{2^\kappa}$. If κ is a $\kappa+2n$-strong cardinal, then $(*)$ is satisfied by $\lambda = (\beth_{2n}(\kappa))^+$. Utilizing a lemma from [7], Zhang proved in [9] a "tail-cone" version which is intermediate between $\mathrm{HL}^+_{n,\theta}(\kappa)$ and $\mathrm{HL}_{n,\theta}(\kappa)$. He then applied the tail-cone version to obtain a polarized partition relation for finite products of κ-rationals, for κ satisfying $(*)$, proving an analogue of Laver's result for finite products of rationals in [6].

In [1], the first author and Hathaway proved that $\mathrm{HL}_{1,n}(\kappa)$ holds for any $2 \leq n < \omega$ when κ is strongly inaccessible. Soon after, Zhang showed in [9] that when κ is weakly compact, then $\mathrm{HL}_{1,\theta}(\kappa)$ holds for all $2 \leq \theta < \kappa$. We will improve both results by proving the following:

Theorem 1.5. *If κ is strongly inaccessible and $2 \leq \theta < \kappa$, then $\mathrm{HL}_{1,\theta}(\kappa)$ holds.*

In [1], the upper bound for the consistency strength of $\mathrm{HL}_{n,\theta}(\kappa)$, for κ strongly inaccessible, $2 \leq n < \omega$, and $2 \leq \theta < \kappa$, was reduced from a $\kappa + 2n$-strong cardinal to a $\kappa + n$-strong cardinal. Our first theorem extends this result to strong limit cardinals κ of countable cofinality. The hypotheses of Theorem 1.6 are satisfied whenever κ is a $\kappa + n$-strong cardinal.

Theorem 1.6. *Let $1 \leq \mathbf{n} < \omega$ and $2 \leq k < \omega$ be given. Suppose $\lambda \geq (\beth_{\mathbf{n}}(\kappa))^+$ and that κ is measurable in the generic extension via $\mathbb{P} = \mathrm{Cohen}(\kappa, \lambda)$ forcing. Let $\widetilde{\mathbb{Q}}$ be a $\mathbb{P}$-name for Prikry forcing. Then $\mathrm{HL}_{n,k}(\kappa)$ holds for all $1 \leq n \leq \mathbf{n}$ in the generic extension forced by $\mathbb{P} * \widetilde{\mathbb{Q}}$.*

Džamonja, Larson, and Mitchell pointed out in [3] that $\mathrm{HL}^+_{2,2}(\kappa)$ implies that κ is weakly compact. In the following theorem we find lower bounds for the weak version $\mathrm{HL}_{2,\theta}[\kappa]$, for all $2 \leq \theta < \kappa$.

Theorem 1.7. (1) *If κ is the first inaccessible, then $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails, for each $2 \leq \theta < \kappa$.*

(2) *Suppose κ is inaccessible and not Mahlo. Then $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails, for each $2 \leq \theta < \kappa$.*
(3) *If κ is a singular strong limit cardinal and $\mathrm{cf}(\kappa) = \mu^+$, with μ regular, then $\mathrm{HL}^+_{2,\mu^+}[\kappa]$ fails.*

Theorem 1.8. *Assume κ is strongly inaccessible, $S \subseteq \kappa$ is a non-reflecting stationary set, and $\diamondsuit_S$ holds. Then $\mathrm{HL}_{2,\theta}[\kappa]$ fails, for each $2 \leq \theta \leq \kappa$.*

Since the hypotheses of the previous theorem hold in L for all strongly inaccessible κ which are not weakly compact, we have the following corollary.

Corollary 1.9. *If $V = L$, then for all strongly inaccessible, non-weakly compact κ and for each $2 \leq \theta \leq \kappa$, $\mathrm{HL}_{2,\theta}[\kappa]$ fails.*

2. Halpern–Läuchli on one tree

In [1], Hathaway and the first author proved that $\mathrm{HL}_{1,n}(\kappa)$ holds for all weakly compact κ and all $2 \leq n < \omega$; Zhang pointed out that the proof in [1] actually implies $\mathrm{HL}_{1,n}(\kappa)$ holds for all strongly inaccessible cardinals κ. In [9], Zhang proved $\mathrm{HL}_{1,\theta}(\kappa)$ holds for all weakly compact κ and all $2 \leq \theta < \kappa$. Zhang also proved two consistency results showing that under certain large cardinal assumptions, it is consistent that there is a strongly inaccessible, not weakly compact cardinal κ such that for all $2 \leq \theta < \kappa$, $\mathrm{HL}_{1,\theta}(\kappa)$ holds. (See Corollary 5.7 and Theorem 5.8 in [9].)

The following theorem shows that the strong tree version of Halpern–Läuchli holds on one tree for all strongly inaccessible κ and all colorings into less than κ many colors.

Theorem 1.5. *If κ is strongly inaccessible and $2 \leq \theta < \kappa$, then $\mathrm{HL}_{1,\theta}(\kappa)$ holds.*

Proof. Suppose that κ is strongly inaccessible and that $2 \leq \theta < \kappa$ is the least ordinal such that $\mathrm{HL}_{1,\theta}(\kappa)$ fails. By a result in [1], θ must be at least ω; furthermore, it is straightforward to see that θ must be a regular cardinal. Let $c : {}^{\kappa>}2 \to \theta$ be a coloring which witnesses failure of $\mathrm{HL}_{1,\theta}(\kappa)$. Without loss of generality, we may assume that θ is the range of the coloring c.

For $\varepsilon < \theta$, define OB_ε to be the set of triples

$$\mathbf{m} = (T, A, \alpha) = (T_{\mathbf{m}}, A_{\mathbf{m}}, \alpha_{\mathbf{m}}) \tag{1}$$

satisfying the following (a)–(f):

(a) $\alpha < \kappa$ and $A \subseteq \alpha$ is unbounded in α.
(b) T is a subtree of ${}^{\alpha\geq}2$.
(c) If $\eta \in T$, then there is a $\nu \in T \cap {}^{\alpha}2$ such that $\eta \trianglelefteq \nu$.
(d) If $\eta \in T$, then ($\eta^\frown 0$ and $\eta^\frown 1$ are both in T) $\iff \lg(\eta) \in A$.
(e) If $\delta \leq \alpha$ is a limit ordinal and $\eta \in {}^{\delta}2$, then $\eta \in T \iff \forall \beta < \delta\, (\eta \restriction \beta \in T)$.
(f) c is constant with value ε on $T \cap (\bigcup_{\beta \in A} {}^{\beta}2)$.

Define $\leq_\varepsilon$ as the following 2-place relation on OB_ε: For $\mathbf{m}, \mathbf{n} \in \mathrm{OB}_\varepsilon$, $\mathbf{m} \leq_\varepsilon \mathbf{n}$ if and only if $\alpha_{\mathbf{m}} \leq \alpha_{\mathbf{n}}$, $A_{\mathbf{m}} = A_{\mathbf{n}} \cap \alpha_{\mathbf{m}}$, and $T_{\mathbf{m}} = T_{\mathbf{n}} \cap {}^{\alpha_{\mathbf{m}}\geq}2$.

The following facts are straightforward.

Fact 2.1. (1) *$\leq_\varepsilon$ is a partial order on OB_ε.*
(2) *If $\langle \mathbf{m}_i : i < \delta \rangle$ is a $\leq_\varepsilon$-increasing sequences, where $\delta < \kappa$, then the sequence has a $\leq_\varepsilon$-least upper bound.*

Proof. (1) is clear. For (2), take $\alpha = \sup_{i<\delta} \alpha_{\mathbf{m}_i}$ and $A = \bigcup_{i<\delta} A_{\mathbf{m}_i}$, and take T to be $\bigcup_{i<\delta} T_{\mathbf{m}_i}$ along with all maximal branches in $\bigcup_{i<\delta} T_{\mathbf{m}_i}$. Then $\mathbf{m} = (\alpha, A, T)$ is a member of OB_ε and is the $\leq_\varepsilon$-least upper bound of $\langle \mathbf{m}_i : i < \delta\rangle$. $\square$

Lemma 2.2. *For each $\varepsilon < \theta$ and each $\mathbf{m} \in \mathrm{OB}_\varepsilon$, there is an $\mathbf{n} \in \mathrm{OB}_\varepsilon$ such that $\mathbf{m} \leq_\varepsilon \mathbf{n}$ and $\mathbf{n}$ is $\leq_\varepsilon$-maximal.*

Proof. Suppose not. Then there are $\varepsilon < \theta$ and $\mathbf{m}_0 \in \mathrm{OB}_\varepsilon$ such that for each $\mathbf{n} \in \mathrm{OB}_\varepsilon$, $\mathbf{m}_0 \leq_\varepsilon \mathbf{n}$ implies that $\mathbf{n}$ is not maximal. Thus, we can build a $\leq_\varepsilon$-strictly increasing sequence $\langle \mathbf{m}_i : i < \kappa\rangle$ as follows: Given $\mathbf{m}_i$, since $\mathbf{m}_0 \leq_\varepsilon \mathbf{m}_i$ there is some $\mathbf{m}_{i+1} \in \mathrm{OB}_\varepsilon$ such that $\mathbf{m}_i <_\varepsilon \mathbf{m}_{i+1}$. If $i < \kappa$ is a limit ordinal, take $\mathbf{m}_i$ to be the least upper bound of $\langle \mathbf{m}_j : j < i\rangle$, guaranteed by Fact 2.1.

Let $A = \bigcup_{i<\kappa}$ and $T = \bigcup_{i<\kappa} T_{\mathbf{m}_i}$. Note that $A \in [\kappa]^\kappa$ since $\sup_{i<\kappa} \alpha_{\mathbf{m}_i} = \kappa$ and each $A_{\mathbf{m}_i}$ is unbounded in $\alpha_{\mathbf{m}_i}$. Thus, (κ, T, A) is a suitable triple. But then c has constant value ε on $\bigcup_{\alpha\in A} T \cap {}^{\alpha}2$, contradicting that c witnesses the failure of $\mathrm{HL}_{1,\theta}(\kappa)$. $\square$

We will choose $(\Lambda_i, \overline{\mathbf{m}}_i, \bar\varepsilon_i)$ by induction on $i < \theta$ satisfying the following.

(a) Λ_i is a nonempty set of pairwise $\triangleleft$-incomparable nodes in ${}^{\kappa>}2$.
(b) If $j < i$, then for each $\eta \in \Lambda_i$ there is a unique $\nu \in \Lambda_j$ such that $\nu \triangleleft \eta$. (It follows that $\Lambda_j \cap \Lambda_i = \emptyset$.)
(c) $\bar\varepsilon_i = \langle \varepsilon_\eta = \varepsilon(\eta) : \eta \in \Lambda_i\rangle$, where ε_η is the minimum ordinal in the range of c on $\{\zeta \in {}^{\kappa>}2 : \eta \trianglelefteq \zeta\}$ which is also above $\sup\{\varepsilon_\nu : \exists j < i\, (\nu \in \Lambda_j) \wedge \nu \triangleleft \eta\}$.
(d) $\overline{\mathbf{m}}_i = \langle \mathbf{m}_\eta = \mathbf{m}(\eta) : \eta \in \Lambda_i\rangle$. Notation: $\mathbf{m}_\eta = (T_\eta, A_\eta, \alpha_\eta = \alpha(\eta))$. (There is no ambiguity using η as an index since the Λ_i will be disjoint.)
(e) $\mathbf{m}_\eta$ is a $\leq_{\varepsilon(\eta)}$-maximal member of $\mathrm{OB}_{\varepsilon(\eta)}$.
(f) $\eta \in T_\eta$ and $\lg(\eta) \leq \min(A_\eta)$. (This implies that $\eta \trianglelefteq \mathrm{stem}(T_\eta)$.)
(g) If i is a limit ordinal and $\langle \eta_j : j < i\rangle$, $\eta_j \in \Lambda_j$, is an $\triangleleft$-increasing sequence, then $\bigcup_{j<i} \eta_j \in \Lambda_i$.

We now carry out the inductive construction.

Case 1: $i = 0$. Let $\Lambda_0 = \{\langle\rangle\}$ and $\varepsilon_{\langle\rangle} = 0$. Take $\mathbf{m}_{\langle\rangle}$ to be any $\leq_0$-maximal member of OB_0, and let $\overline{\mathbf{m}}_0 = \langle \mathbf{m}_{\langle\rangle}\rangle$.

For Cases 2 and 3, we use the following notation. For $\eta \in {}^{\kappa>}2$, define

$$\Theta_\eta = \{\varepsilon \in \theta : \exists \zeta\, (\eta \trianglelefteq \zeta \wedge c(\zeta) = \varepsilon)\}. \tag{2}$$

That is, Θ_η is the range of c on the set of nodes in ${}^{\kappa>}2$ extending η. Note that $|\Theta_\eta| = \theta$, since θ is by assumption the least ordinal for which $\mathrm{HL}_{1,\theta}(\kappa)$ fails and the coloring c witnesses this failure.

Case 2: $i = j + 1$. Let $\Lambda_i = \bigcup_{\nu\in\Lambda_j} T_\nu \cap {}^{\alpha(\nu)}2$. Since Λ_j is an antichain, so is Λ_i. By (f) of the induction hypothesis, for each $\nu \in \Lambda_j$, $\nu \triangleleft \mathrm{stem}(T_\nu)$ and hence $|\nu| < \alpha(\nu)$. Given $\eta \in \Lambda_i$, choose

$$\varepsilon_\eta = \min(\Theta_\eta \setminus \{\varepsilon_\nu : \nu \triangleleft \eta \text{ and } \exists k < i\, (\nu \in \Lambda_k)\}). \tag{3}$$

Then choose $\mathbf{m}_\eta$ to be some $\leq_{\varepsilon_\eta}$-maximal member of $\mathrm{OB}_{\varepsilon_\eta}$ such that $\eta \trianglelefteq \mathrm{stem}(T_\eta)$.

Case 3: i is a limit ordinal. Let Λ_i be the set of all nodes $\eta \in T$ such that $\lg(\eta)$ is a limit ordinal and $\forall \alpha < \lg(\eta)$, $\exists j < i\, \exists \nu \in \Lambda_j\, (\nu \triangleleft \eta \wedge \lg(\nu) \geq \alpha)$. In other words, Λ_i is the set of limits of $\triangleleft$-increasing sequences $\langle \nu_j : j < i\rangle$ with each $\nu_j \in \Lambda_j$. For $\eta \in \Lambda_i$, choose ε_η and $\mathbf{m}_\eta$ as in Case 2.

Let $\Lambda = \bigcup_{i<\theta} \Lambda_i$. Note that $|\Lambda| < \kappa$ since $|\Lambda_i| < \kappa$ for each $i < \theta$, and κ is regular. Fix some $\alpha(*) < \kappa$ greater than $\sup\{\alpha_\eta : \eta \in \Lambda\}$. We choose by induction a sequence $\langle \eta_i : i < \theta\rangle$ such that

(a) $\eta_i \in \Lambda_i$;
(b) $j < i$ implies $\eta_j \triangleleft \eta_i$;
(c) $\eta_i \trianglelefteq \nu \in {}^{\alpha(*)}2$ implies $c(\nu) > \varepsilon_{\eta_j}$ for all $j < i$.

Case 1: $i = 0$. Let $\eta_0 \in \Lambda_0$; that is, $\eta_0 = \langle\rangle$.

Case 2: i is a limit ordinal. By the construction of the $(\Lambda_i, \overline{\mathbf{m}}_i, \overline{\varepsilon}_i)$, $\eta_i = \bigcup\{\eta_j : j < i\}$ belongs to Λ_i, where η_j is the member of Λ_j such that $\eta_j \triangleleft \eta_i$. Clearly, (b) holds, and (c) follows from (b) and (c) holding for all $j < i$.

Case 3: $i = j + 1$. Then $\eta_j \in \Lambda_j$. Let

$$\Omega = \{\eta \in T_{\eta_j} \cap {}^{\alpha(\eta_j)}2 : \exists \nu \in {}^{\alpha(*)}2\, (\eta \triangleleft \nu \wedge c(\nu) = \varepsilon_{\eta_j})\}. \tag{4}$$

Now, if $\Omega = T_{\eta_j} \cap {}^{\alpha(\eta_j)}2$ then we get a contradiction to $\mathbf{m}(\eta_j)$ being $\leq_{\varepsilon(\eta_j)}$-maximal. So we can choose $\eta_i \in T_{\eta_j} \cap {}^{\alpha(\eta_j)}2$ which is not in Ω. Then η_i is in Λ_i, so (a) holds, and for all $\nu \in {}^{\alpha(*)}2$ such that $\eta_i \triangleleft \nu$, $c(\nu) \neq \varepsilon_{\eta_j}$.

Recall that by the definition of ε_{η_i}, for each ν above η_i, either $c(\nu) \geq \varepsilon_{\eta_i}$ or else $c(\nu) = \varepsilon_{\eta_k}$ for some $k \leq j$. We have already seen that $c(\nu) \neq \varepsilon_{\eta_j}$, and by the induction hypothesis, $c(\nu) > \varepsilon_{\eta_k}$ for all $k < j$. Thus, (c) holds. Note that (b) holds since $\eta_j \triangleleft \operatorname{stem}(T_{\eta_j})$.

This finishes the construction of a sequence $\langle \eta_i : i < \theta\rangle$ satisfying (a)–(c). Let $\eta = \bigcup_{i<\theta} \eta_i$, noting that $\lg(\eta) \leq \alpha(*)$. Take any $\nu \in {}^{\alpha(*)}2$ such that $\eta \trianglelefteq \nu$. Then for each $i < \theta$, $\eta_{i+1} \trianglelefteq \nu$ so (c) implies that $c(\nu) > \varepsilon_{\eta_i}$. The sequence of ordinals $\langle \varepsilon_{\eta_i} : i < \theta\rangle$ is strictly increasing, so $\sup_{i<\theta} \varepsilon_{\eta_i} = \theta$ since θ is regular, implying that $c(\nu) \geq \theta$. But this contradicts that $c(\nu)$ must be in θ. □

3. Halpern–Läuchli at singular cardinals of countable cofinality

In this section, we prove Theorem 1.6, the consistency of $\mathrm{HL}_{n,\theta}(\kappa)$ for κ a singular cardinal of countable cofinality.

Notation 3.1. Given $1 \leq n < \omega$, we define the function ξ on $\bigcup\{{}^n({}^\alpha 2) : \alpha < \kappa\}$ as follows: For $\alpha < \kappa$ and $\bar{\eta} \in {}^n({}^\alpha 2)$, let

$$\xi(\bar{\eta}) = \min\{\beta \leq \alpha + 1 : \beta \leq \alpha \text{ implies } \langle \eta_\ell \restriction \beta : \ell < n\rangle \text{ has no repetition}\}. \tag{5}$$

Thus, for $\alpha < \kappa$ and $\bar{\eta} \in {}^n({}^\alpha 2)$, if all members of the tuple $\bar{\eta}$ are distinct, then $\xi(\bar{\eta})$ is the least ordinal where they are all distinct; if the members of $\bar{\eta}$ are not all distinct, then $\xi(\bar{\eta}) = \alpha + 1$.

Recall that given $\ell < \omega$, we let $\eta^*_\ell = \langle (0_\ell), 1\rangle$ denote the sequence of length $\ell + 1$ where the last entry is 1 and all other entries are 0. Given $1 \leq n < \omega$, define

$$A_n(\kappa) = \{\bar{\eta} = (\eta_0, \ldots, \eta_{n-1}) \in \bigcup_{\alpha<\kappa} {}^n({}^\alpha 2) : (\forall \ell < n)\, \eta^*_\ell \trianglelefteq \eta_\ell\}, \tag{6}$$

and for $m < \omega$, define

$$A_{n,m}(\kappa) = \{(u, \bar{\eta}) : \exists \gamma < \kappa\, (u \in [\gamma]^m \text{ and } \bar{\eta} \in {}^n({}^\gamma 2))\}. \tag{7}$$

When κ is clear, we omit it and simply write A_n and $A_{n,m}$.

Lemma 3.2. *(A) $\Longrightarrow$ (B), where*
(A) is the statement:

(a) $1 \leq \mathbf{n}$, κ *is inaccessible, and* $\lambda \geq (\beth_{\mathbf{n}}(\kappa))^+$.
(b) $\mathbb{P}$ *is Cohen*(κ, λ).
(c) κ *is* $\mathbb{P}$*-indestructibly measurable. That is, there is a* $\mathbb{P}$*-name* $\widetilde{\mathcal{D}}$ *so that* $\Vdash_{\mathbb{P}}$ "$\widetilde{\mathcal{D}}$ *is a normal ultrafilter on* κ".
(d) *For any* $1 \leq n \leq \mathbf{n}$, $m < \omega$, *and* $2 \leq k_{n,m} < \omega$, $\widetilde{c}_{n,m}$ *is a* $\mathbb{P}$*-name for a function with domain* $A_{n,m}(\kappa)$ *and range* $k_{n,m}$.

(B) is the statement: There exist $(\widetilde{g}, \widetilde{h})$ *such that*

(α) (i) $\widetilde{g}$ *is a* $\mathbb{P}$*-name for an increasing function from* κ *to* κ.
(ii) $\widetilde{h}$ *is a* $\mathbb{P}$*-name for a function from* ${}^{\kappa>}2$ *into* ${}^{\kappa>}2$ *mapping* ${}^{\alpha}2$ *into* ${}^{\widetilde{g}(\alpha)}2$, *for each* $\alpha < \kappa$.
(iii) $\widetilde{h}(\eta)^\frown\langle\ell\rangle \trianglelefteq \widetilde{h}(\eta^\frown\langle\ell\rangle)$, *for* $\eta \in {}^{\kappa>}2$ *and* $\ell < 2$.

(β) *There is a* $\mathbb{P}$*-name* $\widetilde{D}$ *for a set in* $\widetilde{\mathcal{D}}$ *such that given* $1 \leq n \leq \mathbf{n}$ *and* $m < \omega$, *for each* $(u, \bar{\eta}) \in A_{n,m}$ *with* $u \subseteq \widetilde{D}$ *and* $\eta^*_\ell \trianglelefteq \eta_\ell$ *for all* $\ell < \mathbf{n}$, *the value of* $\widetilde{c}_{n,m}(u, h''(\bar{\eta}))$ *in the* $\mathbb{P}$*-generic extension of* V *depends only on* (n, m).

The proof of Lemma 3.2 will use the following Theorem 3.3 and Lemma 3.5.

Theorem 3.3 (Erdős–Rado, [5]). *For* $r \geq 0$ *finite and* κ *an infinite cardinal,* $\beth_r(\kappa)^+ \to (\kappa^+)^{r+1}_{\kappa}$.

Definition 3.4. Let $\kappa < \lambda$ be given and let $\mathbb{P}$ denote Cohen(κ, λ). We say that a subset $X \subseteq \mathbb{P}$ is *image homogenized* if

(a) All members of X have domain with the same order-type: i.e., there is some $\zeta < \kappa$ such that o. t.(dom(p)) $= \zeta$ for all $p \in X$; and
(b) For all $p, q \in X$ and $\xi < \zeta$, if $\alpha < \lambda$ is the ξ-th element of dom(p) and $\beta < \lambda$ is the ξ-th element of dom(q), then $p(\alpha) = q(\beta)$.

The following instance of Lemma 4.3 of [1] allows us to assume only that $\lambda \geq (\beth_{\mathbf{n}}(\kappa))^+$ in (A) of Lemma 3.2.

Lemma 3.5 ([1]). *Let* $1 \leq n < \omega$ *be given and let* κ *be a strongly inaccessible cardinal satisfying* $\kappa \to (\mu_1)^{2n}_{\mu_2}$ *for all* $\mu_1, \mu_2 < \kappa$. *Suppose that* $\{p_{\vec{\alpha}} : \vec{\alpha} \in [\kappa]^n\}$ *is an image homogenized set of conditions in the forcing* $\mathbb{P} = \mathrm{Cohen}(\kappa, \lambda)$, *where* $\lambda \geq \kappa$. *Then for each* $\gamma < \kappa$ *there is are sets* $K_i \subseteq \kappa$, $i < n$, *such that each* o. t.$(K_i) \geq \gamma$, *every element of* K_i *is less than every element of* K_j *whenever* $i < j < n$, *and* $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<n} K_i\}$ *is a pairwise compatible set of conditions.*

Proof of Lemma 3.2. Assume (A) and, without loss of generality, assume the conditions of $\mathbb{P}$ have the following form: For $p \in \mathbb{P}$, p is a function from some subset of $\lambda \times \mathbf{n}$ of cardinality less than κ into ${}^{\kappa>}2$, the size of dom$(p) \cap (\lambda \times \{\ell\})$ is the same for all $\ell < \mathbf{n}$, $p(\alpha, \ell)$ extends η^*_ℓ for each $(\alpha, \ell) \in$ dom(p), and all nodes in ran(p) have the same length.

Let $\widetilde{G}$ denote the canonical name for the generic object forced by $\mathbb{P}$ over V, and let V_1 denote $V[G]$. We let A_n denote $A_n(\kappa)$ and $A_{n,m}$ denote $A_{n,m}(\kappa)$, and note that A_n and $A_{n,m}$ remain the same in V and V_1. Given $\vec{\alpha} \in [\kappa]^{\mathbf{n}}$, we write $\vec{\alpha}$ in increasing order as $\{\alpha_i : i < \mathbf{n}\}$. In V_1, given $\vec{\alpha} \in [\kappa]^{\mathbf{n}}$, $1 \leq n \leq \mathbf{n}$, and $\gamma < \kappa$, let

(8) $$G_n(\vec{\alpha}) \restriction \gamma := \langle G(\alpha_i, i) \restriction \gamma : i < n\rangle.$$

Claim 1. *In* V_1, *given* $\vec{\alpha} \in [\lambda]^{\mathbf{n}}$ *there is an* $X_{\vec{\alpha}} \in \mathcal{D}$ *and an integer* $k_{n,m,\vec{\alpha}} < k_{n,m}$, *for all* $1 \leq n \leq \mathbf{n}$ *and* $m < \omega$, *such that* $c_{n,m}$ *is constant on the set*

(9) $$A_{n,m,\vec{\alpha}} := \{(u, G_n(\vec{\alpha}) \restriction \gamma) \in A_{n,m} : u \in [X_{\vec{\alpha}}]^m \text{ and } \gamma \in X_{\vec{\alpha}}\},$$

with value $k_{n,m,\vec{\alpha}}$.

Proof. Work in V_1 and fix $\vec{\alpha} \in [\lambda]^{\mathbf{n}}$. Given $1 \le n \le \mathbf{n}$, notice that since $G_n(\vec{\alpha}) \restriction \gamma$ is completely determined by γ, for each $u \in [\kappa]^{<\omega}$ the function $c_{n,|u|}(u, \cdot)$ restricted to the set $\{G_n(\vec{\alpha}) \restriction \gamma : \max(u) < \gamma < \kappa\}$ is essentially a function on the ordinals in the interval $(\max(u), \kappa)$. Since $\mathcal{D}$ is a normal ultrafilter on κ, there is a set $X_{n,u} \in \mathcal{D}$ such that $c_{n,|u|}(u, \cdot)$ is constant on $\{G_n(\vec{\alpha}) \restriction \gamma : \gamma \in X_{n,u}\}$, say with value $k_{n,u,\vec{\alpha}}$. Without loss of generality, we may assume that $\max(u) < \min(X_{n,u})$.

Now for each $\beta < \kappa$, let

$$X_\beta = \bigcap \{X_{n,u} : 1 \le n \le \mathbf{n},\ u \in [\beta]^{<\omega}\}, \tag{10}$$

and let $X = \Delta_{\beta<\kappa} X_\beta$. Since $\mathcal{D}$ is normal, X is in $\mathcal{D}$. Without loss of generality, we may further assume that all ordinals in X are limit ordinals.

Let $u \in [\kappa]^{<\omega}$ and $\gamma \in X$ be given with $\max(u) < \gamma$. Since all members of X are limit ordinals, $\max(u) + 1 < \gamma$; let $\beta_u = \max(u) + 1$. Note that $\gamma \in X_{\beta_u}$ since $\beta_u < \gamma$. It follows that for each $1 \le n \le \mathbf{n}$, $\gamma \in X_{\beta_u} \subseteq X_{n,u}$, which implies that $c_{n,|u|}(u, G_n(\vec{\alpha}) \restriction \gamma) = k_{n,u,\vec{\alpha}}$.

For each $1 \le n \le \mathbf{n}$, define a coloring $\psi_n : [X]^{<\omega} \to k$ by setting $\psi_n(u) = k_{n,u,\vec{\alpha}}$. Then there is a $Y_n \subseteq X$ in $\mathcal{D}$ such that for each $m < \omega$, ψ_n is constant on $[Y_n]^m$; denote its value by $k_{n,m,\vec{\alpha}}$. Letting $X_{\vec{\alpha}} = \bigcap_{1 \le n \le \mathbf{n}} Y_n$, we see that for each $u \in [X_{\vec{\alpha}}]^{<\omega}$, $k_{n,u,\vec{\alpha}} = k_{n,|u|,\vec{\alpha}}$. $\square$

In V, for each $1 \le n \le \mathbf{n}$, $m < \omega$, and $\vec{\alpha} \in [\lambda]^{\mathbf{n}}$, there are $\mathbb{P}$-names $\widetilde{X}_{\vec{\alpha}}$ and $\widetilde{A}_{n,m,\vec{\alpha}}$ for the sets $X_{\vec{\alpha}}$ and $A_{n,m,\vec{\alpha}}$ guaranteed by Claim 1, and a condition $p_{\vec{\alpha}} \in \mathbb{P}$ which forces the following: "$\widetilde{X}_{\vec{\alpha}} \in \widetilde{\mathcal{D}}$ and for all $1 \le n \le \mathbf{n}$ and $m < \omega$, $\widetilde{c}_{n,m}$ takes value $k_{n,m,\vec{\alpha}}$ on the set $\widetilde{A}_{n,m,\vec{\alpha}}$." Without loss of generality, we may assume that $p_{\vec{\alpha}}$ forces "$\lg(\mathrm{ran}(p_{\vec{\alpha}})) \in \widetilde{X}_{\vec{\alpha}}$ and $\widetilde{c}_{n,m}(u, \mathrm{ran}(p_{\vec{\alpha}})) = k_{n,m,\vec{\alpha}}$ for all $u \in [\widetilde{X}_{\vec{\alpha}}]^m$ with $\max(u) < \lg(\mathrm{ran}(p_{\vec{\alpha}}))$."

Now we find an image homogenized collection of $p_{\vec{\alpha}}$'s. For $\vec{\alpha} \in [\lambda]^{\mathbf{n}}$, recall that $\mathrm{dom}(p_{\vec{\alpha}})$ is a subset of $\lambda \times \mathbf{n}$ of cardinality less than κ. Fix a bijection $b : \lambda \times \mathbf{n} \to \lambda$. For $\vec{\alpha} \in [\lambda]^{\mathbf{n}}$, let $b[\mathrm{ran}(p_{\vec{\alpha}})]$ denote the range of $p_{\vec{\alpha}}$ ordered as the sequence $\langle p_{\vec{\alpha}}(b^{-1}(\beta)) : \beta \in b[\mathrm{dom}(p_{\vec{\alpha}})]\rangle$. Let f be the coloring on $[\lambda]^{\mathbf{n}}$ into κ many colors defined as follows: For $\vec{\alpha} \in [\lambda]^{\mathbf{n}}$,

$$\begin{aligned} f(\vec{\alpha}) = {} & \langle \mathrm{o.\,t.}(b[\mathrm{dom}(p_{\vec{\alpha}})])\rangle^\frown \lg(\mathrm{ran}(p_{\vec{\alpha}}))^\frown b[\mathrm{ran}(p_{\vec{\alpha}})]^\frown \\ & \langle k_{n,m,\vec{\alpha}} : 1 \le n \le \mathbf{n},\ m < \omega\rangle^\frown \langle p_{\vec{\alpha}}(\alpha_\ell, \ell) : \ell < \mathbf{n}\rangle. \end{aligned} \tag{11}$$

Since $\lambda \to (\kappa)^{\mathbf{n}}_{\kappa}$, there are $J \in [\lambda]^\kappa$, $\gamma^*, \delta^* < \kappa$, nodes $\langle t^*_\beta : \beta < \delta^*\rangle$ in ${}^{\kappa>}2$, $k^*_{n,m} < k_{n,m}$, and $\vec{\zeta^*} \in {}^{\mathbf{n}}({}^{\gamma^*}2)$ with $\eta^*_\ell \trianglelefteq \zeta^*_\ell$ for each $\ell < \mathbf{n}$ such that for all $\vec{\alpha} \in [J]^{\mathbf{n}}$, the following hold:

1. $\mathrm{o.\,t.}(b[\mathrm{dom}(p_{\vec{\alpha}})]) = \delta^*$;
2. $\lg(\mathrm{ran}(p_{\vec{\alpha}})) = \gamma^*$;
3. $b[\mathrm{ran}(p_{\vec{\alpha}})] = \langle t^*_\beta : \beta < \delta^*\rangle$;
4. $k_{n,m,\vec{\alpha}} = k^*_{n,m}$, for each $1 \le n \le \mathbf{n}$ and $m < \omega$;
5. $\langle p_{\vec{\alpha}}(\alpha_\ell, \ell) : \ell < \mathbf{n}\rangle = \vec{\zeta^*}$.

In particular, $|J| = \kappa$, and the set $\{p_{\vec{\alpha}} : \vec{\alpha} \in [J]^{\mathbf{n}}\}$ is image homogenized. By Lemma 3.5, for each $\gamma < \kappa$, there are sets $K_n \subseteq J$, $n < \mathbf{n}$, with $K_0 < K_1 < \cdots < K_{\mathbf{n}-1}$ such that each $\mathrm{o.\,t.}(K_n) \ge \gamma$ and, letting $\vec{K}$ denote $\prod_{n<\mathbf{n}} K_n$, the set $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{K}\}$ is pairwise compatible.

Claim 2. *In V_1, there is a $D \in \mathcal{D}$, a strictly increasing function $g : \kappa \to \kappa$ with $\mathrm{ran}(g) \subseteq D$, and a strong subtree $T \subseteq {}^{\kappa>}2$ with splitting levels in $\mathrm{ran}(g)$ such that given $1 \leq n \leq \mathbf{n}$ and $m < \omega$, for all $u \in [D]^m$ and $\bar\eta \subseteq T$ in A_n with $\max(u) < \lg(\bar\eta)$, we have*

$$c_{n,m}(u, \bar\eta) = k^*_{n,m}.$$

Proof. The function g will be constructed recursively and will give the levels of the strong subtree $T \subseteq {}^{\kappa>}2$ which is being built. For ease of notation, we will let $T(\gamma)$ denote ${}^{g(\gamma)}2 \cap T$.

For the base case, let $u = \emptyset$. Take $\vec\alpha$ to be any increasing sequence in J. By Claim 1 and following exposition, we may take $g(0) = \lg(\mathrm{ran}(p_{\vec\alpha})) \in X_{\vec\alpha}$ so that for all $1 \leq n \leq \mathbf{n}$,

$$c_{n,0}(\emptyset, \langle p_{\vec\alpha}(\alpha_\ell, \ell) : \ell < n\rangle) = k^*_{n,0}. \tag{12}$$

Let $T(0) = \{p_{\vec\alpha}(\alpha_\ell, \ell) : \ell < \mathbf{n}\}$ and let $X_0 = X_{\vec\alpha}$.

Given $0 < \gamma < \kappa$, suppose that for all $\delta < \gamma$, $g(\delta)$ and $T(\delta)$ have been defined and satisfy the claim. If γ is a limit ordinal, let B denote the set of branches through $\bigcup_{\delta<\gamma} T(\delta)$. If γ is a successor ordinal, let B denote the set of immediate successors in ${}^{\kappa>}2$ of the branches through $\bigcup_{\delta<\gamma} T(\delta)$. Take sets $K_\ell \subseteq J$, $\ell < \mathbf{n}$, such that K_ℓ has the same cardinality as the set of nodes in B extending η^*_ℓ, $K_0 < \cdots < K_{\mathbf{n}-1}$, and $\{p_{\vec\alpha} : \vec\alpha \in \vec K\}$ is pairwise compatible, where $\vec K = \prod_{\ell<\mathbf{n}} K_\ell$. Let q be a condition in $\mathbb{P}$ such that, for each $\ell < \mathbf{n}$, $\{q(\beta, \ell) : \beta \in K_\ell\}$ is in one-to-one correspondence with $\{\eta \in B : \eta^*_\ell \trianglelefteq \eta\}$. In particular, q extends $\bigcup\{p_{\vec\alpha} : \vec\alpha \in \vec K\}$.

Let

$$X_\gamma = \bigcap\{X_\delta : \delta < \gamma\} \cap \bigcap\{X_{\vec\alpha} : \vec\alpha \in \vec K\}, \tag{13}$$

noting that this set is in $\mathcal{D}$. Extend q to some $r \in \mathbb{P}$ so that $\lg(\mathrm{ran}(r)) \in X_\gamma$ and, utilizing Claim 1 again, for each $m < \omega$, $u \in [\{g(\delta) : \delta < \gamma\}]^m$, and $\vec\alpha \in \vec K$, $c_{n,m}(u, \cdot)$ has color $k_{n,m,\vec\alpha} = k^*_{n,m}$ on $\langle r(\alpha_\ell, \ell) : \ell < n\rangle$. We let $g(\gamma) = \lg(\mathrm{ran}(r))$ and $T(\gamma) = \{r(\alpha_\ell, \ell) : \ell < \mathbf{n} \text{ and } \vec\alpha \in \vec K\}$.

In this way, we construct g and T and a decreasing sequence of sets $X_\gamma \in \mathcal{D}$. Let $D = \Delta_{\gamma<\kappa} X_\gamma$. Each $g(\gamma) \in X_\gamma \setminus (\gamma + 1)$, so $\mathrm{ran}(g) \subseteq X$. Then for each $1 \leq n \leq \mathbf{n}$, for any $\gamma < \kappa$ and $u \in [\{g(\delta) : \delta < \gamma\}]^{<\omega}$, for each $\bar\eta \in A_n \cap T$ with $\lg(\bar\eta) \geq g(\gamma)$, we have $c_{n,m}(u, \bar\eta) = k^*_{n,m}$. □

Letting $\widetilde{D}$ be a $\mathbb{P}$-name for D, and letting $\widetilde{h}$ be a $\mathbb{P}$-name for the tree isomorphism from ${}^{<\kappa}2$ to T finishes the proof of (β) of Lemma 3.2. □

Remark 3.6. In fact, we get more than the Lemma claims: The statement (β) actually holds for all $(u, \bar\eta) \in A_{n,m}$ with $u \subseteq \widetilde{D}$ and $\bar\eta$ with $\lg(\eta_\ell) \in \widetilde{D}$ (rather than just $\lg(\eta_\ell) \in \mathrm{ran}(\widetilde{g})$).

We now restate Theorem 1.6 and prove it.

Theorem 1.6. *Let $1 \leq \mathbf{n} < \omega$ and $2 \leq k < \omega$ be given. Suppose $\lambda \geq (\beth_{\mathbf{n}}(\kappa))^+$ and that κ is measurable in the generic extension via $\mathbb{P} =$ Cohen(κ, λ) forcing. Let $\widetilde{\mathbb{Q}}$ be a $\mathbb{P}$-name for Prikry forcing. Then $\mathrm{HL}_{n,k}(\kappa)$ holds for all $1 \leq n \leq \mathbf{n}$ in the generic extension forced by $\mathbb{P} * \widetilde{\mathbb{Q}}$.*

Proof. Let $1 \leq \mathbf{n} < \omega$ and $2 \leq k < \omega$ be given. Suppose $\lambda \geq (\beth_{\mathbf{n}}(\kappa))^+$ and that κ is measurable in the generic extension via $\mathbb{P} = \mathrm{Cohen}(\kappa, \lambda)$ forcing. Let $\widetilde{\mathcal{D}}$ be a

$\mathbb{P}$-name in V for a normal ultrafilter on $\check{\kappa}$. Let G be $\mathbb{P}$-generic over V and let V_1 denote $V[G]$. In V_1, let $\mathbb{Q}$ denote Prikry forcing with tails in $\mathcal{D}$. Let H be $\mathbb{Q}$-generic over V_1, and let V_2 denote $V_1[H]$.

Notice that A_n and $A_{n,m}$ remain the same in V, V_1, and V_2. In V_2, for each $1 \le n \le \mathbf{n}$, let $c_n : A_n \to k$ be a function. In V_1, let $\widetilde{c}_n$ be a $\mathbb{Q}$-name for c_n. For each $m < \omega$, define in V_1 a function $c_{n,m} : A_{n,m} \to k$ by

$$c_{n,m}(u, \bar{\eta}) = j \iff \exists X \in \mathcal{D}\ (u, X) \Vdash_{\mathbb{Q}} \widetilde{c}_n(\bar{\eta}) = j. \tag{14}$$

Let $\widetilde{c}_{n,m}$ be a $\mathbb{P}$-name in V for $c_{n,m}$. By Lemma 3.2, there are $\mathbb{P}$-names $(\widetilde{g}, \widetilde{h})$, $\widetilde{T}$, and $\widetilde{D}$ in V and integers $k^*_{n,m}$ denoting the value in V_1 of $c_{n,m}(u, \bar{\eta})$ for $(u, \bar{\eta}) \in A_{n,m}$ with $\bar{\eta}$ from T.

Claim 3. *In V_1, there is an $E \subseteq D$ in $\mathcal{D}$ such that for each $u \in [E]^{<\omega}$ with $\max(u) < \alpha \in E$ and each $1 \le n \le \mathbf{n}$, we have*

$$(u, E \setminus (\alpha + 1)) \Vdash_{\mathbb{Q}} \widetilde{c}_n(\bar{\eta}) = k^*_{n,|u|},$$

for each $\bar{\eta} \in A_n \cap T$ such $\max(u) < \lg(\bar{\eta}) < \alpha$.

Proof. Given $\alpha < \kappa$ and $u \in [\alpha]^{<\omega}$, let $Y_{\alpha,u} \subseteq D$ be in $\mathcal{D}$ so that for all $1 \le n \le \mathbf{n}$ and $\bar{\eta} \in A_n \cap T$ with $\max(u) < \lg(\bar{\eta}) < \alpha$,

$$(u, Y_{\alpha,u}) \Vdash_{\mathbb{Q}} \widetilde{c}_n(\bar{\eta}) = c_n(u, \bar{\eta}). \tag{15}$$

By Claim 2, the value of $c_n(u, \eta)$ in equation (15) is $k^*_{n,|u|}$. Let $Y_\alpha = \bigcap\{Y_{\alpha,u} : u \in [\alpha]^{<\omega}\}$, and let $E = \Delta_{\alpha<\kappa} Y_\alpha$. Then for each $m < \omega$, whenever $\alpha \in E$ and $u \in [E \cap \alpha]^m$, we have $E \setminus (\alpha + 1) \subseteq Y_{\alpha,u}$. Thus, $(u, E \setminus (\alpha + 1)) \Vdash_{\mathbb{Q}} \widetilde{c}_n(\bar{\eta}) = k^*_{n,m}$, for all $\bar{\eta}$ in T with $\lg(\bar{\eta}) < \alpha$. □

In V_2, let $\bar{\lambda} = \langle \lambda_i : i < \omega \rangle$ be the generic Prikry generic sequence over V_1 given by H. By genericity, for each $Z \in \mathcal{D}$ in V_1, all but finitely many members of $\bar{\lambda}$ are contained in Z. In particular, $\bar{\lambda} \subseteq^* E$, so without loss of generality we may assume $\bar{\lambda} \subseteq E$. We may assume (by genericity) that the Prikry sequence has the property that $|\lambda_{i+1} \setminus \lambda_i|$ is strictly increasing with limit κ. Moreover, we may assume that the sequence $|\operatorname{ran}(g) \cap (\lambda_i, \lambda_{i+1})|$ is strictly increasing with limit κ.

In V_2, we now construct a strong subtree U of T with κ many levels so that c_n is constant for each $1 \le n \le \mathbf{n}$. By Claim 3, for all $1 \le n \le \mathbf{n}$ and all $\bar{\eta} \in A_n \cap T$ with $\lg(\bar{\eta}) < \lambda_0$, we have $(\emptyset, D \setminus (\lambda_0 + 1)) \Vdash_{\mathbb{Q}} \widetilde{c}_n(\bar{\eta}) = k^*_{\emptyset,n}$. In general, given $i < \omega$, for all $1 \le n \le \mathbf{n}$ and all $\bar{\eta} \in A_n \cap T$ with $\lg(\bar{\eta}) \in (\lambda_i, \lambda_{i+1})$, we have $(\{\lambda_0, \dots, \lambda_i\}, Y \setminus (\lambda_{i+1} + 1)) \Vdash_{\mathbb{Q}} \widetilde{c}_n(\bar{\eta}) = k^*_{n,i+1}$. Since each $k^*_{n,i} < k$, in V_2 there is a set $I \in [\omega]^\omega$ such that for all $1 \le n \le \mathbf{n}$, c_n is constant for all $\bar{\eta}$ in T with length in $\bigcup_{i\in I}(\lambda_i, \lambda_{i+1})$. Then we may take a strong subtree U of T which has splitting levels in $\bigcup_{i\in I}(\lambda_i, \lambda_{i+1})$, so that U has κ many splitting levels. Then this tree U witnesses that $\mathrm{HL}_{n,k}(\kappa)$ holds in V_2 for each $1 \le n \le \mathbf{n}$. □

Remark 3.7. If the colorings $c_n : A_n \to k$ are in V, then the functions $\widetilde{h}$ and $\widetilde{g}$ in conclusion of Lemma 3.2 can be found in V. Hence, the strong subtree T witnessing $\mathrm{HL}_{n,k}(\kappa)$ for such c_n in Theorem 1.6 exists in V.

We now consider a version of Halpern–Läuchli for infinite colorings. For $2 \le \theta_0 \le \theta_1$, let $\mathrm{HL}_{n,(1,\theta_0,\theta_1)}[\kappa]$ abbreviate the following statement: Given a coloring $c : \bigcup\{{}^n({}^\alpha 2) : \alpha < \kappa\} \to \theta_1$, then there is a suitable triple (κ, T, A) and a subset $u \subseteq \theta_1$

such that $|u| < \theta_0$ and for all $\alpha \in A$ and $\overline{\nu} = (\nu_0, \dots \nu_{n-1})$ with $\eta^*_\ell \trianglelefteq \nu_\ell \in T \cap {}^\alpha 2$ for each $\ell < n$, then $c(\overline{\nu}) \in u$.

A minor straightforward modification of the proof of Theorem 1.6 yields the following theorem.

Theorem 3.8. *Under the assumptions of Theorem 1.6, if $\aleph_0 < \theta < \kappa$, we get that* $\Vdash_{\mathbb{P} * \widetilde{\mathbb{Q}}} \mathrm{HL}_{n,(1,\aleph_1,\theta)}(\kappa)$.

4. Consistent Failures of Halpern–Läuchli

This section provides conditions under which various versions of the Halpern–Läuchli Theorem fail. Our first theorem provides conditions which imply strong failure of Halpern–Läuchli; that is, failure of $\mathrm{HL}^+_{2,\theta}[\kappa]$ for all $2 \le \theta < \kappa$. For this, we will use negative square bracket partition relations. Given cardinals $1 \le n < \omega$ and $\theta, \mu \le \kappa$, the square bracket partition relation

$$\kappa \to [\mu]^n_\theta \tag{16}$$

holds if for every function $c : [\kappa]^n \to \theta$, there is a subset $A \subseteq \kappa$ with $|A| = \mu$ such that $\mathrm{ran}(c \restriction [A]^n)$ is a proper subset of θ. The negation

$$\kappa \nrightarrow [\mu]^n_\theta \tag{17}$$

holds if there is a function $c : [\kappa]^n \to \theta$ so that for each subset $A \subseteq \kappa$ with $|A| = \mu$, $\mathrm{ran}(c \restriction [A]^n) = \theta$. The following lemma will aid in the proof of Theorem 1.7.

Lemma 4.1. *Suppose κ is a strong limit cardinal and either*

(a) *κ is strongly inaccessible, $\theta < \kappa$, and $\kappa \nrightarrow [\kappa]^2_\theta$; or*
(b) *$\theta < \mathrm{cf}(\kappa)$ and $\mathrm{cf}(\kappa) \nrightarrow [\mathrm{cf}(\kappa)]^2_\theta$.*

Then $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails.

Proof. To prove (a), suppose κ is strongly inaccessible and $\theta < \kappa$, and let $c : [\kappa]^2 \to \theta$ be a function witnessing $\kappa \nrightarrow [\kappa]^2_\theta$.

Claim 4. *For each $A \in [\kappa]^\kappa$, there is an $\alpha \in A$ such that c has range θ on the set $\{\{\alpha, \beta\} : \beta \in A \cap (\alpha, \kappa)\}$.*

Proof. Suppose not. Then there is an $A \in [\kappa]^\kappa$ such that for each $\alpha \in A$, there is some ordinal $e(\alpha) \in \theta$ which is not in the range of c on the set $\{\{\alpha, \beta\} : \beta \in A \cap (\alpha, \kappa)\}$. Since $\theta < \kappa$, there is an $A' \in [A]^\kappa$ such that e is constant on A'. But then $\mathrm{ran}(c \restriction [A']^2) \neq \theta$, contradicting that c witnesses $\kappa \to [\kappa]^2_\theta$. □

Let $\overline{<}$ be any sequence of well-orderings of the levels of ${}^{\kappa >}2$. Define the function $d : \bigcup_{\alpha<\kappa} [{}^\alpha 2]^2 \to \theta$ by

$$\{\nu_0, \nu_1\} \in [{}^\alpha 2]^2 \implies d(\{\nu_0, \nu_1\}) = c(\{\lg(\nu_0 \cap \nu_1), \alpha\}). \tag{18}$$

Let (κ, T, A) be a suitable triple. Take $A' \in [A]^\kappa$ such that between any two consecutive ordinals in A', there are ω many ordinals in A. Fix $\alpha \in A'$ as in Claim 4, and fix any node $\eta \in T \cap {}^\alpha 2$ and distinct nodes $\eta_0, \eta_1 \in {}^{\alpha+1}2$ such that $\eta_0 \cap \eta_1 = \eta$. For each $\beta \in A' \cap (\alpha, \kappa)$ there are at least 2^ω many pairs of distinct nodes ν_0, ν_1 in $T \cap {}^\beta 2$ such that $\eta_0 \lhd \nu_0$ and $\eta_1 \lhd \nu_1$ and $\nu_0 <_\beta \nu_1$. By Claim 4, d has range θ on the set

$$\{\{\nu_0, \nu_1\} \in [T]^2 : \eta_0 \lhd \nu_0,\ \eta_1 \lhd \nu_1,\ \lg(\nu_0) = \lg(\nu_1) \in A',\ \text{and } \nu_0 <_{\lg(\nu_0)} \nu_1\}. \tag{19}$$

Thus, $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails.

The proof of (b) is similar. Suppose κ is a strong limit cardinal, let $\mu = \mathrm{cf}(\kappa)$, and let $c : [\mu]^2 \to \theta$ be a function witnessing that $\mu \nrightarrow [\mu]^2_\theta$. Let $\langle \kappa_i : i < \mu \rangle$ be an increasing continuous sequence with limit κ, and assume $\kappa_0 = 0$. Define $h : \kappa \to \mu$ by

$$h(\alpha) = i \iff \alpha \in [\kappa_i, \kappa_{i+1}). \tag{20}$$

A proof similar to the one given for Claim 4 yields the following:

Claim 5. *Given $B \in [\mu]^\mu$, there is an $a \in B$ such that c has range θ on the set $\{\{a, b\} : b \in B \cap (a, \mu)\}$.*

Define the function $d : \bigcup_{\alpha<\kappa} [{}^\alpha 2]^2 \to \theta$ by

$$\{\eta, \nu\} \in [{}^\alpha 2]^2 \Longrightarrow d(\{\eta, \nu\}) = c(\{h(\mathrm{lg}(\eta \cap \nu)), h(\alpha)\}) \tag{21}$$

if $\mathrm{lg}(\eta \cap \nu)$ and α are in different intervals of the partition $[\kappa_i, \kappa_{i+1})$ $(i < \mu)$ of κ, and let $d(\{\eta, \nu\}) = 0$ otherwise. Let (κ, T, A) be a suitable triple. Again take $A' \in [A]^\kappa$ such that between any two consecutive ordinals in A', there are ω many ordinals in A.

Let $B = h[A']$ and take $a \in B$ satisfying Claim 5. Let $\alpha \in A'$ be such that $h(\alpha) = a$, and fix any node $\eta \in T \cap {}^\alpha 2$ and nodes $\eta_0, \eta_1 \in {}^{\alpha+1}2$ such that $\eta_0 \cap \eta_1 = \eta$. For each $\beta \in A' \cap (\alpha, \kappa)$ there are at least 2^ω many pairs of distinct nodes ν_0, ν_1 in $T \cap {}^\beta 2$ such that $\eta_0 \lhd \nu_0$ and $\eta_1 \lhd \nu_1$ and $\nu_0 <_\beta \nu_1$. By Claim 5, d has range θ on the set

$$\{\{\nu_0, \nu_1\} \in [T]^2 : \eta_0 \lhd \nu_0,\ \eta_1 \lhd \nu_1,\ \mathrm{lg}(\nu_0) = \mathrm{lg}(\nu_1) \in A',\ \text{and } \nu_0 <_{\mathrm{lg}(\nu_0)} \nu_1\}. \tag{22}$$

Thus, $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails. □

The next Lemma 4.3 will also aid in the proof of Theorem 1.7. For this, we need the following.

Definition 4.2 ([8], Definition 1.2 page 418)**.** $\mathrm{Pr}_1(\kappa, \mu, \sigma, \theta)$, where $\sigma + \theta \le \mu \le \kappa$ and κ is an infinite cardinal, means the following: There is a symmetric function $c : \kappa \times \kappa \to \sigma$ such that

- (∗) If $\xi < \theta$ and for all $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals less than κ with the $\alpha_{i,\zeta}$ being distinct, and if $\gamma < \sigma$, then there are $i < j < \mu$ such that
  - ⊗ $\zeta_1 < \zeta_2 < \xi \Longrightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = \gamma$.

$\mathrm{Pr}_1(\kappa, \sigma, \theta)$ denotes $\mathrm{Pr}_1(\kappa, \kappa, \sigma, \theta)$, which implies $\kappa \nrightarrow [\kappa]^\sigma_\theta$. The following lemma is Conclusion 4.8 in Chapter III Section 4 of [8]:

Lemma 4.3. (1) *Suppose that either*
  - (a) *μ, θ are regular cardinals and $\mu > \theta^+$, or*
  - (b) *μ is singular, $\mu < \aleph_\mu$, and $\theta = \mathrm{cf}(\mu)$.*

  Then $\mathrm{Pr}_1(\mu^+, \mu^+, \theta)$.

(2) *Suppose $\kappa > \aleph_0$ is inaccessible, there is a stationary set $S \subseteq \kappa$ which reflects in no inaccessible, $\delta \in S$ implies $\mathrm{cf}(\delta) \ge \theta$, and $\sigma < \kappa$ and $\aleph_0 < \theta < \kappa$. Then $\mathrm{Pr}_1(\kappa, \sigma, \theta)$ holds.*

Note that $\mathrm{Pr}_1(\kappa, \sigma, \theta)$ for any $2 \le \theta$ implies $\mathrm{Pr}_1(\kappa, \sigma, 2)$, which implies $\kappa \nrightarrow [\kappa]^2_\sigma$, which is what we will use in order to apply Lemma 4.1.

Theorem 1.7. (1) *If κ is the first inaccessible, then $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails, for each $2 \leq \theta < \kappa$.*
(2) *Suppose κ is inaccessible and not Mahlo. Then $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails, for each $2 \leq \theta < \kappa$.*
(3) *If κ is a singular strong limit cardinal and $\mathrm{cf}(\kappa) = \mu^+$, with μ regular, then $\mathrm{HL}^+_{2,\mu^+}[\kappa]$ fails.*

Proof. (1) Suppose κ is the first inaccessible and $2 \leq \theta < \kappa$, and let $\mu = \max(\theta, \aleph_1)$. Then the set $S = \{\delta < \kappa : \mathrm{cf}(\delta) \geq \mu\}$ is stationary and trivially does not reflect in any inaccessible. Thus, by Lemma 4.3 (2), $\mathrm{Pr}_1(\kappa, \sigma, \mu)$ holds, for any $\sigma < \kappa$. It follows that $\mathrm{Pr}_1(\kappa, \theta, 2)$ holds, and hence, $[\kappa] \nrightarrow [\kappa]^2_\theta$ holds. Then Lemma 4.1 (a) implies that $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails.

(2) Now suppose κ is inaccessible and not Mahlo. By Lemma 4.1 (A) $\Rightarrow$ (C) in Chapter III of [8], there is a stationary set $S \subseteq \kappa$ which does not reflect in inaccessible cardinals, and such that $\delta \in S$ implies $\mathrm{cf}(\delta) \geq \mu$, where $\mu = \max(\theta, \aleph_1)$. By Lemma 4.3 (2), $\mathrm{Pr}_1(\kappa, \theta, 2)$ holds, and hence, $[\kappa] \nrightarrow [\kappa]^2_\theta$ holds. Then $\mathrm{HL}^+_{2,\theta}[\kappa]$ fails, by Lemma 4.1 (a).

(3) Lastly, suppose κ is a singular strong limit cardinal, with $\mathrm{cf}(\kappa) = \mu^+$, where μ is regular. By Lemma 4.3 (1), $\mathrm{Pr}_1(\mu^+, \mu^+, 2)$ holds, and hence, $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ holds. Then $\mathrm{HL}^+_{2,\mu^+}(\kappa)$ fails, by Lemma 4.1 (b). □

Džamonja, Larson, and Mitchell point out in Section 8 of [3] that $\mathrm{HL}^+_{2,2}(\kappa)$ implies that κ must be weakly compact. Weakly compact cardinals are Mahlo and hence not the least strongly inaccessible. Theorem 1.7 showed that if $\mathrm{HL}^+_{2,\theta}[\kappa]$ holds for any $2 \leq \theta < \kappa$ where κ is strongly inaccessible, then κ must be Mahlo. Recall that $\mathrm{HL}^+_{2,2}[\kappa]$ is exactly $\mathrm{HL}^+_{2,2}(\kappa)$, leading to the following question.

Question 4.4. Can $\mathrm{HL}^+_{2,\theta}[\kappa]$ consistently hold for some $2 < \theta \leq \kappa$ when κ is a Mahlo, non-weakly compact cardinal?

We can only ask for consistency because Corollary 1.9 to the next theorem shows that $\mathrm{HL}_{2,\theta}[\kappa]$ fails in L for all $2 \leq \theta \leq \kappa$ whenever κ is strongly inaccessible and not weakly compact.

The following theorem shows that the failure of $\mathrm{HL}_{2,\theta}[\kappa]$, for all $2 \leq \theta \leq \kappa$, follows from $\diamondsuit_S$ for a non-reflecting stationary subset $S \subseteq \kappa$.

Theorem 1.8. *Assume κ is strongly inaccessible, $S \subseteq \kappa$ is a non-reflecting stationary set, and $\diamondsuit_S$ holds. Then $\mathrm{HL}_{2,\theta}[\kappa]$ fails, for each $2 \leq \theta \leq \kappa$.*

Proof. Let κ be inaccessible and $S \subseteq \kappa$ be a non-reflecting stationary set, and suppose that $\diamondsuit_S$ holds. By possibly thinning S, we may assume that S is a set of strong limit cardinals, and that there is a sequence

$$\{(\mu, T^0_\mu, T^1_\mu, A_\mu, F_\mu, \xi_\mu) : \mu \in S\} \tag{23}$$

such that

(a) for each $\mu \in S$:

$\oplus_\mu$
- $\bullet_1$ For each $\ell < 2$, (μ, T^ℓ_μ, A_μ) is a suitable triple,
- $\bullet_2$ F_μ is a 1-1 function from T^0_μ onto T^1_μ,
- $\bullet_3$ $\eta \in T^0_\mu$ implies $\lg(F_\mu(\eta)) = \lg(\eta)$,
- $\bullet_4$ $(\ell < 2 \ \wedge\ \eta \in T^0_\eta \ \wedge\ \lg(\eta) \in A_\mu)$ implies $F_\mu(\eta^\frown\langle\ell\rangle) = F_\mu(\eta)^\frown\langle\ell\rangle$,

- $\bullet_5$ $\eta \triangleleft \nu \in T^0_\mu$ implies $F_\mu(\eta) \triangleleft F_\mu(\nu)$,
- $\bullet_6$ $\xi_\mu < \theta$,

and such that

(b) if $(\kappa, T^0_\kappa, T^1_\kappa, A_\kappa, F_\kappa, \xi_\kappa)$ are as above, i.e., $\oplus_\kappa$ holds, then for stationarily many $\mu \in S$, we have

$$(\mu, T^0_\mu, T^1_\mu, A_\mu, F_\mu, \xi_\mu) = (\mu, T^0_\kappa \cap {}^{\mu>}2, T^1_\kappa \cap {}^{\mu>}2, A_\kappa \cap \mu, F_\kappa \restriction T^0_\mu, \xi_\kappa).$$

Note that for each $\mu \in S$, A_μ is an unbounded subset of μ, since (μ, A_μ, T^ℓ_μ) is suitable, for each $\ell < 2$.

Claim 6. *Given $\beta < \gamma < \kappa$ with $\beta \notin S$, then there is a function $g = g_{\beta,\gamma}$ such that*

(1) *g is one-to-one;*
(2) *$\mathrm{dom}(g) = S \cap \gamma \setminus (\beta + 1)$;*
(3) *$\mu \in \mathrm{dom}(g)$ implies $g(\mu) \in A_\mu \setminus (\beta + 1)$.*

Proof. The proof is by induction on $\gamma < \kappa$.

Case 1. γ is a successor ordinal, say $\gamma = \gamma_1 + 1$. Let $\beta \in \gamma \setminus S$ be given. If $\beta = \gamma_1$, then $S \cap \gamma \setminus (\beta + 1) = \emptyset$ and the empty function trivially satisfies (1)–(3); so now assume that $\beta < \gamma_1$. If $\gamma_1 \notin S$, then $S \cap \gamma \setminus (\beta + 1) = S \cap \gamma_1 \setminus (\beta + 1)$ and we let $g_{\beta,\gamma} = g_{\beta,\gamma_1}$.

If $\gamma_1 \in S$, since A_{γ_1} is unbounded in γ_1 we can choose $\beta_1 \in A_{\gamma_1} \setminus (\beta + 1)$. By the induction hypothesis, we have $g_1 := g_{\beta,\beta_1+1}$ on $S \cap (\beta_1 + 1) \setminus (\beta + 1)$ satisfying (1)–(3) and $g_2 := g_{\beta_1+1,\gamma_1}$ on $S \cap \gamma_1 \setminus (\beta_1 + 2)$ satisfying (1)–(3). Then $\mathrm{dom}(g_1) \cap \mathrm{dom}(g_2) \subseteq (\beta, \beta_1] \cap (\beta_1 + 1, \gamma_1) = \emptyset$, so $g_1 \cup g_2$ is a function. Further. $\mathrm{ran}(g_1) \subseteq \beta_1$ and $\mathrm{ran}(g_2) \cap (\beta_1 + 2) = \emptyset$, so $g_1 \cup g_2$ is one-to-one and does not contain β_1 in its range. Define $g_{\beta,\gamma} = g_1 \cup g_2 \cup \{(\gamma_1, \beta_1)\}$. Then $g_{\beta,\gamma}$ is a one-to-one function. Moreover,

$$\begin{aligned} \mathrm{dom}(g_{\beta,\gamma}) &= \{\gamma_1\} \cup \big(S \cap ((\beta, \beta_1] \cup (\beta_1 + 1, \gamma_1))\big) \\ &= S \cap (\beta, \gamma_1] \\ &= S \cap \gamma \setminus (\beta + 1) \end{aligned} \tag{24}$$

since $\gamma_1 \in S$ and $\beta_1 + 1 \notin S$ as S consists only of limit ordinals. As $\beta_1 \in A_{\gamma_1} \setminus (\beta + 1)$, this along with (3) of the induction hypothesis for g_1 and g_2 imply that $g_{\beta,\gamma}$ satisfies (1)–(3).

Case 2. γ is a limit ordinal. Let $\beta \in \gamma \setminus S$. As S does not reflect, $S \cap \gamma$ is not stationary in γ. Recalling that S consists only of limit ordinals, there is an increasing continuous sequence $\langle \beta_\iota : \iota < \mathrm{cf}(\gamma) \rangle$ of ordinals in $\gamma \setminus S$ such that $\gamma = \sup\{\beta_\iota : \iota < \mathrm{cf}(\gamma)\}$, with $\beta_0 = \beta$. Choose g_ι for $(\beta_\iota, \beta_{\iota+1})$ according to the induction hypothesis, and let $g_{\beta,\gamma} = \bigcup\{g_\iota : \iota < \mathrm{cf}(\gamma)\}$. Then $g_{\beta,\gamma}$ is a function satisfying (1)–(3). ☐

Using Claim 6, we define a function $c : [{}^\alpha 2]^2 \to \theta$ for each $\alpha < \kappa$ as follows:

$(*)_2$ For $\gamma \in \mathrm{ran}(g_{0,\alpha})$ and μ such that $g_{0,\alpha}(\mu) = \gamma$, if $\{\nu_0, \nu_1\} \in [{}^\alpha 2]^2$ satisfy $\gamma = \lg(F_\mu(\nu_0 \restriction \gamma') \cap \nu_1)$, where γ' is the least ordinal in A_μ above γ, then define $c(\{\nu_0, \nu_1\}) = \xi_\mu$; otherwise, $c(\{\nu_0, \nu_1\}) = 0$. Let $c = \bigcup_{\alpha<\kappa} c_\alpha$.

To finish the proof, let $\xi < \theta$ be given and let T^ℓ, $\ell < 2$, and $A \subseteq \kappa$ be such that (κ, T^ℓ, A) is a suitable triple for each $\ell < \kappa$, and let F be the isomorphism from T^0

onto T^1. Then $\oplus_\kappa$ holds for the sequence $(\kappa, T^0, T^1, A, F, \xi)$, so the set S' of those $\mu \in S$ for which

$$(\mu, T^0_\mu, T^1_\mu, A_\mu, F_\mu, \xi_\mu) = (\mu, T^0 \cap {}^{\mu>}2, T^1 \cap {}^{\mu>}2, A \cap \mu, F \restriction T^0_\mu, \xi)$$

holds is stationary. Fix $\mu < \sigma$, both in S'. Then $F_\sigma = F \restriction T^0_\sigma$ and $F_\sigma \restriction T^0_\mu = F_\mu = F \restriction T^0_\mu$. Choose $\alpha \in A_\sigma$ (which equals $A \cap \sigma$) such that $\mu + \omega \leq \alpha$. This is possible since $\sigma \in S$ implies that σ is a strong limit, and A_σ is unbounded in σ. Note that $T^\ell \cap {}^\alpha 2 = T^\ell_\sigma \cap {}^\alpha 2$, for each $\ell < 2$,

Let $\gamma = g_{0,\alpha}(\mu)$. Then $\gamma \in A_\mu = A \cap {}^\mu 2$, since $\mu \in S'$ and by (3) of Claim 6. Fix a node $\eta \in T^1 \cap {}^\gamma 2$ and let $\eta_0 = \eta^\frown 0$ and $\eta_1 = \eta^\frown 1$. Extend η_0 to some ν_0' in ${}^\alpha 2 \cap T^1_\sigma$, and extend η_1 to some ν_1 in ${}^\alpha 2 \cap T^1_\sigma$. Let $\nu_0 = F^{-1}_\sigma(\nu_0')$. Then ν_0 is in $T^0 \cap {}^\alpha 2$. Note that, letting γ' be the least ordinal in A_μ above γ,

$$F_\mu(\nu_0 \restriction \gamma') = F_\sigma(\nu_0 \restriction \gamma') = F_\sigma(\nu_0) \restriction \gamma' = \nu_0' \restriction \gamma'.$$

In particular, $\gamma = \lg(F_\mu(\nu_0 \restriction \gamma') \cap \nu_1)$. By the definition $(*)_2$ of c it follows that $c(\{\nu_0, \nu_1\}) = \xi_\mu$, which is ξ. Since ξ was an arbitrary ordinal less than θ, we see that $HL_{2,\theta}[\kappa]$ fails. □

Corollary 1.9 follows immediately.

5. Open Problems

We conclude by stating some of the multitude of open problems regarding various versions of Halpern–Läuchli at uncountable cardinals and their consistency strengths.

Question 5.1. For κ weakly compact, if $\mathrm{HL}^+_{2,\theta}[\kappa]$ holds for some $2 < \theta < \kappa$, then must $\mathrm{HL}^+_{2,2}(\kappa)$ hold?

A similar question can be asked for Halpern-Läuchli on products of two trees:

Question 5.2. Given $2 < \theta < \kappa$, is $\mathrm{HL}_{2,\theta}[\kappa]$ strictly weaker than $\mathrm{HL}_{2,\theta}(\kappa)$?

Question 5.3. For $2 \leq n < \omega$ and $2 \leq \theta < \kappa$, how do $\mathrm{HL}_{n,\theta}(\kappa)$ and $\mathrm{HL}^+_{n,\theta}[\kappa]$ compare? Are there models of ZFC where one holds but the other does not?

References

1. Natasha Dobrinen and Daniel Hathaway, *The Halpern-Läuchli Theorem at a measurable cardinal*, Journal of Symbolic Logic **82** (2017), no. 4, 1560–1575.
2. ______, *Forcing and the Halpern-Läuchli Theorem*, Journal of Symbolic Logic **85** (2020), no. 1, 87–102.
3. M. Džamonja, J. Larson, and W. J. Mitchell, *A partition theorem for a large dense linear order*, Israel Journal of Mathematics **171** (2009), 237–284.
4. ______, *Partitions of large Rado graphs*, Archive for Mathematical Logic **48** (2009), no. 6, 579–606.
5. Paul Erdős and Richard Rado, *A partition calculus in set theory*, Bulletin of the American Mathematical Society **62** (1956), 427–489.
6. Richard Laver, *Products of infinitely many perfect trees*, Journal of the London Mathematical Society (2) **29** (1984), no. 3, 385–396.
7. Saharon Shelah, *Strong partition relations below the power set: consistency – was Sierpinski right? II*, Sets, Graphs and Numbers (Budapest, 1991), vol. 60, Colloq. Math. Soc. János Bolyai, North-Holland, 1991, pp. 637–688.
8. ______, *Cardinal arithmetic*, Oxford Science Publications, 1994.
9. Jing Zhang, *A tail cone version of the Halpern-Läuchli theorem at a large cardinal*, Journal of Symbolic Logic **84** (2019), no. 2, 473–496.

University of Notre Dame, Department of Mathematics, 255 Hurley Bldg, Notre Dame, IN 46556 USA
Email address: ndobrine@nd.edu

The Hebrew University of Jerusalem, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, Jerusalem, 91904, Israel, and Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854 USA
Email address: shelah@math.huji.ac.il