MORE FORCING FOR NO ULTRAFILTERS
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Abstract. We introduce a forcing for making an ultrafilter extendable to an ultrafilter which is a relative of P-point. This forcing is related to the original forcing used in the consistency proof of "there is no nowhere ultratilters".

This seem not to help to intend to show this give more. We hope to used it to ....

We prove the consistency of "no α-ultrafilters" for $\alpha \geq 1$ a countable ordinal and (?) no van-Dowen ultrafilter on $\mathbb{Q}$. This continues [She98b] where we prove the consistency of "there is no NWD (nowhere dense) ultrafilter on $\mathbb{N}$".

But we first deal with relatives of the forcing from [She98b]; of self interest in a self contained way.

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Annotated Content

§ 0. Introduction

§ 0(A). **0A.**

In [She98b] we prove the consistency of "there is no NWD-ultrafilter on ω" (see below, non-principal, of course). This answers a question of van Douwen [vD81] which appears as question 31 of [Bau95]. Baumgartner [Bau95] considering this question, dealt more generally with J-ultrafilters where:

Definition 0.1. 1) An ultrafilter D, say on ω, is called a J-ultrafilter where J is an ideal on some set X (to which all singletons belong, to avoid trivialities) when for every function $f : \omega \to X$ for some $A \in D$ we have $f''(A) \in J$.

2) The NWD-ultrafilters are the J-ultrafilters for $J = \{B \subseteq \mathbb{Q} \colon B$ is nowhere dense$\}$, ($\mathbb{Q}$ is the set of all rationals[1]; we may use an equivalent version, see [She98b, 2.4] and here in §3).

3) An ultrafilter D is called a J^1_α-ultrafilter for a (countable) ordinal $\alpha \geq 1$ when it is J^1_α-ultrafilter where $J^1_\alpha = \{A \subseteq \omega^\alpha \colon \mathrm{otp}(A) < \omega^\alpha\}$, where ω^α is the ordinal exponentiation.

3A) An ultrafilter D is called an α-ultrafilter when it is a J^0_α-ultrafilter where $J^0_\alpha = \{A \subseteq \alpha \colon \mathrm{otp}(A) < \alpha\}$.

4) A van-Dowen ultrafilter is one on $\mathbb{Q}$ such that the family of $A \subseteq D$ dense in themselves is dense in D.

The non-existence of NWD ultrafilters is also relevant for the consistency of "every (non-trivial) σ-centered forcing notion adds a Cohen real", see [BS01].

The most natural approach to a proof of the consistency of "there is no NWD-ultrafilter" was to generalize the proof of CON(there is no P-point) (see [She82, Ch.VI,§4] or [She98a, Ch.VI,§4]), but for some time I (and probably others) had not seen how.

Professor Shelah, you did not tell me where to put the content of page 4A: but by the referee request in §1, we concentrate on $\ell = 1$ and comment in the end on $\ell = 2, 3$.

We use in [She98b] an idea taken from [She92], which is to replace the given maximal ideal I on ω by a quotient; moreover, we allow ourselves to change the quotient. In fact, the forcing here is simpler than the one in [She92]. A related earlier work is Goldstern Shelah [GS90], it uses a "one real" version of $\mathbb{Q}^1_{\mathbf{i}}$ from §1.

We similarly may consider the consistency of "no J^1_α–ultrafilter" for non-zero $\alpha < \omega_1$ (see [Bau95] for discussion of α–ultrafilters). This question and the problems of preservation of ultrafilters and distinguishing existence properties of ultrafilters was promised in [She98b] to be dealt with in the subsequent work [S$^+$a]; we try to deal here with the first and with van-Dowen ultrafilters; the second is still in preparation (see [S$^+$b] continuing [She]).

We have not try to deal with having different answers to different such α's.

Discussion 0.2. In §1 we use $\mathbf{i} \in \mathrm{FP}_1$, a forcing parameter. We define $\mathbb{Q}_{\mathbf{i}} := \mathbb{Q}^\ell_{\mathbf{i}}$ for $\ell = 1$ later we deal with $\ell = 1, 2, 3$ and $\mathbb{Q}^4_{\mathbf{i}}$. Now $\ell = 1, 2$ are as in [She98b]; for

[1]Notice the difference between the symbols "$\mathbb{Q}$" and "$\mathbb{Q}$", the last one will be used to denote forcing notions.

$\ell = 1$, a kind of power of [GS90], for $\ell = 2$ a kind of power of [She92]. but by the referee request we concentrate on $\ell = 1$. Now for $\ell = 1$, all n/E^p behave as in n, hence $\le_n^{\otimes}$ does not work. For $\ell = 3$ we make $\le_n^{\otimes}$ work but it is not good enough for our purpose (no suitable ultrafilter).

In §2 introduce $\mathbb{Q}_{\mathbf{i}}^4$, the forcing when use creatures, see [RS99]. The $\mathbb{Q}_{\mathbf{i}}$ for $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}$ from there combine the desired properties mentioned above.

How do we prove in [She98b] that by suitable iteration $\mathbf{q}$ of $\mathbb{Q}_{\mathbf{i}}^1$'s in $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$ there is no nowhere dense ultrafilter? In the end, i.e. in $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$ toward proving "no nowhere dense ultrafilter" for a candidate $\underset{\sim}{D}$ we try $\langle \underset{\sim}{\eta}_n : n < \omega \rangle$, so toward contradiction $p_* \Vdash$ "$\underset{\sim}{X} \in \mathrm{fil}(I_{\mathbf{m}})$ satisfies $\{\underset{\sim}{\eta}_n : n \in \underset{\sim}{X}\}$ is nowhere dense". Without loss of generality above p_* we can read $\underset{\sim}{X}$ promptly, i.e. if $n \in A^{p_*}$ and $f \colon \{x_i^m : m \in A^{p_*} \cap (n+1), i < h(n)\} \Rightarrow \{-1, 1\}$ <u>then</u> $p^{[f]}$ read (i.e. forces a value to) $\underset{\sim}{X} \cap (n+1)$ and even p_* forces $u_n \cap \underset{\sim}{X} \neq \emptyset$ and moreover if $n_1 < n_2$ are from A then $\underset{\sim}{X} \cap [n_1, n_2] \neq \emptyset$ and moreover the members of u_n are not in $\cup\{m/E^p : m \in A^p \cap (n+1)$ and $u_n \subseteq \min(A^p \backslash (n+1))\}$. Then find q above p_* forcing density.

We continue this in the later section.

In what way is the proof here better than the one in [She98b]? There, proving $\mathbb{Q}_{\mathbf{i}}^2$ $\mathbb{Q}_{\mathbf{i}}^2$ is "nice", we prove that the "good guy" does not lose in the game. Here prove that he wins. We intend to use then to get new consistency results.

<u>Question 1</u>: In general we do not know whether, if α is limit, there is a J_{ω^α}-ultrafilter that is not a β-ultrafilter for some $\beta < \omega^\alpha$, even if CH or MA is assumed. See §3B for solution.

<u>Question 2</u>: It is consistent that there are no P-point but ω^ω-ultrafilters exists?

<u>Question 3</u>: Assuming CH or MA, does it follow that there exsits an ω^ω-ultrafilter u such that $\forall v \le_{\mathrm{RK}} u$ if v is non-principal, then v is a proper ω^ω-ultraafilter?

<u>Question 4</u>: If u is a proper $\omega^{\alpha+\omega}$-ultrafilter, does it follow that there is a proper ω^ω-ultrafilter $v \le_{\mathrm{RK}} u$?]

§ 1. The basic forcing

§ 1(A). A new version.

In Definition 1.2 below we define the forcing notion $\mathbb{Q}^1_{\mathbf{i}}$ which was the one used in the proof of the main result of [She98b, §3], here we shall use the one from §2. Among the other forcing notions defined later, $\mathbb{Q}^3_{\mathbf{i}}$ is the closest relative of $\mathbb{Q}^1_{\mathbf{i}}$. Various properties may be easier to check for some relatives but it is more complicated to define, anyhow unfortunately it does not help concerning ω^α-ultrafilters but we feel are of interest. In [She98b] we have eventually used only $\mathbb{Q}^1_{\mathbf{i}}$.

Definition 1.1. Let I be an ideal on ω containing the family $[\omega]^{<\aleph_0}$ of finite subsets of ω.

1) We say that an equivalence relation E is an I–equivalence relation <u>when</u>:

(a) $\mathrm{dom}(E) \subseteq \omega$,
(b) $\omega \setminus \mathrm{dom}(E) \in I$,
(c) each E-equivalence class belongs to I.

2) For I-equivalence relations E_1, E_2 we write $E_1 \leq E_2$ if:

(i) $\mathrm{dom}(E_2) \subseteq \mathrm{dom}(E_1)$,
(ii) $E_1 \restriction \mathrm{dom}(E_2)$ refines E_2,
(iii) $\mathrm{dom}(E_2)$ is the union of a family of E_1-equivalence classes.

3) We say I is a P-c.c.c. ideal <u>when</u>:

(a) I is an ideal on ω containing all the finite subsets
(b) I is a P-ideal, i.e. if $A_n \in I$ for $n < \omega$ then for some $A \in I$ we have $n < \omega \Rightarrow A_n \subseteq^* A$
(c) $\mathscr{P}(\omega)/I$ is a c.c.c. Boolean Algebra.

Definition 1.2. 1) Let FP_1 be the set of (forcing parameters) $\mathbf{i}$ which means it consists of:

(a) I, an ideal on ω to which all finite subsets of ω belong; let $D_{\mathbf{i}} = \mathrm{dual}(I)$
(b) $h\colon \omega \to \omega$ be a non-decreasing function
(c) h goes to infinity, (if $h(n) = n$ we may omit it).

2) For $\mathbf{i} \in \mathrm{FP}_1$ we define a forcing notion $\mathbb{Q}^1_{\mathbf{i}}$ intended to add $\langle y^n_i : i < h(n), n < \omega\rangle$ with $y^n_i \in \{-1, 1\}$. We use x^n_i as variables.

3) $p \in \mathbb{Q}_{\mathbf{i}}$ if and only if $p = (H, E, A) = (H^p, E^p, A^p)$ and:

(a) E is an I-equivalence relation, so on $\mathrm{dom}(E) \subseteq \omega$,
(b) $A = \{n \in \mathrm{dom}(E) : n = \min(n/E)\}$,
(c) H is a function with range $\subseteq \{-1, 1\}$ and domain

$B^p_1 = \{x^n_i\colon i < h(n) \wedge n \in \omega \setminus \mathrm{dom}(E)$ <u>or</u> $n \in \mathrm{dom}(E) \wedge i \in [h(\min(n/E)), h(n))\}$.

4) For $a \subseteq A^p$ we define $\mathscr{F}_a = \{f\colon f$ is a function with domain $\{x^n_i : i < h(n), n \in a\}$ such that $f(x^n_i) \in \{-1, 1\}\}$.

5) For a finite set $\mathbf{u} \subseteq \omega$ we let $\mathrm{var}(\mathbf{u}) := \{x^n_i : i < h(n), n \in \mathbf{u}\}$.

6) We say that a function $f\colon \{x^n_i\colon i < h(n),\ n < \omega\} \to \{-1, 1\}$ satisfies a condition $p \in \mathbb{Q}_{\mathbf{i}}$ <u>when</u>:

(a) $f(x^n_i) = H^p(x^n_i)$ <u>when</u> $x^n_i \in B^p_1$,

(b) $f(x_i^n) = (f(x_i^{\min(n/E^p)}))$ <u>when</u> $\ell = 1$, $n \in \mathrm{dom}(E^p)$ and $i < h(\min(n/E^p))$

7) The partial order $\leq = \leq_{\mathbb{Q}_1}$ is defined by: $p \leq q$ if and only if:

(α) $E^p \leq E^q$, i.e.
- $\mathrm{dom}(E^q) \subseteq \mathrm{dom}(E^p)$
- if $n \in \mathrm{dom}(E^q)$ then $n/E^p \subseteq \mathrm{dom}(E^q)$
- $E^p \restriction \mathrm{dom}(E^q)$ refines E^q

(β) every function $f : \{x_i^n : i < h(n),\ n < \omega)\} \to \{-1, 1\}$ satisfying q satisfies p.

Proposition 1.3. *$(\mathbb{Q}^1_{\mathbf{i}}, \leq_{\mathbb{Q}^1_{\mathbf{i}}})$ is a partial order.*

Remark 1.4. We may reformulate the definition of the partial order $\leq_{\mathbb{Q}_{\mathbf{i}}}$, making them perhaps more direct. Thus, in particular, if $p, q \in \mathbb{Q}^1_{\mathbf{i}}$ then $p \leq_{\mathbb{Q}^1_{\mathbf{i}}} q$ if and only if the demand (α) of 1.2(7) holds and

($*$) for each $x_i^n \in B_1^q$:
  (i) if $x_i^n \in B_1^p$ then $H^q(x_i^n) = H^p(x_i^n)$,
  (ii) if $n \in \mathrm{dom}(E^p) \setminus \mathrm{dom}(E^q), i < h(\min(n/E^p))$ and $n \notin A^p$ then $H^q(x_i^n) = H^q(x_i^{\min(n/E^p)})$,
  (iii) if $n \in \mathrm{dom}(E^q) \setminus A^p, \min(n/E^p) > \min(n/E^q)$ and $h(\min(n/E^q)) \leq i < h(\min(n/E^p))$ then $H^q(x_i^n) = H^q(x_i^{\min(n/E^p)})$.

One may wonder why we have h in the definition of $\mathbb{Q}^1_{\mathbf{i}}$ and we do not fix that, e.g. $h(n) = n$. The reason is to be able to describe nicely what is the forcing notion $\mathbb{Q}^1_{\mathbf{i}}$ above a condition p like. The point is that $\mathbb{Q}^1_{\mathbf{i}} \restriction \{q \colon q \geq p\}$ is like $\mathbb{Q}^1_{\mathbf{i}}$ but we replace I by its quotient by E^p and we change the function h.

More precisely:

Proposition 1.5. *Assume that $p \in \mathbb{Q}_{\mathbf{i}}$, $A^p = \{n_k \colon k < \omega\}$, where $n_k < n_{k+1}$, $h^* \colon \omega \to \omega$ is defined by $h^*(k) = h(n_k)$,*

$$I^* = \left\{ B \subseteq \omega \colon \bigcup_{k \in B} (n_k/E) \in I \right\}$$

and $\mathbf{i}(*) := (I^*, h^*)$. Then, $\mathbb{Q}^1_{\mathbf{i}} \restriction \{q \colon p \leq_{\mathbb{Q}^1_{\mathbf{i}}} q\}$ is isomorphic to $\mathbb{Q}^1_{\mathbf{i}(*)}$.

Proof. Natural. $\square_{1.5}$

Definition 1.6. We define a $\mathbb{Q}_{\mathbf{i}}$-name $\bar{\underset{\sim}{\eta}} = \langle \underset{\sim}{\eta}_n \colon n < \omega \rangle$ by: $\underset{\sim}{\eta}_n$ is a sequence of length $h(n)$ of members from $\{-1, 1\}$, such that

$$\underset{\sim}{\eta}_n[\mathbf{G}_{\mathbb{Q}_{\mathbf{i}}}](i) = 1 \Leftrightarrow (\exists p \in \mathbf{G}_{\mathbb{Q}_{\mathbf{i}}})(H^p(x_i^n) = 1 \wedge n < \min(A^p)).$$

[Note that, even if we omit "$n < \min(A^p)$", if $x_i^n \in \mathrm{dom}(H^p)$, $H^p(x_i^n) = 1$ and $q \geq p$, then $H^q(x_i^n) = 1$; remember 1.2(7).]

Proposition 1.7. *1) If $n < \omega$, $p \in \mathbb{Q}_{\mathbf{i}}$ and $A^p \cap (n+1) = \emptyset$, (or just $n \notin \mathrm{dom}(E^p)$) <u>then</u>*

$$p \Vdash \text{“}\underset{\sim}{\eta}_n = \langle H^p(x_i^n) \colon i < h(n) \rangle\text{”}.$$

2) For each $n < \omega$, the set $\{p \in \mathbb{Q}_{\mathbf{i}} \colon A^p \cap (n+1) = \emptyset\}$ is dense in $\mathbb{Q}_{\mathbf{i}}$.

3) If $p \in \mathbb{Q}_{\mathbf{i}}$, $a \subseteq A^p$ is finite or at least $\bigcup_{n \in a}(n/E^p) \in I$, and $f \in \mathscr{F}_a$, <u>then</u> for some unique q which we denote by $p^{[f]}$, we have:

- (a) $p \le q \in \mathbb{Q}^1_{\mathbf{i}}$,
- (b) $E^q = E^p \restriction \bigcup\{n/E^p : n \in A \setminus a\}$,
- (c) *for $n \in a$ and $i < h(n)$, we have $H^q(x^n_i)$ is $f(x^n_i)$.*

Proof. Straightforward. $\square_{1.7}$

Definition 1.8. 1) $p \le_n q$ (in $\mathbb{Q}_{\mathbf{i}}$) <u>iff</u> $p \le q$ and: if $k \in A^p$ and $|A^p \cap k| < n$ then $k \in A^q$.

2) $p \le^*_n q$ <u>iff</u> $p \le q$ and: if $k \in A^p$ and $|A^p \cap k| < n$ then $k \in A^q$ and $k/E^p = k/E^q$.

3) $p \le^{\otimes}_n q$ <u>iff</u> $p \le_{n+1} q$ and: $n > 0 \Rightarrow p \le^*_n q$ and $\mathrm{dom}(E^q) = \mathrm{dom}(E^p)$.

Proposition 1.9. *1) If $p \le q, \mathbf{u}$ is a finite initial segment of A^p and $A^q \cap \mathbf{u} = \emptyset$, <u>then</u> for some unique $f \in \mathscr{F}_{\mathbf{u}}$ we have $p \le p^{[f]} \le q$ (where $p^{[f]}$ is from 1.7(3), so $A^q = A^p \setminus \mathbf{u}$).*

2) If $p \in \mathbb{Q}^1_{\mathbf{i}}$ and $\mathbf{u}$ is a finite initial segment of A^p <u>then</u>:

- $(*)_1$ *$f \in \mathscr{F}_{\mathbf{u}}$ implies $p \le p^{[f]}$ and $p^{[f]} \Vdash$ "$(\forall n \in \mathbf{u})(\forall i < h(n))(\underset{\sim}{\eta}_n(i) = f(x^n_i))$",*
- $(*)_2$ *the set $\{p^{[f]} : f \in \mathscr{F}_{\mathbf{u}}\}$ is predense above p (in $\mathbb{Q}^1_{\mathbf{i}}$).*

3) $\le_n$ is a partial order on $\mathbb{Q}_{\mathbf{i}}$, and $p \le_{n+1} q \Rightarrow p \le_n q$. Similarly for $<^*_n$ and $<^{\otimes}_n$. Also

- $(*)_1$ $p \le^{\otimes}_n q \Rightarrow p \le^*_n q \Rightarrow p \le_n q \Rightarrow p \le q$
- $(*)_2$ $p \le^{\otimes}_n q \Rightarrow p \le_{n+1} q$.

4) If $p \in \mathbb{Q}^1_{\mathbf{i}}, \mathbf{u}$ is a finite initial segment of $A^p, |\mathbf{u}| = n$ and $f \in \mathscr{F}_{\mathbf{u}}$ and $p^{[f]} \le q \in \mathbb{Q}_{\mathbf{i}}$ <u>then</u> for some $r \in \mathbb{Q}_{\mathbf{i}}$ we have $p \le^*_n r \le q$ and $r^{[f]} = q$.

Proof. 1) Define $f \in \mathscr{F}_{\mathbf{u}}$ by: $f(x^n_i)$ is the value of $H^q(x^n_i)$;

2) By 1.7(3) and 1.9(1).

3) Check.

4) First let us define the required condition r in the case $\ell = 1$. So we let $\mathrm{dom}(E^r) = \bigcup_{n \in \mathbf{u}}(n/E^p) \cup \mathrm{dom}(E^q), E^r = \{(n_1, n_2) : n_1 E^q n_2$ or for some $n \in \mathbf{u}$ we have: $\{n_1, n_2\} \subseteq (n/E^p)\}, A^r = \mathbf{u} \cup A^q$ (note that if $n_1 E^q n_2$ then $n_1 \notin \mathbf{u}$). Next, for $x^n_i \in B^r_1$ (where B^r_1 is given by 1.2(3)(c)) we define

$$H^r(x^n_i) = \begin{cases} H^q(x^n_i) & \text{if } n \notin \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^q), \\ H^p(x^n_i) & \text{if } n \in \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^p). \end{cases}$$

It should be clear that $r = (H^r, E^r, A^r) \in \mathbb{Q}^\ell_{\mathbf{i}}$ is as required.

$\square_{1.9}$

Corollary 1.10. *If $p \in \mathbb{Q}^1_{\mathbf{i}}, n < \omega$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}^1_{\mathbf{i}}$-name of an ordinal, <u>then</u> there are $\mathbf{u}, q$ and $\bar{\alpha} = \langle \alpha_f : f \in \mathscr{F}_{\mathbf{u}} \rangle$ such that:*

- (a) $p \le^*_n q \in \mathbb{Q}_{\mathbf{i}}$,

(b) $\mathbf{u} = \{\ell \in A^p : |\ell \cap A^p| < n\}$,
(c) *for* $f \in \mathscr{F}_{\mathbf{u}}$ *we have* $q^{[f]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_f$",
(d) $q \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_f : f \in \mathscr{F}_{\mathbf{u}}\}$" *(which is a finite set).*

Proof. Let $k = \prod_{\ell \in \mathbf{u}} 2^{h(\ell)}$. Let $\{f_\ell : \ell < k\}$ enumerate $\mathscr{F}_{\mathbf{u}}$. By induction on $\ell \leq k$ define r_ℓ, α_{f_ℓ} such that:

- $r_0 = p$
- $r_\ell \leq^*_n r_{\ell+1} \in \mathbb{Q}^\ell_{\mathbf{i}}$
- $r^{[f_\ell]}_{\ell+1} \Vdash_{\mathbb{Q}^\ell_{\mathbf{i}}}$ "$\underset{\sim}{\tau} = \alpha_{f_\ell}$".

The induction step is by 1.9(4). Now $q = r_k$ and $\langle \alpha_f : f \in \mathscr{F}_{\mathbf{u}}\rangle$ are as required. $\square_{1.10}$

Definition 1.11. Let I be an ideal on ω containing $[\omega]^{<\aleph_0}$ and let E be an I-equivalence relation.

1) We define a game $GM_I(E)$ between two players. The game lasts ω moves. Both players choose I-equivalence relations, where those of player I are denoted by E^1_n and those of player II are denoted by E^2_n.

In the n-th move the first player chooses an I-equivalence relation E^1_n such that $E^1_0 = E, [n > 0 \Rightarrow E^2_{n-1} \leq E^1_n]$, and the second player chooses an I-equivalence relation E^2_n such that $E^1_n \leq E^2_n$. In the end, the second player wins if

$$\bigcup\{\mathrm{dom}(E^2_n) \setminus \mathrm{dom}(E^1_{n+1}) : n \in \omega\} \in I$$

(otherwise the first player wins).

2) For a countable elementary submodel N of $(\mathscr{H}(\chi), \in, <^*)$ such that $I, E \in N$ we define a game $GM^N_I(E)$ in a similar manner as $GM_I(E)$, but we demand additionally that the relations played by both players are from N (i.e. $E^1_n, E^2_n \in N$ for $n \in \omega$).

The following propositions 1.12,1.13 are needed in [She98b] for the case $\mathbb{Q}^2_{\mathbf{i}}$ but not for $\mathbb{Q}^1_{\mathbf{i}}$.

Proposition 1.12. *Assume that I is a maximal (non-principal) ideal on ω and E is an I-equivalence relation. Then the game $GM_I(E)$ is not determined. Moreover, for each countable $N \prec (\mathscr{H}(\chi), \in, <^*)$ such that $I, E \in N$ the game $GM^N_I(E)$ is not determined.*

Proof. Clearly if a player has a winning strategy in $\mathrm{GM_I(E)}$, then it has one in $\mathrm{GM^N_I(E)}$, so fix N. First, assume that player I has a winning strategy stat_1, then we define a strategy stat_2 for player II: it choose $E^2_0 := E$ (which by 1.10 is equal to (E^1_0)) and ensure that:

(∗) if the play $\langle (E^1_n, E^2_n) : n < \omega\rangle$ is played according to stat_2, then $\langle (\check{E}^2_n, \check{E}^2_n) : n < \omega\rangle$ is a play played according to stat_1: $\check{E}^1_0 = \check{E}^2_0 = E$, $\check{E}^1_1 = E^2_0$, $\check{E}^2_1 = E^1_1$, $\check{E}^2_2 = E^1_0$, etc.

Hence player I cannot have a winning strategy.

Now, this is not enough when we fix E, however we actually have proved that player I has no winning strategy in $GM^N_{\mathscr{J}}(E')$ for every I-equivalence relation $E' \in N$. So if player II has a winning strategy, let it choose E^2_0, but now as J is a maximal ideal, we are actually playing $GM^N_I(E^2_0)$ with the players interchanged, and so player I wins it, contradiction. $\square_{1.12}$

Proposition 1.13. *1) Let $p \in \mathbb{Q}^1_{\mathbf{i}}$. Suppose that the first player has no winning strategy in $GM_I(E^p)$ where, of course, $I = I_{\mathbf{i}}$. Then in the following game, $\partial^1_{\mathbf{i}}$, Player I has no winning strategy:*

(A) *in the n-th move, Player I chooses a $\mathbb{Q}^1_{\mathbf{i}}$-name, $\underset{\sim}{\tau}_n$ of an ordinal and Player II chooses $p_n, \mathbf{u}_n, w_n$ such that: w_n is a set of $\leq \prod_{\ell \in \mathbf{u}_n} 2^{h(\ell)}$ ordinals, $p \leq p_n \leq^*_n p_{n+1}$, $p_n \leq_{n+1} p_{n+1}$, $\mathbf{u}_n$ is a finite initial segment of A^{p_n} with n elements and $p_n \Vdash$ "$\underset{\sim}{\tau}_n \in w_n$", moreover $f \in \mathscr{F}_{\mathbf{u}_n} \Rightarrow p^{[f]}_n$ forces a value to $\underset{\sim}{\tau}_n$*

(B) *In the end, the second player wins if for some $q \geq p$ we have $q \Vdash$ "$(\forall n \in \omega)(\underset{\sim}{\tau}_n \in w_n)$".*

2) The result of part (1) still holds when we let Player II choose $k_n < \omega$ and demand $|\mathbf{u}_n| \leq k_n$, and in the end Player II wins if $\liminf\langle k_n : n < \omega\rangle < \omega$ or there is q as above.

3) Let $p \in \mathbb{Q}^1_{\mathbf{i}}$ and let N be a countable elementary submodel of $(\mathscr{H}(\chi), \in, <^*)$ such that $p, I, h \in N$. If the first player has no winning strategy in $GM^N_I(E^p)$ then Player I has no winning strategy in the game like above but with restriction that $\underset{\sim}{\tau}_n, p_n \in N$.

Proof. 1) As in [She92, 1.11,p.436], but we elaborate.

Let $\mathbf{St}_p$ be a strategy for Player I in the game $\partial^1_{\mathbf{i}}$ from 1.13. Our goal is to show that $\mathbf{St}_p$ cannot be a winning strategy. We shall define a strategy $\mathbf{St}$ for the first player in $GM_I(E^p)$ during which the first player, on a side, plays a play of the game $\partial^1_{\mathbf{i}}$ from 1.13, using $\mathbf{St}_p$, with $\langle p_\ell : \ell < \omega\rangle$ and he also chooses $\langle q_\ell : \ell < \omega\rangle$.

Then, as $\mathbf{St}$ cannot be a winning strategy in $GM_I(E^p)$, in some play in which the first player uses his strategy $\mathbf{St}$ he loses, and then $\langle p_\ell : \ell < \omega\rangle$ will have an upper bound which shows that $\mathbf{St}_p$ is not a winning strategy for player I, as required.

In the n-th move (so $E^1_\ell, E^2_\ell, q_\ell, p_\ell, \mathbf{u}_\ell, w_\ell$ for $\ell < n$ are defined), the first player in addition to choosing E^1_n chooses $q_n, p_n, \mathbf{u}_n$, such that:

(a) $p = p_{-1} \leq q_0 = p_0$, $p_n \in \mathbb{Q}_{\mathbf{i}}, q_n \in \mathbb{Q}_{\mathbf{i}}$,
(b) $p_n \leq^*_n p_{n+1} \in \mathbb{Q}_{\mathbf{i}}$,
(c) $\mathbf{u}_0$ is $\emptyset$,
(d) $\mathbf{u}_{n+1} = \mathbf{u}_n \cup \{\min(A^{q_{n+1}} \setminus \mathbf{u}_n)\}$, so $|\mathbf{u}_{n+1}| = n+1$,
(e) $E^1_0 = E^p$, $E^1_{n+1} = E^{p_n} \restriction (\mathrm{dom}(E^{p_n}) \setminus \bigcup_{i \in \mathbf{u}_n} i/E^{p_n})$,
(f) q_n is defined as follows:
 (f_0) if $n = 0$ then $E^{q_n} = E^2_0$,
 (f_1) if $n > 0$ then $\mathrm{dom}(E^{q_n}) = \mathrm{dom}(E^{p_{n-1}})$ and $xE^{q_n}y$ if and only if either $xE^2_n y$, or for some $k \in \mathbf{u}_{n-1}$ we have $x, y \in k/E^{p_{n-1}}$ or $x, y \in (\mathrm{dom}(E^1_n) \setminus \mathrm{dom}(E^2_n)) \cup \min(\mathrm{dom}(E^2_n))/E^2_n$,
 (f_2) H^{q_n} is such that $p_{n-1} \leq q_n$,
(g) $p_n \leq^*_n q_{n+1} \leq^*_{n+1} p_{n+1}$, $p_n \leq_{n+1} q_{n+1}$ (so $p_n \leq_{n+1} p_{n+1}$),
(h) if $f \in \mathscr{F}_{\mathbf{u}_n}$ then $p^{[f]}_n$ forces a value to $\underset{\sim}{\tau}_n$.

In the first move, when $n = 0$, the first player plays $E^1_0 = E^p$ (as the rules of the game require, according to (e)). The second player answers choosing an I-equivalence relation $E^2_0 \geq E^1_0$. Now, on a side, Player I starts to play the game of 1.13 using his strategy $\mathbf{St}_p$. The strategy instructs him to play a name $\underset{\sim}{\tau}_0$ of an ordinal. He defines q_0 by (f) (so $q_0 \in \mathbb{Q}^\ell_{\mathbf{i}}$ is a condition stronger than p and such that $E^{q_0} = E^2_0$) and chooses a condition $p_0 \geq q_0$ deciding the value of the name $\underset{\sim}{\tau}_0$, say p_0 forces $\underset{\sim}{\tau}_0 = \alpha$. He pretends that the second player answered (in the game of 1.13) by: p_0, $\mathbf{u}_0 = \emptyset$, $w_0 = \{\alpha\}$. Next, in the play of $GM_I(E^p)$, he plays $E^1_1 = E^{p_0}$ as declared in (e).

Now suppose that we are at the $(n+1)^{\text{th}}$ stage of the play of $GM_I(E^p)$, the first player has played E^1_{n+1} already and on a side he has played the play of the game 1.13 as defined by (a)–(h) and $\mathbf{St}_p$ (so in particular he has defined a condition p_n and $E^1_{n+1} = E^{p_n} \restriction (\mathrm{dom}(E^{p_n}) \setminus \bigcup_{i \in \mathbf{u}_n} i/E^{p_n})$ and $\mathbf{u}_n$ is the set of the first n elements of A^{p_n}). The second player plays an I-equivalence relation $E^2_{n+1} \geq E^1_{n+1}$.

Now the first player chooses (on a side, pretending to play in the game of 1.13): a name $\underset{\sim}{\tau}_{n+1}$ given by the strategy $\mathbf{St}_p$, a condition $q_{n+1} \in \mathbb{Q}^\ell_{\mathbf{i}}$ determined by (f) (check that (g) is satisfied), $\mathbf{u}_{n+1}$ as in (d) and a condition $p_{n+1} \in \mathbb{Q}^\ell_{\mathbf{i}}$ satisfying (g), (h) (the last exists by 1.10). Note that, by (g) and 1.9, the condition p_{n+1} determines a suitable set w_{n+1}. Thus, Player I pretends that his opponent in the game of 1.13 played $p_{n+1}, \mathbf{u}_{n+1}, w_{n+1}$ and he passes to the actual game $GM_I(E^p)$. Here he plays E^1_{n+2} defined by (e).

The strategy $\mathbf{St}$ described above cannot be a winning one by the assumptions of the theorem. Consequently, there is a play in $GM_I(E^p)$ in which Player I uses $\mathbf{St}$, but he looses. During the play he constructed a sequence $\langle (p_n, \mathbf{u}_n, w_n) : n \in \omega \rangle$ of legal moves of Player II in the game of 1.13 against the strategy $\mathbf{St}_p$. Let $E^q = \lim_{n<\omega} E^{p_n}$ (i.e. $\mathrm{dom}(E^q) = \bigcap_{n<\omega} \mathrm{dom}(E^{p_n})$, $x\, E^q\, y$ if and only if for every large enough n, $x\, E^{p_n}\, y$) and let $H^q(x^m_i)$ be $H^{p_n}(x^m_i)$ for any large enough n (it is eventually constant). It follows from the demand (g) that E^q-equivalence classes are in I. Moreover, $\mathrm{dom}(E^1_{n+1}) \setminus \mathrm{dom}(E^2_{n+1}) \subseteq k/E^q$, where k is the $(n+1)^{\text{th}}$ member of A^q.

Therefore $\omega \setminus \mathrm{dom}(E^q) = \omega \setminus \bigcap_{n\in\omega} \mathrm{dom}(E^{p_n}) \subseteq \omega \setminus \mathrm{dom}(E^{p_0}) \cup \bigcup\{\mathrm{dom}(E^2_n) \setminus \mathrm{dom}(E^1_{n+1}) : n \in \omega\} \in I$ (remember, Player I lost in $GM_I(E^p)$). Now it should be clear that $q \in \mathbb{Q}^\ell_{\mathbf{i}}$ and it is stronger than every p_n (even $p_n \leq^*_n q$). Hence Player II wins the corresponding play of 1.13, showing that $\mathbf{St}_p$ is not a winning strategy.

2),3) The same proof. $\square_{1.13}$

Definition 1.14. [See [She98a, Ch.VI,2.12,A-F].] 1) A forcing notion $\mathbb{P}$ has the PP-property <u>when</u>:

$\otimes^{\mathrm{PP}}$ for every $\eta \in {}^\omega\omega$ from $\mathbf{V}^{\mathbb{P}}$ and a strictly increasing $x \in {}^\omega\omega \cap \mathbf{V}$ there is a closed subtree $T \subseteq {}^{<\omega}\omega$ such that $T \in \mathbf{V}$ of incompand

- (α) $\eta \in \lim(T)$, i.e. $(\forall n < \omega)(\eta \restriction n \in T)$,
- (β) $T \cap {}^n\omega$ is finite for each $n < \omega$,
- (γ) for arbitrarily large n there are k, and $n < i(0) < j(0) < i(1) < j(1) < \cdots < i(k) < j(k) < \omega$ and for each $\ell \leq k$, there are $m(\ell) < \omega$

and $\eta^{\ell,0},\dots,\eta^{\ell,m(\ell)} \in T \cap {}^{j(\ell)}\omega$ such that $j(\ell) > x(i(\ell)+m(\ell))$ and $(\forall \nu \in T \cap {}^{j(k)}\omega)(\exists \ell \le k)(\exists m \le m(\ell))(\eta^{\ell,m} \trianglelefteq \nu)$.

2) We say that a forcing notion $\mathbb{P}$ has the strong PP-property when:

$\oplus^{\rm sPP}$ for every function $g\colon \omega \to \mathbf{V}$ from $\mathbf{V}^{\mathbb{P}}$ there exist a set $B \in [\omega]^{\aleph_0} \cap \mathbf{V}$ and a sequence $\langle w_n : n \in B\rangle \in \mathbf{V}$ such that for each $n \in B, |w_n| \le n$ and $g(n) \in w_n$.

Observation 1.15. *Of course, if a proper forcing notion has the strong* PP-*property then it has the* PP-*property.*

Conclusion 1.16. *Assume that for each $p \in \mathbb{Q}^1_{\mathbf{i}}$ and for each countable $N \prec (\mathscr{H}(\chi), \in, <^*)$ such that $p, I, h \in N$, the first player has no winning strategy in $GM^N_I(E^p)$ (e.g. if I is a maximal ideal).*

Then

(∗) *$\mathbb{Q}^\ell_{\mathbf{i}}$ is proper, α-proper, strongly α-proper for every $\alpha < \omega_1$, is ${}^\omega\omega$-bounding and it has the* PP-*property, even the strong* PP-*property.*

By [She98a, Ch.VI,2.12] we know

Theorem 1.17. *Suppose that $\langle \mathbb{P}_i, \mathbb{Q}_j : j < \alpha, i \le \alpha\rangle$ is a countable support iteration such that $\Vdash_{\mathbb{P}_j}$ "$\mathbb{Q}_j$ is proper and has the* PP-*property".*

Then $\mathbb{P}_\alpha$ has the PP-*property.*

§ 1(B). **Variants of $\mathbb{Q}^1_{\mathbf{i}}$.**

Those are comments on variants of $\mathbb{Q}^1_{\mathbf{i}}$, not used later.

Definition 1.18. For $\ell = 2,3$, we define $\mathbb{Q}^\ell_{\mathbf{i}}$ similarly to 1.2(d): if $\ell = 2$, then:

(a) (α) H is a function with domain $\mathrm{dom}(H) = B^p_2 \cup B^p_3$, where $B^p_2 = \{x^m_i : m < \omega, A^p \cap (m+1) = \emptyset, i < h(m)\}$ and $B^p_3 = \{x^m_i : i < h(m) \wedge m \in \mathrm{dom}(E^p) \setminus A^p$ or $m \notin \mathrm{dom}(E^p)$ but $A^p \cap m \ne \emptyset, i < h(m)\}$,

(β) for $x^m_i \in B^p_3, H(x^m_i)$ is a function in the variables $\{x^n_j : (n,j) \in w_p(m,i)\}$ to $\{-1,1\}$, where $w_p(m) = w_p(m,i) = \{(\ell,j) : \ell \in A^p \cap m$ and $j < h(\ell)\}$; for $n \in A^p$ we stipulate $H^p(x^n_i) = x^n_i$

(γ) $H \restriction B^p_2$ is a function to $\{-1,1\}$.

(b) if $\ell = 2$ and $n \in \mathrm{Dom}(E^p), x^n_i \in B^p_3, n^* = \min(n/E^p) < n$ and $y^m_i \in \{-1,1\}$ for $m \in A^p \cap n\backslash\{n^*\}, i < h(m)$ and $z^n_j \in \{-1,1\}$ for $j < h(n^*)$, then for some $y^{n^*}_j \in \{-1,1\}$ for $j < h(n^*)$ we have $j < h(n^*) \Rightarrow z^n_j = (H^p(x^n_j))(\dots, y^m_i, \dots)_{(m,i)\in w_p(n,j)}$

(c) if $\ell = 3$ then H^p is a function from $B^p_0 \cup B^p_1$ into $\{-1,1\}$ where (B^p_1 is as defined above and) $B^p_0 = \{x^n_i : n \in \mathrm{Dom}(E^p)\backslash A^p$ and $i < h(\min(n/E^p)\}$.

Definition 1.19. For $\ell = 1,2,3$, we say that a function $f : \{x^n_i : i < h(n),\ n < \omega\} \to \{-1,1\}$ satisfies a condition $p \in \mathbb{Q}^\ell_{\mathbf{i}}$ when:

(a) $f(x^n_i) = H^p(x^n_i)$ when one of the following occurs:

(α) $x^n_i \in B^p_1$ and $\ell = 1,3$

(β) $x^n_i \in B^p_2$ and $\ell = 2$,

(b) $f(x^n_i) = H^p(x^n_i)(\dots, f(x^m_j), \dots)_{(m,j)\in w_p(n,i)}$ when $\ell = 2$ and $x^n_i \in B^p_3$

(c) $f(x_i^n) = (f(x_i^{\min(n/E^p)}))$ <u>when</u> $\ell = 1$, $n \in \mathrm{dom}(E^p)$ and $i < h(\min(n/E^p))$
(d) $f(x_i^n) = (H^p(x_i^n))\cdot(f(x_i^{\min(n/E^p)}))$ when $\ell = 3, x_i^n \in B_0^p$, i.e. $n \in \mathrm{dom}(E^p)\backslash A^p$ and $i < h(\min(n/E^p))$, i.e. $x_i^n \in B_3^p$.

We can add to 1.9:

Proposition 1.20. *1) We can in 1.9(4) allow $\ell \in \{2, 3\}$.*
2) Let $\ell = 3$. If $p \in \mathbb{Q}_{\mathbf{i}}, \mathbf{u} = w^p \cap \mathbf{k}_p(n+1)$ and $f \in \mathscr{F}_{p,\mathbf{u}}$ is rich (see Definition 2.8(4)) and $p^{[f]} \le q$, <u>then</u> for some $r \in \mathbb{Q}_{\mathbf{i}}^\ell$ we have $p <_n^\otimes r \le q$ and $r^{[f]} = q$.
3) Define $f\colon \{x_i^n \colon i < h(n) \text{ and } n \in \mathbf{u}\} \to \{-1, 1\}$ by: $f(x_i^n)$ is the value of $H^q(x_i^n)$.

Proof. 1) If $\ell = 2$ then we define r in a similar manner, but we have to be more careful defining the function H^r. Thus E^r and A^r are defined as above, B_2^r, B_3^r and $w_r(m, i)$ for $x_i^m \in B_3^r$ are given by 1.2(3)(f) and 1.2(3)(d)(β). Note that $B_2^r = B_2^p$ and $B_3^r \subseteq B_3^p$.
Next we define:
if $x_i^m \in B_2^r$ then $H^r(x_i^m) = H^p(x_i^m)$,
if $x_i^m \in B_3^r$, $m \cap A^r \subseteq \mathbf{u}$ then $H^r(x_i^m) = H^p(x_i^m)$,
if $x_i^m \in B_3^r$ and $\min(\mathrm{dom}(E^q)) < m$ then
$H^r(x_i^m)(\ldots, x_j^k, \ldots)_{(k,j)\in w_r(m,i)} =$
$H^p(x_i^m)(x_j^k \cdot H^q(x_{j'}^{k'})(\ldots, x_{j''}^{k''}, \ldots)_{(k'',j'')\in w_q(k',j')}))_{(k,j)\in w_r(m,i),(k',j')\in w_p(m,i)\backslash w_r(m,i)}$.
Note that if $(k', j') \in w_p(m, i) \setminus w_r(m, i)$, $x_i^m \in B_3^r$ then $k' \in A^p \setminus (\mathbf{u} \cup A^q)$ and $w_q(k', j') \subseteq w_r(m, i)$.
For $\ell = 3$ similarly and in part (5) we say more.
5) Like the proof of (4). Let $n^* = \max(\mathbf{u})$. Put $\mathrm{dom}(E^r) = \mathrm{dom}(E^p)$ and declare that $n_1 E^r n_2$ if one of the following occurs:

(a) for some $n \in \mathbf{u} \setminus \{n^*\}$ we have $\{n_1, n_2\} \subseteq (n/E^p)$, or
(b) $n_1\, E^q\, n_2$ (so $n \in \mathbf{u} \Rightarrow \neg n E^p n_1$), or
(c) $\{n_1, n_2\} \subseteq B$, where $B := n^*/E^p \cup \bigcup\{m/E^p : m \in \mathrm{dom}(E^p)\backslash\mathrm{dom}(E^q), \min(m/E^p) > n^*\}$.

We let $A^r = \mathbf{u} \cup A^q$ (in fact A^r is defined from E^r). Finally the function H^r is defined exactly in the same manner as in (4) above:

(d) $H^r(x_j^m) = H^q(x_j^m)$ when $x_j^m \in \omega\backslash\mathrm{Dom}(E^p)$ or $n := \min(m/E^p) < m \wedge j \in [h(n), h(m))$
(e) $H^r(x_j^m) = H^p(x_j^m)$ if $n \in \cup\{m/E^p : m \in \mathbf{u}\}$
(f) $H^r(x_j^m) = f(x_j^{n^*})H^q(x_j^m)$ if $m \in (n^*/E^r)\backslash(n^*/E^p)$.

$\square_{1.20}$

An addition to corollary 1.10 is:

Corollary 1.21. *If $\ell = 3$, then in 1.10(a) we may require $p \le_n^\otimes q \in \mathbb{Q}_{\mathbf{i}}^\ell$.*

Proof. Similar: just use 1.20, instead of 1.9(4), 1.20(1). $\square_{1.21}$

<u>After 1.13 we can add</u>:

Proposition 1.22. *If in 1.13 we assume $\ell = 3$ and demand $p_n \leq^{\otimes}_n p_{n+1}$ instead of $p_n \leq^*_n p_{n+1}$* then *Player II has a winning strategy.*

Proof. Using 1.8A, the second player can find suitable conditions p_n (in the game of 1.13) such that $p_n \leq^{\otimes}_{n+1} p_{n+1}$. But note that the partial orders $\leq^{\otimes}_n$ have the fusion property, so the sequence $\langle p_n : n < \omega\rangle$ will have an upper bound in $\mathbb{Q}^3_{\mathbf{i}}$. $\square_{1.22}$

Remark 1.23. We could have used $<^{\otimes}_n$ also in [She92].

§ 2. Creature forcing

§ 2(A). On $\mathbb{Q}^4_{\mathbf{i}}$.

We try to combine the good properties of the $\mathbb{Q}^\ell_{\mathbf{i}}$'s from §1 by putting a creature on finite intervals of A^p defining $\mathbb{Q}_{\mathbf{i}} = \mathbb{Q}^4_{\mathbf{i}}$. Recall that an element of FP_1 consists only of an ideal I and a function $h\colon \omega \to \omega$, see 1.2, so no creatures are involved.

Definition 2.1. 1) Let FP_2 (forcing parameters) be the set of objects $\mathbf{i}$ consisting of (so $I = I_{\mathbf{i}} = I[\mathbf{i}]$, etc.):

(a) I be an ideal on ω to which all finite subsets of ω belong and $D = \mathrm{dual}(I)$ its dual, a filter,
(b) let $h : \omega \to \omega$ be a non-decreasing function,
(c) h goes to infinity,
(d) $\bar{S} = \langle S_k : k < \omega\rangle$ is a partition of ω to intervals (each interval is finite non-empty)
(e) $\min(S_{k+1}) = \max(S_k) + 1$ for every k,; actually follows,
(f) each $h{\restriction}S_k$ is constant and let $h'\colon \omega \to \omega$ be such that $n \in S_k \Rightarrow h'(k) = h(n)$,
(g) $\liminf_k |S_k|/2^{h'(k)} = \infty$,
(h) notation: let $E_{\mathbf{i}} = E_{\bar{S}} = \{(m, n) : (\exists k)[n, m \in S_k]\}$

2) Let FP_3 be the set of $\mathbf{i}$ consisting of:

(A) as in part (1)
$(\mathrm{B})_1$ the simple creature version:
  (a) $\mathrm{CR}_n \coloneqq \mathcal{P}(S_n)$,
  (b) we let $\mathrm{val}_n\colon \mathrm{CR}_n \to \mathrm{CR}_n$ be the identity,
  (c) $\sum(u) = \mathcal{P}(u)$ for $u \in \mathrm{CR_n}$,
  (d) $\mathrm{nor}_n(f)$ incomp $\mathrm{nor}_{\mathfrak{c}_n}(f) = |u|/2^{h'(k)}$,
  (e) $\mathrm{nor}_n(f)$ $\langle \mathrm{nor}_n(\emptyset)\colon n < \omega\rangle$ goes to infinity,
  (f) Let $\mathrm{free_n(f) \coloneqq S_n \setminus dom(f)}$.
$(\mathrm{B})_2$ full creature version:
  (a) $\langle\, \mathbf{CR}_n : n < \omega\rangle$ where $\mathbf{CR}_n = (\mathrm{CR}_n, \mathrm{val}_n, \mathrm{nor}_n, \Sigma_n)$,
  (b) the CR_n's are pairwise disjoint, each finite,
  (c) $\mathrm{val}_n(\mathfrak{c}) \in \mathrm{func(S_n)}$ for $n < \omega, \mathfrak{c} \in \mathrm{CR}_n$,
  (d) $\mathrm{nor}_n(\mathfrak{c}) \in \mathbb{R}_{>0}$ for $\mathfrak{c} \in \mathrm{CR}_n$ and $\langle \mathrm{nor}_n(\emptyset)\colon n < \omega\rangle$ goes to infinity,
  (e) if $\mathfrak{c} \in \mathrm{CR}_n$ then $\Sigma_n(\mathfrak{c}) \subseteq \mathrm{CR}_n$ and $\mathfrak{c} \in \Sigma_n(\mathfrak{c})$,
  (f) if $\mathfrak{d} \in \Sigma_n(\mathfrak{c})$ then $\mathrm{val}_n(\mathfrak{d}) \subseteq \mathrm{val}_n(\mathfrak{c})$ and $\Sigma_n(\mathfrak{d}) \subseteq \Sigma_n(\mathfrak{c})$ for $\mathfrak{c}, \mathfrak{d} \in \mathrm{CR}_n$.

3) We say $\mathbf{i} \in \mathrm{FP}_3$ has the *DICH* (divide and choose) if

$(\mathrm{B})_1$ (g) for $\mathfrak{c} \in \mathrm{CR}_n$, if $S_{\mathfrak{c}} = S' \cup S''$, then for some $\mathfrak{d} \in \Sigma_n(\mathfrak{c})$ we have $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$ and $(S_{\mathfrak{d}} \subseteq S') \vee (S_{\mathfrak{d}} \subseteq S'')$.

4) We say that $\mathbf{i}$ has the strong DICH property when in addition: ($\mathbb{Q}_{\mathbf{i}}$ is defined below):

(h) if $p \in \mathbb{Q}_{\mathbf{i}}$ and $B \in D_{\mathbf{i}}/E^p$ (i.e. $B \in D_{\mathbf{i}}$ and $(\forall n \in \mathrm{dom}(E^p))(B \cap (n/E^p) \in \{\emptyset, n/E^p\})$ then there is $q \in \mathbb{Q}_{\mathbf{i}}$ above p such that:
  - $E^q = E^p{\restriction}\mathrm{dom}(E^q)$,
  - if $k \in w^q$ then $\mathrm{nor}(\mathfrak{c}_{q,k}) \geq \mathrm{nor}(c_{p,k,\mathfrak{c}}) - 1$.

5) For $\iota = \{1, 2, 3\}$:

(A) let $\mathrm{FP}^\iota_{\mathrm{uf}}$ be the set of $\mathbf{i} \in \mathrm{FP}_\iota$ such that $D_\mathbf{i} := \mathrm{dual}(I_\mathbf{i}) := \{\omega \backslash A \colon A \in I_\mathbf{i}\}$ is an ultrafilter on ω, necessarily non-principal

(B) let $\mathrm{FP}^\iota_{\mathrm{cc}}$ be the set of $\mathbf{i} \in \mathrm{FP}_\iota$ such that $D_\mathbf{i} = \mathrm{dual}(I_\mathbf{i})$ is a filter on ω such that the Boolean algebra $\mathscr{P}(\mathbb{N})/D_\mathbf{i}$ satisfies the c.c.c.

(C) let $\mathrm{FP}^\iota_{\mathrm{nn}}$ be the set of $\mathbf{i} \in \mathrm{FP}_\iota$; this is just so that we can write "for each $x \in \{\mathrm{uf}, \mathrm{cc}, \mathrm{nn}\}$ in FP^ι_x we have ... ".

6) If $(\forall n)(h'(n) = n)$ then we may omit h.

7) We say $\mathbf{i} \in \mathrm{FP}_3$ is fast <u>when</u>: if $\mathfrak{c} \in \mathrm{CR}_k, \mathrm{nor}(\mathfrak{c}) \geq 1$ then we can find $\bar{\mathfrak{d}} = \langle \mathfrak{d}_i : i < k \rangle$ such that:

(a) $\mathfrak{d}_i \in \Sigma_k(\mathfrak{c})$ and $\mathrm{nor}(\mathfrak{d}_i) \geq \mathrm{nor}(\mathfrak{c}) - 1$

(b) $\mathrm{val}(\mathfrak{c}) \backslash \mathrm{val}(\mathfrak{d}_i)$ has at least $2^{h'(k)}$ members

(c) $\langle \mathrm{val}(\mathfrak{c}) \backslash \mathrm{val}(\mathfrak{d}_\ell) : \ell < k \rangle$ are pairwise disjoint.

Remark 2.2. 1) The "$\mathbf{i}$ is fast" is used for the bounded game for $\mathbb{Q}_\mathbf{i}$, see 2.14 below assuming $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{cc}}$.

2) For $\mathbf{i} \in \mathrm{FP}_3$ alternatively to the assumptions on "$\mathbf{i}$ is fast and $\mathscr{P}(\omega)/I_\mathbf{i}$ satisfies the c.c.c." used in 2.14 we can use:

(a) if $u_k \in [S_k]^{\leq k}$ for $k < \omega$ then $\cup\{u_k \colon k < \omega\} \in I_\mathbf{i}$,

(b) if $\mathfrak{c} \in \mathrm{CR}_k, \mathrm{nor}(\mathfrak{c}) \geq 1$ <u>then</u> there is $\mathfrak{d} \in \Sigma_k(\mathfrak{c})$ such that $\mathrm{val}(\mathfrak{c}) \backslash \mathrm{val}(\mathfrak{d})$ has $\geq 2^{h'(k)}$ elements and $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$.

Instead of "fast" and later "rich", we can change the definition of norm and order adding:

(c) $\mathbb{Q}_\mathbf{i} \models$"$p \leq q$" <u>iff</u> as before and:
  - if $k \in w^q$ and $\mathfrak{c}_{p,k} \neq \mathfrak{c}_{q,k}$ and $\eta \in {}^{h'(k)}\{1, -1\}$, then for some $m \in \mathrm{val}(\mathfrak{b}_{p,k}) \setminus \mathrm{val}(\mathfrak{b}_{q,n})$ we have $\eta = \langle H(x^n_i) \colon i < h'(k) \rangle$.

(d) norm′: if $\mathrm{nor}_k(\mathfrak{c}) \geq 1$, then for some $\mathfrak{d} \in \Sigma(\mathfrak{c})$, $\mathrm{nor}_k(\mathfrak{d}) \supseteq \mathrm{nor}_k(\mathfrak{c}) - 1$ and $|\mathrm{val}(\mathfrak{c}) \setminus \mathrm{val}(\mathfrak{d})| \geq 2^{h'(k)}$.

3) Is it helpful to allow non-unary Σ (in Definition 2.1(2))? In the terms of [RS99], this means allowing so called glueing. This mean that $\mathrm{CR} = \bigcup\{\mathrm{CR}_{\mathrm{m,n}} := \mathrm{m} \leq \mathrm{n}\}$ and $\mathfrak{c} \in \mathrm{CR}_{\mathrm{m,n}} \Rightarrow \mathrm{val}(\mathfrak{c}) \subseteq \mathrm{func}(\bigcup\{\mathrm{S}_\ell \colon \ell \in [\mathrm{m}, \mathrm{n})\})$.

4) The property "$\mathbf{i}$ is fast" is crucial; the ultrafilter property (and the stronger one) presently are not but, they are for definining worthwhile families of ultrafilters.

Claim 2.3. *1) In Definition 2.1(2),(3), clause (B)$_1$ is a special case of clause (B)$_2$.*

2) In Definition 2.1, if $S_n = \{n\}$ for every n and clause (B)$_1$ holds with $\mathrm{nor}_n(\{n\}) = n$, <u>then</u> $\mathbf{i}$ essentially belongs to FP_1. Also every $\mathbf{i} \in \mathrm{FP}_1$ can be interpreted as a member of FP_3 in this way.

3) Any $\mathbf{i} \in \mathrm{FP}_2$ is a special case of FP_3.

Proof. Read the definitions. $\square_{2.3}$

Definition 2.4. For $\mathbf{i} \in \mathrm{FP}_3$ we define the forcing notion $\mathbb{Q}_\mathbf{i} = \mathbb{Q}^4_\mathbf{i}$ and some auxiliary notions as follows:

(A) $p \in \mathbb{Q}_\mathbf{i}$ if and only if $p = (H, E, A, \bar{\mathfrak{c}}) = (H^p, E^p, A^p, \bar{\mathfrak{c}}^p)$ satisfies:

(a) E is an $I_{\mathbf{i}}$-equivalence relation, so on a set called $\mathrm{dom}(E)$ which belongs to $D_{\mathbf{i}}$ hence is $\subseteq \omega$,
(b) $A = A^p := \{n \in \mathrm{dom}(E) : n = \min(n/E)\}$,
(c) H is a function with range $\subseteq \{-1, 1\}$ and domain $B^p_1 = \{x^n_i : i < h(n) \wedge n \in (\omega \setminus \mathrm{dom}(E))$ or $n \in \mathrm{dom}(E) \wedge i \in [h(\min(n/E)), h(n))\}$,
(d) (α) $\bar{\mathfrak{c}} = \langle \mathfrak{c}_k : k \in w\rangle$
(β) $w = w^p \subseteq \omega$ is infinite
(γ) $\mathfrak{c}_k \in \mathrm{CR}_k$
(δ) $A = \cup\{\mathrm{val}(\mathfrak{c}_k) \colon k \in w\}$
(e) [2] $\infty = \limsup_{I[\mathbf{i}]}\langle \mathrm{nor}_n(\mathfrak{c}_n) \colon n \in w\rangle$ which means that for every $\mathscr{U} \in I_{\mathbf{i}}$, the set $\{\mathrm{nor}_k(\mathfrak{c}_{p,k}) \colon k \in w^p \setminus \mathscr{U}\}$ is unbounded (in ω).

(B) For $a \subseteq \omega$ we define $\mathscr{F}_a = \{f \colon f$ is a function with domain $\{x^n_i \colon i < h(n), n \in a\}$ such that $f(x^n_i) \in \{-1, 1\}\}$; let $a \triangleleft w^p$ mean that a is a finite initial segment of w^p; later in 2.8 we shall define $f_{p,a}$

(C) We say that a function $f \in \mathscr{F}_\omega$ satisfies a condition $p \in \mathbb{Q}_{\mathbf{i}}$ when:
(a) $f(x^n_i) = H^p(x^n_i)$ when $x^n_i \in B^p_1$
(b) $f(x^n_i) = (f(x^{\min(n/E^p)}_i))$ when $n \in \mathrm{dom}(E^p)$ and $i < h(\min(n/E^p))$.

(D) The partial order $\le = \le_{\mathbb{Q}^4_{\mathbf{i}}}$ is defined by $p \le q$ if and only if:
(α) $E^p \le E^q$, i.e.
- $\mathrm{dom}(E^q) \subseteq \mathrm{dom}(E^p)$
- if $n \in \mathrm{dom}(E^q)$ then $n/E^p \subseteq \mathrm{dom}(E^q)$
- $E^p{\restriction}\mathrm{dom}(E^q)$ refines E^q

(β) every function $f \in \mathscr{F}_\omega$ satisfying q satisfies p
(γ) $w^q \subseteq w^p$ and if $k \in w^q$, then $\mathfrak{c}_{q,k} \in \Sigma_k(\mathfrak{c}_{p,k})$.

Proposition 2.5. *$(\mathbb{Q}_{\mathbf{i}}, \le_{\mathbb{Q}_{\mathbf{i}}})$ is a partial order.*

Proof. Easy. $\square_{2.5}$

Remark 2.6. One may wonder why we have h in the definition of $\mathbb{Q}_{\mathbf{i}}$ and we do not fix that, e.g. $h'(n) = n$. This is to be able to describe nicely what is the forcing notion $\mathbb{Q}_{\mathbf{i}}$ above a condition p. The point is that $\mathbb{Q}_{\mathbf{i}}{\restriction}\{q : q \ge p\}$ is like $\mathbb{Q}_{\mathbf{i}}$ but we replace I by its quotient by E^p and we change the function h.

More precisely:

Claim 2.7. *Assume $\mathbf{i} \in \mathrm{FP}_3$ and $p \in \mathbb{Q}_{\mathbf{i}}$. Then $\mathbb{Q}_{\mathbf{j}}$ is isomorphic to $\mathbb{Q}_{\mathbf{i}}{\restriction}\{q : q$ is $\le_{\mathbb{Q}_{\mathbf{i}}}$-above $p\}$ and $\mathbf{j} \in \mathrm{FP}_3$ and $\mathbf{j}$ belongs to $\mathrm{FP}_4/\mathrm{FP}^3_{\mathrm{uf}}/\mathrm{FP}^3_{cc}$ when $\mathbf{i}$ does, provided that:*

$\boxplus$ *$\mathbf{j}$ is defined by: letting $g_0 : w^p \to \omega$ be increasing and onto ω and $g_1 : A^p \to \omega$ be increasing and onto ω, we have:*
(a) *$h'_{\mathbf{j}}$ is defined by: if $g_0(k) = \ell$ so $k \in w^p$ then $h'_{\mathbf{j}}(\ell) = h'_{\mathbf{i}}(\ell)$*
(b) *if $g_0(k) = \ell$ then*
- *$S_{\mathbf{j},\ell} = \{g_1(n) : n \in \mathrm{val}_{\mathbf{i}}(\mathfrak{c}_{p,k})\}$*
- *$\mathrm{CR}_{\mathbf{j},\ell} = \Sigma_{\mathbf{i}}(\mathfrak{c}_{p,k}), \Sigma_{\mathbf{j},\ell} = \Sigma_{\mathbf{i},k}{\restriction}\mathrm{CR}_{\mathbf{j},\ell}$ and $\mathrm{val}_{\mathbf{j}}(\mathfrak{c}) = \{g_1(n) : n \in \mathrm{val}_{\mathbf{i}}(\mathfrak{c}_{p,w})\}$*

(c) *$I_{\mathbf{j}} = \{C \subseteq \omega : \cup\{n/E^p : n \in A^p$ and $g_1(n) \in C\} \in I\}$.*

[2]The reader may wonder why $I_{\mathbf{i}}$ does not appear in 2.4(e) The point is that by glueing, i.e., increasing $\bigcup\{m/E^p \colon m \in \mathrm{val}(\mathfrak{c}_{p,k})\}$ the original $\bigcup\{m/E^p \colon n \in \mathrm{val}(\mathfrak{c}_{p,k})\}$ becomes irrelevant.

Proof. Straightforward (as in 1.5). $\square_{2.7}$

Definition 2.8. 1) We define a $\mathbb{Q}_{\mathbf{i}}$-name $\underset{\sim}{\bar\eta} = \langle \underset{\sim}{\eta}_n : n < \omega\rangle$ by: $\underset{\sim}{\eta}_n$ is a sequence of length $h(n)$ of members of $\{-1,1\}$ such that $\underset{\sim}{\eta}_n[\underset{\sim}{G}_{\mathbb{Q}_{\mathbf{i}}}](i) = 1 \Leftrightarrow (\exists p \in \underset{\sim}{G}_{\mathbb{Q}_{\mathbf{i}}})(H^p(x^n_i) = 1 \wedge x^n_i \in B^p_1))$.

2) For $p \in \mathbb{Q}_{\mathbf{i}}$ and $a \subseteq w^p$ let:

- $\mathscr{F}_{p,a} = \{f{\restriction}\{x^n_i : n \in \cup\{(m/E_{\mathbf{i}}) : (\exists k \in a)(m \in \mathrm{val}_{\mathfrak{c}_h} \in S_k)\} \cap A^p$ and $i < h(\min(n/E_p))\} : f$ satisfies $p\}$,
- equivalently $\{f : f$ is a function from $\{x^n_i : n \in A^p$ and $n \in S_k$ for some $k \in a$ and $i < h(\min(n/E_p))\}$ into $\{1,-1\}\}$.

3) For $p \in \mathbb{Q}_{\mathbf{i}}$ and $\mathbf{u} \subseteq w_p$, let $A_{p,\mathbf{u}} = \mathrm{dom}(f)$ for every $f \in \mathscr{F}_{p,\mathbf{u}}$.

4) We say $f \in \mathscr{F}_{p,a}$ is p-rich <u>when</u> $a \subseteq w^p$ has a last element and $\{\langle f(x^n_i) : i < h'(\max(a)\rangle : n \in S_{\max(a)} \cap A^p\}$ is equal to ${}^{h'(\max(a))}2$, that is, all possibilities occur.

Proposition 2.9. *1) If $n < \omega, A^p \cap (n+1) = \emptyset$ <u>then</u> $p \Vdash$ "$\underset{\sim}{\eta}_n = \langle H^p(x^n_i) : i < h(n)\rangle$".*

2) For each $n < \omega$ the set $\{p \in \mathbb{Q}_{\mathbf{i}} : A^p \cap (n+1) = \emptyset\}$ is dense open in $\mathbb{Q}_{\mathbf{i}}$.

3) If $p \in \mathbb{Q}_{\mathbf{i}}$ and $a \subseteq w^p$ is finite or at least $b = \cup\{n/E_p : n \in (\bigcup\limits_{k\in a} S_k) \cap A^p\} \in I$, and $f \in \mathscr{F}_{p,b}$ <u>then</u> for some unique q which we denote by $p^{[f]}$, we have:

(a) $p \le q \in \mathbb{Q}_{\mathbf{i}}$,
(b) $E^q = E^p \restriction \bigcup\{n/E^p : n \in A^p \setminus b\}$,
(c) *for $n \in b$, $k \in n/E_p$, $i < h(n)$ we have $H^q(x^n_i)$ is $f(x^n_i)$*
(d) $k \in w^q \backslash a \Rightarrow \mathfrak{c}_{q,k} = \mathfrak{c}_{p,k}$.

4) For every $p \in \mathbb{Q}_{\mathbf{i}}$ there is $q \in \mathbb{Q}_{\mathbf{i}}$ above p such that:

$\oplus_q$ *the sequence $\langle \mathrm{nor}_k(\mathfrak{c}_{q,k}) : k \in w^q\rangle$ is increasing and $k \in w^p \setminus \{\min(w^p)\} \Rightarrow \mathrm{nor}(\mathfrak{c}^p_k) > k + |\mathscr{F}_{p,w^p\cap k}|$ and $\mathrm{dom}(E^q) = \mathrm{dom}(E^p)$.*

5) Moreover, if $p \in \mathbb{Q}_{\mathbf{i}}, \mathbf{u} \subseteq w^p$ is a finite initial segment of w^p and $\langle \mathrm{nor}(\mathfrak{c}_{p,\ell}) : \ell \in \mathbf{u}\rangle$ is increasing and $\mathrm{nor}(\mathfrak{c}_p, \max(\mathbf{u})) \ge 1$, <u>then</u> for some $q \in \mathbb{Q}^1_{\mathbf{i}}$ we have $p \le^{\otimes}_{|\mathbf{u}|} q$, see Definition 2.10(3) below and $\oplus_q$ above holds.

Proof. 1), 2), 3) Easy.

4) We choose $k_i \in A^p$ by induction on $i < \omega$ such that:

- $k_0 = \min(w^p)$,
- $i > 0 \Rightarrow \mathrm{nor}(\mathfrak{c}_{p,k_i}) > k + |\mathscr{F}_{p,\{k_j : j<i\}}|$.

Now, choose q such that:

$(*)$ (a) $\mathrm{dom}(E^q) = \mathrm{dom}(E^p)$,
(b) $A^q = \bigcup\{\mathrm{val}(\mathfrak{c}_{p,k_i}) : i < \omega\}$,
(c) $\mathfrak{c}_{q,k_i} = \mathfrak{c}_{p,k_i}$,
(d) $\bigcup\{m/E^q : m \in \mathrm{val}(\mathfrak{c}_{p,k_i}) = \bigcup\{m/E^p : (\exists k \in [k_i, k_{i+1}))[m \in \mathrm{val}(\mathfrak{c}_{p,k})]\}\}$,
(e) $p \le q$.

5) Similarly. $\square_{2.9}$

Definition 2.10. 0) For $p \in \mathbb{Q}_{\mathbf{i}}$ and $n < \omega$ let $\mathbf{k}_p(n) = \mathbf{k}(n,p)$ be the minimal k (actually unique k) such that:

(a) $k \in w^p$
(b) $|k \cap w^p| = n$

1) $p \leq_n q$ (in $\mathbb{Q}_{\mathbf{i}}$) iff:

(a) $p \leq q$
(b) if $k \in w^p \cap \mathbf{k}_p(n)$ then $k \in w^q$ and $\mathfrak{c}_{q,k} = \mathfrak{c}_{p,k}$.

2) $p \leq^*_n q$ iff $p \leq_n q$ and:

($*$) if $k \in w^p \cap \mathbf{k}_p(n)$, then not only $k \in w^q, \mathfrak{c}_{q,k} = \mathfrak{c}_{p,k}$ but also $m \in \mathrm{val}(\mathfrak{c}_{q,k}) \Rightarrow m/E^p = m/E^q$.

3) $p \leq^{\otimes}_n q$ iff $p \leq_{n+1} q$ and: $n > 0 \Rightarrow p \leq^*_n q$ and $\mathrm{dom}(E^q) = \mathrm{dom}(E^p)$.

Proposition 2.11. *1) If $p \leq q, \mathbf{u}$ is an initial segment of w^p and $w^q \cap \mathbf{u} = \emptyset$, then for some unique $f \in \mathscr{F}_{p,\mathbf{u}}$ we have $p \leq p^{[f]} \leq q$ (where $p^{[f]}$ is defined in 2.9(3)).*
2) If $p \in \mathbb{Q}_{\mathbf{i}}$ and $\mathbf{u} \triangleleft w^p$, i.e. is a finite initial segment of w^p then:

$(*)_1$ *$f \in \mathscr{F}_{p,\mathbf{u}}$ implies $p \leq p^{[f]}$ and $p^{[f]} \Vdash$ "$(\forall n \in \mathbf{u})(\forall i < h(n))(\underset{\sim}{\eta}_n(i) = f(x^n_i))$",*
$(*)_2$ *the set $\{p^{[f]} : f \in \mathscr{F}_{p,\mathbf{u}}\}$ is predense above p (in $\mathbb{Q}_{\mathbf{i}}$).*

3) $\leq_n$ is a partial order on $\mathbb{Q}_{\mathbf{i}}$, and $p \leq_{n+1} q \Rightarrow p \leq_n q$. Similarly for $<^*_n$ and $<^{\otimes}_n$. Also

$(*)_1$ $p \leq^{\otimes}_n q \Rightarrow p \leq^*_n q \Rightarrow p \leq_n q \Rightarrow p \leq q$
$(*)_2$ $p \leq^{\otimes}_n q \Rightarrow p \leq_{n+1} q$.

4) If $p \in \mathbb{Q}_{\mathbf{i}}, \mathbf{u} = w^p \cap \mathbf{k}_p(n)$ and $f \in \mathscr{F}_{p,\mathbf{u}}$ and $p^{[f]} \leq q \in \mathbb{Q}_{\mathbf{i}}$ then for some $r \in \mathbb{Q}_{\mathbf{i}}$ we have $p \leq^*_n r \leq q$ and $r^{[f]} = q$.
5) If $p \in \mathbb{Q}_{\mathbf{i}}, \mathbf{u} = w^p \cap \mathbf{k}_p(n+1)$ and $f \in \mathscr{F}_{p,\mathbf{u}}$ is rich (see Definition 2.8(4)) and $p^{[f]} \leq q$, then for some $r \in \mathbb{Q}^{\ell}_{\mathbf{i}}$ we have $p <^{\otimes}_n r \leq q$ and $r^{[f]} = q$.

Proof. 1) Define $f : \{x^n_i : i < h(n) \text{ and } n \in \mathbf{u}\} \rightarrow \{-1, 1\}$ by: $f(x^n_i)$ is the value of $H^q(x^n_i)$.
2) By 2.9(3) for $(*)_1$ and direct inspection for $(*)_2$.
3) Check.
4) We define $r \in \mathbb{Q}_{\mathbf{i}}$ by: $w^r = \mathbf{u} \cup w^q, \mathrm{dom}(E^r) = \cup\{(n/E^p) : n \in S_k \cap A^p$ for some $k \in \mathbf{u}\} \cup \mathrm{dom}(E^q), E^r = \{(n_1, n_2) : n_1 E^q n_2$ or some $k \in \mathbf{u}$ satisfies $\min(n_1/E^p) = \min(n_2/E^p) \in S_k \cap A^p\}, A^r = \cup\{S_k \cap A^p : k \in \mathbf{u}\} \cup A^q$
Next, we define $\mathfrak{c}_{r,k}$ for $k \in w^r$ by:

- $\mathfrak{c}_{r,k} = \mathfrak{c}_{p,k}$ if $k \in \mathbf{u}$
- $\mathfrak{c}_{r,k} = \mathfrak{c}_{q,k}$ if $k \in w^q$.

Lastly, for $x^n_i \in B^r_1$ (where B^r_1 is defined in 1.2(1)(e)) we define

$$H^r(x^n_i) = \begin{cases} H^q(x^n_i), & \text{if } n \notin \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^q), \\ H^p(x^n_i), & \text{if } n \in \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^p). \end{cases}$$

It should be clear that $r = (H^r, E^r, A^r, \bar{\mathfrak{c}}^r) \in \mathbb{Q}_{\mathbf{i}}$ is as required.

5) We choose $\mathbf{n}(\eta) \in \mathrm{val}(\mathfrak{c}_{p,\max(\mathbf{u})})$ for $\eta \in {}^{h'(\max(\mathbf{u}))}2$ such that $\eta = \langle f(x_i^{\mathbf{n}(\eta)}) : i < h(\max(\mathbf{u}))\rangle$; note that there is such $\mathbf{n}(\eta) \in \mathrm{val}(\mathfrak{c}_{p,\max(\mathbf{u})})$ because f is p-rich, see Definition 2.8(4). Now we define r.

Put $\mathrm{dom}(E^r) = \mathrm{dom}(E^p)$ and declare that X is an E^r-equivalence class iff (at least) one of the following occurs:

(a) for some $m \in \mathrm{val}(\mathfrak{c}_\ell^p)$ and $\ell \in \mathbf{u} \setminus \{\max(\mathbf{u})\}$ we have $X = (m/E^p)$
(b) $X = m/E^q$ for some $m \in A^q$
(c) $X = m/E^p$ for some $m \in \mathrm{val}(\mathfrak{c}_{p,\mathbf{k}_p(n)})$ which $\notin \{\mathbf{n}(\eta) : \eta \in {}^{h'(\max(\mathbf{u}))}2\}$
(d) for some $\eta \in {}^{h'(\max(\mathbf{u}))}2, X$ is equal to $X_\eta := (\mathbf{n}(\eta)/E^p) \cup \bigcup\{m/E^p : m \in A^p \backslash \mathbf{u}$ and $m \notin \mathrm{Dom}(E^q)$ and $\langle H^q(x_i^n) : i < h'(\max(\mathbf{u}))\rangle = \eta\}$.

We let $A^r = \mathbf{u} \cup A^q$ (in fact A^r is defined from E^r). Finally the function H^r is defined naturally:

$(*)$ $H^r(x_j^m) = H^q(x_j^m)$ when $m \in \omega \backslash \mathrm{Dom}(E^r)$ or $m \in \mathrm{Dom}(E^r) \wedge m' := \min(m/E^r) < m \wedge j \in [h(m'), h(m))$.

The reader may wonder: how come H^p does not appear in the definition of H^r? The answer is that $H^p \subseteq H^q$. $\square_{2.11}$

Corollary 2.12. *If $p \in \mathbb{Q}_\mathbf{i}, n < \omega$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}_\mathbf{i}$-name of an ordinal, then there are $\mathbf{u}, q$ and $\bar\alpha = \langle \alpha_f : f \in \mathscr{F}_{p,\mathbf{u}}\rangle$ such that:*

(a) $p \le_n^* q \in \mathbb{Q}_\mathbf{i}$,
(b) $\mathbf{u} = w^p \cap \mathbf{k}_p(n)$
(c) *for $f \in \mathscr{F}_{p,\mathbf{u}}$ we have $q^{[f]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_f$",*
(d) *$q \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_f : f \in \mathscr{F}_\mathbf{u}\}$" (which is a finite set).*

Proof. Let $k = \prod_{\ell \in \mathbf{u}} 2^{h(\ell)\cdot|S_\ell|}$. Let $\{f_\ell : \ell < k\}$ enumerate $\mathscr{F}_\mathbf{u}$. By induction on $\ell \le k$ define r_ℓ, α_{f_ℓ} such that:

$$r_0 = p, r_\ell \le_n^* r_{\ell+1} \in \mathbb{Q}_\mathbf{i}, r_{\ell+1}^{[f_\ell]} \Vdash_{\mathbb{Q}_\mathbf{i}} \text{“}\underset{\sim}{\tau} = \alpha_{f_\ell}\text{”}.$$

The induction step is by 2.11(4). Notice that $q^{[f_\ell]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_{f_\ell}$" since $r_{\ell+1}^{[f_\ell]} \le q^{[f_\ell]}$. Now $q = r_k$ and $\langle \alpha_f : f \in \mathscr{F}_\mathbf{u}\rangle$ are as required. $\square_{2.12}$

Corollary 2.13. *As in 2.12 but replacing (a)-(d) there by:*

(a) $p \le_n^\otimes q \in \mathbb{Q}_\mathbf{i}$
(b) $\mathbf{u} = w^p \cap \mathbf{k}_p(n+1)$
(c) *if $f \in \mathscr{F}_{p,\mathbf{u}}$ is p-rich then $q^{[f]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_f$", see 2.9(3)*
(d) *$q \Vdash_{\mathbb{Q}_\mathbf{i}}$ "if $\{\underset{\sim}{\eta}_n : n \in \mathrm{val}(\mathfrak{c}_{p,\mathbf{k}_n(p)})\} = {}^{\mathbf{k}(n,p)}2$ then $q \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_f : f \in \mathscr{F}_{p,\mathbf{u}}$ is p-rich$\}$.*

Proof. Similarly to the proof of 2.12 using 2.11(5) instead of 2.11(4). A major point (instead $\mathcal{P}(\omega)/I_\mathbf{r} \models$, we can assume less). $\square_{2.13}$

Claim 2.14. *1) Assume $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{cc}}$ is fast (see part (7) of 2.1; alternatively see 2.2(2) and $\mathbf{i} \in \mathrm{FP}_3$).*

The COM player has a winning strategy in the bounding game $\Game^{\mathrm{bd}}_{\mathbb{Q}_\mathbf{i},p}$ for $p \in \mathbb{Q}_\mathbf{i}$ recalling:

(a) *a play lasts ω-moves,*
(b) *in the n-th move.*
- INC *chooses a $\mathbb{Q}_{\mathbf{i}}$-name $\underset{\sim}{\tau}_n$ of an ordinal*[3],
- *then the* COM *player chooses a finite set $\mathscr{U}_n$ of ordinals*

(c) *in the end the* COM *player wins* *<u>iff</u>* *there is $q \in \mathbb{Q}_{\mathbf{i}}$ above p forcing $\underset{\sim}{\tau}_n \in \mathscr{U}_n$ for every n.*

2) This is true even for the game $\partial^{\rm be}_{\mathbb{Q}_{\mathbf{i}},p}$ defined similarly but we change clause (b) to:

$(b)'$ *in the n-th move*
- *first, the* COM *player chooses $m^\bullet_n$*
- *second, the* INC *player chooses a $\mathbb{Q}_{\mathbf{i}}$-name $\underset{\sim}{\tau}_n$ of an ordinal*
- *third, the* COM *player chooses a set $\mathscr{U}_n$ of $\leq m^\bullet_n$ ordinals*

3) This is true even for the game $\partial_{\mathbb{Q}_{\mathbf{i}},p}$ defined similarly but we change clause (b) to:

$(b)''$ *in the n-th move*
(α) *first, the* COM *player chooses $m^\bullet_n$*
(β) *second, the* INC *player chooses $\ell^\bullet_n$*
(γ) *third, they play a subgame with $\ell^\bullet_n$ moves, in the ℓ-th move*
$\bullet_1$ *the INC player chooses a $\mathbb{Q}_{\mathbf{i}}$-name $\underset{\sim}{\tau}_{n,\ell}$ of an ordinal*
$\bullet_2$ *then, the* COM *player chooses a set $\mathscr{U}_{n,\ell}$ of $\leq m^\bullet_n$ ordinals.*

4) Moreover, $\mathbb{Q}_{\mathbf{i}}$ is strongly bounding (see [She, 4.1] *or 2.17 below).*

5) If $\mathbb{Q}$ is a forcing notion as in part (3) or (4), *<u>then</u>* $\Vdash_{\mathbb{Q}}$ *'every no-where dense subset of $\mathbb{Q}$ (the rationals) is included in an old such set).*

Proof. 1), 2) Follows by part (3).

3) Since if $p \leq_{\mathbb{Q}_{\mathbf{i}}} q$, then a winning strategy for COM is $\partial_{\mathbb{Q}_{\mathbf{i}},q}$is also a winning strategy in $\partial_{\mathbb{Q}_{\mathbf{i}},p}$, we may assume without loss of generality, that $p_0 = p(0) = p$ be as in q in 2.9(4),(5). Now on the side, COM chooses in the n-th move also $\langle p_{n,\ell} : \ell \leq \ell^\bullet_n\rangle, p_{n+1} = p(n+1), k_n = k(n)$ such that:

$(*)^1_n$ (a) $p_n \in \mathbb{Q}_{\mathbf{i}}$ is above p and $\langle \mathrm{nor}(\mathfrak{c}_{p_n,n}) : k \in w^p\rangle$ is increasing
(b) if $n = m+1$ then $p_m <^{\otimes}_{k_n} p_n$ and $k_m < k_n$
(c) $p_{n+1} \Vdash$ "if $\langle \eta_m : m \in S_{k(n)} \cap A^{p_n}\rangle$ is p_n-rich then $\ell < \ell^\bullet_n \Rightarrow \underset{\sim}{\tau}_{n,\ell} \in \mathscr{U}_{n,\ell}$"
(d) $p_n = p_{n,0} \leq^{\otimes}_n p_{n,1} \leq^{\otimes}_n \ldots <^{\otimes}_n p_{n,\ell^\bullet_n} = p_{n+1}$
(e) $p_{n,\ell+1} \Vdash$ "$\underset{\sim}{\tau}_{n,\ell} \in \mathscr{U}_{n,\ell}$"
(f) p_n is as q is in 2.9(4),(5)
(g) $h_{p_n}(\ell) = k_\ell$ for $\ell < n$
(h) $p_0 = p$ and $k_0 = \min(w^p)$.

Why is it possible? That is, why is it a legal strategy for COM?

In the n-th move so p_n is well defined, let $k^\bullet_n = \sup\{\mathrm{nor}(\mathfrak{c}_{p,\ell}) : \ell \in w^{p(n)} \cap \mathbf{k}_{p_n}(k_i+1)$ for every $i < n\} + 1$. Let k_n be $\min(w^{p_0}) = \min(w^p)$ when $n = 0$, otherwise let $k = k_n > \mathbf{k}_{p_n}(n)$ be from w^{p_n} such that $k \geq k_n \wedge k \in w^{p_n} \Rightarrow \mathrm{nor}(\mathfrak{c}_{p_n,k}) \geq k^\bullet_n + 2$ and $k_n > \sup\{k_\ell : \ell < n\}$.

[3] Can use just "of and old element".

Let $p^\bullet_n$ be such that $p_n \le^\otimes_n p^\bullet_n, w^{p^\bullet_n} \cap k_n = \{k_\ell : \ell < n\}$ and $w^{p^\bullet_n} \backslash k_n = w^{p_n} \backslash k_n$ and $\mathfrak{c}_{p^\bullet_n,k} = \mathfrak{c}_{p_n,k}$ for $k \in w^{p^\bullet_n}$, clearly exists. Let $\mathbf{u} = \{k_\ell : \ell \le n\}$ so $\mathbf{u} \triangleleft w^{p^\bullet_n}$ and for parts (2),(3) let $m^\bullet_n = |\mathscr{F}_{p_n,\mathbf{u}_n}|$ be the move of COM. For parts (1),(2) after INC choose $\mathcal{I}_n$ let $p_{n+1} \in \mathbb{Q}_{\mathbf{i}}$ be as in 2.13 for the triple $(p^\bullet_n, k_n, \mathcal{I}_n)$. For part (3), p_n, k_n are well defined and we choose $p_{n,0} = p^\bullet_n$ and $(p_{n,\ell+1}, \mathscr{U}_{n,\ell})$ such that $p_{n,\ell} \le^\otimes_n p_{n,\ell+1}$ and $p_{n,\ell+1} \Vdash_{\mathbb{Q}_{\mathbf{i}}}$ "$\mathcal{I}_{n,\ell} \in \mathscr{U}_{n,\ell}$" by 2.13.

Why this is a winning strategy? So assume $\langle (k_n, p_n) : n < \omega\rangle$ and (for part (1),(2)) $\langle (\mathcal{I}_n, \mathscr{U}_n) : n < \omega\rangle$ and (for parts (2),(3)) $\langle m_n : n < \omega\rangle$ and (for part (3)) $\langle \mathcal{I}_{n,\ell}, \mathscr{U}_{n,\ell} : \ell < \ell^\bullet_n\rangle$ were chosen. Let $q = \lim\langle p_n : n < \omega\rangle \in \mathbb{Q}_{\mathbf{i}}$ be naturally defined. Clearly $p, p_n \le_{\mathbb{Q}_{\mathbf{i}}} q$ for $n < \omega$.

Now we use "$\mathbf{i}$ is fast", consider $\mathfrak{c}_{p_n,k_n} = \mathfrak{c}_{q,k_n}$ and choose $m_{n,\eta,\iota} \in \mathrm{val}(\mathfrak{c}_{p_n,k_n})$ for $\eta \in {}^{h'(k(n))}2, \iota < k_n$ and $\mathfrak{d}^\iota_n \in \Sigma_n(\mathfrak{c}_{p_n,k_n})$ such that:

- $\mathrm{nor}(\mathfrak{d}^\iota_n) \ge n+1$
- $\mathrm{val}(\mathfrak{d}^\iota_n)$ is disjoint to $\{m_{n,\eta,\iota} : \eta \in {}^{h'(k)}2\}$
- for each $\iota < k_n$ the sequence $\langle m_{n,\eta,\iota} : \eta \in {}^{h'(k(n))}2\rangle$ is without repetition
- $\langle \mathrm{val}(\mathfrak{c}^\iota_n) \backslash \mathrm{val}(\mathfrak{d}^\iota_n) : \iota < k_n\rangle$ are pairwise disjoint.

For $\nu \in \prod\limits_n k_n$, let $A^\bullet_\nu = \mathrm{Dom}(E^q) \backslash (\cup\{m/E^q\text{: for some } n, m \in \mathrm{val}(\mathfrak{d}^{\nu(n)}_n)\})$. Clearly we can find $\Lambda \subseteq \prod\limits_n k_n$ of cardinality $2^{\aleph_0}$ such that $\nu \ne \rho \in \Lambda \Rightarrow |\{n : \nu(n) = \rho(n)\}| < \aleph_0$.

Also $\langle \mathrm{Dom}(q) \backslash A^\bullet_\nu : \nu \in \Lambda\rangle$ has pairwise finite intersection so as $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}$ or just $\mathbf{i} \in \mathrm{FP}^3_{cc}$, for some $\nu \in A, A^\bullet_\nu = \omega \mod I_{\mathbf{i}}$.

Now we can define r as desired:

$(*)$
- (a) $\mathrm{dom}(E^r) = A^\bullet_\nu$
- (b) $E^r = E^q{\restriction}A^\bullet_\nu$
- (c) $w^r = w^q$
- (d) $\mathfrak{c}_{r,m}$ is
  - $\mathfrak{d}^{\nu(k)}_{k_n}$ if $m = k_n, n > 0$
  - $\mathfrak{c}_{q,m}$ otherwise
- (e) H^r extends H^q
- (f) if $n < \omega, k = k_n, \eta \in {}^{h'(k(n))}2, m = m_{n,\eta,\nu(n)}$ and $\ell \in m/E^q$ <u>then</u> $\langle H^r(x^\ell_i) : i < h(k_n)\rangle = \eta$.

Now check. $\square_{2.14}$

Conclusion 2.15. *Assume $\mathbf{i} \in \mathrm{FP}^3_{cc}$ is fast (see 2.1(7), alternatively use 2.2(2)).*

1) Then $\mathbb{Q}_{\mathbf{i}}$ is bounding (i.e. $\Vdash$ "every $f \in {}^\omega\omega$ is $\le g$ for some $g \in ({}^\omega\omega)^{\mathbf{V}}$"). Hence this holds, in particular, whenever $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}, \mathbf{i}$ is fast.

2) Moreover, $\mathbb{Q}_{\mathbf{i}}$ has the PP*-property (even the strong one) see* [She98a, Ch.VI,2.12] *or Definition f50 below.*

3) Each of the properties from part (1) and (2) is preserved by CS *iteration.*

Proof. 1) By 2.14(1).

2) By 2.14(2).

3) For bounding by [She98a, Ch.V], for the PP-property by [She98a, Ch.VI,2.12A-F] $\square_{2.15}$

Claim 2.16. *Let an ideal $I \supseteq [\omega]^{<\aleph_0}$ on ω be given.*

1) For any function $h' \colon \mathbb{N} \to \mathbb{N}$ going to infinity, if $\bar{n} = \langle n_k \colon k < \omega \rangle$ satisfies $n_0 = 0$ and $n_{k+1} - n_k > 2^{h'(k)}k$, then letting $S_k = [n_k, n_{k+1})$, there is $\mathbf{i} \in \mathrm{FP}_4$ which is fast.

2) If $\mathrm{dual}(I)$ is an ultrafilter then some $\mathbf{i}$ as above has the uf-property, i.e., belongs to FP_4.

3) With stronger bound on n_{k+1},we can demand that every $(\mathrm{CR}_k, \Sigma_k)$ has bigness (see [RS99]) which means: if $\mathrm{cf} \in \mathrm{CR}_n$, $\mathrm{val}(\mathfrak{c}) = u_1 \cap U_2$ then for $\mathfrak{d} \in \Sigma_n(\mathfrak{c})$ and $\iota \in \{1, 2\}$ we have that $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$ and $\mathrm{val}(\mathfrak{d}) \subseteq u_\iota$.

Proof. As, ignoring the numerical bounds but are not important here, part (3) implies that (1), (2) we do elaborate in their proof.

1) We use 2.1(2)(B)$_1$ and we define $\mathrm{nor}_k \colon \mathscr{P}^-(S_k)$ by $\mathrm{nor}_k(X) = \lfloor |X|/2^{h'(k)} \rfloor$.

Now check.

2) We use $2.1(B)_1$ choosing:

(∗) if $X \subseteq S_k$ then $\mathrm{nor}_k(X) = \lfloor \log_2(\mathrm{nor}_{k,0}(X)) \rfloor$ where $\mathrm{nor}_{k,0}(X) = \lfloor |X|/2^{h'(k)} \rfloor$ the use of $\log_2$ is to help prove "the uf property". Easy to check.

3) For $k < \omega$ we define $\mathbf{CR}_n$ as follows:

$(*)_1$ (a) $\mathfrak{c} \in \mathrm{CR}_k$ iff $\mathfrak{c} \subseteq {}^{S(k)}2$ is not empty
(b) for $\mathfrak{c} \in \mathrm{CR}_k$ let $\mathrm{val}(\mathfrak{c}) = \{n \in S_k\colon$ for some $\eta, \nu \in \mathfrak{c}$ we have $\eta(n) \neq \nu(n)\}$
(c) for $\mathfrak{c} \in \mathrm{CR}_k$ let $\mathrm{nor}_k(\mathfrak{c}) = \frac{1}{(k+1)\cdot 2^{h'(k)}} \log_2(\log_2(|\mathfrak{c}|))$
(d) $\Sigma_n(\mathfrak{c}) = \{\mathfrak{d} \in \mathrm{CR}_n : \mathfrak{d} \subseteq \mathfrak{c}\}$

$(*)_2$ $\mathbf{CR}_k$ is as required in Definition 2.1.

[Why? Easy.]

$(*)_3$ $\mathbf{CR}_k$ has bigness.

[Why? Obvious by the definitions.]

$(*)_4$ $\mathbf{CR}_k$ has the uf-property.

[Why? Assume $\mathfrak{c} \in \mathrm{CR}_k, \mathrm{nor}_k(\mathfrak{c}) \geq 1$. For $S \subseteq S_k$ let $m_{\mathfrak{c},S} = \max\{|\{\rho \in \mathfrak{c} : \rho \supseteq \nu\}| \colon \nu \in {}^{S(k)\setminus S}\{-1, 1\}\}$ hence,

$(*)_{4.1}$ Why $S = u \cup v \Rightarrow |\mathfrak{c}| \leq |m_{\mathfrak{c},u}| \times |m_{\mathfrak{c},v}|$?

$(*)_{4.2}$ if $u \subseteq w \subseteq S_k$, then $m_{\mathfrak{c},u} \leq m_{\mathfrak{c},v}$

[Why? if $v \in {}^{(S_k \setminus u)}\{1, -1\}$ witness $m_{\mathfrak{c},u} \geq m$, then $v \restriction (s_k \setminus w)$ witness $m_{\mathfrak{c},w} \geq m$].

$(*)_{4.3}$ to prove $(*)_{4.1}$ without loss of generality, $u \cap v = \emptyset$.

$(*)_{4.3}$ let $O \coloneqq \{-1, 1\}$, (Professor Shelah, I do this so that the equations do not exceed the margin of the page. If you don't like it, I can split it into several more lines.) then

$$\begin{aligned}|\mathfrak{c}| &= |\{(\eta,\nu)\colon \eta \in {}^uO,\, \nu \in {}^vO \wedge \eta \cup \nu \in \mathfrak{c}\}| \\ &= incomp_{\eta\in {}^uO}|\{\nu \in {}^uO\colon (\eta,\nu) \in \mathfrak{c}\}| \\ &= |\{\eta \in {}^uC\colon (\exists \nu \in {}^vO)(\eta \cup \nu \in \mathfrak{c})\}| \times \max_{\eta \in {}^uO} |\{\nu \in {}^uO\colon \eta \cup \nu \in \mathfrak{c}\}| \\ &\leq m_{\mathfrak{c},u} \times m_{\mathfrak{c},u}\end{aligned}$$

So if $h\colon S \to \{0,1\}$, then for some $\iota < 2$ we have $|\mathfrak{c}_\iota| \geq \sqrt{\mathfrak{c}}$ where $\mathfrak{c}_\iota = \{\rho \in {}^{S(k)}2 : \rho \supseteq \nu_\iota\}$ where $v_\iota : \{n \in S_k : h(n) = \iota\} \to \{-1,1\}$ is chosen such that $|\mathfrak{c}_\iota|$ is maximal. Now compute.]

$(*)_5$ $\mathbf{CR}_k$ is fast.

[Why? Assume $\mathfrak{c} \in \mathrm{CR}_k$ and $\mathrm{nor}_k(\mathfrak{c}) \geq 1$.

Now we try to choose $(n_\ell, \iota_\ell, \mathfrak{c}_\ell)$ by induction on $\ell < m = k \cdot 2^{h'(k)}$

$(*)_{5.1}$
(a) $\mathfrak{c}_\ell = \{\eta \in \mathfrak{c}\colon \text{if } k < \ell \text{ then } \eta(n_k) = \iota_\ell\}$
(b) $n_\ell \in \mathrm{val}_k(\mathfrak{c})\backslash\{n_j : j < \ell\}$
(c) $|\mathfrak{c}_\ell| \geq |\mathfrak{c}| \cdot 2^{-\ell}$.

Now as $\mathrm{nor}_k(\mathfrak{c}) \geq 1$, clearly $|\mathfrak{c}| \geq 2^m$ hence $\mathrm{val}_k(\mathfrak{c}) \geq m$, so we can carry the induction. For $\iota < k$ we let $\mathfrak{d}_\iota = \mathfrak{d} = \mathfrak{c}_{k-h'(k)}$ and for each $\ell < k$ let $\langle m_{\eta,\ell} : \eta \in {}^{h'(k)}2\rangle$ list $\{n_j : j \in [2^{h'(k)}\ell, 2^{h'(k)}(\ell+1))\}$.
Lastly, $\mathrm{nor}_k(\mathfrak{d}_\iota) = \mathrm{nor}_k(\mathfrak{c}_m) \geq \frac{1}{(k+1)\cdot 2^{h'(k)}} \log_2(\log_2(|\mathfrak{c}| \cdot 2^{-m}) \geq \mathrm{nor}_k(\mathfrak{c}) - 1$.

$(*)_6$ if $\mathrm{nor}_k(\mathfrak{c}) \geq 1$ then for some partition u_1, u_2 of S_k and $\mathfrak{c}_1, \mathfrak{c}_2 \in \Sigma(\mathfrak{c})$, we have $\mathrm{val}(\mathfrak{c}_\iota) \subseteq u_\iota, \mathrm{nor}_k(\mathfrak{c}_\iota) \geq \mathrm{nor}_k(\mathfrak{c}) - 1$.

[Why? We can find a maximal $u \subseteq S_k$ such that $|\mathscr{P}_{\mathfrak{c},u}| \leq \sqrt{|\mathfrak{c}|}$, so $u \subsetneqq S_k$. Let $n \in S_k\backslash u$ hence $|\mathscr{P}_{\mathfrak{c},u}| \leq \sqrt{|\mathfrak{c}|} < |\mathscr{P}_{\mathfrak{c},u\cup\{n\}}| \leq |\mathscr{P}_{\mathfrak{c},u}| \cdot 2 \leq 2\sqrt{|\mathfrak{c}|}$, so

- $|\mathscr{P}_{\mathfrak{c},u}| \in [\sqrt{|\mathfrak{c}|}, 2\sqrt{|\mathfrak{c}|})$.

Let $v = S\backslash u$. By $(*)_{4.1}$,

- $|\mathscr{P}_{\mathfrak{c},v}| \geq [\sqrt{\mathfrak{c}}.]$

$\square_{2.16}$

Definition 2.17. 1) For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define $\partial_{\mathrm{sb}} = \partial_p^{\mathrm{sb}} = \partial_p^{\mathrm{sb}}(\mathbb{Q}) = \partial_{\mathbb{Q},p}^{\mathrm{sb}}$, the strong bounding game between the null player NU and the bounding player BND as follows:

(a) a play last ω moves and
(b) in the n-th move:
 (α) first the NU player gives a (non-empty) tree $\mathscr{T}_n$ with ω levels and no maximal node and a $\mathbb{Q}$-name $\underset{\sim}{\mathscr{F}}_n$ of a function with domain $\mathscr{T}_n$ such that $\eta \in \mathscr{T}_n \Rightarrow p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathscr{F}}_n(\eta) \in \mathrm{suc}_{\mathscr{T}_n}(\eta)$"
 (β) then BND player chooses $\eta_n \in \mathscr{T}_n$
(c) in the end, the BND player wins the play $\langle \mathscr{T}_n, \eta_n : n < \omega\rangle$ <u>iff</u> there is $q \in \mathbb{Q}$ above p forcing that "$(\forall n < \omega)(\exists k < \mathrm{level}(\eta_n))(\underset{\sim}{\mathscr{F}}_n(\eta_n{\restriction}k) \leq_{\mathscr{T}_n} \eta_n \wedge k$ is even)" where $\eta_n{\restriction}k$ is the unique $\nu \leq_{\mathscr{T}_n} \eta_n$ of level k.

2) Omitting p means NU chooses it in his first move.

3) A forcing notion $\mathbb{Q}$ is strongly bounding if for every condition $p \in \mathbb{Q}$ player BND has a winning strategy in the game $\eth^{\rm sb}_{\mathbb{Q},p}$.

Conclusion 2.18. *Assume that for each $p \in \mathbb{Q}_{\mathbf{i}}$ the first player has no winning strategy in $\eth^{\rm sb}_p(\mathbb{Q}_{\mathbf{i}})$, see 2.17 (e.g. if I is a maximal ideal).*

Then

- $(*)$ *$\mathbb{Q}_{\mathbf{i}}$ is proper, α-proper, strongly α-proper for every $\alpha < \omega_1$, is ${}^\omega\omega$-bounding and it has the* PP*-property, even the strong* PP*-property.*

§ 2(B). **On $\mathbb{Q}^5_{\mathbf{i}}$.**

We suggest a relative of $\mathbb{Q}^4_{\mathbf{i}}$, the creature are more complicated but in the proof of COM winning the games for $\mathbb{Q}^5_{\mathbf{i}}$ we do not need incomp assumption on $\mathbf{i}$. We just indicate the difference.

Convention 2.19. When we say "as in a place in §2" we mean replacing $\mathbb{Q}^4_{\mathbf{i}}$ by $\mathbb{Q}^5_{\mathbf{i}}$, FP_3 by FP_4 (and if it is a claim, also the proof is similar).

Definition 2.20. 1) For a set S and $m < \omega$ such that $|S|$ is divisible by 3 let $\mathrm{val}_{S,m}$ be the set of f such that:

- (α) f is a function from $\{x^n_i \colon n \in S_k \text{ and } i < h'(k)\}$,
- (β) $\mathrm{Range}(f) \subseteq \{-1, 1\}$,
- (γ) for every $\eta \in {}^m\{-1,1\}$ the following set has exactly $|S|/2^m$ members:
$$\{s \in S \colon \langle f(x^n_i) \colon i < m\rangle = \eta\}.$$

2) Let FP_3 be the set of $\mathbf{i}$ such that:

- (A) as in 2.1(1) replacing clause (g) by:
  (g)$'$ $|S_n|$ is divisible by 2^n and $\infty = \liminf\langle |S_n|/2^n \colon n < \omega\rangle$.
- (B)$_1$ <u>the simple creature version</u>:
  - (a) $\mathfrak{c} \in \mathrm{CR}_n$ iff $\mathfrak{c}$ incomp $(k, m, S, F, \mathrm{val}, \mathrm{nor}) = (k_{\mathfrak{c}}, m_{\mathfrak{c}}, S_{\mathfrak{c}}, F_{\mathfrak{c}}, \mathrm{val}_{\mathfrak{c}}, \mathrm{nor}_{\mathfrak{c}})$, we may write $\mathrm{val}(\mathfrak{c})$, etc.
    - (α) $m \leq h`(k)$,
    - (β) $S \subseteq S_k$ and $|S|$ is divisible by $2^{h`}(k)$,
    - (γ) F is a functions from $\mathrm{val}_{S,m}$ into $\mathrm{val}_{S_k,h'(k)}$,
    - (δ) if $\eta \in \mathrm{val}_{S,m}$, then $\eta \subseteq F(\eta)$,
    - (ε) $\mathrm{val}(\mathfrak{c}) = \mathrm{Range}(F)$,
    - (ζ) $\mathrm{nor}_{\mathfrak{c}} = \log_2(15/2^n)$.
- (C) $\mathrm{cg} \in \Sigma(\mathfrak{c})$ when:
  - (α) $\mathfrak{c}, \mathfrak{d} \in \mathrm{CR}_k$,
  - (β) $S_{\mathfrak{d}} \subseteq S_{\mathfrak{c}}$,
  - (γ) $m_{\mathfrak{d}} \leq m_{\mathfrak{c}}$,
  - δ $\mathrm{val}_{\mathfrak{d}} \subseteq \mathrm{val}_{\mathfrak{c}}$.
- (B)$_2$ <u>full creature version</u>:
  - (a) $\langle \mathrm{CR}_n \colon n < \omega\rangle$ where $\mathrm{CR}_n = (\mathrm{CR}_n, \mathrm{val}_n, \mathrm{nor}_n, \Sigma_n)$,
  - (b) the CR_n's are pairwise disjoint, each finite,
  - (c) $\mathrm{val}_n(\mathfrak{c}) = \mathrm{val}(\mathfrak{d})$ for some $\mathfrak{c}$ from (B)$_1$, so $S_{\mathfrak{c}}, m_{\mathfrak{c}}$ are $S_{\mathfrak{d}}$ and $m_{\mathfrak{d}}$ respectively, $\mathfrak{c} \in \mathrm{CR}_n$,
  - (d) $\mathrm{nor}_n(\mathfrak{c}) \in \mathbb{R}_{>0}$ for $\mathfrak{c} \in \mathrm{CR}_n$ and $\langle \mathrm{nor}_n(\emptyset) \colon n < \omega\rangle$ goes to infinity,
  - (e) if $\mathfrak{c} \in \mathrm{CR}_n$ then $\Sigma_n(\mathrm{cg}) \subseteq \mathrm{CR}_n$ and $\mathfrak{c} \in \Sigma_n(\mathfrak{c})$,

(f) if $\mathfrak{c} \in \Sigma_n(\mathfrak{c})$ then $\mathrm{val}_n(\mathfrak{d}) \subseteq \mathrm{val}_n(\mathfrak{c})$ and $\Sigma_n(\mathfrak{c}) \subseteq \Sigma_n(\mathfrak{c})$, hence $S_{\mathfrak{d}} \subseteq S_{\mathfrak{c}}$ and $m_{\mathfrak{d}} \leq m_{\mathfrak{c}}$ for $\mathfrak{c}, \mathfrak{d} \in \mathrm{CR}_n$.

3) - 7) As in 2.1.

Definition 2.21. As in 2.3.

Definition 2.22. For $\mathbf{i} \in \mathrm{FP}_4$ we define the forcing notion $\mathbb{Q}^5_{\mathbf{i}}$ as follows:

(A) $p \in \mathbb{Q}^5_{\mathbf{i}}$ if, and only if, $p = (H, E, A, \bar{\mathfrak{c}}) = (H^o, E^p, A^p, \bar{\mathfrak{c}}^p)$ satisfies:
  (a) E is an $I_{\mathbf{i}}$-equivalence relation, so on a set called $\mathrm{dom}(E)$ which belongs to $D_{\mathbf{i}}$ hence is $\subseteq \omega$,
  (b) $A = A^p := \{n \in \mathrm{dom}(E) \colon n = \min(E/E)\}$,
  (c) H is a function with range $\subseteq \{-1, 1\}$ and domain
$$B^p_0 = \{x^n_i \colon i < h(n) \wedge n \in (\omega \setminus \mathrm{dom}(E))\},$$
 (d)-(e) as in 2.4(A),

(C) We say that a function $f \in \mathscr{F}_\omega$ satisfies a condition $p \in \mathbb{Q}_{\mathbf{i}}$ <u>when</u>:
  (a) $f(x^n_i) = H^p(x^n_i)$ <u>when</u> $x^n_i \in B^p_0$,
  (b) if $k \in \omega$, then $f \restriction \{x^n_i \colon n \in S_k, i < h'(k)\} = F_{\mathfrak{c}_p,k}(f \restriction \{x^n_i \colon n \in S_{\mathfrak{c}_p,k}, i < m_{\mathfrak{c}_p,k}\})$.

(D) As in 2.4(D).

Claim 2.23. *As in 2.5*

Claim 2.24. *As in 2.7.*

Definition 2.25. 1), 2), 3) as in 2.8.

Proposition 2.26. *As in 2.9.*

Definition 2.27. As in 2.10.

Proposition 2.28. *1) - 4) As in incomp.*

5) If $p \in \mathbb{Q}^5_{\mathbf{i}}$, $\mathbf{u} = w^p \cap \mathbf{k}_p(n+1)$, $\mathrm{nor}(\mathfrak{c}_p, \max()) \geq \mathbf{1}$ and $f \in \mathscr{F}_{p,\mathbf{u}}$ and $p^{[f]} \leq q$ <u>then</u>, for some $r \in \mathbb{Q}_{\mathbf{i}}$ we have $p \leq^{\otimes}_n r \leq q$ and $r^{[f]} = q$.

Proof. 1) - 4) As in 2.9.

Similar recalling our definitions. $\square_{2.28}$

Corollary 2.29. *As in 2.12, but:*

(b)′ $\mathbf{u} = w^p \cap \mathbf{k}_p(n+1)$ *and* $\mathrm{nor}_{\mathfrak{c}_p,\max \mathbf{u}} \geq 1$.

Corollary 2.30. *As in 2.13.*

Claim 2.31. *Assume $\mathbf{i} \in \mathrm{FP}_4$ is fast. Then:*

1) The COM player has a winning strategy in the bounding game $\partial^{\mathrm{bd}}_{\mathbb{Q}_{\mathbf{i}},p}$ for $p \in \mathbb{Q}^5_{\mathbf{i}}$ recalling clauses (a), (b), (c) of 2.14.

2) - 5) Also as in 2.14.

Proof. As in 2.14 recalling our definitions. $\square_{2.31}$

Conclusion 2.32. *Assume $\mathbf{i} \in \mathrm{FP}_4$.*

1), 2), 3) As in 2.15.

Claim 2.33. *As in 2.16.*

§ 3. On no α-ultrafilter

Recall (and we shall use freely).

Definition 3.1. For ordinals α, β, let their *natural sum* $\alpha \oplus \beta$, defined as follows:

$$\alpha \oplus \beta := \min\{\gamma\colon \text{if } u, v \subseteq \text{Ord}, \text{otp}(u) = \alpha \text{ and } \text{otp}(v) = \beta, \text{ then } \text{otp}(u \cup v) \subseteq \gamma\}.$$

It is well-founded.

Fact 3.2. The following condition on the ordinal α_*, are equivalent:

(a) $\alpha_* = \omega^\alpha$ for some ordinal α (recall that, $\omega^0 = 1$).
(b) $(\forall \beta, \gamma < \alpha_*)(\beta + \gamma < \alpha_*)$.
(c) $(\forall \beta, \gamma < \alpha^*)(\beta \oplus \gamma < \alpha_*)$.

§ 3(A). **Covering countable sets of ordinals.**

In the April version, we try to replace J^1_α by an ideal on a suitable well-founded sub-tree of ${}^{\omega >}\omega$, there was a problem which I think is overcomed.

This equivalence seem to be of interest, but have not looked at what I wrote again.

Then there was a problem in the proof that the iteration works. DEBT.

Here we restart still has to preserve the basic properties of J^1_α.

The following may help in iteration proving it suffice to deal one iterand when we try "no J^1_α-ultra-filter".

Claim 3.3. *If $\alpha < \omega_1^{\mathbf{V}_1}$, $u \in \mathbf{V}_2$ is a subset of α, __then__ for some $v \in \mathbf{V}_1$ such that $u \subseteq \alpha$ and $\text{otp}(w) = \text{otp}(v)$, __when__ either (A) or (B), where:*

(A) *$\mathbf{V}_1 = \mathbf{V}$, $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P}}$, $\mathbb{P}$ is a proper bounding forcing (i.e., every $f \in ({}^\omega\omega)^{\mathbf{V}[\mathbb{P}]}$ is bounded by some $g \in ({}^\omega\omega)^{\mathbf{V}_2}$.)*
(B) *$\mathbf{V}_1$ is a transitive class of $\mathbf{V}_2$, both models of* ZFC, *$(\mathbf{V}_1, \mathbf{V}_2)$ is bounding and $[[{}^\omega 2]^{\aleph_0}]^{\mathbf{V}_1}$ is cofinal in $\mathbf{V}_2$.*

Proof. We prove this by induction on $\text{otp}(u)$:

__Case 1__: $\text{otp}(u)$ is finite.

This case is trivial.

__Case 2__: $\text{otp}(u)$ is additively descomposable, i.e., $\text{otp}(u) = \beta_1 + \beta_2$, where $\beta_1, \beta_2 < \text{otp}(u)$.

Let α_* be the β_1-th member of u and let $u_1 := u \cap \alpha_1$, $u_2 := u \setminus b_1$, so $\text{otp}(u_1) = \beta_1$ and $\text{otp}(u_2) = \beta_2$.

By the hypothesis induction, for $\ell = 1, 2$, there is $v_\ell \in \mathbf{V}_1$ such that $u_\ell \subseteq v_\ell \subseteq \alpha_1$ andotp$(v_\ell) = \beta_\ell$.

Let $v = (v_1 \cap \alpha_1) \cup (v_2 \setminus \alpha_1) \in \mathbf{V}_1$. Clearly $v \in \mathbf{V}_1$ and $v \subseteq \text{otp}(v) = \beta_1 + \beta_2 = \text{otp}(u)$, so we are done.

__Case 3__: *opt*(u) is additively indescomposable (that is, no case 2).

So $\delta := \text{otp}(u)$ is a limit ordinal and without loss of generality, $\sup(u) = \alpha$. If $\alpha = \delta$, let $\nu = \delta$ so, without loss of generality, $\alpha < \delta$, and by the induction hypothesis it is enough to prove

$(*)_3$ there is $v \in \mathbf{V}_1$ such that $u \subseteq v \subseteq \delta$, $\text{otp}(v) < \delta$.

In $\mathbf{V}_1$, let $\langle \beta_m \colon n < \omega \rangle$ be increasing with limit α such that $\beta_0 = 0$.

Now,

(1) Without loss of generality, one of the following occurs:
  (a) $\alpha < \beta_1$ and $\alpha \cdot \omega < \delta$, but $n_{incomp} = 0$,
  (b) β_1 is additively indescomposable and $\mathrm{otp}(v \cap [\beta_n, \beta_{n+1})]) < \beta_1$ for $n <$ *incomp*.

[Why? If $\alpha \cdot \omega < \delta$ omitting finitely many β_n-s are get clause (a). Otherwise, letting β be minimal such that $\beta \cdot \omega = \delta$, we have that β is additively indescomposable, and as $\mathrm{otp}(n) = \alpha < \delta$, there is $n(*)$ such that

- $n \geq n(*) \Rightarrow \mathrm{otp}(u \cap [\beta_n, \beta_{n+1}]) < \beta$.

]

But clearly it suffice to deal with *incomp* $= u \setminus \beta_{n(*)}$. As without loss of generality $\beta_{incomp} = \beta$ we are done proving ....

By properness (if (A), or use (B)) and the induction hypothesis, we have:

$(*)_1$ there is a list $\langle v_{n,\ell} \colon n, \ell < \omega \rangle \in \mathbf{V}_1$ such that:
  (a) $v_{n,\ell} \subseteq [\beta_n, \beta_{n+1}]$ has order type $< \beta_1$,
  (b) $\bigwedge_{n<\omega} \big(\bigvee_{\ell<\omega} [u \cap [\beta_n, \beta_{n+1}) \subseteq v_{n,\ell}] \wedge \mathrm{otp}(v_{n,\ell}) = \mathrm{otp}(u \cap [\beta_n, \beta_{n+1})) \big)$

By bounding,

$(*)_4$ there is some $f \in {}^{\omega}(\omega \setminus \{u\})$, such that $n < \omega \Rightarrow \bigvee_{\ell < f(m)} u \cap [\beta_n, \beta_{n+1}]$.

For any $n < \omega$, let $v_n := \bigcup \{ v_{n,\ell} \colon \ell < f(n) \} \in \mathbf{V}_1$. By Theorem 3.1 and Theorem 3.2, we have that $\mathrm{otp}(v_n) < \beta_1$. Similarly, $\bigwedge_{k<\omega} \mathrm{otp}(\bigcup_{n<k} v_n) < \delta = \mathrm{otp}(u)$. Now, let $v := \bigcup_{n<\omega} v_n$, so $v \in \mathbf{V}_1$, $v \subseteq \alpha$ and $\mathrm{otp}(v) \leq \mathrm{otp}(u)$ because $\bigwedge_{n<\omega} (v \cap \beta_n \triangleleft v)$.

$\square_{3.3}$

§ 3(B). Proof of CON(no nowhere dense ultrafilters).

Hypothesis 3.4. 1) We assume that $2^{\aleph_0} = \aleph_1$ and $\lambda = \lambda^{\aleph_1}$ (e.g. $\lambda = \aleph_2$).

2) $\mathbb{Q}_{\mathbf{i}}$ mean $\mathbb{Q}^4_{\mathbf{i}}$, $\mathbf{i} \in \mathrm{FP}_3$ (see §2A)

Remark 3.5. We like to have also "$2^{\aleph_0}$ arbitrary large". We may in 3.6 have $\underset{\sim}{\mathbb{Q}}_\alpha$-s of different forcing notions, similar enough to the incomp $\mathbb{Q}_{\mathbf{i}}$.

Definition 3.6. Let $\mathbf{Q}^3_\gamma$ be the set of $\mathbf{q}$ that consists of:

(a) $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \colon \alpha \leq \gamma, \beta < \gamma$ is a countable-support iteration,
(b) $\underset{\sim}{\eta}_\beta$ is a generic of $\mathbb{Q}_\eta$,
(c) $\bar{u} = \langle u_\alpha \colon \alpha < \lambda \rangle$, where for any $\alpha < \lambda$, $u_\alpha \in [\alpha]^{\leq \aleph_1}$, and $\beta \in u_\alpha \Rightarrow u_\beta \subseteq u_\alpha$,
(d) $\underset{\sim}{D}_\alpha$ is a $\mathbb{P}_\alpha$-name, computable from $\underset{\sim}{\bar{\eta}} \restriction u_\alpha$, of a non-principal ultrafilter on ω in $\mathbf{V}[\underset{\sim}{\eta} \restriction u_\alpha]$,
(e) $\mathbb{Q}_\alpha = \underset{\sim}{\mathbb{Q}}_{\mathbf{i}(\alpha)}$ in $\mathbf{V}[\underset{\sim}{\bar{\eta}} \restriction u_\alpha]$ with $\mathbf{i}(\alpha) \in \mathrm{FP}_3$ and $\underset{\sim}{D}_{\mathbf{i}(\alpha)} = \underset{\sim}{\alpha}$. [One can just demand $\underset{\sim}{\mathbb{Q}}_\alpha$ has some properties: the games from 2.14 or 2.31,
(f) $\gamma = \lg(\mathbf{q})$ and $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_\gamma$.

Definition 3.7. 1) If $\mathbf{q} \in \mathbf{Q}^3_\lambda$, we say $u \subseteq \lg(\mathbf{q})$ is $\bar{u}_{\mathbf{q}}$-*closed*, when $\beta \in u$ implies $u_\beta \subseteq u$.

2) If $u \subseteq \lg(\mathbf{q})$, we define $\mathbb{P}_u := \{p \in \mathbb{P}\colon \operatorname{dom}(p) \subseteq \alpha$ and each $p(\beta)$ is a Borel function of $\underset{\sim}{\bar\eta} \restriction$ for some countable $v \subseteq u_\beta\}$.

Claim 3.8. *For $\mathbf{q} \in \mathbf{Q}^3_\gamma$,*

1) (a) *$\mathbb{P}_\alpha$ is proper,*
   (b) *$\mathbb{P}$ is $\aleph_2$-cc,*
   (c) *$\mathbb{P}_\alpha$ has continuous reading of names.*

2) $\mathbb{P}^* := \{p \in \mathbb{P}\colon$ if $\alpha \in \operatorname{dom}(p)$, then $p(\alpha) = \mathbf{B}(\underset{\sim}{\bar\eta} \restriction (\operatorname{dom}(p) \cap u_\alpha)$, for some Borel function $\mathbf{B}$ from $\mathbf{V}\}$.

3) If $u \subseteq \gamma$ is $\bar{\mathbf{q}}$-closed, then $\mathbb{P}_u \lessdot \mathbb{P}$.

Proof. 1) As in §2, by fusion, see Theorem 2.14.

2), 3) As of old. $\square_{3.8}$

Claim 3.9. *1) Any $\mathbf{i} \in \mathrm{FP}_3$, $\mathbb{Q}_{\mathbf{i}}$ satisfies: every no-where dense subset of $({}^{\omega>}2, \lhd)$ (e.g. the rationals) is included in an old one.*

2) Moreover, this holds for CS iteration of $\mathbb{Q}_{\mathbf{i}}$-s.

3) Even for $\mathbb{P}_{\mathbf{q}}$, <u>when</u> $\mathbf{q} \in \mathbf{Q}^3_\lambda$.

4) There is $\mathbf{q} \in \mathbf{Q}^3_\lambda$ such that:

(a) *if $\underset{\sim}{D}$ is a $\mathbb{P}_{\mathbf{q}}$-name of an ultrafilter, <u>then</u> for some $\bar u_{\mathbf{q}}$-closed, $u \in [\lambda]^{\aleph_1}$, $\Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$D' = \underset{\sim}{D} \cap \mathbf{V}[\underset{\sim}{\bar\eta}\restriction u]$ is definable in (hence belongs to) $\mathbf{V}[\underset{\sim}{\bar\eta} \restriction u]$",*
(b) *if $(\underset{\sim}{D}, \underset{\sim}{D}')$ is as above, then for some α, $(u, \underset{\sim}{D}') = (u_{\alpha, \underset{\sim}{D}_\alpha})$.*

5) If $\lambda = \aleph_2$ and $\diamondsuit(S^{\aleph_2}_{\aleph_1})$, <u>then</u> without loss of generality, $\bigwedge_\alpha u_{\mathbf{q},\alpha} = \alpha$.

Proof. 1) Easy by §2.

2) See [She98a, Ch. 6, pg. 305, 2.15D], which says: for CS iteration $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j \colon i \leq \alpha, j < \alpha\rangle$, if $\vDash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is proper and any new open dense subset of ${}^{\omega>}\omega$ includes an old one", then $\mathbb{P}_\alpha$ is proper and any dense open subset $A \in \mathbf{V}^{\mathbb{P}}$ of ${}^{\omega>}\omega$ includes a dense open subset $A \in \mathbf{V}$ of ${}^{\omega>}\omega$.

3) Similarly by f4(5).

4) Easy by book-keeping.

5) Easy by $\diamondsuit(S^{\aleph_2}_{\aleph_1})$. $\square_{3.9}$

Claim 3.10. *Assume $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}$, $\mathbf{G} \subseteq \mathbb{Q}_{\mathbf{i}}$ is generic over $\mathbf{V}$, $h = h_{\mathbf{i}}$ and the generic is $\underset{\sim}{\eta} = \langle \underset{\sim}{\nu}_\ell \colon \ell < \omega\rangle$, $\nu_\ell \in {}^{h(\ell)}\{1, -1\}$.*

If $A \subseteq ({}^{\omega>}\{1, -1\}, \lhd)$ is no-where-dense and $p \in \mathbb{Q}_{\mathbf{i}}$, then for some q, we have that:

(a) $\mathbb{Q}_{\mathbf{i}} \models p \leq q$,
(b) $C_{q,B} = \emptyset \mod D$, *where* $C_{q,B} := \{n < \omega\colon$ *if* $\eta \in {}^{h(n)}\{-1, 1\}$ *and* $(i \in [\min(n/E^q), h(n)) \Rightarrow \eta(i) = B^q(x^n_i$ <u>*then*</u> $(\forall \nu \in {}^{\omega>}\{-1, 1\})(\eta \trianglelefteq \nu \Rightarrow \nu \notin A)\}$.

Proof. For every $\eta \in {}^{\omega>}\{-1, 1\}$ choose ν_η such that $\eta \lhd \nu_\eta \in {}^{\omega>}\{-1, 1\}$ and $\neg(\exists \rho)(\nu_\eta \trianglelefteq \rho \in X)$. Hence,

$(*)_1$ for every $n < \omega$ there are $J^{>n}_{(n)}$ and $\rho_n \in {}^{[n, j(n)]}\{-1, 1\}$ such that: if $\eta \in {}^n\{-1, 2\}$ and $\eta^\frown \rho_n \trianglelefteq \varrho \in {}^{\omega>}\{-1, 1\} \Rightarrow \varrho \notin X$.

We next define a function $g\colon m^p \to w^p$ as follows:

$(*)_2$ if $k \in w^p$, then $g(k)$ is the maximal $m \in w^p \cup \{0\}$ such that: if $m > 0$, then $m < k$ and $g(n) < h'(k)$.

Easily,

$(*)_3$ we can find $k_i \in w^p$ increasing with i such that:
- (a) if $k \in w^p \setminus k_{i+1}$, then $g(k) > k_i$,
- (b) $\bigcup\{\mathrm{val}(\mathfrak{c}_{p,k_i})\colon i < \omega\}$ belong to $I_{\mathbf{i}}$.

Now we define $q = (H, E, A, \bar{\mathfrak{c}}) \in \mathbb{Q}^4_{\mathbf{i}}$.

$(*)_4$
- (a) $w = \{k_i\colon i < \omega\}$,
- (b) $A = \bigcup\{\mathrm{val}(\mathfrak{c}_{p,k_i})\colon i < \omega\}$,
- (c) E is defined as follows: if $n \in A$ so $n \in \mathrm{val}(\mathfrak{c}_{p,k_i})$ then n/E is:
  - (α) n/E^p if $n > \min(\mathrm{val}(\mathfrak{c}_p, k_i))$,
  - (β) $(n/E^p) \cup \{\mathrm{val}(\mathfrak{c}_{p,k})\colon k \in w^p \cap (k_{i+1},\, k_{i+1})\}$.
- (d) $\mathfrak{c}_{k_i} = \mathfrak{c}_{p,k_i}$,
- (e) H is defined as follows:
  - (α) $H(x^n_i) = H^p(x^n_i)$ when $x^n_2 \in \mathrm{dom}(H^p)$,
  - (β) $H(x^n_j) = \rho_{h'(k_i)}(j)$ when $k \in w^p \wedge (k_{,i}\,, k_{+1})$ and $j \in [h'(k_i),\, h'(k)]$,
  - (γ) $H(x^n_j) = 0$ if for some $k \in w^p \cap k_0$ and $n \in \mathrm{val}(\mathfrak{c}_{p,incomp})$ and $j < h`(k)$.

Easily,

$(*)_5$
- (a) $q \in \mathbb{Q}_{\mathbf{i}}$,
- (b) $p \leq_{\mathbb{Q}_{\mathbf{i}}} q$,
- (c) $q \Vdash$"if $n \in \mathrm{dom}(E^q \setminus \bigcup\{S_{k_i}: i < \omega\})$, then $\underset{\sim}{\eta}_n \notin X$.

As $\bigcup\{S_{k_i}: i < \omega\} \in I_{\mathbf{i}}$ we are done. $\square_{3.10}$

Claim 3.11. *If* $\mathbf{q}$ *is as in 3.9(4) (so* $\mathbf{q}$ *exits),* *<u>then</u> in* $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$ *there is no nowhere-dense ultrafilter* D *on* ω *(and naturally* $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}} \models$ "$2^{\aleph_0} = \lambda$"*).*

Proof. Towards contradiction, suppose that $\Vdash_{\mathbb{P}}$"$\underset{\sim}{D}$ is a nowhere dense ultrafilter on ω". By the assumption on $\mathbf{q}$, there is $\alpha < \lambda$ such that $\Vdash_{\mathbb{P}}$"$\underset{\sim}{D}_\alpha \subseteq \underset{\sim}{D}_\alpha$". Now, the generic $\eta_\alpha = \langle \underset{\sim}{\nu}_{\alpha,n}\colon n < \omega\rangle$, where $\underset{\sim}{\nu}_{\alpha,n} \in {}^{h(n)}\{1,-1\}$ is a function from ω to ${}^\omega\{-1,1\}$, hence some $p_1 \in \mathbb{P}_{\mathbf{q}}$ forces "$\underset{\sim}{Y} = \{\underset{\sim}{\nu}_{\alpha,n}\colon n \in \underset{\sim}{A}\}$ is nowhere dense in ${}^{\omega>}\{-1,1\}$ and $\underset{\sim}{A} \in \underset{\sim}{D}$".

By 3.9(1), (2), (3), know that $\Vdash$"there is a nowhere dense $X \subseteq {}^{\omega>}\{-1,1\}$ from $\mathbf{V}$ including $\underset{\sim}{Y}$". So for some such X and $p_2 \geq p_1$, $p_2 \Vdash$"$\underset{\sim}{Y} \subseteq X$" and, without loss of generality, $p_2 \in \mathbb{P}'_{\mathbf{q}}$.

As $\underset{\sim}{Y}$ is a $\mathbb{P}_{u_\alpha \cup \{\alpha\}}$-name and $\mathbb{P}_{u_\alpha \cup \{\alpha\}} \lessdot \mathbb{P}_{\mathbf{q}}$, we get $p_2 \restriction \Vdash_{\mathbb{P}}$"$\langle n\colon \underset{\sim}{\nu}_{\alpha,n} \in X\rangle$ is not disjoint to any $A \in \underset{\sim}{D}_\alpha$ (a $\mathbb{P}_{n_\alpha}$-name).

This contradicts 3.10. $\square_{3.11}$

So we are done proving the promised consistency, an alternative to [She98b].

§ 3(C). Existence of J_{ω^α}-ultrafilters from P-points.

Maybe see later for more on trees (maybe some exists).

Choice 3.12. Let $\mathbf{T}_5 := \bigcup\{\mathbf{T}_{5,\alpha} \colon \alpha \in [1, \omega_1)\}$ and $\operatorname{level}(\mathbf{t}) = \alpha$ when $\mathbf{t} \in \mathbf{T}_{5,\alpha}$, where

Definition 3.13. For $\alpha < \omega_1$, define $\mathbf{T}_{5,\alpha}$ as the set of $\mathbf{t} = (\mathscr{T}, h, f)$ such that:

(a) $\mathscr{T}$ is a well founded subtree of ${}^{\omega>}\omega$,
(b) $f \colon \max(\mathscr{T}) \to \omega^\alpha$ is an isomorphism from $(\max T, <_{\text{lex}})$ onto $(\omega^\alpha, <)$,
(c) if $\eta \in \operatorname{inn}(\mathscr{T}) = \mathscr{T} \setminus \max(\mathscr{T})$, <u>then</u> $n < \omega \Rightarrow \eta^\frown\langle n\rangle \in \mathscr{T}$,
(d) $h \colon \mathscr{T} \to \alpha + 1$ satisfies that $\eta \triangleleft \nu \in \mathscr{T} \Rightarrow h(\eta) > h(\nu)$.
(we may write $h_{\mathbf{t}}(n)$ or $h(n, \mathbf{t}$.)
(e) $h(\langle\rangle) = \alpha$, $h(n) = 0 \Leftrightarrow \eta \in \max(T)$,
(f) if $\eta \in \operatorname{inn}(\mathscr{T})$ and $h(\eta) = \beta + 1$, <u>then</u> $(\forall n < \omega)(h(\eta^\frown\langle n\rangle)) = \beta$,
(g) if $\eta \in \operatorname{inn}(\mathscr{T})$ and $h(n) = \delta$ is a limit ordinal, <u>then</u> $\langle h(\eta)^\frown\langle n\rangle \colon n < \omega\rangle$ is an increasing sequence of ordinals with limit δ.

Fact 3.14. 1) If $\alpha \in [1, \omega_1)$, <u>then</u> $\mathscr{T}_{5,\alpha} \neq \emptyset$.

2) If $\alpha \in [1, \omega_1)$, $(\mathscr{T}, h, f) \in \mathbf{T}_5$ and $\eta \in \mathscr{T}$, <u>then</u> f maps $\max(\mathscr{T}^{\leq n})$ to a set of ordinals of order type $\omega^{h(\alpha)}$, where $\mathscr{T}^{[\geq \eta]} := \mathscr{T}[\leq \eta] := \{\nu \in T_\alpha \colon \eta \trianglelefteq \nu\}$.

Definition 3.15. 1) For $\mathbf{t} = (\mathscr{T}, h, f) \in \mathbf{T}_5$, let $\operatorname{nf}(\mathbf{t}) := \{\bar{\mathrm{D}} \colon \bar{\mathrm{D}} = \langle \mathrm{D}_\eta \colon \eta \in \operatorname{inn}(\mathrm{T})$, where D_η is a non-principal ultra-filter on $\omega\}$.

2) For $\bar{D} \in \operatorname{nf}(\mathbf{t})$, let $\bar{E}_{\bar{D}} = \langle E_{\bar{D},\eta} \colon \eta \in T_\alpha\rangle$ be defined by defining $E_{D,\eta}$ by induction on $h_\alpha(\eta)$ such that, it is an ultrafilter on $\max(T^{[\geq \eta]})$ and

(a) if $h(\eta) = 0$, then $E_{\bar{D},\eta}$ is the principal ultrafilter on $h(\eta)$,
(b) if $h(\eta) > 0$, then $E_{\bar{D},\eta} = \sum_{D_\eta} \langle E_{\bar{D},\eta^\frown\langle n\rangle} \colon n < \omega\rangle$, that is, $\{A \subseteq \max(\mathscr{T}^{[\geq \eta]}) \colon \{n < \omega \colon A \cap T^{[\geq \eta^\frown\langle n\rangle]} \in E_{\bar{D},\eta^\frown\langle n\rangle}\} \in D_n\}$.

Claim 3.16. *1) Assume* $\mathbf{t} \in \mathbf{T}_5$, $\eta \in \operatorname{inn}(\mathscr{T}_{\mathbf{t}}), \bar{\mathrm{D}} \in \operatorname{nf}(\mathscr{T})$ *and* $A \in E_{\bar{D},\eta}$, *then* $\operatorname{otp}(f''_{\mathbf{t}}(A)) = \omega^{h_{\mathbf{t}}(\eta)}$.

2) So $E_{\bar{D},\eta}$ *is not a* $J^1_{h_{\mathbf{t}}[\eta]}$*-ultrafilter.*

Proof. Should be clear. $\square_{3.16}$

Claim 3.17. *1) Assume* $D \in \beta(\mathbb{N}) \setminus \mathbb{N}$ *is a P-point. If* $f \colon \omega \to \operatorname{Ord}$, <u>*then*</u> *for some* $A \in D$, *either* $f''(A)$ *has order type* ω *or* 1 *(i.e., constant).*

2) D *is a P-point iff* D *is a* J_{ω^ν}*-ultrafilter.*

Proof. 1) Let $A \in D$ be such that $\beta = \operatorname{otp}(f''(A))$ is minimal. As D is an ultrafilter necessarily is a limit ordinal moreover additively indescomposable (i.e. $\alpha_1, \alpha_2 < \beta \Rightarrow \alpha_1 + \alpha_2 < \beta$). If $\beta = 1$, we are done, so assume not.

Hence, $\beta = \omega^\alpha$ for some α, so letting $\beta = f''(A)$, there is an increasing sequence $\langle \alpha_n \colon n < \omega\rangle$ of members of B, which is unbounded in $\sup(B)$. Clearly, for any $n < \omega$, $A_n := \{k \in A \colon f(k) \geq \alpha_n\}$ belongs to D and $\bigcap_{n<\omega} A_n = \emptyset$. As D is a P-point, for some $A_* \in D$ and we $k_m < \omega$ for $n < k$, have $A_* \subseteq A$ and $A_* \setminus A_m \subseteq k_m < \omega$, so it is finite, without loss of generality, k_n is increasing with n. Therefore, $B_* := f''(A_*)$ is a subset of B, hence of $\bigcup_{n<\omega} \beta_n$. Also, for each $n < \omega$, we have that $B_* \cap \beta_n \subseteq \{f(i) \colon i \in A_*, f(i) < \beta_n\}$, hence it is finite.

So, we are done.

2) Should be clear. $\square_{3.17}$

Claim 3.18. *Assume* $\mathbf{t} \in \mathbf{T}_5$, $\eta \in \mathrm{inn}(\mathscr{T}_{\mathbf{t}})$, $\bar{\mathrm{D}} \in \mathrm{nf}(\mathscr{T})$ *and every* D_η *is a* P*-point. If* $f\colon \max(\mathscr{T}_{\mathbf{t}}[\geq \eta]) \to \mathrm{Ord}$, *then for some* $A \in E_{\bar{D},\eta}$, *we have that* $\mathrm{otp}(f''(A)) \leq \omega^{h(\eta,\mathbf{t})}$.

Proof. We prove this by induction on $\alpha = h_{\mathbf{t}}(\eta)$. As $\eta \in \mathrm{inn}(\mathrm{T}_{\mathbf{t}})$, we have $\alpha \in [1, \omega_1)$. If $\alpha = 1$, this holds by 3.17, so assume $\alpha > 1$. For any $n < \omega$, choose $A_n \in E_{\bar{D},\eta^\frown\langle n\rangle}$ and $\beta_n = \mathrm{otp}(f''(A_n))$ is minimal, so again it is an ordinal power of ω (including $\omega^0 = 1$).

Now, for any $n < \omega$, let $\gamma_n := \sup(f''(A_n))$. We can also demand on A_n:

$(*)_1$ if $m < n$ and $\beta_m < \beta_n$, then $f''(A_n) \cap \gamma_m = \emptyset$.

$(*)_2$ if $\beta \in \{\beta_n : n < \omega\}$ and $u_\beta = \{n < \omega : \beta_n = \beta\}$ is infinite, let $\langle \varepsilon(n,\beta) : n < \omega\rangle$ be increasing with limit β, and demand:
- if $\beta_n = \beta$, then $f''(A_n) \cap \varepsilon(n,\beta) = \emptyset$, for any $n < \omega$.

Also, as D_η is a P-point, we have that

$(*)_3$ there is $A \in D_\eta$ such that $\{\beta_n : n \in A\}$ satisfies either $n \in A \Rightarrow \aleph_0 > |\{k \in A : \beta_k = \beta_n\}|$ or $\{\beta_n : n \in A\}$ is a singleton.

Recalling "natural sum of ordinals" we know:

$\otimes$ if $n < \omega$, $\mathrm{otp}(A_\ell) < \omega^\alpha$ for $\ell < n$, then $\mathrm{otp}\left(\bigcup_{\ell<n} A_n\right) < \omega^\alpha$.

The rest should be clear. $\square_{3.18}$

Conclusion 3.19. *If there is a* P*-point, then for every* $\alpha \in [1, \omega_1)$, *there is* D *such that:*

$(*)$ (a) $D \in \beta(\mathbb{N}) \setminus \mathbb{N}$,
(b) *there is a function* $f\colon \omega \to \omega^\alpha$ *such that* $A \in D$ *implies that* $\mathrm{otp}(f''(A)) = \omega^\alpha$ *(so* D *is not* J_{ω^α}*-ultrafilter),*
(c) *if* $f\colon \omega \to \mathrm{Ord}$, *then for some* $A \in D$, $\mathrm{otp}(f(A)) \leq \omega^\alpha$ *(so* D *is* J_{ω^γ}*-ultrafilter* $\gamma \in (\alpha, \omega_1)$*),*
(d) *if* $1 \leq \beta < \alpha$, *then there is a function* $f\colon \omega \to \omega^\beta$ *such that* $A \in D \Rightarrow \mathrm{otp}(f''(A)) = \omega^\beta$.

Proof. Let $\mathbf{t} \in \mathbf{T}_{3,\alpha}$, which exists by virtue of 3.14(1) and choose $\bar{D} = \langle D_\eta : \eta \in \mathrm{inn}(\mathrm{T}_{\mathbf{t}})\rangle$, such that each D_η is a P-point, which clearly exists. As D_η is an ultrafilter on $\max(T_{\mathbf{t}})$ and $\max(T_{\mathbf{t}}) = \aleph_0$, up to renaming, clause (a) holds.

Now, clause (b) holds by 3.16.

Clause (c) holds by 3.18. $\square_{3.19}$

Remark 3.20. 1) This solve (if phrased correctly). In general, we do not know whether, if α is limit, there is a J_{ω^α}-ultrafilter that is not a β-ultrafilter for some $\beta < \omega^\alpha$, even if CH or MA is assumed. We shall consider under weaker assumptions.

2) Starting with a J_{ω^α}-ultrafilter (instead of a P-point), equivalently J_{ω^2}-ultrafilter). We get such result, well sort out elaborate, if you recommend.

Discussion 3.21. 1) Claim 3.24 seem to became obsolete by 3.19.

2) We may work out how the existence of ω^α ultra-filter implies other. DEBT.

§ 3(D). The other direction: a try.

Definition 3.22. 1) For $\alpha \in [1, \omega_1)$, let $\mathbf{T}_6$ is the set of $\mathscr{T}$ such for some $\varrho = \mathrm{rt}(T)$, the root and,

- (a) $\mathscr{T} \subseteq \{\nu \in {}^{\omega>}\omega \colon \eta \trianglelefteq \nu\}$,
- (b) $\varrho \in T$,
- (c) if $\varrho \trianglelefteq \rho \trianglelefteq \nu$ and $\nu \in \mathscr{T}$, then $\rho \in \mathscr{T}$,
- (d) T is well founded, i.e., no infinite branches.

1A) We may use the tree $\mathscr{T}_\alpha$, and as in [She, 4.7 = k12], prove by quoting but see later.

2) We define $\mathrm{rk}_T \colon \mathscr{T} \to \mathrm{Ord}$ by (and $\mathrm{rk}(\mathscr{T}) = \mathrm{rk}(\mathrm{rt}(\mathscr{T}))$), $\mathrm{rk}_{\mathscr{T}}(\nu)$ is the minimal α such that:

- (a) $\rho \in \mathrm{suc}_T(\nu) \Rightarrow \mathrm{rk}_T(\rho) \subseteq \alpha$,
- (b) $\{n < \omega \colon \mathrm{rk}_T(\nu^\frown\langle n\rangle) \geq \alpha\}$ is finite.

3) $\mathrm{isp}(\mathscr{T}) \coloneqq \{\eta \in \mathscr{T} \colon \mathrm{suc}_{\mathscr{T}}(\eta) \text{ is infinite}\}$.

4) Let $\leq_6 \coloneqq \{(\mathscr{T}_1, \mathscr{T}_2) \colon \mathscr{T}_1 \supseteq \mathscr{T}_2$ are from $\mathbf{T}_6$, $\max(\mathscr{T}_1) \supseteq \max(\mathscr{T}_2)$ and $\mathrm{rk}(\mathscr{T}_1) = \mathrm{rk}(\mathscr{T}_2)\}$.

5) Let $\mathrm{otp}(\mathscr{T}) \coloneqq \mathrm{otp}(\max(\mathscr{T}), <_{\mathrm{lex}})$.

6) We say that $\mathscr{T}$ has *uniqueness*, if $\eta \in \mathscr{T} \Rightarrow |\mathrm{suc}_{\mathscr{T}}(\eta)| \in \{0, 1, \omega\}$.

Claim 3.23. *Assume $\mathscr{T} \in \mathbf{T}_6$ and $\mathbf{c} \colon \mathrm{isp}(\mathscr{T}) \to 2$. Then there are $\mathscr{T}_1, \mathscr{T}_2$ such that:*

- (a) *$\mathscr{T} \leq_6 \mathscr{T}_\ell$ for $\ell = 1, 2$,*
- (b) *$\mathbf{c} \restriction \mathrm{isp}(\mathscr{T}_\ell)$ is constantly ℓ (e.g. $\mathrm{isp}(T_\ell) = 0$),*
- (c) *$\mathscr{T}_\ell$ has uniqueness,*
- (d) *$\mathrm{rk}(\mathscr{T}) \leq \mathrm{rk}(\mathscr{T}_1) \oplus \mathrm{rk}(\mathscr{T}_2)$, where $\oplus$ is the natural sum (see 3.1).*

Proof. We prove this by induction on $\alpha = \mathrm{rk}(T)$:

Case $\alpha = 0$: obvious.

Case $\alpha > 0$:

Let $\eta \coloneqq \mathrm{rt}(\mathscr{T})$. Without loss of generality, we can assume that $\langle n\rangle \in \mathscr{T} \Rightarrow \mathrm{rk}_{\mathscr{T}}(\eta^\frown\langle n\rangle) > 0$ (otherwise, let $\nu \in \mathscr{T}$ be $\triangleleft$-minimal such that $\mathrm{rk}_T(\nu) = \mathrm{rk}(\mathscr{T})$ and $\mathrm{rt}(\mathscr{T})[\geq \nu]$).

Let $\bar\alpha = \langle \alpha_n \colon n < \omega\rangle$ be such that either for some β, $\alpha\beta + 1$, $\bigwedge_{n<\omega} \alpha_n = \beta$ or $\bar\alpha$ is increasing with limit α.

Let $u \coloneqq \{\eta \colon \eta^\frown\langle n\rangle \in \mathscr{T}\}$, so it is infinite. If $\mathrm{rk}(T) = \beta + 1$, so without loss of generality, $(\forall n \in u)(\mathrm{rk}_{\mathscr{T}}(\eta^\frown\langle n\rangle) = \beta)$, so if $n \in u$, then $\mathrm{rk}_{\mathscr{T}}(\eta^\frown\langle n\rangle) = \beta$.

On the other hand, if $\mathrm{rk}(\mathscr{T}) = \delta$ limit, without loss of generality $\langle \mathrm{rk}_T(\eta^\frown\langle n\rangle) \colon n \in u\rangle$ is increasing and $\mathrm{rk}_{\mathscr{T}}(\eta^\frown\langle n\rangle) = \beta_n$.

For each $n \in u$, let $\mathscr{T}_n \coloneqq \mathscr{T}[\geq \eta^\frown\langle n\rangle]$, so there are $(\mathscr{T}_{n,1}, \mathscr{T}_{n,2})$ by induction hypothesis for β_n. As we can replace η by any infinite set, without loss of generality, for $\ell = 1, 2$, we have that:

- $(*)$ $\langle \mathrm{rk}(\mathscr{T}_{n,f}) \colon n \in u\rangle$ is constant or increasing.

The rest should be clear. $\square_{\text{Theorem 3.23}}$

Claim 3.24. *[seem obsolete as above more is done] Assume* CH. *Let $\alpha < \omega_1$ be a limit ordinal. Then there is $D \in \beta(\mathbb{N}) \setminus \mathbb{N}$ such that:*

(a) *for any ordinal $\beta \in [\omega, \omega_1)$, D is a J_{ω^β}-ultrafilter iff $\beta \geq \alpha$,*
(b) *if $g\colon \omega \to \omega_1$, then for some $A \in D$, $g''(A)$ has order type $1, \omega, \omega^2$ or ω^α.*

Proof. For transparency, we shall construct an ultrafilter D on $\alpha(*) = \alpha_* = \omega^\alpha$ such that:

$(*)_1$ $A \subseteq \alpha_* \wedge \mathrm{otp}(A) < \alpha_* \Rightarrow A \in D$.

$(*)_2$ Let $\langle \alpha_n = \alpha(n)\colon n < \omega \rangle$ be increasing with limit α and $\alpha_n(n) = \omega^{\alpha(n)}$.

$(*)_3$ (a) $\mathrm{AP} = \{A \subseteq \alpha_*\colon \mathrm{otp}(A) = \alpha_*,\ A$ has no limit members (nec?)$\}$
(b) $\leq_{\mathrm{AP}}$ is the following partial order on AP: $A \leq_{\mathrm{AP}} B$ iff $A, B \in \mathrm{AP}$ and $A \setminus B$ is bounded.

$(*)_4$ if $\langle A_n\colon n < \omega \rangle$ is $\leq_{\mathrm{AP}}$-increasing, then it has an upper bound (used when $2^{\aleph_0} = \aleph_1$).

[Why? Choose by induction a bounded subset B_n of $\bigcap_{\ell<\omega} A_\ell$ of order type $\geq \alpha_m(*)$ such that $\min(B_n) > \sup(\bigcup\{B_\ell\colon \ell < n\})$. Easy to do and $B = \bigcup\{B_n\colon n < \omega\}$ is as required.]

$(*)_5$ if $A_1 \in \mathrm{AB}$ and $f\colon A_1 \to \{0,1\}$, then for some A_2, we have that:
- $\bullet_1$ $A_2 \subseteq A_1$,
- $\bullet_2$ $A_2 \in \mathrm{AP}$, i.e., $\mathrm{otp}(A_2) = \alpha_*$,
- $\bullet_3$ $f \restriction A_2$ is constant.

[Why? As α is a limit ordinal, ω^α is additively indescomposable, and also under natural sums.]

$(*)_6$ $\beta \neq \alpha$, $\alpha < \beta < \omega_1$ (yes! $2 \neq \beta$), $A_1 \in \mathrm{AP}$ and $g\colon A_1 \to \omega^\beta$, then for some A_2, we have that:
- $A_2 \subseteq A_1$,
- $\mathrm{otp}(A_2) = \alpha_* (= \omega^\alpha)$,
- $\mathrm{otp}(g''(A_2)) < \omega^\beta$.

[Why? If $\beta > \alpha$, then this holds by §3C, see xyz. If $\beta < \alpha$, then this holds by a claim below **r · r**.]

$(*)_7$ Assume CH. There is an ultrafilter as promised.

[Why? Let $\langle g_\varepsilon\colon \varepsilon < \omega_1 \rangle$ lists the set of functions from α_* into ω_1. We choose A_ε by induction on $\varepsilon < \omega_1$ such that:

(a) $A_\varepsilon \in \mathrm{AP}$,
(b) if $\zeta < \varepsilon$, then $A_\zeta \leq_{\mathrm{AP}} A_\varepsilon$,
(c) if $\varepsilon = \zeta + 1$, then $g''_\zeta(A_\varepsilon)$ has order type $\in \{1, \omega, \omega^2, \omega^\alpha\}$.

Why can carry the induction?

For $\varepsilon = 0$, let $A_\varepsilon := \alpha_*$.

When ε is a limit ordinal, use $(*)_4$.

For $\varepsilon = \zeta+1$, let A_ε be a subset of A_ζ of order type α_* such that $g''_\zeta(A_\varepsilon)$ has minimal order type. By α_* being indescomposable under natural sums, e.g., ordinal sums,

$\mathrm{otp}(g''_\zeta(A_\varepsilon))$ is of the form ω^γ for some $0 \le \gamma < \omega_1$ (for $\gamma = 0$, this is one). By Theorem 3.31 below, γ is as required in $(*)_5$.]

By $(*)_1$-$(*)_7$, we are done proving Theorem 3.24. $\square_{3.24}$

Discussion 3.25. 1) What about $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$?

2) Note that, if $\langle \alpha_n : n < \omega\rangle$ is increasing with limit α and $\alpha_0 = 0$, then D constructed as in 3.24 such that $D/\langle \alpha_{n+1} \setminus \alpha_n : n < \omega\rangle$ is a P-point, so 3.40 below does no apply.

Convention 3.26. (for the rest of §3D).

1) α a is countable ordinal.

2) $\alpha_* = \alpha(*) = \omega^\alpha$.

3) $\alpha_n(*) = \alpha_n$??.

4) $2 < \beta < \alpha$.

5) $\beta_* = \beta(*) = \omega^\beta$.

Definition 3.27. 1) $\mathbf{S}_{7,\alpha,\beta}$ be the set of pairs $\mathbf{s} = (\mathscr{T}, g) = (\mathbf{T}_\mathbf{s}, g_\mathbf{s})$ such that:

- (a) $\mathscr{T} \in \mathbf{T}_{G,\alpha}$ has uniqueness (so, $\mathrm{rt}_\mathbf{s}$, $\mathrm{suc}_\mathrm{T}(\eta)$ are $\mathrm{rt}_\mathbf{s}$, $\mathrm{suc}_\mathbf{s}(\eta)$ respectively) similarly in (2)),
- (b) $\mathrm{rk}(\mathscr{T}) = \alpha$,
- (c) $g\colon \min(\mathscr{T}) \to \omega_1$,
- (d) $\mathrm{rang}(g)$ has order type ω^β,
- (f) $u_\eta := u_{\mathscr{T},\eta} = \{k < \omega \colon \eta^\frown\langle k\rangle \in \mathscr{T}\}$.

2) Let $\mathbf{T}_{8,\alpha,\beta}$ be the set of triples $r = (T, g, h) = (T_\mathbf{r}, g_\mathbf{r}, h_\mathbf{r})$ such that:

- (a) $\mathscr{T} \in \mathbf{T}_6$ has uniqueness, so $u_\mathbf{r}, \mathrm{suc}_\mathbf{r}(\eta), u_{\mathbf{r},\eta}$ as above,
- (b) $h\colon T \to \beta + 1$,
- (c) if $\eta \in \mathrm{inn}(\mathscr{T})$, <u>then</u> $g_0(\eta) = \min(g''(\mathrm{suc}_\mathscr{T}(\eta')))$ and $g(\eta) = g_1(\eta) = \sup(g''(\mathrm{suc}_\mathscr{T}(\eta)))$,
- (d) if $\eta \in \mathscr{T}$, <u>then</u> $\mathrm{otp}(g''(\max(\mathscr{T})[\ge \eta])) = \omega^{h(\eta)}$,
- (e) if $\eta \in \mathrm{inn}(\mathscr{T})$, then
  - (α) $\langle g(\eta^\frown\langle n\rangle) \colon n \in u_n\rangle$ is constant or increasing,
  - (β) if it increasing, then:
    - $\bullet_1$ if $n < m$ are from u_η, then $g_1(\eta^\frown\langle m\rangle) < g_0(\eta)^\frown\langle n\rangle$,
    - $\bullet_2$ if it constant, then $\langle g_0(\nu) \colon \nu \in \mathrm{suc}_\mathscr{T}(\eta)\rangle$ is increasing with limit $g_1(\eta)$.
- (f) if $\eta \in \mathrm{inn}(\mathscr{T})$, then:
  - (α) $\langle h(\eta^\frown\langle n\rangle) \colon n \in u_n\rangle$ is constant or increasing,
  - (β) if it is increasing, then $h(\eta) = \sup\{h(\nu) \colon \nu \in \mathrm{suc}_\mathscr{T}(\eta)\}$ and $\langle g(\eta^\frown\langle n\rangle) \colon n \in u_n\rangle$ is increasing,
  - (γ) if it is constantly ζ, then $h(\eta) = \zeta + 1$.

Claim 3.28. *If $(T, g) \in \mathscr{T}_{7,\alpha,\beta}$, <u>then</u> for some $\mathbf{r}$ we have that:*

- (a) $\mathbf{r} \in \mathbf{S}_{8,\alpha,\beta}$,
- (b) $\mathscr{T} \le \mathscr{T}_\mathbf{r}$,
- (c) $g_\mathbf{r}(\max(\mathscr{T}_\mathbf{r})) = g \restriction \max(\mathscr{T})$.

Proof. We prove it by induction on $\mathrm{rk}(\mathscr{T})$ (as in §3C). $\square_{3.28}$

Claim 3.29. *If* $\mathbf{r} \in \mathbf{T}_{8,\alpha,\beta}$, *then:*

(a) $\text{otp}(g_{\mathbf{r}}(\mathscr{T}_{\mathbf{r}}))$ *is additively indescomposable and* $\leq \omega^{h(\text{tr}(T_{\mathbf{r}}))}$,
(b) *if* $\eta \in \mathscr{T}_{\mathbf{r}}$, *then* $\text{otp}(g_{\mathbf{r}}(\max(\mathscr{T}_{\mathbf{r}}[\geq \eta])))$ *is additively indescomposable and* $\leq \omega^{h(\eta,\mathbf{r})}$

Proof. By induction on $\text{rk}(\eta, T_{\mathbf{r}})$. $\square_{3.29}$

We need:

Claim 3.30. *If* $\mathbf{r} \in \mathbf{T}_{8,\alpha,\beta}$ *and* $\gamma < \alpha$, *then there is* $\mathscr{T}$ *such that:*

$\boxplus_{\mathbf{r},\mathscr{T}}$ (a) $\mathscr{T}_{\mathbf{r}} \subseteq \mathscr{T}$ *(so* $\text{rk}(\mathscr{T}) = \text{rk}(\mathscr{T}_{\mathbf{r}})$*),*
(b) $\text{otp}(g_{\mathbf{r}}(\max(\mathscr{T}))) \oplus h(\text{rt}_{\mathscr{T}}) \geq \gamma$,
(c) $\text{otp}(g_{\mathbf{r}}(\max(\mathscr{T}))) = \omega^2$,
(d) $g_{\mathbf{r}}(\max(\mathscr{T}))$ *is unbounded in* $g_{\mathbf{r}}(\max)(\mathscr{T}_{\mathbf{r}})$.

Proof. We try to prove it by induction on $\text{rk}(\mathscr{T}_{\mathbf{r}})$.

If $\beta = 2$, then we can let $\mathbf{r}_1 = 2$, so without loss of generality, $\beta >$ *incomp*.

Now for every $\nu \in \text{suc}_{\mathscr{T}}(\text{rk}_{\mathscr{T}})$ we can apply the induction hypothesis to $\mathbf{r}_\nu = \mathbf{r}[\nu] = \mathbf{r}[\geq \nu]$ naturally defined and it belongs to $\mathbf{T}_{8,g(\nu),h(\nu)}$ (maybe replace claim?), so by the induction hypothesis, there is some T_ν such that $\boxplus_{\mathbf{r}[\nu],\mathscr{T}}$ holds.

We shall use $T = \bigcup\{\mathscr{T}_\nu \colon \nu \in \text{suc}_{\mathbf{r}}(\text{rk}\mathscr{T})\} \cup \{\text{rk}_T\}$.

Now, we consider several cases:

<u>Case 1</u>: $\langle g_{\mathbf{r}}(\nu) \colon \nu \in \text{suc}_T(\eta)\rangle$ is constant.

The main point is why $g_{\mathbf{r}}(\max(\mathscr{T}))$ has order type $\subseteq \omega^\beta$, e.g., for every $\gamma < g_{\mathbf{r}}(\text{rk}_{\mathbf{r}})$, $(g_{\mathbf{r}}(\max(\mathscr{T}))) \cap \gamma$ has order type $< \omega^\beta$, which holds by the natural sum.

<u>Case 2</u>:not case 1, but $\langle g_{\mathbf{r}}(\nu) \colon \nu \in \text{suc}_{\mathbf{r}}(\text{rk}_{\mathbf{r}})\rangle$ is increasing. So,

$(*)$ (a) if $m < n$ are in $u_{\mathscr{T}_1,\text{rk}(\mathbf{r})}$, then

$$\sup(g(\max(T^{[\geq \text{rt}(\mathbf{r})^\frown\langle m\rangle]}))) < \min(g(\min(T^{[\geq \text{rt}(\mathbf{r})^\frown\langle n\rangle]}))),$$

(b) if $\nu \in \text{suc}_{\mathbf{r}}(\text{rk}_{\mathbf{r}})$, then $h_{\mathbf{r}}(\nu) < h_{\mathbf{r}}(\nu)$ clearly we are done.

$\square_{3.30}$

Claim 3.31. *If* $\alpha < \omega_1$ *is a limit ordinal and* $A_1 \subseteq \omega^\alpha$ *has order type* ω^α *and* $\beta_* < \omega^\alpha$ *and* $g_* \colon A_1 \to \beta$, <u>*then*</u> *for some* $A_2 \subseteq A_1$ *of order type* ω^α *we have* $\text{otp}(A_2) \in \{1, \omega, \omega^2\}$.

Proof. Without loss of generality we can assume that:

$(*)$ $A \subseteq A_1$ and $\text{otp}()A) = \omega^\alpha \Rightarrow \text{otp}(A) = \beta_*$.

Hence, $\beta_* = \omega^\beta$, $\beta < \alpha$ for some β.

Let $\mathbf{t}$ (see §3C) be such that:

$(*)_6$ (a) $\mathbf{t} \in \mathbf{T}_{5,\alpha}$,
(b) $\text{rk}(\mathscr{T}_{\mathbf{t}}) = \langle\rangle$,
(c) $p_{\mathbf{t}}$ is an isomorphism from $(\max(\mathscr{T}_{\mathbf{t}}), <_{\text{lex}})$ onto ω^α.

Now, we use 3.28 and 3.29. $\square_{3.31}$

§ 3(E). Relatives of nowhere dense ultrafilters.

Definition 3.32. Let $h\colon \omega \to (\omega \setminus \{0\})$, $\lim_n(h(n)) = \infty$, $\mathbf{k} = \langle \mathbf{k}_n : n < \omega \rangle$, $\mathbf{k}_n > 0$ and:

1) Let,

- $\mathbf{S}_{h,\mathbf{k},n} := \mathbf{S}_{\mathbf{k},n} = \prod_{\ell < h(n)} \mathbf{k}_\ell$,
- $\mathbf{S} := \mathbf{S}_{\mathbf{k}} := \mathbf{S}_{h,\mathbf{k}} := \bigcup_{n<\omega} S_{h,\mathbf{k},n}$,
- $\mathbf{A} := A_{\mathbf{k}} := A_{h,\mathbf{k}} := \prod_{n<\omega} \mathbf{S}_{\mathbf{k},n}$.

2) We say $S \subseteq \mathbf{S}_{\mathbf{k}}$ is 0-*small*, <u>when</u> for some $\nu \in \mathbf{A}_{\mathbf{k}}$ we have:

(∗) for every $m < \omega$, for all but finitely many $\eta \in S$ we have that

$$|\{\ell < \lg(\eta) \colon \eta(\ell) = \nu(\ell)\}| \leq m.$$

3) $S \subseteq S_{\mathbf{k}}$ is 1-*small*, when above $m = 1$.

4) $S \subseteq \mathbf{S}_{\mathbf{k}}$ is 2-*small* when there is some F such that:

(a) $\mathrm{dom}(F) = \bigcup_{n<\omega} \left(\prod_{\ell<n} \mathbf{k}_\ell\right)$,
(b) $F(\eta) < \mathbf{k}_{\lg(\eta)}$,
(c) as above in "0-small" replacing $\eta(\ell) = \nu(\ell)$ by $\eta(\ell) = F(\eta \restriction \ell)$.

5) $S \subseteq \mathbf{S}_{\mathbf{k}}$ is 3-*small* mean combining (4) and (3).

6) Let $J^\iota_{\mathbf{k}}$ is the family of subsets of $\mathbf{S}_{\mathbf{k}}$ which are ι-small $S \subseteq \mathbf{S}_{\mathbf{k}}$ (this is a semi-ideal; that is, $A \subseteq^* B \subseteq \mathbf{S}_{\mathbf{k}} \cap B \in J \Rightarrow A \in J$).

7) Let $\mathrm{Id}^{J^\iota}_{\mathbf{k}}$ the ideal that $J^\iota_{\mathbf{k}}$ generates.

8) Let We define $J^\iota_{\mathbf{k},\mathscr{F}}$ where $\mathscr{F} \subseteq \prod_{n<\omega} \mathbf{k}_n$ similarly, but:

(∗) for $\iota = 0, 2$ instead of $\nu \in \prod_\ell k_\ell$ we have $\Omega \subseteq \prod_\ell \mathbf{k}_\ell$ (closed without loss of generality) such that ...

Definition 3.33. 1) D is a semi-filter on ω <u>iff</u> $D \subseteq \mathcal{P}(\omega)$, $\omega \in D$, $A \subseteq^* B \wedge A \subseteq \omega \wedge B \in D \Rightarrow A \in D$.

2) A filter D on ω is a J-filter <u>when</u> (J is a semi-filter, here on ω and): for every $f\colon \omega \to \omega$ for some $A \in D$ we have $f''(A) = \{f(n) \colon n \in A\} \in J$.

Observation 3.34. *For an ultrafilter D, D is J^ι_k-ultrafilter <u>iff</u> it is a $J^\iota_{\mathbf{k}}$-filter and is a ultrafilter.*

Definition 3.35. 1) $J \subseteq \mathcal{P}(\omega)$ is a semi-ideal when $A \subseteq B \in J \Rightarrow A \in J$.

Definition 3.36. Let D a non-principal ultrafilter on ω. Then,

1) If D is $J^\iota_{\mathbf{k}}$-ultrafilter, <u>then</u> D is a nowhere dense ultrafilter.

2) The natural implication among the $\{J^\iota_{\mathbf{k}} \colon \iota, \mathbf{k}\}$ holds.

Claim 3.37. *We can forces "there is no $J^0_{\mathbf{k}_1}$-ultrafilter but, there is a $J^0_{\mathbf{k}_2}$-ultrafilter".*

Remark 3.38. 1) See Goldstern-Shelah [GS93], Kellner - Shelah [KS09], [KS12] for the exsitence of a $J^0_{\mathbf{k}_2}$-ultrafilter and §3 (or like [She98b]) for the "no $J^0_{\mathbf{k}_4}$-ultrafilter". Fill?

Proof. As in §2B. Details? $\square_{3.38}$

§ 3(F). **Further comments.**

Claim 3.39. *Let D be an ultrafilter as constructed in* [She]*. Then D is a Van-Douwen ultrafilter (see end of* [She98b]*).*

Proof. Debt. $\square_{3.39}$

Claim 3.40. *Assume $D \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and no $D' \leq_{\mathrm{RK}} D$ is a a P-point. Then for every $n \geq 1$, D is not a $J_{(\omega^n)}$-ultrafilter.*

Proof. By induction on $n < \omega$, for all D-s (or D/E.) For $n = 1$, $f = \mathrm{id}_\omega$ is a witness. For $n = k + 1 \geq 1$, there is a partition $\bar{A} = \langle A_n : n < \omega\rangle$ to infinite sets such that $(\forall A \in D)(\exists^\infty n < \omega)(A_n \cap A$ is infinite) because D is not a P-point. By the induction hypothesis, there is $h_g : \omega \to \omega^k$, a witness for "$D/\bar{A}$ is not $J_{(\omega^k)}$-ultrafilter". Now define $h : \omega \to \omega^n$ by: if $i \in A_n$, then $h(i) := \omega h_0(n) + i$. Finally, check. $\square_{3.40}$

Claim 3.41. *The forcing $\mathbb{Q}_{\mathbf{i}}$ from §2, add a set $\underset{\sim}{A} \subseteq \omega$ which divide every old $A \subseteq [\omega]^{\aleph_0}$.*

Proof. Let $\underset{\sim}{A} := \{n < \omega : \prod\{\eta_m(0) : m \leq n\} = 1\}$. $\square_{3.41}$

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