Classification over a predicate - the general case Part I - structure theory

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Abstract

We begin a systematic development of structure theory for a first order theory, which is stable over a monadic predicate. We show that stability over a predicate implies quantifier free definability of types over stable sets, introduce an independence notion and explore its properties, prove stable amalgamation results, and show that every type over a model, orthogonal to the predicate, is generically stable.

1 Introduction

The classical Classification Theory [She90] deals with a first order theory T, how complicated its models can be, and to which extent they can be characterized by cardinal invariants.

For algebraically closed fields, or for divisible abelian groups, there is such a structure theory. In general, there is a division into theories that have a structure and such for which we have a non-structure theorem.

Consider now vector spaces over a field. Do we have a structure theory? It depends on how you ask the question: We have a structure theory for the vector space over the field, but not necessarily for the field.

Problem: Classify pairs (T, P) where T is a first order theory and P is a monadic predicate. We want to know how much $M \models T$ is determined by $M|P^M$. So in case T is the two-sorted theory of a vector space over a field, and P is the predicate for the field, we ask how much we can know about a vector space over a field F once F is flixed (obviously, we know quite a bit, especially if the cardinality of the vector space is fixed).

Although at the first glance the problem above may appear close to classical (first order) model theory, this context actually exhibits behavior which

*Publication no. 322 on Shelah's list of publications. Shelah thanks the Israel Science Foundation (Grants 1053/11, 1838/19), and the European Research Council (Grant 338821) for partial support of this research.

†Research partially supported by Marie Sklodowska Curie CIG 321915 "ModStabBan". Usvyatsov thanks the Austrian Science Foundation (FWF), projects P33895 and P33420, and the Portuguese Science Foundation (FCT), Development grant IF/01726/2012, for their partial support at different stages of this research.

is more similar to that of some non-elementary classes (classes of models of a sentence in an infinitary logic, or abstract elementary classes). See e.g. Hart and Shelah [HS90]. An intuitive "explanation" for this is that fixing P is similar to insisting on omitting a certain type (the type enlarging P), which immediately puts one in a non-elementary context.

Much work has already been done on classification theory over P. Relative categoricity for particular theories was investigated by e.g. Hodges et al [Hod99, HY09, Hod02]; countable categoricity over P was studied by Pillay [Pil83]. Pillay and the first author laid the foundations for the study of stability and related properties in this context in [PS85]. Here we are going to continue their investigation and build upon their results.

In [She86] the first author proved an analogue of Morley's Theorem over P under the assumption of "no two-cardinal models", which means that for all $M \models T$, $|P^M| = |M|$. However, this assumption is very strong. Even for the the case of uncountable categoricity (a notion that we discuss below in Definition 1.3), it would be nice to be able to prove an analogue of Morley's Theorem without making this assumotion it *a priori*. Furthermore, the "no two cardinal model" assumption does not hold in many natural examples that should "morally" be quite tame. For example, the general theory developed in [She86] does not even cover the example of a vector space over a field, already mentioned briefly above (Example 1.2). Apparently, from the point of view of model theory, this example is not that easy (but is quite instructive).

A much more interesting example that we hope will eventually be included in our treatment is the theory of so-called "Zilber's field", more precisely, the theory of exponentially closed fields of characteristic 0 [Zil05, KZ14, Hen14]. In our context, one considers the theory of an exponentially closed field (F, e^z) over P, which is the kernel of the exponential function e^z. For example, it follows from the results in [KZ14] that this theory is stable over P.

Another natural example to consider in this setting is the theory of algebraically closed field of characteristic 0 with a generic automorphism σ, where P is the fixed field of σ. This theory has been studied by Chatzidakis in a recent preprint [Cha20]. This example is different than the ones mentioned above, since in this case P is model theoretically "tame" (it is pseudo-finite), and the full theory T is also very well understood (it is simple, has QE, etc). Nevertheless, considering it in this new framework, offers further insight.

Some results in [Cha20] can be derived directly from the more general analysis obtained in our paper. For example, Chatzidakis proves (Theorem 3.14 in [Cha20]) the existence of a κ-prime κ-atomic model over an algebraically closed difference field of characteristic 0 with a κ-saturated and pseudo-finite P-part (for κ uncountable or $\aleph_\varepsilon$). We prove an analogous result in our general context (see Theorem 6.5 and Corollary 6.6 thereafter) for the case that the P-part is saturated (of cardinality κ). So in particular, the case $\kappa = \aleph_\varepsilon$ is not covered. A more nuanced analysis, which would imply stronger results closer to Theorem 3.14 in [Cha20] goes beyond the scope of this article. This will be investigated in a future work.

Let us make the above discussion a bit more precise.

One rough measure of complexity of a theory is (as usual) the number of its non-isomorphic models in different cardinalities:

Definition 1.1. (i) $I(\lambda, N)$ = cardinality of $\{M/_{\cong_N} : M \models T, M|_P = N, |M| = \lambda\}$, where $\cong_N$ means isomorphic over N and $M/_{\cong_N}$ denotes the isomorphism class.

(ii) $I(\lambda, \mu) = \sup\{I(\lambda, N) : |N| = \mu\}$.

(iii) $I(\lambda) =: I(\lambda, \lambda)$.

Example 1.2. (Back to vector spaces over a field). Let T be the theory of two-sorted models (V, F) where V is a vector space over the field F, and let P be a predicate for the field F. If the cardinality of F is $\aleph_\alpha$, then $I(\aleph_\alpha, F) = |\omega + \alpha|$ and for $\beta > \alpha$, $I(\aleph_\beta, F) = 1$.

So we want to divide the pairs (T, P) according to how much freedom we have to determine M knowing $M|P$.

Definition 1.3. (i) (T, P) is *categorical in* (λ_1, λ_2) when the following holds: If $M_1|P = M_2|P, M_1$ and M_2 are models of T and $|M_l| = \lambda_1$, $|P^{M_l}| = \lambda_2$ for $l = 1, 2$, then M_1 and M_2 are isomorphic over P^{M_l}.

(ii) We write *categorical in* λ instead of (λ, λ).

(iii) We say *totally categorical* if (ii) holds for all λ.

For our purpose, categorical pairs are the simplest. However, we will deal here with a more general context of *stability* over P. For example, the class of vector spaces over a field is not categorical in λ the sense of the definition above; still, it is *almost* categorical, and it would obviously be desirable to develop a general theory that covers this example.

Recall that much of the work in classification theory follows the following general recipe. First, we assume that the theory T (or, more generally, the class of models under investigation) has a particular "bad" model theoretic property: e.g., is unstable. Under this assumption, we prove a non-structure theorem: e.g., T has many non-isomorphic models of some big enough cardinality λ (normally, in many, if not all, such cardinalities). Since we are ultimately interested in the "good case" – for example, if we are trying to prove an analogue of Morley's Categoricity Theorem, we only care about theories (or classes) with few models – we may assume from now on that T falls on the "right" side of the dividing line: e.g., is stable. This way we can use good properties of stability in order to investigate properties of our class further.

As perhaps should be clear from the title, in this paper we focus on the development of *structure theory.* In particular, we do not prove any new non-structure results here. However, we do follow "recipe" described above. That is, we recall (mostly from [PS85] and [She86]) that certain "bad" properties (e.g. instability) imply non-structure, by which we normally mean many non-isomorphic models over P – sometimes only in a forcing extension of the universe, or under mild set-theoretic assumptions. Non-structure theorems become quite hard in this context, so at the moment we are happy with just consistency results. Since we are ultimately interested in absolute properties (e.g., stability, categoricity), there does not seem to be much loss in this approach (although non-structure results in ZFC are definitely on our to-do list). Then we restrict our attention to theories on the "good side" of all the dividing lines, and focus on proving structure results for this case. There is, of course, also a need for better and stronger non-structure theorems, that would "justify" restricting onelsef to even nicer contexts; this is a topic for future work.

This paper is organized as follows.

In Section 2 we recall non-structure results from [PS85] and make our most basic working assumptions, e..g., every type over P is definable.

Section 3 is devoted to some of the less obvious consequences of the assumptions mentioned above (already discussed to a large extent in [PS85]). In particular, we explain why we may assume that T has quantifier elimination.

In Section 4 we revisit some basic stability theory over P originally developed in [PS85] and [She86] and take it further. We introduce some of the major players necessary for analyzing models over their P-part: complete sets and "good" types, which we call *-types; these are types "orthogonal to P", that is, types, realizing which do not increase the P-part. Based on these notions, we define the relevant notion of stability: a set A is called stable (over P) if there are "few" *-types over all sets elementarily equivalent to A. This concept and many of its basic properties already appear in the previous works mentioned above. However, it is our goal to make this paper reasonably self-contained, so we include all the definitions as well as details of proofs.

Section 5 revisits a notion of rank (already discussed in [PS85, She86]) for *-types that captures stability over P.

Section 6 is devoted to the existence of "small" (atomic, primary) saturated models over stable sets with a saturated P-part.

In Section 7 we make an additional working assumption: every model of T is stable over P (this hypothesis is justified by a non-structure result proved in [She86]). We then use stability of models together with the existence of primary models (the results of section 6) in order to obtain quantifier-free internal definitions for *-types. This is, in a sense, the main technical result of this paper, on which the rest of our analysis is built. In particular, we immediately conclude that *-types over stable sets are definable (in $\mathfrak{C}$).

In section 8 we define stationarization – a version of non-forking independence appropriate for our context – and examine its basic properties.

Finally, in Section 9 we prove the main structure results of this article. In particular, we establish stable amalgamation for models followed by a similar result for types over stable sets, show symmetry of independence over models, and draw several conclusions.

One basic corollary of our analysis is that if every model is stable over P (as mentioned earlier, a non-structure result from [She86] implies that this is a natural assumption in our context; in addition, it holds in many interesting examples), then every $*$-type over a model is generically stable. In particular, we conclude that over models, independence derived from stationarization coincides with non-forking.

Acknowledgements

The authors are grateful to Mariana Vicaria for a careful reading, and for numerous comments, questions, suggestions, and corrections. We also thank Zoé Chatzidakis for helpful comments, clarifications, and instructive questions, which led to improving the readability and clarity of this paper.

The second author thanks Jakob Kellner for several conversations, motivating the authors to expand the background and the introductory sections, hopefully making them more clear and accessible to non-experts, and to eventually finalize the paper. Usvyatsov is also thankful to Alf Onshuus for inspiring conversations at early stages of the writing of this paper, and to Jonathan Kirby for bringing the paper [KZ14] to the authors' attention.

2 The Gross Non-structure Cases

Convention 2.1. *Let T be a complete first order theory, P a monadic predicate in its vocabulary.*

Let $\mathfrak{C}$ be the monster model of T. From now on, we assume that all models of T are elementary submodels of $\mathfrak{C}$, and all sets are subsets of $\mathfrak{C}$.

For $M \models T$, we denote by $M|_P$ the set P^M viewed as a substructure of M. Similarly, for a subset $A \subseteq M$, we denote by $M|_A$ the substructure of M with universe A. We write $A \equiv B$ if $Th(\mathfrak{C}|_A) = Th(\mathfrak{C}|_B)$.

We also denote $T^P = Th(\mathfrak{C}|_P)$. For a set A, we denote $P^A = A \cap P^{\mathfrak{C}}$.

Our first dividing line concerns the connection between subsets of $P^{\mathfrak{C}}$ that are 0-definable externally (in $\mathfrak{C}$) and internally (in $\mathfrak{C}|_P$, that is, in $P^{\mathfrak{C}}$ viewed as a substructure of $\mathfrak{C}$). One direction of this correspondence is straightforward:

Remark 2.2. *For every formula $\theta(\bar{x})$ there exists a formula $\theta^P(\bar{x})$ such that for every $\bar{b} \in P$ we have $\mathfrak{C}|_P \models \theta(\bar{b})$ if and only $\mathfrak{C} \models \theta^P(\bar{b})$. Moreover, for every $M \models T$ we have $M|_P \models \theta(\bar{b})$ if and only $M \models \theta^P(\bar{b})$*

Proof: By induction on θ, by replacing quantifiers in θ by quantifiers restricted to P (e.g., $\exists y \in P$). ■

Corollary 2.3. *Let A be a set.*

(i) *If A is a model (so $A \prec \mathfrak{C}$), then $(A|_P =)\mathfrak{C}|_{P^A} \prec \mathfrak{C}|_P$.*

(ii) *Every (partial) T^P-type $q(\bar{x})$ over P^A is equivalent to a T-type $q^P(\bar{x})$ (over P^A) with $[\bar{x} \subseteq P] \in q^P$.*

In particular, if A is a λ-saturated model, then $A|_P = \mathfrak{C}|_{P^A}$ is a λ-saturated model of T^P.

Proof:

(i) Assume $\mathfrak{C}|_P \models \theta(\bar{b})$ with $\bar{b} \in P^A$. Then (see Remark 2.2) $\mathfrak{C} \models \theta^P(\bar{b})$, hence $A \models \theta^P(\bar{b})$. Again, by Remark 2.2, $A|_P \models \theta(\bar{b})$.

(ii) Let $\theta(\bar{x}, \bar{b})$ be a formula such that $A|_P \models \exists \bar{x} \theta(\bar{x}, \bar{b})$. Then

$$A \models \exists \bar{x} \in P\ \theta^P(\bar{x}, \bar{b})$$

Hence given a T^P-type $q(\bar{x})$ over P^A, the collection $q^P = \{\theta^P(\bar{x}) \colon \theta(\bar{x}) \in q\}$ is indeed a T-type, and the rest should be clear.

$\blacksquare$

On the other hand, it is not clear (and, in general, not true) that any externally definable subset of $P^{\mathfrak{C}}$ is definable internally in $\mathfrak{C}|_P$.

Question 2.4. *If $M \models T$, and $\psi(\bar{x})$ is a relation on P^M which is first order definable in M, is ψ definable in $M|_P$, possibly with parameters?*

If the answer to this question is "no", then, as shown in [PS85], for every $\lambda \geq |T|$, $I(\lambda) \geq Ded(\lambda)$, where $Ded(\lambda) = \sup\{|I| : I$ is a linear order with dense subset of power $\lambda\}$. The proof relies on earlier papers by Chang, Makkai, Reyes and Shelah.

Following the "general recipe of Classification Theory" outlined in the introduction, we therefore assume that the answer to the Question 2.4 is "yes".

Furthermore we expand T by the necessary individual constants (maybe working in $\mathfrak{C}^{eq}$) in order to assume that all such relations are parameter-free definable in $\mathfrak{C}|_P$. Note that we only have to add elements in $dcl(P^{\mathfrak{C}})$ inside $\mathfrak{C}^{eq}$.

Specifically, if $\mathfrak{C} \models (\exists \bar{y} \in P)(\forall \bar{x} \in P)(\psi(\bar{x}) = \theta^P(\bar{x}, \bar{y}))$, we define an equivalence relation on tuples of sort $\bar{y}$ as follows:

$$\bar{y}_1 \equiv \bar{y}_2 \iff P(\bar{y}_1) \wedge P(\bar{y}_2) \wedge \forall \bar{x} \left[\theta^P(\bar{x}, y_1) \leftrightarrow \theta^P(\bar{x}, y_2)\right]$$

and expand $\mathfrak{C}^{eq}$ with names for the relevant equivalence classes; that is, for each $\psi(\bar{x})$ as above, expand the language by a constant $[\bar{y}]_\psi$ for the class of a tuple $\bar{b}$ for which $\theta^P(\bar{x}, \bar{b})$ is equivalent to $\psi(\bar{x})$. Clearly, this does not increase the cardinality of the language.

As a result, given a formula $\psi(\bar{x})$ defining a relation on P and a formula $\theta(\bar{x}, \bar{b})$ defining this relation in $\mathfrak{C}|_P$, we have that e.g. the formula $\exists \bar{y}\,([\bar{y}] = [\bar{y}]_\psi) \wedge \theta(\bar{x}, \bar{y})$ defines the same relation without parameters. Note that we add the new constants to P and to the language of T^P.

Remark 2.5. See [PS85] for the number of such expansions for models of T. If the new theory is T^+, note that $I_T(\lambda, N) = \sup\{I_{T^+}(\lambda, N^+) : N^+$ an expansion of N as described above$\}$. In particular $I_T(\lambda, \mu)^{|T|} \geq I_T(\lambda, \mu) \geq I_{T^+}(\lambda, \mu)$, so our non-structure results are not affected.

Question 2.6. *For $\psi = \psi(\bar{x}, \bar{y})$, $\bar{c} \in \mathfrak{C}$, is $tp_\psi(\bar{c}/P^{\mathfrak{C}})$ definable?*

Recall that $tp_\psi(\bar{c}/A) = \{\psi(\bar{x}, \bar{a}) : \bar{a} \in A,\ \mathfrak{C} \models \psi(\bar{c}, \bar{a})\}$ and $p = tp_\psi(\bar{c}/A)$ is definable if it is definable by some $\theta(\bar{y}, \bar{d})$ with $\bar{d} \in A$, which means that for every $\bar{a} \in A$ we have $\psi(\bar{x}, \bar{a}) \in p \iff \mathfrak{C} \models \theta(\bar{a}, \bar{d})$. We restrict ourselves to ψ-types in order to be able to use compactness arguments.

Again, it is shown in [PS85] that if the answer to this question is "no", then $I(\lambda) \geq Ded(\lambda)$. The two questions are closely related (the second one is a version of the first one with external parameters), and the proofs are similar, and, in fact, both work for pseudo-elementary classes.

In conclusion, for the rest of the paper we make the following assumptions:

Hypothesis 2.7. *(Hypothesis 1)*
T is a complete first order theory , P a monadic predicate in the language of T, $\mathfrak{C}$ is the monster model of T such that

(i) *Subsets $P^{\mathfrak{C}}$ that are 0-definable (in $\mathfrak{C}$), are already 0-definable in $\mathfrak{C}|_P$.*

(ii) *Every type over $P^{\mathfrak{C}}$ is definable.*

Later we shall add an additional clause to this hypothesis regarding quantifier elimination of T; see Hypothesis 4.14.

For simplicity we also make the set theoretic assumption that for arbitrarily large λ, we have $\lambda^{<\lambda} = \lambda$ (in particular, T has arbitrarily large saturated models). Note that for any conclusion we draw which says something about every λ, this hypothesis can be eliminated.

3 Internal and external definability

In this section we observe several basic consequences of Hypothesis 1.

First we reformulate the assumption that every type over $P^{\mathfrak{C}}$ is definable in the following obviously equivalent way: every externally definable subset of $P^{\mathfrak{C}}$ is also internally definable in $\mathfrak{C}|_P$ (with parameters in $P^{\mathfrak{C}}$).

Observation 3.1. (i) *Let $\varphi(\bar{x}, \bar{a})$ define (in $\mathfrak{C}$) a subset of $P^{\mathfrak{C}}$. Then there exists a formula $\hat{\theta}(\bar{x}, \bar{b})$ with $\bar{b} \in P^{\mathfrak{C}}$ that defines the same set. Moreover, there exists a formula $\theta(\bar{x}, \bar{b})$ that defines the same set in $\mathfrak{C}|_P$.*

(ii) *Furthermore, for any $A \subseteq P^{\mathfrak{C}}$ such that $\mathrm{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over A, $\theta(\bar{x}, \bar{b})$ in the previous clause can be chosen so that $\bar{b} \in P^A$.*

Proof: By Hypothesis 2.7(ii), $\mathrm{tp}(a/P^{\mathfrak{C}})$ is definable. Hence there is $\hat{\theta}(\bar{x}, \bar{b})$ with $\bar{b} \in P^{\mathfrak{C}}$ such that $\mathfrak{C} \models \forall \bar{x} \left(\hat{\theta}(\bar{x}, \bar{b}) \longleftrightarrow \varphi(\bar{x}, \bar{a})\right)$. By Hypothesis 2.7(i), there exists $\theta(\bar{x}, \bar{y})$ such that for every $\bar{c}, \bar{d} \in P^{\mathfrak{C}}$ we have $\mathfrak{C} \models \hat{\theta}(\bar{c}, \bar{d})$ if and only if $\mathfrak{C}|_P \models \theta(\bar{c}, \bar{d})$. Clearly $\theta(\bar{x}, \bar{b})$ is as required (in both clauses of the Observation). $\blacksquare$

Corollary 3.2. (i) *Let $p(\bar{x})$ be a (partial) type over $\mathfrak{C}$ such that $P(\bar{x}) \in p$. Then p is equivalent to a T^P-type p' over $P^{\mathfrak{C}}$.*

(ii) *Let A be a set. Assume that for every $\bar{a} \in A$ the type $\mathrm{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over $P^A = P^{\mathfrak{C}} \cap A$. Let p be a (partial) type over A with $P(x) \in p$. Then p is equivalent to a (T-)type $\hat{p}$ over P^A, and to a T^P-type p' over P^A.*

The following corollary again follows immediately, but is very useful.

Observation 3.3. *Let A be a set such that for every $\bar{a} \in A$ the type $\mathrm{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over P^A. Let λ be a cardinal.*

(i) *Assume that $\mathfrak{C}|_{P^A}$ is a λ-saturated (or just λ-compact) model of T^P. Then every T-type of size $< \lambda$ over P^A which is finitely satisfiable in P^A is realized in P^A.*

In fact, it is enough to assume that any T^P-type of size $< \lambda$ over P^A which is finitely satisfiable in P^A is realized in P^A (so $\mathfrak{C}|_{P^A}$ does not need to be itself a model of T^P).

(ii) *If P^A satisfies the conclusion of the clause (i), then every type p of size $< \lambda$ over A with $P(x) \in p$ is realized in P^A.*

Proof: For clause (i), let p is a T-type of size $< \lambda$ over P^A finitely satisfiable in P^A. By Corollary 3.2(ii), it is equivalent to a T^P-type over P^A, and so is realized in P^A. For clause (ii), note that, again by Corollary 3.2(ii), such a type p is equivalent to a type of size $< \lambda$ over P^A (and clearly it is finitely satisfiable in P^A). ■

Let us formulate a simple version of the last Observation, which will be particularly useful:

Corollary 3.4. *Let A be a set such that for every $\bar{a} \in A$ the type $\mathrm{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over P^A. Let λ be a cardinal, and assume that $\mathfrak{C}|_{P^A}$ is a λ-saturated model of T^P. Then every T-type p of size $< \lambda$ over A with $P(x) \in p$ is realized in P^A.*

Remark 3.5. In the rest of the paper, we will prove claims about sets and models under the assumption that their P-part is λ-saturated. As a matter of fact, all we'll need in these claims is the conclusion of Observation 3.3 (i). So in particular assuming that P^A is a "λ-compact subset" of $\mathfrak{C}$ (every T-type of size $< \lambda$ which is finitely satisfiable in P^A is realized in P^A) is enough. But we will not focus on this.

A perhaps less obvious consequence of Hypothesis 1 is that T can be Morleyrized without much additional cost.

In classical Classification Theory (classifying first order theories), assuming that T has quantifier elimination often comes without much cost (e.g., by Morleyzation). However, in our case, this assumption may appear less harmless. Indeed, in general, if $\mathfrak{C}$ is expanded with new predicates, then so is

$\mathfrak{C}|_P$, potentially enhancing the theory T^P, hence changing the original classification problem. Fortunately, since Morleyzation of T does not add new 0-definable sets to $\mathfrak{C}$, if Hypothesis 1 is true, no new 0-definable sets are added to $\mathfrak{C}|_P$ either. In other words:

Observation 3.6. *Let T' is the Morleyzation of T, and let $\mathfrak{C}'$ be the expanded monster model (of T'). Let T'^P be the theory of $\mathfrak{C}'|_P$. Then T'^P is a trivial expansion of the Morleyzation of T^P (that is, it is an expansion that adds no new 0-definable sets).*

Proof: On the one hand, if $\theta(\bar{x})$ be a T_P-formula, let $\theta^P(\bar{x})$ be the formula that defines the same subset of $P^{\mathfrak{C}}$ in $\mathfrak{C}$ (as in Remark 2.2). Let $R(\bar{x})$ be a T'-predicate equivalent (modulo T') to $\theta^P(\bar{x})$; then it also defines the same subset of $\mathfrak{C}'_P$, hence is equivalent to $\theta(\bar{x})$ modulo T'^P. In conclusion, T'_P *expands* the Morleyzation of T^P (this part is, of course, always true).

On the other hand, let $R(\bar{x})$ be a new predicate in T'^P. Then it is equivalent modulo T' to a T-formula

$$\bar{x} \subseteq P \bigwedge \varphi(\bar{x})$$

By Hypothesis 1, there is a T^P-formula $\theta(\bar{x})$ (without parameters) defining the same subset of $P^{\mathfrak{C}}$. Therefore any T'_P formula $\theta'(\bar{x})$ is equivalent to a T_P formula, as required. ■

We conclude this section with a few trivial consequences of quantifier elimination in T. Some more interesting ones will be discussed in the next section (e.g., Lemma 4.5).

First we note that some of the observations above become trivially true for any substructure of $\mathfrak{C}$ (which in our case just means a set containing all the individual constants), for example:

Remark 3.7. *Assume that T has QE and let A be a substructure of $\mathfrak{C}$.*

(i) *Any externally definable subset of A is definable internally in $\mathfrak{C}|_A$. A T-type over A is also a type in $\mathfrak{C}|_A$.*

(ii) *If $\mathfrak{C}|_A$ is λ-compact, that is, it is a λ-compact model of the theory $Th(\mathfrak{C}|_A)$. Then every T-type of size $< \lambda$ which is finitely satisfiable in A is realized in A.*

Quantifier elimination also adds to the understanding of T^P:

Remark 3.8. *Assume that T has QE and let A be a substructure of $\mathfrak{C}$.*

(i) *T^P has QE.*

(ii) *Every subset of $P^{\mathfrak{C}}$ definable in $\mathfrak{C}$ (with or without parameters) is definable by the same quantifier free formula (with or without parameters, respectively) both in $\mathfrak{C}$ and in $\mathfrak{C}|_P$.*

(iii) *Corollary 3.2(ii) can be strengthened to: both $\hat{p}$ and p' are equivalent to the same quantifier free type.*

From now on, to simplify the notation, when no confusion should arise, we will write P for $P^{\mathfrak{C}}$. Also, for a set A, we will often denote by A both the set and the substructure of $\mathfrak{C}$ with universe A. So for example, when we write that $A \cap P^{\mathfrak{C}}$ is λ-saturated, or just that $A \cap P$ is λ-saturated, we mean that the substructure $\mathfrak{C}|_{A \cap P^{\mathfrak{C}}}$ is a λ-saturated model of the appropriate theory.

4 Completeness and relevant types

In trying to reconstruct M from $M|P^M$ one needs to work with sets A satisfying $P^M \subseteq A \subseteq M$. Such A have the following property (which under a certain assumption on saturation of P^M characterizes such sets; see Proposition 4.13 below), that can be viewed as an analogue to Tarski-Vaught Criterion for being an elementary submodel:

Definition 4.1. $A \subseteq \mathfrak{C}$ is *complete* if for every formula $\psi(\bar{x}, \bar{y})$ and $\bar{b} \subseteq A$, $\models (\exists \bar{x} \in P)\psi(\bar{x}, \bar{b})$ implies $(\exists \bar{a} \subseteq P \cap A) \models \psi(\bar{a}, \bar{b})$.

The following useful characterization offers a different understanding of the notion of completeness:

Observation 4.2. *A set A is complete if and only if for every $\bar{a} \subseteq A$ and $\phi(\bar{x}, \bar{y})$ the ϕ-type $tp_\phi(\bar{a}/P^{\mathfrak{C}})$ is definable over $A \cap P^{\mathfrak{C}}$ and $A \cap P^{\mathfrak{C}} \prec P^{\mathfrak{C}}$.*

Proof: Assume that A is complete. First we use the Tarski-Vaught criterion in order to show that $A \cap P^{\mathfrak{C}} \prec P^{\mathfrak{C}}$. Indeed, if $P^{\mathfrak{C}} \models \exists x \theta(x, \bar{b})$ with $\bar{b} \in A$, then (recall Remark 2.2) $\mathfrak{C} \models \exists x P(x) \wedge \hat{\theta}(x, \bar{b})$. By completeness of A, there exists $c \in A$ such that $\mathfrak{C} \models P(c) \wedge \hat{\theta}(c, \bar{b})$, hence $P^{\mathfrak{C}} \models \theta(c, \bar{b})$, as required.

Now let $\bar{a} \subseteq A$. We know that $tp_\phi(\bar{a}/P^{\mathfrak{C}})$ is definable over $P^{\mathfrak{C}}$; so suppose $\mathfrak{C} \models \forall \bar{y} \theta(\bar{y}, \bar{c}) \iff \phi(\bar{a}, \bar{y})$ for some $\bar{c} \in P^{\mathfrak{C}}$. By completeness, such a $\bar{c}$ exists already in $A \cap P^{\mathfrak{C}}$.

For the other direction, assume that $\mathfrak{C} \models (\exists \bar{x} \in P)\phi(\bar{x}, \bar{a})$ with $\bar{a} \in A$. Since $tp_\phi(\bar{a}/P^{\mathfrak{C}})$ is definable over $A \cap P^{\mathfrak{C}}$, the formula $\phi(\bar{x}, \bar{a})$ is equivalent to a formula $\theta(\bar{x}, \bar{d})$ for some $\bar{d} \in A \cap P^{\mathfrak{C}}$. So $\mathfrak{C} \models (\exists \bar{x} \in P)\theta(\bar{x}, \bar{a})$, hence $P^{\mathfrak{C}} \models \hat{\theta}(\bar{c}, \bar{d})$ for some $\bar{c}$. Since $A \cap P^{\mathfrak{C}} \prec P^{\mathfrak{C}}$, such $\bar{c}$ exists also in $A \cap P^{\mathfrak{C}}$; and we have $\models \theta(\bar{c}, \bar{d})$, as required.

$\blacksquare$

Fact 4.3. For any complete A there are $\langle \Psi_\psi : \psi(\bar{x}, \bar{y}) \in L(T) \rangle$ (depending on A) such that for all $\bar{a} \subseteq A$, $tp_\psi(\bar{a}/P \cap A)$ is definable by $\Psi_\psi(\bar{y}, \bar{c})$ for some $\bar{c} \subseteq A \cap P$.

Proof: By compactness, go to a $|T|^+$-saturated model. So, for each ψ we have but finitely many candidates $\Psi^1_\psi, \ldots, \Psi^n_\psi$ (or else, by compactness, there is an undefinable type). As without loss of generality $|P^{\mathfrak{C}}| \geq 2$, we can manipulate these as in [She90] II§2 to an Ψ_ψ. $\blacksquare$

The following properties of complete sets are clear:

Fact 4.4. (i) If $M \prec \mathfrak{C}$ and $P^M \subseteq A \subseteq M$, then A is complete.

(ii) If $\langle B_i : i < \delta \rangle$ is an increasing sequence of complete sets, then $\bigcup_{i<\delta} B_i$ is complete.

Furthermore, if T has QE, then the property of completeness for a set A depends only on its first order theory (as a substructure of $\mathfrak{C}$):

Lemma 4.5. *(T has QE)*

(i) If $A_1 \equiv A_2$, then A_1 is complete iff A_2 is complete.

(ii) A is complete iff whenever the sentence

$$\theta =: (\forall \bar{y})[S(\bar{y}) \leftrightarrow (\exists x \in P)R(x, \bar{y})]$$

for quantifier free R, S is satisfied in $\mathfrak{C}$, then A satisfies θ.

Furthermore, if T has QE down to the level of predicates (e.g., T has been Morleyized), then it is enough to consider all the formulas above with R, S predicates.

Proof:

(i) Follows easily from (ii), but we also give a direct proof. Assume that $A_1 \equiv A_2$ and A_1 is complete. Let us prove this for A_2.

Let $\varphi(\bar{x}, \bar{y})$ be a formula. Since T has QE, there are quantifier free formulae $\theta(\bar{y})$ and $\theta'(\bar{x}, \bar{y})$ such that:

(1)
$$\mathfrak{C} \models \forall \bar{y} \left[(\exists \bar{x} \in P\varphi(\bar{x}, \bar{y})) \longleftrightarrow \theta(\bar{y}) \right]$$

(2)
$$\mathfrak{C} \models \forall \bar{x}\bar{y} \left[\varphi(\bar{x}, \bar{y}) \leftrightarrow \theta'(\bar{x}, \bar{y}) \right]$$

Since A_1 is complete (combining the definition with (1) and (2) above), we have that whenever $\mathfrak{C} \models \theta(\bar{a}_1)$ for $\bar{a}_1 \in A_1$, there is some $\bar{c} \in A_1 \cap P$ such that $\mathfrak{C} \models \theta'(\bar{c}, \bar{a}_1)$. But since these formulae are quantifier free, clearly A_1 (as a substructure) satisfies

$$A_1 \models \forall \bar{y}\theta(\bar{y}) \longrightarrow \exists \bar{x} \in P\theta'(\bar{x}, \bar{y})$$

Since $A_1 \equiv A_2$, so does A_2, which implies completeness.

(ii) Similar [and easier]: If A is complete and $\mathfrak{C} \models \theta$, assume that $A \models S(\bar{b})$ for some $\bar{b}$; then (since S is quantifier free), so does $\mathfrak{C}$, hence $\mathfrak{C} \models (\exists \bar{x} \in P)R(\bar{x}, \bar{b})$, and by completeness of A, there is such $\bar{x}$ already in $P \cap A$. So $A \models \theta$. Conversely, assume that whenever $\mathfrak{C} \models \theta$, so does A. Let $\varphi(\bar{x}, \bar{y})$ be a formula, and assume that $\mathfrak{C} \models (\exists \bar{x} \in P)\varphi(\bar{x}, \bar{b})$.

As in the proof of (iv), let $R(\bar{x}, \bar{y})$ and $S(\bar{y})$ be quantifier free such that

(1)
$$\mathfrak{C} \models \forall \bar{y} \left[(\exists \bar{x} \in P\varphi(\bar{x}, \bar{y})) \longleftrightarrow S(\bar{y}) \right]$$

(2)

$$\mathfrak{C} \models \forall \bar{x}\bar{y}\,[\varphi(\bar{x}, \bar{y}) \leftrightarrow R(\bar{x}, \bar{y})]$$

Clearly, $\mathfrak{C} \models \theta$ (with these specific R and S), hence so that A. But $\mathfrak{C} \models S(\bar{b})$, hence so does A; so $A \models (\exists \bar{x} \in P)R(\bar{x}, \bar{b})$, hence $A \models R(\bar{a}, \bar{b})$ for some $a \in A \cap P$, hence so does $\mathfrak{C}$, and we are clearly done.

The "futhermore" part is trivial.

∎

We make a few remarks on the relation between P and the algebraic closure in complete sets.

Observation 4.6. *Let A be a complete set. Then:*

(i) $A \cap \operatorname{acl}(P^{\mathfrak{C}}) \subseteq \operatorname{acl}(P^A)$

(ii) $\operatorname{acl}(A) \cap P^{\mathfrak{C}} = P^A$

Proof:

(i) Let $d \in A \cap \operatorname{acl}(P^{\mathfrak{C}})$ (in this proof we will distinguish between tuples and singletons). Then there is a formula $\varphi(x, \bar{a})$ with $\bar{a} \in P^{\mathfrak{C}}$ which has $k < \omega$ solutions in $\mathfrak{C}$ such that $\models \varphi(d, \bar{a})$. Since A is complete (see Observation 4.2), the type $\operatorname{tp}(d/P^{\mathfrak{C}})$ is definable over P^A, so the set

$$\{\bar{y} \in P \colon \varphi(d, \bar{y}) \wedge \exists^k x \varphi(x, \bar{y})\}$$

is definable in $\mathfrak{C}|_P$ by a formula $\theta(\bar{y})$ over P^A. Again, since A is complete, $P^A \prec P^{\mathfrak{C}}$, so $\theta(\bar{y})$ has a solution in P^A, which finishes the proof.

(ii) Let $d \in \operatorname{acl}(A) \cap P$. So there is $\varphi(x, \bar{a})$ with $\bar{a} \in A$ with $k < \omega$ solutions in $\mathfrak{C}$ such that $\varphi(d, \bar{a})$ holds. Just like in the proof of clause (i), the set $D = \{x \in P \colon \models \varphi(x, \bar{a})\}$ is definable in $\mathfrak{C}|_P$ over P^A and is non-empty. Clearly, D is finite. Since $P^A \prec P^{\mathfrak{C}}$, $D \subseteq P^A$, as required.

∎

The characterization of completeness (Observation 4.2), in combination with Corollaries 3.2 and 3.4, yields the following important property of complete sets:

Corollary 4.7. *Let A be complete and let p be a (partial) type over A with "$\bar{x} \subseteq P$" $\in p$. Then p is equivalent both to a T-type and a T^P-type over P^A. If, in addition, P^A is λ-compact, and $|p| < \lambda$, then p is realized in P^A.*

We are now ready to define one of the most basic objects of our study: the relevant notion of type for this context.

Definition 4.8. (i) For a complete set A, let

$$S_*(A) = \{tp(\bar{c}/A) : P \cap (A \cup \bar{c}) = P \cap A \text{ and } A \cup \bar{c} \text{ is complete}\}$$

(ii) A is *stable over* P, or simply *stable*, if (it is complete and) for all A' with $A' \equiv A$, $|S_*(A')| \leq |A'|^{|T|}$.

Remark 4.9. (i) *Even though "stability over P" is a more appropriate and accurate name for our notion of stability of a set (and the term "stable set" exists in literature, and has a different meaning), since we have only one notion of stability in this article (stability over P), we will mostly omit "over P" and simply write "stable".*

(ii) *Sometimes we refer to types in $S_*(A)$ as* complete types over A which are orthogonal to P.

Remark 4.10. *Note that by Observation 4.6, if A is complete and $\bar{c}$ a tuple such that $tp(\bar{c}/A) \in S_*(A)$, then* $\mathrm{acl}(A \cup \bar{c}) \cap P^{\mathfrak{C}} = P^A$.

The last remark also follows from the following more general criterion for a complete type being a *-type:

Observation 4.11. *Let A be a complete set and $\bar{c}$ a tuple. Then $tp(\bar{c}/A) \in S_*(A)$ if and only if for every formula $\psi(\bar{x}, \bar{a}, \bar{y})$ over A we have*

$$\models \exists \bar{x} \in P\, \psi(\bar{x}, \bar{a}, \bar{c}) \Longrightarrow \exists \bar{b} \in P \cap A \textit{ such that} \models \psi(\bar{b}, \bar{a}, \bar{c})$$

Proof: Let $tp(\bar{c}/A) \in S_*(A)$, and let $\psi(\bar{x}, \bar{a}, \bar{y})$ be a formula as above satisfying $\models \exists \bar{x} \in P\, \psi(\bar{x}, \bar{a}, \bar{c})$. Since the set $A \cup \bar{c}$ is complete, the type $\mathrm{tp}(\bar{a}\bar{c}/P^{\mathfrak{C}})$ is definable over $P \cap (A \cup \bar{c}) = P \cap A$. Hence the set $D = \{\bar{x} \in P\colon \models \psi(\bar{x}, \bar{a}, \bar{c})\}$ is definable in $\mathfrak{C}|_P$ over P^A. This set is not empty by the assumption. Since A is complete, $P^A \prec P^{\mathfrak{C}}$, hence the set $P^A \cap D$ is nonempty as well, as required for the "only if" direction.

Now assume that the right hand of the equivalence holds. First we note that $P \cap (A \cup \bar{c}) = P \cup A$. Indeed, if $d \in P \cup (A \cup \bar{c})$, then the formula $x = d$ has a solution in P, hence in P^A, so $d \in P^A$. Now obviously (since A is complete) $P^{A \cup \bar{c}} \prec P^{\mathfrak{C}}$. By Observation 4.2, $A \cup \bar{c}$ is complete. ∎

The following lemma shows that, although the definition of types orthogonal to P may seem quite strong, such types are not very hard to come by.

Lemma 4.12. *(The Small Type Extension Lemma) If $A \prec \mathfrak{C}$ is saturated (or just $A \cap P$ is $|A|$-compact) and $p(\bar{x})$ is an $L(T)$-type over A of cardinality $< |A|$, then there is some $p^*(\bar{x}) \in S_*(A)$ extending p.*

Proof: To prove this, we have to show that p is realized by some $\bar{c} \in \mathfrak{C}$ such that $P^{\mathfrak{C}} \cap (A \cup \bar{c}) = P^{\mathfrak{C}} \cap A$ and $A \cup \bar{c}$ is complete. So we have to extend p in such a way that whatever part of p that can be realized inside P, has to be realized in $P \cap A$.

Let $\langle \psi_i(\bar{z}, \bar{b}, \bar{x}) \colon i < |A| \rangle$ list all the formulas over A. Now define inductively for $i < |A|$ and $\psi_i(\bar{z}, \bar{b}, \bar{x})$ (so $\bar{b} \in A$), a consistent T-type $p_i(\bar{x}), |p_i| < |p|^+ + |i|^+ + \aleph_0$ and p_i increasing continuously, making sure that the requirements are met.

Let $p_0 = p$. If $p_i(\bar{x}) \cup \{(\nexists \bar{z} \in P)\psi(\bar{z}, \bar{b}, \bar{x})\}$ is consistent, let it be p_{i+1}. If not, consider the type $q(\bar{z}) = \{\exists \bar{x}\phi(\bar{x}) \wedge \psi(\bar{z}, \bar{b}, \bar{x})\colon \phi(\bar{x}) \in p_i\} \cup \{\bar{z} \in P\}$. By compactness and saturation of A there is some $\bar{c} \subseteq P \cap A$ such that $p_{i+1} := p_i(\bar{x}) \cup \{\psi(\bar{c}, \bar{b}, \bar{x})\}$ is consistent.

In fact, since A is complete, by Corollary 4.7, q is equivalent to a T^P-type over $P \cap A$. Hence (since $|q| < |A|$), it is enough to assume that $P \cap A$ is $|A|$-compact.

By compactness $p^* = \bigcup_i p_i$ is realized by some $\bar{a} \in \mathfrak{C}$. By Observation 4.11, $\operatorname{tp}(\bar{a}/A) \in S_*(A)$, as required.

■

This Lemma yields another characterization of completeness, justifying, in some sense, the original motivation behind this definition (see the discussion in the very beginning of this section).

Proposition 4.13. *Suppose that $A \cap P$ is $|A|$-compact. Then A is complete if and only if there exists an $M \prec \mathfrak{C}$ with $P^M \subseteq A \subseteq M$. If $|A| = |A|^{<|A|} > |T|$, we can add "M saturated".*

Proof: The direction $\Leftarrow$ is trivial. For the other direction we construct a model inductively, using Fact 4.4 for the limit stages and the previous lemma for the successor stages. ■

As we want to reconstruct M from P^M using some complete A as an approximation, it makes sense to look at $S_*(A)$ as candidates for types to be realized. So with $S_*(A)$ small (A stable), this task appears to be easier, as there is less choice.

The rest of the paper is devoted to the study of stable (in particular complete) sets.

In light of Lemma 4.5, since, in the definition of a stable set, instead of looking at a specific set A, we need to consider the class of all subsets A' of $\mathfrak{C}$ with $A' \equiv A$, it would be very useful to ensure that T has quantifier elimination. Fortunately, by the discussion in the previous section (specifically, by Observation 3.6), we can Morleyize T with not much cost. Therefore for the rest of the paper we strengthen Hypothesis 1 and add

Hypothesis 4.14. *(Hypothesis 1')*

(iii) T has quantifier elimination, even to the level of predicates.

5 Stability and rank

In this section we begin the investigation of stable sets, and introduce a notion of rank that "captures" stability.

Definition 5.1. We say that a type $p \in S(A)$ is *internally definable* if for every formula $\phi(x, y)$, the set $\{a \in A\colon \varphi(x, a) \in p\}$ is internally definable in $\mathfrak{C}|_A$. We say that p is internally definable over $B \subseteq A$ if the set above is internally definable in $\mathfrak{C}|_A$ with parameters in B.

Definition 5.2. For a complete set A, a (partial) n-type $p(\bar{x})$ (with parameters in $\mathfrak{C}$), sets Δ_1, Δ_2 of formulas $\psi(\bar{x}, \bar{y})$, and a cardinal λ, we define when $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$. We usually omit n.

(i) $R_A(p, \Delta_1, \Delta_2, \lambda) \geq 0$ if $p(\bar{x})$ is consistent.

(ii) For α a limit ordinal: $R_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$ if $R_A(p, \Delta_1, \Delta_2, \lambda) \geq \beta$ for every $\beta < \alpha$.

(iii) For $\alpha = \beta + 1$ and β even: For $\mu < \lambda$ and finite $q(\bar{x}) \subseteq p(\bar{x})$ we can find $r_i(\bar{x})$ for $i \leq \mu$ such that;

1. Each r_i is a Δ_1-type over A,
2. For $i \neq j, r_i$ and r_j are explicitly contradictory (i.e. for some ψ and $\bar{c}$, $\psi(\bar{x}, \bar{c}) \in r_i, \neg\psi(\bar{x}, \bar{c}) \in r_j$).
3. $R_A(q(\bar{x}) \cup r_i(\bar{x}), \Delta_1, \Delta_2, \lambda) \geq \beta$ for all i.

(iv) For $\alpha = \beta + 1 : \beta$ odd: For $\mu < \lambda$ and finite $q(\bar{x}) \subseteq p(\bar{x})$ and $\psi_i \in \Delta_2, \bar{d}_i \in A$ $(i \leq \mu)$, there are $\bar{b}_i \in A \cap P$ such that $R(r_i, \Delta_1, \Delta_2, \lambda) \geq \beta$ where $r_i = q(\bar{x}) \cup \{(\forall \bar{z} \subseteq P)\left[\psi_i(\bar{x}, \bar{d}_i, \bar{z}) \equiv \Psi_{\psi_i}(\bar{z}, \bar{b}_i)\right] : i < \mu\}$ where Ψ_{ψ_i} is as in Fact 4.3.

$R^n_A(p, \Delta_1, \Delta_2, \lambda) = \alpha$ if $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$ but not $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha + 1$. $R^n_A(p, \Delta_1, \Delta_2, \lambda) = \infty$ iff $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$ for all α.

The main case for applications will be $\lambda = 2$. Note that the larger $R^n_A(p, \Delta_1, \Delta_2, \lambda)$, the more evidence there is for the existence of many types $q(\bar{x}) \in S_*(A)$ consistent with $p(\bar{x})$.

Fact 5.3. (i) The rank $R^n_A(p, \Delta_1, \Delta_2, \lambda)$ is increasing in A, Δ_1 and decreasing in p, Δ_2, λ.

(ii) For every p there is a finite $q \subseteq p$, such that $R_A(p, \Delta_1, \Delta_2, \lambda) = R_A(q, \Delta_1, \Delta_2, \lambda)$.

(iii) For any A and any finite $\Delta_1, \Delta_2, \lambda, m$ and $\psi(\bar{x}, \bar{y})$ there is a formula $\theta(\bar{y}) \in L(\mathfrak{C}|A)$ such that for $\bar{b} \subseteq A$ $R_A(\psi(\bar{x}, \bar{b}), \Delta_1, \Delta_2, \lambda) \geq m$ iff $A \models \theta(\bar{b})$. The formula θ doesn't really depend on A but has quantifiers ranging on A.

Fact 5.4. Let A be complete, $p \in S_*(A)$, $q^* \subseteq p$, and assume

$$R^n_A(q^*, \Delta_1, \Delta_2, \lambda) = R^n_A(p, \Delta_1, \Delta_2, \lambda) = k < \infty$$

Then k is even.

Proof: Assume k is odd; we shall show that $R^n_A(q^*, \Delta_1, \Delta_2, \lambda) \geq k$ implies $R^n_A(q^*, \Delta_1, \Delta_2, \lambda) \geq k + 1$.

As in Definition 5.2(iv), let $\mu < \lambda$, $q(\bar{x}) \subseteq q^*(\bar{x})$ finite, $\psi_i \in \Delta_2, \bar{d}_i \in A$ $(i \leq \mu)$.

Let $\bar{c} \models p$; so $A \cup \{\bar{c}\}$ is complete by the assumption $p \in S_*(A)$.

As $\bar{d}_i \in A$, clearly $A \cap P = (A \cup \bar{c}\bar{d}_i) \cap P$, hence $tp(\bar{c}\bar{d}_i/A) \in S_*(A)$. Hence $tp_{\psi_i}(\bar{c}\bar{d}_i/P^{\mathfrak{C}})$ is defined by $\Psi_{\psi_i}(\bar{y}, \bar{b}_i)$ with $\bar{b}_i \subseteq P \cap (A \cup \bar{c}\bar{d}_i) = A \cap P$, where Ψ_{ψ_i} is as in Fact 4.3.

So $\theta_i(\bar{x}) := (\forall \bar{y} \subseteq P)[\psi_i(\bar{x}, \bar{d}_i, \bar{y}) \equiv \Psi_{\psi_i}(\bar{y}, \bar{b}_i)]$ belongs to $tp(\bar{c}/A) = p$; hence

$R_A(q \cup \{\theta_i(\bar{x})\}, \Delta_1, \Delta_2, \lambda) \geq R_A(p, \Delta_1, \Delta_2, \lambda) = k$.

Now by (iv) of Definition 5.2, $R_A(p, \Delta_1, \Delta_2, \lambda) \geq k+1$), and we are done. ∎

Theorem 5.5. *The following are equivalent:*

(i) A is stable.

(ii) For every finite Δ_1 and finite n there are some finite Δ_2 and finite m such that $R^n_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) \leq m$.

Proof: $(ii) \Rightarrow (i)$: Suppose (ii) holds. Since condition (ii) speaks only about $Th(\mathfrak{C}|_A)$, it suffices to prove $|S_*(A)| \leq |A|^{|T|}$ as the same proof works for every $A' \equiv A$.

Let $\lambda := |A|^{|T|}$ and assume that for $i < \lambda^+$ there are distinct types $p_i = tp(\bar{c}_i/A) \in S_*(A)$. By Fact 5.3 (ii) for every $i < \lambda^+$ and finite Δ_1, Δ_2 we can find a finite $p = p_{i,\Delta_1,\Delta_2} \subseteq p_i$ such that $R_A(p, \Delta_1, \Delta_2, 2) = R_A(p_i, \Delta_1, \Delta_2, 2)$. Let $q_i = \bigcup_{\Delta_1,\Delta_2} p_{i,\Delta_1,\Delta_2}$. So $q_i \subseteq p_i, |q_i| \leq |T|$ and by Fact 5.3 (i) $R_A(p_i, \Delta_1, \Delta_2, 2) = R_A(q_i, \Delta_1, \Delta_2, 2)$ for every finite Δ_1, Δ_2.

The function F with $domF = \lambda^+, F(i) = \langle p_i|_\psi : \psi \in L(T)\rangle$ is one-to-one where $p|_\psi = \{\pm\psi(\bar{x}, \bar{y}) : \pm\psi(\bar{x}, \bar{z}) \in p\}$. Hence, $\lambda^+ \leq \prod_{\psi \in L(T)} |\{p_i|\psi : i < \lambda^+\}|$. If for every ψ, $|\{p_i|\psi : i > \lambda^+\}| \leq \lambda$, we get a contradiction as $\lambda^{|T|} = \lambda$. Choose ψ^* such that $|\{p_i|\psi^* : i < \lambda^+\}| = \lambda^+$. By renaming we can assume that $\{p_i|\psi^* : i < \lambda^+\}$ are pairwise distinct.

There cannot be more than $(|A| + |T|)^{|T|} = |A|^{|T|} = \lambda$ different q_i, so without loss of generality, $q_i = q^*$ for all i. Also we can assume that all p_i's are n-types for some fixed n. Applying condition (ii) to n and $\Delta_1 := \{\psi^*\}$ there is a finite set Δ_2 and $m < \omega$ such that $R^n_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) \leq m$.

Let $k := R^n_A(q^*, \Delta_1, \Delta_2, 2)$. Since q^* is consistent and by monotonicity (Fact 5.3 (i)) and the fact that $\{\bar{x} = \bar{x}\} \subseteq q^*$, it follows that $0 \leq k \leq m < \omega$.

Recall also that by the construction of q^*, for all i

$$R^n_A(p_i, \Delta_1, \Delta_2, 2) = R^n_A(q^*, \Delta_1, \Delta_2, 2) = k$$

We are now going to show that $R^n_A(q^*, \Delta_1, \Delta_2, 2) \geq k+1$, hence obtaining a contradiction. Note that by 5.4, k is even.

From now on, for simplicity of notation, let R denote R^n_A.

As $p_0|\psi^* \neq p_1|\psi^*$, there is $\psi^*(\bar{x}, \bar{b}), \bar{b} \in A$ such that $\psi^*(\bar{x}, \bar{b}) \in p_0$ and $\neg\psi^*(\bar{x}, \bar{b}) \in p_1$ (or conversely). Now $R(q^* \cup \{\pm\psi^*(\bar{x}, \bar{b})\}, \Delta_1, \Delta_2, 2) \geq k$ as this type is contained in p_0 or p_1, hence by monotonicity $R(q^*, \Delta_1, \Delta_2, 2) = R(q^* \cup \{\pm\psi^*(\bar{x}, \bar{b})\}, \Delta_1, \Delta_2, 2)$. So $R(q^*, \Delta_1, \Delta_2, 2) > k$, a contradiction.

This finishes the proof of one direction.

In order to prove the other direction, assume that condition (ii) fails. We will prove a strong version of $\neg(i)$. Let (ii) fail through Δ_1. So for all finite Δ_2, $R(\bar{x}=\bar{x},\Delta_1,\Delta_2,2)\geq\omega$.

Let $\lambda=\lambda^{<\lambda}>|T|$ (which exists by our set theoretic assumption). Let $B\equiv A$ be saturated, $|B|=\lambda$.

An m-type $p(\bar{x})$ over B (in $L(T)$) is called *large* (for Δ_1) if for all finite Δ_2, $R(p(\bar{x}),\Delta_1,\Delta_2,2)\geq\omega$.

So $\neg(ii)$ says that $\bar{x}=\bar{x}$ is large for Δ_1. It will be enough to prove the following claims:

Assume $p(\bar{x})$ is over B and is large, $|p|<\lambda$. Then the following holds:

(a) For any $\bar{b}\in B,\psi=\psi(\bar{x},\bar{y},\bar{z})$ there exists $\bar{d}\subseteq P\cap B$ such that $p(\bar{x})\cup\{\forall\bar{z}\subseteq P[\psi(\bar{x},\bar{b},\bar{z})\equiv\Psi_\psi(\bar{z},\bar{d})]\}$ is large.

(b) For $\bar{b}\subseteq B$ at least one of $p(\bar{x})\cup\{\pm\psi(\bar{x},\bar{b})\}$ is large.

(c) For some $\psi(\bar{x},\bar{y})\in\Delta_1$ and $\bar{b}\in B$ we have $p(\bar{x})\cup\{\pm\psi(\bar{x},\bar{b})\}$ are large.

Note that from $(a)-(c)$ (and Fact 4.4(ii)) it follows that $|S_*(B)|=2^\lambda$; in fact, even $|\{p|_{\Delta_1}:p\in S_*(B)\}|=2^\lambda$.

This already contradicts stability. But we can say more. For at least one $p\in S_*(B)$, $p|_{\Delta_1}$ is not definable. Hence (by [Sh8]) there exists some B, $B\equiv A$, $|B|=\lambda$ such that $|S_*(B)|\geq Ded(\lambda)$, assuming for simplicity that $Ded(\lambda)$ is obtained. This is a strong negation of (i).

It is left to show $(a)-(c)$:

(a): Without loss of generality assume that $p(\bar{x})$ is closed under conjunction. So we have to find $\bar{d}$ such that for all $\rho=\langle\Delta_2,n,\theta(\bar{x},\bar{e})\rangle$ where $n<\omega,\theta(\bar{x},\bar{e})\in p(\bar{x})$, and $\Delta_2\subseteq L(T)$ finite,

$$(*_\rho)\quad R(\theta(\bar{x},\bar{e}))\wedge(\forall\bar{z}\subseteq P)(\psi(\bar{x},\bar{b},\bar{z})\equiv\Psi_\psi(\bar{z},\bar{d})),\Delta_1,\Delta_2,2)\geq n.$$

For every such ρ there are $\bar{e}^*_\rho\subseteq B,\chi_\rho\in L(T$ such that for $\bar{d}\subseteq P\cap B,B\models\chi_\rho(\bar{d},\bar{e}^*_\rho)$ iff $*_\rho$ holds for $\bar{d}$.

As B is λ-saturated, $|p(\bar{x})|<\lambda$ it suffices to show that for every relevant $\rho_1,\dots,\rho_n$ there is $\bar{d}\subseteq P\cap B$ satisfying $*_{\rho_1}\wedge\cdots\wedge*_{\rho_n}$.

By monotonicity properties of rank and the fact that p is closed under conjunction, it is enough to consider one ρ. But $R(\theta(\bar{x},\bar{e}),\Delta_1,\Delta_2,2)=\omega>n+2$. So by the definition of rank there is suitable $\bar{d}\subseteq P\cap B$ satisfying $*_\rho$.

(b) follows from (a) with $\bar{z}$ empty.

For (c) assume first that $\Delta_1=\{\psi\}$. Repeat the proof of (a) conjuncting over $\pm\psi$, using the other clause in the definition of rank.

If $|\Delta_1|>1$, assume (c) doesn't hold. So for every $\psi\in\Delta_1$ there is some finite $q_\psi\subseteq p(\bar{x})$ such that stops $p(\bar{x})\cup\{\pm\psi(\bar{x},\bar{b}\}$ from being large. Now use $R(\bigcup_{\psi\in\Delta_1}q_\psi,\Delta_1,\Delta_2,2)\geq n+2$ to get a contradiction. ■

The following two corollaries follow from the proof of (the second direction of) Theorem 5.5.

Corollary 5.6. *In Definition 4.1(iv), it is not necessary to consider all $A' \equiv A$. More specifically, a complete set A is stable if and only if $|S_*(A')| \leq |A'|^{|T|}$ for some $A' \equiv A$ saturated, $|A'| > |T|$.*

Corollary 5.7. *Let A be complete unstable and saturated of cardinality $\lambda > |T|$. Then $|S_*(A)| = 2^\lambda$. In fact, $|S_{*,\Delta}(A)| = 2^\lambda$, where Δ is some finite set of formulas, and*

$$S_{*,\Delta}(A) = \{p{\restriction}\Delta \colon p \in S_*(A)\}$$

From now on, we will often omit the superscript and the subscript in the rank R^n_A, and write simply R (at least when n and A are easily deduced from the context).

In conclusion, we observe that every type $p \in S_*(A)$ over a stable set A is internally definable (see Definition 5.1).

Corollary 5.8. (i) *If A is stable, then for every $\psi(\bar{x}, \bar{y}) \in L(T)$ there is Ψ_ψ in $L(A)$ such that if $p \in S_*(A)$, then for some $\bar{b} \subseteq A$, $\Psi_\psi(\bar{y}, \bar{b})$ defines $p|\psi$ in $\mathfrak{C}_A$.*

Specifically, for every $\bar{c} \in A$, $\psi(\bar{x}, \bar{c}) \in p$ if and only if $A \models \Psi_\psi(\bar{c}, \bar{b})$.

(ii) *Moreover, if $|A| \geq 2$, then for every $\psi(\bar{x}, \bar{y})$, there is a definition $\Psi_\psi(\bar{x}, \bar{y})$ as above which works uniformly for all $B \equiv A$ and $p \in S_*(B)$.*

Proof:

(i) Let $\Delta_1 = \{\psi\}$. Then there is some finite Δ_2 such that $R_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) = n^* < \omega$. Let $\theta(\bar{x}) \in p(\bar{x})$ such that $n = R_A(\theta(\bar{x}), \Delta_1, \Delta_2, 2) = R_A(p(\bar{x}), \Delta_1, \Delta_2, 2) \leq n^*$.

Recall that n is even (Fact 5.4). Since $\Delta_1 = \{\psi\}$, there is no $\bar{b} \in A$ such that $R(\theta(\bar{x}) \wedge \pm\psi(\bar{x}, \bar{b}), \Delta_1, \Delta_2, 2) = n$.

But $\psi \in p(\bar{x}) \Rightarrow R(\theta(\bar{x}) \wedge \psi, \Delta_1, \Delta_2, 2) = n$.

So for $\bar{b} \subseteq A$, we have

$$\psi(\bar{x}, \bar{b}) \in p(\bar{x}) \Leftrightarrow R(\theta(\bar{x}) \wedge \psi(\bar{x}, \bar{b}), \Delta_1, \Delta_2, 2) \geq n.$$

By Fact 5.3(iii) the right hand side is definable in A. More precisely, the predicate

$$\Psi(\bar{y}) = [R_A(\theta(\bar{x}) \wedge \psi(\bar{x}, \bar{y}), \Delta_1, \Delta_2, 2) \geq n]$$

is definable in A; note that it may have parameters in A.

This proves that for each type $p \in S_*(A)$, there is some Ψ_ψ.

(ii) Now use a compactness argument to find a uniform definition Ψ_ψ for all $B \equiv A$ and $p \in S_*(B)$. The argument is pretty standard, but we have chosen to include it due to the (unusual) additional requirement of uniformity for all B.

Let $\psi(\bar{x}, \bar{y}) \in L(T)$. Expand $L(T)$ with a new monadic predicate $Q(y)$, which is interpreted in $\mathfrak{C}$ as the set A. Let T^* be the theory of the expanded monster model. Consider the following set of formulae in the expanded language:

$$T^* \bigwedge \{\forall \bar{z} \in Q \exists \bar{y} \in Q \; \psi(\bar{x}, \bar{y}) \not\leftrightarrow [Q \models \Psi(\bar{y}, \bar{z})] \colon \Psi(\bar{y}, \bar{z}) \in L(T)\}$$

Assume that this set is finitely satisfiable. Let M^* be a model of T^* in which the tuple $\bar{a}$ realize the above set, and let $M \models T$ be reduct of M^* to $L(T)$. Denote $B = Q^{M^*}$. We now have (all formulas and types below are in the original language $L(T)$):

- $B \equiv A$
- For all $\Psi(\bar{y}, \bar{z}) \in L(T)$ and $\bar{c} \in B$, $\Psi(\bar{y}, \bar{c})$ does not internally (in B) define $\mathrm{tp}_\psi(\bar{a}/B)$.

So for every $\psi(\bar{x}, \bar{y})$ there are finitely many $\Psi_i(\bar{y}, \bar{z})$ such that for any $B \equiv A$ and any $\bar{a} \in \mathfrak{C}$, the type $\mathrm{tp}_\psi(\bar{a}/B)$ is defined by $\Psi_i(\bar{x}, \bar{b})$ for some i and $\bar{b} \in B$. Now (since $|A| \geq 2$, hence so is any $B \equiv A$), we can combine these Ψ_i's into one formula $\Psi(\bar{x}, \bar{y})$ that works for all $B \equiv A$ and $\bar{a} \in \mathfrak{C}$, as required.

■

So we have obtained a uniform (for all $B \equiv A$) notion of definability of types in $S_*(A)$ for a stable set A. Note, however, that what we got is not the "usual" notion of definability of types; this is different than saying that p is definable (in $\mathfrak{C}$). Specifically, unless $A \prec \mathfrak{C}$, Ψ_ψ might have quantifiers. In order to obtain quantifier free definitions, we will need to make yet another "structure" assumption; see Hypothesis 7.1.

6 Stability and primary models

In this section our goal is to obtain a characterization of stable sets that strengthens the characterization of complete sets with a saturated P-part (Proposition 4.13). Specifically, we will show that if A is (also) stable, then, in addition, one can have the model M in Proposition 4.13 be "constructible" over A in a nice way.

First, we strengthen the Small Type Extension Lemma (Lemma 4.12) in this context.

Lemma 6.1. (i) *Assume B is stable, $|B| = |P^B| = \lambda$, P^B is saturated. Let $p(\bar{x})$ an m-type over B, $|p(\bar{x})| < \lambda$, then there is $q(\bar{x})$ such that $|q(\bar{x})| \leq |T|$, $p(\bar{x}) \cup q(\bar{x})$ consistent and there is $r \in S_*(B)$ such that $p(\bar{x}) \cup q(\bar{x}) \equiv r(\bar{x})$.*

In particular, $r(\bar{x})$ is λ-isolated.

(ii) *The previous clause is also true if $\bar{x}$ is an infinite tuple with $< \lambda$ variables, but in this case we can only require that $|q| < \lambda$.*

Specifically, if $|\bar{x}| = \kappa < \lambda$, then there exists $|q| \leq |T| \cdot \kappa$ as above.

Proof:

(i) Let $\{\psi_i(\bar{x}, \bar{y}_i) : i < |T|\}$ list all formulas of $L(T)$. Let Δ_i be finite such that $R(\bar{x} = \bar{x}, \{\psi_i\}, \Delta_i, 2) < \omega$ (where $R = R^m_B$). Define $q_i(\bar{x})$ by induction on $i < |T|$ such that

(a) q_i is finite and is over B,

(b) $p(\bar{x}) \cup \bigcup_{j \le i} q_j(\bar{x})$ is consistent, and

(c) $R(p \cup \bigcup_{j \le i} q_j, \{\psi_i\}, \Delta_i, 2)$ is minimal with respect to (a) and (b).

By Lemma 4.12, there is $p^* \in S_*(B)$ extending $p \cup \bigcup_{j \le i} q_j$. Clearly, $R(p^*, \{\psi_i\}, \Delta_i, 2) \le R(p \cup \bigcup_{j \le i} q_j, \{\psi_i\}, \Delta_i, 2)$, and for some finite $q' \subseteq p^*$ we have $R(p^*, \{\psi_i\}, \Delta_i, 2) = R(q', \{\psi_i\}, \Delta_i, 2)$. If $R(p^*, \{\psi_i\}, \Delta_i, 2) < R(p \cup \bigcup_{j \le i} q_j, \{\psi_i\}, \Delta_i, 2)$, setting $q'_i = q_i \cup q'$ would contradict the "minimality" of q_i (clause (c) above). Hence $R(p^*, \{\psi_i\}, \Delta_i, 2) = R(p \cup \bigcup_{j \le i} q_j, \{\psi_i\}, \Delta_i, 2)$.

By Fact 5.4, $R(p^*, \{\psi_i\}, \Delta_i, 2)$ is even, hence so is $R(p \cup \bigcup_{j \le i} q_j, \{\psi_i\}, \Delta_i, 2)$. In particular, we have:

(d) For no $\bar{b} \subseteq B$ do we have $R(p \cup \bigcup_{j \le i} q_j \cup \{\pm\psi_i(\bar{x}, \bar{b})\}, \{\psi_i\}, \Delta_i, 2) \ge R(p \cup \bigcup_{j \le i} q_j, \{\psi_i\}, \Delta_i, 2)$

Clearly $q = |\bigcup_{j \le |T|} q_j| \le |T|$. By Lemma 4.12 there is some $r \in S_*(B)$ such that $p \cup \bigcup_{j < |T|} q_j \subseteq r$. By (c) and (d) above it follows that $p \cup \bigcup_{j < |T|} q_j \vdash r$.

(ii) Let $\langle \psi_i(\bar{x}_i, \bar{y}_i) \colon i < \kappa \rangle$ list all the formulas where $\bar{x}_i$ is a finite tuple from $\bar{x}$ (so $|T| \le \kappa < \lambda$), and define q_i on induction on κ just as in the proof of the previous clause. Since each q_i is finite, $q = |\bigcup_{j \le |\kappa|} q_j| \le \kappa < \lambda$, and we can again use Lemma 4.12 in order to obtain $r \in S_*(B)$ as required.

■

Proposition 6.2. *Assume that B is stable, $|B| = |P^B| = \lambda$, P^B is saturated. Let $C \supset B$ such that:*

- *C is complete*
- *$P^C = P^B$*
- *$|C \setminus B| < \lambda$*
- *$\mathrm{tp}(C/B)$ is λ-isolated*

Let $p(\bar{x})$ an m-type over C, $|p(\bar{x})| < \lambda$. Then there is a λ-isolated $r \in S_*(C)$ extending p.

Proof: Let $\bar{c} = \langle c_i \colon i < \kappa \rangle$ list $C \setminus B$ (so $\kappa < \lambda$), and let $B_0 \subseteq B$ be such that $|B_0| < \lambda$ and $\mathrm{tp}(\bar{c}/B_0) \equiv \mathrm{tp}(\bar{c}/B)$.

Separating the parameters of p, we can think of $p(\bar{x})$ as the type $p(\bar{x}, \bar{c})$ over $B\bar{c}$. By replacing all occurrences of c_i by a new variable y_i, we therefore obtain a type $p(\bar{x}, \bar{y})$ over B.

Denote $\hat{p}(\bar{x}, \bar{y}) = p(\bar{x}, \bar{y}) \cup \mathrm{tp}(\bar{c}/B_0)$. By Lemma 6.1(ii), there is $\hat{r} \in S^*(B)$ extending $\hat{p}$ which is λ-isolated. Let $\hat{\bar{a}}\hat{\bar{c}}$ realize $\hat{r}$ in $\mathfrak{C}$. Note that $\hat{\bar{c}} \equiv_B \bar{c}$. Let

$\bar{a} \in \mathfrak{C}$ such that $\bar{a}\bar{c} \equiv_B \hat{\bar{a}}\hat{\bar{c}}$. Note that it is still the case that $\bar{a}C \cap P = P^B$ and $\bar{a}C$ is complete, so in particular $\mathrm{tp}(\bar{a}/C) \in S_*(C)$.

Finally, clearly $\mathrm{tp}(\bar{a}/C)$ is λ-isolated: indeed, if $B_1 \subseteq B$ is such that $\mathrm{tp}(\bar{a}\bar{c}/B_1) \vdash \mathrm{tp}(\bar{a}\bar{c}/B)$, then $\mathrm{tp}(\bar{a}/\bar{c}B_1) \vdash \mathrm{tp}(\bar{a}/\bar{c}B) = \mathrm{tp}(\bar{a}/C)$, so we are done. ∎

Definition 6.3. Let N be a model, $P^N \subseteq B \subseteq N$.

(i) We say that a model N is λ-*prime* over a B if N is λ-saturated, and it can be elementarily embedded over B into any λ-saturated model containing B.

(ii) We say that N is λ-*atomic* over B if for every $\bar{d} \subseteq N$, $tp(\bar{d}, B)$ is λ-isolated over some $B_{\bar{d}} \subseteq B, |B_{\bar{d}}| < \lambda$.

(iii) We say that the sequence $\bar{d} = \{d_i : i < \alpha\} \subseteq N$ is a λ-construction over B in N if for all $i < \alpha$, the type $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated.

(iv) We say that a set $C \subseteq N$ is N is λ-*constructible* over B in N if there is a λ-construction $\bar{d}$ over B in N.

In particular, we say that N is λ-*constructible* over B if there is a construction $N = B \cup \{d_i : i < \lambda\}$ such that for all $i < \lambda$ the type $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated.

(v) We say that a model N is λ-*primary* over B if it is λ-constructible and λ-saturated.

Remark 6.4. (i) *If N is λ-primary over B, then it is λ-prime over B.*

(ii) *(λ regular) If N is λ-constructible over B witnessed by a construction $N = B \cup \{d_i : i < \lambda\}$, then for every $\alpha < \lambda$,* $\mathrm{tp}(\{d_i : i < \alpha\}/B)$ *is λ-isolated. Hence N is λ-atomic over B.*

Recall that given a complete set B satisfying $|B| = \lambda = \lambda^{<\lambda}$ with a saturated P-part, Proposition 4.13 implies that B can be extended to a saturated model of cardinality λ with the same P-part. We now show that in case B is stable, this model can be chosen to be λ-primary (hence λ-prime) over B.

Theorem 6.5. *Assume that B is a stable set, $|P^B| = |B| = \lambda = \lambda^{<\lambda}$, P^B is saturated. Then there is $N \supseteq B$ which is λ-primary over B.*

Proof: By using Proposition 6.2 repeatedly, one constructs $N = B \cup \{d_i : i < \lambda\}$ by induction on $i < \lambda$ such that $\mathrm{tp}(d_i/B \cup \{d_j : j < i\})$ is λ-isolated, while making sure that for all $i < \lambda$, all types over subsets of $B_i = B \cup \{d_j : j < i\}$ of cardinality smaller that λ are realized by some d_j for $j < i$.

Specifically, we construct by induction on i a sequence $\langle \bar{d}_i : i < \lambda\rangle$ of sequences such that:

- $\bar{d}_i = \langle d_\alpha : \alpha < \alpha_i\rangle$ (where d_α is a singleton) with $\alpha_i < \lambda$ (so in particular $|\bar{d}_i| < \lambda$)
- $\bar{d}_i$ is increasing and continuous with i (so in particular $\alpha_i \leq \alpha_j$ for $i < j$)

- For every $i < \lambda$, every $A \subseteq B$, $|A| < \lambda$, every type over $A \cup \bar{d}_i$ is realized by some d_α
- For every $i < \lambda$, the set $B_i = B \cup d_i$ is complete, and $P^{B_i} = P^B$
- For every $\alpha < \lambda$, the type $\mathrm{tp}(d_\alpha / B \cup \{d_\beta \colon \beta < \alpha\})$ is λ-isolated

If we succeed, then clearly $\bar{d}_\lambda = \bigcup_{i<\lambda} \bar{d}_i$ is a λ-construction of a λ-saturated model N over B.

Let $\bar{d}_0 = \langle\rangle$.

For i limit, take unions. For $i = j+1$, let $B_j = B \cup \bar{d}_j$. Let the sequence $\langle p_{j,\gamma} \colon \gamma \in [j,\lambda)\rangle$ list all the types over subsets of B_j of cardinality $< \lambda$ (recall that $\lambda = \lambda^{<\lambda}$). Now consider the sequence $\langle p_{\ell,j} \colon \ell < i\rangle$.

Recall that by induction, $\bar{d}_j = \langle d_\alpha : \alpha < \alpha_j\rangle$. Now for $\ell < i$ and $\alpha = \alpha_j + \ell$, let d_α realize $p_{\ell,j}$ such that $\mathrm{tp}(d_\alpha / B \cup \{d_\beta \colon \beta < \alpha\}) \in S_*(B \cup \{d_\beta \colon \beta < \alpha\})$ is λ-isolated (this is possible by Proposition 6.2).

Clearly, setting $\alpha_i = \alpha_j + i$, the sequence $\bar{d}_i = \langle d_\alpha : \alpha < \alpha_i\rangle$ is as required. ■

In the proof of Theorem 6.5, the assumption that $\lambda = \lambda^{<\lambda}$ was only used in in order to ensure that at any stage $i < \lambda$ of the construction, the number of types over small subsets of B_i is bounded by λ. For a specific theory T, this assumption may hold for other cardinals λ (for instance, if T has a saturated model of cardinality λ).

So for example, the same proof as above gives the following stronger result:

Corollary 6.6. *Let A be a stable set such that there exists a saturated model N of cardinality λ containing A with $P^N = P^A$. Then there exists M, $A \subseteq M \prec N$, M is λ-primary over A.*

7 From Stability of Models: Quantifier Free Definitions

The goal of this section is to establish the major technical tool of this paper: quantifier free definability of types orthogonal to P over stable sets. However, as we have already pointed out at the end of section 5, for this we will need an additional hypothesis.

In [She86], the first author has shown (see Theorems 2.10 and 2.12 there) that if there is an unstable *model*, then there is a forcing extension in which there are many M_i pairwise non-isomorphic with $M_i|P = M_j|P$ (all of cardinality $|P| > \aleph_0$). We will, therefore, following the Classification Theory guidelines, add yet another hypothesis to Hypothesis 2.7. Specifically, from now on we assume the following:

Hypothesis 7.1. *(Hypothesis 2). Every $M \prec \mathfrak{C}$ is stable over P (as in Definition 4.8(ii))*

Now we are ready to prove that *-types over all stable sets (not just models) are quantifier free internally definable, and therefore are also definable in $\mathfrak{C}$ in the usual sense.

Theorem 7.2. *If A is stable, $|A| \geq 2$, then for every $\psi(\bar{x}, \bar{y})$ there is a quantifier free $\Psi_\psi(\bar{y}, \bar{z}) \in L(T)$ such that whenever $B \equiv A$ and $p \in S_*(B)$ then $p|\psi$ is defined by $\Psi_\psi(\bar{y}, \bar{d})$ for some $\bar{d} \subseteq B$, i.e. $p|\psi = \{\psi(\bar{x}, \bar{a}) : \bar{a} \in B, B \models \Psi_\psi(\bar{a}, \bar{d})\}$.*

Proof: Let $\lambda = \lambda^{<\lambda}, \lambda > |A| + |T|$ to make things simple.

Also note that if A is stable and $\bar{c}$ is finite with $tp(\bar{c}/A) \in S_*^n(A)$, then $A \cup \bar{c}$ is also stable (as every $p(x) \in S_*^1(A \cup \bar{c})$ gives rise to some type $q(\bar{x}) \in S_*^{n+1}(A)$.)

Let A, p be a counterexample, and $\bar{c}$ realize p. We can find B saturated of power λ such that $(\mathfrak{C}|(A \cup \bar{c}), A, \bar{c}) \prec (\mathfrak{C}|(B \cup \bar{c}), B, \bar{c})$. Clearly $B, \bar{c}$ form a counterexample too, and in particular $tp(\bar{c}/B) \in S_*(B)$. We will arrive at a contradiction by showing how to construct the required quantifier free definition. By Proposition 4.13 there is a model $M, P^M \subseteq B \cup \bar{c} \subseteq M$. $Th(M, B, \bar{c})$ has a saturated model of power λ preserving the relevant properties. So without loss of generality $(M, B, \bar{c})$ is saturated and $|M| = |B| = \lambda$.

By Theorem 6.5 and Remark 6.4, there is a λ-saturated model $N, P^N \subseteq B \subseteq N$ such that for every $\bar{d} \subseteq N$, $tp(\bar{d}, B)$ is λ-isolated, say over $B_{\bar{d}} \subseteq B, |B_{\bar{d}}| < \lambda$, and a construction $N = \{d_i : i < \lambda\}$ such that $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated. In particular, N is λ-prime over B.

Hence we can embed N into M over B. So without loss of generality $N \prec M$, and in particular $P^M = P \cap B = P \cap N = P^N$ and $tp(\bar{c}/N) \in S_*(N)$.

Hence there are formulas $\Psi_\psi(\bar{x}, \bar{e}_\psi) \in L(T), \bar{e}_\psi \subseteq N$ defining $tp_\psi(\bar{c}/N)$ for $\psi \in L$. Let $E = \bigcup_{\psi \in L(T)} \bar{e}_\psi \subseteq N$ and $B^* = \bigcup_{\bar{d} \subseteq E} B_{\bar{d}}$. So $|E| \leq |T|$ and $|B^*| < \lambda$.

Now, if $\bar{b}_1, \bar{b}_2 \in B$ realize the same type over B^* (in $\mathfrak{C}$), then they realize the same type over $B^* \cup E$ by choice of E. Hence, they realize the same type over $B^* \cup E \cup \bar{c}$.

For $\psi \in L(M, B)$ let

$$\Gamma_\psi = \{\psi(\bar{c}, \bar{y}_1) \equiv \neg\psi(\bar{c}, \bar{y}_2)\} \cup \{\chi(\bar{y}_1, \bar{d}) \equiv \chi(\bar{y}_2, \bar{d}) : \chi \in L(T), \bar{d} \subseteq B^*\} \cup \{\bar{y}_1\bar{y}_2 \subseteq B\}$$

.

By the previous observation Γ_ψ is not realized in (M, B). By the fact that (M, B) is λ-saturated, and $|\Gamma_\psi| < \lambda$, it is inconsistent. By compactness there are $\chi_1, \ldots, \chi_n \in L(T)$ and $\bar{d}_1, \ldots, \bar{d}_n \in B^*$ such that

$$\Gamma_\psi^1 = \{\psi(\bar{c}, \bar{y}_1) \equiv \neg\psi(\bar{c}, \bar{y}_2)\} \cup \{\bar{y}_1\bar{y}_2 \subseteq B\} \cup \{\chi_l(\bar{y}_1, \bar{d}_l) \equiv \chi_l(\bar{y}_2, \bar{d}_l) : l = 1, \ldots, n\}$$

is inconsistent.

So we can define $tp_\psi(\bar{c}/B)$ since

$$\models \psi(\bar{c}, \bar{b}) \Leftrightarrow [\{l : 1 \leq l \leq n, \models \chi_l(\bar{b}, \bar{d}_l)\} \text{ is in} P^*]$$

for some appropriate $P^* \subseteq \mathcal{P}\{1, \ldots, n\}$. Now apply compactness as in [Sh:c,II§2]. ■

Note that we have used the assumption that models are stable in the proof.

Theorem 7.3. *Let A be complete and $\lambda = \lambda^{<\lambda}$. The following are equivalent:*

(i) *A is stable.*

$(ii)_\lambda$ *If $A' \equiv A$ is λ-saturated, $\lambda = |A'| > |T|$, then over A' there is a λ-primary model M.*

$(iii)_\lambda$ *If $A' \equiv A$ is λ-saturated, $\lambda > |T|$, then every m-type p over A, $|p| < \lambda$ can be extended to a λ-isolated $q \in S_*(A')$.*

(iv) *For every $A' \equiv A$ and $p \in S_*(A)$ and $\phi \in L(T)$, $p|\phi$ is definable by some $\Psi_\phi(\bar{y}, \bar{a}), \bar{a} \subseteq A, \Psi_\phi \in L(T)$.*

(v) *There is some collection $\langle \Psi_\phi; \phi \in L \rangle$ such that for every $A' \equiv A, p \in S_*(A')$ and $\psi \in L(T), p|\psi$ is definable by $\Psi_\psi(\bar{y}, \bar{a})$ for some $\bar{a} \in A'$.*

So $(ii)_\lambda, (iii)_\lambda$ do not depend on λ.

Proof: Included in the proofs of Theorem 7.2, Lemma 6.1, and Theorem 6.5. ■

Theorem 7.4. *(T countable) If A is stable, $\bar{a} \in A$, and $\models \exists x \theta(\bar{x}, \bar{a})$, then there is $p \in S_*(A)$ such that $\theta(\bar{x}, \bar{a}) \in p$ and for every $\phi \in L(T)$ there is $\psi(\bar{x}, \bar{a}') \in p$ such that $\psi(\bar{x}, \bar{a}') \vdash p|\phi$ (i.e. p is locally isolated, i.e $\mathbf{F}^l_{\aleph_0}$-isolated. So the locally isolated types are dense in $S_*(A)$.)*

Proof: Again this is contained in the proofs of Theorem 7.2 and Lemma 6.1. ■

8 Stationarization and Independence

The following definition mimics the Tarski-Vaught criterion, when one does not demand $A, B \prec \mathfrak{C}$.

Definition 8.1. $A \subseteq_t B$ if for every $\bar{a} \in A, \bar{b} \in B$ and $\psi \in L(T)$ such that $\models \psi(\bar{b}, \bar{a})$ there is some $\bar{b}' \subseteq A$ such that $\models \psi(\bar{b}', \bar{a})$

As a simple example, note that A is complete if and only if $A \cap P \subseteq_t P$.

We can now define "free" ("non-forking") extensions for *-types over stable sets. Such extensions will be defined only to supersets that are "elementary extensions" in the sense defined above.

The use of the term "non-forking" above is not just by analogy with classical stability theory, but (at least under certain circumstances, e.g., when A

is a model) has a precise technical meaning, as the notion of independence defined below coincides with the usual non-forking independence; see Corollary 9.14.

Definition 8.2. Suppose A is stable, $p \in S_*(A)$ and $A \subseteq_t B$. Then $q \in S(B)$ is a *stationarization* of p over B if for every $\psi \in L$ there is some definition $\Psi_\psi(\bar{y}, \bar{a}_\psi)$ with $\bar{a}_\psi \subseteq A$ that defines both p_ψ and q_ψ.

Notation 8.3. (i) *We write $\bar{a} \downarrow_A B$ if A is stable and $q = \mathrm{tp}(\bar{a}/B)$ is a stationarization of $p = \mathrm{tp}(a/A)$ (so in particular $p \in S_*(A)$ and $A \subseteq_t B$). In this case, will also write $q = p|B$.*

(ii) *We write $C \downarrow_A B$ if for every $\bar{a} \in C$ we have $\bar{a} \downarrow_A B$.*

If $\bar{a} \downarrow_A B$ or $C \downarrow_A B$, we say that $\bar{a}$ (or C) is *independent* from B over A.

Let us point out some basic properties of the notions defined above.

Lemma 8.4. (i) *$A \subseteq_t B$ if and only for every quantifier free formula $\varphi(\bar{x})$ over A, if there exists $\bar{b} \in B$ such that $\models \varphi(\bar{b})$, then $A \models \exists \bar{x} \varphi(\bar{x})$.*

(ii) *If A is λ-saturated then $A \subseteq_t B$ if and only for every (partial) type $p(\bar{x})$ over a subset of A of size $< \lambda$, if p is realised by some $\bar{b} \in B$, then it is realised by some $\bar{a} \in A$.*

Proof:

(i) *By quantifier elimination and the assumption that there are no function symbols (so every subset is a substructure).*

(ii) *By quantifier elimination, $p(x)$ is equivalent (in $\mathfrak{C}$) to a quantifier free type $\Delta(\bar{x})$. By part (i), Δ is finitely satisfiable in A, and by saturation there exists $\bar{a} \in A$ such that $A \models \theta(\bar{a})$ for all $\theta(\bar{x}) \in \Delta$. Since Δ is quantifier free, the truth value of $\theta(\bar{a})$ is preserved between A and $\mathfrak{C}$; so $\Delta(\bar{x})$ is realised by $\bar{a}$, hence so is $p(\bar{x})$.*

$\blacksquare$

Lemma 8.5. *Assume A is stable, $A \subseteq_t B$ and $p \in S_*(A)$. Then:*

(i) *p has a stationarization q over B.*

(ii) *It is unique: We can replace "some $\Psi_\psi(\bar{y}, \bar{a}_\psi)$" by "every...", so q does not depend on its choice.*

(iii) *If B is complete, $q \in S_*(B)$.*

Proof: (i) By Theorem 7.2 there are quantifier free formulas $\Psi_\psi(\bar{y}, \bar{a}_\psi)$ with $\bar{a}_\psi \in A$ defining $p|_\psi$. Let $q = \{\psi(\bar{x}, \bar{b}); \bar{b} \subseteq B \text{ and } \models \Psi_\psi(\bar{b}, \bar{a}_\psi)\}$.

q is consistent: If not, there are $n < \omega, \psi_l(\bar{x}, \bar{b}_l) \in q (l = 1, \ldots, n)$ such that $\models \neg \exists \bar{x} (\bigwedge_{l=1}^n \psi_l(\bar{x}, \bar{b}_l))$. So $\models \bigwedge_{l=1}^n \Psi_{\psi_l}(\bar{b}_l, \bar{a}_{\psi_l}) \wedge \neg \exists \bar{x} (\bigwedge_{l=1}^n \psi_l(\bar{x}, \bar{b}_l))$. This is a formula in $L(T)$. So there are $\bar{b}_l \in A, (l = 1, \ldots, n)$ such that $\models \bigwedge_{l=1}^n \Psi_{\psi_l}(\bar{b}'_l, \bar{a}_{\psi_l}) \wedge \neg \exists \bar{x} (\bigwedge_{l=1}^n \psi_l(\bar{x}, \bar{b}'_l))$. But $\bigwedge_l \psi_l(\bar{x}, \bar{b}'_l) \in p$, contradicting the consistency of p.

q is complete: Assume $\bar{b} \subseteq B, \psi \in L$ and $\psi(\bar{x}, \bar{b}), \neg\psi(\bar{x}, \bar{b}) \notin q$. So $\models \neg\Psi_\psi(\bar{b}, \bar{a}_\psi) \wedge \neg\Psi_{\neg\psi}(\bar{b}, \bar{a}_{\neg\psi})$. By the definition of q and $A \subseteq_t B$ there is some $\bar{b}' \subseteq A$ so that $\models \neg\Psi_\psi(\bar{b}', \bar{a}_\psi) \wedge \neg\Psi_{\neg\psi}(\bar{b}', \bar{a}_{\neg\psi})$. But then $\psi(\bar{x}, \bar{b}'), \neg\psi(\bar{x}, \bar{b}') \notin p$ and p is complete, a contradiction.

(ii) Same proof: Let Ψ, q be as in (i), and suppose that $\Psi'_\psi(\bar{y}, \bar{a}'_\psi)$ (not necessarily quantifier free) with $\bar{a}'_\psi \in A$ also defines $p|_\psi$, and assume that it defines a ψ-type q'_ψ over B. If $q' \neq q$, that is, for example, $\psi(\bar{x}, \bar{b}) \in q$, $\psi(\bar{x}, \bar{b}) \notin q'$, then $B \models \Psi_\psi(\bar{b}) \wedge \neg\Psi'_\psi(\bar{b})$. Since $A \subseteq_t B$, there exists $\bar{a} \in A$ such that $\Psi_\psi(\bar{a}) \wedge \neg\Psi'_\psi(\bar{a})$, which is clearly absurd, since both schemata Ψ and Ψ' define the same type $p \in S(A)$.

(iii) Let B be complete. We consider $\phi(\bar{x}, \bar{y}, \bar{z})$ with $\phi'(\bar{x}, \bar{b}) := (\exists \bar{z} \in P)\phi(\bar{x}, \bar{b}, \bar{z}) \in q, \bar{b} \in^{\omega>} B$. We have to show that for some $\bar{c} \in^{\omega>} (B \cap P), \phi(\bar{x}, \bar{b}\bar{c}) \in q$. Without loss of generality $\phi(\bar{x}, \bar{y}, \bar{z}) \vdash \bar{z} \subseteq P$. So in the problematic case $\bar{b} \in^{\omega>} B, \models \Psi_{\phi'}(\bar{b})$, but for no $\bar{c} \subseteq {}^{\omega>}(B \cap P)$ is $\models \Psi_\phi(\bar{b}, \bar{c}, \bar{a}_\phi)$. But since B is complete this implies $\models \Psi_{\phi'}(\bar{b}) \wedge (\neg\exists \bar{z} \subseteq P)\Psi'_\phi(\bar{b}, \bar{z})$, so this is satisfied by some $\bar{b}' \in A$ (since the definitions are over A and $A \subseteq_t B$), and we get a contradiction to $p \in S_*(A)$. ■

Corollary 8.6. *(of the proof): If $A \subseteq_t B, A$ stable, $\bar{c} \subseteq \bar{b}, tp(\bar{b}/A) \in S_*(A)$, then $tp(\bar{c}/A) \in S_*(A)$ and the stationarization of $tp(\bar{b}/A)$ over B includes the stationarization of $tp(\bar{c}/A)$ over B.*

Corollary 8.7. *Let $q \in S_*(B)$ definable over $A \subseteq_t B$, A a stable set. Then q is the stationarization of $q\restriction A$.*

The following example illustrates the importance of the condition $A \subseteq_t B$ in the definition of the stationarization (and in Lemma 8.5(i)).

Example 8.8. *In [Cha20] Chatzidakis explores the theory $T = ACFA_0$ over $P = Fix(\sigma)$ and proves that for all uncountable λ, if A is a substructure with P^M λ-saturated, there exists a λ-primary model N of T over A. This result immediately implies (by e.g. Theorem 7.3) that $ACFA_0$ is stable over P. In this example, in the construction of the primary model over A, one has to address the following situation: $A \subseteq B$, $B^P = M^P$, B is complete and stable (this is not specifically stated in [Cha20], but is a posteriori clear, since the construction in particular yields a λ-primary model N over B), $d \in B$, and the type (extending) the difference equation $\sigma(x) = d \cdot x$ is not realized in B. Clearly (by e.g. Lemma 4.12), this types extends to $p \in S_*(B)$, and one has to realize p in order to complete the construction of N.*

Now consider $a \models p$. One may ask: can we have $b \in N$ such that $b \models p|Ba$ (so $a, b \models p$ such that $a \mathop{\smile\!\!\!\!\!\mid}_B b$)? The answer is clearly no, since in this case $\frac{a}{b}$ would be a new element in P. So why does N not realize the stationarization of p to Ba? The issue is that if B does not realize p, then $B \not\subseteq_t Ba$. And indeed in this case such a stationarization does not exist.

Let us make a few further remarks.

Remark 8.9. (i) *If $A \subseteq B \subseteq C$ and $A \subseteq_t C$, then $A \subseteq_t B$. In particular, if M is a model, then $M \subseteq_t C$ for any $C \supseteq M$. So $p \in S_*(M)$ has a (unique) stationarization over any superset.*

(ii) *If, under the assumptions of* (i), *q is the stationarization of $p \in S_*(A)$ over C (so in particular A is stable), and B is stable, then $q{\restriction}B$ is the stationarization of p over B.*

(iii) *If $A \subseteq_t B$, $tp(ab/B)$ is the stationarization of $tp(ab/A) \in S_*(A)$, and Ab is stable, then $tp(a/Bb)$ is the stationarization of $tp(a/Ab)$*

(iv) *In the previous clause, if b is finite, then the assumption on Ab is redundant, that is, it follows from the other assumptions.*

Proof: Easy. For clause (iv), note that since $tp(ab/A) \in S_*(A)$, the set Ab is complete; and since A is stable and b is finite, Ab is stable as well. ■

Lemma 8.10. *Let A, B, C be sets such that $A \subseteq_t B$, $A \subseteq C$, $C \downarrow_A B$ (see Notation 8.3). Let F be an elementary map from B onto B', G be an elementary map from C onto C' such that $F{\restriction}A = G{\restriction}A$. Then $F \cup G$ is elementary.*

Proof: Let $A' = F(A) = G(A)$. Let C'' be such that $tp(C''/B)$ is the stationarization of $tp(C'/A)$. Then clearly there is an elementary map G' such that $G'(C') = C''$, and $G''{\restriction}A'$ is the identity. First we claim that $F \cup (G' \circ G)$ is elementary.

Indeed, let $\bar{c} \in C$. Then for every $\varphi(\bar{x}, \bar{y})$ there exists $\bar{d} \in A$ such that $\Psi_\varphi(\bar{y}, \bar{d})$ defines the φ-type of $\bar{c}$ over A. In other words, $\varphi(\bar{c}, \bar{a})$ if and only if $\Psi_\varphi(\bar{a}, \bar{d})$ for all $\bar{a} \in A$. Then (since G is elementary) we have $\varphi(\bar{c}', \bar{a}')$ if and only if $\Psi_\varphi(\bar{a}', \bar{d}')$ for all $\bar{a}' \in A'$, where $\bar{c}'\bar{d}' = G(\bar{c}\bar{d})$. Hence (since G' is elementary) we have $\varphi(\bar{c}'', \bar{a}')$ if and only if $\Psi_\varphi(\bar{a}', \bar{d}')$ for all $\bar{a}' \in A'$, where $\bar{c}'' = G'(c)$. Recalling that $G'(\bar{a}'\bar{d}) = \bar{a}'\bar{d}$, we get:

$$\varphi(\bar{c}'', \bar{a}'') \iff \Psi_\varphi(\bar{a}'', \bar{d}'')$$

for all $\bar{a}'' \in A''$, where $\bar{c}''\bar{d}'' = G' \circ G(\bar{c}\bar{d})$.

In other words, $tp(\bar{c}''/A')$ is definable. Since $tp(c''/B')$ is the stationarization of $tp(\bar{c}''/A')$, by Lemma 8.5, the same definition works for since $tp(c''/B')$. Hence for all $\bar{b}'' \in B'$, letting (as before) $\bar{c}''\bar{d}'' = G' \circ G(\bar{c}\bar{d})$

$$\varphi(\bar{c}'', \bar{b}'') \iff \Psi_\varphi(\bar{b}'', \bar{d}'')$$

In particular, the equivalence above holds for $\bar{b}'' = F(\bar{b})$. Recall that since $\bar{d} \in A$, we have $F(\bar{d}) = G(d) = G' \circ G(\bar{d})$. Hence

$$\varphi(\bar{c}'', \bar{b}'') \iff \Psi_\varphi(\bar{b}'', \bar{d}'') \iff \Psi_\varphi(F(\bar{b}), F(\bar{d}))$$

(the rightmost equivalence holds since F is elementary). Now, by the choice of Ψ_φ, we also have

$$\varphi(G' \circ G(c), F(b)) = \varphi(\bar{c}'', \bar{b}'') \iff \Psi_\varphi(F(\bar{b}), F(\bar{d})) \iff \varphi(b, d)$$

which proves that $F \cup (G' \circ G)$ is elementary.

Now clearly

$$F \cup G = (G')^{-1} \circ \left[F \cup (G' \circ G)\right]$$

is also elementary, and we are done.

■

Lemma 8.11. *Let A be λ-saturated and stable, $A \subseteq_t B$, N a λ-saturated model λ-atomic over A such that $N \perp_A B$.*

Then $tp(N/A) \vdash tp(N/B)$; so the types $tp(N/A)$ and $tp(B/A)$ are weakly orthogonal.

Proof: Let $\bar{c} \in N$ and $\bar{c}' \in \mathfrak{C}$ such that $tp(\bar{c}/A) = tp(\bar{c}'/A)$. Assume towards contradiction that for some formula $\varphi(\bar{z}, \bar{b})$ (with $\bar{b} \subseteq B$) so that $\models \varphi(\bar{c}, \bar{b}) \wedge \neg\varphi(\bar{c}', \bar{b})$.

The type $tp(\bar{c}/A)$ is isolated by $\Theta(\bar{z})$, a partial type over a subset of A of cardinality less than λ.

Consider the following partial type:

$$\pi(\bar{y}) = \left\{\exists \bar{z}\bar{z}' \left[\theta(\bar{z}) \wedge \theta(\bar{z}') \wedge \varphi(\bar{z}, \bar{y}) \wedge \neg\varphi(\bar{z}', \bar{y})\right] : \ \theta \in \Theta\right\}$$

It is realized by $\bar{b} \in B$, hence, by 8.4(ii), it is also realized by some $\bar{a} \in A$.

Now consider the following type:

$$\left\{\theta(\bar{z}) \wedge \theta(\bar{z}') \wedge \varphi(\bar{z}, \bar{a}) \wedge \neg\varphi(\bar{z}', \bar{a}) : \ \theta \in \Theta\right\}$$

It is finitely satisfiable in N (precisely because $\bar{a} \models \pi(\bar{y})$), and since N is λ-saturated, it is realized by some $\bar{c}_1, \bar{c}_2$ in N (recall that Θ is over a "small" set). But $\Theta(\bar{z})$ implies a complete type over A; a contradiction.

■

9 Main consequences

Theorem 9.1. *(Stable Amalgamation for models). If $M_l, l = 1, 2$ is saturated of power λ (or just P^{M_0} is saturated), $P^{M_l} \subseteq M_0 \prec M_l$, then we can find $M \supseteq M_0 \supseteq P^M$ and elementary embeddings f_l of M_l into M over M_0 such that $tp(\bar{c}/f_2(M_2)) \in S_*(f_2(M_2))$ for all $\bar{c} \in f_1(M_1)$, and, moreover it is the stationarization of $tp(\bar{c}/M_0)$ over $f_2(M_2)$; that is, $f(M_1) \perp_{M_0} f(M_2)$.*

If $\lambda = \lambda^{<\lambda}$, then M can be chosen to be saturated.

Proof: We can find an elementary mapping f_1 from M_1 to $\mathfrak{C}$ such that $f_1|_{M_0} = id$ and for all $\bar{c} \subseteq M_1$, $tp(f_1(\bar{c})/M_2)$ is the stationarization of $tp(\bar{c}/M_0)$: Since for $\bar{c} \in M_1, P^{M_1} \subseteq M_0 \cup \bar{c} \subseteq M_1$, $M \cup \bar{c}$ is complete, hence $tp(\bar{c}/M_0)$ is in $S_*(M_0)$, and by Lemma 8.5 has a stationarization $q_{\bar{c}}$ over M_2 ($M_0 \subseteq_t M_2$, of course). By the previous corollary all these types $q_{\bar{c}}$ are compatible (being a directed system), so we can define f_1 as an elementary map so that $f_1|_{M_0} = id, dom f_1 = M_1$ and $f_1(\bar{c})$ realizes $q_{\bar{c}}$: so for $c_1, \ldots, c_n \in M_1$,

$M_2 \cup \{f_1(c_1), \ldots, f_1(c_n)\}$ is complete, $P \cap (M_2 \cup \{f_1(c_1), \ldots, f_1(c_n)\}) = P \cap M_2$ by Lemma 8.5 (iii). Hence by Fact 4.4(i) $M_2 \cup f_1(M_1)$ is complete, $P \cap (M_2 \cup \{f_1(c) : c \in M_1\}) = P \cap M_2 = P \cap M_0$. As $P \cap (M_2 \cup f_1(M_1))$ is saturated of power λ, by Proposition 4.13 there is some M with $M_2 \cup f_1(M_1) \subseteq M$ and $P^M = P^M \cap (M_2 \cup f_1(M_1)) = P^{M_0}$ as required.

If $\lambda = \lambda^{<\lambda}$, then in Proposition 4.13, M can be chosen to be saturated. ■

We can now deduce amalgamation over stable sets with a saturated P-part.

Theorem 9.2. *(Stable Amalgamation for types over stable sets). Let A be stable and saturated (or just $A \cap P$ is $|A|$-compact). If $|T| < \lambda^{<\lambda} = \lambda = |A|$, then we have amalgamation in $S_*(A)$. That is, if $tp(\bar{a}\bar{b}/A) \in S_*(A), tp(\bar{a}\bar{c}/A) \in S_*(A), \bar{a}, \bar{b}, \bar{c}$ of length $< \lambda$, then for some $\bar{a}', \bar{b}', \bar{c}', tp(\bar{a}'\bar{b}'/A) = tp(\bar{a}\bar{b}/A), tp(\bar{a}'\bar{c}'/A) = tp(\bar{a}\bar{c}/A)$ and $tp(\bar{a}'\bar{b}'\bar{c}'/A) \in S_*(A)$.*

Proof: Note that $P \cap A\bar{a}\bar{b} = P \cap A$ is saturated, $A\bar{a}\bar{b}$ is complete. Hence by Proposition 4.13 we can find a model $M_{\bar{b}}$, λ-saturated of cardinality λ such that $A \cup \bar{a}\bar{b} \subseteq M_{\bar{b}}$, and $P^{M_{\bar{b}}} \subseteq A$. Similarly, we can choose $M_{\bar{c}}$, λ-saturated of cardinality λ with $A \cup \bar{a}\bar{c} \subseteq M_{\bar{c}}$, and $P^{M_{\bar{c}}} \subseteq A$. By Theorem 7.3 (ii) there is a model $M_{\bar{a}}$ of cardinality λ, $A \cup \bar{a} \subseteq M_{\bar{a}}$, and $P^{M_{\bar{a}}} \subseteq A$ such that $M_{\bar{a}}$ is λ-primary over $A \cup \bar{a}$. Hence there is an elementary embedding $f_{\bar{b}} : M_{\bar{a}} \to M_{\bar{b}}, f_{\bar{b}}|_{(A \cup \bar{a})} = id$. Similarly, there is an elementary embedding $f_{\bar{c}} : M_{\bar{a}} \to M_{\bar{c}}, f_{\bar{c}}|_{(A \cup \bar{a})} = id$. By the previous theorem, there are elementary mappings $g_{\bar{b}}, g_{\bar{c}}$ and a model M with $g_{\bar{b}} : M_{\bar{b}} \to M, g_{\bar{c}} : M_{\bar{c}} \to M$ and $g_{\bar{b}} \circ f_{\bar{b}} = g_{\bar{c}} \circ f_{\bar{c}}$. So in particular $g_{\bar{b}}|(A \cup \bar{a}) = g_{\bar{c}}|(A \cup \bar{a}) = id$ and $P^M \subseteq A$. Now $a^\frown g_{\bar{b}}(\bar{b})^\frown g_{\bar{c}}(\bar{c})$ is as required. ■

Definition 9.3. (i) We say that a model N is λ-full over a set A if: N is saturated, $P^N \subseteq A$, and for every $B \subseteq N$, $|B| < \lambda$, every $p \in S_*(A \cup B)$ is realized in N.
If $\lambda = |A|$, we omit it.

(ii) We say that a model N is λ-homogenous for sequences if whenever $\langle a_i : i < \alpha \rangle, \langle b_i : i < \alpha \rangle, \alpha < \lambda$, realize the same type in N, then for every $a_\alpha \in N$ there is $b_\alpha \in N$ such that $\langle a_i : i \leq \alpha \rangle, \langle b_i : i \leq \alpha \rangle$ realize the same type in N.

Remark 9.4. *If N is λ-full over A, then $(N, a)_{a \in A}$ is λ-homogenous for sequences.*

Proof: Note that $P^N \subseteq A$, so for every sequence $\langle b_i : i \leq \alpha \rangle$, the type $tp(b_\alpha/A \cup \{b_i : i < \alpha\})$ is in $S_*(A \cup \{b_i : i < \alpha\})$. ■

Lemma 9.5. *Let A is stable, P^A λ-saturated with $\lambda = |A| = \lambda^{<\lambda} > |T|$, then there is M such that:*

(i) $P^M \subseteq A$*; and moreover*

(ii) Every $p \in S_*(A)$ *is realized.*

(iii) M *is* λ*-saturated of cardinality* λ*.*

Proof: Let $\langle p_i : i < \lambda\rangle$ list $S_*(A)$ (note: A is stable, $\lambda = \lambda^{|T|}$).

By induction on $i \leq \lambda$ choose A_i increasing continuously with $A_0 = A$ and $|A_{i+1} \smallsetminus A_i| < \lambda$, such that A_i is complete, and A_{i+1} realizes p_i. Use Amalgamation over A (Theorem 9.2) to amalgamate A_i and $a_i \models p_i$ at successor stages (recall that $\lambda = \lambda^{<\lambda}$) and Fact 4.4(ii) for limit stages.

Since A_λ is complete, $P \cap A_\lambda = P \cap A$ saturated, by Proposition 4.13 there is M as required (since $\lambda = \lambda^{<\lambda}$, M is also saturated).

∎

Corollary 9.6. *If* A *is stable and* λ*-saturated with* $\lambda = |A| = \lambda^{<\lambda} > |T|$*, there is* M *of cardinality* λ *which is full over* A*.*

Proof: Let M_0 be as in the previous Lemma. Now construct M_i increasing (for $i < \lambda$) such that M_i satisfies the requirements (i) – (iii) of the Lemma with A there replaced with $\bigcup_{j<i} M_j$ (note that all models are stable, and $P^{M_i} = P^{M_0} = P \cap A$).

Clearly M_λ is as required (note that λ is regular). ∎

Corollary 9.7. *If* A *is stable and* λ*-saturated with* $\lambda = |A| = \lambda^{<\lambda} > |T|$*, there is* M *such that:*

(i) M *is* λ*-saturated of cardinality* λ *with* $P^M \subseteq A$*; and moreover*

(ii) $(M, a)_{a\in A}$ *is* λ*-homogenous for sequences and every* $p \in S_*(A)$ *is realized.*

Our next goal is a Symmetry Lemma for stationarizations over a model.

We begin by showing that every "Morley sequence" (that is, a sequence of stationarizations of a given type) has a certain weak convergence property (which may remind the reader of the behaviour of indiscernible sequences in dependent theories). After having proved symmetry, we will conclude true convergence (since we will know that every such sequence is in fact an indiscernible set). However, we need the weak convergence property for the proof of symmetry, hence we deal with it first.

Lemma 9.8. *(Weak Convergence over stable sets). Let* $\langle A_i \colon i \leq \mu\rangle$ *be a sequence of stable sets increasing continuously,* $A \subseteq_t A_i$*,* $\bar{a}_i \subseteq A_{i+1}$ *and* $tp(\bar{a}_i/A_i)$ *is the stationarization of* $tp(\bar{a}_0/A)$*. Let* $\bar{c} \in A_\mu$ *and* $\psi(\bar{x}, \bar{z}, \bar{w})$ *a formula,* $\theta(\bar{z}, \bar{x}, \bar{w}) := \psi(\bar{x}, \bar{z}, \bar{w})$*. Let* Δ_2 *be finite such that* $n_\theta := R_{A_\mu}(\bar{x} = \bar{x}, \{\theta\}, \Delta_2, 2) < \omega$ *(*A_μ *is stable, so such* Δ_2*,* n_θ *exist).*

Then there are $n \leq n_\theta$*,* $0 = i_0 < i_1 < \cdots < i_n = \lambda$*, and* $p_0(\bar{x}), \ldots, p_{n-1}(\bar{x})$ *such that for all* $m < n, i_m < i < i_{m+1}$ *implies* $tp_\psi(\bar{a}_i/\bar{c} \cup A) = p_m$*.*

Proof: By Fact 5.3 (i), $\langle R_{A_\mu}(tp(\bar{c}/A_\alpha), \{\theta\}, \Delta_2, 2) : \alpha < \mu\rangle$ is a non-increasing sequence of natural numbers $\leq n_\theta$.

So there are $n \leq n_\theta, 0 = i_0 < \cdots < i_n = \mu$ such that

$$i_l \leq \alpha \leq \beta < i_{l+1} \Rightarrow R_{A_\mu}(tp_\theta(\bar{c}/M_\alpha), \{\theta\}, \Delta_2, 2) = R_{A_\mu}(tp_\theta(\bar{c}/M_\beta), \{\theta\}, \Delta_2, 2).$$

Again by Fact 5.3 (more specifically, by the proof of Corollary 5.8 – the nature of the defining scheme), for each l there is $\bar{d}_l \in A_{i_l}$ and $\Psi_\theta(\bar{x}, \bar{w}, \bar{d}_l)$ which defines $tp_\theta(\bar{c}/A_{i_l})$ so that $\Psi_\theta(\bar{x}, \bar{w}, \bar{d}_l)$ actually defines $tp_\theta(\bar{c}/A_\alpha)$ for $i_l \leq \alpha < i_{l+1}$. But if $i_l \leq \alpha < \beta < i_{l+1}$, then $tp(\bar{a}_\alpha/A_{i_l}) = tp(\bar{a}_\beta/A_{i_l})$, hence for every $\bar{m} \subseteq A$, $\models \Psi_\theta(\bar{a}_\alpha, \bar{m}, \bar{d}_l) \equiv \Psi_\theta(\bar{a}_\beta, \bar{m}, \bar{d}_l)$, hence $\models \theta(\bar{a}_\alpha, \bar{m}, \bar{c}) \equiv \theta(\bar{a}_\beta, \bar{m}, \bar{c})$, as required.

$\blacksquare$

Theorem 9.9. *(The symmetry theorem for models). If $tp(\bar{a}\bar{b}/M) \in S_*(M)$ where $tp(\bar{b}/M \cup \bar{a})$ is the stationarization of $tp(\bar{b}/M)$, then $tp(\bar{a}/M \cup \bar{b})$ is the stationarization of $tp(\bar{a}/M)$.*

Proof: Assume we have a counterexample. Let $\lambda = \lambda^{<\lambda} \geq |M| + |T|^+$. Without loss of generality M is saturated of cardinality λ. We define $\bar{a}_i, \bar{b}_i, M_i$ by induction on $i < \lambda$ such that $P^{M_i} = P^{M_0}$, M_{i+1} is λ-saturated of power λ, M_i increasing continuously, $\bar{a}_i\bar{b}_i \subseteq M_{i+1}$ and $tp(\bar{a}_i\bar{b}_i/M_i)$ is the stationarization of $tp(\bar{a}\bar{b}/M)$. This is straightforward. Let $M_\lambda = \bigcup_{i<\lambda} M_i$.

Since all models are stable, we are clearly in the situation of Lemma 9.8. Note that $P^{M_\lambda} = P^M \subseteq M$, hence by conclusion of the Lemma we get:

For every $\bar{c} \in M_\lambda$ and $\psi(\bar{x}, \bar{y}, \bar{z})$ there are $n \leq n_\theta$ and $0 = i_0 < i_1 < \cdots < i_n = \lambda$ and $p_0, \ldots, p_{n-1}$ such that for all $m < n, i_m < i < i_{m+1}$ implies $tp_\psi(\bar{a}_i\bar{b}_i/\bar{c} \cup P^M) = p_m$.

From the uniqueness and definability of stationarizations it follows that $\langle \bar{a}_i\bar{b}_i : i < \lambda\rangle$ is indiscernible over M_0. That is, if $i_0 < i_1 < \cdots < i_n < \lambda$ and $j_0 < \cdots < j_n < \lambda$ then $tp(\bar{a}_{i_0}\bar{b}_{i_0} \ldots \bar{a}_{i_n}\bar{b}_{i_n}/M_0) = tp(\bar{a}_{j_0}\bar{b}_{j_0} \ldots \bar{a}_{j_n}\bar{b}_{j_n}/M_0)$.

Now let $R = \{\bar{a}_i\bar{b}_i : i < \lambda\}$ and let $<$ be the order on R defined so that $\bar{a}'\bar{b}' < \bar{a}"\bar{b}"$ iff there are $i < j$ with $\bar{a}'\bar{b}' = \bar{a}_i\bar{b}_i$ and $\bar{a}"\bar{b}" = \bar{a}_j\bar{b}_j$. The model $(M_\lambda, R, <)$ has a saturated extension, $(M^*, R^*, <^*)$ of power λ. So for some linear order I, $R^* = \{\bar{a}_t\bar{b}_t : t \in I\}$ where $\bar{a}_s\bar{b}_s < \bar{a}_t\bar{b}_t$ whenever $I \models s < t$.

Note that n depends on ψ and $\bar{c}$, but the bound n_θ depends on ψ only. Therefore the following is true:

Fact 9.10. For every $\bar{c} \in M^*$ and $\psi(\bar{x}, \bar{y}, \bar{z})$ there are $n < n_\theta < \omega$ and $t_0 < \cdots < t_n$ where $t_0 \in I$ is the first element, and $p_0, \ldots p_n$ so that $t_l < t < t_{l+1}$ or $t_l < t, l = n$ implies $tp_\psi(\bar{a}_t\bar{b}_t/P^{M^*} \cup \bar{c}) = p_l$.

Now we shall finish proving the Symmetry Lemma: Since I is a λ-saturated linear order of power λ, it has 2^λ Dedekind cuts $\{(I_\alpha, J_\alpha) : \alpha < 2^\lambda\}$. Let $p_\alpha = \{\psi(\bar{x}, \bar{y}, \bar{d}); \bar{d} \in M^*$ and for some $s_\alpha \in I_\alpha, t_\alpha \in J_\alpha$, if $v \in I$ and $s_\alpha < v < t_\alpha$ then $\models \psi(\bar{a}_v, \bar{b}_v, \bar{d})\}$.

By Fact 9.10, p_α is a complete type over M^*. In fact, $p_\alpha \in S_*(M^*)$: indeed,if $[\exists \bar{z} \in P\psi(\bar{z}, \bar{m}, \bar{x}, \bar{y})] \in p_\alpha$ (where $\bar{m} \subseteq M^*$), then (by definition of p_α) there exist s_α, t_α such that for all $v \in (s_\alpha, t_\alpha)$, we have $\models \exists \bar{z} \in P\psi(\bar{z}, \bar{m}, \bar{a}_v, \bar{b}_v)$; hence M^* satisfies this sentence as well, so there exists such $\bar{d} \subseteq P^{M^*}$. A priori perhaps d depends on v; however, recall (Fact 9.10) that for some s'_α, t'_α all a_v, b_v (for $v \in (s'_\alpha, t'_\alpha)$) have the same ψ-type over $\bar{m} \cup P^M*$, so choosing d for any such v, we get that for all $v \in (s'_\alpha, t'_\alpha)$ the formula $\psi(\bar{d}, \bar{m}, \bar{a}_v, \bar{b}_v)$ holds, hence by the definition $\psi(\bar{d}, \bar{m}, \bar{x}, \bar{y}) \in p_\alpha$. Now by Observation 4.11, $p_\alpha \in S_*(M^*)$, as required.

If $i < j, tp(\bar{b}_j/M \cup \bar{a}_i) \subseteq tp(\bar{b}_j/M_j)$ is a stationarization of $tp(\bar{b}/M)$. By the uniqueness of stationarizations and the assumption that $tp(\bar{b}/M \cup \bar{a})$ is the stationarization of $tp(\bar{b}/M)$ it follows that $tp(\bar{b}_j\bar{a}_i/M) = tp(\bar{b}\bar{a}/M)$.

Similarly $tp(\bar{a}_1/M \cup \bar{b}_0)$ is the stationarization of $tp(\bar{a}_1/M)$ and hence $\neq tp(\bar{a}_0/M \cup \bar{b}_0)$ as we assumed that $\bar{a}\bar{b}$ form a counter-example to symmetry. So for some $\bar{e} \in M$ and θ we have $\models \theta(\bar{a}, \bar{b}, \bar{e}) \wedge \neg\theta(\bar{a}_1, \bar{b}_0, \bar{e})$. Therefore we get:

$$j \geq i \Rightarrow \models \theta(\bar{a}_i b_j); \qquad j < i \Rightarrow \models \neg\theta(\bar{a}_i b_j)$$

So $\theta(\bar{x}, \bar{b}_t) \in p_\alpha(\bar{x}, \bar{y})$ if and only if $t \in J_\alpha$. Hence if $\alpha \neq \beta$, then $p_\alpha \neq p_\beta$. So we have too many types in $S_*(M^*)$, contradicting stability of M^*. ∎

Definition 9.11. (i) We say that $I = \{a_\alpha : \alpha < \alpha^*\}$ is *convergent* over a set A if it is an infinite indiscernible set such that for every ψ and $\bar{d} \in A$ there is some n_ψ such that the type $tp_\psi(\bar{a}_\alpha/P^{\mathfrak{C}} \cup \bar{d})$ is the same for all but $\leq n_\psi$ many α's.

(ii) For any such I let the *average* of I over A be $Av(I/A) = \{\phi(\bar{x}, \bar{b}) : \bar{b} \in A, (\exists^\infty \alpha)\phi(\bar{a}_\alpha, \bar{b})\}$.

Corollary 9.12. *If $\alpha \geq \omega$ and for $i \leq \alpha, tp(\bar{a}_i/M \cup \bigcup_{j<i} \bar{a}_j)$ is a stationarization of $tp(\bar{a}/M)$, then $\{\bar{a}_i : i < \alpha\}$ is an indiscernible set over M. That is, if $i_1, \ldots, i_n < \alpha$ are distinct, then $tp(\bar{a}_{i_1} \ldots \bar{a}_{i_n}/M) = tp(\bar{a}_1 \ldots \bar{a}_n/M)$. In addition, if $I = \{\bar{a}'_\alpha : \alpha < \alpha_0\}$ is an indiscernible set with the same type over M, it is convergent over any $M' \succ M$, and for $M' \succ M, Av(I/M')$ is the stationarization of $tp(\bar{a}_0/M)$ over M'.*

Proof: It follows from Symmetry (Theorem 9.9) that any such sequence is an indiscernible set over M. Now a standard argument shows that for indiscernible sets weak convergence (Lemma 9.8) implies convergence. ∎

We can therefore conclude:

Corollary 9.13. *The global stationarization of a type orthogonal to P over a model $p \in S_*(M)$ is a generically stable type, as defined in [PT11, GOU13].*

Proof: First note that since M is a model, p has a (unique) global stationarization p^*, which is definable (hence invariant) over M. By the previous Corollary, every Morley sequence in p^* is convergent over $\mathfrak{C}$, in particular, p^* is generically stable (as defined in Definition 1 of [PT11]). ∎

One can now use some basic machinery of generic stability to throw more light on *-types over models and the concept of stationarization, for example:

Corollary 9.14. *Let M be a model and $p \in S_*(M)$. Then the unique global stationarization of p is also the unique global non-forking extension of p.*

Proof: By Proposition 1 in [PT11]. ■

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