STABLE FRAMES AND WEIGHTS
SH839

SAHARON SHELAH

Abstract. We would like to generalize imaginary elements, weight of $\mathrm{ortp}(a, M, N)$, **P**-weight, **P**-simple types, etc. from [She90, Ch.III,V,§4] to the context of good frames. This requires allowing the vocabulary to have predicates and function symbols of infinite arity, but it seems that we do not suffer any real loss.

Date: March 7, 2023.

2020 *Mathematics Subject Classification.* Primary 03C45, 03C48.

Key words and phrases. model theory, classification theory, stability, AEC, stability, orthogonality, weight, main gap.

For earlier versions, the author thanks Alice Leonhardt for the beautiful typing up to 2019. The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 242/03). First Typed 2002-Sept.-10.
For later versions, the author would like to thank the NSF and BSF for partially supporting this research — NSF-BSF 2021: grant with M. Malliaris, NSF 2051825, BSF 3013005232 (2021/10-2026/09). The author is also grateful to an individual who wishes to remain anonymous for generously funding typing services, and thanks Matt Grimes for the careful and beautiful typing.
References like [Sh:950, Th0.2=Ly5] mean that the internal label of Th0.2 is y5 in Sh:950. The reader should note that the version in my website is usually more up-to-date than the one in arXiv.

Annotated Content

§ 0. Introduction

We assume $\mathfrak{s}$ is a good λ-frame with some extra properties from [She09d] (e.g., as in the assumption of [She09d, §12]) so we shall assume knowledge of [She09d] and the basic facts on good λ-frames from [She09c].

We can look at results from [She90] which were not regained in beautiful λ-frames. Well, of course, we are far from the main gap for the original $\mathfrak{s}$ ([She90, Ch.XIII]) and there are results which are obviously more strongly connected to elementary classes, particularly ultraproducts. This leaves us with parts of type theory: regular and semi-regular types, weight, **P**-simple[1] types, "hereditarily orthogonal to **P**" (the last two were defined and investigated in [She78, Ch.V,§0 + Def4.4-Ex4.15], [She90, Ch.V,§0,pg.226,Def4.4-Ex4.15,pg.277-284]).

Some of Hrushovski's profound works are a continuation of [She78, §4] and [She90, §4], but note that "a type q is p-simple (or **P**-simple)" and "q is hereditarily orthogonal to p (or **P**)" here are essentially the[2] "internal" and "foreign" there.

For more on understanding regular types in the first order case, see both [She04] and Laskowski and the author in [LS15].

* * *

This paper was Part I of the original [Shec], which has existed (and circulated to some extent) since 2002. The second and third parts have been split off into [Shea], [Sheb]. They have been continued in [LS06] and [LS11], respectively.

Notation 0.1. As in [She90], [Shear], M and N are models, M has vocabulary τ_M, $|M|$ is its universe and $\|M\|$ its cardinality. We write $\text{ortp}(-)$ for the orbital type.

---

[1]The motivation is that for suitable **P** (e.g. a single regular type), on the one hand

$\text{stp}(a, A) \not\perp \mathbf{P} \Rightarrow$ "$\text{stp}(a/E, A)$ is **P**-simple for some equivalence relation definable over A"

and on the other hand, if $\text{stp}(a_i, A)$ is **P**-simple for $i < \alpha$ then $\Sigma\{w(a_i, A) \cup \{a_j : j < i\}) : i < \alpha\}$ does not depend on the order in which we list the a_i-s. Note that **P** here is $\mathscr{P}$ there.

[2]Note, "foreign to **P**" and "hereditarily orthogonal to **P**" are equivalent. Now (with $\mathbf{P} = \{p\}$ for simplicity)

(a) $q(x)$ is $p(x)$-simple when for some set A, in $\mathfrak{C}$ we have $q(\mathfrak{C}) \subseteq \text{acl}(A \cup \bigcup p_i(\mathfrak{C}))$.
(b) $q(x)$ is $p(x)$-internal when for some set A, in $\mathfrak{C}$ we have $q(\mathfrak{C}) \subseteq \text{dcl}(A \cup p(\mathfrak{C}))$.

Note

(α) Internal implies simple.
(β) If we aim at computing weights, it is better to stress acl as it covers more.
(γ) But the difference is minor, and in existence it is better to stress dcl.
(δ) Also, it is useful that

$$\{F \restriction (p(\mathfrak{C}) \cup q(\mathfrak{C}) : F \text{ an automorphism of } \mathfrak{C} \text{ over } p(\mathfrak{C}) \cup \text{dom}(p)\}$$

is trivial when $q(x)$ is p-internal but not so for p-simple (though form a pro-finite group).

§ 1. Weight and **P**-weight

On 'good$^+$,' see Definition [She09d, 1.3(1), pg.7] and Claim [She09d, 1.5(1), pg.7], which relies on [She09c, §3], [She01].
On 'type-full,' see Definition [She09c, 6.35, pg.112]: it means $\mathcal{S}_{\mathfrak{s}}(M) = \mathcal{S}^1_{\mathbf{k}}(M)$.
On primes and $K^{3,\mathrm{qr}}_{\mathfrak{s}}$, see [She09d, 5.15, pg.73].
On $K^{3,\mathrm{vq}}_{\mathfrak{s}} = K^{3,\mathrm{qr}}_{\mathfrak{s}}$, see Definition [She09d, 5.9, pg.69].
On orthogonality, see [She09d, §6].

Context 1.1. 1) $\mathfrak{s}$ is a type-full good$^+$ λ-frame with primes, $K^{3,\mathrm{vq}}_{\mathfrak{s}} = K^{3,\mathrm{qr}}_{\mathfrak{s}}$, $\perp = \underset{\mathrm{wk}}{\perp}$ and $p \perp M \Leftrightarrow p \underset{\mathrm{a}}{\perp} M$. Note that as $\mathfrak{s}$ is full, $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(M)$; also, $\mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}[\mathfrak{s}] = (K^{\mathfrak{s}}, \leq_{\mathfrak{k}_{\mathfrak{s}}})$ is the AEC.
2) $\mathfrak{C}$ is an $\mathfrak{s}$-monster so it is $K^{\mathfrak{s}}_{\lambda^+}$-saturated over λ, and $M <_{\mathfrak{s}} \mathfrak{C}$ means $M \leq_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M \in K_{\mathfrak{s}}$. As $\mathfrak{s}$ is full, it has regulars.

Observation 1.2. $\mathfrak{s}^{\mathrm{reg}}$ *satisfies all the above except being full.*

Remark 1.3. Recall $\mathfrak{s}^{\mathrm{reg}}$ is derived from $\mathfrak{s}$, replacing $\mathcal{S}_{\mathfrak{s}}(M)$ by $\{p \in \mathcal{S}_{\mathfrak{s}}(M) : p \text{ regular}\}$ (see [She09d, 10.18, pg.164]).

Proof. See [She09d, 10.18=L10.p19tex] and Definition [She09d, 10.17=L10.p18tex]. □

Claim 1.4. *1) If $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>then</u> we can find b, N and a finite $\mathbf{J}$ such that:*

⊛ (a) $M \leq_{\mathfrak{s}} N$
(b) $\mathbf{J} \subseteq N$ *is a finite independent set in* (M, N).
(c) $c \in \mathbf{J} \Rightarrow \mathrm{ortp}(c, M, N)$ *is regular (recalling that* ortp *stands for 'orbital type').*
(d) $(M, N, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$
(e) $b \in N$ *realizes* p.

2) If M is brimmed, we can add

(f) $(M, N, b) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.

3) In (2), $|\mathbf{J}|$ depends only on (p, M).
4) If M is brimmed, <u>then</u> we can work in $\mathfrak{s}(\mathrm{brim})$ and get the same $\|\mathbf{J}\|$ and N (so $N \in K_{\mathfrak{s}}$ is brimmed).

Proof. 1) By induction on $\ell < \omega$, we try to choose N_ℓ, a_ℓ, q_ℓ such that:

(∗) (a) $N_0 = M$
(b) $N_\ell \leq_{\mathfrak{s}} N_{\ell+1}$
(c) $q_\ell \in \mathcal{S}_{\mathfrak{s}}(N_\ell)$, so possibly $q_\ell \notin \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N_\ell)$.
(d) $q_0 = p$
(e) $q_{\ell+1} \restriction N_\ell = q_\ell$
(f) $q_{\ell+1}$ forks over N_ℓ, so now <u>necessarily</u> $q_\ell \notin \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N_\ell)$.
(g) $(N_\ell, N_{\ell+1}, a_\ell) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
(h) $r_\ell = \mathrm{ortp}(a_\ell, N_\ell, N_{\ell+1})$ is regular.
(i) r_ℓ either is $\perp M$ or does not fork over M.

If we succeed to carry the induction for all $\ell < \omega$, let $N = \bigcup\{N_\ell : \ell < \omega\}$. As this is a countable chain (recalling $\mathfrak{K}_{\mathfrak{s}}$ has amalgamation), there is $q \in \mathcal{S}(N)$ such that $\ell < \omega \Rightarrow q \restriction N_\ell = q$ and as q is not algebraic (because each q_n is not), and $\mathfrak{s}$ is full, clearly $q \in \mathcal{S}_{\mathfrak{s}}(N)$; but q contradicts the finite character of non-forking. So for some $n \geq 0$ we are stuck, but this cannot occur if $q_n \in \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}}(N_n)$.

[Why? Because we are assuming that $\mathfrak{s}$ is type-full. Alternatively, we can use $\mathfrak{s}^{\rm reg}$, recalling that by 1.2, we know that $\mathfrak{s}^{\rm reg}$ has enough regulars and then we can apply [She09d, 8.3=L6.1tex].]

So for some $b \in N_n$ we have $q_n = \text{ortp}(b, N_n, N_n)$; i.e., b realizes q_n hence it realizes p.

Let $\mathbf{J} = \{a_\ell : \text{ortp}(a_\ell, N_\ell, N_{\ell+1})$ does not fork over $N_0\}$. By [She09d, 6.2] we have $(M, N_n, \mathbf{J}) = (N_0, N_n, \mathbf{J}) \in K^{3,\rm vq}_{\mathfrak{s}}$ hence $\in K^{3,\rm qr}_{\mathfrak{s}}$ by [She09d] so we are done.
2) Let $N, b, \mathbf{J}$ be as in part (1) with $|\mathbf{J}|$ minimal. We can find $N' \le_{\mathfrak{s}} N$ such that $(M, N', b) \in K^{3,\rm pr}_{\mathfrak{s}}$ and we can find $\mathbf{J}'$ such that $\mathbf{J}' \subseteq N'$ is independent regular in (M, N') and maximal under those demands. Then we can find $N'' \le_{\mathfrak{s}} N'$ such that $(M, N'', \mathbf{J}') \in K^{3,\rm qr}_{\mathfrak{s}}$. If $\text{ortp}_{\mathfrak{s}}(b, N'', N') \in \mathcal{S}^{\rm na}_{\mathfrak{s}}(N'')$ is not orthogonal to M we can contradict the maximality of $\mathbf{J}'$ in N' as in the proof of part (1), so $\text{ortp}_{\mathfrak{s}}(b, N'', N') \perp M$ (or $\notin \mathcal{S}^{\rm na}_{\mathfrak{s}}(N)$). Also without loss of generality $(N'', N', b) \in K^{3,\rm pr}_{\mathfrak{s}}$, so by [She09d] we have $(M, N', \mathbf{J}') \in K^{3,\rm qr}_{\mathfrak{s}}$. Hence there is an isomorphism f from N' onto N'' which is the identity of $M \cup \mathbf{J}'$ (by the uniqueness for $K^{3,\rm qr}_{\mathfrak{s}}$). So using $(N', f(b), \mathbf{J}')$ for $(N, b, \mathbf{J})$ we are done.
3) If not, we can find $N_1, N_2, \mathbf{J}_1, \mathbf{J}_2, b$ such that $M \le_{\mathfrak{s}} N_\ell \le_{\mathfrak{s}} N$ and the quadruple $(M, N_\ell, \mathbf{J}_\ell, b)$ is as in (a)-(e)+(f) of part (1)+(2) for $\ell = 1, 2$. Assume toward contradiction that $|\mathbf{J}_1| \neq |\mathbf{J}_2|$, so without loss of generality $|\mathbf{J}_1| < |\mathbf{J}_2|$.

By "$(M, N_\ell, b) \in K^{3,\rm pr}_{\mathfrak{s}}$," without loss of generality, $N_2 \le_{\mathfrak{s}} N_1$.

By [She09d, 10.15=L10b.11tex(3)] for some $c \in \mathbf{J}_2 \setminus \mathbf{J}_1$, $\mathbf{J}_1 \cup \{c\}$ is independent in (M, N_1), in contradiction to $(M, N, \mathbf{J}_1) \in K^{3,\rm vq}_{\mathfrak{s}}$ by [She09d, 10.15=L10b.11tex(4)].
4) Similarly. $\square_{1.4}$

Definition 1.5. 1) For $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$, let the *weight* of p, $w(p)$, be the unique natural number such that if $M \le_{\mathfrak{s}} M'$, M' is brimmed, and $p' \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M')$ is a non-forking extension of p <u>then</u> it is the unique $|\mathbf{J}|$ from Claim 1.4(3). (It is a natural number.)
2) Let $w_{\mathfrak{s}}(a, M, N) = w(\text{ortp}_{\mathfrak{s}}(a, M, N))$.

Claim 1.6. *1) If $p \in \mathcal{S}^{\rm bs}(M)$ is regular, <u>then</u> $w(p) = 1$.*
2) If $\mathbf{J}$ is independent in (M, N) and $c \in N$, <u>then</u> for some $\mathbf{J}' \subseteq \mathbf{J}$ with $\le w_{\mathfrak{s}}(c, M, N)$ elements, $\{c\} \cup (\mathbf{J} \setminus \mathbf{J}')$ is independent in (M, N).

Proof. Easy by now. $\square_{1.6}$

Note that the use of $\mathfrak{C}$ in Definition 1.7 is for transparency only and can be avoided; see 1.11 below.

Definition 1.7. 1) We say that $\mathbf{P}$ is an M^*-based family (inside $\mathfrak{C}$) <u>when</u>:

(A) $M^* <_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M^* \in K_{\mathfrak{s}}$
(B) $\mathbf{P} \subseteq \bigcup \{\mathcal{S}^{\rm bs}_{\mathfrak{s}}(M) : M \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M \in K_{\mathfrak{s}}\}$
(C) $\mathbf{P}$ is preserved by automorphisms of $\mathfrak{C}$ over M^*.

2) Let $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$, where $M \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$.

(A) We say that p is hereditarily orthogonal to $\mathbf{P}$ (or $\mathbf{P}$-foreign) <u>when</u>: if $M \le_{\mathfrak{s}} N \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$, $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$, and $q \restriction M = p$, <u>then</u> q is orthogonal to p.
(B) We say that p is $\mathbf{P}$-regular <u>when</u> p is regular, not orthogonal to $\mathbf{P}$ and if $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M')$, $M \le_{\mathfrak{s}} M' <_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$, and q is a forking extension of p <u>then</u> q is hereditarily orthogonal to $\mathbf{P}$.
(C) p is weakly $\mathbf{P}$-regular <u>if</u> it is regular and is not orthogonal to some $\mathbf{P}$-regular p'.

3) $\mathbf{P}$ is normal <u>when</u> $\mathbf{P}$ is a set of regular types and each of them is $\mathbf{P}$-regular.
4) For $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $M <_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$, let $w_{\mathbf{P}}(q)$ be defined as the natural number satisfying the following:

- $\circledast$ If $M \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2 \le_{\mathfrak{s}} \mathfrak{C}$, M_ℓ is $(\lambda,*)$-brimmed, $b \in M_2$, $\mathrm{ortp}_{\mathfrak{s}}(b, M_1, M_2)$ is a non-forking extension of q, $(M_1, M_2, b) \in K^{3,\rm pr}_{\mathfrak{s}}$, $(M_1, M_2, \mathbf{J}) \in K^{3,\rm qr}_{\mathfrak{s}}$, and $\mathbf{J}$ is regular in (M_1, M_2) (i.e. independent and $c \in \mathbf{J} \Rightarrow \mathrm{ortp}_{\mathfrak{s}}(c, M_1, M_2)$ is a regular type) <u>then</u>

$$w_{\mathbf{P}}(q) = \big|\{c \in \mathbf{J} : \mathrm{ortp}_{\mathfrak{s}}(c, M_1, M) \text{ is weakly } \mathbf{P}\text{-regular}\}\big|.$$

5) We replace $\mathbf{P}$ by p if $\mathbf{P} = \{p\}$, where $p \in \mathcal{S}^{\rm bs}(M^*)$ is regular (see 1.8(1)).

Claim 1.8. *1) If $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is regular <u>then</u> $\{p\}$ is an M-based family and is normal.*
2) Assume $\mathbf{P}$ is an M^*-based family. If $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $M^* \le_{\mathfrak{s}} M \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ <u>then</u> $w_{\mathbf{P}}(q)$ is well defined (and is a natural number).
3) In Definition 1.7(4) we can find $\mathbf{J}$ such that for every $c \in \mathbf{J}_1$ we have:

$$\mathrm{ortp}(c, M_1, M) \textit{ is weakly } \mathbf{P}\textit{-regular} \Rightarrow \mathrm{ortp}(c, M_1, M) \textit{is } \mathbf{P}\textit{-regular.}$$

Proof. Should be clear. $\square_{1.8}$

Discussion 1.9. 1) It is tempting to try to generalize the notion of $\mathbf{P}$-simple ($\mathbf{P}$-internal in Hrushovski's terminology) and semi-regular. An important property of those notions in the first order case is that: e.g.

- $(*)$ If $(\bar a/A) \not\perp p$ and p regular, then for some equivalence relation E definable over A, $\mathrm{ortp}(\bar a/E, A) \not\perp p$ and is $\{p\}$-simple.

The aim of defining $\{p\}$-simple is:

- (A) For an element a realizing p over $A \subseteq \mathfrak{C}^{\rm eq}_T$, we can define the p-weight $w_p(a, A)$.
- (B) The p-weight of such elements behaves like finite sequences from a vector space — so they behave like dimensions of vector spaces.
- (C) We have appropriate density results.

2) First attempt towards (C) above: assume that $p, q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ are not orthogonal, and we can define an equivalence relation $\mathscr{E}^{p,q}_M$ on $\{c \in \mathfrak{C} : c \text{ realizes } p\}$, defined by

$$c_1 \ \mathscr{E}^{p,q}_M \ c_2 \quad \underline{\text{iff}} \quad \begin{array}{l}\text{for every } d \in \mathfrak{C} \text{ realizing } q, \text{ we have} \\ \mathrm{ortp}_{\mathfrak{s}}(c_1 d, M, \mathfrak{C}) = \mathrm{ortp}_{\mathfrak{s}}(c_2 d, M, \mathfrak{C}).\end{array}$$

This (the desired property) may fail even in the first order case: suppose p, q are definable over $a^* \in M$ (on getting this, see later) and we have $\langle c_\ell : \ell \le n\rangle$, $\langle M_\ell : \ell < n\rangle$ such that $\mathrm{ortp}(c_\ell, M_\ell, \mathfrak{C}) = p_\ell$, each p_ℓ is parallel to p, $c_\ell \ \mathscr{E}^{p,q}_{M_\ell} \ c_{\ell+1}$ but c_0, c_n realize p and q respectively, and $\{c_0, c_n\}$ is independent over M_0. Such a situation defeats the attempt to define a $\mathbf{P}$-$\{q\}$-simple type $p/\mathscr{E}$ as in [She90, Ch.V].

However (see [She90, V, §4]), in first order logic we can find a saturated N and $a^* \in N$ such that

$$\mathrm{ortp}\big(M, \bigcup_\ell M_\ell \cup \{c_0, \dots, c_n\}\big)$$

does not fork over a^* and use "average on the type with an ultrafilter c over $q(\mathfrak{C}) + a^*_t$" (for suitable a^*_t-s). See more below.

Discussion 1.10. 1) Assume ($\mathfrak{s}$ is full and) every $p \in \mathcal{S}^{\rm na}_{\mathfrak{s}}(M)$ is representable by some $a_p \in M$ (e.g., in [She90], the canonical base $\mathrm{Cb}(p)$). We can define for $\bar{a}, \bar{b} \in {}^{\omega>}\mathfrak{C}$ when $\mathrm{ortp}(\bar{a}, \bar{b}, \mathfrak{C})$ is stationary (and/or non-forking). We should check the basic properties. See §3.
2) Assume $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is regular, definable over $\bar{a}^*$ (in the natural sense). We may wonder if the niceness of the dependence relation holds for $p \restriction \bar{a}^*$?

If you feel that the use of a monster model is not natural in our context, how do we "translate" a set of types in $\mathfrak{C}^{\rm eq}$ preserved by every automorphism of $\mathfrak{C}$ which is the identity on A? By using a "place" defined by:

Definition 1.11. 1) A *local place* is a pair $\mathbf{a} = (M, A)$ such that $A \subseteq M \in K_{\mathfrak{s}}$ (compare with [Sheb, §1]).
2) The places $(M_1, A_1), (M_2, A_2)$ are equivalent if $A_1 = A_2$ and there are n and $N_\ell \in K_{\mathfrak{s}}$ for $\ell \leq n$ satisfying $A \subseteq N_\ell$ for $\ell = 0, \ldots, n$ such that $M_1 = N_0$, $M_2 = N_n$, and for each $\ell < n$, $N_\ell \leq_{\mathfrak{s}} N_{\ell+1}$ or $N_{\ell+1} \leq_{\mathfrak{s}} N_\ell$. We write $(M_1, A_1) \sim (M_2, A_2)$ or $M_1 \sim_{A_1} M_2$.
3) For a local place $\mathbf{a} = (M, A)$, let $K_{\mathbf{a}} = K_{(M,A)} = \{N : (N, A) \sim (M, A)\}$, so in $(M, A)/\sim$ we fix both A as a set and the type it realizes in M over $\varnothing$.
4) We call such class $K_{\mathbf{a}}$ a *place.*
5) We say that $\mathbf{P}$ is an invariant set[3] of types in a place $K_{(M,A)}$ <u>when</u>:

- (A) $\mathbf{P} \subseteq \{\mathcal{S}^{\rm bs}_{\mathfrak{s}}(N) : N \sim_A M\}$
- (B) Membership in $\mathbf{P}$ is preserved by isomorphism over A.
- (C) If $N_1 \leq_{\mathfrak{s}} N_2$ are both in $K_{(M,A)}$ and $p_2 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_2)$ does not fork over N_1 then $p_2 \in \mathbf{P} \Leftrightarrow p_2 \restriction N_1 \in \mathbf{P}$.

6) We say $M \in K_{\mathfrak{s}}$ is brimmed over A <u>when</u> for some N we have $A \subseteq N \leq_{\mathfrak{s}} M$ and M is brimmed over N.

Claim/Definition 1.12. *1) If $A \subseteq M \in K_{\mathfrak{s}}$ and $\mathbf{P}_0 \subseteq \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ <u>then</u> there is at most one invariant set $\mathbf{P}'$ of types in the place $K_{(M,A)}$ such that $\mathbf{P}^+ \cap \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M) = \mathbf{P}_0$ and*

$$M \leq_{\mathfrak{s}} N \wedge p \in \mathbf{P}^+ \cap \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N) \Rightarrow \text{"}p \textit{ does not fork over } M\text{"}.$$

2) If in addition, M is brimmed[4] over A <u>then</u> we can omit the last demand in part (1).
3) If $\mathbf{a} = (M_1, A)$ and $(M_2, A) \in K_{\mathbf{a}}$ then $K_{(M_2,A)} = K_{\mathbf{a}}$.

Proof. Easy. $\square_{1.12}$

Definition 1.13. 1) If in 1.12 there are such $\mathbf{P}$, we denote it by $\mathrm{inv}(\mathbf{P}_0) = \mathrm{inv}(\mathbf{P}_0, M)$.
2) If $\mathbf{P}_0 = \{p\}$, then let $\mathrm{inv}(p) = \mathrm{inv}(p, M) = \mathrm{inv}(\{p\})$.
3) We say $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ *does not split* (or *is definable*) over A <u>when</u> $\mathrm{inv}(p)$ is well defined.

[3] Really a class.
[4] M is brimmed over A means that for some M_1, $A \subseteq M_1 \leq_{\mathfrak{s}} M$ and M is brimmed over M_1.

§ 2. Imaginary elements, an essential-(μ,λ)-AEC, and frames

§ 2(A). **Essentially AEC.**

We consider revising the definition of an AEC $\mathfrak{k}$, by allowing function symbols in $\tau_{\mathfrak{k}}$ with infinite number of places while retaining local characters, e.g., if $M_n \le M_{n+1}$ and $M = \bigcup\{M_n : n < \omega\}$ is uniquely determined. Before this, we introduce the relevant equivalence relations. In this context, we can give name to equivalence classes for equivalence relations on infinite sequences.

Definition 2.1. We say that $\mathfrak{k}$ is an essentially $[\lambda,\mu)$-AEC or ess-$[\lambda,\mu)$-AEC (or $[\lambda,\mu)$-EAEC[5]) <u>if</u> ($\lambda < \mu$ and) it is an object consisting of:

I. (a) A vocabulary $\tau = \tau_{\mathfrak{k}}$, which has predicates and function symbols of possibly infinite arity but $\le \lambda$.
(b) A class $K = K_{\mathfrak{k}}$ of τ-models.
(c) A two-place relation $\le_{\mathfrak{k}}$ on K.
(Note that we allow $\mu = \infty$).

such that

II. (a) If $M_1 \cong M_2$ <u>then</u> $M_1 \in K \Leftrightarrow M_2 \in K$.
(b) If $(N_1, M_1) \cong (N_2, M_2)$ <u>then</u> $M_1 \le_{\mathfrak{k}} N_1 \Leftrightarrow M_2 \le_{\mathfrak{k}} N_2$.
(c) Every $M \in K$ has cardinality $\in [\lambda,\mu)$.
(d) $\le_{\mathfrak{k}}$ is a partial order on K.

III$_1$. If $\langle M_i : i < \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing and the cardinality of $\bigcup\limits_{i<\delta} M_i$ is less than μ <u>then</u> there is a unique $M \in K$ such that $|M| = \bigcup\big\{|M_i| : i < \delta\big\}$ and $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} M$.

III$_2$. If in addition, $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} N$ <u>then</u> $M \le_{\mathfrak{k}} N$.

IV. If $M_1 \subseteq M_2$ and $M_\ell \le_{\mathfrak{k}} N$ for $\ell = 1,2$ <u>then</u> $M_1 \le_{\mathfrak{k}} M_2$.

V. If $A \subseteq N \in K$, <u>then</u> there is M satisfying $A \subseteq M \le_{\mathfrak{k}} N$ and $\|M\| \le \lambda + |A|$. (Here it is enough to restrict ourselves to the case $|A| \le \lambda$.)

Definition 2.2. 1) We say $\mathfrak{k}$ is an ess-λ-AEC <u>if</u> it is an ess-$[\lambda,\lambda^+)$-AEC.
2) We say $\mathfrak{k}$ is an ess-AEC <u>if</u> there is λ such that it is an ess-$[\lambda,\infty)$-AEC, so $\lambda = \mathrm{LST}(\mathfrak{k})$.
3) If $\mathfrak{k}$ is an ess-$[\lambda,\mu)$-AEC and $\lambda \le \lambda_1 < \mu_1 \le \mu$ then let

$$K^{\mathfrak{k}}_{\lambda_1} = (K_{\mathfrak{k}})_{\lambda_1} = \{M \in K_{\mathfrak{k}} : \|M\| = \lambda_1\}$$

and $K^{\mathfrak{k}}_{\lambda_1,\mu_1} = \{M \in K_{\mathfrak{k}} : \lambda_1 \le \|M\| < \mu_1\}$.
4) We define $\Upsilon^{\mathrm{or}}_{\mathfrak{k}}$ as in [She09b, 0.8=L11.1.3A(2)].
5) We may omit the "essentially" <u>when</u> $\mathrm{arity}(\tau_{\mathfrak{k}}) = \aleph_0$ where $\mathrm{arity}(\mathfrak{k}) = \mathrm{arity}(\tau_{\mathfrak{k}})$ and for vocabulary τ,

$$\mathrm{arity}(\tau) = \min\{\kappa : \text{ every predicate and function symbol has arity } < \kappa\}.$$

We now consider the claims on ess-AECs.

Claim 2.3. *Let $\mathfrak{k}$ be an ess-$[\lambda,\mu)$-AEC.*
1) The parallel of $\mathrm{Ax}(III)_1, (III)_2$ *holds with a directed family $\langle M_t : t \in I\rangle$.*
2) If $M \in K$ we can find $\langle M_{\bar a} : \bar a \in {}^{\omega>}M\rangle$ such that:

(A) $\bar a \subseteq M_{\bar a} \le_{\mathfrak{k}} M$
(B) $\|M_{\bar a}\| = \lambda$
(C) If $\bar b$ is a permutation of $\bar a$ then $M_{\bar a} = M_{\bar b}$.
(D) if $\bar a$ is a subsequence of $\bar b$ then $M_{\bar a} \le_{\mathfrak{k}} M_{\bar b}$.

[5] And we may write (μ,λ) instead of $[\lambda,\mu)$.

3) If $N \le_{\mathfrak{k}} M$ we can add in (2) that $\bar{a} \in {}^{\omega>}N \Rightarrow M_{\bar{a}} \subseteq N$.
4) If for simplicity

$$\lambda_* = \lambda + \sup\Big\{ \sum \{|R^M| : R \in \tau_{\mathfrak{k}}\} + \sum \{|F^M| : F \in \tau_{\mathfrak{k}}\} : M \in K_{\mathfrak{k}} \text{ has cardinality } \lambda \Big\}$$

then $K_{\mathfrak{k}}$ and $\{(M,N) : N \le_{\mathfrak{k}} M\}$ are essentially $\mathrm{PC}_{\chi,\lambda_*}$-classes, where

$$\chi = \big|\{M/\cong : M \in K^{\mathfrak{k}}_\lambda\}\big|$$

noting that $\chi \le 2^{2^\theta}$. That is, a sequence $\langle M_{\bar{a}} : \bar{a} \in {}^{\omega>}A\rangle$ satisfying clauses (b),(c),(d) of part (2) such that $A = \bigcup\{|M_{\bar{a}}| : \bar{a} \in {}^{\omega>}A\}$ will represent a unique $M \in K_{\mathfrak{k}}$ with universe A — and similarly for $\le_{\mathfrak{k}}$. (On the Definition of $\mathrm{PC}_{\chi,\lambda_*}$, see [She09a, 1.4(3)].) Note that if, in $\tau_{\mathfrak{k}}$, there are no two distinct symbols with the same interpretation in every $M \in K_{\mathfrak{k}}$ then $|\tau)k_*| \le 2^{2^\lambda}$.
5) The results on omitting types in [She99] or [She09b, 0.9=L0n.8,0.2=0n.11] hold, i.e., if $\alpha < (2^{\lambda_*})^+ \Rightarrow K^{\mathfrak{k}}_{\beth_\alpha} \neq \varnothing$ <u>then</u> $\theta \in [\lambda,\mu) \Rightarrow K_\theta \neq \varnothing$ and there is an EM-model, i.e., $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{k}}$ with $|\tau_\Phi| = |\tau_{\mathfrak{k}}| + \lambda$ and $\mathrm{EM}(I,\Phi)$ having cardinality $\lambda + |I|$ for any linear order I.
6) The lemma on the equivalence of being universal model homogeneous and of being saturated (see [She09e, 3.18=3.10] or [She09c, 1.14=L0.19]) holds.
7) We can generalize the results of [She09c, §1] on deriving an ess-(∞,λ)-AEC from an ess-λ-AEC.

Proof. The same proofs. On the generalization in 2.3(7), see [Shea, §1]. The point is that, in the language of [Shea, §1], our $\mathfrak{k}$ is a (λ,μ,κ)-AEC (automatically with primes). $\square_{2.3}$

Remark 2.4. 1) In 2.3(4) we can decrease the bound on χ if we have more nice definitions of K_λ; e.g., if $\mathrm{arity}(\tau) \le \kappa$ then $\chi = 2^{(\lambda^{<\kappa}+|\tau|)}$, where $\mathrm{arity}(\tau) = \min\{\kappa :$ every predicate and function symbol of τ has arity $< \kappa\}$.
2) We may use above $|\tau_{\mathfrak{s}}| \le \lambda$, $\mathrm{arity}(\tau_{\mathfrak{k}}) = \aleph_0$ to get that

$$\big\{(M,\bar{a})/\cong : M \in K^{\mathfrak{k}}_\lambda,\ \bar{a} \in {}^\lambda M \text{ lists } M\big\}$$

has cardinality $\le 2^\lambda$. See also 2.18.
3) In 2.10 below, if we omit "$\mathbb{E}$ is small" and $\lambda_1 = \sup\{|\mathrm{seq}(M)/\mathbb{E}_M| : M \in K^{\mathfrak{k}}_\lambda\}$ is $< \mu$ then $\mathfrak{k}_{[\lambda_1,\mu)}$ is an ess-$[\lambda_1,\mu)$-AEC.
4) In Definition 2.1, we may omit axiom V and define $\mathrm{LST}(\mathfrak{k}) \in [\lambda,\mu]$ naturally, and if $M \in K^{\mathfrak{k}}_\lambda \Rightarrow \mu > |\mathrm{seq}(M)/\mathbb{E}_M|$ <u>then</u> in 2.10(1) below we can omit "$\mathbb{E}$ is small".
5) Can we preserve in such "transformation" the arity finiteness? A natural candidate is trying to code $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ by $\{\bar{a} : \bar{a} \in {}^{\omega>}M\}$, where there are $M_0 \le_{\mathfrak{s}} M_1$ such that $M \le_{\mathfrak{s}} M_1$, $\mathrm{ortp}(a_\ell, M_0, M_1)$ is parallel to p, and $\bar{a}$ is independent in (M_0,M_1). If e.g., $K_{\mathfrak{s}}$ is saturated this helps but still we suspect it may fail.
6) What is the meaning of ess-$[\lambda,\mu)$-AEC? Can we look just at $\langle M_t : t \in I\rangle$, I directed, $t \le_I s \Rightarrow M_t \le_{\mathfrak{s}} M_s \in K_\lambda$? For isomorphism types we take a kind of completion and so make more pairs isomorphic, but $\bigcup_{t\in I} M_t$ does not determine $\overline{M} = \langle M_t : t \in I\rangle$, and the completion may depend on this representation.
7) If we like to avoid this and this number is λ', <u>then</u> we should change the definition of $\mathrm{seq}(N)$ (see 2.5(b)) to

$$\mathrm{seq}'(N) = \big\{\bar{a} : \ell g(\bar{a}) = \lambda,\ a_0 < \mu_*, \text{ and for some } M \le_{\mathfrak{s}} N \text{ from } K^{\mathfrak{k}}_\lambda, \\ \langle a_{1+\alpha} : \alpha < \lambda\rangle \text{ lists the members of } M\big\}.$$

§ 2(B). Imaginary Elements and Smooth Equivalence Relations.

Now we return to our aim of getting canonical base for orbital types.

Definition 2.5. Let $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ be a λ-AEC or just ess-$[\lambda,\mu)$-AEC (if $\mathfrak{k}_\lambda = \mathfrak{k}_{\mathfrak{s}}$ we may write $\mathfrak{s}$ instead of $\mathfrak{k}_\lambda$, see 2.11). We say that $\mathbb{E}$ is a smooth $\mathfrak{k}_\lambda$-equivalence relation <u>when</u>:

(A) $\mathbb{E}$ is a function with domain $K_{\mathfrak{k}}$ mapping M to $\mathbb{E}_M$.

(B) For $M \in K_{\mathfrak{k}}$, $\mathbb{E}_M$ is an equivalence relation on a subset of

$$\mathrm{seq}(M) = \{\bar a \in {}^\lambda M : M \restriction \mathrm{Rang}(\bar a) \le_{\mathfrak{k}} M\}$$

so $\bar a$ is not necessarily without repetitions. Note that $\mathfrak{k}$ determines λ (pedantically, when non-empty).

(C) If $M_1 \le_{\mathfrak{k}} M_2$ then $\mathbb{E}_{M_2} \restriction \mathrm{seq}(M_1) = \mathbb{E}_{M_1}$.

(D) If f is an isomorphism from $M_1 \in K_{\mathfrak{s}}$ onto M_2 <u>then</u> f maps $\mathbb{E}_{M_1}$ onto $\mathbb{E}_{M_2}$.

(E) If $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous <u>then</u>

$$\big\{\bar a/\mathbb{E}_{M_\delta} : \bar a \in \mathrm{seq}(M_\delta)\big\} = \big\{\bar a/\mathbb{E}_{M_\delta} : \bar a \in \bigcup_{\alpha<\delta} \mathrm{seq}(M_\alpha)\big\}.$$

2) We say that $\mathbb{E}$ is small <u>if</u> each $\mathbb{E}_M$ has $\le \|M\|$ equivalence classes.

Remark 2.6. 1) Note that <u>if</u> we have $\langle \mathbb{E}_i : i < i^*\rangle$, where each $\mathbb{E}_i$ is a smooth $\mathfrak{k}_\lambda$-equivalence relation and $i^* < \lambda^+$, <u>then</u> we can find a smooth $\mathfrak{k}_\lambda$-equivalence relation $\mathbb{E}$ such that the $\mathbb{E}_M$-equivalence classes are essentially the $\mathbb{E}_i$-equivalence classes for $i < i^*$. In detail: without loss of generality $i^* \le \lambda$, and $\bar a \ \mathbb{E}_M \ \bar b$ <u>iff</u> $\ell g(\bar a) = \ell g(\bar b)$ and

$\circledast_1$ $i(\bar a) = i(\bar b)$, and if $i(\bar a) < i^*$ then

$$\bar a \restriction [1 + i(\bar a) + 1, \lambda) \ \mathbb{E}_{i(\bar a)} \ \bar b \restriction [1 + i(\bar b) + 1, \lambda)$$

where $i(\bar a) = \min\{j : j + 1 < i(*) \wedge a_0 \ne a_{1+j} \text{ or } j = \lambda\}$.

2) In fact $i^* \le 2^\lambda$ is OK: e.g., choose a function $\mathbf{e}$ from $\{e : e \text{ an equivalence relation on } \lambda\}$ to i^*. For $\bar a, \bar b \in \mathrm{seq}(M)$ we let $i(\bar a) = \mathbf{e}\big(\{(i,j) : a_{2i+1} = a_{2j+1}\}\big)$ and

$\circledast_2$ $\bar a \ \mathbb{E}_M \ \bar b$ <u>iff</u> $i(\bar a) = i(\bar b)$ and $\langle a_{2i} : i < \lambda\rangle \ \mathbb{E}_{i(\bar a)} \ \langle b_{2i} : i < \lambda\rangle$.

3) We can redefine $\mathrm{seq}(M)$ as ${}^{\lambda \ge}M$, but then we have to make minor changes above.

Definition 2.7. Let $\mathfrak{k}$ be a λ-AEC or just ess-$[\lambda,\mu)$-AEC and $\mathbb{E}$ a small smooth $\mathfrak{k}$-equivalence relation and the reader may assume for simplicity that the vocabulary $\tau_{\mathfrak{k}}$ has only predicates. Also assume $F_*, c_*, P_* \notin \tau_{\mathfrak{k}}$. We define τ_* and $\mathfrak{k}_* = \mathfrak{k}\langle\mathbb{E}\rangle = (K_{\mathfrak{k}_*}, \le_{\mathfrak{k}_*})$ as follows:

(A) $\tau^* = \tau \cup \{F_*, c_*, P_*\}$ with P_* a unary predicate, c_* an individual constant and F_* a λ-place function symbol.

(B) $K_{\mathfrak{k}_*}$ is the class of $\tau^*_{\mathfrak{k}}$-models M^* such that for some model $M \in K_{\mathfrak{k}}$ we have:

  (α) $|M| = P_*^{M^*}$

  (β) If $R \in \tau$ then $R^{M^*} = R^M$.

  (γ) If $F \in \tau$ has arity α then $F^{M^*} \restriction M = F^M$ and for any $\bar a \in {}^\alpha(M^*)$, $\bar a \notin {}^\alpha M$ we have $F^{M^*}(\bar a) = c_*^{M^*}$ (or allow partial functions, or use $F^{M^*}(\bar a) = a_0$ when $\alpha > 0$ and $F^{M^*}(\langle\ \rangle)$ when $\alpha = 0$; i.e. F is an individual constant);

  (δ) F_* is a λ-place function symbol, and:

i. If $\bar{a} \in \mathrm{seq}(M)$ then $F_*^{M^*}(\bar{a}) \in |M^*| \setminus |M| \setminus \{c_*^{M^*}\}$.
ii. If $\bar{a}, \bar{b} \in \mathrm{dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ then $F_*^{M^*}(\bar{a}) = F_*^{M^*}(\bar{b}) \Leftrightarrow \bar{a}\ \mathbb{E}_M\ \bar{b}$.
iii. If $\bar{a} \in {}^{\lambda}(M^*)$ and $\bar{a} \notin \mathrm{dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ then $F_*^{M^*}(\bar{a}) = c_*^{M^*}$.

(ε) $c_*^{M^*} \notin |M|$, and if $b \in |M^*| \setminus |M| \setminus \{c_*^{M^*}\}$ then for some $\bar{a} \in \mathrm{dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ we have $F_*^{M^*}(\bar{a}) = b$.

(C) $\leq_{\mathfrak{k}_*}$ is the two-place relation on $K_{\mathfrak{k}_*}$ defined by: $M^* \leq_{\mathfrak{k}_*} N^*$ if
(α) $M^* \subseteq N^*$ and
(β) for some $M, N \in \mathfrak{k}$ as in clause (B) we have $M \leq_{\mathfrak{k}} N$.

Definition 2.8. 1) In 2.7(1) we call $M \in \mathfrak{k}$ a *witness* for $M^* \in K_{\mathfrak{k}_*}$ if it is as in clause (B) above.
2) We call $M \leq_{\mathfrak{k}} N$ a witness for $M^* \leq_{\mathfrak{k}^*_\lambda} N^*$ if they are as in clause (C) above.

Discussion 2.9. Up to now we have restricted ourselves to vocabularies with each predicate and function symbol of finite arity, and this restriction seems very reasonable. Moreover, it seems *a priori* that for a parallel to superstable, it is quite undesirable to have infinite arity. Still, our desire to have imaginary elements (in particular, canonical basis for types) forces us to accept them. The price is that for a general class of τ-models the union of increasing chains of τ-models is not a well defined τ-model; more accurately, we can show its existence but not smoothness. However, inside the class $\mathfrak{k}\langle\mathbb{E}\rangle$ defined above, it will be.

Claim 2.10. *1) If $\mathfrak{k}$ is a $[\lambda, \mu)$-AEC or just an ess-$[\lambda, \mu)$-AEC and $\mathbb{E}$ a small smooth $\mathfrak{k}$-equivalence relation then $\mathfrak{k}\langle\mathbb{E}\rangle$ is an ess-$[\lambda, \mu)$-AEC.*
2) If $\mathfrak{k}$ has amalgamation and $\mathbb{E}$ is a small $\mathfrak{k}$-equivalence class then $\mathfrak{k}\langle\mathbb{E}\rangle$ has the amalgamation property.

Proof. The same proofs. Left as an exercise to the reader. $\square_{2.10}$

§ 2(C). **Good Frames.**

Now we return to good frames.

Definition 2.11. We say that $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame if Definition [She09c, 2.1=L1.1tex] is satisfied, except that:

(a) in clause (A), $\mathfrak{K}_{\mathfrak{s}} = (K_{\mathfrak{s}}, \leq_{\mathfrak{s}})$, $\mathfrak{k}$ is an ess-$[\lambda, \mu)$-AEC, and $\mathfrak{K}[\mathfrak{s}]$ is an ess-(∞, λ)-AEC.
(b) $K_{\mathfrak{s}}$ has a superlimit model in χ in every $\chi \in [\lambda, \mu)$.
(c) $K_\lambda^{\mathfrak{s}}/\cong$ has cardinality $\leq 2^\lambda$, for convenience.

Discussion 2.12. We may consider other relatives as our choice and mostly have similar results. In particular:

(A) We can demand less: as in [SV, §2] we may replace $\mathcal{S}_{\mathfrak{s}}^{\mathrm{bs}}$ by a formal version of $\mathcal{S}_{\mathfrak{s}}^{\mathrm{bs}}$.
(B) We may demand goodness only for $\mathfrak{s}_\lambda$, i.e. $\mathcal{S}_{\mathfrak{s}}$ and $\bigcup_{\mathfrak{s}}$ apply only to models in $K_\lambda^{\mathfrak{s}}$ (hence we can use their nice properties from [She09c, 2.1=L1.1tex]) and amalgamation and JEP are required only for models of cardinality λ.

Claim 2.13. *All the definitions and results in* [She09c], [She09d] *and §1 here work for good ess-$[\lambda, \mu)$-frames.*

Proof. No problem. $\square_{2.13}$

Definition 2.14. If $\mathfrak{s}$ is a $[\lambda, \mu)$-frame (see 1.1), or just an ess-$[\lambda, \mu)$-frame, and $\mathbb{E}$ a small smooth $\mathfrak{s}$-equivalence relation <u>then</u> let $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$ be defined by:

(A) $\mathfrak{k}_\mathfrak{t} = \mathfrak{k}_\mathfrak{s}\langle\mathbb{E}\rangle$

(B) $\mathcal{S}^{\rm bs}_\mathfrak{t}(M^*) = \{\text{ortp}_{\mathfrak{k}_\mathfrak{t}}(a, M^*, N^*) : M^* \le_{\mathfrak{k}_\mathfrak{t}} N^*$, and if $M \le_\mathfrak{k} N$ witness $M^*, N^* \in \mathfrak{k}_\mathfrak{t}$ then $a \in N \setminus M$ and $\text{ortp}_\mathfrak{s}(a, M, N) \in \mathcal{S}^{\rm bs}_\mathfrak{s}(M)\}$

(C) Non-forking is defined similarly.

Remark 2.15. We may add: if $\mathfrak{s}$ is[6] an NF-frame we define $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$ as an NF-frame similarly, see [She09d].

Claim 2.16. *1) If $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame, $\mathbb{E}$ a small, smooth $\mathfrak{s}$-equivalence relation <u>then</u> $\mathfrak{s}\langle\mathbb{E}\rangle$ is a good ess-$[\lambda, \mu)$-frame.*
2) In part (1), for every κ, $\dot{I}(\kappa, K^{\mathfrak{s}\langle\mathbb{E}\rangle}) = \dot{I}(\kappa, K^\mathfrak{s})$.
3) If $\mathfrak{s}$ has primes/regulars <u>then</u> $\mathfrak{s}\langle\mathbb{E}\rangle$ does as well.

Remark 2.17. We may add: if $\mathfrak{s}$ is an NF-frame <u>then</u> so is $\mathfrak{s}\langle\mathbb{E}\rangle$, hence $(\mathfrak{s}\langle\mathbb{E}\rangle)^{\rm full}$ is a full NF-frame; see [She09d].

Proof. Straightforward. $\square_{2.16}$

Our aim is to change $\mathfrak{s}$ inessentially such that for every $p \in \mathcal{S}^{\rm bs}_\mathfrak{s}(M)$ there is a canonical base, etc. The following claim shows that in the context we have presented this can be done.

Claim 2.18. ***The imaginary elements Claim***
Assume $\mathfrak{s}$ a good λ-frame or just a good ess-$[\lambda, \mu)$-frame.
1) If $M_* \in K_\mathfrak{s}$ and $p^* \in \mathcal{S}^{\rm bs}_\mathfrak{s}(M_*)$, <u>then</u>[7] there is a small, smooth $\mathfrak{k}_\mathfrak{s}$-equivalence relation $\mathbb{E} = \mathbb{E}_{\mathfrak{s}, M_*, p^*}$ and function $\mathbf{F}$ such that:

$(*)$ *If $M_* \le_\mathfrak{s} N$, $\bar{a} \in \text{seq}(N)$ (so $M := N \restriction \text{Rang}(\bar{a}) \le_\mathfrak{s} N$), and $M \cong M_*$, <u>then</u>*
- (α) *$\mathbf{F}(N, \bar{a})$ is well defined iff $\bar{a} \in \text{dom}(\mathbb{E}_N)$. If this is the case, then $\mathbf{F}(N, \bar{a})$ belongs to $\mathcal{S}^{\rm bs}_\mathfrak{s}(N)$.*
- (β) *$S \subseteq \{(N, \bar{a}, p) : N \in K_\mathfrak{s},\ \bar{a} \in \text{dom}(\mathbb{E}_N)\}$ is the minimal class such that:*
  - (i) *If $\bar{a} \in \text{seq}(M_*)$ and p does not fork over $M_* \restriction \text{Rang}(\bar{a})$ then $(M_*, \bar{a}, p) \in S$.*
  - (ii) *S is closed under isomorphisms.*
  - (iii) *If $N_1 \le_\mathfrak{s} N_2$, $p_2 \in \mathcal{S}^{\rm bs}_\mathfrak{s}(N_2)$ does not fork over $\bar{a} \in \text{seq}(N_1)$ then $(N_2, \bar{a}, p_2) \in S \Leftrightarrow (N_1, \bar{a}, p_2 \restriction N_1) \in S$.*
  - (iv) *If $\bar{a}_1, \bar{a}_2 \in \text{seq}(N)$ and $p \in \mathcal{S}^{\rm bs}_\mathfrak{s}(N)$ does not fork over $N \restriction \text{Rang}(\bar{a}_\ell)$ for $\ell = 1, 2$ then $(N_2, \bar{a}_1, p) \in S \Leftrightarrow (N_2, \bar{a}_2, p) \in S$.*
- (γ) *$\mathbf{F}(N, \bar{a}) = p$ iff $(N, \bar{a}, p) \in S$; hence if $\bar{a}, \bar{b} \in \text{seq}(N)$ then $\bar{a}\ \mathbb{E}_N\ \bar{b}$ iff $\mathbf{F}(\bar{a}, N) = \mathbf{F}(\bar{b}, N)$.*

2) There are unique small[8] smooth $\mathbb{E}$-equivalence relations $\mathbb{E}_\mathfrak{s}$ and a function $\mathbf{F}$ such that:

$(**)(\alpha)$ *$\mathbf{F}(N, \bar{a})$ is well defined iff $N \in K_\mathfrak{s}$ and $\bar{a} \in \text{seq}(N)$.*
(β) *$\mathbf{F}(N, \bar{a})$, when defined, belongs to $\mathcal{S}^{\rm bs}_\mathfrak{s}(N)$.*

[6] The reader may ignore this version.
[7] Note that there may well be an automorphism of M^* which maps p^* to some $p^{**} \in \mathcal{S}^{\rm bs}_\mathfrak{s}(M^*)$ such that $p^{**} \neq p^*$.
[8] For 'small' we use stability in λ.

(γ) *If* $N \in K_{\mathfrak{s}}$ *and* $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$ *<u>then</u> there is* $\bar{a} \in {\rm seq}(N)$ *such that* ${\rm Rang}(\bar{a}) = N$ *and* $\mathbf{F}(N, \bar{a}) = p$.

(δ) *If* $\bar{a} \in {\rm seq}(M)$ *and* $M \le_{\mathfrak{s}} N$ *<u>then</u>* $\mathbf{F}(N, \bar{a})$ *is (well defined and is) the non-forking extension of* $\mathbf{F}(M, \bar{a})$.

(ε) *If* $\bar{a}_\ell \in {\rm seq}(N)$ *and* $\mathbf{F}(N, \bar{a}_\ell)$ *is well defined for* $\ell = 1, 2$ *<u>then</u>* $\bar{a}_1 \ \mathbb{E}_N \ \bar{a}_2 \Leftrightarrow \mathbf{F}(N, \bar{a}_1) = \mathbf{F}(N, \bar{a}_2)$.

(ζ) $\mathbf{F}$ *commutes with isomorphisms.*

3) For $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$*, where* $\mathbb{E}$ *is as in part (2), and whenever* $M^* \in K_{\mathfrak{t}}$ *as witnessed by* $M \in K_{\mathfrak{s}}$*, and* $p^* \in \mathcal{S}^{\rm bs}_{\mathfrak{t}}(M^*)$ *is projected to* $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$*, we let* ${\rm bas}(p^*) = {\rm bas}(p) = \mathbf{F}(\bar{a}, M^*)/\mathbb{E}$ *whenever* $\mathbf{F}(M, \bar{a}) = p$.

(α) *If* M_ℓ *witnesses that* $M^*_\ell \in K_{\mathfrak{t}}$ *for* $\ell = 1, 2$ *and* $(M^*_1, M^*_2, a) \in K^{3,{\rm bs}}_{\mathfrak{t}}$ *then* $(M_1, M_2, a) \in K^{3,{\rm bs}}_{\mathfrak{s}}$ *and* $p^* = {\rm ortp}_{\mathfrak{t}}(a, M^*_1, M^*_2)$, $p = {\rm ortp}_{\mathfrak{s}}(a, M_1, M_2)$.

(β) *If* $M^*_\ell \le_{\mathfrak{s}} M^*$ *and* $p_\ell \in \mathcal{S}^{\rm bs}_{\mathfrak{t}}(M^*_\ell)$ *then* $p^*_1 \parallel p^*_2 \Leftrightarrow {\rm bas}(p^*_1) = {\rm bas}(p^*_2)$.

(γ) $p^* \in \mathcal{S}^{\rm bs}_{\mathfrak{t}}(M^*)$ *does not split over* ${\rm bas}(p^*)$ *(see Definition 1.13(3) or* [She09d, §2 end]*).*

Proof. 1) Let $M^{**} \le_{\mathfrak{s}} M^*$ be of cardinality λ such that p^* does not fork over M^{**}. Let $\bar{a}^* = \langle a_\alpha : \alpha < \lambda\rangle$ list the elements of M^{**}.

We say that $p_1 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_1)$ is a *weak copy* of p^* when there is a witness (M_0, M_2, p_2, f), which means:

$\circledast_1$ (a) $M_0 \le_{\mathfrak{s}} M_2$ and $M_1 \le_{\mathfrak{s}} M_2$.
(b) if $\|M_1\| = \lambda$ then $\|M_2\| = \lambda$.
(c) f is an isomorphism from M^{**} onto M_0.
(d) $p_2 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_2)$ is a non-forking extension of p_1.
(e) p_2 does not fork over M_0.
(f) $f(p^* \restriction M^{**})$ is $p_2 \restriction M_0$.

For $M_1 \in K^{\mathfrak{s}}_\lambda$, $p_1 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_1)$ a weak copy of p^*, we say that $\bar{b}$ explicates its being a weak copy <u>when</u>, for some witness (M_0, M_2, p_2, f) and $\bar{c}$,

$\circledast_2$ (a) $\bar{b} = \langle b_\alpha : \alpha < \lambda\rangle$ lists the elements of M_1.
(b) $\bar{c} = \langle c_\alpha : \alpha < \lambda\rangle$ lists the elements of M_2.
(c) $\{\alpha : b_{2\alpha} = b_{2\alpha+1}\}$ codes the following sets:
  (α) The isomorphic type of $(M_2, \bar{c})$.
  (β) $\{(\alpha, \beta) : b_\alpha = c_\beta\}$
  (γ) $\{(\alpha, \beta) : f(a^*_\alpha) = c_\beta\}$

Now

$\circledast_3$ If $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is a weak copy of p^* then for some $\bar{a} \in {\rm seq}(M)$, there is a $M_1 \le_{\mathfrak{s}} M$ over which p does not fork such that $\bar{a}$ lists M_1 and explicates '$p \restriction M_1$ is a weak copy of p^*.'

$\circledast_4$ (a) If $M \in K^{\mathfrak{s}}_\lambda$ and $\bar{b}$ explicates '$p_1 \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is a weak copy of p^*,' <u>then</u> we can reconstruct p_1 from M and $\bar{b}$. (Call it $p_{M,\bar{b}}$.)
(b) If $M \le_{\mathfrak{s}} N$, let $p_{N,\bar{b}}$ be its non-forking extension in $\mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$. We also call it $\mathbf{F}(N, \bar{b})$.

Now we define $\mathbb{E}$. First, for $N \in K_{\mathfrak{s}}$ we define a two-place relation $\mathbb{E}_N$.

$\circledast_5$ (α) $\mathbb{E}_N$ is on $\{\bar{a}$: for some $M \le_{\mathfrak{s}} N$ of cardinality λ and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ which is a copy of p^*, the sequence $\bar{a}$ explicates p being a weak copy of $p^*\}$.
(β) $\bar{a}_1 \ \mathbb{E}_N \ \bar{a}_2$ iff ($\bar{a}_1, \bar{a}_2$ are as above and) $p_{N,\bar{a}_1} = p_{N,\bar{a}_2}$.

Now

- $\odot_1$ For $N \in K_{\mathfrak{s}}$, $\mathbb{E}_N$ is an equivalence relation on $\mathrm{dom}(E_N) \subseteq \mathrm{seq}(N)$.
- $\odot_2$ If $N_1 \leq_{\mathfrak{s}} N_2$ and $\bar{a} \in \mathrm{seq}(N_1)$ <u>then</u> $\bar{a} \in \mathrm{dom}(\mathbb{E}_{N_1}) \Leftrightarrow \bar{a} \in \mathrm{dom}(\mathbb{E}_{N_2})$.
- $\odot_3$ If $N_1 \leq_{\mathfrak{s}} N_2$ and $\bar{a}_1, \bar{a}_2 \in \mathrm{dom}(\mathbb{E}_{N_1})$ then $\bar{a}_2 \ \mathbb{E}_{N_1} \ \bar{a}_2 \Leftrightarrow \bar{a}_1 \ \mathbb{E}_{N_2} \ \bar{a}_2$.
- $\odot_4$ If $\langle N_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous and $\bar{a}_1 \in \mathrm{dom}(\mathbb{E}_{N_\delta})$ then, for some $\alpha < \delta$ and $\bar{a}_2 \in \mathrm{dom}(\mathbb{E}_{N_\alpha})$, we have $\bar{a}_1 \ \mathbb{E}_{N_\delta} \ \bar{a}_2$.

[Why? Let $\bar{a}_2$ list the elements of $M_1 \leq_{\mathfrak{s}} N_\delta$ and let $p = p_{N_\delta, \bar{a}_1}$ so $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N_\delta)$. Hence for some $\alpha < \delta$, p does not fork over M_α; hence for some $M'_1 \leq_{\mathfrak{s}} M_\alpha$ of cardinality λ, the type p does not fork over M'_1. Let $\bar{a}_2$ list the elements of M'_1 such that it explicates $p \restriction M'_1$ being a weak copy of p^*. So clearly $\bar{a}_2 \in \mathrm{dom}(\mathbb{E}_{N_\alpha}) \subseteq \mathrm{dom}(\mathbb{E}_{N_\delta})$ and $\bar{a}_1 \ \mathbb{E}_{N_\delta} \ \bar{a}_2$.]

Clearly we are done.

2) Similar, only we vary (M^*, p^*) but it suffices to consider 2^λ such pairs.

3) Should be clear. $\square_{2.18}$

Definition/Claim 2.19. Assume that $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame, so without loss of generality it is full. We can repeat the operations in 2.18(3) and 2.16(2), so after ω times we get $\mathfrak{t}_\omega$ which is full (that is, $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{t}_\omega}(M^\omega) = \mathcal{S}^{\mathrm{na}}_{\mathfrak{t}_\omega}(M^\omega)$) and $\mathfrak{t}_\omega$ has canonical type-bases as witnessed by a function $\mathrm{bas}_{\mathfrak{t}_\omega}$ (see Definition 2.20).

Proof. Should be clear. $\square_{2.19}$

Definition 2.20. We say that $\mathfrak{s}$ has type bases <u>if</u> there is a function $\mathrm{bas}(-)$ such that:

- (A) If $M \in K_{\mathfrak{s}}$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>then</u> $\mathrm{bas}(p)$ is (well defined and is) an element of M.
- (B) p does not split over $\mathrm{bas}(p)$; that, is any automorphism[9] of M over $\mathrm{bas}(p)$ maps p to itself.
- (C) If $M \leq_{\mathfrak{s}} N$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ then $\mathrm{bas}(p) \in M$ <u>iff</u> p does not fork over M.
- (D) If f is an isomorphism from $M_1 \in K_{\mathfrak{s}}$ onto $M_2 \in K_{\mathfrak{s}}$ and $p_1 \in \mathcal{S}^{\mathrm{bs}}(M_1)$ then $f(\mathrm{bas}(p_1)) = \mathrm{bas}(f(p_1))$.

Remark 2.21. 1) In §3 we can add:

- (E) Strong uniqueness: if $A \subseteq M \leq_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$ and $p \in \mathcal{S}(A, \mathfrak{C})$ is well defined <u>then</u> $\mathrm{bas}(p) \in A$ and there is at most one $q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ such that q extends p. (Needed for non-forking extensions).

2) In 2.22 we can work in $\mathfrak{C}$.

Definition 2.22. We say that $\mathfrak{s}$ is equivalence-closed <u>when</u>:

- (A) $\mathfrak{s}$ has type bases $p \mapsto \mathrm{bas}(p)$.
- (B) If $\mathbb{E}_M$ is a definition of an equivalence relation on ${}^{\omega>}M$ preserved by isomorphisms and $\leq_{\mathfrak{s}}$-extensions (i.e. $M \leq_{\mathfrak{s}} N \Rightarrow \mathbb{E}_M = \mathbb{E}_N \restriction {}^{\omega>}M$) <u>then</u> there is a definable function F from ${}^{\omega>}M$ to M such that $F^M(\bar{a}) = F^M(\bar{b})$ iff $\bar{a} \ \mathbb{E}_M \ \bar{b}$.

To phrase the relation between $\mathfrak{k}$ and $\mathfrak{k}'$ we define the following.

Definition 2.23. Assume $\mathfrak{k}_1, \mathfrak{k}_2$ are ess-$[\lambda, \mu)$-AECs.

1) We say $\mathbf{i}$ is an interpretation in $\mathfrak{k}_2$ <u>when</u> $\mathbf{i}$ consists of

[9] There are reasonable stronger versions, but it follows that the function $\mathrm{bas}(-)$ satisfies them.

(A) A predicate $P^*_{\mathbf{i}}$.
(B) A subset $\tau_{\mathbf{i}}$ of $\tau_{\mathfrak{k}_2}$.

2) In this case, for $M_2 \in K_{\mathfrak{k}_2}$, let $M_2^{[\mathbf{i}]}$ be the $\tau_{\mathbf{i}}$-model $M_1 = M_2^{[\mathbf{i}]}$ with

- universe $P_{\mathbf{i}}^{M_2}$, and
- $R^{M_1} = R^{M_2} \restriction |M_1|$ for $R \in \tau_{\mathbf{i}}$.
- F^{M_1} is defined similarly, so it can be a partial function even if F^{M_2} is full.

3) We say that $\mathfrak{k}_1$ is $\mathbf{i}$-interpreted (or interpreted by $\mathbf{i}$) in $\mathfrak{k}_2$ <u>when</u>:

(A) $\mathbf{i}$ is an interpretation in $\mathfrak{k}_1$.
(B) $\tau_{\mathfrak{k}_1} = \tau_{\mathbf{i}}$
(C) $K_{\mathfrak{k}_1} = \{M_2^{[\mathbf{i}]} : M_2 \in K_{\mathfrak{k}_2}\}$
(D) If $M_2 \le_{\mathfrak{k}_2} N_2$ then $M_2^{[\mathbf{i}]} \le_{\mathfrak{k}_1} N_2^{[\mathbf{i}]}$.
(E) If $M_1 \le_{\mathfrak{k}_1} N_1$ and $N_1 = N_2^{[\mathbf{i}]}$ (so $N_2 \in K_{\mathfrak{k}_2}$) <u>then</u> for some $M_2 \le_{\mathfrak{k}} N_2$ we have $M_1 = M_2^{[\mathbf{i}]}$.
(F) If $M_1 \le_{\mathfrak{k}_1} N_1$ and $M_1 = M_2^{[\mathbf{i}]}$ (so $M_2 \in K_{\mathfrak{k}_2}$) <u>then</u> (possibly replacing M_2 by a model isomorphic to it over M_1) there is $N_2 \in K_{\mathfrak{k}_2}$ such that $M_2 \le_{\mathfrak{k}_2} N_2$ and $N_1 = N_2^{[\mathbf{i}]}$.

Definition 2.24. 1) Assume $\mathfrak{k}_1$ is interpreted by $\mathbf{i}$ in $\mathfrak{k}_2$. We say *strictly* interpreted when: if $M_2^{[\mathbf{i}]} = N_2^{[\mathbf{i}]}$ then M_2 and N_2 are isomorphic over $M_2^{[\mathbf{i}]}$.
2) We say $\mathfrak{k}_1$ is equivalent to $\mathfrak{k}_2$ if there are n and $\mathfrak{k}'_0, \dots, \mathfrak{k}'_n$ such that $\mathfrak{k}_1 = \mathfrak{k}'_0$, $\mathfrak{k}_2 = \mathfrak{k}'_n$ and for each $\ell < n$, $\mathfrak{k}_\ell$ is strictly interpreted in $\mathfrak{k}_{\ell+1}$ or vice versa. Actually, we can demand $n = 2$ and that k_ℓ is strictly interpreted in $\mathfrak{k}'_1$ for $\ell = 1, 2$.

Definition 2.25. As above for (good) ess-$[\lambda, \mu)$-frames.

Claim 2.26. *Assume $\mathfrak{s}$ is a good* ess-$[\lambda, \mu)$*-frame. <u>Then</u> there exists $\mathfrak{C}$ (called a* μ-saturated monster *for $K_{\mathfrak{s}}$) such that:*

(a) *$\mathfrak{C}$ is a $\tau_{\mathfrak{s}}$-model of cardinality $\le \mu$.*
(b) *$\mathfrak{C}$ is a union of some $\le_{\mathfrak{s}}$-increasing continuous sequence $\langle M_\alpha : \alpha < \mu \rangle$.*
(c) *if $M \in K_{\mathfrak{s}}$ (so $\lambda \le \|M\| < \mu$) <u>then</u> M is $\le_{\mathfrak{s}}$-embeddable into some M_α from clause (b).*
(d) *$M_{\alpha+1}$ is brimmed over M_α for $\alpha < \mu$.*

§ 3. P-simple types

We define the basic types over sets not necessary models. Note that in Definition 3.5(2) there is no real loss using C of cardinality $\in (\lambda, \mu)$, as we can replace λ by $\lambda_1 = \lambda + |C|$ and so replace $K_{\mathfrak{k}}$ to $K^{\mathfrak{k}}_{[\lambda_1,\mu)}$.

Hypothesis 3.1. 1) $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame (see Definition 2.11).
2) $\mathfrak{s}$ has type bases (see Definition 2.20).
3) $\mathfrak{C}$ will denote some μ-saturated model for $K_{\mathfrak{s}}$ of cardinality $\leq \mu$; see 2.26.
4) But $M, A, \ldots$ will be $<_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$ and $\subseteq \mathfrak{C}$, respectively, but of cardinality $< \mu$.

Definition 3.2. Let $A \subseteq M \in K_{\mathfrak{s}}$.
1) $\mathrm{dcl}(A, M) = \{a \in M :$ if $M' \leq_{\mathfrak{s}} M''$, $M \leq_{\mathfrak{s}} M''$, and $A \subseteq M'$ then $a \in M'$ and for every automorphism f of M', $f \restriction A = \mathrm{id}_A \Rightarrow f(a) = a\}$.
2) $\mathrm{acl}(A, M)$ is defined similarly, but only with the first demand.

Definition 3.3. 1) For $A \subseteq M \in K_{\mathfrak{s}}$ let

$$\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M) = \{q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) : \mathrm{bas}(q) \in \mathrm{dcl}(A, \mathfrak{C})\}.$$

2) We call $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M)$ regular <u>if</u> p as a member of $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is regular.

Definition 3.4. 1) $\mathbb{E}_{\mathfrak{s}}$ is as in Claim 2.18(2).
2) If $A \subseteq M \in K_{\mathfrak{s}}$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, then $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M)$ <u>iff</u> p is definable over A (see 1.13(3)) <u>iff</u> $\mathrm{inv}(p)$ from Definition 1.13 is $\subseteq A$ and well defined.

Definition 3.5. Let $A \subseteq \mathfrak{C}$.
1) We define a dependency relation on $\mathrm{good}(A, \mathfrak{C}) = \{c \in \mathfrak{C}$: for some $M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$, $A \subseteq M$ and $\mathrm{ortp}(c, M, \mathfrak{C})$ is definable over some finite $\bar{a} \subseteq A\}$ as follows:

- ⊛ c depends on $\mathbf{J}$ in $(A, \mathfrak{C})$ <u>iff</u> there is no $M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$ such that $A \cup \mathbf{J} \subseteq M$ and $\mathrm{ortp}(c, M, \mathfrak{C})$ is the non-forking extension of $\mathrm{ortp}(c, \bar{a}, \mathfrak{C})$, where $\bar{a}$ witnesses $c \in \mathrm{good}(A, \mathfrak{C})$.

2) We say that $C \in {}^{\mu>}[\mathfrak{C}]$ is *good* over (A, B) <u>when</u> there is a brimmed $M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$ such that $B \cup A \subseteq M$ and $\mathrm{ortp}(C, M, \mathfrak{C})$ (see Definition 1.13(3)) is definable over A. (In the first order context we could say $\{c, B\}$ is independent over A, but here this is problematic as $\mathrm{ortp}(B, A, \mathfrak{C})$ is not necessarily basic.)
3) We say $\langle A_\alpha : \alpha < \alpha^* \rangle$ is independent over A in $\mathfrak{C}$ (see [She09d, L8.8,6p.5(1)]) <u>if</u> we can find M and $\langle M_\alpha : \alpha < \alpha^* \rangle$ such that:

⊛ (a) $A \subseteq M \leq_{\mathfrak{k}(\mathfrak{s})} M_\alpha <_{\mathfrak{s}} \mathfrak{C}$ for $\alpha < \alpha^*$.
(b) M is brimmed.
(c) $A_\alpha \subseteq M_\alpha$
(d) $\mathrm{ortp}(A_\alpha, M, \mathfrak{C})$ definable over A (= does not split over A).
(e) $\langle M_\alpha : \alpha < \alpha^* \rangle$ is independent over M.

3A) Similarly for "over (A, B)".
4) We define 'locally independent' naturally; that is, every finite subfamily is independent.

Claim 3.6. *Assume $a \in \mathfrak{C}$, $A \subseteq \mathfrak{C}$.*
1) $a \in \mathrm{good}(A, \mathfrak{C})$ iff a realizes $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ for some M satisfying $A \subseteq M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$.

Claim 3.7. *1) If $A_\alpha \subseteq \mathfrak{C}$ is good over $(A, \bigcup_{i<\alpha} A_i)$ for $\alpha < \alpha^* < \omega$ then $\langle A_\alpha : \alpha < \alpha^* \rangle$ is independent over A.*
2) Independence is preserved by reordering.
3) If $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ is regular <u>then</u> on $p(\mathfrak{C}) = \{c : c$ realizes $p\}$ the independence relation satisfies:

(*a*) *Like clause (1).*

(*b*) *If* b^1_ℓ *depends on* $\{b^0_0, \dots, b^0_{n-1}\}$ *for* $\ell < k$ *and* b^2 *depends on* $\{b^1_\ell : \ell < k\}$ *then* b^2 *depends on* $\{b^0_\ell : \ell < n\}$.

(*c*) *If* b *depends on* $\mathbf{J}$, $\mathbf{J} \subseteq \mathbf{J}'$ *then* b *depends on* $\mathbf{J}'$.

Remark 3.8. 1) We have not mentioned finite character, but the local independence satisfies it trivially.

Proof. Easy. $\square_{3.7}$

Definition 3.9. 1) Assume $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a, \mathfrak{C})$. We say that q is explicitly (p,n)-simple when:

⊛ There are $b_0, \dots, b_{n-1}, c$ such that:[10]
- (a) b_ℓ realizes p.
- (b) c realizes q.
- (c) b_ℓ is not good[11] over $(\bar a, c)$ for $\ell < n$.
- (d) $\langle b_\ell : \ell < n\rangle$ is independent over $\bar a$.
- (e) $\langle c, b_0, \dots, b_{n-1}\rangle$ is good over $\bar a$.
- (f) If[12] c' realizes q then $c = c'$ iff for every $b \in p(\mathfrak{C})$ we have that b is good over $(\bar a, c)$ iff b is good over $(\bar a, c')$.

1A) We say that a is explicitly (p,n)-simple over A if $\text{ortp}(a, A, \mathfrak{C})$ is; similarly, in the other definitions replacing (p,n) by p will mean "for some n."
2) Assume $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a, \mathfrak{C})$ and $\mathbf{P}$ as in Definition 1.7. We say that q is $\mathbf{P}$-simple if we can find n and explicitly $\mathbf{P}$-regular types $p_0, \dots, p_{n-1} \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a, \mathfrak{C})$ such that each $c \in p(\mathfrak{C})$ is definable by its type over $\bar a \cup \bigcup\limits_{\ell<n} p_\ell(\mathfrak{C})$.

3A) In part (1) we say weakly (p,n)-simple if in ⊛, clause (f) is replaced by

(f)$'$ If b is good over $(\bar a, a^*_m)$ then c and c' realize the same type over $\bar a{}^\frown\langle a^*_m, b\rangle$.

3B) In part (1) we say (p,n)-simple if for some $\bar a^* \in {}^{\omega>}\mathfrak{C}$ good over $\bar a$, for every $c \in q(\mathfrak{C})$, there are $b_0, \dots, b_{n-1} \in p(\mathfrak{C})$ such that $c \in \text{dcl}(\bar a, \bar a^*, b_0, \dots, b_{n-1})$ and $\bar a{}^\frown\langle b_0, \dots, b_{n-1}\rangle$ is good over $\bar a$ if simple.
4) Similarly in (2).
5) We define $\text{gw}_p(b, \bar a)$ for p regular and parallel to some $p' \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a)$. (Here gw stands for 'general weight.') Similarly for $\text{gw}_p(q)$.

We first list some obvious properties.

Claim 3.10. *1) If* c *is* $\mathbf{P}$*-simple over* $\bar a$, *with* $\bar a \subseteq A \subset \mathfrak{C}$, *then* $w_p(c, A)$ *is finite.*
2) The obvious implications.

Claim 3.11. *1)* [*Closures of the simple* bs].
2) Assume $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a, \mathfrak{C})$. *If* $\bar b_1, \bar b_2$ *are* p*-simple over* A *then*

(*a*) $\bar b_1{}^\frown\bar b_2$ *is* p*-simple (of course,* $\text{ortp}_{\mathfrak{s}}(\bar b_2 \bar b_2, \bar a, \mathfrak{C})$ *is not necessary in* $\mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a, \mathfrak{C})$ *even if* $\text{ortp}_{\mathfrak{s}}(\bar b_\ell, \bar a, \mathfrak{C}) \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(\bar a, \mathfrak{C})$ *for* $\ell = 1, 2$*).*

(*b*) *Also,* $\text{ortp}(\bar b_2, \bar a b_1, \mathfrak{C})$ *is* $\mathbf{P}$*-simple.*

---

[10] Clauses (c) + (e) are replacements for 'c is algebraic over $\bar a + \{b_\ell : \ell < n\}$' and 'each b_ℓ is necessary.'

[11] 'Not good' here is a replacement to "$\text{ortp}(b_\ell, \bar a + c, \mathfrak{C})$ does not fork over $\bar a$."

[12] This seems a reasonable choice here but we can take others; this is an unreasonable choice for first order.

2) If $\bar{b}_\alpha$ is p-simple over $\bar{a}$ for $\alpha < \alpha^*$ and $\pi : \beta^* \to \alpha^*$ one to one and onto, then

$$\sum_{\alpha<\alpha^*} \mathrm{gw}_p\big(b_\alpha, \bar{a}_* \cup \bigcup_{\ell<\alpha} b_\alpha\big) = \sum_{\beta<\beta^*} \mathrm{gw}\big(b_{\pi(\beta)}, \bar{a} \cup \bigcup_{i<\beta} \bar{b}_{\pi(i)}\big).$$

The following definition comes from [Shea, 6.9(1)=Lg29].

Definition 3.12. Assume $p_1, p_2 \in \mathcal{S}^{\mathrm{bs}}(M)$. We say p_1, p_2 are weakly orthogonal (and denote it $p_1 \underset{\mathrm{wk}}{\perp} p_2$) when the following implication holds: if $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$, $(M_0, M_\ell, a_\ell) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ and $\mathrm{ortp}_{\mathfrak{s}}(a_\ell, M_0, M_\ell) = p_\ell$ for $\ell = 1, 2$ then $\mathrm{ortp}_{\mathfrak{s}}(a_2, M_1, M_3)$ does not fork over M_0. (this is symmetric by **Ax.E**(f).)

Claim 3.13. [$\mathfrak{s}$ *is equivalence-closed.*]
Assume that $p, q \in \mathcal{S}^{\mathrm{bs}}(M)$ are not weakly orthogonal (see 3.12). Then for some $\bar{a} \in {}^{\omega>}M$ we have: p, q are definable over $\bar{a}$ (works without being stationary) and for some $\mathfrak{k}_{\mathfrak{s}}$-definable function $\mathbf{F}$, for each $c \in q(\mathfrak{C})$, $\mathrm{ortp}_{\mathfrak{s}}(\mathbf{F}(c, \bar{a}), \bar{a}, \mathfrak{C}) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ and is explicitly (p, n)-simple for some n. (If, e.g., M is $(\lambda, *)$-brimmed then $n = w_p(q)$.)

Proof. We can find n and $c_1, b_0, \ldots, b_{n-1} \in \mathfrak{C}$ with c realizing q, b_ℓ realizing p, $\{b_\ell, c\}$ not independent over M, and n maximal. Choose $\bar{a} \in {}^{\omega>}M$ such that

$$\mathrm{ortp}_{\mathfrak{s}}(\langle c, b_0, \ldots, b_{n-1}\rangle, M, \mathfrak{C})$$

is definable over $\bar{a}$. Define $E_{\bar{a}}$, an equivalence relation on $q(\mathfrak{C})$: $c_1\ E_{\bar{a}}\ c_2$ iff for every $b \in p\mathfrak{C}$, we have "b is good over (a, c_1)" $\Rightarrow$ "b is good over $(\bar{a}, c_2)$." By "$\mathfrak{s}$ is eq-closed," we are done. $\square_{3.13}$

Claim 3.14. *1) Assume $p, q \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ are weakly orthogonal (see 3.12) but not orthogonal. Then we can find $\bar{a} \in {}^{\omega>}M$ over which p, q are definable and $r_1 \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ such that letting $p_1 = p \restriction \bar{a}$, $q_1 = q \restriction \bar{a}$, $n := w_p(q) \geq 1$ we have:*[13]

$\circledast^n_{\bar{a},p_1,q_1,r_2}$
- *(a) $p_1, q_1, r_1 \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a})$, $\bar{a} \in {}^{\omega>}\mathfrak{C}$.*
- *(b) p_1, q_1 are weakly orthogonal.*
- *(c) If $\{a^*_n : n < \omega\} \subseteq r_1(\mathfrak{C})$ is independent over $\bar{a}$ and c realizes q then for infinitely many $m < \omega$ there is $b \in p(\mathfrak{C})$ such that b is good over $(\bar{a}, a^*_n)$ but not over $(\bar{a}, a^*_n, c)$.*
- *(d) In (c) there really are n independent $b_0, \ldots, b_{n-1}$ which are all good over $(\bar{a}, a^*_n)$ but not over $(\bar{a}, a^*_n, c)$ (but we cannot find $n+1$ such b-s.).*

2) If $\circledast^n_{\bar{a},p_1,q_1,r_1}$ then (see Definition 3.9(3) for some definable function $\mathbf{F}$, if c realizes q_1, $c^* = F(c, \bar{a})$ and $\mathrm{ortp}_{\mathfrak{n}}(c^*, \bar{a}, \mathfrak{C})$ is (p_1, n)-simple.

See proof below.

Claim 3.15. *0) Assuming $A \subseteq \mathfrak{C}$ and $a \in A$, we say* $\mathrm{ortp}(a, A, \mathfrak{C})$ *is* finitary *when it is definable over $A \cup \{a_0, \ldots, a_{n-1}\}$ for some n, where each a_ℓ is in $\mathfrak{C}$ and is good over A inside $\mathfrak{C}$.*
1) If $a \in \mathrm{dcl}\big(\bigcup\{A_i : i < \alpha\} \cup A, \mathfrak{C}\big)$, $\mathrm{ortp}(a, A, \mathfrak{C})$ is finitary, and $\{A_i : i < \alpha\}$ is independent over A then for some finite $u \subseteq \alpha$ we have

$$a \in \mathrm{dcl}\big(\bigcup\{A_i : i \in u\} \cup A, \mathfrak{C}\big).$$

2) If $\mathrm{ortp}(b, \bar{a}, \mathfrak{C})$ is $\mathbf{P}$-simple, then it is finitary.
3) If $\{A_i : i < \alpha\}$ is independent over A and a is finitary over A then for some finite $u \subseteq \alpha$ (even $|u| < \mathrm{wg}(c, A)$), $\{A_i : i \in \alpha \setminus u\}$ is independent over $A, A \cup \{c\}$. (Or use (A', A''), $(A', A'' \cup \{c\})$.)

[13] We can say more concerning simple types.

Definition 3.16. 1) $\mathrm{dcl}(A) = \{a : f(a) = a$ for every automorphism f of $\mathfrak{C}\}$.
2) $\mathrm{dcl}_{\mathrm{fin}}(A) = \bigcup\{\mathrm{dcl}(B) : B \subseteq A$ finite$\}$.
3) a is finitary over A <u>if</u> there are $n < \omega$ and $c_0, \ldots, c_{n-1} \in \mathrm{good}(A)$ such that $a \in \mathrm{dcl}(A \cup \{c_0, \ldots, c_{n-1}\})$.
4) For such A, let $\mathrm{wg}(a, A)$ be $w(\mathrm{tp}(a, A, \mathfrak{C}))$ when well defined.
5) Strongly simple implies simple.

Claim 3.17. *In Definition 3.9(3), for some $m, k < \omega$ large enough, for every $c \in q(\mathfrak{C})$ there are $b_0, \ldots, b_{m-1} \in \bigcup\limits_{\ell<n} p_\ell(\mathfrak{C})$ such that*

$$c \in \mathrm{dcl}\big(\bar a \cup \{a^*_\ell : \ell < k\} \cup \{b_\ell : \ell < m\}\big).$$

Proof. Let $M_1, M_2 \in K_{\mathfrak{s}(\mathrm{brim})}$ be such that $M \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2$, M_1 is $(\lambda, *)$-brimmed over M, $p_\ell \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\ell)$ a non-forking extension of p, $q_\ell \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\ell)$ is a non-forking extension of q, $c \in M_2$ realizes q_1, and $(M_1, M_2, c) \in K^{3,\mathrm{pr}}_{\mathfrak{s}(\mathrm{brim})}$. Let $b_\ell \in p_1(M_2)$ for $\ell < n^* := w_p(q)$ be such that $\{b_\ell : \ell < n^*\}$ is independent in (M_1, M_2); let $\bar a^* \in {}^{\omega>}(M_1)$ be such that $\mathrm{ortp}_{\mathfrak{s}}(\langle c, b_0, \ldots, b_{n-1}\rangle, M_1, M_2)$ is definable over $\bar a^*$ and $r = \mathrm{ortp}_{\mathfrak{s}}(\bar a^*, M_1, M_2)$, $r^+ = \mathrm{ortp}(\bar a^{*\frown}\langle b_0, \ldots, b_{n-1}\rangle, M, M_2)$.

Let $\bar a \in {}^{\omega>}M$ be such that $\mathrm{ortp}_{\mathfrak{s}}(\bar a^*, \langle c, b_0, \ldots, b_{n-1}\rangle, M, M_2)$ is definable over $\bar a$. As M_1 is $(\lambda, *)$-saturated over M, there is $\{\bar a^*_f : f < \omega\} \subseteq r(\mathfrak{C})$ independent in (M, M_1). Moreover, letting $a^*_\omega = \bar a^*$, we have $\langle a^*_\alpha : \alpha \le \omega\rangle$ is independent in (M, M_1). Clearly $\mathrm{ortp}_{\mathfrak{s}}(c\bar a^*_n, M, M_2)$ does not depend on n hence we can find $\big\langle\langle b^\alpha_\ell : \ell < n\rangle : \alpha \le \omega\big\rangle$ such that $b^\alpha_\ell \in M_2$, $b^\omega_\ell = b_\ell$, and $\{c\bar a^*_\alpha, b^\alpha_0 \ldots b^\alpha_{n-1} : \alpha \le \omega\}$. (As usual, this is because the index set is independent in (M_1, M_2).)

The rest should be clear. $\square_{3.14}$

Definition 3.18. Assume $\bar a \in {}^{\omega>}\mathfrak{C}$, $n < \omega$, and $p, q, r \in \mathcal{S}^{\mathrm{bs}}(M)$ are as in the definition of p-simple$^{[-]}$ but p and q are weakly orthogonal (see e.g. Definition 3.12(1)). Let p be a definable related function such that for any $\bar a^\nu_\ell \in r(\mathfrak{C})$, $\ell < k^*$, the independent mapping $c \mapsto \big\{b \in q(\mathfrak{C}) : R\mathfrak{C} \models R(b, c, \bar a^*_\ell)\big\}$ is a one-to-one function from $q(\mathfrak{C})$ into

$$\big\{\langle J_\ell : \ell < k^*\rangle : J_\ell \subseteq p(\mathfrak{C}) \text{ is closed under dependence and has } p\text{-weight } n^*\big\}.$$

1) We can define $E = E_{p,q,r}$, a two-place relation over $r(\mathfrak{C})$: $\bar a^*_1 \, E \, \bar a^*_2$ <u>iff</u> $\bar a_1, \bar a_2 \in r(\mathfrak{C})$ have the same projection common to $p(\mathfrak{C})$ and $q(\mathfrak{C})$.
2) Define [a / the] unitless group on r/E and its action on $q(\mathfrak{C})$.

Remark 3.19. 1) A major point: as q is p-simple, $w_p(-)$ acts "nicely" on $p(\mathfrak{C})$, so if $c_1, c_2, c_3 \in q(\mathfrak{C})$ then $w_p(\langle c_1, c_2, c_3\rangle \bar a) \le 3n^*$. This enables us to define averages using a finite sequence in a quite satisfying way. Alternatively, look more at averages of independent sets.
2) **Silly Groups**: Concerning interpreting groups, note that in our present context, for every definable set P^M we can add the group of finite subsets of P^M with symmetric difference (as addition).
3) The axiomatization above has prototype $\mathfrak{s}$ where $K_{\mathfrak{s}} = \{M : M$ a κ-saturated model of $T\}$, $\le_{\mathfrak{s}} = \prec\restriction K_{\mathfrak{s}}$, $\bigcup\limits_{\mathfrak{s}}$ is non-forking, T a stable first order theory with $\kappa(T) \le \mathrm{cf}(\kappa)$. But we may prefer to formalize the pair $(\mathfrak{t}, \mathfrak{s})$ with $\mathfrak{s}$ as above, $K_{\mathfrak{t}} =$ models of T, $\le_{\mathfrak{t}} = \prec\restriction K_{\mathfrak{t}}$, $\bigcup\limits_{\mathfrak{t}}$ is non-forking.

From $\mathfrak{s}$ we can reconstruct a $\mathfrak{t}$ by closing $\mathfrak{k}_{\mathfrak{s}}$ under direct limits, but in interesting cases we end up with a bigger $\mathfrak{t}$.

References

[LS06] Michael Chris Laskowski and Saharon Shelah, *Decompositions of saturated models of stable theories*, Fund. Math. **191** (2006), no. 2, 95–124. MR 2231058

[LS11] ———, *A trichotomy of countable, stable, unsuperstable theories*, Trans. Amer. Math. Soc. **363** (2011), no. 3, 1619–1629, arXiv: 0711.3043. MR 2737280

[LS15] ———, **P**-*NDOP and* **P**-*decompositions of* $\aleph_\epsilon$-*saturated models of superstable theories*, Fund. Math. **229** (2015), no. 1, 47–81, arXiv: 1206.6028. MR 3312115

[Shea] Saharon Shelah, *AEC for strictly stable.*

[Sheb] ———, *AEC for strictly stable II.*

[Shec] ———, *AEC: weight and p-simplicity.*

[She78] ———, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam-New York, 1978. MR 513226

[She90] ———, *Classification theory and the number of nonisomorphic models*, 2nd ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990, Revised edition of [Sh:a]. MR 1083551

[She99] ———, *Categoricity for abstract classes with amalgamation*, Ann. Pure Appl. Logic **98** (1999), no. 1-3, 261–294, arXiv: math/9809197. MR 1696853

[She01] ———, *Categoricity of an abstract elementary class in two successive cardinals*, Israel J. Math. **126** (2001), 29–128, arXiv: math/9805146. MR 1882033

[She04] ———, *Characterizing an* $\aleph_\epsilon$-*saturated model of superstable NDOP theories by its* $\mathbb{L}_{\infty,\aleph_\epsilon}$-*theory*, Israel J. Math. **140** (2004), 61–111, arXiv: math/9609215. MR 2054839

[She09a] ———, *Abstract elementary classes near* $\aleph_1$, Classification theory for abstract elementary classes, Studies in Logic (London), vol. 18, College Publications, London, 2009, arXiv: 0705.4137 Ch. I of [Sh:h], pp. vi+813.

[She09b] ———, *Categoricity and solvability of A.E.C., quite highly*, 2009, arXiv: 0808.3023 Ch. IV of [Sh:h].

[She09c] ———, *Categoricity in abstract elementary classes: going up inductively*, 2009, arXiv: math/0011215 Ch. II of [Sh:h].

[She09d] ———, *Toward classification theory of good* λ *frames and abstract elementary classes*, 2009, arXiv: math/0404272 Ch. III of [Sh:h].

[She09e] ———, *Universal Classes: Axiomatic Framework [Sh:h]*, 2009, Ch. V (B) of [Sh:i].

[Shear] ———, *Non-structure theory*, Oxford University Press, to appear.

[SV] Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, arXiv: 1805.06291.

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at