AEC FOR STRICTLY STABLE
SH1238

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ABSTRACT. Good frames were suggested in [She09d] as the (bare-bones) parallel, in the context of AECs, to superstable (among elementary classes). Here we consider (μ, λ, κ)-frames as candidates for being (in the context of AECs) the correct parallel to the class of $|T|^+$-saturated models of a strictly stable theory (among elementary classes). One thing we lose compared to the superstable case is that going up by induction on cardinals is problematic (for stages of small cofinality). But this arises only when we try to lift such classes to higher cardinals. Also, we may use, as a replacement, the existence of prime models over unions of increasing chains. For this context we investigate the dimension.

Annotated Content

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[Smooth out: generalize [She09b, §1].]

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References like [Sh:950, Th0.2=Ly5] mean that the internal label of Th0.2 is y5 in Sh:950. The reader should note that the version in my website is usually more up-to-date than the one in arXiv.

§ 0. Introduction

In this part we try to deal with classes like "$\aleph_1$-saturated models of a first order theory T, and even strictly stable ones" rather than of "a model of T," in the AEC framework. The parallel problem for "model of T, even superstable one" is the subject of [She09d].

Now, some constructions go well by induction on cardinality (say, by dealing with a $(\lambda, \mathcal{P}(n))$-system of models) but not all. E.g., starting with $\aleph_0$ we may consider $\lambda > \aleph_0$, so we can find $F : [\lambda]^{\aleph_0} \to \lambda$ such that there is no infinite decreasing sequence of F-closed subsets of λ such that $u \in [\lambda]^{<\aleph_0} \Rightarrow F(u) = \varnothing$, but maybe such that $u \in [\lambda]^{\leq \aleph_0} \Rightarrow |c\ell_F(u)| \leq \aleph_0$. Let $\langle u_\alpha : \alpha < \alpha_* \rangle$ list $\{c\ell_F(u) : u \in [\lambda]^{\leq \aleph_0}\}$ such that $c\ell_F(u_\alpha) \subseteq c\ell_F(u_\beta) \Rightarrow \alpha \leq \beta$. We try to choose M_{u_α} by induction on α.

Another approach is to consider strictly stable theories with 'nice enough' type theory, like superstable. See [PS18] on so-called 'flat' first order theories.

* * *

This was the middle part of [Sheb] and was divided by editor request; the third part is [Shea]. The original full paper has existed (and to some extent, has circulated) since 2002.

Notation 0.1. Let $\lambda^{<\kappa} := \sum\{\lambda^\sigma : \sigma < \kappa\}$; in subscripts we may use $\lambda[<\kappa]$.

We shall use the following freely.

Claim 0.2. *If* $\lambda = \lambda^{<\kappa}$ *then* $\chi \geq \lambda \Rightarrow (\chi^{<\kappa})^{<\kappa} = \chi^{<\kappa}$.

§ 1. Axiomatizing AEC without full continuity

§ 1(A). **DAEC.** Classes like "the $\aleph_1$-saturated models of a first order T which is not superstable", do not fall under AEC — still, they are close, and below we suggest a framework for them. So for increasing sequences of short length the union is not necessarily in the class, but we have weaker demands. In the main case, as 'compensation,' we have that prime models exist; in particular, over short increasing chains of models.

We shall lift a (μ, λ, κ)-AEC to $(\infty, \lambda, \kappa)$-AEC (see below), so actually $\mathfrak{k}_\lambda$ will suffice. But for our main objects – good frames – this is more complicated, as their properties (e.g., the amalgamation property) are not necessarily preserved by the lifting.

This section generalizes [She09b, §1]; in some cases the differences are minor, whereas sometimes the differences are the whole point.

Convention 1.1. In this section, if not said otherwise, $\mathfrak{k}$ will denote a 1-DAEC (i.e. a directed AEC; see Definition 1.2). We may write DAEC (the D stands for directed).

Definition 1.2. Assume $\lambda < \mu$, $\lambda^{<\kappa} = \lambda$ (for notational simplicity), $\alpha < \mu \Rightarrow |\alpha|^{<\kappa} < \mu$, and κ is regular.

We say that $\mathfrak{k}$ is a (μ, λ, κ)-1-DAEC when $\boxplus$ and all the axioms below hold.

(We may omit or add the '1' and/or the '(μ, λ, κ)' by $\boxplus$(a) below; similarly in similar definitions. Instead of $\mu = \mu_1^+$ we may write $\leq \mu_1$.) We write pre-DAEC or 0-DAEC when we omit **Ax.III**(b), **IV**(b).

$\boxplus$ (=**Ax.O**) $\mathfrak{k}$ consists of the objects in clauses (a)-(d), having the properties listed in (e)-(g).

- (a) The cardinals $\mu = \mu_{\mathfrak{k}} = \mu(\mathfrak{k})$, $\lambda = \lambda_{\mathfrak{k}} = \lambda(\mathfrak{k})$ and $\kappa = \kappa_{\mathfrak{k}} = \kappa(\mathfrak{k})$, satisfying $\mu > \lambda = \lambda^{<\kappa} \geq \kappa = \mathrm{cf}(\kappa)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\kappa} < \mu$ (but possibly $\mu = \infty$).
- (b) $\tau_{\mathfrak{k}}$, a vocabulary with each predicate and function symbol of arity $\leq \lambda$.
- (c) K a class of τ-models.
- (d) A two-place relation $\leq_{\mathfrak{k}}$ on K.
- (e) If $M_1 \cong M_2$ then $M_1 \in K \Leftrightarrow M_2 \in K$.
- (f) if $(N_1, M_1) \cong (N_2, M_2)$ <u>then</u> $M_1 \leq_{\mathfrak{k}} N_1 \Rightarrow M_2 \leq_{\mathfrak{k}} N_2$.
- (g) Every $M \in K$ has cardinality $\lambda \leq \|M\| < \mu$.

Ax.I(a) $M \leq_{\mathfrak{k}} N \Rightarrow M \subseteq N$

Ax.II(a) $\leq_{\mathfrak{k}}$ is a partial order.

Ax.III Assume that $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathfrak{k}}$-increasing sequence and $\left\| \bigcup \{M_i : i < \delta\} \right\| < \mu$. <u>Then</u>:

- (a) **Existence of unions**
  If $\mathrm{cf}(\delta) \geq \kappa$ then there is $M \in K$ such that $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M$ and $|M| = \bigcup \{|M_i| : i < \delta\}$ (but not necessarily $M = \bigcup\limits_{i<\delta} M_i$).
- (b) **Existence of limits**
  There is $M \in K$ such that $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M$.

Ax.IV(a) **Weak uniqueness of limit** (= weak smoothness)
For $\langle M_i : i < \delta \rangle$ as above,

- (a) If $\mathrm{cf}(\delta) \geq \kappa$, M is as in **Ax.III**(a), and $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} N$, then $M \leq_{\mathfrak{k}} N$. (This implies the uniqueness of M.)

(b) If $N_\ell \in K$ and $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} N_\ell$ for $\ell = 1, 2$ <u>then</u> there are $N \in K$ and f_1, f_2 such that f_ℓ is a $\le_{\mathfrak{k}}$-embedding of N_ℓ into N for $\ell = 1, 2$ and $i < \delta \Rightarrow f_1 \restriction M_i = f_2 \restriction M_i$.

Ax.V If $N_\ell \le_{\mathfrak{k}} M$ for $\ell = 1, 2$ and $N_1 \subseteq N_2$ <u>then</u> $N_1 \le_{\mathfrak{k}} N_2$.

Ax.VI L.S.T. property
If $A \subseteq M \in K$ and $|A| \le \lambda$ <u>then</u> there is $M \le_{\mathfrak{k}} N$ of cardinality λ such that $A \subseteq M$.

Remark 1.3. There are some more axioms listed in 1.4(5), but we shall mention them in any claim in which they are used so no need to memorize. Note that 1.4(1)-(4) assumes some of them.

Definition 1.4. 1) We say $\mathfrak{k}$ is a 4-DAEC or DAEC$^+$ <u>when</u> it is a (λ, μ, κ)-1-DAEC and satisfies **Ax.III**(d), **Ax.IV**(e) below.
2) We say $\mathfrak{k}$ is a 2-DAEC or DAEC$^\pm$ <u>when</u> it is a (λ, μ, κ)-0-DAEC and **Ax.III**(d), **Ax.IV**(d) below hold.
3) We say $\mathfrak{k}$ is 5-DAEC <u>when</u> it is 1-DAEC and **Ax.III**(d),(f) holds.
4) We say $\mathfrak{k}$ is 6-DAEC <u>when</u> it is a 1-DAEC and **Ax.III**(d),(f) + **Ax.IV**(f).
5) Concerning Definition 1.2, we consider the following axioms:

Ax.III (c) If I is κ-directed and $\overline{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing (that is, $s \le_I t \Rightarrow M_s \le_{\mathfrak{k}} M_s$), and $\sum\{\|M_s\| : s \in I\} < \mu$ <u>then</u> $\overline{M}$ has a $\le_{\mathfrak{k}}$-upper bound M (i.e. $s \in I \Rightarrow M_s \le_{\mathfrak{k}} M$).

(d) **Union of directed systems**
If I is κ-directed, $|I| < \mu$, $\langle M_t : t \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing, and $\left\| \bigcup\{M_t : t \in I\} \right\| < \mu$ <u>then</u> there is one and only one M with universe $\bigcup\{|M_t| : t \in I\}$ such that $M_s \le_{\mathfrak{k}} M$ for every $s \in I$. (We call it the $\le_{\mathfrak{k}}$-union of $\langle M_t : t \in I \rangle$.)

(e) Like **Ax.III**(c), but I is just directed.

(f) If $\overline{M} = \langle M_i : i < \delta \rangle$ is $\le_{\mathfrak{k}}$-increasing, $\mathrm{cf}(\delta) < \kappa$, and

$$\left| \bigcup\{M_i : i < \delta\} \right| < \mu$$

<u>then</u> there is M which is $\le_{\mathfrak{k}}$-prime over $\overline{M}$; i.e.

- If $N \in K_{\mathfrak{k}}$ and $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} N$ <u>then</u> there is a $\le_{\mathfrak{k}}$-embedding of M into itself over $\bigcup\{|M_i| : i < \delta\}$.

Ax.IV (c) If I is κ-directed and $\overline{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing and N_1, N_2 are $\le_{\mathfrak{k}}$-upper bounds of $\overline{M}$ <u>then</u> for some (N_2', f) we have $N_2 \le_{\mathfrak{k}} N_2'$ and f is a $\le_{\mathfrak{k}}$-embedding of N_1 into N_2 which is the identity on M_s for every $s \in I$. (This is a weak form of uniqueness.)

(d) If I is a κ-directed partial order, $\overline{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing, $s \in I \Rightarrow M_s \le_{\mathfrak{k}} M$ and $|M| = \bigcup\{|M_s| : s \in I\}$, <u>then</u> $\bigwedge\limits_s M_s \le_{\mathfrak{k}} N \Rightarrow M \le_{\mathfrak{k}} N$.

(e) Like **Ax.IV**(c), but I is just directed.

(f) If I is directed and $\overline{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing <u>then</u> there is M which is a $\le_{\mathfrak{k}}$-prime over $\overline{M}$, defined as in **Ax.III**(f).

Claim 1.5. *Assume*[1] $\mathfrak{k}$ *is a DAEC.*
1) ***Ax.III****(d) implies* ***Ax.III****(c). Also,* ***Ax.III****(c) implies* ***Ax.III****(a).*
2) ***Ax.III****(e) implies* ***Ax.III****(c) and also* ***Ax.III****(b). Also,* ***Ax.III****(b) implies* ***Ax.III****(a).*
3) ***Ax.IV****(d) implies* ***Ax.IV****(a).*

[1]By 1.1, it is not necessary to say this.

4) ***Ax.IV****(e) implies* ***Ax.IV****(c) and also* ***Ax.III****(b).*
5) In all the axioms in Definition 1.4 it is necessary that $\left|\bigcup\{M_s : s \in I\}\right| < \mu_{\mathfrak{k}}$.
6) ***Ax.IV****(b) implies that* $\mathfrak{k}$ *has amalgamation.*

Proof. Easy. $\square_{1.5}$

Example 1.6. The first order case

Let T be a stable complete first order theory, and $\kappa = \kappa_r(T) \in \left[\aleph_1, |T|^+\right]$. (Equivalently, κ is the minimal regular cardinal such that $\lambda = \lambda^{<\kappa} \geq 2^{|T|} \Rightarrow$ T stable in λ.) We shall define $\mathfrak{k} = \mathfrak{k}_T$:

$\boxplus$ (a) $K = K_{\mathfrak{k}}$ is the class of κ-saturated models of T (equivalently, $\mathbf{F}^a_\kappa$-saturated).
(b) $\leq\, = \,\leq_{\mathfrak{k}}$ means "is an elementary submodel of."

Example 1.7. Existentially closed

Let T be a universal first order theory with the JEP, for transparency.[2] We shall define $\mathfrak{k}$:

$\boxplus$ (a) $K = K_{\mathfrak{k}}$ is the class of existentially closed models of T.
(b) $\leq\, = \,\leq_{\mathfrak{k}}$ means "is a submodel of."

See, in [She75], what was called the "kind III context"; recall that kind II was for such T with amalgamation and JEP. Much more was done by Hrushovski.

Example 1.8. Metric Spaces

1) (a) We say that τ is a *metric vocabulary* if if has the distinguished 2-place predicates R_q (q a positive rational) and nmet:[3] let

$$\mathrm{nmet}(\tau) = \tau \setminus \{R_q : q \in \mathbb{Q}^+\}$$

and τ is finitary. That is, each predicate and function symbol has finitely many places.

(b) M is a *metric model* when its vocabulary τ_M is a metric vocabulary and there is a metric $\mathbf{d}_M(-,-)$ on M such that
- $\bullet_1$ $\mathbf{d}_M(a,b) = \inf\{q \in \mathbb{Q}^+ : (a,b) \in R^+_q\}$
- $\bullet_2$ For any predicate $R \in \tau$, R^M is closed.
- $\bullet_3$ For any function symbol F, F^M is a continuous function.
- $\bullet_4$ M is complete as a metric space.

(c) Without clause $\bullet_4$, we say M is an *almost metric* model.

(d) We say that the metric models M_1, M_2 are *topologically isomorphic* when there is a π such that
- $\bullet_1$ π is an isomorphism from $M_1 \restriction \mathrm{nmet}(\tau)$ onto $M_2 \restriction \mathrm{nmet}(\tau)$.
- $\bullet_2$ $\mathrm{dist}_\pi(M_1, M_2) :=$

$$\sup\left\{\frac{\mathbf{d}_{M_2}\big(\pi(a),\pi(b)\big)}{\mathbf{d}_{M_1}(a,b)}, \frac{\mathbf{d}_{M_1}(a,b)}{\mathbf{d}_{M_2}\big(\pi(a),\pi(b)\big)} : a \neq b \in M_1\right\}$$

is finite.

(Note that this is the meaning of isomorphism for Banach space theorists; what we call isomorphism they would call isometry.)

2) (a) We say $\mathfrak{k}$ is a *metric* AEC (or MAEC) <u>when</u>:
- $\bullet_1$ $\tau_{\mathfrak{k}}$ is a metric vocabulary.
- $\bullet_2$ $\mathfrak{k}$ is a DAEC with $\mu_{\mathfrak{k}} = \infty$, $\kappa = \aleph_1$, and $\lambda = \lambda^{\aleph_0}$ (and for convenience $|\tau_{\mathfrak{k}}| \leq \lambda$).
- $\bullet_3$ Each $M \in K_{\mathfrak{k}}$ is a metric model.

[2] Otherwise the class of existentially closed models of T is divided into $\leq 2^{|T|}$ subclasses, each of them of this form.

[3] nmet stands for 'non-metric.'

- $\bullet_4$ If I is a directed partial order and $\overline{M} = \langle M_s : s \in I\rangle$ is $\leq_{\mathfrak{k}}$-increasing then the completion M of $\bigcup\{M_s : s \in I\}$, naturally defined, is a $\leq_{\mathfrak{k}}$-l.u.b. of $\overline{M}$.

(b) We say $\mathfrak{k}$ is an *almost metric* AEC when we omit the completeness demand in (1)(b), and add
- $\bullet$ If N is the completion of $M \in K_{\mathfrak{k}}$ (so necessarily $N \in K_{\mathfrak{k}}$, $M \leq_{\mathfrak{k}} N$) then $M \subseteq M' \subseteq N \Rightarrow M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} N$.

3) (a) If $\mathfrak{k}$ is a metric AEC then all the axioms in Definitions 1.2, 1.4 hold.

(b) If $\mathfrak{k}$ is an almost metric AEC then

$$\mathrm{comp}(\mathfrak{k}) := \mathfrak{k} \restriction \{M \in K_{\mathfrak{k}} : (|M|, \mathbf{d}_M) \text{ is complete}\}$$

is a metric AEC; also, $\mathfrak{k}$ is an AEC. In this case, "the completion of $M \in K_{\mathfrak{k}}$" is naturally defined.

(c) **The representation theorem**.
If $\mathfrak{k}$ is a metric AEC then for some τ_1, T_1, Γ, we have:
- $\bullet_1$ $\tau_1 \supseteq \tau_{\mathfrak{k}}$ and $|\tau_1| \leq \lambda_{\mathfrak{k}}$.
- $\bullet_2$ T_1 is a universal f.o. theory in $\mathbb{L}(\tau_1)$.
- $\bullet_3$ Γ is a set of $\mathbb{L}(\tau_1)$-types consisting of formulas (so they are m-types for some m).
- $\bullet_4$ Every $M \in \mathrm{EC}(T_1, \Gamma)$ is a weak metric model.
- $\bullet_5$ $K_{\mathfrak{k}} = \{M : M$ is the completion of $M_1 \restriction \tau_{\mathfrak{k}}$ for some $M_1 \in \mathrm{EC}(T_1, \Gamma)\}$
- $\bullet_6$ $\leq_{\mathfrak{k}}$ is defined as

$$\big\{(M, N) : \text{there are } M_1 \subseteq N_1 \text{ from } \mathrm{EC}(T_1, \Gamma) \text{ such that}$$
$$M \subseteq N \text{ are the completions of } M_1 \restriction \tau_{\mathfrak{k}},\ N_1 \restriction \tau_{\mathfrak{k}} \text{ resp.}\big\}$$

[Why is this true? As in the AEC case.]

Regarding metric model theory and topological model theory, the field was started by Chang and Keisler in [CK66]; for an introduction see a recent survey [Kei20].

Definition 1.9. We say $\langle M_i : i < \alpha\rangle$ is $\leq_{\mathfrak{k}}$-increasing $(\geq \kappa)$-continuous when it is $\leq_{\mathfrak{k}}$-increasing and $\delta < \alpha$ and $\mathrm{cf}(\delta) \geq \kappa \Rightarrow |M_\delta| = \bigcup\{|M_j| : j < \delta\}$.

As an exercise we consider directed systems with mappings.

Definition 1.10. 1) We say that $\overline{M} = \langle M_t, h_{t,s} : s \leq_I t\rangle$ is a $\leq_{\mathfrak{k}}$-directed system when

(A) I is a directed partial order.
(B) If $s \leq_I t$ then $h_{t,s}$ is an isomorphism from M_s onto some $M' \leq_{\mathfrak{k}} M_t$.
(C) If $t_0 \leq_I t_1 \leq_I t_2$ then $h_{t_2,t_0} = h_{t_2,t_1} \circ h_{t_1,t_0}$.

1A) We say that $\overline{M} = \langle M_t, h_{t,s} : s \leq_I t\rangle$ is a $\leq_{\mathfrak{k}}$-θ-directed system when in addition I is θ-directed.

2) We may omit $h_{t,s}$ when $s \leq_I t \Rightarrow h_{t,s} = \mathrm{id}_{M_s}$ and write $\overline{M} = \langle M_t : t \in I\rangle$.

3) We say $(M, \bar{h})$ is a $\leq_{\mathfrak{k}}$-limit of $\overline{M}$ when $\bar{h} = \langle h_s : s \in I\rangle$, h_s is a $\leq_{\mathfrak{k}}$-embedding of M_s into M, and $s \leq_I t \Rightarrow h_s = h_t \circ h_{t,s}$.

4) We say $\overline{M} = \langle M_\alpha : \alpha < \alpha^*\rangle$ is $\leq_{\mathfrak{k}}$-semi-continuous when: (see **Ax.III**(f) in 1.4)

(A) $\overline{M}$ is $\leq_{\mathfrak{k}}$-increasing.
(B) If $\alpha < \alpha^*$ has cofinality $\geq \kappa$ then $M_\alpha = \bigcup\{M_\beta : \beta < \alpha\}$.
(C) If $\alpha < \alpha^*$ has cofinality $< \kappa$ then M_δ is $\leq_{\mathfrak{k}}$-prime over $\overline{M} \restriction \alpha$.

Observation 1.11. [$\mathfrak{k}$ *is a DAEC.*]

1) If $\overline{M} = \langle M_t, h_{t,s} : s \leq_I t\rangle$ is a $\leq_{\mathfrak{k}}$-directed system, then we can find a $\leq_{\mathfrak{k}}$-directed system $\langle M'_t : t \in I\rangle$ (so $s \leq_I t \Rightarrow M'_s \leq_{\mathfrak{k}} M'_t$) and $\bar{g} = \langle g_t : t \in I\rangle$ such that:

(a) g_t *is an isomorphism from* M_t *onto* M'_t.
(b) *If* $s \leq_I t$ *then* $g_s = g_t \circ h_{t,s}$.

2) So in the axioms ***III****(a),(b),*___IV___*(a) from Definition 1.2 as well as those of 1.4 we can use* $\leq_{\mathfrak{k}}$*-directed system* $\langle M_s, h_{t,s} : s \leq_I t \rangle$ *with* I *as there.*

3) If $\mathfrak{k}$ *is an ess-*(μ, λ)*-AEC (see* [Sheb, §1]*)* *then* $\mathfrak{k}$ *is a* $(\mu, \lambda, \aleph_0)$*-DAEC and satisfies all the axioms from 1.4.*

4) If $(M, \bar{h})$ *is prime over* $\overline{M} = \langle M_t, h_{t,s} : s \leq_I t \rangle$ *and* $\chi = \sum\{\|M_t\| : t \in I\}$ *then* $\|M\| \leq \chi^{<\kappa}$.

Proof. Straightforward; e.g. we can use "$\mathfrak{k}$ has $(\chi^{<\kappa})$-LST" (i.e. Observation 1.12 below). $\square_{1.11}$

More serious is proving the LST theorem in our context (recall that in the axioms, see **Ax.VI**, we demand it only down to λ).

Claim 1.12. *1)* $\mathfrak{k}$ *is a* (μ, λ, κ)*-2-DAEC — see Definition 1.4(2).*
If $\lambda_{\mathfrak{k}} \leq \chi = \chi^{<\kappa} < \mu_{\mathfrak{k}}$, $A \subseteq N \in \mathfrak{k}$, *and* $|A| \leq \chi \leq \|N\|$ *then there is* $M \leq_{\mathfrak{k}} N$ *of cardinality* χ *such that* $\|M\| = \chi$ *and* $A \subseteq M$.

2) If $\mathfrak{k}$ *satisfies* ***Ax.III****(e),* ***Ax.IV****(e)* *then in part (1) we do not need the assumption "*$\chi = \chi^{<\kappa}$*."*

Proof. 1) As $\chi \leq \|N\|$,

$(*)_0$ Without loss of generality $|A| = \chi$.

Let $\langle u_\alpha : \alpha < \alpha(*) \rangle$ list $[A]^{<\kappa(\mathfrak{k})}$ and let I be the following partial order:

$(*)_1$ (a) The set of elements is $\{\alpha < \chi : \text{for no } \beta < \alpha \text{ do we have } u_\alpha \subseteq u_\beta\}$.
(b) $\alpha \leq_I \beta$ iff $u_\alpha \subseteq u_\beta$ (hence $\alpha \leq \beta$).

Easily

$(*)_2$ (a) I is κ-directed.
(b) For every $\alpha < \alpha(*)$, for some $\beta < \alpha(*)$, we have $u_\alpha \subseteq u_\beta \wedge \beta \in I$.
(c) $\bigcup\{u_\alpha : \alpha \in I\} = A$.

Now we choose M_α by induction on $\alpha < \chi$ such that

$(*)_3$ (a) $M_\alpha \leq_{\mathfrak{k}} N$
(b) $\|M_\alpha\| = \lambda_{\mathfrak{k}}$
(c) M_α includes $\bigcup\{M_\beta : \beta <_I \alpha\} \cup u_\alpha$.

Note that

$$\big|\{\beta \in I : \beta <_I \alpha\}\big| \leq \big|\{u : u \subseteq u_\alpha\}\big| = 2^{|u_\alpha|} \leq 2^{<\kappa(\mathfrak{k})} \leq \lambda_{\mathfrak{k}},$$

and by the induction hypothesis $\beta < \alpha \Rightarrow \|M_\beta\| \leq \lambda_{\mathfrak{k}}$. Recall $|u_\alpha| < \kappa(\mathfrak{k}) \leq \lambda_{\mathfrak{k}}$ hence the set $\bigcup\{M_\beta : \beta < \alpha\} \cup u_\alpha$ is a subset of N of cardinality $\leq \lambda$, hence by **Ax.VI** there exists M_α as required.

Having chosen $\langle M_\alpha : \alpha \in I \rangle$, clearly by **Ax.V** it is a $\leq_{\mathfrak{k}}$-increasing $(< \kappa)$-directed system; hence by **Ax.III**(d), $M = \bigcup\{M_\alpha : \alpha \in I\}$ is well defined with universe $\bigcup\{|M_\alpha| : \alpha \in I\}$ and by **Ax.IV**(d) we have $M \leq_{\mathfrak{k}} N$.

Clearly $\|M\| \leq \sum\{\|M_\alpha\| : \alpha \in I\} \leq |I| \cdot \lambda_{\mathfrak{k}} = \chi$, and by $(*)_2(c) + (*)_3(c)$ we have

$$A \subseteq \bigcup\{u_\alpha : \alpha < \chi\} = \bigcup\{u_\alpha : \alpha \in I\} \subseteq \bigcup\{|M_\alpha| : \alpha \in I\} = M$$

and so M is as required.

2) Similarly. $\square_{1.12}$

Notation 1.13. 1) For $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$ let $K_\chi = K^{\mathfrak{k}}_\chi = \{M \in K : \|M\| = \chi\}$ and $K_{<\chi} = \bigcup\limits_{\mu<\chi} K_\mu$.

2) $\mathfrak{k}_\chi = (K_\chi, \leq_{\mathfrak{k}} \restriction K_\chi)$.

3) If $\lambda_{\mathfrak{k}} \leq \lambda_1 < \mu_1 \leq \mu_{\mathfrak{k}}$, $\lambda_1 = \lambda_1^{<\kappa}$, and $(\forall \alpha < \mu_1)\big[|\alpha|^{<\kappa} < \mu_1\big]$ <u>then</u> we define $K_{[\lambda_1,\mu_1]} = K^{\mathfrak{k}}_{[\lambda_1,\mu_1]}$ and $\mathfrak{k}_1 = \mathfrak{k}_{[\lambda_1,\mu_1]}$ similarly:

- (A) $K_{\mathfrak{k}_1} = \big\{M \in K_{\mathfrak{k}} : \|M\| \in [\lambda_1, \mu_1)\big\}$
- (B) $\leq_{\mathfrak{k}_1} = \leq_{\mathfrak{k}} \restriction K_{\mathfrak{k}_1}$
- (C) $\lambda_{\mathfrak{k}_1} = \lambda_1$, $\mu_{\mathfrak{k}_1} = \mu_1$, $\kappa_{\mathfrak{k}_1} = \kappa_{\mathfrak{k}}$.

4) Let $\mathfrak{k}_{[\lambda_1,\mu_1]} := \mathfrak{k}_{[\lambda_1,\mu_1^+)}$.

Definition 1.14. The embedding $f : N \to M$ is a $\mathfrak{k}$-embedding or a $\leq_{\mathfrak{k}}$-embedding <u>when</u> its range is the universe of a model $N' \leq_{\mathfrak{k}} M$ (so $f : N \to N'$ is an isomorphism, hence it is onto).

Claim 1.15. [$\mathfrak{k}$ *is a 2-DAEC.*]

1) For every $N \in K$ there is a $\kappa_{\mathfrak{k}}$-directed partial order I of cardinality $\leq \|N\| < \kappa$ and $\overline{M} = \langle M_t : t \in I\rangle$ such that $t \in I \Rightarrow M_t \leq_{\mathfrak{k}} N$, $\|M_t\| \leq \mathrm{LST}(\mathfrak{k}) = \lambda_{\mathfrak{k}}$,

$$I \models \text{“}s < t\text{”} \Rightarrow M_s \leq_{\mathfrak{k}} M_t,$$

and $N = \bigcup\limits_{t\in I} M_t$.

*1A) If $\mathfrak{k}$ satisfies **Ax.III**(e), **Ax.IV**(e) <u>then</u> in part (1) we can add $|I| \leq \|M\|$.*

2) For every $N_1 \leq_{\mathfrak{k}} N_2$ we can find $\langle M^\ell_t : t \in I^*\rangle$ as in part (1) for N_ℓ such that $I_1 \subseteq I_2$ and $t \in I_1 \Rightarrow M^2_t = M^1_t$.

Proof. 1), 1A) As in the proof of 1.12.

2) Similarly. $\square_{1.15}$

Claim 1.16. *Assume $\lambda_{\mathfrak{k}} \leq \lambda_1 = \lambda_1^{<\kappa} < \mu_1 \leq \mu_{\mathfrak{k}}$ and $(\forall \alpha < \mu_1)\big[|\alpha|^{<\kappa} < \mu_1\big]$.*

1) <u>Then</u> $\mathfrak{k}^*_1 := \mathfrak{k}_{[\lambda_1,\mu_1)}$ as defined in 1.13(3) is a $(\lambda_1, \mu_1, \kappa_{\mathfrak{k}})$-DAEC.

*2) For each of the following axioms, if $\mathfrak{k}$ satisfies it then so does $\mathfrak{k}_1$: **Ax.III**(d),(e), **Ax.IV**(c),(d),(e).*

*3) In part (2), its conclusion also applies to **Ax.III**(f), **Ax.IV**(f).*

Proof. Read the definitions. $\square_{1.16}$

Claim 1.17. *1) If $\mathfrak{k}$ satisfies **Ax.IV**(e) <u>then</u> $\mathfrak{k}$ satisfies **Ax.III**(e) provided that $\mu_{\mathfrak{k}}$ is regular or at least the relevant I has cardinality $< \mathrm{cf}(\mu_{\mathfrak{k}})$.*

*2) If **Ax.III**(d),**IV**(d) hold, we can waive ‘$\mu_{\mathfrak{k}}$ is regular.’*

Proof. Recall $\mathfrak{k}$ is a DAEC. We prove this by induction on $\theta = |I|$.

Let $\chi = \lambda + \theta + \sum\big\{\|M_s\| : s \in I\big\}$, which is in $[\lambda, \mu)$.

Case 1: I is finite.

So there is $t^* \in I$ such that $t \in I \Rightarrow t \leq_I t^*$, so this is trivial.

Case 2: I is countable.

So we can find a sequence $\langle t_n : n < \omega\rangle$ such that $t_n \in I$, $t_n \leq_I t_{n+1}$, and $s \in I \Rightarrow \bigvee\limits_{n<\omega} s \leq_I t_n$. Now we can apply **Ax.III**(b) to $\langle M_{t_n} : n < \omega\rangle$.

Case 3: I uncountable.

First, we can find an increasing continuous sequence $\langle I_\alpha : \alpha < |I|\rangle$ such that $I_\alpha \subseteq I$ is directed of cardinality $\leq |\alpha| + \aleph_0$ and $I_{|I|} = I = \bigcup\{I_\alpha : \alpha < |I|\}$.

Second, by the induction hypothesis for each $\alpha < |I|$ we choose N_α and $\bar{h}^\alpha = \langle h_{\alpha,t} : t \in I_\alpha\rangle$ such that:

(A) $N_\alpha \in \mathfrak{k}^{\mathfrak{s}}_{\leq\chi} \subseteq \mathfrak{k}$

(B) $h_{\alpha,t}$ is a $\leq_{\mathfrak{k}}$-embedding of M_t into N_α.

(C) If $s <_I t$ are in I_α then $h_{\alpha,s} = h_{\alpha,t} \circ h_{t,s}$.

(D) If $\beta < \alpha$ then $N_\beta \leq_{\mathfrak{k}} N_\alpha$ and $t \in I_\beta \Rightarrow h_{\alpha,t} = h_{\beta,t}$.

For $\alpha = 0$ use the induction hypothesis.

For α a limit ordinal, by **Ax.III**(a) there is N_α as required; as $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$, there are no new h_t-s. (Well, we have to check $\sum\{\|N_\beta\| : \beta < \alpha\} < \mu_{\mathfrak{k}}$, but as we assume $\mu_{\mathfrak{k}}$ is regular this holds.)

For $\alpha = \beta + 1$, by the induction hypothesis there is $(N'_\alpha, \bar{g}^\alpha)$ which is a limit of $\langle M_s, h_{t,s} : s \leq_{I_\alpha} t\rangle$. Now apply **Ax.IV**(e): well, apply the directed system version with $\langle M_s, h_{t,s} : s \leq_{I_\beta} t\rangle$, $(N'_\alpha, \bar{g}_\alpha)$, $(N_\beta, \langle h_s : s \in I_\beta\rangle)$ here standing for $\overline{M}, N_1, N_2$ there.

So there are N_α, f^α_s (with $s \in I_\beta$) such that $N_\beta \leq_{\mathfrak{k}} N_\alpha$ and $s \in I_\beta \Rightarrow f^\alpha_s \circ g_s = h_s$. Lastly, for $s \in I_\alpha \setminus I_\beta$ we choose $h_s = f^\alpha_s \circ g_s$, so we are clearly done.

2) Similarly, noting that in the last case, the result has cardinality $\leq \chi$ by 1.12(2) or 1.16. $\square_{1.17}$

§ 1(B). **Basic Notions.**

As in [She09b, §1], we now recall the definition of orbital types (note that it is natural to look at types only over models which are amalgamation bases recalling **Ax.IV**(b) implies every $M \in K_{\mathfrak{k}}$ is).

Definition 1.18. 1) For $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$ and $M \in K_\chi$ we define $\mathcal{S}(M)$ as

$$\{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{k}} N \in K_{\leq\chi^{<\kappa}} \text{ and } a \in N\},$$

where $\mathrm{ortp}(a, M, N) = (M, N, a)/\mathscr{E}_M$, where $\mathscr{E}_M$ is the transitive closure of $\mathscr{E}^{\mathrm{at}}_M$, and the two-place relation $\mathscr{E}^{\mathrm{at}}_M$ is defined as follows.

⊛ $(M, N_1, a_1)\ \mathscr{E}^{\mathrm{at}}_M\ (M, N_2, a_2)$ <u>iff</u>:

- (a) $M \leq_{\mathfrak{k}} N_\ell$ and $a_\ell \in N_\ell$ for $\ell = 1, 2$.
- (b) $\|M\| \leq \|N_\ell\| \leq \chi^{<\kappa}$ for $\ell = 1, 2$.
- (c) There exists an $N \in K_{\leq\chi^{<\kappa}}$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell : N_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \restriction M = \mathrm{id}_M = f_2 \restriction M$ and $f_1(a_1) = f_2(a_2)$.

2) We say "a realizes p in N" for $a \in N$ and $p \in \mathcal{S}(M)$ <u>when</u> (letting $\chi = \|M\|$) for some N' we have $M \leq_{\mathfrak{k}} N' \leq_{\mathfrak{k}} N$, $a \in N'$, and $p = \mathrm{ortp}(a, M, N')$. So necessarily $M, N' \in K_{\chi^{<\kappa}}$, but possibly $N \notin K_{\leq\chi^{<\kappa}}$.

3) We say "a_2 strongly[4] realizes $(M, N^1, a_1)/\mathscr{E}^{\mathrm{at}}_M$ in N" <u>when</u> for some N^2 we have $M \leq_{\mathfrak{k}} N^2 \leq_{\mathfrak{k}} N$ and $a_2 \in N^2$ and $(M, N^1, a_1)\ \mathscr{E}^{\mathrm{at}}_M\ (M, N^2, a_2)$.

4) We say M_0 is a $\leq_{\mathfrak{k}[\chi_0,\chi_1)}$-amalgamation base if this holds in $\mathfrak{k}_{[\chi_0,\chi_1)}$; see below.

4A) We say $M_0 \in \mathfrak{k}$ is an amalgamation base or $\leq_{\mathfrak{k}}$-amalgamation base <u>when</u>: for every $M_1, M_2 \in \mathfrak{k}$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in \mathfrak{k}_\lambda$ and $\leq_{\mathfrak{k}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.

5) We say $\mathfrak{k}$ is stable in χ <u>when</u>:

- (a) $\lambda_{\mathfrak{k}} \leq \chi < \mu_{\mathfrak{k}}$
- (b) $M \in K_\chi \Rightarrow |\mathcal{S}(M)| \leq \chi$
- (c) $\chi = \chi^{<\kappa}$
- (d) $\mathfrak{k}_\chi$ has amalgamation.

[4] Note that $\mathscr{E}^{\mathrm{at}}_M$ is not an equivalence relation, and $\mathscr{E}_M$ certainly isn't, in general.

6) We say $p = q \restriction M$ if $p \in \mathcal{S}(M)$, $q \in \mathcal{S}(N)$, $M \leq_{\mathfrak{k}} N$, and for some N^+ such that $N \leq_{\mathfrak{k}} N^+$ and $a \in N^+$ we have $p = \mathrm{ortp}(a, M, N^+)$ and $q = \mathrm{ortp}(a, N, N^+)$. Note that $p \restriction M$ is well defined if $M \leq_{\mathfrak{k}} N$ and $p \in \mathcal{S}(N)$.

7) For finite m, for $M \leq_{\mathfrak{k}} N$ and $\bar{a} \in {}^m N$, we can define $\mathrm{ortp}(\bar{a}, N, N)$ and $\mathcal{S}^m(M)$ similarly, and let $\mathcal{S}^{<\omega}(M) = \bigcup\limits_{m<\omega} \mathcal{S}^m(M)$. (But we shall not use this in any essential way, hence we choose $\mathcal{S}(M) = \mathcal{S}^1(M)$.)

Remark 1.19. We may replace 1.18(5)(c) by

(c)$'$ $\chi \in \mathrm{Car}_{\mathfrak{k}}$, which means $\chi = \chi^{<\kappa}$ or at least the conclusion of 1.12 holds.

In 1.20(1) we change the default value of χ to $\chi = \mathrm{rnd}_{\mathfrak{k}}(\|N\|)$ (where $\mathrm{rnd}_{\mathfrak{k}}(\theta) := \min(\mathrm{Car}_{\mathfrak{k}} \setminus \theta)$) so it is $\leq \|N\|^{<\kappa(\mathfrak{k})}$ (similarly in 1.21(1)).

Definition 1.20. 1) We say N is χ-*universal above or over* M <u>when</u> $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$, $M \in K_{\leq\chi}$, and for every M' with $M \leq_{\mathfrak{k}} M' \in K^{\mathfrak{k}}_{\chi}$, there is a $\leq_{\mathfrak{k}}$-embedding of M' into N over M. If we omit χ we mean $\|N\|^{<\kappa(\mathfrak{k})}$; clearly this implies that M is a $\leq_{\mathfrak{k}_{[\chi_0,\chi_1]}}$-amalgamation base, where $\chi_0 = \|M\|$ and $\chi_1 = \|N\|^{<\kappa}$.

2) $K^3_{\mathfrak{k}} = \{(M, N, a) : M \leq_{\mathfrak{k}} N,\ a \in N \setminus M$ and $M, N \in K_{\mathfrak{k}}\}$, with the partial order $\leq = \leq_{\mathfrak{k}}$ defined by $(M, N, a) \leq (M', N', a')$ iff $a = a'$, $M \leq_{\mathfrak{k}} M'$ and $N \leq_{\mathfrak{k}} N'$.

3) We say (M, N, a) is minimal if $(M, N, a) \leq (M', N_\ell, a) \in K^3_{\mathfrak{k}}$ for $\ell = 1, 2$ implies $\mathrm{ortp}(a, M', N_1) = \mathrm{ortp}(a, M', N_2)$ and moreover, $(M', N_1, a)\ \mathscr{E}^{\mathrm{at}}_{\lambda}\ (M', N_2, a)$ (this is not needed if every $M' \in K_\lambda$ is an amalgamation basis).

4) $K^{3,\mathfrak{k}}_{\lambda}$ is defined similarly using $\mathfrak{k}_{[\lambda, \mathrm{rnd}_{\mathfrak{k}}(\lambda)]}$.

Generalizing superlimit, we have more than one reasonable choice.

Definition 1.21. 1) For $\ell = 1, 2$ and $\chi = \chi^{<\kappa} \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$ we say $M^* \in K^{\mathfrak{k}}_{\chi}$ is superlimit$_\ell$ (or $(\chi, \geq \kappa)$-superlimit$_\ell$) <u>when</u>: (we may omit ℓ in the case $\ell = 2$)

- (a) it is universal, (i.e., every $M \in K^{\mathfrak{k}}_{\chi}$ can be properly $\leq_{\mathfrak{k}}$-embedded into M^*), and
- (b) **Case 1**: $\ell = 1$. If $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing, $\mathrm{cf}(\delta) \geq \kappa$, $\delta < \chi^+$, and $i < \delta \Rightarrow M_i \cong M^*$ then $M_\delta \cong M^*$.

  Case 2: $\ell = 2$. If I is a $(<\kappa)$-directed partial order of cardinality $\leq \chi$, $\langle M_t : t \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing, and $t \in I \Rightarrow M_t \cong M^*$ then $\bigcup\{M_t : t \in I\} \cong M^*$.

2) M is χ-saturated above θ <u>if</u> $\|M\| \geq \chi > \theta \geq \mathrm{LST}(\mathfrak{k})$, and also $N \leq_{\mathfrak{k}} M$, $\theta \leq \|N\| < \chi$, $p \in \mathcal{S}_{\mathfrak{k}}(N)$ imply p is strongly realized in M. Let "M is χ^+-saturated" mean that "M is χ^+-saturated above χ." Let

$$K(\chi^+\text{-saturated}) = \{M \in K : M \text{ is } \chi^+\text{-saturated}\}$$

and let "M is saturated" mean "M is $\|M\|$-saturated above some $\theta < \|M\|$".

Definition 1.22. 1) We say N is (χ, σ)-*brimmed* over M <u>when</u> we can find a sequence $\langle M_i : i < \sigma \rangle$ which is $\leq_{\mathfrak{k}}$-increasing semi-continuous, $M_i \in K_\chi$, $M_0 = M$, M_{i+1} is $\leq_{\mathfrak{k}}$-universal over M_i, and $\bigcup\limits_{i<\sigma} M_i = N$. We say N is (χ, σ)-brimmed over A if $A \subseteq N \in K_\chi$ and we can find $\langle M_i : i < \sigma \rangle$ as in part (1) such that $A \subseteq M_0$; if $A = \varnothing$ we may omit "over A".

2) We say N is $(\chi, *)$-brimmed over M if for every $\sigma \in [\kappa, \chi)$, N is (χ, σ)-brimmed over M. We say N is $(\chi, *)$-brimmed if N is $(\chi, *)$-brimmed over M for some M.

3) If $\alpha < \chi^+$, let "N is (χ, α)-brimmed over M" mean $M \leq_{\mathfrak{k}} N$ are from K_χ and $\mathrm{cf}(\alpha) \geq \kappa \Rightarrow N$ is $(\chi, \mathrm{cf}(\alpha))$-brimmed over M.

Recall

Claim 1.23. *1) If $\mathfrak{k}$ is a DAEC (or just 0-DAEC with amalgamation), stable in χ, and $\sigma = \mathrm{cf}(\sigma)$ (so $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$) then for every $M \in K^{\mathfrak{k}}_{\chi}$ there is $N \in K^{\mathfrak{k}}_{\chi}$ universal over M which is (χ, σ)-brimmed over M (hence is S^{χ}_{σ}-limit: see* [She09a], *not used).*

2) If N_ℓ is (χ, θ)-brimmed over M for $\ell = 1, 2$ and $\kappa \leq \theta = \mathrm{cf}(\theta) \leq \chi^+$ then N_1, N_2 are isomorphic over M.

3) If M_2 is (χ, θ)-brimmed over M and $M_0 \leq_{\mathfrak{s}} M_1$ then M_2 is (χ, θ)-brimmed over M_0.

Proof. Straightforward for part (1): recall clause (c) of Definition 1.23(5).
2),3) As in [She09b]. $\square_{1.23}$

* * *

§ 1(C). **Lifting such classes to higher cardinals.** Here we deal with lifting; there are two aspects. First, if $\mathfrak{k}^1, \mathfrak{k}^2$ agree in λ they agree in every higher cardinal. Second, given $\mathfrak{k}$ we can find $\mathfrak{k}_1$ with $\mu_{\mathfrak{k}_1} = \infty$ and $(\mathfrak{k}_1)_\lambda = \mathfrak{k}_\lambda$.

Theorem 1.24. *1) If $\mathfrak{k}^\ell$ is a (μ, λ, κ)-AEC for $\ell = 1, 2$ and $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$ then $\mathfrak{k}^1 = \mathfrak{k}^2$.*

*2) If $\mathfrak{k}_\ell$ is a $(\mu_\ell, \lambda, \kappa)$-DAEC for $\ell = 1, 2$ and $\mathfrak{k}^1$ satisfies **Ax.IV**(d), $\mu_1 \leq \mu_2$, and $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$ then $\mathfrak{k}_1 = \mathfrak{k}_2[\lambda, \mu_1)$.*

Proof. By 1.15. $\square_{1.24}$

Theorem 1.25. The lifting-up Theorem

1) If $\mathfrak{k}_\lambda$ is a $(\lambda^+, \lambda, \kappa)$-DAEC$^\pm$ then the pair $\mathfrak{k}' = (K', \leq_{\mathfrak{k}'})$ defined below is an $(\infty, \lambda, \kappa)$-DAEC$^\pm$, where

- (A) *K' is the class of M such that M is a $\tau_{\mathfrak{k}_\lambda}$-model, and for some I and $\overline{M}$ we have:*
  - (a) *I is a κ-directed partial order,*
  - (b) *$\overline{M} = \langle M_s : s \in I \rangle$,*
  - (c) *$M_s \in K_\lambda$,*
  - (d) *$I \models$ "$s < t$" $\Rightarrow M_s \leq_{\mathfrak{k}_\lambda} M_t$.*
  - (e) *If $J \subseteq I$ has cardinality $\leq \lambda$ and is κ-directed, and M_J is the union of $\langle M_t : t \in J \rangle$ (in the sense of **Ax.III**(d) from Definition 1.4) then M_J is a submodel of M.*
  - (f) *$M = \bigcup \{M_J : J \subseteq I$ is κ-directed of cardinality $\leq \lambda\}$ (in the sense of **Ax.IV**(d) of Definition 1.4).*
- $(A)'$ *We call such $\langle M_s : s \in I \rangle$ a* witness *for $M \in K'$, and we call it* reasonable *if $|I| \leq \|M\|^{<\kappa}$.*
- (B) *$M \leq_{\mathfrak{k}'} N$ iff for some $I, J, \overline{M}$ we have:*
  - (a) *J is a κ-directed partial order,*
  - (b) *$I \subseteq J$ is κ-directed,*
  - (c) *$\overline{M} = \langle M_s : s \in J \rangle$ and is $\leq_{\mathfrak{k}_\lambda}$-increasing,*
  - (d) *$\langle M_s : s \in J \rangle$ is a witness for $N \in K'$,*
  - (e) *$\langle M_s : s \in I \rangle$ is a witness for $M \in K'$.*
- $(B)'$ *We call such $I, \langle M_s : s \in J \rangle$ witnesses for $M \leq_{\mathfrak{k}'} N$, or say $\big(I, J, \langle M_s : s \in J \rangle\big)$* witnesses *$M \leq_{\mathfrak{k}'} N$.*

*2) If $\mathfrak{k}_\lambda$ satisfies **Ax.III**(a) then so does $\mathfrak{k}'$.*

Proof. 1) Let us check the axioms one by one.

Ax.O(a),(b),(c) and (d): K' is a class of $\tau_{\mathfrak{k}_\lambda}$-models, $\le_{\mathfrak{k}'}$ a two-place relation on K, K' and $\le_{\mathfrak{k}'}$ are closed under isomorphisms, and $M \in K' \Rightarrow \|M\| \ge \lambda$, etc.
[Why? trivially.]

Ax.I(a): If $M \le_{\mathfrak{k}'} N$ then $M \subseteq N$.
[Why? We use smoothness for κ-directed unions; i.e., **Ax.IV**(x).]

Ax.II(a): $M_0 \le_{\mathfrak{k}'} M_1 \le_{\mathfrak{k}'} M_2$ implies $M_0 \le_{\mathfrak{k}'} M_2$ and $M \in K' \Rightarrow M \le_{\mathfrak{k}'} M$.
Why? The second phrase is trivial. For the first phrase let for $\ell \in \{1,2\}$ the κ-directed partial orders $I_\ell \subseteq J_\ell$ and $\overline{M}^\ell = \langle M^\ell_s : s \in J_\ell\rangle$ witness $M_{\ell-1} \le_{\mathfrak{k}'} M_\ell$.
We first observe

- $\odot$ In clause (A)(f) of this theorem, if $J_\bullet \subseteq J_{\bullet\bullet} \subseteq J$ are $(<\kappa)$-directed <u>then</u> $M_{J_\bullet} \le_{\mathfrak{k}_\lambda} M_{J_{\bullet\bullet}}$.

[Why? By **Ax.VI**(d).]

- $\boxdot$ If I is a κ-directed partial order, $\langle M^\ell_t : t \in I\rangle$ is a $\le_{\mathfrak{k}_\lambda}$-increasing sequence witnessing $M_\ell \in K'$ for $\ell = 1,2$, and $t \in I \Rightarrow M^1_t \le_{\mathfrak{k}_\lambda} M^2_t$ <u>then</u> $M_1 \le_{\mathfrak{k}} M_2$.

[Why? Let I_1 be the partial order with set of elements $I \times \{1\}$ ordered by $(s,1) \le_{I_1} (t,1) \Leftrightarrow s \le_I t$. Let I_2 be the partial order with set of elements $I \times \{1,2\}$ ordered by $(s_1,\ell_1) \le_{I_2} (s_2,\ell_2) \Leftrightarrow s_1 \le_I s_2 \wedge \ell_1 \le \ell_2$. Clearly $I_1 \subseteq I_2$ are both κ-directed.

Let $M_{(s,1)} = M^1_s$ and $M_{(s,2)} = M^2_s$, so clearly $\overline{M} = \langle M_t : t \in I_2\rangle$ is a $\le_{\mathfrak{k}_\lambda}$-increasing, I-directed sequence witnessing $M_2 \in K'$, and $(I_1, I_2, \overline{M})$ witnesses $M_1 \le_{\mathfrak{k}'} M_2$, so we have proved $\boxdot$.]

Without loss of generality J_1, J_2 are disjoint. Let $\chi = \big(|J_1|+|J_2|\big)^{<\kappa}$ so $\lambda \le \chi < \mu_{\mathfrak{k}} = \infty$, and let

$$\begin{aligned}\mathscr{U} := \Big\{u \subseteq J_1 \cup J_2 : {} & |u| \le \lambda,\ u \cap I_\ell \text{ is } \kappa\text{-directed under } \le_{I_\ell} \text{ for } \ell = 1,2,\\ & u \cap J_\ell \text{ is } \kappa\text{-directed under } \le_{I_\ell} \text{ for } \ell = 1,2,\\ & \text{and } \bigcup\big\{|M^2_t| : t \in u \cap I_2\big\} = \bigcup\big\{|M^1_t| : t \in u \cap J_1\big\}\Big\}.\end{aligned}$$

Let $\langle u_\alpha : \alpha < \alpha^*\rangle$ list $\mathscr{U}$, and we define a partial order I as follows:

- (a)$'$ Its set of elements is $\{\alpha < \alpha^* :$ for no $\beta < \alpha$ do we have $u_\beta \subseteq u_\alpha\}$.
- (b)$'$ For $\alpha, \beta \in I$, $\alpha \le_I \beta$ iff $u_\alpha \subseteq u_\beta$.

Note that the set I may have cardinality $\big(\sum\limits_{i<\delta} \|M_i\|\big)^{<\kappa}$ which may be $> \lambda$.

As in the proof of 1.12, I is κ-directed.
For $\ell = 0,1,2$ and $\alpha \in I$, let $M_{\ell,\alpha}$ be

- (A) The $\le_{\mathfrak{k}}$-union of $\langle M^0_t : t \in u_\alpha \cap I_1\rangle$ if $\ell = 0$.
- (B) The $\le_{\mathfrak{k}}$-union of the $\le_{\mathfrak{k}_\lambda}$-directed sequence $\langle M^1_t : t \in J_1\rangle$ when $\ell = 1$.
- (C) The $\le_{\mathfrak{k}}$-union of the $\le_{\mathfrak{k}_\lambda}$-directed sequence $\langle M^2_t : t \in J_2\rangle$ when $\ell = 2$.

Now

- $(*)_1$ If $\ell = 0,1,2$ and $\alpha \le_I \beta$ then $M^\ell_\alpha \le_{\mathfrak{k}_\lambda} M^\ell_\beta$.
- $(*)_2$ If $\alpha \in I$ then $M^0_\alpha \le_{\mathfrak{k}_\lambda} M^1_\alpha \le_{\mathfrak{k}_\lambda} M^2_\alpha$.
- $(*)_3$ $\langle M_{\ell,\alpha} : \alpha \in I\rangle$ is a witness for $M_\ell \in K'$.

$(*)_4$ $M_{0,\alpha} \leq_{\mathfrak{k}_\lambda} M_{2,\alpha}$ for $\alpha \in I$.

Together by $\boxdot$ we get that $M_0 \leq_{\mathfrak{k}'} M_2$ as required.

Ax.III(a): In general.

Let $(I_{i,j}, J_{i,j}, \overline{M}^{i,j})$ witness $M_i \leq_{\mathfrak{k}'} M_j$ when $i \leq j < \delta$, and without loss of generality $\langle J_{i,j} : i < j < \delta\rangle$ are pairwise disjoint. Let $\mathscr{U}$ be the family of sets u such that for some $v \in [\delta]^{\leq \lambda}$,

(A) $v \subseteq \delta$ has cardinality $\leq \lambda$ and has order type of cofinality $\geq \kappa$.
(B) $u \subseteq \bigcup\{J_{i,j} : i < j \text{ are from } v\}$ has cardinality $\leq \lambda$.
(C) For $i \leq j$ from v, the set $u \cap J_{i,j}$ is κ-directed under $\leq_{J_{i,j}}$ and $u \cap I_{i,j}$ is κ-directed under $\leq_{I_{i,j}}$.
(D) If $i(0) \leq i(1) \leq i(2)$ are from v then
$$\bigcup\left\{M_s^{i(0),i(1)} : s \in u \cap J_{i(0),i(1)}\right\} = \bigcup\left\{M_s^{i(1),i(2)} : s \in u \cap I_{i(1),i(2)}\right\}.$$
(E) If $i(0) \leq j(1)$ and $i(1) \leq j(1)$ are from v then
$$\bigcup\left\{M_s^{i(0),j(1)} : s \in u \cap J_{i(0),j(1)}\right\} = \bigcup\left\{M_s^{i(1),j(1)} : s \in u \cap J_{i(1),j(1)}\right\}.$$

Let the rest of the proof be as in the proof of **Ax.II**(a).

Ax.IV(a):
Similar, but $\mathscr{U} = \{u \subseteq I : u \text{ has cardinality} \leq \lambda \text{ and is } \kappa\text{-directed}\}$.

Ax.III(d):
Recall that we are assuming $\mathfrak{k}$ satisfies **Ax.III**(d). Similar proof.

Ax.IV(d):
Again, we are assuming $\mathfrak{k}$ satisfies **Ax.IV**(d).

Ax.V: Assume $N_0 \leq_{\mathfrak{k}'} M$ and $N_1 \leq_{\mathfrak{k}'} M$.
If $N_0 \subseteq N_1$, then $N_0 \leq_{\mathfrak{k}'} N_1$.
[Why? Let $\big(I_\ell, J_\ell, \langle M_s^\ell : s \in J_\ell\rangle\big)$ witness $N_\ell \leq_{\mathfrak{k}} M$ for $\ell = 0, 1$; without loss of generality J_0, J_1 are disjoint.

Let
$$\mathscr{U} := \Big\{u \subseteq J_0 \cup J_1 : |u| \leq \lambda,\ u \cap J_\ell \text{ and } u \cap I_\ell \text{ are } \kappa\text{-directed for } \ell = 0,1, \text{ and } \bigcup\big\{|M_s^0| : s \in u \cap J_0\big\} = \bigcup\big\{|M_s^0| : s \in u \in J_1\big\}\Big\}.$$

For $u \in \mathscr{U}$ let

- $M_u = M \restriction \bigcup\{M_s^\ell : s \in u \cap J_\ell\}$ for $i = 0, 1$.
- $N_{\ell,u} = N_\ell \restriction \{M_s^\ell : s \in u \cap I_\ell\}$.

Let

$(*)$ (a) $(\mathscr{U}, \subseteq)$ is κ-directed.
(b) $N_{\ell,u} \leq_{\mathfrak{k}} M$
(c) $M_{\ell,u} \leq_{\mathfrak{k}} M_{\ell,v}$ when $u \subseteq v$ are from $\mathscr{U}$ and $\ell = 0, 1$.
(d) $M_{0,u} \leq_{\mathfrak{k}} M_{1,u}$
(e) $N_\ell = \bigcup\{N_{\ell,u} : u \in \mathscr{U}\}$

By $\boxdot$ above we are done.

Ax.VI: $\mathrm{LST}(\mathfrak{k}') = \lambda$.

[Why? Let $M \in K'$, $A \subseteq M$, $|A| + \lambda \le \chi < \|M\|$, and let $\langle M_s : s \in I\rangle$ witness $M \in K'$. Without loss of generality $|A| = \chi^{<\kappa}$. Now choose a directed $I \subseteq J$ of cardinality $\le |A| = \chi^{<\kappa}$ such that $A \subseteq M' := \bigcup\limits_{s \in I} M_s$ and so $(I, J, \langle M_s : s \in J\rangle)$ witnesses $M' \le_{\mathfrak{k}'} M$. So as $A \subseteq M'$ and $\|M'\| \le |A| + \mu$ we are done.]

2) **Ax.III**(b): Assume that $\langle M_i : i < \delta\rangle$ is a $\le_{\mathfrak{k}}$-increasing sequence and

$$\left\| \bigcup \{M_i : i < \delta\} \right\| < \mu.$$

We have to prove that there is $M \in K$ such that $i < \delta \Rightarrow M_i \le_{\mathfrak{k}'} M$ (as always, assuming that $\mathfrak{k}_\lambda$ satisfies **Ax.III**(b)).

If $\mathrm{cf}(\delta) \ge \kappa$, we use **Ax.III**(a) (which was proved above), so we may assume $\mathrm{cf}(\delta) < \kappa$. By renaming, without loss of generality $\delta < \kappa$, and we continue in the proof of **Ax.II**(a). $\square_{1.25}$

Also, if two such DAECs have some cardinal in common then we can put them together.

Claim 1.26. *Let $\iota \in \{0, 1, 2, 4\}$, assume $\lambda_1 < \lambda_2 < \lambda_3$, and*

(a) *$\mathfrak{k}^1$ is an $(\lambda_2^+, \lambda_1, \kappa)$-2-DAEC and $K^1 = K_{\mathfrak{k}^1}$.*
(b) *$\mathfrak{k}^2$ is a $(\lambda_3, \lambda_2, \kappa)$-$\iota$-DAEC.*
(c) *$K^{\mathfrak{k}^1}_{\lambda_2} = K^{\mathfrak{k}^2}_{\lambda_2}$ and $\le_{\mathfrak{k}^2} \restriction K^{\mathfrak{k}^2}_{\lambda_2} = \le_{\mathfrak{k}^1} \restriction K^{\mathfrak{k}^1}_{\lambda_2}$.*
(d) *We define $\mathfrak{k}$ as follows: $K_{\mathfrak{k}} = K_{\mathfrak{k}^1} \cup K_{\mathfrak{k}^2}$, $M \le_{\mathfrak{k}} N$ iff $M \le_{\mathfrak{k}^1} N$ or $M \le_{\mathfrak{k}^2} N$ or for some M', $M \le_{\mathfrak{k}^1} M' \le_{\mathfrak{k}^2} N$.*

<u>Then</u> $\mathfrak{k}$ is an $(\lambda_3, \lambda_1, \kappa)$-$\iota$-DAEC.

Proof. Straightforward. E.g.:

Ax.III(d): $\langle M_s : s \in I\rangle$ is a $\le_{\mathfrak{s}}$-κ-directed system.

If $\|M_s\| \ge \lambda_2$ for some s, use $\langle M_t : s \le t \in I\rangle$ and clause (b) of the assumption. If $\bigcup\{M_s : s \in I\}$ has cardinality $\le \lambda_2$ use clause (a) in the assumption. If neither one of them holds, recall $\lambda_2 = \lambda_2^{<\kappa}$ by clause (b) of the assumption, and let

$$\mathscr{U} = \big\{u \subseteq I : |u| \le \lambda_2,\ u \text{ is } \kappa\text{-directed (in I), and } \bigcup\{M_s : s \in u\} \text{ has cardinality } \lambda\big\}.$$

Easily, $(\mathscr{U}, \subseteq)$ is λ_2-directed. For $u \in J$, let M_u be the $\le_{\mathfrak{s}}$-union of $\langle M_s : s \in u\rangle$. Now by clause (a) of the assumption

$(*)_1$ $M_u \in K^{\mathfrak{k}^1}_{\lambda_2} = K^{\mathfrak{k}^2}_{\lambda_2}$
$(*)_2$ If $u_1 \subseteq v$ are from $\mathscr{U}$ then $M_u \le_{\mathfrak{k}^1} M_v$, $M_u \le_{\mathfrak{k}^2} M_v$.

Now use clause (b) of the assumption.

Axiom V: We shall freely use

$(*)$ $\mathfrak{k}^2_{\lambda_2} = \mathfrak{k}^1_{\lambda_2} = \mathfrak{k}_{\lambda_2}$

So assume $N_0 \le_{\mathfrak{k}} M$, $N_1 \le_{\mathfrak{k}} M$, $N_0 \subseteq N_1$.

Now if $\|N_0\| \ge \lambda_2$ use assumption (b), so we can assume $\|N_0\| < \lambda_2$. If $\|M\| \le \lambda_2$ we can use assumption (a), so assume $\|M\| > \lambda_2$; by the definition of $\le_{\mathfrak{k}}$ there is $M_0' \in K^{\mathfrak{k}^1}_{\lambda_2} = K^{\mathfrak{k}^2}_{\lambda_2}$ such that $N_0 \le_{\mathfrak{k}^1} M_0' \le_{\mathfrak{k}^2} M$. First assume $\|N_1\| \le \lambda_2$, so we can find $M_1' \in K^{\mathfrak{k}^1}_{\lambda_2}$ such that $N_1 \le_{\mathfrak{k}^1} M_1' \le_{\mathfrak{k}^2} M$.

[Why? If $N_1 \in K^{\mathfrak{k}^1}_{<\lambda_2}$ by the definition of $\le_{\mathfrak{k}}$, and if $N_1 \in K^{\mathfrak{k}^1}_{\lambda_2}$ just choose $M_1' = N_1$.]

Now we can, by assumption (b), find $M'' \in K^{\mathfrak{k}^1}_{\lambda_2}$ such that $M'_0 \cup M'_1 \subseteq M'' \leq_{\mathfrak{k}} M$, hence by assumption (b) (i.e. **Ax.V** for $\mathfrak{k}^2$) we have $M'_0 \leq_{\mathfrak{k}} M''$, $M'_1 \leq_{\mathfrak{k}} M''$. As $N_0 \leq_{\mathfrak{k}} M'_0 \leq_{\mathfrak{k}} M'' \in K^{\mathfrak{k}}_{\leq \lambda_2}$ by assumption (a) we have $N_0 \leq_{\mathfrak{k}} M''$, and similarly we have $N_1 \leq_{\mathfrak{k}} M''$. So $N_0 \subseteq N_1$, $N_0 \leq_{\mathfrak{k}} M''$, $N_1 \leq_{\mathfrak{k}} M'$, so by assumption (b) we have $N_0 \leq_{\mathfrak{k}} N_1$.

We are left with the case $\|N_1\| > \lambda$. By assumption (b) there is $N'_1 \in K_{\lambda_2}$ such that $N_0 \subseteq N'_1 \leq_{\mathfrak{k}^2} N_2$. Also by assumption (b), we have $N'_1 \leq_{\mathfrak{k}} M$, so by the previous paragraph we get $N_0 \leq_{\mathfrak{k}} N'_1$; together with the previous sentence we have $N_0 \leq_{\mathfrak{k}^1} N'_1 \leq_{\mathfrak{k}^2} N_1$ so by the definition of $\leq_{\mathfrak{k}}$ we are done. $\square_{1.26}$

Definition 1.27. If $M \in K_\chi$ is $(\chi, \geq \kappa)$-superlimit$_1$ let

$$K^{[M]}_\chi = \{N \in K_\chi : N \cong M\}$$

and $\mathfrak{k}^{[M]}_\chi = (K^{[M]}_\chi, \leq_{\mathfrak{k}} \restriction K^{[M]}_\chi)$, and $\mathfrak{k}^{[M]}$ is the $\mathfrak{k}'$ we get in 1.25(1), with $(\mathfrak{k}^{[M]}, \mathfrak{k})$ here standing in for $(\mathfrak{k}_\lambda, \mathfrak{k}')$ there.

Claim 1.28. *1) If $\mathfrak{k}$ is an (μ, λ, κ)-AEC, $\lambda \leq \chi < \mu$, $M \in K_\chi$ is $(\chi, \geq\kappa)$-superlimit$_1$ (see Definition 1.21) <u>then</u> $\mathfrak{k}^{[M]}_\chi$ is a (χ^+, χ, κ)-DAEC.*

2) If in addition $\mathfrak{k}$ is a (μ, λ, κ)-DAEC$^\pm$ <u>then</u> $\mathfrak{k}^{[M]}_\chi$ is a (χ^+, χ, κ)-DAEC$^\pm$.

*3) [$\mathfrak{k}$ satisfies **Ax.IV**(d).]*

M is $(\chi, \geq \kappa)$-superlimit$_1$ <u>iff</u> M is $(\chi, \geq \kappa)$-superlimit$_2$.

Proof. Easy. $\square_{1.28}$

§ 2. PR FRAMES

Below, the main case is $\iota = 4$.

Definition 2.1. Here $\iota = 0,1,2,3,4$. We say that $\mathfrak{s}$ is a good (μ,λ,κ)-ι-frame when $\mathfrak{s}$ consists of the following objects satisfying the following condition: μ,λ,κ (so we should write $\mu_{\mathfrak{s}},\lambda_{\mathfrak{s}},\kappa_{\mathfrak{s}}$ but we may ignore them when defining $\mathfrak{s}$) and

(A) $\mathfrak{k} = \mathfrak{k}_{\mathfrak{s}}$ is a (μ,λ,κ)-4-DAEC (see 1.4(4)), so we may write $\mathfrak{s}$ instead of $\mathfrak{k}$, e.g. $\leq_{\mathfrak{s}}$-increasing, etc., and $\chi \in [\lambda,\mu) \Rightarrow \mathrm{LST}(\chi^{<\kappa})$.

(B) $\mathfrak{k}$ has a $(\lambda,\geq\kappa)$-superlimit model M^* which[5] is not $<_{\mathfrak{k}}$-maximal — i.e.:
  - (a) $M^* \in K^{\mathfrak{s}}_\lambda$
  - (b) If $M_1 \in K^{\mathfrak{s}}_\lambda$ then for some M_2, $M_1 <_{\mathfrak{s}} M_2 \in K^{\mathfrak{s}}_\lambda$ and M_2 is isomorphic to M^*.
  - (c) If $\langle M_i : i<\delta\rangle$ is $\leq_{\mathfrak{s}}$-increasing, $i<\delta \Rightarrow M_i \cong M$, and $\mathrm{cf}(\delta)\geq\kappa$, $\delta<\lambda^+$ then $\bigcup\{M_i : i<\delta\}$ is isomorphic to M^*.

(C) $\mathfrak{k}$ has the amalgamation property, the JEP (joint embedding property), and has no $\leq_{\mathfrak{k}}$-maximal member. If $\iota \geq 1$ then $\mathfrak{k}$ has primes over chains (i.e. **Ax.III**(f)), and if $\iota \geq 4$, $\mathfrak{k}$ has primes over $\leq_{\mathfrak{s}}$-directed sequences (i.e. **Ax.IV**(f)).

(D) (a) $\mathcal{S}^{\mathrm{bs}} = \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}$ (the class of basic types for $\mathfrak{k}_{\mathfrak{s}}$) is included in $\bigcup\{\mathcal{S}(M) : M \in K_{\mathfrak{s}}\}$ and is closed under isomorphisms including automorphisms. For $M \in K_\lambda$, let $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}} \cap \mathcal{S}(M)$; no harm in allowing types of finite sequences.
  - (b) If $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, then p is non-algebraic (i.e., not realized by any $a \in M$).
  - (c) **Density**:
  If $M \leq_{\mathfrak{k}} N$ are from $K_{\mathfrak{s}}$ and $M \neq N$, then for some $a \in N\setminus M$ we have $\mathrm{ortp}(a,M,N) \in \mathcal{S}^{\mathrm{bs}}$. The intention is that examples are minimal types in [She01] (i.e. [She09c]) and regular types for superstable theories.
  - (d) **bs-Stability**: $\mathcal{S}^{\mathrm{bs}}(M)$ has cardinality $\leq \|M\|^{<\kappa}$ for $M \in K_{\mathfrak{s}}$.

(E) (a) $\bigcup = \bigcup_{\mathfrak{s}}$ is a four place relation called *non-forking*, with $\bigcup(M_0,M_1,a,M_3)$ implying $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3$ are from $K_{\mathfrak{s}}$, $a \in M_3\setminus M_1$, $\mathrm{ortp}(a,M_0,M_3) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_0)$, and $\mathrm{ortp}(a,M_1,M_3) \in \mathcal{S}^{\mathrm{bs}}(M_1)$. Also, $\bigcup$ is preserved under isomorphisms.
  We may also write $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} a$ and demand that if $M_0 = M_1 \leq_{\mathfrak{k}} M_3$ are both in K_λ then $\bigcup(M_0,M_1,a,M_3)$ is equivalent to "$\mathrm{ortp}(a,M_0,M_3) \in \mathcal{S}^{\mathrm{bs}}(M_0)$". We may state $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} a$ as "$\mathrm{ortp}(a,M_1,M_3)$ does not fork over M_0 (inside M_3)." (This is justified by clause (b) below.)
  [Explanation: The intention is to axiomatize non-forking of types, but we allow ourselves to deal only with basic types. Note that in [She01] (i.e. [She09c]) we know something on minimal types but other types are something else.]
  - (b) **Monotonicity:**

$$\text{If } M_0 \leq_{\mathfrak{k}} M_0' \leq_{\mathfrak{k}} M_1' \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3 \leq_{\mathfrak{k}} M_3'$$

[5] Follows by (C), in fact.

and $M_1 \cup \{a\} \subseteq M_3'' \leq_{\mathfrak{k}} M_3'$, with all of them in K_λ, then

$$\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup(M_0', M_1', a, M_3') \Leftrightarrow \bigcup(M_0', M_1', a, M_3'')$$

so it is legitimate to just say "ortp(a, M_1, M_3) does not fork over M_0".

[Explanation: non-forking is preserved by decreasing the type, increasing the basis (i.e. the set over which it does not fork) and increasing or decreasing the model inside which all this occurs. The same holds for stable theories, only here we restrict ourselves to "legitimate" types.]

(c) **Local Character:**
Case 1: $\iota = 1, 2, 3$.
If $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-semi-continuous, $p \in \mathcal{S}^{\rm bs}(M_\delta)$, and $\mathrm{cf}(\delta) \geq \kappa$ then for every $\alpha < \delta$ large enough, p does not fork over M_α.

Case 2: $\iota = 4$.
If I is a κ-directed partial order, $\overline{M} = \langle M_t : t \in I \rangle$ is a $\leq_{\mathfrak{s}}$-directed system, M is its $\leq_{\mathfrak{k}}$-union, $M \leq_{\mathfrak{s}} N$, and ortp$(a, M, N) \in \mathcal{S}^{\rm bs}(M_\delta)$ then for every $s \in I$ large enough ortp(a, M, N) does not fork over M_s.

Case 3: $\iota = 0$.
Like Case 1, using $(\geq\kappa)$-continuity.

[Explanation: This is a replacement for $\kappa \geq \kappa_r(T)$: if $p \in \mathcal{S}(A)$ then there is a $B \subseteq A$ of cardinality $< \kappa$ such that p does not fork over A.]

(d) **Transitivity**:
If $M_0 \leq_{\mathfrak{k}} M_0' \leq_{\mathfrak{k}} M_0'' \leq_{\mathfrak{k}} M_3$ and $a \in M_3$ and ortp(a, M_0'', M_3) does not fork over M_0' and ortp(a, M_0', M_3) does not fork over M_0 (all models are in K_λ, of course, and necessarily the three relevant types are in $\mathcal{S}^{\rm bs}$), then ortp(a, M_0'', M_3) does not fork over M_0.

(e) **Uniqueness**:
If $p, q \in \mathcal{S}^{\rm bs}(M_1)$ do not fork over $M_0 \leq_{\mathfrak{k}} M_1$ (all in $K_{\mathfrak{s}}$) and $p \restriction M_0 = q \restriction M_0$ then $p = q$.

(f) **symmetry**:
Case 1: $\iota \geq 3$.
If $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$ and $(M_0, M_\ell, a_\ell) \in K^{3,\rm pr}_{\mathfrak{s}}$ (see clause (j) below) for $\ell = 1, 2$ then $\mathrm{ortp}_{\mathfrak{s}}(a_2, M_1, M_3)$ does not fork over M_0 iff $\mathrm{ortp}_{\mathfrak{s}}(a_1, M_2, M_3)$ does not fork over M_0.

Case 2: $\iota = 0, 1, 2$.
If $M_0 \leq_{\mathfrak{k}} M_3$ are in $\mathfrak{k}_\lambda$, and for $\ell = 1, 2$ we have $a_\ell \in M_3$ and ortp$(a_\ell, M_0, M_3) \in \mathcal{S}^{\rm bs}(M_0)$, then the following are equivalent:

(α) There are M_1, M_3' in $K_{\mathfrak{s}}$ such that $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{K}} M_3'$, $a_1 \in M_1$, $M_3 \leq_{\mathfrak{k}} M_3'$ and ortp(a_2, M_1, M_3') does not fork over M_0.

(β) There are M_2, M_3' in K_λ such that $M_0 \leq_{\mathfrak{k}} M_2 \leq_{\mathfrak{k}} M_3'$, $a_2 \in M_2$, $M_3 \leq_{\mathfrak{k}} M_3'$ and ortp(a_1, M_2, M_3') does not fork over M_0.
[Explanation: this is a replacement to "ortp$(a_1, M_0 \cup \{a_2\}, M_3)$ forks over M_0 iff ortp$(a_2, M_0 \cup \{a_1\}, M_3)$ forks over M_0," which is not well defined in our context.]

(g) **Existence**:
If $M \leq_{\mathfrak{s}} N$ and $p \in \mathcal{S}^{\rm bs}(M)$ then there is $q \in \mathcal{S}^{\rm bs}(N)$ which is a non-forking extension of p.

(h) **Continuity**:
Case 1: $\iota = 1, 2, 3$.

If $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing and $\le_{\mathfrak{s}}$-semi-continuous, $M_\delta = \bigcup\limits_{\alpha<\delta} M_\alpha$ (which holds if $\mathrm{cf}(\delta) \ge \kappa$), $p \in \mathcal{S}_{\mathfrak{s}}(M_\delta)$, and $p \restriction M_\alpha$ does not fork over M_0 for $\alpha < \delta$ <u>then</u> $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$ and it does not fork over M_0.

Case 2: $\iota = 4$.
Similarly, but for $\overline{M} = \langle M_t : t \in I\rangle$, I directed, and $M = \bigcup\{M_t : t \in I\}$ is a $\le_{\mathfrak{s}}$-upper bound of $\overline{M}$.

Case 3: $\iota = 0$.
Like Case 1 for $\overline{M}$ $(\ge\kappa)$-continuous.

- $\bullet_1$ If $\iota \ge 1$, $\mathfrak{s}$ has $K^{3,\mathrm{pr}}_{\mathfrak{s}}$-primes (see 2.7 below).
- $\bullet_2$ If $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ <u>then</u> p does not fork over M for some $M \le_{\mathfrak{s}} N$ from K_λ.

(i) **Strong continuity**:
Case 1: $\iota = 1, 2, 3$.
We have that $\mathrm{ortp}(b, M_\delta, M_{\delta+1})$ does not fork over M_0, where

- $\bullet_1$ $\overline{M} = \langle M_i : i \le \delta + 1\rangle$ is $\le_{\mathfrak{s}}$-increasing.
- $\bullet_2$ M_δ is prime over $\overline{M} \restriction \delta$.
- $\bullet_3$ $b \in M_{\delta+1} \setminus M_\delta$
- $\bullet_4$ $\mathrm{ortp}(b, M_i, M_{\delta+1})$ does not fork over M_0 for $i < \delta$.

Case 2: $\iota = 4, 5$.
We have that $\mathrm{ortp}(b, M_\delta, M_{\delta+1})$ does not fork over M_0, where

- $\bullet_1$ $\overline{M} = \langle M_s : s \in I\rangle$ is $\le_{\mathfrak{s}}$-increasing, I a partial order with $0 \in I$ minimal.
- $\bullet_2$ N_0 is prime over $\overline{M}$.
- $\bullet_3$ $b \in N_1 \setminus N_0$, where $N_0 \le_{\mathfrak{s}} N_1$.
- $\bullet_4$ $\mathrm{ortp}(b, M_s, N_1)$ does not fork over M_0 for all $s \in I$.

Claim 2.2. *1) If $\langle M_i : i < \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing, $\left(\sum\left\{\|M_i\| : i < \delta\right\}\right) < \mu$, $p_i \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ does not fork over M_0 for $i < \delta$, and $i < j \Rightarrow p_j \restriction M_i = p_i$ <u>then</u>:*

- (a) *We can find M_δ such that $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} M_\delta$.*
- (b) *For any such M_δ, we can find $p_\delta \in \mathcal{S}_{\mathfrak{s}}(M_\delta)$ such that $\bigwedge\limits_{i<\delta} p_\delta \restriction M_i = p_i$ and p_δ does not fork over M_0.*
- (c) *In clause (b), p_δ is unique.*
- (d) *If $\ell \ge \kappa \wedge \mathrm{cf}(\delta) \ge \kappa$, we can add $M = \bigcup\{M_\alpha : \alpha < \delta\}$.*

2) Similarly for $\overline{M} = \langle M_t : t \in I\rangle$, I directed.

Proof. 1) First, choose M_δ by 2.1, Clause (A). Second, choose $p_\delta \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$, a non-forking extension of p_0, which exists by Axiom (g) of 2.1(E). Now $p_\delta \restriction M_i \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ does not fork over M_0 by 2.1(E)(b) and it extends p_0, so it is equal to p_i by (E)(e). Third, p_δ is unique by (E)(e).
2) Should be clear, too. $\square_{2.2}$

Definition 2.3. 1) Assume $M_\ell \le_{\mathfrak{s}} N$ and $p_\ell \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\ell)$ for $\ell = 1, 2$. We say that p_1, p_2 are parallel <u>when</u> some $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ is a non-forking extension of p_ℓ for $\ell = 1, 2$.
2) We say $\mathfrak{s}$ is *type-full* when $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathcal{S}^{\mathrm{na}}_{\mathfrak{k}_{\mathfrak{s}}}(M)$ for $M \in K_{\mathfrak{s}}$.
3) We say $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is based on $\bar{\mathbf{a}}$ when:

- (A) $\bar{\mathbf{a}}$ is a sequence from M.

(B) If $M \le_{\mathfrak{s}} N$, $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$ is a non-forking extension of p, and π is an automorphism of N over $\bar{\mathbf{a}}$ then $\pi(q) = q$. (See [Sheb] for how we can guarantee there is such $\bar{\mathbf{a}} \in {}^\lambda M$, and even $\bar{\mathbf{a}} \in {}^1 M$.)

3A) Similarly for $p \in \mathcal{S}^\varepsilon_{\mathfrak{s}}(M)$; similarly for part (4).
4) We say $\mathfrak{s}$ is $(<\theta)$-based when in clause 2.3(3) above there is such $\bar{\mathbf{a}} \in {}^{\theta >}M$.

Definition 2.4. 1) We say that NF is a non-forking relation on a (μ, λ, κ)-1-DAEC $\mathfrak{k}$ when, in addition to 2.1(A)-(C):

(F) (a) NF is a four-place relation on $\mathfrak{k}_{\mathfrak{s}}$, and $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ implies $M_0 \le_{\mathfrak{k}} M_\ell \le_{\mathfrak{k}} M_1$ and $\mathrm{NF}_{\mathfrak{s}}$ is preserved by isomorphisms.

$(b)_1$ **Monotonicity**:
If $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$, $M_0 \le_{\mathfrak{s}} M'_\ell \le_{\mathfrak{s}} M_\ell$ for $\ell = 1, 2$, and $M'_1 \cup M'_2 \subseteq M'_3 \le_{\mathfrak{s}} M$ then $\mathrm{NF}_{\mathfrak{s}}(M_0, M'_1, M'_2, M'_3)$.

(c) **Symmetry**:
$\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ implies $\mathrm{NF}_{\mathfrak{s}}(M_0, M_2, M_1, M_3)$.

$(d)_1$ **Transitivity**:
If $\mathrm{NF}_{\mathfrak{s}}(M_{2\ell}, M_{2\ell+1}, M_{2\ell+3}, M_{2\ell+4})$ for $\ell = 0, 1$ then $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_4, M_5)$.

$(d)_2$ **Long transitivity**:
If $\langle (N_i, M_i) : i < \delta \rangle$ is an $\mathrm{NF}_{\mathfrak{s}}$-sequence (i.e. M_i is $\le_{\mathfrak{s}}$-increasing, N_i is $\le_{\mathfrak{s}}$-increasing, $M_i \le_{\mathfrak{s}} N_i$, $i < j < \delta \Rightarrow \mathrm{NF}_{\mathfrak{s}}(M_i, N_i, M_j, N_j)$, and $\sum\{\|N_i\| : i < \delta\} < \mu$) then we can find (N_δ, M_δ) such that $\langle (M_i, N_i) : i \le \delta \rangle$ is an NF-sequence.

$(d)_2^+$ Like $(d)_2$, for directed systems.

$(d)_3$ Moreover, in $(d)_2$, if $\langle M_i : i < \delta \rangle$ and $\langle N_i : i < \delta \rangle$ are $(\ge\kappa)$-continuous and $\mathrm{cf}(\delta) \ge \kappa_{\mathfrak{s}}$ then we can demand $M_\delta = \bigcup\{M_i : i < \delta\}$, $N_\delta = \bigcup\{N_i : i < \delta\}$.

Definition 2.5. 1) Let $\mathfrak{s}$ be a good λ-frame and NF a non-forking relation on $\mathfrak{k}$. We say NF respects $\mathfrak{s}$ when: if $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ and $a \in M_2$, $\mathrm{ortp}_{\mathfrak{s}}(a, M_0, M_3) \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_0)$ then $\mathrm{ortp}_{\mathfrak{s}}(a, M_1, M_3)$ is a non-forking extension of $\mathrm{ortp}_{\mathfrak{s}}(a, M_0, M_2)$.
2) We say $\mathfrak{s}$ is a good (λ, μ, κ)-NF-frame when it is a good (λ, μ, χ)-frame and $\mathrm{NF}_{\mathfrak{s}}$ is a non-forking relation on $\mathfrak{k}_{\mathfrak{s}}$ which respects $\mathfrak{s}$.

Definition 2.6. We say that $\mathfrak{s}$ is a *very good* (μ, λ, κ)-NF-frame when it is a good (μ, λ, κ)-NF-frame and

(G) (a) If $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ then there is $M^*_3 \le_{\mathfrak{s}} M_3$ which is prime over $M_1 \cup M_2$. That is,
- If $\mathrm{NF}_{\mathfrak{s}}(M'_0, M'_1, M'_2, M'_3)$ and f_ℓ is an isomorphism from M_ℓ onto M'_ℓ for $\ell = 0, 1, 2$ such that $f_0 \subseteq f_1$ and $f_0 \subseteq f_2$ then there is a $\le_{\mathfrak{s}}$-embedding f_3 of M^*_3 into M'_3 extending $f_1 \cup f_2$.

(b) $\mathfrak{k}_{\mathfrak{s}}$ has $K^{3,\rm pr}_{\mathfrak{s}}$-primes (see 2.7(3) below).

Definition 2.7. 0) $K^{3,\rm bs}_{\mathfrak{s}} = \{(M, N, a) : M \le_{\mathfrak{s}} N,\ a \in N$, and $\mathrm{ortp}_{\mathfrak{s}}(a, M, N) \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)\}$.
1) $K^{3,\rm pr}_{\mathfrak{s}} = \{(M, N, a) \in K^{3,\rm bs}_{\mathfrak{s}}$: if $M \le N'$, $a' \in N'$, $\mathrm{ortp}_{\mathfrak{s}}(a', M, N') = \mathrm{ortp}(a, M, N)$ then there is a $\le_{\mathfrak{k}}$-embedding of N into N' extending id_M and mapping a to $a'\}$.
2) $\mathfrak{k}_{\mathfrak{s}}$ has $K^{3,\rm pr}_{\mathfrak{s}}$-primes if, for every $M \in K_{\mathfrak{s}}$ and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$, there are (N, a) such that $(M, N, a) \in K^{3,\rm pr}_{\mathfrak{s}}$ and $\mathrm{ortp}_{\mathfrak{s}}(a, M, N) = p$.

Definition 2.8. $[\iota \ge 3]$
1) Assume $p_1, p_2 \in \mathcal{S}^{\rm bs}(M)$. We say p_1, p_2 are weakly orthogonal (and denote it $p_1 \underset{\rm wk}{\perp} p_2$) when the following implication holds: if $M_0 \le_{\mathfrak{s}} M_\ell \le_{\mathfrak{s}} M_3$, $(M_0, M_\ell, a_\ell) \in$

$K^{3,\mathrm{pr}}_{\mathfrak{s}}$, and $\mathrm{ortp}_{\mathfrak{s}}(a_\ell, M_0, M_\ell) = p_\ell$ for $\ell = 1, 2$ then $\mathrm{ortp}_{\mathfrak{s}}(a_2, M_1, M_3)$ does not fork over M_0 (this is symmetric by Axiom (f) of 2.1(E)).

2) We say p_1, p_2 are orthogonal (denoted $p_1 \perp p_2$) when: if $M \leq_{\mathfrak{s}} M_2$, $M_1 \leq_{\mathfrak{s}} M_2$ and $q_\ell \in \mathcal{S}^{\mathrm{bs}}(M_2)$ is a non-forking extension of p_ℓ and q_ℓ does not fork over M_1 then $q_1 \underset{\mathrm{wk}}{\perp} q_2$.

3) We say that $\{a_t : t \in I\}$ is independent in (M_0, M_1, M_2) when:

- (A) $a_t \in M_2 \setminus M_1$
- (B) $\mathrm{ortp}_{\mathfrak{s}}(a_t, M_1, M_2)$ does not fork over M_0.
- (C) There is a sequence $\langle t(\alpha) : \alpha < \alpha(*)\rangle$ listing I with no repetitions, and a $\leq_{\mathfrak{s}}$-increasing sequence $\langle M_{1,\alpha} : \alpha \leq \alpha(*) + 1\rangle$ with $M_1 \leq_{\mathfrak{s}} M_{1,0}$ and $M_2 \leq M_{1,\alpha(*)+1}$ such that $a_{t(\alpha)} \in M_{1,\alpha+1}$ and $\mathrm{ortp}_{\mathfrak{s}}(a_{t(\alpha)}, M_{1,\alpha}, M_{1,\alpha+1})$ does not fork over M_0.

4) Let $(M, N, \mathbf{J}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ if $M \leq_{\mathfrak{s}} N$ and $\mathbf{J}$ is independent in (M, N).

5) Let $(M, N, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$ if:

- (A) $M \leq_{\mathfrak{s}} N$
- (B) $\mathbf{J}$ is independent in (M, N).
- (C) If $M \leq_{\mathfrak{s}} N'$ and h is a one-to-one function from $\mathbf{J}$ into N' such that $(M, N', h''(\mathbf{J})) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ then there is a $\leq_{\mathfrak{s}}$-embedding g of N into N' over M extending h.

Remark 2.9. We now can imitate relations of the axioms (as in [She09b, §2]), and basic properties of the notions introduced in 2.8.

Definition 2.10. 1) We say p is *strongly dominated* by $\{p_t : t \in I\}$ and write $p \leq_{\mathrm{st}} \{p_t : t \in I\}$ (this set may contain repetitions[6]) when:

- (A) $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$, $p_t \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N_t)$, $N_t \leq_{\mathfrak{s}} N^+ \in K_{\mathfrak{s}}$, $N \leq_{\mathfrak{s}} N^+$ and
- (B) If $N^+ \leq_{\mathfrak{s}} N^*$, $a_t \in N^*$, $\mathrm{ortp}(a_t, N^+, N^*) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N^+)$ is parallel to p_t and $p' \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N^+)$ is parallel to p (see Definition 2.3), and $\{a_t : t \in I\}$ is independent in (N^+, N^*) then some $a \in N^*$ realizes p'.

2) We say p is weakly dominated by $\{p_t : t \in I\}$ and write $p \leq_{\mathrm{wk}} \{p_t : t \in I\}$ when for some set J and function h from J onto I we have $p \leq_{\mathrm{st}} \{p_{h(t)} : t \in J\}$.

3) Let 'dominated' mean strongly dominated.

4) We say $\mathfrak{s}$ is a strongly good ι-frame when

- (A) It is a good ι-frame.
- (B) If $\mathbf{J}$ is the disjoint union of $\mathbf{J}_1$ and $\mathbf{J}_2$, $(M, N, \mathbf{J}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$, $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N$, and $(M, M_1, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$ then $(M_1, N, \mathbf{J}_2) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ and $\mathrm{ortp}(a, M_1, N)$ does not fork over M for all $a \in \mathbf{J}_2$.

Claim 2.11. *Assume $\mathfrak{s}$ is strongly good.*

1) If p is strongly dominated by $\{p_t : t \in I\}$ then p is weakly dominated by $\{p_t : t \in I\}$.

2) If p is strongly dominated by $\{p_t : t \in I\}$ then for some $J \subseteq I$ of cardinality $< \kappa_{\mathfrak{s}}$, p is strongly dominated by $\{p_t : t \in J\}$.

3) p is weakly dominated by $\{p_t : t \in I\}$ iff for some $\langle i_t : t \in I\rangle$, p is strongly dominated by $\big\{p'_s : s \in \{(t, i) : t \in I,\ i < i_t\}\big\}$, where $p'_{(t,i)} = p_t$ and $i_t < \kappa_{\mathfrak{s}}$ for each $t \in I$.

4) In Definition 2.10(2) without loss of generality $(\forall s \in I)(\exists^{<\kappa} t \in J)[h(t) = s]$.

[6] So pedantically, we should use a sequence and write $p \leq_{\mathrm{st}} \langle p_t : t \in I\rangle$.

5) Preservation by parallelism.

6) $(M, N, \mathbf{J}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *iff for every finite* $\mathbf{I} \subseteq \mathbf{J}$ *we have* $(M, N, \mathbf{I}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$*.*

7) If $\mathbf{J} = \{a_s : s \in I\}$ *is independent in* (M, N) *as witnessed by* $\langle M_{1,\alpha} : \alpha \le \alpha(*)\rangle$, $\langle t(\alpha) : \alpha < \alpha(*)\rangle$ *(see 2.8(5)) then* $(M_{1,0}, M_{1,\alpha(*)}, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$*.*

Proof. 1) Easy.

2) By 2.11(2), following some manipulation.

3) By 2.12 and clause (i) of Definition 2.1.

4),5) Easy.

6) The $\Rightarrow$ direction follows from the definition; the $\Leftarrow$ direction can be proved by induction on $\alpha(*)$.

7) By Definition 2.10(4). $\square_{2.11}$

Claim 2.12. *If* $(M, N, \mathbf{J}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *and* $b \in N$ *then there exist* $\mathbf{I} \subseteq \mathbf{J}$ *and* M_1 *such that:*

- *(A)* $M \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} N$
- *(B)* $|\mathbf{I}| < \kappa_{\mathfrak{s}}$
- *(C)* $b \in M_1$
- *(D)* $(M, M_1, \mathbf{I}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$

Proof. Without loss of generality, $b \notin \mathbf{J}$. We try, by induction on $i \le \kappa$, to choose N_i (and if possible, $\mathbf{I}_i$) such that

$(*)$
- (a) $N_i \le_{\mathfrak{s}} N$ and $\mathbf{I}_i \subseteq \mathbf{J} \setminus \bigcup\limits_{j<i} \mathbf{I}_j$, with $|\mathbf{I}_i| < \kappa$.
- (b) If $j < i$ then $N_j \le_{\mathfrak{s}} N_i$ and $(N_j, N_{j+1}, \mathbf{I}_j) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.
- (c) $N_0 = M$
- (d) If i is a limit ordinal then N_i is prime over $\langle N_j : j < i\rangle$.
- (e) If $i = j+1$ and N_j has already been defined with $b \notin N_j$, and there is $\mathbf{I} \subseteq \mathbf{J} \setminus \bigcup\limits_{\ell<j} \mathbf{I}_\ell$ of cardinality $< \kappa$ (or simply finite) such that
$$\big(N_j, N, \mathbf{I} \cup \{b\}\big) \notin K^{3,\mathrm{bs}}_{\mathfrak{s}}$$
<u>then</u> we can choose such $\mathbf{I}$ as our $\mathbf{I}_j$ and choose $N_i \le_{\mathfrak{s}} N$ such that $(N_j, N_i, \mathbf{I}_j) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.

If we carry the induction for all $i < \kappa$ we get a contradiction (see 2.1(E)(c)), so for some $i(*) < \kappa$ we will hit a point where $N_{i(*)}$ is well defined, but $\mathbf{I}_{i(*)}$ is not.

We prove, by induction on $\theta \le |\mathbf{J}|$, that if $\mathbf{I} \subseteq \mathbf{J}' = \mathbf{J} \setminus \bigcup_{j<i(*)} \mathbf{I}_j$ has cardinality θ then $(N_{i(*)}, N, \mathbf{I} \cup \{b\}) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$. So, using Case 1 of Definition 2.1(E)(i), we are finished. $\square_{2.12}$

Claim 2.13. *1) If* $p \le_{\mathrm{wk}} \{p_i : i < i^*\}$ *and* $i < i^* \Rightarrow q \perp p_i$ *then* $q \perp p$ *(see Definition 2.7(3)).*

2) If $p \le_{\mathrm{wk}} \{p_i : i < i^*\}$ *and* $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ <u>*then*</u> $p \not\perp p_i$ *for some* $i < i^*$*.*

3) If $p \le_{\mathrm{st}} \{p_i : i < \alpha\}$ *then* $p \le_{\mathrm{st}} \{p_i : i < \alpha,\ p_i \not\perp p\}$ *(see Definition 2.10).*

Proof. 1) By induction on i^*: for i^* limit we use 2.1(E)(i), and for i^* successor use $q \perp p_{i^*-1}$.

2) By part (1) and 2.11(3).

3) Easy. $\square_{2.13}$

Claim 2.14. *Assume* $\mathfrak{s}$ *is type-full.*

If $\chi = \chi^{<\kappa} \in [\lambda, \mu)$*, the following is impossible:*

- *(a)* $\langle M_i : i < \chi^+\rangle$ *is* $\le_{\mathfrak{s}}$*-increasing* $\le_{\mathfrak{s}}$*-semi-continuous,*

(b) $\langle N_i : i < \chi^+ \rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing,* $\leq_{\mathfrak{s}}$*-semi-continuous,*

(c) $M_i \leq_{\mathfrak{s}} N_i \in K_{\leq \chi}$,

(d) *for some stationary* $S \subseteq \{\delta < \chi^+ : \mathrm{cf}(\gamma) \geq \kappa\}$, *for every* $i \in S$,

- *There is* $a_i \in M_{i+1} \setminus M_i$ *such that* $\mathrm{ortp}(a_i, N_i, N_{i+1})$ *is not the non-forking extension of* $\mathrm{ortp}(a_i, M_i, M_{i+1}) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$.

Proof. For some club E of χ^+, we have $i \in E \wedge j \in [i, \chi^+) \Rightarrow N_i \cap M_j = M_i$. For each $i \in S \cap E$, by 2.1(E)(c), there is a $j_i < i$ such that $\mathrm{ortp}(a_i, M_i, M_{i+1})$ does not fork over M_{j_i}. By clause (E)(i) of 2.1, for some $j \in [j_i, i)$, we have that $\mathrm{ortp}(a_i, N_j, N_{i+1})$ is not the non-forking extension of $\mathrm{ortp}(a_i, M_{j_i}, M_{i+1})$, so without loss of generality this holds for $j = j_i$.

By Fodor's Lemma, for some $j(*) < j$ the set $S' = \{i \in S \cap E : j_i = j(*)\}$ is stationary. So $\{b_i : i \in S'\}$ is independent in $\big(\bigcup\limits_j M_j, M_{j+1}\big)$. By part (3) we are done.

Also, there is a sequence $\langle M_{j(*),\varepsilon} : \varepsilon \leq \varepsilon_* \leq \kappa \rangle$ which is $\varepsilon_{\mathfrak{s}}$-increasing continuous, with $M_{j(*),0} = M_{j(*)}$, $M_{j(*),\varepsilon} = N_{j(*)}$, and $(M_{j(*),\varepsilon}, M_{j(*),\varepsilon+1}, c_\varepsilon) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$. Now we can choose $\zeta_\varepsilon < \chi^+$ by induction on $\varepsilon < \varepsilon *$, increasing continuous, such that $\big\{a_i : i \in [\zeta_i, \chi^+)\big\}$ is independent in $(M_{j(*),\varepsilon}, \bigcup\limits_j N_j)$ and $\mathrm{ortp}(a_i, M_{j(*),\varepsilon}, N_{i+1})$ does not fork over $M_{j(*)}$ for $i \in [\zeta, \chi^+)$ — an easy contradiction. The induction works for $\varepsilon = 0$ trivially, for ε limit by 2.11(6), and for $\varepsilon = \xi + 1$ we use 2.12. $\square_{2.14}$

Example 2.15. 1) For a complete f.o. strictly stable T and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, we define $\mathfrak{k}$ by

(A) $K_{\mathfrak{k}}$ is the class of κ-saturated (equivalently, $\mathbf{F}^a_\kappa$-saturated) models of T.

(B) $\leq_{\mathfrak{k}}$ is defined by $M \leq_{\mathfrak{k}} N$ iff $M, N \in K_{\mathfrak{k}}$ and $M \prec N$.

Claim 2.16. *If* $p, p_i \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ *for* $i < \kappa_{\mathfrak{s}}$ *and* $i < j \Rightarrow p_i \perp p_j$ *then* $p \perp p_i$ *for every* $i < \kappa$ *large enough.*

Proof. Similar to the proof of [Shea, 1.6=Lj20]. $\square_{2.16}$

Definition 2.17. 1) We say that a good frame $\mathfrak{s}$ is θ-based$_1$ when:

(A) If $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ then for some $\bar{\mathbf{a}} \in {}^{\theta>}M$, p is based on $\bar{\mathbf{a}}$ (see Definition 2.3(4)).

2) We say that $\mathfrak{s}$ is θ-based$_2$ when:

(A) Is as in part (1).

(B) $\mathfrak{s}$ is type-full.

(C) If $M_1 \leq_{\mathfrak{s}} M_2$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_2)$ then, for some $\bar{a}_\ell \in {}^{\theta>}(M_\ell)$, the types p and $\mathrm{ortp}_{\mathfrak{s}}(\bar{a}_2, M_1, M_2)$ are based on $\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_1$, respectively.

§ 3. Thoughts on the main gap

We address here two problems: type theory (i.e. dimension, orthogonality, etc.) for strictly stable classes, and the main gap concerning somewhat saturated models. The hope always was that advances in the first will help the second.

Concerning the first order case, work started in [She90, Ch.V] (particularly §5) and [She91] and was much advanced in Hernandes [Her92]; but this was not enough for the main gap for somewhat saturated models.

Here we are dealing with the type dimension in a general framework.

* * *

The main gap for $\aleph_1$-saturated models of a countable first order theory is open. *A priori*, it has looked easier than the one for models (which was preferred, being "the original question") because of the existence of prime models over any, but is still open. (The problem for uncountable first order $|T|^+$-saturated models is as well).

Why doesn't the proof in [She90, Ch.XII] work? What's missing is, in $\mathfrak{C}^{\rm eq}$,

⊛ If $M_0 \prec M_1 \prec M_2$ are $\aleph_1$-saturated, $a \in M_2 \setminus M_1$ and $(a/M_1) \not\perp M_0$ <u>then</u> for some $b \in M_2 \setminus M_1$ we have $b \underset{M_0}{\bigcup} M_1$.

The central case is when a/M_1 is orthogonal to q if $q \perp M_0$.

Possible Approach 1: We use T being first order countable, stable NDOP (even shallow) to understand types. See [LS06].

Possible Approach 2: We use the context dealt with in this paper. We are poorer in knowledge on the class <u>but</u> we have a richer $\mathfrak{C}^{\rm eq}$, so we may prove ⊛ even if it fails for T in the elementary case (this is a connection between [Sheb] and this work).

Possible Approach 3: We start with the context here. If things are not OK, we define such a derived DAEC; this was done in [She09e] and [She09b]. It may have non-structure properties — enough to get the maximal number of models up to isomorphism. If not, we arrive to a finer $\mathfrak{k}$, but still a case of our context. Similarly in limit. If we succeed enough times we shall prove that all is OK.

Possible Approach 4: Now we have a maximal non-forking tree $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ inside a somewhat saturated model; for [She90], e.g. $\|M_\eta\| \le \lambda$, the models are λ^+-saturated but we use models from here. If M is prime over $\bigcup\{M_\eta : \eta \in \mathcal{T}\}$ we are done, but maybe there is a residue. This appears in the following way: for $\eta \in \mathcal{T}$ and $p \in \mathcal{S}^{\rm bs}(M_\eta)$, the dimension of p is not exhausted by

$$\{a_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in \mathcal{T} \text{ and } (a_{\eta\hat{}\langle\alpha\rangle}/M_\eta) \not\perp p\}$$

but the lost part is not infinite! This imposes $\le \lambda$ unary functions from $\mathcal{T}$ to $\mathcal{T}$. Now it seems to us that the question of whether this possible non-exhaustion can arise[7] is not a good dividing line, as though its negation is informative it is not clear whether it has any consequence. However, there are two candidates for dividing lines (actually, their disjunction seems to be what we want).

(A) $(*)$ We can find M, $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ as above and $\eta_* \in \mathcal{T}$, $\ell g(\eta) = 2$, $\nu_* \in \mathcal{T}$, $\ell g(\nu_*) = 1$, $\eta_* \restriction 1 \ne \nu_*$, and $p \in \mathcal{S}^{\rm bs}(M_{\eta_*})$, $p \perp M_{\eta\restriction 1}$ with a residue as above such that we need M_{ν_*} to explicate it.

[7]Essentially: there is a non-algebraic $p \in (M^\perp)^\perp$ which do not 1-dominate any $q \in \mathcal{S}(M)$.

More explicitly,

$(*)'$ If $M' \leq_{\mathfrak{s}} M$ is prime over $\bigcup\{M_\eta : \eta \in \mathcal{T}\}$ and we can find $a_{\eta_*,\nu_*} \in M \setminus M'$ such that $\mathrm{ortp}\big(\mathscr{C}(a_{\eta^*,\nu^*}, M'), \bigcup\{M_\eta : \eta \in \mathcal{T}\}\big)$ mark (M_{η_*}, M_{ν_*}).

Even in $(*)'$ we have to say more in order to succeed in using it.

From $(*)'$ we can prove a non-structure result: on $\mathcal{T}$ we can code any two-place relation R on $\{\eta \in \mathcal{T} : \ell g(\eta) = 1,\ M_\eta, M_{\eta_* \restriction 1}$ isomorphic over $M_{\langle\ \rangle}\}$ which is of the form $\nu_1\ R\ \nu_2 \Leftrightarrow (\exists \nu) \bigwedge\limits_{\ell}$ [there is $\eta',\ \eta_\ell \triangleleft \eta' \in \mathcal{T},\ \ell g(\eta') = 2$ and $\nu \in T,\ \ell g(\nu) = 1$ and there is $a_{\eta',\nu}$ as above].

More complicated is the case

(B) $(**)$ We can fix M, $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ as above, $\eta_* \in T$, $\nu_*, \nu_{**} \in \mathcal{T}$, $\ell g(\eta_*) = \ell g(\nu_*) = \ell g(\nu_{**}) = 1$ such that $(\eta_*, \nu_*), (\eta_*, \nu_{**})$ are as above.

But whereas for (A) we have to make both η_* and ν_* not redundant in (B), in order to get non-structure we have to use a case of (B) which is not "a faking;" e.g. we cannot replace (M_{η_*}, a_{η_*}) by two such pairs.

That is, the "faker" is a case where we can find $M'_{\eta_*}, M''_{\eta_*}$ such that:

- $\mathrm{NF}(M_{\langle\ \rangle}, M'_{\eta_*}, M''_{\eta_*}, M_{\eta_*})$
- M_{η_*} is prime over $M'_{\eta_*} \cup M''_{\eta_*}$.
- Only (M'_{η_*}, M_{ν_*}) and $(M''_{\eta_*}, M_{\nu_{**}})$ relate.

(C) If both (A) and (B), in the right formulation, do not appear then
 (α) **A good possibility**
 We can prove a structure theory: for M, $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$ as above; that is, on each $\mathrm{suc}_{\mathcal{T}}(\eta)$ we have a two-place relation, but it is very simple: you have to glue some together or expand the set of successors by a tree structure.

If this fails we may fall back to approach (3).

We may consider (see [She08], [PS18]):

Question 3.1. 1) For an AEC $\mathfrak{k}$, when does the theory of a model in the logic $\mathscr{L} = \mathbb{L}_{\infty,\kappa}[\mathfrak{k}]$ enriched by dimension quantifiers, characterize models of $\mathfrak{k}$ up to isomorphism? Similarly enriching also by game quantifiers of length $\leq \kappa$.

2) Prove the main gap theorem in the version: if $\mathfrak{s}$ is n-beautiful **[or $n+1$?]** then for $K_{\lambda^{+n}}$ the main gap holds. In particular, if $\mathfrak{s}$ has NDOP, then every $M \in K_{\lambda^{+n}}$ is prime over some non-forking tree of $\leq_{\mathfrak{K}[\mathfrak{s}]}$-submodels $\langle M_\eta : \eta \in \mathcal{T}\rangle$, each M_η of cardinality $\leq \lambda$, where $\mathcal{T} \subseteq {}^{\omega>}(\lambda^{+n})$. If $\mathfrak{s}$ is shallow then the tree has depth $\leq \mathrm{Depth}(\mathfrak{s}) < \lambda^+$ and we can draw a conclusion on the number of models.

Discussion 3.2. Assume stability in $\lambda_{\mathfrak{s}}$.

Let $M_0 \in K_{\mathfrak{s}}$, $\lambda^+_{\mathfrak{s}}$-saturated, at least for the time being.

1) Assume

$\boxplus_1$ $N_0 \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{s}} M$, $N_\ell \in K^{\mathfrak{s}}_\lambda$, $a \in N_0$, and $(N_0, N_1, a) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.

We choose $(N^+_{1,i}, N_{1,i}, \mathbf{I}_i)$ and also, if possible, (M_1, a_i) by induction on $i \leq \lambda^+_{\mathfrak{s}}$ such that

$(*)$ (a) $N_{0,i} \leq_{\mathfrak{s}} N_{1,i} \leq_{\mathfrak{s}} N^+_{1,i} \leq_{\mathfrak{s}} M$
 (b) $\mathbf{I}_i \subseteq \{c \in M : \mathrm{ortp}(c, N_{1,i}, M_0) \perp N_0\}$ is independent in $(N_{1,i}, N^+_{1,i}, M)$ and minimal.
 (c) $\langle N_j : j \leq i\rangle$ is $\leq_{\mathfrak{s}}$-semi-continuous; also, $\langle N^+_j : j \leq i\rangle$ is as well.
 (d) If $i = j+1$ then $N^+_{1,i}$ is $\leq_{\mathfrak{s}}$-universal over $N^+_{1,j}$ and $(N_0, N_{1,i}, a) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.

(e) If $j < i$ then $\mathbf{I}_j \setminus (N_i \cap \mathbf{I}_j) \subseteq \mathbf{I}_i$.
(f) If possible:
 (α) $N_i \leq_{\mathfrak{s}} M_i^+ \leq_{\mathfrak{s}} M$
 (β) $(\mathbf{I}_i \setminus M_i)$ is independent in (M_i, M).
 (γ) $a_i \in M \setminus (\mathbf{I}_i)$
 (δ) $\mathrm{ortp}(a_i, M_1^*, M) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N_i^+)$ is $\perp N_i$.
 (ε) $N_i^* \leq N_{1,i+1}$
(g) If $i = j+1$ and there are (b, N_*^+, N_{**}) such that $b \in N_{1,j}^+ \setminus N_{1,j}$,

$$N_{1,i} \leq_{\mathfrak{s}} N_* \leq_{\mathfrak{s}} N_{**} \in K^{\mathfrak{s}}_{\lambda_{\mathfrak{s}}},$$

$N_{1,i}^+ \leq_{\mathfrak{s}} N_{**}$, and $\mathrm{ortp}_{\mathfrak{s}}(b, N_*, N_{**})$ forks over $N_{1,j}$ then, for some $b \in N_{1,j}^+ \setminus N_{1,j}$, the type $\mathrm{ortp}_{\mathfrak{s}}(N_{1,i}, N_{1,i}^+)$ forks over $N_{1,j}$.

There is no problem to carry the induction.

$\boxplus_2$ The following subset of $\lambda_{\mathfrak{s}}^+$ is not stationary — say, disjoint to the club C:
- $S = \{i < \lambda_{\mathfrak{s}}^+ : \mathrm{cf}(i) \geq \kappa_{\mathfrak{s}}$ and (M_i, a_i) is well defined$\}$
- $S_2 = \big\{i : \mathrm{cf}(i) \geq \kappa_{\mathfrak{s}}$ and for some $b \in N_{1,i}^+,\ \mathrm{tp}(b, N_{1,i}, N_{1,i}^+) = N_0\big\}$.

2) Similarly without (N_0, a) hence without "$\perp N_0$;" it's just simpler.

Definition 3.3. We say $(\overline{N}, \bar{a}, \bar{I})$ is a decreasing pair for M when for some n:

(A) $\overline{N} = \langle N_\ell : \ell \leq n\rangle$ is $\leq_{\mathfrak{s}}$-increasing.
(B) $N_\ell \leq_{\mathfrak{s}} M$, $N_\ell \in K^{\mathfrak{s}}_{\lambda_{\mathfrak{s}}}$
(C) $\bar{a} = \langle a_\ell : \ell < n\rangle$
(D) $(N_\ell, N_{i+1}, a_\ell) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
(E) $\bar{\mathbf{I}} = \langle \mathbf{I}_\ell : \ell \leq n\rangle$
(F) $\mathbf{I}_\ell$ is independent in (N_ℓ, M).
(G) $\mathbf{I}_\ell \subseteq \{c \in M : \mathrm{ortp}(c, N_\ell, M) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N_\ell)$ is $\perp N_k$ if $k < \ell\}$
(H) If $N_\ell \leq_{\mathfrak{s}} N \leq_{\mathfrak{s}} M$, $b \in M \setminus N_0 \setminus \mathbf{I}_\ell$, and $\mathrm{ortp}(b, N, M)$ is $\not\perp N_\ell$ but is orthogonal to N_k for $k < \ell$ then b depends on $\mathbf{I}_\ell$ in (N_ℓ, M).

Attempt to prove decomposition

We assume dimensional continuity to prove decomposition. If we would like to get rid of "M is $\lambda_{\mathfrak{s}}^+$-saturated", we must assume we have a somewhat weaker version $\mathfrak{s}_*$ of $\mathfrak{s}$ where $\lambda_{\mathfrak{s}_*} < \lambda_{\mathfrak{s}}$ and $\langle N_{0,i} : i < \lambda_{\mathfrak{s}}\rangle \leq_{\mathfrak{s}_*}$-represent N_0, and work with that. Assuming CH, $|T| = \aleph_0$ is fine. Without dimensional discontinuity we call 'nice' any $(\bar{N}, \bar{a}, \bar{\mathbf{I}})$ of length $\leq \kappa_{\mathfrak{s}}$!

* * *

Definition 3.4. We say $\mathbf{d} = (I, N, \bar{a}, \bar{\mathbf{I}}) = (I_{\mathbf{d}}, \overline{N}_d, \bar{a}_{\mathbf{d}}, \bar{\mathbf{I}}_{\mathbf{d}})$ is a partial decomposition of when:

$\boxplus$ (a) $I \subseteq {}^{\omega>}\mathrm{Ord}$ is closed under initial segments.
(b) $\overline{N} = \langle N_\eta : \eta \in I\rangle$, so $N_\eta = N_{\mathbf{d},\eta}$.
(c) $\bar{a} = \big\langle a_\eta : \eta \in I \setminus \{\langle\ \rangle\}\big\rangle$, so $a_\eta = a_{\mathbf{d},\eta}$.
(d) $\bar{\mathbf{I}} = \langle \mathbf{I}_\eta : \eta \in I\rangle$, so $\mathbf{I}_\eta = \mathbf{I}_{\mathbf{d},\eta}$.
(e) If $\eta \in I$ then

$$\big(\big\langle N_{\eta\restriction\ell} : \ell \leq \ell g(\eta)\big\rangle, \big\langle \bar{a}_{\eta\restriction(\ell+1)} : \ell < \ell g(\eta)\big\rangle, \big\langle \mathbf{I}_{\eta\restriction\ell} : \ell \leq \ell g(\eta)\big\rangle\big)$$

is nice in M.

(f) If $\eta \in I$ then $\langle a_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in I\rangle$ is a sequence of members of $\mathbf{I}_\eta$ with no repetitions.

Definition 3.5. Let $\leq_\mu$ be the following two-place relation on the set of decompositions of M:
$\bar{\mathbf{d}}_1 \leq_M \mathbf{d}_2$ iff

(A) $I_{\mathbf{d}_1} \subseteq \mathbf{I}_{d_1}$
(B) $\bar{N}_{\mathbf{d}_1} = \bar{N}_{\mathbf{d}_2} \restriction I_{d_1}$
(C) $\bar{a}_{\mathbf{d}_1} = \bar{a}_{\mathbf{d}_2} \restriction (I_{\mathbf{d}_1} \setminus \{< j\})$
(D) $\bar{\mathbf{I}}_{\mathbf{d}_1} = \bar{\mathbf{I}}_{d_2} \restriction I_{\mathbf{d}_1}$

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