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On the non-existence of κ-mad families

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Abstract
Starting from a model with a Laver-indestructible supercompact cardinal κ, we construct a model of $ZF + DC_\kappa$ where there are no κ-mad families.

Keywords Generalized descriptive set theory · Mad families · Supercompact cardinals

Mathematics Subject Classification 03E15 · 03E25 · 03E35 · 03E55

1 Introduction

The study of the definability and possible non-existence of mad families has a long tradition, originating with the paper [6] of Mathias where it was proven that mad families can't be analytic and that there are no mad families in the Solovay model constructed from a Mahlo cardinal (as always, by "mad families" we refer to infinite such families). It was later shown by Toernquist that an inaccessible cardinal suffices for the consistency of this statement [8], and it was then shown by the authors that the non-existence of mad families (in $ZF + DC$) is actually equiconsistent with ZFC [2].

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The current paper can be seen as a continuation of the line of investigation of [2], as well as of [5], where the definability of κ-mad families was considered. Recall the following definition:

Definition 1 Let κ be an infinite regular cardinal. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is κ-almost disjoint if $|A \cap B| < \kappa$ for every $A \neq B \in \mathcal{A}$. $\mathcal{A}$ will be called κ-maximal almost disjoint (κ-mad) if $\mathcal{A}$ is κ-almost disjoint and can't be extended to a larger κ-almost disjoint family.

Assuming the existence of a Laver-indestructible supercompact cardinal κ, we constructed in [5] a generic extension where κ remained supercompact and there are no $\Sigma^1_1(\kappa) - \kappa-$mad families, thus obtaining a higher analog of Mathias' result. Our current main goal is to obtain a higher analog of the main result of [2], i.e. for an uncountable cardinal $\theta > \aleph_0$, we would like to construct a model of $ZF + DC_\theta$ where there are no θ-mad families. As opposed to [2], we only achieve this goal assuming the existence of a supercompact cardinal. The main result of the paper is the following:

Theorem 2 *a. Suppose that* $\aleph_0 < cf(\theta) = \theta < cf(\kappa) = \kappa \leq \lambda = \lambda^{<\kappa}$ *and* θ *is a Laver indestructible supercompact cardinal, then there is a model of* $ZF + DC_{<\kappa} +$ *"there exist no* θ*-mad families" (note that* θ *here has the role of* κ *in the abstract).*
b. If we start from a universe V*, then the final model* V_1 *will have the same cardinals and same* $H(\theta)$ *as* V*.*

We remark that during the time that the current paper was being reviewed, a newer result was announced by Chan, Jackson and Trang [1], where they show the non-existence of certain mad families on uncountable cardinals under AD^+.

We note that while their result requires a weaker large cardinal assumption, it's incompatible with DC_{ω_1}. This should be contrasted with our result which provides us with many high instances of dependent choice.

Finally, we briefly describe our proof strategy. We shall force with a partial order $\mathbb{P}$ where the conditions themselves are forcing notions (this is somewhat similar to [3, 7] and [4], as well as to the recent work of Viale in [9], where a similar approach is applied to the study of generic absoluteness). Forcing with $\mathbb{P}$ will generically introduce the forcing notion $\mathbb{Q}$ that will give us the desired results. More specifically, we shall fix a Laver-indestructible supercompact cardinal θ. The conditions in $\mathbb{P}$ will be elements from a suitable $H(\lambda^+)$ that are $(< \theta)$-support iterations along wellfounded partial orders of $(< \theta)$-directed closed forcing notions satisfying a strong version of θ^+-cc. Given $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{P}$, we will have $\mathbf{q}_1 \leq_{\mathbb{P}} \mathbf{q}_2$ when the iteration given by $\mathbf{q}_1$ is an "initial segment" (in an adequate sense) of the iteration given by $\mathbf{q}_2$. Forcing with $\mathbb{P}$ will introduce a generic iteration $\mathbf{q}_G$ given by the union of $\mathbf{q} \in \mathbb{P}$ that belong to the generic set. In the further generic extension given by $\mathbf{q}_G$, we shall consider $V_1 = HOD(\mathcal{P}(\theta)^{<\kappa} \cup V)$ (for an adequate fixed κ). We shall then prove that there are no θ-mad families in V_1. In order to prove this fact, we shall consider towards contradiction a condition $(\mathbf{q}_0, \underset{\sim}{p}_0)$ that forces a counterexample $\mathcal{A}$, where $\mathbf{q}_0$ will be "sufficiently closed". The filter that's dual to the ideal generated by $\mathcal{A}$ will then be extended to a θ-complete ultrafilter (using the Laver-indestructibility of θ), and we

Sh:1168

shall obtain a contradiction with the help of an amalgamation argument over $\mathbf{q}_0$ using a higher analog of Mathias forcing relative to this ultrafilter.

The rest of the paper will be devoted to the proof of Theorem 2.

2 Proof of the main result

Definition 3 A. Let K be the class of pairs $(\mathbf{q}, U_{\mathbf{q}})$ that consist of the following objects with the following properties:

a. $U = U_{\mathbf{q}}$ a well-founded partial order whose elements are ordinals. We let $U^+ = U \cup \{\infty\}$ where ∞ is a new element above all elements from U, and for $\alpha \in U^+$, we let $U_{<\alpha} = \{\beta \in U : \beta <_U \alpha\}$.

b. An iteration $(\mathbb{P}_{\mathbf{q},\alpha}, \underset{\sim}{\mathbb{Q}}_{\mathbf{q},\beta} : \alpha \in U^+, \beta \in U) = (\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \in U^+, \beta \in U)$.

We shall often denote the iteration itself by $\mathbf{q}$.

c. $\mathbf{q}$ is a $(< \theta)$-support iteration, and in addition:

(α) Each $\underset{\sim}{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name of a forcing notion whose set of elements is an object X_β from V.

(β) Given $\alpha \in U^+$, $p \in \mathbb{P}_\alpha$ iff p is a function with domain $dom(p) \in [U_{<\alpha}]^{<\theta}$ such that $p(\beta)$ is a canonical $\mathbb{P}_\beta$-name for every $\beta \in dom(p)$.

(γ) $\leq_{\mathbb{P}_\alpha}$ is defined as usual.

(δ) If $w \subseteq U$ is downward closed (i.e. $\alpha <_U \beta \in w \rightarrow \alpha \in w$) and $\mathbb{P}_{\mathbf{q},w} = \mathbb{P}_w = \mathbb{P}_\infty \restriction w = \{p \in \mathbb{P}_\infty : dom(p) \subseteq w\}$, then $\mathbb{P}_w \lessdot \mathbb{P}_\infty$.

d. In $V^{\mathbb{P}_\beta}$, $\underset{\sim}{\mathbb{Q}}_\beta$ satisfies $*^\epsilon_\theta$ for a fixed limit $\epsilon < \theta$, namely, if $\{p_\alpha : \alpha < \theta^+\} \subseteq \underset{\sim}{\mathbb{Q}}_\beta$, then there is some club $E \subseteq \theta^+$ and a pressing down function $f : E \rightarrow \theta^+$ such that if $\delta_1, \delta_2 \in E$, $cf(\delta_1) = cf(\delta_2)$ and $f(\delta_1) = f(\delta_2)$, then p_{δ_1} and p_{δ_2} have a common least upper bound.

e. For $\beta \in U$, the following holds in $V^{\mathbb{P}_\beta}$: If I is a directed partial order of cardinality $< \theta$ and $(p_s : s \in I) \in \mathbb{Q}_\beta^I$ is $\leq_{\mathbb{Q}_\beta}$-increasing, then $\{p_s : s \in I\}$ has a $\leq_{\mathbb{Q}_\beta}$-least upper bound.

Notational remark: As $U_{\mathbf{q}}$ is implicitly part of the definition of $\mathbf{q}$, we shall often just write $\mathbf{q}$ instead of $(\mathbf{q}, U_{\mathbf{q}})$.

B. Let $\leq_K$ be the following partial order on K:

$\mathbf{q}_1 \leq_K \mathbf{q}_2$ iff the following conditions hold:

a. $U_{\mathbf{q}_1} \subseteq U_{\mathbf{q}_2}$ as partial orders.

b. If $U_{\mathbf{q}_2} \models \alpha < \beta$ and $\beta \in U_{\mathbf{q}_1}$, then $\alpha \in U_{\mathbf{q}_1}$.

c. If $w \subseteq U_{\mathbf{q}_1}$ is downward closed, then $\mathbb{P}_{\mathbf{q}_1,w} = \mathbb{P}_{\mathbf{q}_2,w}$.

d. If $\alpha \in U_{\mathbf{q}_1}$, then $\underset{\sim}{\mathbb{Q}}_{\mathbf{q}_1,\alpha} = \underset{\sim}{\mathbb{Q}}_{\mathbf{q}_2,\alpha}$ (this is well-defined recalling clause (b)).

C. Let K_{wf} be the class of U as in (A)(a), and let $\leq_{wf}$ be the partial order on K_{wf} defined as in clauses (B)(a) and (B)(b).

We shall now observe some easy basic properties of the objects defined above:

Observation 4 a. If $(U_\alpha : \alpha < \delta)$ is $\leq_{wf}$-increasing, then $\bigcup\limits_{\alpha<\delta} U_\alpha$ is a $\leq_{wf}$-least upper bound for $(U_\alpha : \alpha < \delta)$.

b. $\leq_K$ is a partial order on K.
c. If $\mathbf{q}_2 \in K$ and $U_1 \subseteq U_{\mathbf{q}_2}$ is downward closed, then there is a unique $\mathbf{q}_1 \in K$ such that $\mathbf{q}_1 \leq_K \mathbf{q}_2$ and $U_{\mathbf{q}_1} = U_1$.
d. If $(\mathbf{q}_\alpha : \alpha < \delta)$ is $\leq_K$-increasing, then there is a unique $\mathbf{q}_\delta \in K$ such that $\alpha < \delta \to \mathbf{q}_\alpha \leq_K \mathbf{q}_\delta$ and $U_{\mathbf{q}_\delta} = \bigcup_{\alpha<\delta} \mathbf{q}_\alpha$.
e. If $U_0, U_1, U_2 \in K_{wf}$, $U_0 = U_1 \cap U_2$ and $U_0 \leq_{wf} U_l$ $(l = 1, 2)$, then there is a unique $U \in K_{wf}$ such that $\bigwedge\limits_{l=1,2} U_l \leq_{wf} U$, $\alpha \in U$ iff $\alpha \in U_1 \vee \alpha \in U_2$ and $\leq_U = \leq_{U_1} \cup \leq_{U_2}$. We denote this U by $U_1 +_{U_0} U_2$.
f. If $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2 \in K$, $\mathbf{q}_0 \leq_K \mathbf{q}_l$ $(l = 1, 2)$ and $U_{\mathbf{q}_0} = U_{\mathbf{q}_1} \cap U_{\mathbf{q}_2}$, then there is a unique $\mathbf{q} \in K$ such that $\bigwedge\limits_{l=1,2} \mathbf{q}_l \leq_K \mathbf{q}$ and $U_{\mathbf{q}} = U_{\mathbf{q}_1} +_{U_{\mathbf{q}_0}} U_{\mathbf{q}_2}$. We shall denote this $\mathbf{q}$ by $\mathbf{q}_1 +_{\mathbf{q}_0} \mathbf{q}_2$.
g. If $\alpha \in U^+_{\mathbf{q}}$, then $\mathbb{P}_{\mathbf{q},\alpha}$ is a $(< \theta)$-complete forcing satisfying $*^\epsilon_\theta$ (hence θ^+-cc).
h. Suppose that $\mathbf{q} \in K$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{q},\infty}$-name of a forcing notion whose universe is from V, such that the conditions of definitions 3(d) and 3(e) are satisfied, then there is $\mathbf{q}' \in K$ such that $\mathbf{q} \leq_K \mathbf{q}'$, $U_{\mathbf{q}'} = U_{\mathbf{q}} \cup \{\gamma\}$, $U_{\mathbf{q}'} \models \alpha < \gamma$ for every $\alpha \in U_{\mathbf{q}}$ and $\underset{\sim}{\mathbb{Q}}_{\mathbf{q}',\gamma} = \underset{\sim}{\mathbb{Q}}$. □

Definition 5 The forcing notion $\mathbb{P}$ will be defined as follows:

a. The conditions of $\mathbb{P}$ are the elements $\mathbf{q}$ of $K \cap H(\lambda^+)$ such that $U_{\mathbf{q}} \subseteq \lambda^+$, and for every $\beta \in U_{\mathbf{q}}$, $\underset{\sim}{\mathbb{Q}}_\beta$ is a name for a forcing whose underlying set of conditions is some $X_\beta \subseteq \lambda^+$.
b. Given $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{P}$, $\mathbb{P} \models$ "$\mathbf{q}_1 \leq \mathbf{q}_2$" iff $\mathbf{q}_1 \leq_K \mathbf{q}_2$.
c. Given a generic set $G \subseteq \mathbb{P}$, we let $\mathbf{q}_G = \bigcup\{\mathbf{q} : \mathbf{q} \in G\}$.

Before the next claim, we shall remind the reader of the definition of $(< \kappa)$-strategic completeness. Given a forcing $\mathbb{P}$, a condition $p \in \mathbb{P}$ and an ordinal α, the two-player game $G_\alpha(p, \mathbb{P})$ will consist of α moves. In the βth move, player **I** chooses $p_\beta \in \mathbb{P}$ above p and all q_γ $(\gamma < \beta)$ previously chosen by player **II**. Player **II** will respond with a condition $q_\beta \in \mathbb{P}$ above p_β. Player **I** wins the game iff for each $\beta < \alpha$ he has a legal move. $\mathbb{P}$ is α-strategically complete if player **I** has a winning strategy in $G_\alpha(p, \mathbb{P})$ for every $p \in \mathbb{P}$. Finally, $\mathbb{P}$ is $(< \kappa)$-strategically complete if it's α-strategically complete for every $\alpha < \kappa$.

Claim 6 a. $\mathbb{P}$ is $(< \kappa)$-strategically complete. Moreover, it's $(< \lambda^+)$-complete and $(< \theta)$-directed closed.
b. $\Vdash_{\mathbb{P}}$ "$\mathbf{q}_{\underset{\sim}{G}} \in K$", hence $\Vdash_{\mathbb{P}}$ "$\mathbb{P}_{\mathbf{q}_{\underset{\sim}{G}},\infty}$ is $(< \theta)$-directed closed and θ^+-cc".
c. If $\delta < \lambda^+$, $cf(\delta) > \theta$ and $(\mathbf{q}_\alpha : \alpha < \delta)$ is $\leq_{\mathbb{P}}$-increasing, then $\mathbf{q} := \bigcup_{\alpha<\delta} \mathbf{q}_\alpha$ belongs to $\mathbb{P}$ and $\mathbb{P}_{\mathbf{q}} = \bigcup_{\alpha<\delta} \mathbb{P}_{\mathbf{q}_\alpha}$. By θ^+-c.c., $\underset{\sim}{a}$ is a canonical $\mathbb{P}_{\mathbf{q}}$-name of a member of $[\theta]^\theta$ iff $\underset{\sim}{a}$ is a canonical $\mathbb{P}_{\mathbf{q}_\alpha}$-name of a member of $[\theta]^\theta$ for some $\alpha < \delta$.

Proof The claim follows directly from the definitions. The fact that $\Vdash_{\mathbb{P}}$ "$\mathbf{q}_{\underset{\sim}{G}} \in K$" follows from the general fact that if I is a directed set, $\{\mathbf{q}_t : t \in I\} \subseteq K$ and $s \leq_I t \to \mathbf{q}_s \leq_K \mathbf{q}_t$, then $\bigcup\{\mathbf{q}_t : t \in I\}$ is well-defined and belongs to K. This also shows that $\mathbb{P}$ is $(< \theta)$-directed closed. □

We shall now define our desired model:

Definition 7 a. In $V^{\mathbb{P}}$, let $\mathbb{Q} = \mathbb{P}_{\mathbf{q}_{\underset{\sim}{G}},\infty}$.

b. Let $V_2 = V^{\mathbb{P}\star\underset{\sim}{\mathbb{Q}}}$.
c. Let V_1 be $HOD(\mathcal{P}(\theta)^{<\kappa} \cup V)$ inside V_2.

Claim 8 a. $V_1 \models ZF+DC_{<\kappa}$. b. $(Ord^{<\kappa})^{V_1} = (Ord^{<\kappa})^{V_2}$, hence $\mathcal{P}(\theta)^{V_1} = \mathcal{P}(\theta)^{V_2}$.

Proof We shall prove the first part of clause (b), the rest should be clear. Clearly, $(Ord^{<\kappa})^{V_1} \subseteq (Ord^{<\kappa})^{V_2}$. Now let $\eta \in (Ord^{\gamma})^{V_2}$ for some $\gamma < \kappa$, then $\eta = \underset{\sim}{\eta}[G]$ for some name $\underset{\sim}{\eta}$ of a member of Ord^{γ}, where $G \subseteq \mathbb{P}\star\underset{\sim}{\mathbb{Q}}$ is generic. $G = G_1\star G_2$ where $G_1 \subseteq \mathbb{P}$ is generic and $G_2 \subseteq \underset{\sim}{\mathbb{Q}}[G_1]$ is generic. Working in $V[G_1]$, $\underset{\sim}{\eta}/G_1$ is a $\underset{\sim}{\mathbb{Q}}[G_1]$-name. As $\underset{\sim}{\mathbb{Q}}[G_1]$ is θ^+-cc, for every $\beta < \gamma$ there is a maximal antichain $\{p_{\beta,i} : i < \theta\} \subseteq \underset{\sim}{\mathbb{Q}}[G_1]$ of conditions that force a value to $\underset{\sim}{\eta}/G_1(\beta)$. Let $\{\zeta_{\beta,i} : i < \theta\}$ be the set corresponding values forced by the above conditions. Let $\Gamma = \{\underset{\sim}{p}_{\beta,i}, \underset{\sim}{\zeta}_{\beta,i} : \beta < \gamma, i < \theta\}$ be the corresponding $\mathbb{P}$-names for the above objects (so we can regard them as $\mathbb{P}$-names for ordinals). As there are $< \kappa$ such names and $\mathbb{P}$ is $(< \kappa)$-strategically complete, there is a dense set of $\mathbf{q} \in \mathbb{P}$ that force values to all elements of Γ. Therefore, there is some $\mathbf{q} \in \mathbb{P} \cap G_1$ that forces values to all elements of Γ (and the values forced are necessarily $\{p_{\beta,i}, \zeta_{\beta,i} : \beta < \gamma, i < \theta\}$). It follows that $\{p_{\beta,i}, \zeta_{\beta,i} : \beta < \gamma, i < \theta\} \in V$. In V_2, there is a function $f : \gamma \to \theta$ such that for every $\beta < \gamma$, $\eta(\beta) = \zeta_{\beta,f(\beta)}$. As $f \in \mathcal{P}(\theta)^{<\kappa}$ and $\{p_{\beta,i}, \zeta_{\beta,i} : \beta < \gamma, i < \theta\} \in V$, it follows that $\eta \in V_1$. $\square$

Main Claim 9 There are no θ-mad families in V_1.
The rest of the paper will be devoted to the proof of Claim 9.
Suppose towards contradiction that there is a θ-mad family in V_1, so there is some $(\mathbf{q}_0, \underset{\sim}{p}_0) \in \mathbb{P}\star\underset{\sim}{\mathbb{Q}}$ forcing this statement about $\underset{\sim}{\mathcal{A}}$ where $\underset{\sim}{\mathcal{A}}$ is a canonical $\mathbb{P}\star\underset{\sim}{\mathbb{Q}}$-name of a θ-mad family definable using $\underset{\sim}{\eta}$, and $\underset{\sim}{\eta}$ is a canonical $\mathbb{P}\star\underset{\sim}{\mathbb{Q}}$-name of a parameter (so $\underset{\sim}{\eta} = ((\underset{\sim}{a}_\epsilon : \epsilon < \underset{\sim}{\epsilon}(*)), \underset{\sim}{x})$, where $\Vdash$ "$\underset{\sim}{\epsilon}(*) < \kappa$", each $\underset{\sim}{a}_\epsilon$ is a $\mathbb{P}\star\underset{\sim}{\mathbb{Q}}$-name of a subset of θ and $\Vdash$ "$\underset{\sim}{x} \in V$"). Let $G_0 \subseteq \mathbb{P}$ be generic over V such that $\mathbf{q}_0 \in G_0$. In $V[G_0]$, $\underset{\sim}{\eta}$ is a $\mathbb{P}_{\mathbf{q}_{G_0},\infty}$-name, and by increasing $\mathbf{q}_0$, we may assume wlog that $p_0 := \underset{\sim}{p}_0[G_0] \in \mathbb{P}_{\mathbf{q}_0}$, $x = \underset{\sim}{x}[G_0] \in V$, $\epsilon(*) = \underset{\sim}{\epsilon}(*)[G_0] \in \kappa$ and that each $\underset{\sim}{a}_\epsilon$ $(\epsilon < \epsilon(*))$ is a canonical $\mathbb{P}_{\mathbf{q}_0}$-name of a subset of θ. Given $\mathbf{q} \in \mathbb{P}$ above $\mathbf{q}_0$, let $\mathcal{A}_{\mathbf{q}}$ be the set of canonical $\mathbb{P}_{\mathbf{q}}$-names $\underset{\sim}{a}$ such that $(\mathbf{q}, \underset{\sim}{p}_0) \Vdash_{\mathbb{P}\times\underset{\sim}{\mathbb{Q}}}$ "$\underset{\sim}{a} \in \underset{\sim}{\mathcal{A}}$", so $\mathbf{q}_0 \leq \mathbf{q}_1 \leq \mathbf{q}_2 \to \mathcal{A}_{\mathbf{q}_1} \subseteq \mathcal{A}_{\mathbf{q}_2}$. Note that if $\mathbf{q}_0 \leq \mathbf{q}_1$, $\mathbb{P}_{\mathbf{q}_1,\infty} \models$ "$p_0 \leq p_1$" and $(\mathbf{q}_1, p_1) \Vdash$ "$\underset{\sim}{b} \in [\theta]^{\theta}$", then for some $(\mathbf{q}_2, \underset{\sim}{a})$ we have $\mathbf{q}_1 \leq_{\mathbb{P}} \mathbf{q}_2$, $\underset{\sim}{a} \in \mathcal{A}_{\mathbf{q}_2}$ and $(\mathbf{q}_2, p_0) \Vdash$ "$\underset{\sim}{b} \cap \underset{\sim}{a} \in [\theta]^{\theta}$". By extending any given $\mathbf{q}_1 \in \mathbb{P}$ above $\mathbf{q}_0$ in this way sufficiently many times to add witnesses for madness, and recalling Claim 6(c), we establish that the set $\{\mathbf{q}_1 : \mathbf{q}_0 \leq_{\mathbb{P}} \mathbf{q}_1$ and $\Vdash_{\mathbb{P}_{\mathbf{q}_1}}$ "$\mathcal{A}_{\mathbf{q}_1}$ is θ-mad"$\}$ is dense in $\mathbb{P}$ above $\mathbf{q}_0$.

Now, in V_2, let $I = \{A \subseteq \theta : A$ is contained in a union of $< \theta$ members of $\mathcal{A}\}$, then I is a θ-complete ideal and $\theta \notin I$. Let F be the dual filter of I, then F is θ-complete, and as θ is supercompact in V_2 (recalling that θ is Laver indestructible and that $\mathbb{P}\star\underset{\sim}{\mathbb{Q}}$ is $(<\theta)$-directed closed), there is a $\mathbb{P}\star\underset{\sim}{\mathbb{Q}}$-name $\underset{\sim}{D}$ such that $(\mathbf{q}_0, p_0) \Vdash_{\mathbb{P}\star\underset{\sim}{\mathbb{Q}}}$ "$\underset{\sim}{D}$ is a θ-complete ultrafilter on θ that extends F, and hence is disjoint to $\underset{\sim}{\mathcal{A}}$". By Claim 6 and what we observed in the previous paragraph, we may assume wlog that $\mathbf{q}_0 \Vdash_{\mathbb{P}}$ "$\mathcal{A}_{\mathbf{q}_0}$ is θ-mad and $\underset{\sim}{D}_{\mathbf{q}_0} := \underset{\sim}{D} \cap \mathcal{P}(\theta)^{V^{\mathbb{P}_{\mathbf{q}_0,\infty}}}$ is a $\mathbb{P}_{\mathbf{q}_0,\infty}$-name of an ultrafilter on θ".

Given an ultrafilter U on θ, the forcing $\mathbb{Q}_U$ is defined as follows: the conditions of $\mathbb{Q}_U$ have the form (u, A) where $u \in [\theta]^{<\theta}$ and $A \in U$. the order is defined naturally, i.e. $(u_1, A_1) \leq (u_2, A_2)$ iff $u_1 \subseteq u_2$, $u_2 \setminus u_1 \subseteq A_1$ and $A_2 \subseteq A_1$.

We may assume wlog that $\mathbb{P}_{\mathbf{q}_0,\infty}$ forces $2^\theta = \lambda$, hence there is a canonical $\mathbb{P}_{\mathbf{q}_0,\infty}$-name $\underset{\sim}{f}$ of a bijection from $\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$ onto λ. Let $\underset{\sim}{\mathbb{Q}}'$ be a name for the forcing such that $\Vdash_{\mathbb{P}_{\mathbf{q}_0}}$ "$\underset{\sim}{f}$ is an isomorphism from $\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$ onto $\underset{\sim}{\mathbb{Q}}'$". Let $\underset{\sim}{B} = \underset{\sim}{B}_{\underset{\sim}{D}_{\mathbf{q}_0}}$ be the $\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$-name $\bigcup\{u : (u, A) \in G_{\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}}\}$, so $\Vdash_{\mathbb{P}_{\mathbf{q}_0,\infty}\star\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}}$ "$\underset{\sim}{B} \in [\theta]^\theta$ is θ-almost disjoint to $\mathcal{A}_{\mathbf{q}_0}$". Let $\underset{\sim}{B}'$ be the canonical $\mathbb{P}_{\mathbf{q}_0,\infty}\star\mathbb{Q}_{\underset{\sim}{D}_{\mathbf{q}_0}}$-name for the image of $\underset{\sim}{B}$ under $\underset{\sim}{f}$.
Now observe that there is $\mathbf{q}' \in \mathbb{P}$ such that $\mathbf{q}_0 \leq_{\mathbb{P}} \mathbf{q}'$, $U_{\mathbf{q}'} = U_{\mathbf{q}_0} \cup \{\gamma\}$, $\alpha <_{U_{\mathbf{q}'}} \gamma$ for every $\alpha \in U_{\mathbf{q}_0}$ and $\underset{\sim}{\mathbb{Q}}_{\mathbf{q}',\gamma} = \underset{\sim}{\mathbb{Q}}'$. As before, there is $\mathbf{q}'' \in \mathbb{P}$ above $\mathbf{q}'$ such that $p_0 \Vdash_{\mathbb{P}_{\mathbf{q}'',\infty}}$ "$\mathcal{A}_{\mathbf{q}''}$ is θ-mad". Therefore, there is some $\mathbb{P}_{\mathbf{q}'',\infty}$-name $\underset{\sim}{A} \in \mathcal{A}_{\mathbf{q}''}$ such that $p_0 \Vdash_{\mathbb{P}_{\mathbf{q}'',\infty}}$ "$\underset{\sim}{A} \cap \underset{\sim}{B}' \in [\theta]^\theta$, so $\underset{\sim}{A}$ has intersection of size θ with every member of $\underset{\sim}{D}_{\mathbf{q}_0}$ and $\underset{\sim}{A} \notin \mathcal{A}_{\mathbf{q}_0}$".

Now let $(\mathbf{q}_1, \underset{\sim}{B}_1, \underset{\sim}{A}_1) = (\mathbf{q}'', \underset{\sim}{B}', \underset{\sim}{A})$ and let $(\mathbf{q}_2, \underset{\sim}{B}_2, \underset{\sim}{A}_2)$ be an isomorphic copy of $(\mathbf{q}_1, \underset{\sim}{B}_1, \underset{\sim}{A}_1)$ over $\mathbf{q}_0$ such that $U_{\mathbf{q}_1} \cap U_{\mathbf{q}_2} = U_{\mathbf{q}_0}$ and $\mathbf{q}_2 \in \mathbb{P}$.

Claim 10 Let $\mathbf{q}_0$, $(\mathbf{q}_1, \underset{\sim}{B}_1, \underset{\sim}{A}_1)$ and $(\mathbf{q}_2, \underset{\sim}{B}_2, \underset{\sim}{A}_2)$ be as above (so $\mathbf{q}_0 \leq_K \mathbf{q}_l$ $(l = 1, 2)$, $U_{\mathbf{q}_1} \cap U_{\mathbf{q}_2} = U_{\mathbf{q}_0}$ and $\bigwedge\limits_{l=1,2} \Vdash_{\mathbb{P}_{\mathbf{q}_l,\infty}}$ "$\underset{\sim}{A}_l \in \underset{\sim}{\mathcal{A}} \setminus \mathcal{A}''_{\mathbf{q}_0}$) and let $G \subseteq \mathbb{P}_{\mathbf{q}_0,\infty}$ be generic over V, then $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G \times \mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_2 \setminus \underset{\sim}{A}_1, \underset{\sim}{A}_1 \setminus \underset{\sim}{A}_2 \in [\theta]^{\theta}$".

Proof We shall prove the claim for $A_2 \setminus A_1$, the other case is similar. Suppose towards contradiction that (p_1, p_2) forces that $\underset{\sim}{A}_2 \setminus \underset{\sim}{A}_1 \subseteq \gamma < \theta$. For $l \in \{1, 2\}$, let $B_l = \{\epsilon < \theta : p_l \nVdash_{\mathbb{P}_{\mathbf{q}_l,\infty}/G}$ "$\epsilon \notin \underset{\sim}{A}_l$"$\} \in V[G]$. By the assumption of the claim, $B_l \in [\theta]^\theta$. By the θ-madness of $\underset{\sim}{\mathcal{A}}_0[G]$ in $V[G]$, there is some $Y \in \underset{\sim}{\mathcal{A}}_0[G]$ such that $|Y \cap B_2| = \theta$. As $p_1 \Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G}$ "$|\underset{\sim}{A}_1 \cap Y| < \theta$", there are q_1 and $\beta_1 < \theta$ such that $p_1 \leq q_1 \in \mathbb{P}_{\mathbf{q}_1,\infty}/G$ and $q_1 \Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G}$ "$\underset{\sim}{A}_1 \cap Y \subseteq \beta_1$". Let $\beta_2 \in Y \cap B_2$ such that $max\{\gamma, \beta_1\} < \beta_2$ (recalling that $|Y \cap B_2| = \theta$). By the definition of B_2, there is $q_2 \in \mathbb{P}_{\mathbf{q}_2,\infty}/G$ above p_2 that forces "$\beta_2 \in \underset{\sim}{A}_2$". Therefore, $(p_1, p_2) \leq (q_1, q_2) \in$

Sh:1168

$\mathbb{P}_{\mathbf{q}_1,\infty}/G \times \mathbb{P}_{\mathbf{q}_2,\infty}/G$ and $(q_1, q_2) \Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\beta_2 \in \underset{\sim}{A}_2 \setminus \underset{\sim}{A}_1$", a contradiction. It follows that $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_2 \setminus \underset{\sim}{A}_1 \in [\theta]^\theta$". □

Claim 11 Under the assumptions of Claim 10 (recalling that $\Vdash_{\mathbb{P}_{\mathbf{q}_l,\infty}}$ "$\underset{\sim}{A}_l \cap B \neq \emptyset$ for every $B \in \underset{\sim}{D}_{\mathbf{q}_0}$" $(l = 1, 2)$), we have $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_1 \cap \underset{\sim}{A}_2 \in [\theta]^\theta$".

Proof Assume towards contradiction that $(p_1, p_2) \in \mathbb{P}_{\mathbf{q}_1,\infty}/G \times \mathbb{P}_{\mathbf{q}_2,\infty}/G$ forces that $\underset{\sim}{A}_1 \cap \underset{\sim}{A}_2 \subseteq \gamma$ for some $\gamma < \theta$. It's forced by (p_1, p_2) that $\underset{\sim}{A}_l \subseteq B_l$ $(l = 1, 2)$ where B_l is as in the proof of the previous claim, hence it's forced by (p_1, p_2) that each B_l intersects each member of $\underset{\sim}{D}_{\mathbf{q}_0}$. As $B_1, B_2 \in V[G]$, it follows that $B_1, B_2 \in \underset{\sim}{D}_{\mathbf{q}_0}[G]$.

Therefore, there is some $\beta \in (B_1 \cap B_2)\setminus\gamma$, hence there is $q_l \in \mathbb{P}_{\mathbf{q}_l,\infty}/G$ above p_l that forces "$\beta \in \underset{\sim}{A}_l$" $(l = 1, 2)$. It follows that $(p_1, p_2) \leq (q_1, q_2) \in \mathbb{P}_{\mathbf{q}_1,\infty}/G \times \mathbb{P}_{\mathbf{q}_2,\infty}/G$ and $(q_1, q_2) \Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\beta \in \underset{\sim}{A}_1 \cap \underset{\sim}{A}_2$", contradicting the choice of γ and (p_1, p_2). It follows that $\Vdash_{\mathbb{P}_{\mathbf{q}_1,\infty}/G\times\mathbb{P}_{\mathbf{q}_2,\infty}/G}$ "$\underset{\sim}{A}_1 \cap \underset{\sim}{A}_2 \in [\theta]^\theta$". □

Now given $\mathbf{q}_0$, $(\mathbf{q}_1, \underset{\sim}{B}_1, \underset{\sim}{A}_1)$ and $(\mathbf{q}_2, \underset{\sim}{B}_2, \underset{\sim}{A}_2)$ as above, let $\mathbf{q}_3 = \mathbf{q}_1 +_{\mathbf{q}_0} \mathbf{q}_2$. Then $\mathbf{q}_3 \in \mathbb{P}$, $\mathbf{q}_1, \mathbf{q}_2 \leq_K \mathbf{q}_3$, and by claims 10 and 11, we get a contradiction. This completes the proof of Main Claim 9 and hence of Theorem 2. □

We conclude with the following natural question:

Question What's the consistency strength of $ZF + DC_\theta +$ "there are no θ-mad families" for some $\theta > \aleph_0$?

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