EXPRESSIVE POWER OF INFINITARY LOGIC AND ABSOLUTE CO-HOPFIANITY

MOHSEN ASGHARZADEH, MOHAMMAD GOLSHANI, AND SAHARON SHELAH

Abstract. Recently, Paolini and Shelah have constructed absolutely Hopfian torsion-free abelian groups of any given size. In contrast, we show that this is not necessarily the case for absolutely co-Hopfian groups. We apply the infinitary logic to show a dichotomy behavior of co-Hopfian groups, by showing that the first beautiful cardinal is the turning point for this property. Namely, we prove that there are no absolute co-Hopfian abelian groups above the first beautiful cardinal. An extension of this result to the category of modules over a commutative ring is given.

Contents

Date: November 28, 2023.

2020 *Mathematics Subject Classification.* Primary: 20K99, 03C10; Secondary: 03C60, 03C45, 03C75, 20K30.

Key words and phrases. Absolutely co-Hopfian groups; almost isomorphic; beautiful cardinal; infinitary logic; elimination of quantifiers; forcing techniques.

The second author's research has been supported by a grant from IPM (No. 1402030417). The third author's research partially supported by the Israel Science Foundation (ISF) grant no: 1838/19 and (ISF) grant no: 2320/23; Research partially supported by the grant "Independent Theories" NSF-BSF, (BSF 3013005232). This is publication 1246 on Shelah's list.

§ 1. Introduction

An abelian group G is called *Hopfian* (resp. *co-Hopfian*) if its surjective (resp. injective) endomorphisms are automorphisms. In other words, co-Hopfian property of groups is, in some sense, dual to the Hopfian groups. These groups were first considered by Baer in [2], under different names. Hopf [17] himself showed that the fundamental group of closed two-dimensional orientable surfaces are Hopfian.

There are a lot of interesting research papers in this area. Here, we recall only a short list of them. Following the book [24], Hopf in 1932, raised the question as to whether a finitely generated group can be isomorphic to a proper factor of itself. For this and more observations, see [24]. Beaumont [4] proved that if G is an abelian group of finite rank all of whose elements have finite order, then G has no proper isomorphic subgroups. Kaplansky [18] extended this to modules over a commutative principal ideal ring R such that every proper residue class ring of R is finite. Beaumont and Pierce [3] proved that if G is co-Hopfian, then the torsion part of G is of size at most continuum, and further that G cannot be a p-groups of size $\aleph_0$. This naturally left open the problem of the existence of co-Hopfian p-groups of uncountable size $\leq 2^{\aleph_0}$, which was later solved by Crawley [9] who proved that there exist p-groups of size $2^{\aleph_0}$. One may attempt to construct (co-)Hopfian groups of large size by taking a huge direct sum of (co-)Hopfian groups. In this regard, Baumslag [7] asked when is the direct sum of two (co-)Hopfian groups again (co)-Hopfian? Corner [8] constructed two torsion-free abelian Hopfian groups which have non-Hopfian direct sum. See [15], for more on this and its connections with the study of algebraic entropy.

Despite its long history, only very recently the problem of the existence of uncountable (co-)Hopfian abelian groups was solved, see [1] and [22].

The usual construction of such groups is not absolute, in the sense that they may loose their property in some generic extension of the universe, and the problem of giving an explicit and constructive construction of such groups has raised some attention in the literature. For example, while it is known from the work of Shelah [25] that there are indecomposable abelian groups of any infinite size, the problem of the existence of arbitrary large absolutely indecomposable groups is still open, see [21]. It is worth noticing that indecomposability implies the Hopfian property.

In order to be more explicit, call a group G *absolutely co-Hopfian* if it is co-Hopfian in any further generic extension of the universe. Similarly, one may define *absolutely Hopfian* groups. An open problem in this area was as follows:

Problem 1.1. (i) Is it possible to construct absolutely Hopfian torsion-free groups of a given size?

(ii) Is it possible to construct absolutely co-Hopfian torsion-free groups of a given size?

Recently, Paolini and Shelah [22, Theorem 1.3] constructed absolutely Hopfian torsion-free groups of any given size λ, thereby confirming Problem 1.1(i) in positive direction. On the one hand, and as Hopfian and co-Hopfian groups are dual to each ether, one may predict that there is a connection between Problem 1.1 (i) and (ii). But, any such dual functor, may enlarge or collapse the cardinality of the corresponding groups, hence we can not use the ideas behind the duality to answer Problem 1.1(ii). For example, Braun and Strüngmann [6] showed that the existence of infinite abelian p-groups of size $\aleph_0 < |G| < 2^{\aleph_0}$ of the following types are independent of ZFC:

(a) both Hopfian and co-Hopfian,
(b) Hopfian but not co-Hopfian,
(c) co-Hopfian but not Hopfian.

Also, they proved that the above three types of groups of size $2^{\aleph_0}$ exist in ZFC.

On the other hand, there are some partial positive results in the direction of Problem 1.1 (ii). Here, we recall some of these. In [29], Shelah studied and coined the concept of a *beautiful cardinal*, denoted by κ_{beau}, which is a kind of Ramsey cardinal (see Definition 3.12). This cardinal has an essential role in the study of absolutely co-Hopfian groups. Indeed, according to [16], for any infinite cardinal $\lambda < \kappa_{\text{beau}}$, there is an absolutely endo-rigid abelian group, but if $|G| \geq \kappa_{\text{beau}}$, then by [12] it has non-trivial monomorphisms. Let us state the remaining part of Problem 1.1:

Problem 1.2. Is it possible to construct absolutely co-Hopfian torsion-free groups of size $\geq \kappa_{\text{beau}}$?

The Hopfian and co-Hopfian property easily can be extended to the context of modules over commutative rings and even to the context of sheaves over schemes. However, compared to the case of abelian groups, and up to our knowledge, there are very few results for modules. For example, a funny result of Vasconcelos [31, 1.2] says that any surjective $f : M \to M$ is an isomorphism, where M is a noetherian module over a commutative and noetherian ring R. As a geometric sample, let X be an algebraic variety over an algebraically closed field. If a morphism $f : X \to X$ is injective then a result of Ax and Grothendieck indicates that f is bijective, see Jean-Pierre Serre's exposition [23].

In contrast to the the case of Hopfian groups, we prove the following dichotomy property of co-Hopfian groups. Indeed, we prove a more general result in the context of additive τ-models (see Definition 2.3), which includes in particular the cases of abelian groups and R-modules, for an arbitrary commutative ring R:

Theorem 1.3. *The following assertions are valid:*

1. *If M is an abelian group of cardinality greater or equal $\kappa := \kappa_{\text{beau}}$, then M is not absolutely co-Hopfian, indeed, after collapsing the size of M into ω, there is a one-to-one endomorphism $\varphi \in \text{End(M)}$ which is not onto.*
2. *If M is an R-module of cardinality greater or equal $\kappa = \kappa_{\text{beau}}(R, \aleph_0)$, then M is not absolutely co-Hopfian.*
3. *If M is an additive τ-model of cardinality greater or equal $\kappa = \kappa_{\text{beau}}(\tau)$, then M is not absolutely co-Hopfian.*

Putting things together, we conclude that for every infinite cardinal λ, there is an absolutely co-Hopfian abelian group of size λ if and only if λ is less than the first beautiful cardinal.

The organization of this paper is as follows.

Szmielew [30] developed first order theory of abelian groups, see also [28] for further discussions. In Section 2 we review infinitary languages and develop some parts of abelian group theory in this context. For this, we use the concept of θ-models from [27]. We also introduce some sub-languages of the infinitary languages, which play some role in our later investigation.

For Section 3, let us fix a pair (λ, θ) of regular cardinals and let κ be as Theorem 1.3. The new object studied here is called the general frame and its enrichment the *additive frame*. Such a frame is of the form

$$\mathbf{f} := (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega),$$

where M comes from Theorem 1.3, and it has an additive τ_M-model of cardinality $\geq \kappa$ and $\mathscr{L}$ is a class of certain formulas in the vocabulary τ_M. For more details, see Definition 3.1. The main result of Section 3 is Theorem 3.14. This gives us an additive frame $\mathbf{f}$.

Section 4 is about the concept of *algebraic closure* in a frame (see Definition 4.1). This enables us to improve Theorem 3.14, which is needed in the squeal.

In Section 5 we put all things together and present the proof of Theorem 1.3. Let $\chi := |M| \geq \kappa$, and let $\mathbb{P} := \mathrm{Col}(\aleph_0, \chi)$. Forcing with $\mathbb{P}$ enables us to collapse $|M|$ into $\aleph_0$, i.e., for any $\mathbb{P}$-generic filter $G_{\mathbb{P}}$ over V, we have

$$V[G_{\mathbb{P}}] \models \text{“}M \text{ is countable”}.$$

We show in $V[G_{\mathbb{P}}]$, there exists a 1-1 map $\pi : M \to M$ which is not surjective.

Finally, as another application of infinitary logic, we present a new construction of Hopfian groups, see Proposition 6.3.

We close the introduction by noting that all groups (resp. rings) are abelian (resp. commutative), otherwise specialized, and our notation is standard and follows that in Fuchs books [14] and [13] and Eklof-Mekler [10].

§ 2. Infinitary languages

In the first subsection, we briefly review infinitary languages and the concept of additive θ-models, introduced in [27]. In the second subsection, we present basic properties of affine subsets, i.e., ones closed under $x - y + z$.

§ 2(A). A review of infinitary languages.

In this subsection we briefly review the infinitary logic, and refer to [5] and [27] for more information.

Convention 2.1. Given a model M, by τ_M we mean the language (or vocabulary) of the model M.

Notation 2.2. (1) By AB we mean the class of abelian groups.

(2) Given a vocabulary τ which contains two place functions $+, -$, we define the affine operation $\text{Affine}(x, y, z)$ as the three place function $\text{Affine}(x, y, z) := x - y + z$.

Definition 2.3. Let M be a model of vocabulary τ_M.

(1) We say M is an *additive θ-model* when:
  (a) the two place function symbols $+, -$ and the constant symbol 0 belong to τ_M,
  (b) $G_M = (|M|, +^M, -^M, 0^M) \in \text{AB}$,
  (c) R^M is a subgroup of ${}^n(G_M)$, for any predicate symbol $R \in \tau_M$ with arity(R) = n,
  (d) F^M is a homomorphism from ${}^n(G_M)$ into G_M, for any function symbol $F \in \tau_M$ with arity n,
  (e) τ_M has cardinality $\leq \theta$.

(2) For an additive θ-model M, we say $X \subseteq M$ is *affine if X is closed under the affine operation* $\text{Affine}(x, y, z)$. *In other words, $a - b + c \in X$ provided that $a, b, c \in X$.*

(3) We say M is an affine θ-model provided:
  (a) we do not necessarily have $+, -, 0$ in the vocabulary, but only the three place function $\text{Affine}(x, y, z)$,
  (b) if $R \in \tau_M$ is an n-place predicate and $\bar{a}_l = \langle a_{l,i} : i < n\rangle \in R^M$ for $l = 0, 1, 2$ and

$$\bar{b} := \text{Affine}(\bar{a}_0, \bar{a}_1, \bar{a}_2) = \big\langle \text{Affine}(a_{0,i}, a_{1,i}, a_{2,i}) : i < n \big\rangle,$$

  then $\bar{b} \in R^M$,
  (c) for any n-place function symbol $F \in \tau_M$ and $\bar{a}_l = \langle a_{l,i} : i < n\rangle \in {}^nM$, for $l = 0, 1, 2$, we have

$$F^M(\text{Affine}(\bar{a}_0, \bar{a}_1, \bar{a}_2)) = \text{Affine}(F^M(\bar{a}_0), F^M(\bar{a}_1), F^M(\bar{a}_2)),$$

  (d) τ_M has cardinality $\leq \theta$.

(4) Suppose M is an affine θ-model. We say M is truly affine provided for some fixed $a \in M$ and for the following interpretation

- $x + y := \mathrm{Affine}(x, a, y) = x - a + y$,
- $x - y := \mathrm{Affine}(x, y, a) = x - y + a$,
- $0 := a$,

then we get an abelian group, and hence an additive θ-model.

We may omit θ if it is clear from the context.

Remark 2.4. i) A natural question arises: Is any affine θ-model truly affine? Not necessarily this holds, see Example 2.5, below.

ii) More generally, we can replace $\{+, -, \mathrm{Affine}\}$ for a set τ_f of beautiful function from [26]. The corresponding result holds in this frame.

Example 2.5. Let G be an abelian group, H be a proper subgroup of it and $a \in G \setminus H$. Define M as follows:

- the universe of M is $a + H$,
- $\tau_M := \{+, -, \mathrm{Affine}\}$,
- $+^M$ and $-^M$ are $+^G \restriction M$ and $-^G \restriction M$ respectively,
- $\mathrm{Affine}^M := \mathrm{Affine}^G \restriction M$, where $\mathrm{Affine}^G = \{x - y + z : x, y, z \in G\}$.

Then the following two assertions hold:

a) M is an affine $\aleph_0$-model, isomorphic to H.

b) M is not an abelian group.

Definition 2.6. (1) We say a class K of models is an *additive θ-class*, when M is an additive θ-model for all $M \in K$, and

$$\tau_M = \tau_N \quad \forall M, N \in K.$$

We denote the resulting common language by τ_K.

(2) Similarly, one can define affine θ-classes.

Hypothesis 2.7. Let Ω be a set of cardinals with $1 \in \Omega$ and members of $\Omega \setminus \{1\}$ are infinite cardinals.

Notation 2.8. (1) By $\overline{x}_{[u]}$ or $\overline{x}_u$ we mean $\langle x_\alpha : \alpha \in u\rangle$. So, with no repetition.
(2) Suppose $\varphi(\overline{x}_u)$ is a formula. By $\varphi(M)$ we mean $\{\overline{a} \in {}^uM : M \models \varphi[\overline{a}]\}$.
(3) For a formula $\varphi(\overline{x}_{[u]}, \overline{y}_{[v]})$ and $\overline{b} \in {}^vM$, we let

$$\varphi(M, \overline{b}) := \{\overline{a} \in {}^uM : M \models \varphi[\overline{a}, \overline{b}]\}.$$

(4) Given a sequence t, by lg(t) we mean the length of t.

Definition 2.9. Suppose κ and μ are infinite cardinals, which we allow to be ∞. The infinitary language $\mathcal{L}_{\mu,\kappa}(\tau)$ is defined so as its vocabulary is the same as τ, it has the same terms and atomic formulas as in τ, but we also allow conjunction and disjunction of length less than μ, i.e., if ϕ_j, for $j < \beta < \mu$ are formulas, then so are $\bigvee_{j<\beta} \phi_j$ and $\bigwedge_{j<\beta} \phi_j$. Also, quantification over less than κ many variables (i.e., if $\phi = \phi((v_i)_{i<\alpha})$, where $\alpha < \kappa$, is a formula, then so are $\forall_{i<\alpha} v_i \phi$ and $\exists_{i<\alpha} v_i \phi$).

Note that $\mathcal{L}_{\omega,\omega}(\tau)$ is just the first order logic with vocabulary τ. Given κ, μ and τ as above, we are sometimes interested in some special formulas from $\mathcal{L}_{\mu,\kappa}(\tau)$.

Definition 2.10. Suppose κ and λ are infinite cardinals or possibly ∞. We define the logic $\mathscr{L}_{\lambda,\kappa,\Omega}$ as follows:

(1) For a vocabulary τ, the language $\mathscr{L}_{\lambda,\kappa,\Omega}(\tau)$ is defined as the set of formulas with $< \theta$ free variables (without loss of generality they are subsets of $\{x_\zeta : \zeta < \theta\}$, see Discussion 2.11) which is the closure of the set of basic formulas , i.e., atomic and the negation of atomic formulas, under:
  (a) conjunction of $< \lambda$ formulas,
  (b) disjunction of $< \lambda$ formulas,
  (c) For any $\sigma \in \Omega$,
    (c_1) $\varphi(\overline{x}) := (\exists^\sigma \overline{x}')\psi(\overline{x}, \overline{x}')$, or
    (c_2) $\varphi(\overline{x}) := (\forall^\sigma \overline{x}')\psi(\overline{x}, \overline{x}')$,
  where $\psi(\overline{x}, \overline{x}')$ is a formula. We usually omit σ, if $\sigma = 1$.

  We usually omit Ω if it is clear from the context.
(2) Satisfaction is defined as usual, where for the formulas $\varphi(\overline{x}) := (\exists^\sigma \overline{x}')\psi(\overline{x}, \overline{x}')$ and $\varphi(\overline{x}) := (\forall^\sigma \overline{x}')\psi(\overline{x}, \overline{x}')$, it is defined as:

(a) If $\varphi(\overline{x}) := (\exists^\sigma \overline{x}')\psi(\overline{x}, \overline{x}')$, M is a τ-model, and $\overline{a} \in {}^{\lg(\overline{x})}M$, then

$$M \models \varphi(\overline{a})$$

if and only if there are $\overline{b}_\varepsilon \in {}^{\lg(\overline{x}')}M$ for all $\varepsilon < \sigma$ pairwise distinct such that $M \models \psi(\overline{a}, \overline{b}_\varepsilon)$ for all $\varepsilon < \sigma$.

(b) If $\varphi(\overline{x}) := (\forall^\sigma \overline{x}')\psi(\overline{x}, \overline{x}')$, then

$$M \models \varphi(\overline{x}) \iff M \models \neg\big[\exists^\sigma \overline{x}' \neg\big(\psi(\overline{x}, \overline{x}')\big)\big].$$

Note that $\neg(\psi(\overline{x}, \overline{x}'))$ is not necessarily in $\mathscr{L}_{\lambda,\kappa,\Omega}(\tau)$.

Discussion 2.11. Given a formula φ in $\mathcal{L}_{\infty,\theta}(\tau)$ with free variables $\overline{x}_\varphi$, we can always assume that $\overline{x}_\varphi = \langle x_\zeta : \zeta \in u_\varphi \rangle$, for some $u_\varphi \in [\theta]^{<\theta}$. The key point is that if $\varphi = \varphi(\overline{x})$, where $\overline{x} = \langle x_\zeta : \zeta \in w \rangle$, where w is a set of ordinals of size less than θ, and if $f : w \leftrightarrow u$ is a bijection where $u \in [\theta]^{<\theta}$, and

$$\psi(\overline{x}) \equiv \mathrm{Sub}^{\overline{x}}_f(\varphi),$$

where $\mathrm{Sub}^{\overline{x}}_f(\varphi)$ is essentially the formula obtained from φ by replacing the variable x_ζ by $x_{f(\zeta)}$, then if $\bar{a} \in{}^w M$, $\bar{b} \in{}^u M$ and $a_\zeta = b_{f(\zeta)}$, for $\zeta \in w$, then

$$M \models \varphi(\bar{a}) \iff M \models \psi(\bar{b}).$$

We can similarly assume that all bounded variables are from $\{x_i : i < \theta\}$.

Convention 2.12. In what follows, saying closed under $\exists$ (resp. $\forall$) means under all $\exists^\sigma$ (resp. $\forall^\sigma$).

In the next definition, we consider some classes of infinitary formulas that we will work with them latter.

Definition 2.13. Suppose θ is an infinite cardinal, or ∞, and suppose τ is a language. Here, we collect some infinitary sub-classes of the language $\mathscr{L}_{\infty,\theta}(\tau)$:

(1) $\mathscr{L}^{\mathrm{cop}}_{\infty,\theta}(\tau)$ is the class of conjunction-positive formulas, i.e., the closure of atomic formulas under $\bigwedge, \exists, \forall$.

(2) $\mathscr{L}^{\mathrm{cpe}}_{\infty,\theta}(\tau)$ is the class of conjunction-positive existential formulas, i.e., the closure of atomic formulas under $\bigwedge$ and $\exists$.

(3) $\mathscr{L}^{\rm co}_{\infty,\theta}(\tau)$ is the closure of atomic formulas and $x_i \neq x_j$ under $\bigwedge$, $\exists$ and $\forall$.

(4) $\mathscr{L}^{\rm ce}_{\infty,\theta}(\tau)$ is the closure of atomic formulas and $x_i \neq x_j$ under $\bigwedge$ and $\exists$.

We shall use freely the following simple fact.

Fact 2.14. $\mathscr{L}^{\rm co}_{\infty,\theta}(\tau) \supseteq \mathscr{L}^{\rm cop}_{\infty,\theta}(\tau) \cup \mathscr{L}^{\rm ce}_{\infty,\theta}(\tau) \supseteq \mathscr{L}^{\rm cop}_{\infty,\theta}(\tau) \cap \mathscr{L}^{\rm ce}_{\infty,\theta}(\tau) \supseteq \mathscr{L}^{\rm cpe}_{\infty,\theta}(\tau)$.

The following lemma is easy to prove.

Lemma 2.15. *Assume M is an additive θ-model, $\tau = \tau_M$ and $\varphi(\overline{x}_u) \in \mathscr{L}_{\infty,\infty}(\tau)$ with $\varepsilon = \lg(\overline{\rm x})$. The following assertions are valid:*

(1) If $\varphi(\overline{x}_u) \in \mathscr{L}^{\rm cop}_{\infty,\infty}(\tau)$, then $\varphi(M)$ is a sub-group of ${}^uM$.

(2) If $\varphi(\overline{x}_u) \in \mathscr{L}^{\rm cpe}_{\infty,\infty}(\tau)$, $f \in \rm End(M)$ and $M \models \varphi[\overline{a}]$, then $M \models \varphi[f(\overline{a})]$.

(3) If $\varphi(\overline{x}_u) \in \mathscr{L}^{\rm cpe}_{\infty,\theta}(\tau)$, M, N are τ-models and $f : M \to N$ is a homomorphism, then f maps $\varphi(M)$ into $\varphi(N)$.

(4) If $\varphi(\overline{x}_u) \in \mathscr{L}^{\rm ce}_{\infty,\theta}(\tau)$, M, N are τ-models and $f : M \to N$ is a 1-1 homomorphism, then f maps $\varphi(M)$ into $\varphi(N)$.

(5) If $\varphi(\overline{x}_u) \in \mathscr{L}^{\rm co}_{\infty,\theta}(\tau)$, M, N are τ-models and $f : M \to N$ is a bijection, then f is an isomorphism from $\varphi(M)$ onto $\varphi(N)$.

(6) Assume $\psi(\bar{y})$ is obtained from $\varphi(\bar{x})$ by adding dummy variables, permuting the variables and substitution not identifying variables. Then

$$\psi(\bar{y}) \in \mathscr{L}^{*}_{\infty,\theta}(\tau) \iff \varphi(\bar{x}) \in \mathscr{L}^{*}_{\infty,\theta}(\tau),$$

where $* \in \{\rm cop, cpe, ce, co\}$.

Proof. The proof is by induction on the complexity of the formulas. For completeness, we sketch the proof. If the formula is an atomic formula, then it is evident that all of the above items are satisfied. It is also easy to see that each item is preserved under $\bigwedge$, in the sense that if $\psi = \bigwedge_{i \in I} \varphi_i$ is well-defined and the lemma holds for each φ_i, then it holds for ψ.

We now consider the case where $\psi(\bar{x}) = (\exists^\sigma \bar{y})\varphi(\bar{x}, \bar{y})$, and assume the induction hypothesis holds for φ. We consider each clause separately, assuming in each case, the formula φ is in the assumed language.

Clause (1): Suppose $\varphi(M)$ is a subgroup of ${}^{\lg(\bar{x})+\lg(\bar{y})}M$. We show that $\psi(M)$ is a subgroup of ${}^{\lg(\bar{x})}M$. To see this, let $\bar{a}_0, \bar{a}_1 \in \psi(M)$. Then for some $\bar{b}_0$ and $\bar{b}_1$ we have

$$M \models \text{“}\varphi[\bar{a}_0, \bar{b}_0] \textit{ and } \varphi[\bar{a}_1, \bar{b}_1]\text{''}.$$

By induction, $M \models \varphi[\bar{a}_0 \text{ - } \bar{a}_1, \bar{b}_0 \text{ - } \bar{b}_1]$, hence

$$M \models \psi[\bar{a}_0 - \bar{a}_1].$$

Thus $\bar{a}_0$-$\bar{a}_1 \in \psi(M)$.

Clause (2): Suppose $M \models \psi[\bar{a}]$. Then for some $\bar{b}$, we have $M \models \varphi[\bar{a}, \bar{b}]$. By the induction, $M \models \varphi[f(\bar{a}), f(\bar{b})]$, and hence $M \models \psi[f(\bar{a})]$, as requested.

Clause (3): As in clause (2), we can show that if $M \models \psi[\bar{a}]$, then $N \models \psi[f(\bar{a})]$, and this gives the required result.

Clause (4): As in clause (3). The assumption of f being 1-1 is used to show that if $x_i \neq x_j$, then $f(x_i) \neq f(x_j)$.

Clause (5): As in clause (4).

Clause (6): This is easy.

Finally, suppose that $\psi(\bar{x}) = (\forall^\sigma \bar{y})\varphi(\bar{x}, \bar{y})$, and assume the induction hypothesis holds for φ. We only have to consider items (1), (5) and (6).

Clause (1): Suppose $\varphi(M)$ is a subgroup of ${}^{\lg(\bar{x})+\lg(\bar{y})}M$. We show that $\psi(M)$ is a subgroup of ${}^{\lg(\bar{x})}M$. To see this, let $\bar{a}_0, \bar{a}_1 \in \psi(M)$. We have to show that $\bar{a}_0$-$\bar{a}_1 \in \psi(M)$. Thus let $\bar{b} \in {}^{\lg(\bar{y})}M$. By the induction hypothesis,

$$M \models \text{“}\varphi[\bar{a}_0, \bar{b}] \textit{ and } \varphi[\bar{a}_1, \bar{0}]\text{''}.$$

Thanks to induction, $M \models \varphi[\bar{a}_0 \text{ - } \bar{a}_1, \bar{b} \text{ - } \bar{0}]$. As this holds for all $\bar{b}$, we have

$$M \models \psi[\bar{a}_0 - \bar{a}_1].$$

Thus $\bar{a}_0$-$\bar{a}_1 \in \psi(M)$, as requested.

Clause (5): As before, we can easily show that f maps $\psi(M)$ into $\psi(N)$. To see it is onto, let $\bar{c} \in \psi(N)$. Then $N \models \psi(\bar{c})$. As f is onto, for some $\bar{a}$ we have $\bar{c} = f(\bar{a})$. We have to show that $\bar{a} \in \psi(M)$. Thus let $\bar{b} \in {}^{\lg(\bar{y})}M$. Then

$\bar{d} = f(\bar{b}) \in^{\lg(\bar{y})} N$, and by our assumption, $N \models \varphi(\bar{c}, \bar{d})$. As f is an isomorphism, $M \models \varphi(\bar{a}, \bar{b})$. As $\bar{b}$ was arbitrary, $M \models \psi(\bar{a})$, i.e., $\bar{a} \in \psi(M)$.

Clause (6): This is easy.

The lemma follows. □

Let us repeat the above result in the context of R-modules:

Corollary 2.16. *Let M be an R-module.*

(1) If $\varphi(\overline{x}_u) \in \mathscr{L}^{\rm cpe}_{\infty,\theta}$, then $\varphi(M)$ is an abelian subgroup of $({}^u M, +)$.

(2) Similar result holds for formulas $\varphi(\overline{x}_u) \in \mathscr{L}^{\rm cop}_{\infty,\theta}$.

Furthermore, if R is commutative, then in the above, $\varphi(M)$ becomes a sub-module of $({}^u M, +)$.

Remark 2.17. If R is not commutative, then $\varphi(M)$ is not necessarily a submodule. To see this, suppose $a, b, c \in R$ are such that $abc \neq bac$, and suppose M is a left R-module. Define $F_a : M \to M$ as $F_a(x) = ax$. Now note that $(c, ac) \in F_a(M)$. If $F_a(M)$ is a submodule, then we must have $(bc, bac) \in F_a(M)$. By definition, $(bc, bac) = F_a(x) = (x, ax)$ for some $x \in M$. It then follows that $x = bc$ and hence

$$bac = ax = abc,$$

which contradicts $abc \neq bac$.

§ 2(B). **More on affineness.** In this subsection we try replacing subgroups by affine subsets. The main result of this subsection is Proposition 2.22. First, we fix the hypothesis and present the corresponding definitions. Here, affinity demand relates only to the formulas, not the content.

Hypothesis 2.18. (1) R is a ring,

(2) M is an R-module,

(3) κ, θ are as in Definition 3.1.

Definition 2.19. Let Affine_1 be the set of all formulas $\varphi(\overline{x}) \in \mathscr{L}_{\infty,\theta}(\tau_M)$ so that $\lg(\overline{x}) < \theta$ and $\varphi(M)$ is closed under $\overline{x} - \overline{y} + \overline{z}$. In other words, $\overline{a} - \overline{b} + \overline{c} \in \varphi(M)$ provided that $\overline{a}, \overline{b}, \overline{c} \in \varphi(M)$.

We now define another class Affine_2 of formulas of $\mathscr{L}_{\infty,\theta}(\tau_M)$, and show that it is included in Affine_1. To this end, we first make the following definition.

Definition 2.20. Suppose α_* is an ordinal. Let $\varphi(\overline{x}, \overline{y})$ and $\overline{\psi}(\overline{x}, \overline{y}) = \langle \psi_\alpha(\overline{x}, \overline{y}) : \alpha < \alpha_* \rangle$ be a sequence of formulas from $\mathscr{L}_{\infty,\theta}(\tau_M)$. Let $\overline{b} \in {}^{\lg(\overline{x})}M$ and $\overline{a} \in {}^{\lg(\overline{y})}M$. Then we set

(1) $\text{set}_{\overline{\psi}}(\overline{b}, \overline{a})$ stands for the following set

$$\text{set}_{\overline{\psi}}(\overline{b}, \overline{a}) := \big\{\alpha \in \alpha_* : (\overline{b}^\frown \overline{a} \in \psi_\alpha(M))\big\}.$$

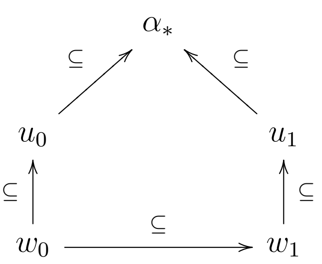
(2) By $\text{Set}_{\varphi,\overline{\psi}}(\overline{a})$ we mean

$$\text{Set}_{\varphi,\overline{\psi}}(\overline{a}) := \big\{u \subseteq \alpha_* : \text{for some } \overline{c} \in \varphi(M, \overline{a}) \text{ we have } u = \text{set}_{\overline{\psi}}(\overline{c}, \overline{a})\big\}.$$

(3) By $\text{inter}_{\varphi,\overline{\psi}}(\overline{a})$ we mean

$$\Big\{(w_0, w_1) : w_0 \subseteq w_1 \subseteq \alpha_* \text{ and } \exists\, u_0, u_1 \in \text{Set}_{\varphi,\overline{\psi}}(\overline{a}) \text{ s.t. } w_1 \subseteq u_1 \text{ and } u_0 \cap w_1 = w_0\Big\}.$$

In particular, we have the following flowchart:



.

We are now ready to define the class Affine_2 of formulas:

Definition 2.21. Let Affine_2 be the closure of the set of atomic formulas by:

(a) arbitrary conjunctions,

(b) existential quantifier $\exists \overline{x}$, and

(c) suppose for a given ordinal α_*, the formulas $\varphi(\overline{x}, \overline{y})$, $\langle \psi_\alpha(\overline{x}, \overline{y}) : \alpha < \alpha_* \rangle$ are taken from Affine_2 such that $\varphi(\overline{x}, \overline{y}) \geq \psi_\alpha(\overline{x}, \overline{y})$ for all $\alpha < \alpha_*$. Also suppose that

$$\Upsilon \subseteq \{(w_0, w_1) : w_0 \subseteq w_1 \subseteq \alpha_*\}.$$

Then $\vartheta(\bar{y}) = \Theta_{\varphi,\bar{\psi},\Upsilon}(\bar{y}) \in \mathrm{Affine}_2$, where $\vartheta(\bar{y})$ is defined such that

$$M \models \vartheta[\bar{a}] \iff \Upsilon \subseteq \mathrm{inter}_{\varphi,\bar{\psi}}(\bar{a}).$$

The main result of this section is the following.

Proposition 2.22. *Adopt the previous notation. Then* $\mathrm{Affine}_2 \subseteq \mathrm{Affine}_1$.

Proof. We prove the theorem in a sequence of claims. We proceed by induction on the complexity of the formula ϑ that if $\vartheta \in \mathrm{Affine}_2$, then $\vartheta \in \mathrm{Affine}_1$. This is clear if ϑ is an atomic formula. Suppose $\vartheta = \bigwedge_{i\in I} \vartheta_i$, and the claim holds for all $\vartheta_i, i \in I$. It is then clear from inductive step that $\vartheta \in \mathrm{Affine}_1$ as well. Similarly, if $\vartheta = \exists \bar{x}\varphi(\bar{x})$, and if the claim holds for $\varphi(\bar{x})$, then clearly it holds for ϑ.

Now suppose that α_* is an ordinal, $\varphi(\bar{x},\bar{y})$, $\langle \psi_\alpha(\bar{x},\bar{y}) : \alpha < \alpha_* \rangle$ are in Affine_2, such that $\varphi(\bar{x},\bar{y}) \geq \psi_\alpha(\bar{x},\bar{y})$ for all $\alpha < \alpha_*$. Also, suppose that

$$\Upsilon \subseteq \{(w_0, w_1) : w_0 \subseteq w_1 \subseteq \alpha_*\}.$$

Assume by the induction hypothesis that the formulas $\varphi(\bar{x},\bar{y})$ and $\psi_\alpha(\bar{x},\bar{y})$, for $\alpha < \alpha_*$, are in Affine_1. We have to show that $\vartheta(\bar{y}) = \Theta_{\varphi,\bar{\psi},\Upsilon}(\bar{y}) \in \mathrm{Affine}_1$ as well.

Now, we bring the following claim:

Claim 2.23. *Adopt the above notation. Assume* $\bar{a}_l \in {}^{\lg(\bar{y})}M$ *for* $l = 0,1,2,3$ *and* $\bar{a}_3 = \bar{a}_0 - \bar{a}_1 + \bar{a}_2$. *If* $u_j \in \mathrm{Set}(\bar{a}_j)$ *for* $j = 0,1$ *and* $u = u_0 \cap u_1$, *then*

$$\{w \cap u : w \in \mathrm{Set}_{\varphi,\bar{\psi}}(\bar{a}_2)\} = \{w \cap u : w \in \mathrm{Set}_{\varphi,\bar{\psi}}(\bar{a}_3)\}.$$

Proof. Let $\bar{b}_0 \in \varphi(M,\bar{a}_0)$ and $\bar{b}_1 \in \varphi(M,\bar{a}_1)$ be such that $u_0 = \mathrm{set}_{\bar{\psi}}(\bar{b}_0,\bar{a}_0)$ and $u_1 = \mathrm{set}_{\bar{\psi}}(\bar{b}_1,\bar{a}_1)$. Suppose that $w \in \mathrm{Set}_{\varphi,\bar{\psi}}(\bar{a}_2)$. Then for some $\bar{b}_2 \in \varphi(M,\bar{a}_2)$ we have $w = \mathrm{set}_{\bar{\psi}}(\bar{b}_2,\bar{a}_2)$. We have to find $w' \in \mathrm{Set}_{\varphi,\bar{\psi}}(\bar{a}_3)$ such that $w' \cap u = w \cap u$.

Set $\bar{b}_3 := \bar{b}_0 - \bar{b}_1 + \bar{b}_2$. Note that

$$l = 0,1,2 \implies \bar{b}_l{}^\frown\bar{a}_l \in \varphi(M),$$

hence, as $\varphi \in \mathrm{Affine}_1$, we have

$$\bar{b}_0{}^\frown\bar{a}_0 - \bar{b}_1{}^\frown\bar{a}_1 + \bar{b}_2{}^\frown\bar{a}_2 \in \varphi(M).$$

Clearly,

$$\bar{b}_3{}^\frown \bar{a}_3 = \bar{b}_0{}^\frown \bar{a}_0 - \bar{b}_1{}^\frown \bar{a}_1 + \bar{b}_2{}^\frown \bar{a}_2 \in \varphi(M).$$

According to its definition, $\bar{b}_3 \in \varphi(M, \bar{a}_3)$.

Let $w' = \mathrm{set}_{\overline{\psi}}(\bar{b}_3, \bar{a}_3)$. We show that $w' \cap u = w \cap u$. Suppose $\alpha \in u$. Then, we have $\alpha \in u_0 \cap u_1$, and hence $\bar{b}_j{}^\frown \bar{a}_j \in \psi_\alpha(M)$, for $j = 0, 1$. Thus as $\psi_\alpha \in \mathrm{Affine}_1$, we have

$$\begin{aligned} \alpha \in w' &\iff \bar{b}_3{}^\frown \bar{a}_3 \in \psi_\alpha(M) \\ &\iff \bar{b}_2{}^\frown \bar{a}_2 \in \psi_\alpha(M) \\ &\iff \alpha \in w. \end{aligned}$$

Suppose $w' \in \mathrm{Set}_{\varphi,\overline{\psi}}(\bar{a}_3)$. By symmetry, $w \cap u = w' \cap u$ for some $w \in \mathrm{Set}_{\varphi,\overline{\psi}}(\bar{a}_2)$. The claim follows. □

Let us apply the previous claim and observe that:

Claim 2.24. *Let $\varepsilon = \lg(\bar{y}) < \theta$ and $\bar{a}_l \in {}^\varepsilon M$ for $l = 0, 1, 2$, and set $\bar{a}_3 := \bar{a}_0 - \bar{a}_1 + \bar{a}_2$. If $\Upsilon \subseteq \bigcap_{l \leq 2} \mathrm{inter}_{\varphi,\overline{\psi}}(\bar{a}_l)$, then $\Upsilon \subseteq \mathrm{inter}_{\varphi,\overline{\psi}}(\bar{a}_3)$.*

Proof. Let $(w_0, w_1) \in \Upsilon$. We shall prove that $(w_0, w_1) \in \mathrm{inter}(\bar{a}_3)$. For $j \leq 2$, as $(w_0, w_1) \in \Upsilon \subseteq \mathrm{inter}_{\varphi,\overline{\psi}}(\bar{a}_j)$, there is a pair $u_{j,0}, u_{j,1} \in \mathrm{set}_{\varphi,\overline{\psi}}(\bar{a}_j)$ witnessing it. Namely, we have

$$w_1 \subseteq u_{j,1} \text{ and } u_{j,0} \cap w_1 = w_0.$$

Now, we can find $\bar{b}_{j,0}, \bar{b}_{j,1}$ such that $\mathrm{set}(\bar{\mathrm{b}}_{\mathrm{j},0}, \bar{\mathrm{a}}_{\mathrm{j}}) = \mathrm{u}_{\mathrm{j},0}$ and $\mathrm{set}(\bar{\mathrm{b}}_{\mathrm{j},1}, \bar{\mathrm{a}}_{\mathrm{j}}) = \mathrm{u}_{\mathrm{j},1}$. Set

- $\bar{b}_{3,0} := \bar{b}_{0,0} - \bar{b}_{1,0} + \bar{b}_{2,0}$,
- $\bar{b}_{3,1} := \bar{b}_{0,1} - \bar{b}_{1,1} + \bar{b}_{2,1}$.

In the light of Claim 2.23, there exist $u_{3,1} \in \mathrm{set}_{\varphi,\overline{\psi}}(\bar{b}_3)$ and $u_{3,0} \in \mathrm{set}_{\varphi,\overline{\psi}}(\bar{b}_3)$ such that the following two equalities are valid:

(1) $u_{3,1} \cap (u_{0,1} \cap u_{1,1}) = u_{2,1} \cap (u_{0,1} \cap u_{1,1})$, and
(2) $u_{3,0} \cap (u_{0,0} \cap u_{1,0}) = u_{2,0} \cap (u_{0,0} \cap u_{1,0})$.

By clause (1), we have $u_{3,1} \supseteq w_1$. Hence

$$\begin{aligned} w_0 &\subseteq u_{3,1} \cap w_1 \\ &= u_{3,1} \cap (u_{0,1} \cap u_{1,1}) \cap w_1 \\ &= \big(u_{3,1} \cap (u_{0,1} \cap u_{1,1})\big) \cap (u_{0,1} \cap u_{1,1}) \cap w_1 \\ &\overset{(2)}{=} \big(u_{2,1} \cap (u_{0,1} \cap u_{1,1})\big) \cap (u_{0,1} \cap u_{1,1}) \cap w_1 \\ &\subseteq w_0, \end{aligned}$$

and so

$$u_{3,1} \cap w_1 = w_0.$$

The claim follows. □

Now, we are ready to complete the proof of Proposition 2.22. To this end, we fix the following data:

- $\bullet_1$ $\vartheta(\overline{y}) = \theta_{\varphi,\overline{\psi},\Upsilon}(\overline{y})$,
- $\bullet_2$ $\overline{a}_0, \overline{a}_1, \overline{a}_2 \in \vartheta(M)$,
- $\bullet_3$ $\overline{a}_3 := \overline{a}_0 - \overline{a}_1 + \overline{a}_2$.

This gives us $\Upsilon \subseteq \text{inter}_{\varphi,\overline{\psi}}(\overline{a}_l)$ for $l \leq 2$. Thanks to Claim 2.24, we know $\Upsilon \subseteq \text{inter}_{\varphi,\overline{\psi}}(\overline{a}_3)$. According to Definition 2.21(c) one has $\overline{a}_3 \in \vartheta(M)$. Consequently, $\vartheta(\overline{y}) \in \text{Affine}_1$, and the proposition follows. □

§ 3. Additive frames

In this section we introduce the concept of an additive frame. Each additive frame contains, among other things, an abelain group. We will show that each abelian group can be realized in this way. In particular, the main result of this section is Theorem 3.14.

The following is one of our main and new frameworks:

Definition 3.1. (A) We say

$$\mathbf{f} := (M_{\mathbf{f}}, \mathscr{L}_{\mathbf{f}}, \lambda_{\mathbf{f}}, \kappa_{\mathbf{f}}, \theta_{\mathbf{f}}, \Omega_{\mathbf{f}}) = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$$

is a *general frame* if:

(1) M is a τ_M-model.

(2) $\mathscr{L}$ is a class or set of formulas in the vocabulary τ_M, such that each $\varphi \in \mathscr{L}$ has the form $\varphi(\overline{x})$, $\overline{x}$ of length $< \theta$.

(3) For every $\overline{a} \in {}^{\varepsilon}M$, $\varepsilon < \theta$, there is a formula $\varphi_{\overline{a}}(\overline{x}) \in \mathscr{L}$ such that:
   (a) $\overline{a} \in \varphi_{\overline{a}}(M)$,
   (b) (the minimality condition) if $\psi(\overline{x}) \in \mathscr{L}$ and $\overline{a} \in \psi(M)$, then $\varphi_{\overline{a}}(M) \subseteq \psi(M)$.

(4) (a) If $\varphi_\alpha(\overline{x}) \in \mathscr{L}$ for $\alpha < \kappa$, then for some $\alpha < \beta < \kappa$, we have $\varphi_\alpha(M) \supseteq \varphi_\beta(M)$,
   (b) if $\varphi_{\alpha,\beta}(\overline{x}, \overline{y}) \in \mathscr{L}$ for $\alpha < \beta < \lambda$, then for some $\alpha_1 < \alpha_2 < \alpha_3 < \lambda$ we have $\varphi_{\alpha_1,\alpha_2}(M) \supseteq \varphi_{\alpha_1,\alpha_3}(M), \varphi_{\alpha_2,\alpha_3}(M)$.

(5) λ, κ are regular and $\lambda \geq \kappa \geq \theta \geq |\Omega| + |\tau_M|$, where Ω is a set of cardinals such that $1 \in \Omega$ and all other cardinals in it are infinite.

(B) We say a general frame $\mathbf{f}$ is an *additive frame* if in addition, it satisfies:
   (6) $(|M|, +^M, -^M, 0^M)$ is an abelian group. Moreover, M is an additive θ-model.
   (7) If $\varphi(\overline{x}_u) \in \mathscr{L}$, then $\varphi(M) \subseteq {}^uM$ is a sub-group of M.

(C) An additive frame $\mathbf{f}$ is an *additive$^+$ frame* if $M_{\mathbf{f}}$ has cardinality greater or equal to λ.

Remark 3.2. Given a general frame $\mathbf{f}$ as above, we always assume that the language $\mathscr{L}$ is closed under permutation of variables, adding dummy variables and finite conjunction.

The next lemma is a criterion for an additive frame to be additive$^+$.

Lemma 3.3. *Suppose* $\mathbf{f} = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$ *is an additive frame. Then it is an additive$^+$ frame if and only if for each* $\varepsilon \in (0, \theta)$, *there exists some* $\bar{a} \in^{\varepsilon} M$ *such that* $\varphi_{\bar{a}}$ *has cardinality* $\geq \lambda$.

Proof. he assumption clearly implies $\mathbf{f}$ is additive$^+$. To see the other direction, suppose $\mathbf{f}$ is an additive$^+$ frame and $\varepsilon \in (0, \theta)$. Suppose by the way of contradiction, $|\varphi_{\bar{a}}| < \lambda$ for all $\bar{a} \in^{\varepsilon} M$. By induction on $\alpha < \kappa$ we can find a sequence $\langle \bar{a}_\beta : \beta < \kappa \rangle$ such that for each $\beta < \kappa$, $\bar{a}_\beta \notin \bigcup_{\alpha < \beta} \varphi_{\bar{a}_\alpha}$. This contradicts Definition 3.1(4)(a). $\square$

The following defines a partial order relation on formulas of a frame.

Definition 3.4. Assume $\mathbf{f}$ is a general frame, and let $\psi(\overline{x}), \varphi(\overline{x})$ be in $\mathscr{L}_{\mathbf{f}}$.

(1) We say $\psi(\overline{x}) \le \varphi(\overline{x})$ if $\varphi(M) \supseteq \psi(M)$.

(2) We say $\psi(\overline{x})$ and $\varphi(\overline{x})$ are equivalent, denoted by $\psi(\overline{x}) \equiv \varphi(\overline{x})$, if $\varphi(M) = \psi(M)$.

(3) Suppose $\overline{a}, \overline{b} \in {}^{\varepsilon}M$. We say $\overline{a} \le \overline{b}$ (resp. $\overline{a} \equiv \overline{b}$) if $\varphi_{\overline{a}} \le \varphi_{\overline{b}}$ (resp. $\varphi_{\overline{a}} \equiv \varphi_{\overline{b}}$).

Notation 3.5. Assume $\mathbf{f} = (M, \lambda, \kappa, \theta, \Omega)$ is an additive frame. Let $\overline{a}_l = \langle a_{l,\zeta} : \zeta < \varepsilon \rangle \in {}^{\varepsilon}M$ for $l < n$. We set:

- $-\overline{a}_l := \langle -a_{l,\zeta} : \zeta < \varepsilon \rangle$,
- $\overline{a}_1 + \overline{a}_2 := \langle a_{1,\zeta} + a_{2,\zeta} : \zeta < \varepsilon \rangle$,
- $\sum_{l<n} \overline{a}_l := \langle \sum_{l<n} a_{l,\zeta} : \zeta < \varepsilon \rangle$,
- $\overline{a} - \overline{b} := \overline{a} + (-\overline{b})$.

Lemma 3.6. *Suppose* $\mathbf{f} = (M, \lambda, \kappa, \theta, \Omega)$ *is an additive frame,* $\overline{a} \in {}^{\varepsilon}M, \varphi = \varphi_{\overline{a}}$ *and* $\overline{a}_\alpha \in \varphi(M)$ *for* $\alpha < \lambda$. *Then for some* $\overline{\beta}, \overline{\gamma}$ *we have:*

(a) $\overline{\beta} = \langle \beta_i : i < \lambda \rangle \in {}^{\lambda}\lambda$ *is increasing,*

(b) $\overline{\gamma} = \langle \gamma_i : i < \lambda \rangle \in {}^{\lambda}\lambda$ *is increasing,*

(c) $\beta_i < \gamma_i < \beta_{i+1}$, *for all* $i < \lambda$,

(d) $\overline{a} - \overline{a}_{\beta_i} + \overline{a}_{\gamma_i}$ *is equivalent to* $\overline{a}$, *for all* $i < \lambda$.

Proof. First, we reduce the lemma to the following claim:

$(*)$ It is suffice to prove, for each sequences $\overline{a}, \langle \overline{a}_\alpha : \alpha < \lambda \rangle$ as above, there are $\beta < \gamma < \lambda$ such that $\overline{a} - \overline{a}_\beta + \overline{a}_\gamma$ is equivalent to $\overline{a}$.

To see this, suppose $(*)$ holds. By induction on $i < \lambda$, we define the increasing sequences $\langle \beta_i : i < \lambda \rangle$ and $\langle \gamma_i : i < \lambda \rangle$ as requested. Thus suppose that $i < \lambda$, and we have defined $\langle \gamma_j, \beta_j : j < i \rangle$. In order to define (β_i, γ_i), we let

$$\alpha_* := \sup\{\gamma_j + \beta_j + 1 : j < i\}.$$

Since λ is regular, $\alpha_* < \lambda$. Now, apply $(*)$ to $\overline{a}$ and $\langle \overline{a}_{\alpha_* + \alpha} : \alpha < \kappa \rangle$. This gives us $\beta < \gamma < \kappa$ such that

$$\overline{a} - \overline{a}_{\alpha_* + \beta} + \overline{a}_{\alpha_* + \gamma} \equiv \overline{a}.$$

Thus it suffices to set $\beta_i = \alpha_* + \beta$ and $\gamma_i = \alpha_* + \gamma$.

So, things are reduced in showing $(*)$ holds. To see this, we define the formula $\varphi_{\beta\gamma}$ as

$$(+) \qquad \varphi_{\beta\gamma} := \varphi_{\bar{a}-\bar{a}_\beta+\bar{a}_\gamma},$$

where $\beta < \gamma < \lambda$. Note that $\bar{a}, \bar{a}_\beta, \bar{a}_\gamma \in \varphi_{\bar{a}}(M)$, hence as $\varphi_{\bar{a}}(M)$ is a submodel, $\bar{a} - \bar{a}_\beta + \bar{a}_\gamma \in \varphi_{\bar{a}}(M)$. Thanks to the minimality condition from Definition 3.1(3)(b), this implies that

$$\varphi_{\bar{a}} \geq \varphi_{\beta,\gamma}.$$

Thus, it is sufficient to find $\beta < \gamma < \kappa$ such that $\varphi_{\bar{a}} \leq \varphi_{\beta,\gamma}$. By the property presented in Definition 3.1(4)(b) there are $\alpha_1 < \alpha_2 < \alpha_3 < \kappa$ such that

$$\varphi_{\alpha_1,\alpha_2} \geq \varphi_{\alpha_1,\alpha_3}, \varphi_{\alpha_2,\alpha_3}.$$

So,

(1) $\bar{a} - \bar{a}_{\alpha_1} + \bar{a}_{\alpha_2} \in \varphi_{\alpha_1,\alpha_2}(M)$,
(2) $\bar{a} - \bar{a}_{\alpha_1} + \bar{a}_{\alpha_3} \in \varphi_{\alpha_1,\alpha_3}(M) \subseteq \varphi_{\alpha_1,\alpha_2}(M)$,
(3) $\bar{a} - \bar{a}_{\alpha_2} + \bar{a}_{\alpha_3} \in \varphi_{\alpha_2,\alpha_3}(M) \subseteq \varphi_{\alpha_1,\alpha_2}(M)$.

Hence

$$\bar{a} \ = (1) - (2) + (3) \in \varphi_{\alpha_1,\alpha_2}(M).$$

Combining this with the minimality property from Definition 3.1(3)(b) we observe that $\varphi_{\bar{a}} \leq \varphi_{\alpha_1,\alpha_2}$. Thus it suffices to take $\beta = \alpha_1$ and $\gamma = \alpha_2$. □

Corollary 3.7. *Suppose* $\mathbf{f} = (M, \lambda, \kappa, \theta, \Omega)$ *is an additive frame. The following assertions are valid:*

(1) Suppose $\varphi_{\bar{a}}(M)$ *has cardinality* $\geq \lambda$. *Then there is* $\bar{b} \in \varphi_{\bar{a}}(M)$ *such that* $\bar{b} \neq \bar{a}$ *and* $\bar{b}$ *is equivalent to* $\bar{a}$.

(2) If for some $\bar{c} \in^\varepsilon M$, $\varphi_{\bar{c}}(M)$ *has cardinality* $\geq \lambda$, *then for all* $\bar{a} \in^\varepsilon M$ *the set*

$$\{\bar{b} \in \varphi_{\bar{a}}(M) : \bar{b} \textit{ is equivalent to } \bar{a}\}$$

has cardinality $\geq \lambda$.

(3) If $\mathbf{f}$ *ia an additive*$^+$ *frame and* $\varepsilon \in (0, \theta)$, *then for all* $\bar{a} \in^\varepsilon M$, *the set* $\{\bar{b} \in \varphi_{\bar{a}}(M) : \bar{b}$ *is equivalent to* $\bar{a}\}$ *has cardinality* $\geq \lambda$.

Proof. (1) Since $\varphi_{\overline{a}}(M)$ has cardinality $\geq \lambda$, we can take a sequence $\langle \overline{a}_\alpha : \alpha < \lambda \rangle$ of length λ of pairwise distinct elements of $\varphi_{\overline{a}}(M)$ with no repetition. We apply Lemma 3.6 to find increasing sequences $\overline{\beta} = \langle \beta_i : i < \lambda \rangle$ and $\overline{\gamma} = \langle \gamma_i : i < \lambda \rangle$ such that for all $i < \lambda$

(a) $\beta_i < \gamma_i$,

(b) $\overline{a} - \overline{a}_{\beta_i} + \overline{a}_{\gamma_i}$ is equivalent to $\overline{a}$.

Set $\overline{b}_i := \overline{a} - \overline{a}_{\beta_i} + \overline{a}_{\gamma_i}$. Since $\overline{a}_\alpha$'s are distinct, we deduce that $\overline{a} \neq \overline{b}_i$, for at least one $i < \lambda$. Thanks to (b), we know $\overline{b}_i$ is equivalent to $\overline{a}$.

(2) Let $X = \{\overline{b} \in \varphi_{\overline{a}}(M) : \overline{b}$ is equivalent to $\overline{a}\}$, and let $\mu = |X|$. Suppose towards contradiction that $\mu < \lambda$, and let $\langle \overline{b}_i : i < \mu \rangle$ be enumerates X. Let also $\langle \overline{c}_\alpha : \alpha < \lambda \rangle$ be a sequence of length λ of pairwise distinct elements of $\varphi_{\overline{c}}(M)$ with no repetition. By induction on $\alpha < \lambda$ we can find $\xi_\alpha < \lambda$ such that

$(*)_\alpha$: $\overline{a}_{\xi_\alpha} \notin \{\overline{b}_i - \overline{c} + \overline{c}_{\xi_\beta} : \beta < \alpha, i < \mu\}$.

By the argument of clause (1), applied to the sequence $\langle \overline{c}_{\xi_\alpha} : \alpha < \lambda \rangle$, we can find some $\beta_\iota < \gamma_\iota < \lambda$ such that $\overline{a} - \overline{c}_{\xi_{\beta_\iota}} + \overline{c}_{\xi_{\gamma_\iota}}$ is equivalent (but not equal) to $\overline{c}$. Thus for some $i < \mu, \overline{a} - \overline{c}_{\xi_{\beta_\iota}} + \overline{c}_{\xi_{\gamma_\iota}} = \overline{b}_i$. But then

$$\overline{c}_{\xi_{\gamma_\iota}} = \overline{b}_i - \overline{a} + \overline{c}_{\xi_{\beta_\iota}},$$

which contradicts $(*)_{\gamma_\iota}$.

(3) By Lemma 3.3 and clause (2). □

Lemma 3.8. *(1) Suppose* $\mathbf{f} = (M, \lambda, \kappa, \theta, \Omega)$ *is a general frame,* $\varepsilon < \theta$ *and* $\overline{a}_\alpha \in {}^\varepsilon M$ *for* $\alpha < \kappa$*. Then there is some* $\alpha < \kappa$ *such that the set* $\{\beta < \kappa : \overline{a}_\beta \in \varphi_{\overline{a}_a}(M)\}$ *is unbounded in* κ*.*

(2) In clause (1), we can replace κ *by any cardinal* $\kappa' \geq \kappa$*.*

Proof. We prove clause (2). Suppose on the way of contradiction that, for each $\alpha < \kappa'$, the set $X_\alpha = \{\beta < \kappa' : \overline{a}_\beta \in \varphi_{\overline{a}_a}(M)\}$ is bounded in κ'. So,

$(*)_1$ $\forall \alpha < \kappa', \exists \alpha < \beta_\alpha < \kappa'$ such that $\forall \beta \geq \beta_\alpha$ we have $\varphi_\alpha \ngeq \varphi_\beta$.

We define an increasing and continuous sequence $\langle \zeta_\alpha : \alpha < \kappa \rangle$ of ordinals less than κ', by induction on α as follows:

- $\zeta_0 := 0$,
- $\zeta_{\alpha+1} := \beta_{\zeta_\alpha}$,
- $\zeta_\delta := \lim_{\alpha<\delta} \zeta_\alpha$ for limit ordinal δ.

Consider the sequence $\{\varphi_{\zeta_\alpha} : \alpha < \kappa\}$, and apply the property presented in Definition 3.1(4)(a) to find $\gamma < \delta < \kappa$ such that

$(*)_2$ $\varphi_{\zeta_\gamma} \geq \varphi_{\zeta_\delta}$.

Since $\gamma < \delta$, $\zeta_\delta \geq \zeta_{\gamma+1} = \beta_{\zeta_\gamma}$. We apply $(*)_1$ for $\alpha := \zeta_\gamma$ and $\beta := \zeta_\delta \geq \beta_\alpha$. This gives us $\varphi_{\zeta_\gamma} \ngeq \varphi_{\zeta_\delta}$, which contradicts $(*)_2$. □

In what follows we need to use a couple of results from [29]. To make the paper more self contained, we borrow some definitions and results from it.

Definition 3.9. (1) By a *tree* we mean a partially-ordered set $(\mathscr{T}, \leq)$ such that for all $t \in \mathscr{T}$,

$$pred(t) := \{s \in \mathscr{T} : s < t\},$$

is a well-ordered set; moreover, there is only one element r of $\mathscr{T}$, called the *root* of $\mathscr{T}$, such that $pred(r)$ is empty.

(2) The order-type of $pred(t)$ is called the height of t, denoted by $\mathrm{ht}(t)$.

(3) The height of $\mathscr{T}$ is $\sup\{\mathrm{ht}(t) + 1 : t \in \mathscr{T}\}$.

Definition 3.10. (1) A *quasi-order* $\mathcal{Q}$ is a pair $(\mathcal{Q}, \leq_{\mathcal{Q}})$ where $\leq_{\mathcal{Q}}$ is a reflexive and transitive binary relation on $\mathcal{Q}$.

(2) $\mathcal{Q}$ is called *κ–narrow*, if there is no *antichain* in $\mathcal{Q}$ of size κ, i.e., for every $f : \kappa \to \mathcal{Q}$ there exist $\nu \neq \mu$ such that $f(\nu) \leq_{\mathcal{Q}} f(\mu)$.

(3) For a quasi-order $\mathcal{Q}$, a *$\mathcal{Q}$-labeled tree* is a pair $(\mathscr{T}, \Phi_{\mathscr{T}})$ consisting of a tree $\mathscr{T}$ of height $\leq \omega$ and a function $\Phi_{\mathscr{T}} : \mathscr{T} \to \mathcal{Q}$.

(4) $\mathcal{Q}$ is κ-well ordered if for every sequence $\langle q_i : i < \kappa\rangle$ of elements of $\mathcal{Q}$, there are $i < j < \kappa$ such that $q_i \leq_{\mathcal{Q}} q_j$.

Remark 3.11. On any set of $\mathcal{Q}$-labeled trees we define a quasi-order by: $(\mathscr{T}_1, \Phi_1) \preceq (\mathscr{T}_2, \Phi_2)$ if and only if there is a function $\nu : \mathscr{T}_1 \to \mathscr{T}_2$ equips with the following properties:

a) for all $t \in \mathscr{T}_1$, $\Phi_1(t) \leq_{\mathcal{Q}} \Phi_2(\nu(t))$,

b) $t \leq_{\mathscr{T}_1} t' \Longrightarrow \nu(t) \leq_{\mathscr{T}_2} \nu(t')$,

c) for all $t \in \mathscr{T}_1$, $\mathrm{ht}_{\mathscr{T}_1}(t) = \mathrm{ht}_{\mathscr{T}_2}(\nu(t))$.

Definition 3.12. (1) Given infinite cardinals κ and μ, the notation $\kappa \longrightarrow (\omega)^{<\omega}_{\mu}$ means that: for every function $f : [\kappa]^{<\omega} \to \mu$, there exists an infinite subset $X \subseteq \kappa$ and a function $g : \omega \to \mu$ such that $f(Y) = g(|Y|)$ for all finite subsets Y of X.

(2) Let κ_{beau} denote the first beautiful[1] cardinal. This is defined as the smallest cardinal κ such that $\kappa \longrightarrow (\omega)^{<\omega}_{2}$.

(3) Given a ring R and an infinite cardinal θ, let $\kappa_{\mathrm{beau}}(R, \theta)$ denote the least cardinal κ such that $\kappa \longrightarrow (\omega)^{<\omega}_{|R|+\theta^{<\theta}}$.

(4) Given a vocabulary τ, let $\kappa_{\mathrm{beau}}(\tau, \theta)$ denote the least cardinal κ such that $\kappa \longrightarrow (\omega)^{<\omega}_{|\tau|+\theta^{<\theta}}$. If $\theta = \aleph_0$, we may omit it.

Now, we can state:

Fact 3.13. (Shelah, [29, theorems 5.3+ 2.10]) Let $\mathcal{Q}$ be a quasi-order of cardinality $< \kappa_{\mathrm{beau}}$, and $\mathcal{S}$ be a set of $\mathcal{Q}$-labeled trees with $\leq \omega$ level. Then $\mathcal{S}$ is κ_{beau}-narrow, and even κ-well ordered.

We are now ready to state and prove the main result of this section.

Theorem 3.14. *(i) Assume M is an R-module and $\kappa = \kappa_{\mathrm{beau}}(R, \theta)$ and Ω is such that $1 \in \Omega$ and $|\Omega| \leq \theta$. Also assume that $\lambda \geq \kappa$ is regular and satisfies $\lambda \to (\kappa + 1)^3_4$. The following hold:*

(a) $\mathbf{f} = (M, \lambda, \kappa, \theta, \Omega)$ *is an additive frame, whenever*

$$\mathscr{L} \subseteq \{\varphi(\overline{x}) : \varphi \in \mathscr{L}^{\mathrm{cop}}_{\infty,\theta}(\tau_M), \lg(\overline{\mathrm{x}}) < \theta\}$$

is closed under arbitrary conjunctions.

(b) $\mathbf{f} = (M, \lambda, \kappa, \theta, \Omega)$ *is an additive frame, whenever*

$$\mathscr{L} \subseteq \{\varphi(\overline{x}) : \varphi \in \mathscr{L}^{\mathrm{co}}_{\infty,\theta}(\tau_M), \lg(\overline{\mathrm{x}}) < \theta\}$$

is closed under arbitrary conjunctions.

[1]This also is called the first ω-Erdös cardinal.

(ii) If M is a τ-model, $\kappa = \kappa_{\rm beau}(\tau, \theta)$ and $\lambda \geq \kappa$ is regular and satisfies $\lambda \to (\kappa+1)^3_4$. Then $\mathbf{f} = (M, \lambda, \kappa, \theta, \Omega)$ is a additive frame, whenever

$$\mathscr{L} \subseteq \{\varphi(\overline{x}) : \varphi \in \mathscr{L}^{\rm co}_{\infty,\theta}(\tau_M), \lg(\overline{\rm x}) < \theta\}$$

is closed under arbitrary conjunctions.

(iii) Furthermore, if $|M| \geq \lambda$, then in items (i) and (ii), $\mathbf{f}$ becomes an additive$^+$ frame.

Proof. (i): We have to show that $\mathbf{f}$ satisfies the relevant items of Definition 3.1. Items (1), (2) and (5) are true by definition. As M is an R-module, clause (6) is valid. The validity of (7) follows from Lemma 2.15.

Let us consider clause (3). Thus suppose that $\overline{a} \in {}^{\varepsilon}M$, where $\varepsilon < \theta$. We are going to find some $\varphi \in \mathscr{L}$ such that:

- $\overline{a} \in \varphi(M)$,
- if $\psi \in \mathscr{L}$ and $\overline{a} \in \psi(M)$, then $\varphi(M) \subseteq \psi(M)$.

For any $\overline{b} \in {}^{\varepsilon}M$, if there is some formula $\varphi(\overline{x})$ such that

$$(\dagger)_1 \qquad M \models \varphi(\overline{a}) \wedge \neg\varphi(\overline{b}),$$

then let $\varphi_{\overline{b}} \in \mathscr{L}$ be such a formula. Otherwise, let $\varphi_{\overline{b}} \in \mathscr{L}$ be any true formula such that $\varphi_{\overline{b}}(M) = {}^{\varepsilon}M$. Finally set

$$(\dagger)_2 \qquad \varphi := \bigwedge\{\varphi_{\overline{b}} : \overline{b} \in {}^{\varepsilon}M\}.$$

Now, we claim that φ is as desired. First, we check (3)(a), that is $\overline{a} \in \varphi(M)$. As

$$\varphi(M) = \bigcap_{\overline{b} \in {}^{\varepsilon}M} \varphi_{\overline{b}}(M),$$

it suffices to show that $\overline{a} \in \varphi_{\overline{b}}(M)$, for $\overline{b} \in {}^{\varepsilon}M$. Fix $\overline{b}$ as above. If there is no formula $\varphi(\overline{x})$ as in ($\dagger$), then $\overline{a} \in {}^{\varepsilon}M = \varphi_{\overline{b}}(M)$, and we are done. Otherwise, by its definition, $M \models \varphi_{\overline{b}}(\overline{a})$, and hence, again we have $\overline{a} \in \varphi_{\overline{b}}(M)$.

To see (3)(b) holds, let $\psi(\overline{x})$ be such that $\overline{a} \in \psi(M)$. We have to show that $\varphi(M) \subseteq \psi(M)$. Suppose by the way of contradiction that $\varphi(M) \nsubseteq \psi(M)$. Take $\overline{b} \in \varphi(M) \setminus \psi(M)$.

Now, $M \models \neg\psi(\overline{b})$, and by our assumption $M \models \psi(\overline{a})$. In particular, by our construction, the formula $\varphi_{\overline{b}}$ satisfies

$$(\dagger)_3 \qquad M \models \varphi_{\overline{b}}(\overline{a}) \wedge \neg\varphi_{\overline{b}}(\overline{b}).$$

Now,

- $\bullet_1$ $\overline{b} \in \varphi(M) \overset{(+)}{\subseteq} \varphi_{\overline{b}}(M)$,
- $\bullet_2$ by $(\dagger)_3$, $M \models \neg\varphi_{\overline{b}}(\overline{b})$, and hence $\overline{b} \notin \varphi_{\overline{b}}(M)$.

By $\bullet_1$ and $\bullet_2$ we get a contradiction.

Now we turn to clause (4). First let us consider (4)(a). Thus suppose that $\varphi_\alpha(\overline{x}) \in \mathscr{L}$, for $\alpha < \kappa$, are given. We should find some $\alpha < \beta < \kappa$ such that $\varphi_\alpha \geq \varphi_\beta$, i.e., $\varphi_\alpha(M) \supseteq \varphi_\beta(M)$. To this end, first note that we can restrict ourselves to those formulas such that both free and bounded variables appearing in them are among $\{x_i : i < \theta\}$, the set of free variables of φ has the form $\{x_\zeta : \zeta < \varepsilon\}$, and the quantifiers have the form $\exists\bar{x}_{[\varepsilon_0,\varepsilon_1)}$ and $\forall\bar{x}_{[\varepsilon_0,\varepsilon_1)}$, where $\varepsilon_0 < \varepsilon_1 < \theta$, and $\bar{x}_{[\varepsilon_0,\varepsilon_1)} = \langle x_\xi : \varepsilon_0 \leq \xi < \varepsilon_1\rangle$. In what follows, writing a formula as $\varphi(\bar{x})$, we mean $\bar{x}$ lists the free variables appearing in φ in increasing order.

We can consider a formula $\varphi(\bar{x})$ as a type $(\mathscr{T}, c)$ such that

(a) $\mathscr{T}$ is a tree with $\leq \omega$ levels with no infinite branches,

(b) c is a function with domain $\mathscr{T}$,

(c) if $t \in \mathscr{T} \setminus \max(\mathscr{T})$, then $c(t)$ is in the following set

$$\left\{\wedge, \exists\bar{x}, \forall\bar{x} : \bar{x} \textit{ has form } \bar{x}_{[\varepsilon_0,\varepsilon_1)} \textit{ for some } \varepsilon_0 < \varepsilon_1 < \theta\right\},$$

(d) if $t \in \max(\mathscr{T})$ then $c(t)$ is an atomic formula in τ_M.

Clearly, $|\mathrm{Rang}(c)| \leq \theta + |R|$. For each $\alpha < \kappa$ set $\varphi_\alpha(\bar{x}) = (\mathscr{T}_\alpha, c_\alpha)$.

Let $\mathcal{Q}$ be the range of the function c. Note that it is a quasi-order under the $\leq$ relation defined in Definition 3.4, and clearly, it has cardinality

$$|\mathcal{Q}| \leq |\tau_M| + \theta^{<\theta} < \kappa_{\mathrm{beau}}.$$

In particular, by Fact 3.13, applied to the sequence

$$\langle(\mathscr{T}_\alpha, c_\alpha) : \alpha < \kappa\rangle.$$

we get some $\alpha < \beta$ and a function f equipped with the following property:

(∗) f is a 1-1 function from $\mathscr{T}_\alpha$ into $\mathscr{T}_\beta$, which is level, order and non-order preserving and such that

$$t \in c_\alpha \Rightarrow \big(c_\beta(f(t)) = c_\alpha(t)\big).$$

Let $\mathscr{T}'_\alpha := \mathrm{Rang}(f)$ and $c'_\alpha := c_\beta \restriction \mathscr{T}'_\alpha$. By this notation, $\varphi_\alpha(\bar{x})$ can also be represented by $(\mathscr{T}'_\alpha, c'_\alpha)$. For any $t \in \mathscr{T}'_\alpha$ we define φ^1_t to be the formula represented by

$$\Big(\{s \in \mathscr{T}'_\alpha : t \leq_{\mathscr{T}'_\alpha} s\}, c_\beta \restriction \{s \in \mathscr{T}'_\alpha : t \leq_{\mathscr{T}'_\alpha} s\}\Big),$$

and define φ^2_t to be the formula represented via

$$\Big(\{s \in \mathscr{T}_\beta : t \leq_{\mathscr{T}'_\beta} s\}, c_\beta \restriction \{s \in \mathscr{T}_\beta : t \leq_{\mathscr{T}'_\beta} s\}\Big).$$

Note that the formula φ^1_t may have fewer free variables than φ^2_t, but we can add the remaining variables. So, we may and do assume that

$$\varphi^\ell_t = \varphi^\ell_t(\bar{x}_t) \quad \ell = 1, 2.$$

Now we are going to prove that for every t,

$$\varphi^1_t(M) \supseteq \varphi^2_t(M).$$

This is done by induction on the depth of t inside $\mathscr{T}'_\alpha$ which is possible, as $\mathscr{T}'_\alpha$ is well-founded. By the way we defined our trees, it suffices to deal with the following cases:

Case 1): $c_\beta(t)$ is a basic formal, i.e., an atomic formula or its negation.

Case 2): $c_\beta(t)$ is $\wedge$.

Case 3): $c_\beta(t)$ is $\exists \overline{x}'$ or just $c_\beta(t)$ is $\exists^\sigma \overline{x}'$.

Case 4) $c_\beta(t)$ is $\forall \overline{x}'$ or just $c_\beta(t)$ is $\forall^\sigma \overline{x}'$.

Let discuss each cases separably:

Case 1): Here, t is a maximal node of the tree $\mathscr{T}_\beta$. Hence, necessarily a maximal node of $\mathscr{T}'_\alpha$, and recall that

$$\varphi^1_t = c_\beta(t) = \varphi^2_t.$$

Consequently, the conclusion is obvious.

Case 2): Here, we have

- $\varphi^2_t := \bigwedge\{\varphi^2_s : s \in \mathrm{suc}_{\mathscr{T}_\beta}(t)\}$,
- $\varphi^1_t := \bigwedge\{\varphi^1_s : s \in \mathrm{suc}_{\mathscr{T}'_\alpha}(t)\}$.

By the choice of the $\mathscr{T}'_\alpha$ and the function f, we have

$$\mathrm{suc}_{\mathscr{T}'_\alpha}(t) \subseteq \mathrm{suc}_{\mathscr{T}_\beta}(t).$$

Now, due to the induction hypotheses

$$s \in \mathrm{suc}_{\mathscr{T}'_\alpha}(t) \Longrightarrow \varphi^1_s(M) \supseteq \varphi^2_s(M).$$

According to the definition of sanctification for $\wedge$ we are done, indeed:

$$\begin{aligned}\varphi^1_t(M) \quad &= \bigcap\{\varphi^1_s(M) : s \in \mathrm{suc}_{\mathscr{T}'_\alpha}(t)\}\\ &\supseteq \bigcap\{\varphi^2_s(M) : s \in \mathrm{suc}_{\mathscr{T}'_\alpha}(t)\}\\ &\supseteq \bigcap\{\varphi^s_s(M) : s \in \mathrm{suc}_{\mathscr{T}_\beta}(t)\}.\end{aligned}$$

Case 3): Let $\overline{x}'_t = \overline{x}_s{}^\frown\overline{x}'$, and recall that $\mathrm{suc}_{\mathscr{T}_\beta}(t)$ is singleton, and also $\mathrm{suc}_{\mathscr{T}_\alpha}(f^{(-1)}(t))$ is singleton, because $c_\beta(t) = c_\alpha(f^{-1}(t))$. This implies that $\mathrm{suc}_{\mathscr{T}'_\alpha}(t)$ is singleton, say $\mathrm{suc}_{\mathscr{T}'_\alpha}(t) = \{s\} = \mathrm{suc}_{\mathscr{T}_\beta}(t)$. In order to see $\varphi^1_t(M) \supseteq \varphi^2_t(M)$, we take $\overline{a} \in {}^{\lg(\overline{x})}M$ be such that $M \models \varphi^2_t(\overline{a})$ and shall show that $M \models \varphi^1_t(\overline{a})$. Indeed,

$$\varphi^\ell_t(\overline{x}) := (\exists^\sigma \overline{x}')\varphi^\ell_s(\overline{x}_s, \overline{x}') \quad \ell = 1, 2,$$

and since

$$M \models (\exists^\sigma \overline{x}')\varphi^2_s(\overline{a}, \overline{x}'),$$

necessarily, for some pairwise disjoint $\overline{b}_\zeta \in {}^{\lg(\overline{x}')}M$ we have

$$M \models \varphi^2_s(\overline{a}, \overline{b}_\zeta).$$

Thanks to the inductive hypothesis, we know

$$M \models \varphi^1_s(\overline{a}, \overline{b}_\zeta).$$

According to the definition of sanctification, we have

$$M \models (\exists^\sigma \overline{x}')\varphi^1_s(\overline{a}, \overline{x}'),$$

which means that $M \models \varphi^1_t(\overline{a})$, as promised.

Case 4): Let $\overline{x}'_t = \overline{x}_s{}^\frown\overline{x}'$, and recall that $\mathrm{suc}_{\mathscr{T}_\beta}(t)$ is singleton, and also $\mathrm{suc}_{\mathscr{T}_\alpha}(f^{-1}(t))$ is singleton, because $c_\alpha(f^{-1}(t)) \subseteq c_\beta(t)$. This implies that $\mathrm{suc}_{\mathscr{T}'_\alpha}(t)$ is singleton, say $\mathrm{suc}_{\mathscr{T}'_\alpha}(t) = \{s\} = \mathrm{suc}_{\mathscr{T}_\beta}(t)$. In order to see $\varphi^1_t(M) \supseteq \varphi^2_t(M)$, we take $\overline{a} \in {}^{\lg(\overline{x})}M$ be such that $M \models \varphi^2_t(\overline{a})$ and shall show that $M \models \varphi^1_t(\overline{a})$. Similar to the Case (3), we can write

$$\varphi^\ell_t(\overline{x}) := (\forall^\sigma \overline{x}')\varphi^\ell_s(\overline{x}_s, \overline{x}') \quad \ell = 1, 2.$$

Suppose it is not the case that $M \models \varphi^1_t(\overline{a})$. This means, by Definition 2.10(2)(b), that there are pairwise disjoint $\overline{b}_\zeta \in {}^{\lg(\overline{x}')}M$, for $\zeta < \sigma$ such that

$$M \models \neg\varphi^1_s(\overline{a}, \overline{b}_\zeta).$$

By the induction hypothesis, we have $\varphi^1_s(M) \supseteq \varphi^2_s(M)$, hence for each $\zeta < \sigma$,

$$M \models \neg\varphi^2_s(\overline{a}, \overline{b}_\zeta).$$

The later means that $M \models \varphi^2_t(\overline{a})$ is not true, which contradicts our initial assumption. This completes the proof of clause (4)(a).

Finally, let us turn to check the property presented in clause (4)(b) from Definition 3.1. Suppose $\varphi_{\alpha,\beta}(\overline{x}) \in \mathscr{L}$ for $\alpha < \beta < \lambda$ are given. We need to find some $\alpha_1 < \alpha_2 < \alpha_3 < \kappa$ such that

$$\varphi_{\alpha_1,\alpha_2} \geq \varphi_{\alpha_1,\alpha_3}, \varphi_{\alpha_2,\alpha_3}.$$

To see this, we define a coloring $\mathbf{c} : [\lambda]^3 \to 2 \times 2$ as follows. Fix $\alpha < \beta < \gamma < \lambda$, and define the following pairing function

$$\mathbf{c}(\{\alpha, \beta, \gamma\}) := \Big(\textit{truth value of } \varphi_{\alpha,\beta} \geq \varphi_{\alpha,\gamma},\ \textit{truth value of } \varphi_{\alpha,\gamma} \geq \varphi_{\beta,\gamma}\Big).$$

By the assumption $\lambda \to (\kappa+1)^3_4$, there is $X \subseteq \lambda$ of order type $\kappa + 1$ such that $\mathbf{c} \restriction [X]^3$ is constant. Let $\alpha_i \in X$, for $i \leq \kappa$, be an increasing enumeration of X. Consider the sequence

$$\{\varphi_{\alpha_0,\alpha_i} : i < \kappa\},$$

and applying clause (4)(a) to it. This gives us $i < j < \kappa$ such that

$$\varphi_{\alpha_0,\alpha_i} \geq \varphi_{\alpha_0,\alpha_j}.$$

Note that this implies that $\mathbf{c}(\alpha_0, \alpha_i, \alpha_j) = (1, \iota)$, for some $\iota \in \{0,1\}$. Since $\mathbf{c} \restriction [X]^3$ is constant, it follows that for any $\alpha < \beta < \gamma$ from X, we have $\mathbf{c}(\alpha, \beta, \gamma) = (1, \iota)$, in particular

$(*)_1$ $\varphi_{\alpha,\beta} \geq \varphi_{\alpha,\gamma}$, for all $\alpha < \beta < \gamma$ in X.

Again, applying clause (4)(a) to the sequence $\{\varphi_{\alpha_i,\alpha_\kappa} : i < \kappa\}$, we can find some $i < j < \kappa$ such that

$$\varphi_{\alpha_i,\alpha_\kappa} \geq \varphi_{\alpha_j,\alpha_\kappa}.$$

It follows that $\mathbf{c}(\alpha_i, \alpha_j, \alpha_\kappa) = (1,1)$, hence as $\mathbf{c} \restriction [X]^3$ is constant, we have $\mathbf{c}(\alpha, \beta, \gamma) = (1,1)$, for all $\alpha < \beta < \gamma$ from X. In particular,

$(*)_2$ $\varphi_{\alpha,\gamma} \geq \varphi_{\beta,\gamma}$, for any $\alpha < \beta < \gamma$ in X.

Now, combining $(*)_1$ along with $(*)_2$, we get that for all $\alpha < \beta < \gamma$ from X

$$\varphi_{\alpha,\beta} \geq \varphi_{\alpha,\gamma} \geq \varphi_{\beta,\gamma},$$

and this completes the proof of (i).

(ii): This is similar to case (i). □

Remark 3.15. By the Erdos-Rado partition theorem, see [11], it suffices to take $\lambda = \beth_2(\kappa)^+$.

§ 4. κ-ALGEBRAIC CLOSURE

In this section, and among other things, we define the concept of closure of a sequence inside an additive model and present some properties of it.

Definition 4.1. Suppose $\mathbf{f} = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$ is an additive frame, $\varepsilon < \theta$ and $\overline{a} \in {}^{\varepsilon}M$. We define the closure of $\overline{a}$ in M as the following:

$$\mathrm{cl}(\bar{a}, M) := \{b \in M : \varphi_{\bar{a}^\frown\langle b\rangle}(M, \bar{a}) \text{ has cardinality } < \kappa\}.$$

In what follows we will use the following result several times:

Lemma 4.2. *Suppose* $\mathbf{f} = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$ *is a general frame as in Theorem 3.14,* $\varepsilon < \theta$ *and* $\overline{a} \in {}^{\varepsilon}M$*. We assume in addition that* $\kappa \in \Omega$ *and* $\mathscr{L}$ *is closed under* $\exists^\kappa$*. Then the following three assertions are true:*

(a) the set $\mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$ *has cardinality* $< \kappa$*.*

(b) $\{a_i : i < \varepsilon\} \subseteq \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$*.*

(c) Assume $b \in \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$*. Then* $\mathrm{cl}(\bar{\mathrm{a}}{}^\frown\langle \mathrm{b}\rangle, \mathrm{M}) \subseteq \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$*.*

Proof. (a). Suppose not, and let $\bar{a} \in {}^{\varepsilon}M$ be such that the set $\mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$ has cardinality $\geq \kappa$. This gives us a family $\{b_\alpha : \alpha < \kappa\} \subseteq \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$. Define $\bar{a}_\alpha := \bar{a}{}^\frown\langle b_\alpha\rangle$. In the light of Lemma 3.8, there is $\alpha_* < \kappa$ such that the set

$$X := \{\beta < \kappa : \bar{a}_\beta \in \varphi_{\bar{a}_{\alpha_*}}(M)\}$$

is unbounded in κ. So, $\{\bar{a}_\beta : \beta \in X\} \subseteq \varphi_{\bar{a}_{\alpha_*}}(M, \bar{a})$, which implies that

$$|\varphi_{\bar{a}_{\alpha_*}}(M, \bar{a})| \geq |X| = \kappa.$$

This contradicts the fact that $b_{\alpha_*} \in \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$.

(b). Let $i < \varepsilon$, and define the formula $\psi(\bar{x}, y)$ as

$$\psi(\bar{x}, y) := (y = x_i).$$

Then $\psi(\bar{a}, M) = \{a_i\}$. It is also clear that $\psi(\bar{x}, y)$ is minimal with this property. This implies that

$$\varphi_{\bar{a}^\frown\langle a_i\rangle}(\bar{a}, M) = \psi(\bar{a}, M).$$

In particular, $|\varphi_{\bar{a}^\frown\langle a_i\rangle}(\bar{a}, M)| = 1 < \kappa$, and consequently $a_i \in \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$.

(c). Suppose $d \in \mathrm{cl}(\bar{\mathrm{a}}{}^\frown\langle \mathrm{b}\rangle, \mathrm{M})$. Thanks to Definition 4.1, $\varphi_{\bar{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\bar{a}, b, M)$ has cardinality less that κ. As $b \in \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$, clearly

$$(*)_1 \qquad \textit{the set } B := \varphi_{\bar{a}^\frown\langle b\rangle}(\bar{a}, M) \textit{ has cardinality } < \kappa.$$

For $b_1 \in B$ let $A_{b_1} := \varphi_{\bar{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\bar{a}, b_1, M)$.

We now show that

$$(*)_2 \qquad \textit{if } b_1 \in B \textit{ then } A_{b_1} \textit{ has cardinality } < \kappa.$$

Assume towards a contradiction that $|A_{b_1}| \geq \kappa$ for some $b_1 \in B$. Reformulating this, means that:

$$M \models \varphi_{\bar{a}^\frown\langle b\rangle}(\bar{a}, b_1) \wedge \exists^\kappa z\ \varphi_{\bar{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\bar{a}, b_1, z).$$

Let

$$\psi(\overline{x}, y) := \varphi_{\overline{a}^\frown\langle b\rangle}(\overline{x}, y) \wedge \exists^\kappa z\ \varphi_{\overline{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\overline{x}, y, z),$$

and recall that $\psi \in \mathscr{L}$ and $\overline{a}^\frown\langle b_1\rangle \in \psi(M)$. Note that $\overline{a}^\frown\langle b\rangle \notin \psi(M)$, as

$$|\varphi_{\overline{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\overline{a}, b, M)| < \kappa.$$

Next we bring the following claim:

$$(*)_{2.1} \qquad \overline{a}^\frown\langle b_1\rangle \notin \varphi_{\overline{a}^\frown\langle b\rangle}(M).$$

To see this we argue by the way of contradiction that $\overline{a}^\frown\langle b_1\rangle \in \varphi_{\overline{a}^\frown\langle b\rangle}(M)$. This implies following the minimality condition that

$$\varphi_{\overline{a}^\frown\langle b\rangle}(M) \subseteq \varphi_{\overline{a}^\frown\langle b_1\rangle}(M) \subseteq \psi(M).$$

Consequently, $\overline{a}^\frown\langle b\rangle \in \psi(M)$. This contradiction completes the proof of $(*)_{2.1}$. But, we have $\langle b_1\rangle \in \varphi_{\overline{a}^\frown\langle b\rangle}(\overline{a}, M)$. This yields that

$$(*)_{2.2} \qquad \overline{a}^\frown\langle b_1\rangle \in \varphi_{\overline{a}^\frown\langle b\rangle}(M).$$

But $(*)_{2.1}$ and $(*)_{2.2}$ together lead to a contradiction. In sum, the desired property $(*)_2$ is valid.

Recalling that κ is regular, it follows that

$$(*)_3 \qquad \bigcup_{b_1\in\beta} \varphi_{\overline{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\overline{a}, b_1, M) \textit{ has cardinality } < \kappa.$$

We will show that

$$(\dagger) \qquad \varphi_{\overline{a}^\frown\langle d\rangle}(\overline{a}, M) \subseteq \bigcup_{b_1\in B} \varphi_{\overline{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\overline{a}, b_1, M),$$

from which it will follow that $\varphi_{\overline{a}^\frown\langle d\rangle}(\overline{a}, M)$ has cardinality less that κ, and hence by definition, $d \in \mathrm{cl}(\overline{a}, M)$. Let us prove $(\dagger)$. To this end, let $d_1 \in \varphi_{\overline{a}^\frown\langle d\rangle}(\overline{a}, M)$. This implies that $\overline{a}^\frown\langle d_1\rangle \in \varphi_{\overline{a}^\frown\langle d\rangle}(M)$. Clearly,

$$M \models \exists y \varphi_{\overline{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\overline{a}, y, d),$$

hence,

$$M \models \exists y \varphi_{\overline{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\overline{a}, y, d_1).$$

This gives us some b_1 so that

$$M \models \varphi_{\bar{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\bar{a}, b_1, d_1).$$

Then $b_1 \in B$, and $d_1 \in \varphi_{\bar{a}^\frown\langle b\rangle^\frown\langle d\rangle}(\bar{a}, b_1, M)$, and consequently, (†) holds. We are done. □

Definition 4.3. Suppose $\mathbf{f} = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$ is a general frame. For $\bar{a} \in {}^{<\theta}M$, we introduce the following:

(1) $\mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}}) = \{\mathrm{c} \in \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M}) : \varphi_{\bar{\mathrm{a}}^\frown\langle \mathrm{b}\rangle} \leq \varphi_{\bar{\mathrm{a}}^\frown\langle \mathrm{c}\rangle}\}$.

(2) Suppose $\mathbf{f}$ is abelian. Then $\mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}}) = \{\mathrm{c}_1 - \mathrm{c}_2 : \mathrm{c}_1, \mathrm{c}_2 \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})\}$.

Hypothesis 4.4. In what follows, and up to the end of this section, let us assume that $\mathbf{f} = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$ is an additive frame, $\kappa \in \Omega$ and $\mathscr{L}$ is closed under $\exists^\kappa x$.

Lemma 4.5. *Let $\bar{a} \in {}^{<\theta}M$ and $b \in \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$. Then $\mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$ is a subset of $\mathrm{cl}(\bar{\mathrm{a}})$ and it is affine.*

Proof. First, recall that "$\mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}}) \subseteq \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$" holds by the definition. For the second phrase, we have to show that $\mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$ is closed under $x - y + z$. To see this, let $c_i \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$ for $i = 1, 2, 3$ and set $c := c_1 - c_2 + c_3$. Since $\varphi_{\bar{a}^\frown\langle b\rangle}(M)$ is affine-closed,

$$\bar{a}^\frown\langle c\rangle = \bar{a}^\frown\langle c_1\rangle - \bar{a}^\frown\langle c_2\rangle + \bar{a}^\frown\langle c_3\rangle \in \varphi_{\bar{a}^\frown\langle b\rangle}(M).$$

According to the minimality, $\varphi_{\bar{a}^\frown\langle b\rangle} \leq \varphi_{\bar{a}^\frown\langle c\rangle}$. Thanks to Lemma 4.2, $c \in \mathrm{cl}(\bar{\mathrm{a}}, \mathrm{M})$. Hence $c \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$, and we are done. □

Lemma 4.6. *The following holds:*

(1) $\mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$ *is a subgroup of M.*

(2) $\mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}}) = \{\mathrm{b} + \mathrm{d} : \mathrm{d} \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})\}$.

Proof. For clause (1), let $c_i \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$, where $i = 1, 2$. Following definition, there are some $b_{i,1}, b_{i,2} \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$ such that $c_i = b_{i,1} - b_{i,2}$. So,

$$\begin{aligned} c_1 - c_2 &= (b_{1,1} - b_{1,2}) - (b_{2,1} - b_{2,2}) \\ &= (b_{1,1} - b_{1,2} + b_{2,2}) - b_{2,1}. \end{aligned}$$

According to Lemma 4.5, we know $b^*_{2,1} = b_{1,1} - b_{1,2} + b_{2,2} \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$, and hence by definition of $\mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$,

$$c_1 - c_2 = b^*_{2,1} - b_{2,1} \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}}).$$

To prove clause (2), let $c \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$. As clearly $b \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$, by clause (1), $-b + c \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$, hence

$$c = b + (-b + c) \in \big\{b + d : d \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})\big\}.$$

Conversely, suppose $d \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$. Due to its definition, there are for some $c_1, c_2 \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$ so that $d = c_1 - c_2$. Consequently, in view of Lemma 4.5, we see

$$b + d = b - c_2 + c_1 \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}}).$$

The equality follows. □

Definition 4.7. Let $\mathbf{f} = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$ be an additive frame such that $\kappa \in \Omega$ and $\mathscr{L}$ is closed under $\exists^\kappa x$. We say $\mathbf{f}$ is very nice, if it satisfies the following extra properties:

1. $\mathscr{L}_{\mathbf{f}} = \mathscr{L}^{\mathrm{pe}}_{\infty,\theta}(\tau_M)$, or just $\mathscr{L}_{\mathbf{f}}$ is closed under $\exists \bar{x}_u$, up to equivalence, where $|u| < \theta$.
2. For every $X \subseteq {}^\varepsilon M$, there is a formula $\varphi_X(\bar{x})$, such that $X \subseteq \varphi_X(M)$, and if $\psi(\bar{x})$ is such that $X \subseteq \psi(M)$, then $\varphi_X(M) \subseteq \psi(M)$.

Discussion 4.8. By a repetition of the argument presented in the proof of Theorem 3.14 we know Definition 4.7(2) holds, when M is an R-module and $\mathbf{f}$ is defined as in 3.14.

Proposition 4.9. *Suppose the additive frame $\mathbf{f} = (M, \mathscr{L}, \lambda, \kappa, \theta, \Omega)$ is very nice. The following conditions hold.*

1. *Assume $b \in M$ and $\bar{a} \in {}^\varepsilon M$. There exists some formula $\varphi(\bar{x}, y)$ with $\lg(\bar{x}) = \varepsilon$ such that $\varphi(\bar{a}, M) = \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$.*
2. *If $c \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$ and $b \in \mathrm{cl}(\bar{\mathrm{a}})$ then $c \in \mathrm{cl}(\bar{\mathrm{a}})$. Moreover, $\mathrm{cl}(\bar{\mathrm{a}})$ is a subgroup of M.*
3. *Let $b \in \mathrm{cl}(\bar{\mathrm{a}})$. Then $\varphi_{\mathrm{grp}(\mathrm{b},\bar{\mathrm{a}})}(M)$ is of cardinality $< \kappa$.*

Proof. (1): Define the formula φ as

$$\varphi(\bar{x}, y) = (\exists y^1, y^2)\Big[\varphi_{\bar{a}^\frown\langle b\rangle}(\bar{x}, y^1) \wedge \varphi_{\bar{a}^\frown\langle b\rangle}(\bar{x}, y^2) \wedge y = y_2 - y_1\Big].$$

We show that φ is as required. By Definition 4.7, $\varphi(\bar{x}, y) \in \mathscr{L}$.

First, suppose that $b \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$, and let $b_1, b_2 \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$ be such that $b = b_2 - b_1$. Then b_1, b_2 witness $M \models \varphi(\bar{a}, b)$. Hence

$$(*)_1 \qquad \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}}) \subseteq \varphi(\bar{\mathrm{a}}, \mathrm{M}).$$

In order to prove the reverse inclusion, suppose that $b \in \varphi(\bar{a}, M)$. This implies that $M \models \varphi(\bar{a}, b)$. Take b_1, b_2 be witness it, i.e., $b = b_2 - b_1$ and

$$M \models \varphi_{\bar{a}^\frown\langle b\rangle}(\bar{a}, b_1) \wedge \varphi_{\bar{a}^\frown\langle b\rangle}(\bar{a}, b_2).$$

On the other hand, for $l = 1, 2$ we have

$$M \models \varphi_{\bar{a}^\frown\langle b\rangle}(\bar{a}, b_l) \Rightarrow \varphi_{\bar{a}^\frown\langle b\rangle} \leq \varphi_{\bar{a}^\frown\langle b_l\rangle},$$

hence $b_l \in \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$. Consequently, $b = b_2 - b_1 \in \mathrm{grp}(\mathrm{b}, \bar{\mathrm{a}})$. As b was arbitrary, we conclude that

$$(*)_2 \qquad \varphi(\bar{a}, M) \subseteq \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}}).$$

By $(*)_1$ and $(*)_2$, we have $\varphi(M, \bar{a}) = \mathrm{afn}(\mathrm{b}, \bar{\mathrm{a}})$, and we are done.

(2): In the light of Lemma 4.5, it suffices to show that $\mathrm{cl}(\bar{\mathrm{a}})$ is a subgroup of M. To this end, let $b_1, b_2 \in \mathrm{cl}(\bar{\mathrm{a}})$, $\varepsilon = \lg(\bar{\mathrm{a}})$ and let $\varphi(\bar{x}, y)$ be as in clause (1). It is easily seen that:

- $\bullet_1$ $M \models \varphi(\bar{a}, b_2 - b_1)$,
- $\bullet_2$ $\varphi(\bar{a}, M) = \{b'' - b' : b'' \in \varphi_{\bar{a}^\frown\langle b_2\rangle}(\bar{a}, M) \text{ and } b' \in \varphi_{\bar{a}^\frown\langle b_1\rangle}(\bar{a}, M)\}$,
- $\bullet_3$ $\varphi(\bar{a}, M)$ has cardinality $< \kappa$.

Thanks to the minimality condition,

$$\varphi_{\bar{a}^\frown\langle b_2 - b_1\rangle}(M) \subseteq \varphi(\bar{a}, M).$$

In other words, $|\varphi_{\bar{a}^\frown\langle b_2 - b_1\rangle}(M)| < \kappa$, which implies that $b_2 - b_1 \in \mathrm{cl}(\bar{\mathrm{a}})$. We have proved

$$b_1, b_2 \in \mathrm{cl}(\bar{\mathrm{a}}) \Rightarrow \mathrm{b}_2 - \mathrm{b}_1 \in \mathrm{cl}(\bar{\mathrm{a}}).$$

Therefore, $\mathrm{cl}(\bar{\mathrm{a}})$ is a subgroup of M.

(3): The proof is similar to the proof of clause (2). □

§ 5. Dichotomy behavior of absolutely co-Hopfian property

This section is devoted to the proof of Theorem 1.3 from the introduction. We start by recalling the definition of (co)-Hopfian modules.

Definition 5.1. Let M be an R-module.

(i) M is called *Hopfian* if its surjective R-endomorphisms are automorphisms.
(ii) M is called *co-Hopfian* if its injective R-endomorphisms are automorphisms.

This can be extended to:

Definition 5.2. Let M be a τ-model.

(i) M is called *Hopfian* if its surjective τ-morphisms are τ-automorphisms.
(ii) M is called *co-Hopfian* if its injective τ-morphisms are τ-automorphisms.

For the convenience of the reader, we present the definition of potentially isomorphic, and discuss some basic facts about them which are used in the paper, and only sketch the proofs in most instances.

Definition 5.3. Let M, N be two structures of our vocabulary. Recall that M and N are called *potentially isomorphic provided they are isomorphic in some forcing extension.*

Recall that a group G is called absolutely co-Hopfian (resp. Hopfian) if it is co-Hopfian (resp. Hopfian) in any further generic extension of the universe.

Discussion 5.4. Suppose M and N are potentially isomorphic. According to [19] and [20] this is holds iff for every $\mathscr{L}_{\infty,\aleph_0}$-sentence φ,

$$(M \models \varphi) \iff (N \models \varphi).$$

We denote this property by $M \equiv_{\mathscr{L}_{\infty,\aleph_0}} N$.

The following is a simple variant of Discussion 5.4. We state it in our context.

Lemma 5.5. *Assume M and N are two τ-structures.*

(1) Suppose for every sentence $\varphi \in \mathscr{L}_{\infty,\aleph_0}(\tau)$,

$$(M \models \varphi) \Longrightarrow (N \models \varphi).$$

Then there is an embedding of M *into* N *in* $V[G_{\mathbb{P}}]$, *where* $\mathbb{P}$ *collapses* $|M| + |N|$ *into* $\aleph_0$.

(2) In clause (1), it suffices to consider sentences φ *in the closure of base formulas under arbitrary conjunctions and* $\exists x$.

Proof. We give a proof for completeness. Let $M = \{a_n : n < \omega\}$ be an enumeration of M in $V[G_{\mathbb{P}}]$. By induction on n we define a sequence $\langle b_n : n < \omega\rangle$ of members of N such that for each formula $\varphi(x_0, \cdots, x_n)$ from $\mathscr{L}_{\infty,\aleph_0}$,

$$(*)_n \qquad \big(M \models \varphi(a_0, \cdots, a_n)\big) \Rightarrow \big(N \models \varphi(b_0, \cdots, b_n)\big).$$

Let

$$\Phi_0(x_0) = \bigwedge\{\varphi(x_0) : M \models \varphi(a_0)\} \in \mathscr{L}_{\infty,\aleph_0}.$$

Then $M \models \Phi_0(a_0)$, and by our assumption, there exists some $b_0 \in N$ such that $N \models \Phi_0(b_0)$. Now suppose that $n < \omega$ and we have defined $b_0, \cdots, b_n$. We are going to define b_{n+1}. Let

$$\Phi_{n+1}(x_0, \ldots, x_{n+1}) = \bigwedge\big\{\varphi(x_0, \ldots, x_{n+1}) : M \models \varphi(a_0, \cdots, a_{n+1})\big\}.$$

Clearly, $\Phi_{n+1}(x_0, \cdots, x_{n+1}) \in \mathscr{L}_{\infty,\aleph_0}$. Also,

$$M \models \exists x_{n+1} \Phi_{n+1}(a_0, \cdots, a_n, x_{n+1}).$$

According to the induction hypothesis $(*)_n$, we have

$$N \models \varphi(b_0, \cdots, b_{n+1})$$

for some $b_{n+1} \in N$. This completes the construction of the sequence $\langle b_n : n < \omega\rangle$. The assignment $a_n \mapsto b_n$ defines a map $f : M \to N$ which is an embedding of M into N. □

Fact 5.6. (1) Let $\lambda \geq \kappa_{\rm beau}$ and let $\langle G_\alpha : \alpha < \lambda\rangle$ be a sequence of τ-models with $|\tau| < \kappa_{\rm beau}(\tau)$,. Then in some forcing extension $\mathbf{V}^{\mathbb{P}}, G_\alpha$ is embeddable into G_β, for some $\alpha < \beta < \lambda$. Here, $\mathbb{P}$ collapses $|G_\alpha| + |G_\beta|$ into $\aleph_0$. Moreover, if $x_\gamma \in G_\gamma$

for $\gamma < \lambda$ then for some $\alpha < \beta < \lambda$, in some $\mathbf{V}^{\mathbb{P}}$ there is an embedding of G_α into G_β mapping x_α to x_β[2].

(2) Suppose M and N are abelian groups, or R-modules, and for every sentence $\varphi \in \mathscr{L}^{\mathrm{ce}}_{\infty,\theta}(\tau)$, we have

$$(M \models \varphi) \Longrightarrow (N \models \varphi).$$

Then there is an embedding of M into N in $V[G_{\mathbb{P}}]$, where $\mathbb{P}$ collapses $|M| + |N|$ into $\aleph_0$.

(3) Moreover, we can strengthen the conclusion of part (2) to the following:
   - $(*)$ there is a $\mathbb{P}$-name π satisfying:
     - $(*)_1$ If $\overline{a} \in ({}^{<\theta}M) \cap V$ then $\pi(\overline{a}) \in ({}^{<\theta}N) \cap V$,
     - $(*)_2$ $\Vdash_{\mathbb{P}}$ π maps $\overline{a} \in ({}^{<\theta}M) \cap V$ onto $\{\overline{b} \in {}^{<\theta}M : \mathrm{rang}(\overline{b}) \subseteq \mathrm{rang}(\pi)\}$.

Proof. For (1), see [12]. Parts (2) and (3) are standard, see for example [19]. □

Now, we are ready to prove:

Theorem 5.7. *The following assertions are valid:*

1. *If M is an abelian group of cardinality $\geq \kappa := \kappa_{\mathrm{beau}}$, then M is not absolutely co-Hopfian, indeed, after collapsing the size of M into ω, there is a one-to-one endomorphism $\varphi \in \mathrm{End(M)}$ which is not onto.*
2. *If M is an R-module of cardinality $\geq \kappa = \kappa_{\mathrm{beau}}(R)$, then M is not absolutely co-Hopfian.*
3. *If M is an τ-model of cardinality $\geq \kappa = \kappa_{\mathrm{beau}}(\tau)$, then M is not absolutely co-Hopfian.*

Proof. Let M be an abelian group or an R-module of size $|M| \geq \kappa$. Thanks to Theorem 3.14, there exists an additive frame $\mathbf{f}$ as there such that $M := M_{\mathbf{f}}$, $\mathscr{L}_{\mathbf{f}} = \mathscr{L}^{\mathrm{ce}}_{\infty,\theta}(\tau)$, $\kappa_{\mathbf{f}} = \kappa$, $\lambda_{\mathbf{f}} = \beth_2(\kappa)^+$ and $\theta_{\mathbf{f}} = \aleph_0$.

The proof splits into two cases:

Case 1: for some $\varepsilon < \theta$, and $\overline{a}, \overline{b} \in {}^{\varepsilon}M$, $\varphi_{\overline{a}}(M) \supsetneqq \varphi_{\overline{b}}(M)$.

---

[2]This explains why [14] gets only indecomposable abelian groups (not endo-rigid).

Consider the τ-models $(M, \bar{b})$ and $(M, \bar{a})$. Let $\varphi \in \mathscr{L}^{\mathrm{ce}}_{\infty,\theta}(\tau)$ be a sentence, and suppose that $(M, \bar{a}) \models \varphi$. As $\varphi_{\bar{a}}(M) \supsetneqq \varphi_{\bar{b}}(M)$, it follows that $(M, \bar{b}) \models \varphi$. Thus by Fact 5.6(2), working in the generic extension by $\mathbb{P} = \mathrm{Col}(\aleph_0, |M|)$, there exists a one-to-one endomorphism $\pi \in \mathrm{End(M)}$ such that $\pi(\bar{a}) = \bar{b}$.

There is nothing to prove if π is not onto. So, without loss of generality we may and do assume that π is onto. Then $\pi, \pi^{-1} \in \mathrm{Aut(M)}$ and $\pi^{-1}(\bar{b}) = \bar{a}$. We claim that $\varphi_{\bar{a}} \leq \varphi_{\bar{b}}$. Due to the minimality condition for $\varphi_{\bar{a}}$, it is enough to show that $\bar{a} \in \varphi_{\bar{b}}(M)$. To this end, recall that $\bar{b} \in \varphi_{\bar{b}}(M)$. By definition, $M \models \varphi_{\bar{b}}[\bar{b}]$. In the light of Lemma 2.15(2) we observe that

$$M \models \varphi_{\bar{b}}[\pi^{-1}(\bar{b})] = \varphi_{\bar{b}}[\bar{a}].$$

By definition, this means that $\bar{a} \in \varphi_{\bar{b}}(M)$, as requested. Consequently, $\varphi_{\bar{a}}(M) \subseteq \varphi_{\bar{b}}(M)$, which contradicts our assumption.

Case 2: not case 1.

Given two sets $A, B \subseteq M$, by $A \| B$ we mean that $A \subseteq B$ or $B \subseteq A$. So in this case, the following holds:

$$(*): \qquad \forall \varepsilon < \kappa \text{ and } \forall \bar{a}, \bar{b} \in {}^{\varepsilon}M \text{ we have } \Big(\varphi_{\bar{a}}(M) \| \varphi_{\bar{b}}(M) \Rightarrow \varphi_{\bar{a}}(M) = \varphi_{\bar{b}}(M)\Big).$$

Set $\Gamma = \{\varphi_{\bar{a}} : \bar{a} \in {}^{<\theta}M\}$. Now, we have the following easy claim.

Claim 5.8. Γ *is a set of cardinality* $< \kappa$.

Proof. To see this, set $\Gamma_\varepsilon := \{\varphi_{\bar{a}} : \bar{a} \in {}^{\varepsilon}M\}$. Clearly, $\Gamma = \bigcup_{\varepsilon<\theta} \Gamma_\varepsilon$, and since $\theta < \kappa$ and κ is regular, it suffices to show that $|\Gamma_\varepsilon| < \kappa$ for all $\varepsilon < \theta$. Suppose not and search for a contradiction. Take $\varepsilon < \theta$ be such that $|\Gamma_\varepsilon| \geq \kappa$. This enables us to find a sequence $\langle \bar{a}_\alpha : \alpha < \kappa \rangle$ in ${}^{\varepsilon}M$ such that

- $\bullet_1$ $\forall \alpha < \kappa,\ \varphi_{\bar{a}_\alpha} \in \Gamma_\varepsilon$
- $\bullet_2$ $\forall \alpha \neq \beta,\ \varphi_{\bar{a}_\alpha}(M) \neq \varphi_{\bar{a}_\beta}(M)$.

We apply the property presented in Definition 3.1(5)(a) to the family $\{\varphi_{\bar{a}_\alpha}\}_{\alpha<\kappa}$, to find some $\alpha < \beta < \kappa$ such that $\varphi_{\bar{a}_\beta}(M) \supseteq \varphi_{\bar{a}_\alpha}(M)$. By $(*)$, this implies that $\varphi_{\bar{a}_\beta}(M) = \varphi_{\bar{a}_\alpha}(M)$, which contradicts $\bullet_2$. $\square$

Let $\chi := |M| \geq \kappa$, and let $\mathbb{P} := \mathrm{Col}(\aleph_0, \chi)$. Forcing with $\mathbb{P}$, collapses $|M|$ into $\aleph_0$, i.e., for any $\mathbb{P}$-generic filter $G_{\mathbb{P}}$ over V, we have

$$V[G_{\mathbb{P}}] \models \text{“}M \text{ is countable”}.$$

We are going to show there is in $V[G_{\mathbb{P}}]$, there exists a 1-1 map $\pi : M \to M$ which is not surjective. To this end, we define approximations to the existence of such π:

⊞ Let AP be the set of all triples $(\overline{a}, \overline{b}, c)$ such that:

(a) $\overline{a}, \overline{b} \in {}^{\varepsilon}M$ for some $\varepsilon < \theta$,

(b) $\varphi_{\overline{a}}(\overline{x}) \equiv \varphi_{\overline{b}}(\overline{x})$ (in M),

(c) $c \in M$ is such that $c \notin \mathrm{cl}(\overline{\mathrm{b}}, \mathrm{M})$, i.e., $\varphi_{\overline{b}^\frown\langle c\rangle}(M)$ has cardinality $\geq \kappa$.

Claim 5.9. $\mathrm{AP} \neq \emptyset$.

Proof. According to Lemma 4.2(a), $\mathrm{cl}(\overline{\mathrm{a}})$ has cardinality $< \kappa$. In particular, $|\mathrm{cl}(\emptyset)| < \kappa$, and hence as $|M| \geq \kappa$, we can find some $c \in M \setminus \mathrm{cl}(\emptyset)$, and consequently, $(\langle\rangle\langle\rangle, c) \in \mathrm{AP}$. The claim follows. □

Next, we bring the following claim, which plays the key role in our proof.

Claim 5.10. *Suppose* $(\overline{a}, \overline{b}, c) \in \mathrm{AP}$ *and* $d_1 \in M$. *Then there is some* $d_2 \in M$ *such that*

$$\left(\overline{a}^\frown\langle d_1\rangle, \overline{b}^\frown\langle d_2\rangle, c\right) \in \mathrm{AP}.$$

Proof. Recall that in M we have $\varphi_{\overline{a}}(\overline{x}) \equiv \varphi_{\overline{b}}(\overline{x})$. First, we use this to find $d \in M$ such that

$$(\dagger) \qquad M \models \varphi_{\overline{a}^\frown\langle d_1\rangle}(\overline{b}, d).$$

Indeed, we look at the formula

$$\psi(\overline{x}) := \exists y \varphi_{\overline{a}^\frown\langle d_1\rangle}(\overline{x}, y).$$

Since $\overline{a} \in \psi(M)$, and due to the minimality of $\varphi_{\overline{a}}$ with respect to this property, we should have

$$\overline{b} \in \varphi_{\overline{b}}(M) = \varphi_{\overline{a}}(M) \subseteq \psi(M).$$

In other words,

$$M \models \exists y \varphi_{\overline{a}^\frown\langle d_1\rangle}(\overline{b}, y).$$

Hence, for some $d \in M$ we must have $M \models \varphi_{\bar{a}^\frown\langle d_1\rangle}(\bar{b}, d)$. So, (†) is proved. From this, $\bar{b}^\frown\langle d\rangle \in \varphi_{\bar{a}^\frown\langle d_1\rangle}(M)$. This implies, using the minimality condition on formulas, that $\varphi_{\bar{a}^\frown\langle d_1\rangle}(M) \subseteq \varphi_{\bar{b}^\frown\langle d\rangle}(M)$. Combining this along with $(*)$ yields that

$$\varphi_{\bar{a}^\frown\langle d_1\rangle}(M) = \varphi_{\bar{b}^\frown\langle d\rangle}(M).$$

First, we deal with the case $d \in \mathrm{cl}(\bar{\mathrm{b}}, \mathrm{M})$. Thanks to Lemma 4.2(c), and recalling that Ω contains $\{1, \kappa\}$, we know $\mathrm{cl}(\bar{\mathrm{b}}^\frown\langle \mathrm{d}\rangle) = \mathrm{cl}(\bar{\mathrm{b}})$. Consequently, $c \notin \mathrm{cl}(\bar{\mathrm{b}}^\frown\langle \mathrm{d}\rangle)$. Following definition, one has

$$\left(\bar{a}^\frown\langle d_1\rangle, \bar{b}^\frown\langle\rangle, c\right) \in \mathrm{AP},$$

and we are done by taking $d_2 = d$. So, without loss of generality let us assume that $d \notin \mathrm{cl}(\bar{\mathrm{b}}, \mathrm{M})$. It follows that the set

$$\mathbf{I} := \left\{c' \in M : M \models \varphi_{\bar{b}^\frown\langle d\rangle}[\bar{b}, c']\right\}$$

has cardinality $\geq \kappa$. Therefore, there is $c' \in \mathbf{I} \setminus \mathrm{cl}(\bar{\mathrm{b}}^\frown\langle \mathrm{d}\rangle, \mathrm{M})$. By the minimality condition on $\varphi_{\bar{b}^\frown\langle d\rangle}$ and since $\bar{b}^\frown\langle c'\rangle \in \varphi_{\bar{b}^\frown\langle d\rangle}(M)$, we have $\varphi_{\bar{b}^\frown\langle d\rangle}(M) \subseteq \varphi_{\bar{b}^\frown\langle c'\rangle}(M)$. Now, we use this along with $(*)$ and deduce that

$$\varphi_{\bar{b}^\frown\langle d\rangle}(M) = \varphi_{\bar{b}^\frown\langle c'\rangle}(M).$$

Then, in the same vein as above, we can find some d' such that $\varphi_{\bar{b}^\frown\langle c,d'\rangle}(M) = \varphi_{\bar{b}^\frown\langle c',d\rangle}(M)$. This yields that $c \notin \mathrm{cl}(\bar{\mathrm{b}}^\frown\langle \mathrm{d}'\rangle, \mathrm{M})$, hence $d_2 = d'$ is as required. □

In $V[G_{\mathbb{P}}]$, M is countable. Let us list M as $\{a_i : i < \omega\}$. We define $\pi : M \to M$ by evaluating π at a_i. We do this by induction, in such a way that for some fixed $c \in M$ and all $n < \omega$, if $\pi(a_i) = b_i$, then

$$(\dagger\dagger)_n \qquad \left(\langle a_i : i < n\rangle, \langle b_i : i < n\rangle, c\right) \in \mathrm{AP}.$$

Recall from Claim 5.9 and its proof that there is some c in M such that $(\langle\rangle\langle\rangle, c) \in \mathrm{AP}$. Let us apply Claim 5.10 to

- $\bar{a} := \langle\rangle$
- $\bar{b} := \langle\rangle$
- $d_1 := a_0$.

This gives us an element $b_0 \in M$ such that $\big(\langle a_0\rangle, \langle b_0\rangle, c\big) \in \mathrm{AP}$. Let $f(a_0) = b_0$. Now, suppose inductively we have defined $\pi(a_i) = b_i$ for all $i < n$ such that $(\dagger\dagger)_n$ is true. Let us apply Claim 5.10 to

- $\overline{a} := \langle a_i : i < n\rangle$,
- $\overline{b} := \langle b_i : i < n\rangle$,
- $d_1 := a_n$.

This gives us an element $b_n \in M$ such that

$$\big(\overline{a}^\frown\langle a_n\rangle, \overline{b}^\frown\langle b_n\rangle, c\big) \in \mathrm{AP}.$$

Let $\pi(a_n) = b_n$, and note that $(\dagger\dagger)_{n+1}$ holds as well. This completes the inductive definition of π.

The proof becomes complete if we can show the following three items are satisfied:

$(\maltese)_1$ π is a homomorphism.

$(\maltese)_2$ π is 1-to-1.

$(\maltese)_3$ π is not surjective.

Let us check these:

$(\maltese)_1$ Suppose $\varphi = \varphi_{\overline{a}}$ is a first order formula, hence, without loss of generality, the length of $\overline{a}$ is finite and we can enlarge $\overline{a}$ to $\langle a_i : i < n\rangle$ for some $n < \omega$. Recall that $b_i := \pi(a_i)$. By the construction

$$\big(\langle a_j : j < n\rangle, \langle b_j : j < n\rangle, c\big) \in \mathrm{AP}.$$

Assume $M \models \varphi(\overline{a})$. We show that $M \models \varphi(\overline{b})$. We have $\overline{b} \in \varphi_{\overline{b}}(M)$ and by our construction, $\varphi_{\overline{b}}(M) = \varphi_{\overline{a}}(M) = \varphi(M)$, hence $\overline{b} \in \varphi(M)$, which means that $M \models \varphi(\overline{b})$. From this, it immediately follows that π is a homomorphism.

$(\maltese)_2$ Following the construction given by Claim 5.10, we can always find b_n so that $b_n \neq b_i$ for all $i < n$. So, f is 1-to-1.

$(\maltese)_3$ Suppose by the way of contradiction that π is surjective. In particular, we can find some $n < \omega$ such that $c = \pi(a_n) = b_n$. In the light of Lemma 4.2(b) we observe that

$$c = b_n \in \mathrm{cl}\big(\langle \mathrm{b_i} : \mathrm{i} < \mathrm{n}+1\rangle, \mathrm{M}\big),$$

which contradicts the following fact

$$\big(\langle a_i : i < n+1\rangle, \langle b_i : i < n+1\rangle, c\big) \in \mathrm{AP}.$$

The proof is now complete.

(3): The proof of this item is similar to the first part. □

§ 6. A new construction of Hopfian groups

In this section we give a new construction of absolutely Hopfian groups. Let us start with the following well-known definition.

Definition 6.1. A group G is pure in a torsion-free abelian group H if $G \subseteq H$ and $nG = nH \cap G$ for every $n \in \mathbb{Z}$.

Notation 6.2. i) Let G be an abelian torsion free group and Y be a subset of G. By $\mathrm{PC}_G(Y)$ we mean the minimal pure subgroup of G which includes Y.

ii) The notation $\mathbb{Q}_R$ stands for the field of fractions of the integral domain R.

As another application of infinitary logic, we now present the promised construction of Hopfian groups:

Proposition 6.3. *Suppose* $\mathbf{f}$ *is an endomorphism of* M *such that:*

- (a) R *is a commutative torsion-free ring with* 1_R*,*
- (b) $\mathbf{f}$ *maps* $\varphi_0(M)$ *onto* $\varphi_0(M)$*,*
- (c) M *is a torsion-free* R*-module,*
- (d) $\varphi_\varepsilon(x) \in \mathscr{L}^{ep}_{\infty,\aleph_0}(\tau_R)$*, for* $\varepsilon \le \varepsilon_*$ *where ep stands for existential positive (or just generated from the atomic formulas by* $\exists$ *and* $\wedge$*),*
- (e) $\langle \varphi_\varepsilon(M) : \varepsilon \le \varepsilon_* \rangle$ *is* $\subseteq$*-decreasing continuous of sub-modules of* M*,*
- (f) $\varphi_{\varepsilon_*}(M) = \{0_M\}$*,*
- (g) $\varphi_\varepsilon(M)/\varphi_{\varepsilon+1}(M)$ *is torsion-free of rank* 1*,*
- (h) $x_\varepsilon \in \varphi_\varepsilon(M) \backslash \varphi_{\varepsilon+1}(M)$ *for* $\varepsilon < \varepsilon_*$*,*
- (i) $\varphi_0(M) = \mathrm{PC}_M(\{x_\varepsilon : \varepsilon < \varepsilon_*\})$*.*

Then $\mathbf{f} \restriction \varphi_0(M)$ *is one-to-one.*

Proof. For $\varepsilon < \varepsilon_*$ let $y_\varepsilon := \mathbf{f}(x_\varepsilon)$. As $\varphi_\varepsilon(x) \in \mathscr{L}_{\infty,\aleph_0}(\tau_R)$ is existential positive, clearly $\mathbf{f}$ maps $\varphi_\varepsilon(M)$ into $\varphi_\varepsilon(M)$, hence $y_\varepsilon \in \varphi_\varepsilon(M)$. Set

$$\mathscr{U} := \{\varepsilon < \varepsilon_* : y_\varepsilon \in \varphi_{\varepsilon+1}(M)\}.$$

Claim 6.4. $\mathscr{U} = \emptyset$.

Proof. Assume towards a contradiction that $\mathscr{U} \neq \emptyset$ and let ζ be the first member of $\mathscr{U}$. As we are assuming "$\mathbf{f}$ maps $\varphi_0(M)$ onto $\varphi_0(M)$" there is $z \in \varphi_0(M)$ such that $\mathbf{f}(z) = x_\zeta$. Let $R^+ := R \setminus \{0\}$. Since $z \in \varphi_0(M)$ by items (i) and (c) of the assumption of the claim we can find

- $b \in R^+, n \in \mathbb{N}$
- $\varepsilon_0 < \ldots < \varepsilon_n < \varepsilon_*$ and
- $a_0, \ldots, a_n \in R^+$,

such that

$$M \models \text{“}bz = \sum_{\ell \leq n} a_\ell x_{\varepsilon_\ell}\text{”}.$$

Now applying $\mathbf{f}$ we have

$$M \models \text{“}bx_\zeta = \sum_{\ell \leq n} a_\ell y_{\varepsilon_\ell}\text{”}.$$

We consider two cases:

<u>Case 1</u>: $\varepsilon_0 < \zeta$.

To see this, note that $bx_\zeta \in \varphi_\zeta(M) \subseteq \varphi_{\varepsilon_0+1}(M)$ and for $\ell > 0$ we have $a_\ell y_{\varepsilon_\ell} \in \varphi_{\varepsilon_\ell}(M) \subseteq \varphi_{\varepsilon_0+1}(M)$, so as $bx_\zeta = \sum\limits_{\ell \leq n} a_\ell y_{\varepsilon_\ell}$ we get $a_0 y_{\varepsilon_0} \in \varphi_{\varepsilon_0+1}(M)$ and as $a_0 \in R^+$ by clause (f) we get $y_{\varepsilon_0} \in \varphi_{\varepsilon_0+1}(M)$. This contradicts $\varepsilon_0 < \zeta = \min(\mathscr{U})$.

<u>Case 2</u>: $\varepsilon_0 \geq \zeta$.

On the one hand as $b \in R^+$ clearly $bx_\zeta \notin \varphi_{\zeta+1}(M)$. On the other hand

(2.1) $\varepsilon_\ell > \zeta \Rightarrow a_\ell y_{\varepsilon_\ell} \in \varphi_{\varepsilon_\ell}(M) \subseteq \varphi_{\varepsilon_0+1}(M) \subseteq \varphi_{\zeta+1}(M)$ and

(2.2) $\varepsilon_\ell = \zeta \Rightarrow \ell = 0 \Rightarrow y_{\varepsilon_\ell} = y_\zeta \in \varphi_{\zeta+1}(M)$.

Hence $\sum\limits_{\ell \leq n} a_\ell y_{\varepsilon_\ell} \in \varphi_{\zeta+1}(M)$. Combining these together we get a contradiction to

$$M \models \text{“}bx_\zeta = \sum_{\ell \leq n} a_\ell y_{\varepsilon_\ell}\text{”},$$

and we are done. The claim follows. □

Now, if $x \in \varphi_0(M)\backslash\{0_M\}$, then by clause (i) of the assumption for some $b \in R^+, n$ and $\varepsilon_0 < \ldots < \varepsilon_n < \varepsilon_*$ and $a_\ell \in R^+$ for $\ell \leq n$ we have

$$M \models \text{“}bx = \sum_{\ell \leq n} a_\ell x_{\varepsilon_\ell}\text{”}$$

hence

$$b\mathbf{f}(x) = \sum_{\ell \leq n} a_\ell \mathbf{f}(x_{\alpha_\ell}) \in a_0 y_{\varepsilon_0} + \varphi_{\varepsilon_0+1}(M).$$

As $a_0 \in R^+$, and in the light of clause (h) we observe $y_{\varepsilon_0} \in \varphi_{\varepsilon_0}(M_{\varepsilon_0})\backslash\varphi_{\varepsilon_0+1}(M)$. By items (f) and (g) of the assumption, $a_0 y_{\varepsilon_0} \notin \varphi_{\varepsilon_0+1}(M)$. Due to the previous sentence, we know $b\mathbf{f}(x) \neq 0_M$. Hence $\mathbf{f}(x) \neq 0_M$. So, we have proved that $\mathbf{f}$ maps any non-zero member of $\varphi_0(M)$ into a non-zero member of $\varphi_0(M)$. In other words, $\mathbf{f} \restriction \varphi_0(M)$ is one-to-one as promised. □

References

[1] M. Asgharzadeh, M. Golshani, and Saharon Shelah, *Co-Hopfian and boundedly endo-rigid mixed abelian groups*, Pacific Journal of math, to appear.

[2] Reinhold Baer, *Groups without proper isomorphic quotient groups*, Bull. Amer. Math. Soc. **50** (1944), 267-278.

[3] R. A. Beaumont and R. S. Pierce, *Partly transitive modules and modules with proper isomorphic submodules*, Trans. Amer. Math. Soc. **91** (1959), 209-219.

[4] R. A. Beaumont, Groups with isomorphic proper subgroups, Bull. Amer. Math. Soc. **51** (1945) 381-387.

[5] Jon Barwise, *Back and forth through infinitary logic*, Studies in Model Theory, Math. Assoc. Amer., Buffalo, N.Y., , MAA Studies in Math., Vol. **8**, 1973, 5-34.

[6] G. Braun and L. Strüngmann, *The independence of the notions of Hopfian and co-Hopfian abelian p-groups*, Proc. Amer. Math. Soc. **143** (2015), no. 8, 3331-3341.

[7] G. Baumslag, *Hopficity and Abelian Groups*, in Topics in Abelian Groups (Proc. Sympos., New Mexico State Univ.), Scott Foresman and Co., Chicago, (1963) 331-335.

[8] A. L. S. Corner, *Three examples on hopficity in torsion-free abelian groups*, Acta Math. Hungarica **16**, No. 3-4(1965), 303-310.

[9] Peter Crawley, *An infinite primary abelian group without proper isomorphic subgroups* , Bull. Amer. Math. Soc. **68** (1962), no. 5, 463-467.

[10] Paul C. Eklof and Alan Mekler, *Almost free modules: Set theoretic methods*, North–Holland Mathematical Library, vol. **65**, North–Holland Publishing Co., Amsterdam, 2002, Revised Edition.

[11] P. Erdos, A. Hajnal, A. Mate and R. Rado, *Combinatorial set theory: partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics, vol. **106**, (1984), Amsterdam: North-Holland Publishing Co.

[12] Paul C. Eklof and Saharon Shelah, *Absolutely rigid systems and absolutely indecomposable groups*, abelian groups and modules (Dublin, 1998), Trends in Mathematics, Birkhäuser, Basel, 1999, 257–268.

[13] Laszlo Fuchs, *Abelian groups*, Springer Monographs in Mathematics. Springer, Cham, 2015.

[14] László Fuchs, *Infinite abelian Groups*, vol. I, II, Academic Press, New York, 1970, 1973.

[15] B. Goldsmith and K. Gong. *Algebraic entropies, Hopficity and co-Hopficity of direct sums of Abelian groups*, Topol. Algebra Appl. **3** (2015), 75-85.

[16] Rüdiger Göbel and Saharon Shelah, *Absolutely Indecomposable Modules, Proceedings of the American Mathematical Society*, **135** (2007), 1641-1649.

[17] H. Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv., **14** (1941), 257-309.

[18] I. Kaplansky, *A note on groups without isomorphic subgroups*, Bull. Amer. Math. Soc. **51** (1945) 529-530.

[19] D. Marker, *Lectures on infinitary model theory*, Lecture Notes in Logic, **46** Association for Symbolic Logic, Chicago, IL; Cambridge University Press, Cambridge, 2016.

[20] M.E. Nadel, J. Stavi, *$L_{\infty\lambda}$-equivalence, isomorphism and potential isomorphism*, Trans. Amer. Math. Soc. **236** (1978), 51-74.

[21] M.E. Nadel, *Scott heights of abelian groups*, Journal of Symbolic Logic **59** (1994), 1351–1359.

[22] G. Paolini and S. Shelah, *On the existence of uncountable Hopfian and co-Hopfian abelian groups*, to appear Israel J. Math. arXiv: 2107.11290.

[23] Jean-Pierre Serre, *How to use finite fields for problems concerning infinite fields*, arXiv:0903.0517 [math.AG].

[24] W. Magnus, A. Karras and D. Solitar, *Combinatorial group theory*, Interscience, New York, 1966.

[25] Saharon Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel Journal of Mathematics **18** (1974), 243–256.

[26] Saharon Shelah, *Interpreting set theory in the endomorphism semi-group of afree algebra or in a category* Ann. Sci. Univ. Clermont, (**60** Math. No. 13), (1976). 1-29.

[27] Saharon Shelah, *Modules and infinitary logics, Groups and model theory*, Contemp. Math., vol. **576**, Amer. Math. Soc., Providence, RI, (20120) 305-316.

[28] Saharon Shelah, *The lazy model theorist's guide to stability*, Logique et Analyse, 18 Anne, Vol. **71-72** (1975), 241-308.

[29] Saharon Shelah, *Better quasi-orders for uncountable cardinals*, Israel J. Math. **42** (1982), no. 3, 177-226.

[30] W. Szmielew, *Elementary properties of abelian groups*, Fundamenta Mathematica, **41** (1954), 203-271.

[31] Wolmer V. Vasconcelos, *On finitely generated flat modules*, Trans. Amer. Math. Soc. **138** (1969), 505-512.

Hakimiyeh, Tehran, Iran.

E-mail address: mohsenasgharzadeh@gmail.com

School of Mathematics,, Institute for Research in Fundamental Sciences (IPM),, P.O. Box: 19395–5746, Tehran, Iran.

E-mail address: golshani.m@gmail.com

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 9190401, Jerusalem, Israel; and, Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

URL: https://shelah.logic.at/

E-mail address: shelah@math.huji.ac.il